Mathematics Level 9

Chapter 4 — Space: similarity, the trigonometric ratios and geometric algorithms · 3 subtopics

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Contents

Similar right-angled triangles and the constancy of the sine, cosine and tangent ratios

4A · Chapter 4 — Space: similarity, the trigonometric ratios and geometric algorithms · Level 9

Content description `VC2M9SP01` (word for word): *recognise the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles using properties of similarity*

Achievement standard anchors (AS 9 and AS 11): *They solve problems involving ratio, similarity and scale in two-dimensional situations.* (AS 9) and *Students apply Pythagoras' theorem and use trigonometric ratios to solve problems involving right-angled triangles.* (AS 11)

Note on the double anchor (D7): AS 11 is the Level 9 space sentence this section begins to earn: 4A builds the constancy of the ratios that the later sections of this chapter use to solve right-angled triangles. AS 9 anchors here as well, because the whole argument is an argument about ratio, similarity and scale in two dimensions.

Elaboration ids for linkage: `VC2M9SP01.E01`, `VC2M9SP01.E02`, `VC2M9SP01.E03`, `VC2M9SP01.E04`

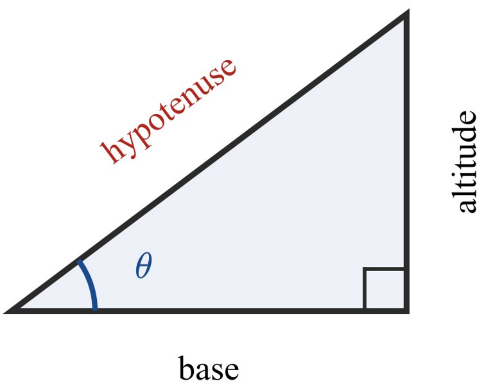
Prerequisite pointer: 2B (the gradient of a straight line) and the Level 8 tools — similarity (`VC2M8SP01`) and Pythagoras' theorem (`VC2M8M06`) — rehearsed here as tools, not re-taught.

Note on figures: every triangle in this section is drawn from its stated side lengths and every line from its own rule (R7); every context is a generic one, so no real data ships and no source line is owed (R1, R2).

Theory

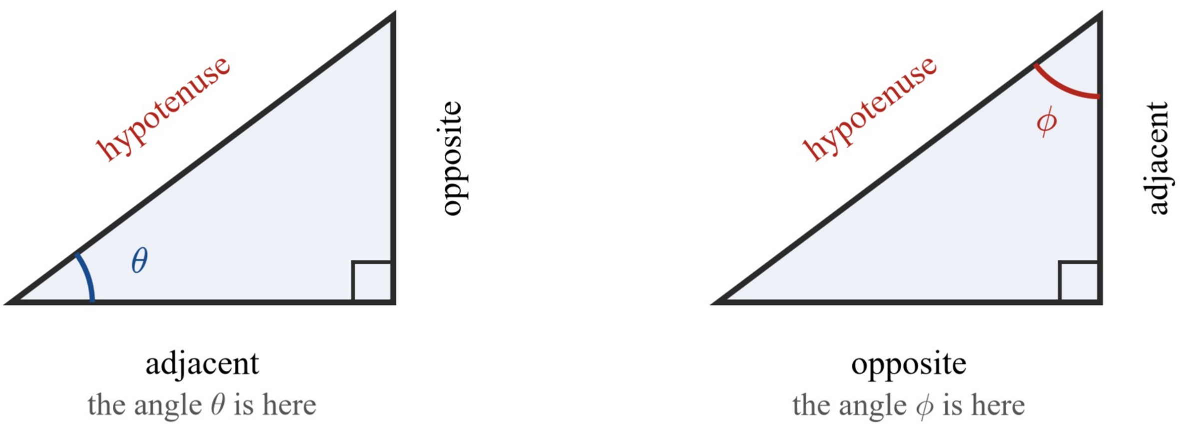
Naming the sides of a right-angled triangle (E01)

In a right-angled triangle, the side opposite the right angle is the **hypotenuse** — always the longest side. The two sides that make the right angle are called the **base** (the side the triangle rests on) and the **altitude** (the side standing up from the base). Pythagoras' theorem connects the three: $\text{base}^2 + \text{altitude}^2 = \text{hypotenuse}^2$.



A right-angled triangle resting on its base, with the base labelled below, the altitude labelled at the right, the hypotenuse labelled along the slanting side, a small square marking the right angle and the angle theta marked at the left

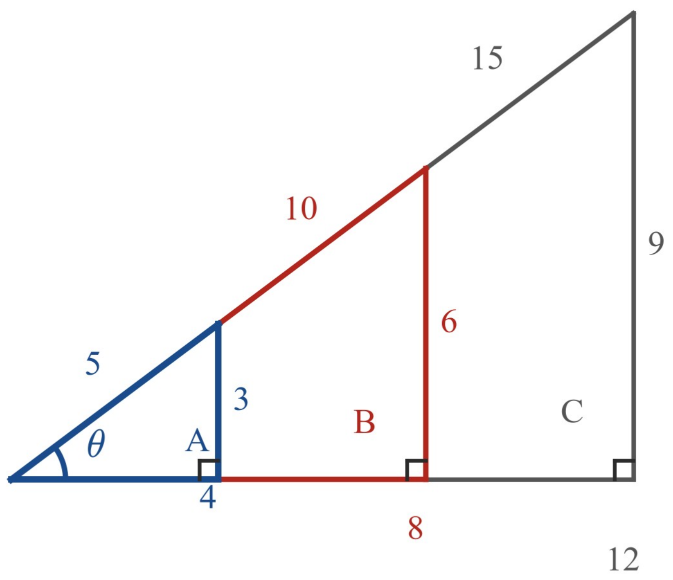
Mark one of the acute angles, and the base and altitude take a second pair of names. The **opposite** side is the side across from the marked angle; the **adjacent** side is the side next to the marked angle that is not the hypotenuse. The hypotenuse keeps its name no matter which angle is marked; opposite and adjacent swap when the marked angle changes.



The same right-angled triangle drawn twice: on the left the angle theta is marked at the bottom left, so the base is adjacent and the altitude is opposite; on the right the angle phi is marked at the top, so the altitude is adjacent and the base is opposite; the hypotenuse keeps its name in both

Nested similar triangles on a coordinate plane (E02)

Now nest three right-angled triangles inside one another, all sharing the angle θ at the origin, all with their bases along the horizontal axis and their altitudes vertical. Because the three altitudes are all vertical, they are parallel, so each triangle has the same three angles as the others: the shared angle θ, a right angle, and therefore the same third angle. Same angles means similar triangles (Level 8 similarity), and the hypotenuses of all three lie along one line through the origin.



Three nested similar right-angled triangles A, B and C sharing the angle theta at the origin: A with sides 3, 4 and 5 in blue, B with sides 6, 8 and 10 in red, C with sides 9, 12 and 15 in grey, each with a small square at its right angle

Read the side lengths off the figure and form the three ratios of corresponding sides:

Triangle	opposite (altitude)	adjacent (base)	hypotenuse	$\frac{\text{opposite}}{\text{hypotenuse}}$	$\frac{\text{adjacent}}{\text{hypotenuse}}$	$\frac{\text{opposite}}{\text{adjacent}}$
A	3	4	5	$\frac{3}{5}=0.6$	$\frac{4}{5}=0.8$	$\frac{3}{4}=0.75$

Triangle	opposite (altitude)	adjacent (base)	hypotenuse	$\frac{\text{opposite}}{\text{hypotenuse}}$	$\frac{\text{adjacent}}{\text{hypotenuse}}$	$\frac{\text{opposite}}{\text{adjacent}}$
B	6	8	10	$\frac{6}{10}=0.6$	$\frac{8}{10}=0.8$	$\frac{6}{8}=0.75$
C	9	12	15	$\frac{9}{15}=0.6$	$\frac{12}{15}=0.8$	$\frac{9}{12}=0.75$

B is A enlarged by a scale factor of 2, and C is A enlarged by a scale factor of 3 — the sides grow, but every ratio stays exactly the same. That is the big idea of this section: **for a fixed angle, the ratios of the sides are constant, no matter how large the triangle is.** The ratios belong to the angle, not to the triangle.

Naming the three constant ratios: sine, cosine, tangent

Because the ratios depend only on the angle, they earn names. For a marked angle θ in any right-angled triangle:

$$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}\ \cos\theta=\frac{\text{adjacent}}{\text{hypotenuse}}\ \tan\theta=\frac{\text{opposite}}{\text{adjacent}}$$

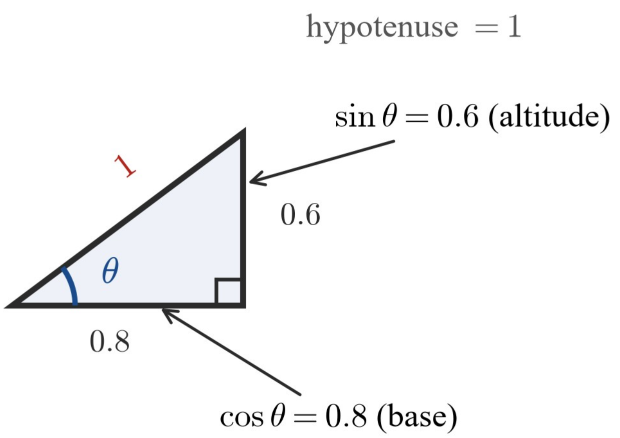
For the whole nested family above, $\sin\theta=0.6$, $\cos\theta=0.8$ and $\tan\theta=0.75$ — the same three numbers in A, in B and in C. A different angle would give a different triple of numbers; the same angle always gives the same triple. (Later sections of this chapter put these names to work finding missing sides; the job here is to see why the ratios are constant in the first place.)

The triangle with hypotenuse one (E03)

Take any right-angled triangle with angle θ and shrink or enlarge it until the hypotenuse is exactly 1. In that triangle the definitions become short:

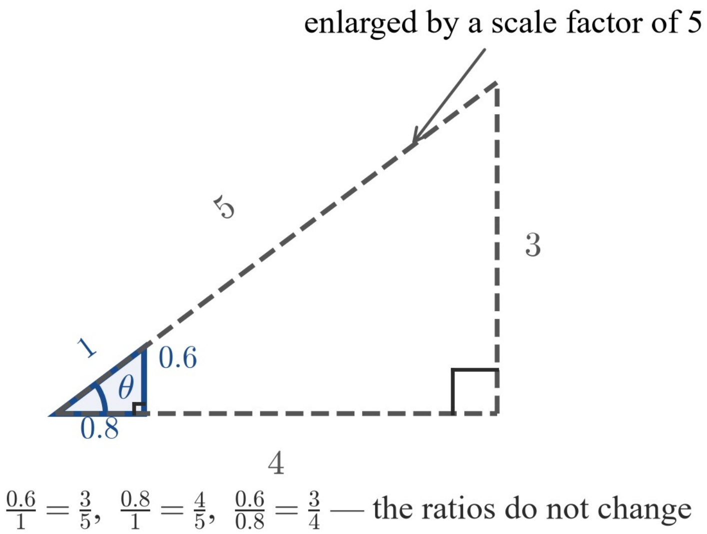
$\sin\theta=\frac{\text{altitude}}{1}=\text{altitude}$ and $\cos\theta=\frac{\text{base}}{1}=\text{base}$.

So the sine of θ is exactly the altitude of the right-angled triangle with hypotenuse 1, and the cosine of θ is exactly the base of that same triangle.



The right-angled triangle with hypotenuse 1: the base is 0.8 and the altitude is 0.6, with arrows labelling the altitude as sin theta equals 0.6 and the base as cos theta equals 0.8

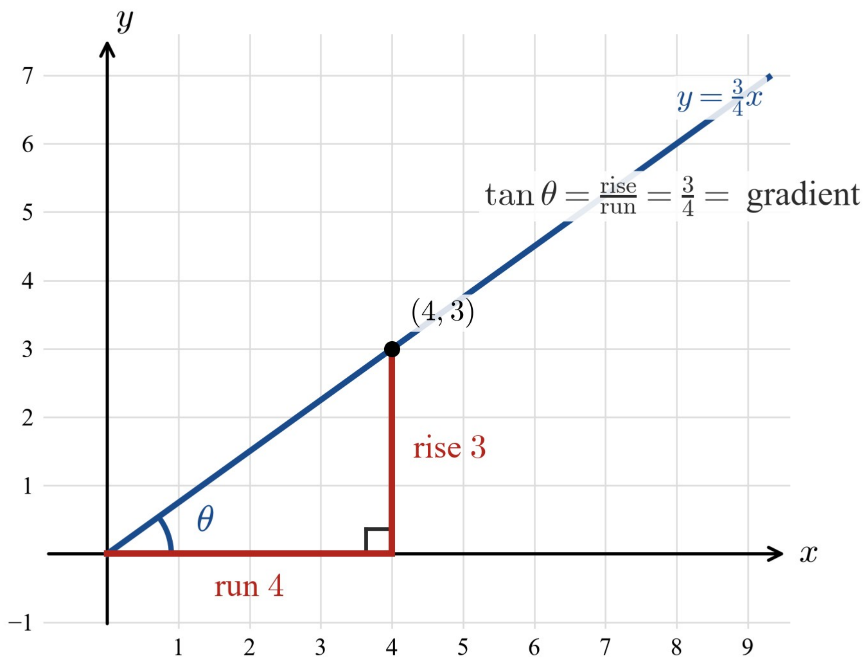
Enlarging the unit triangle changes its sides but not its ratios, because enlargement makes a similar triangle. Enlarge the 0.6–0.8–1 triangle by a scale factor of 5 and the sides become 3, 4 and 5; the ratios $\frac{0.6}{1}=\frac{3}{5}$, $\frac{0.8}{1}=\frac{4}{5}$ and $\frac{0.6}{0.8}=\frac{3}{4}$ do not change.



The 0.6, 0.8, 1 triangle in solid blue nested inside its enlargement by a scale factor of 5, the 3, 4, 5 triangle in dashed grey, sharing the angle theta at the origin, with the equal ratios written below

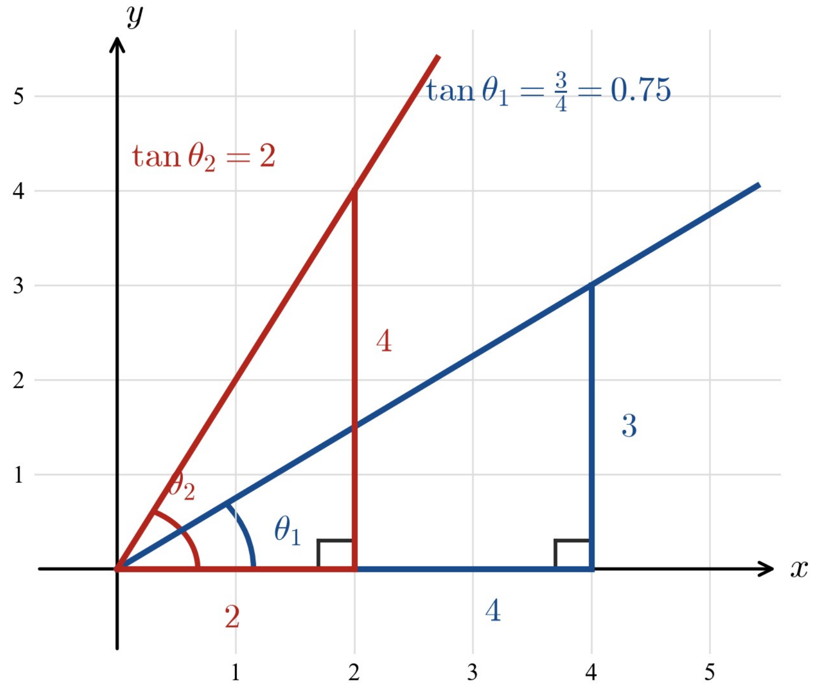
The tangent of the angle a line makes with the horizontal axis is its gradient (E04)

Return to the nested figure. In every triangle of the family, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{altitude}}{\text{base}}$, and altitude over base is exactly rise over run. The hypotenuses lie on the line $y = \frac{3}{4}x$, and 2B says the gradient of that line is rise over run $= \frac{3}{4}$. So in this figure, $\tan\theta = \frac{3}{4}$ = the gradient of the line.



The line y equals three-quarters x on a coordinate plane, with the right-angled triangle from the origin to the point (4, 3): the run of 4 along the x-axis in red, the rise of 3 in red, the angle theta at the origin, and the statement tan theta equals rise over run equals three-quarters equals gradient

That connection is general: **the tangent of the angle at which a line meets the positive direction of the horizontal axis is the gradient of the line.** A steeper line makes a larger angle and has a larger gradient, so a larger tangent goes with a larger angle.



Two lines through the origin on one set of axes: y equals three-quarters x in blue making the angle theta 1 with run 4 and rise 3, and y equals 2x in red making the larger angle theta 2 with run 2 and rise 4; tan theta 1 equals 0.75 and tan theta 2 equals 2

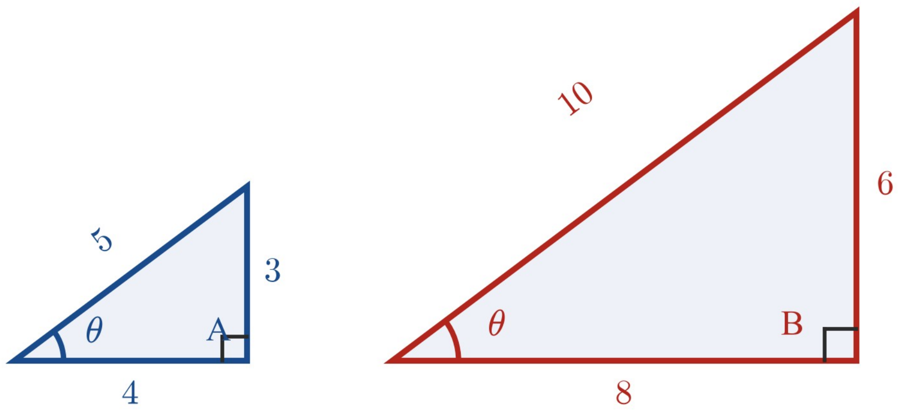
Worked examples

Example 1 — scaffolded

Triangles A (sides $3, 4, 5$) and B (sides $6, 8, 10$) each have the angle θ opposite the shortest side, as in the figure. Show that A and B are similar, then find $\sin\theta$, $\cos\theta$ and $\tan\theta$ in each and compare.

Working	Comment
$\frac{6}{3}=2,\ \frac{8}{4}=2,\ \frac{10}{5}=2$	Every pair of corresponding sides has the same ratio, 2, so B is A enlarged by scale factor 2: the triangles are similar (Level 8).
In A: $\sin\theta=\frac{3}{5},\ \cos\theta=\frac{4}{5},\ \tan\theta=\frac{3}{4}$	θ is opposite the 3, so opposite $=3$, adjacent $=4$, hypotenuse $=5$.
In B: $\sin\theta=\frac{6}{10}=\frac{3}{5},\ \cos\theta=\frac{8}{10}=\frac{4}{5},\ \tan\theta=\frac{6}{8}=\frac{3}{4}$	Same angle, larger triangle: each fraction cancels to the value from A.

The two triangles give the same three ratios: $\sin\theta=0.6$, $\cos\theta=0.8$, $\tan\theta=0.75$. Similar triangles, same angle, same ratios — that is constancy.



Triangle A with sides 3, 4 and 5 in blue beside triangle B with sides 6, 8 and 10 in red, both with the angle theta opposite the shortest side and a small square at the right angle

Example 2 — scaffolded

A right-angled triangle has sides 5, 12 and 13, with θ opposite the side of length 5. Write $\sin\theta$, $\cos\theta$ and $\tan\theta$ as fractions, and as decimals to 4 decimal places.

Working	Comment
Check: $5^2+12^2=25+144=169=13^2$	Pythagoras' theorem holds, so the right angle is opposite the 13: hypotenuse $=13$.
opposite $=5$, adjacent $=12$	θ is opposite the 5; the 12 is next to θ and is not the hypotenuse.
$\sin\theta=\dfrac{5}{13}\approx0.3846$	Opposite over hypotenuse.
$\cos\theta=\dfrac{12}{13}\approx0.9231$	Adjacent over hypotenuse.
$\tan\theta=\dfrac{5}{12}\approx0.4167$	Opposite over adjacent.

Example 3 — unscaffolded

In a right-angled triangle with hypotenuse 1 and angle θ, the base is 0.8. Use Pythagoras' theorem to find the altitude, write the three ratios for θ, then enlarge the triangle by a scale factor of 20 and write the ratios again.

$\text{altitude}^2=1^2-0.8^2=1-0.64=0.36$, so altitude $=0.6$. With hypotenuse 1: $\sin\theta=\dfrac{0.6}{1}=0.6$, $\cos\theta=\dfrac{0.8}{1}=0.8$, $\tan\theta=\dfrac{0.6}{0.8}=0.75$. Enlarging by 20: the sides are $0.6\times20=12$, $0.8\times20=16$, $1\times20=20$. In the enlarged triangle: $\sin\theta=\dfrac{12}{20}=0.6$, $\cos\theta=\dfrac{16}{20}=0.8$, $\tan\theta=\dfrac{12}{16}=0.75$.

$\therefore$ The ratios are 0.6, 0.8, 0.75 in both triangles: enlarging a triangle does not change the ratios of its sides, because it produces a similar triangle.

Example 4 — unscaffolded

The line $y=\dfrac{3}{4}x$ meets the positive direction of the horizontal axis at the angle θ. Find $\tan\theta$ twice: once from a triangle, once from the gradient.

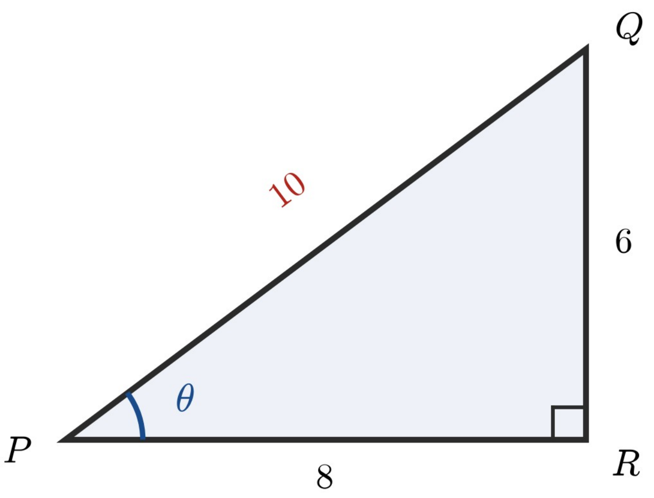
Triangle method: the point $(4,3)$ is on the line. Drop the altitude to the axis: rise $=3$, run $=4$, so $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$. Gradient method (2B): the gradient of $y = \frac{3}{4}x$ is $\frac{3}{4}$; check with a second point, $(8,6)$: rise over run $= \frac{6}{8} = \frac{3}{4}$, the same number.

$\therefore \tan\theta = \frac{3}{4} =$ the gradient of the line — the tangent of the angle a line makes with the positive horizontal direction is the gradient of that line.

Practice questions

Scaffolded

Q1. Triangle PQR has its right angle at R, with $PR = 8$, $QR = 6$ and $PQ = 10$, and θ marked at P, as in the figure. a Name the hypotenuse. b For θ: which side is opposite, and which is adjacent? c Now look at the angle at Q instead. Which side is opposite it, and which is adjacent? Which name never changes?



Triangle PQR with the right angle at R: P at the left, R at the bottom right, Q at the top right, the base PR of 8, the altitude RQ of 6, the hypotenuse PQ of 10, and the angle theta marked at P

Q2. Use the nested triangles A, B and C of the theory figure (sides 3–4–5, 6–8–10, 9–12–15, all sharing θ). a Write down the opposite, adjacent and hypotenuse of B and of C. b Work out $\frac{\text{opposite}}{\text{hypotenuse}}$ for A, for B and for C. c What do you notice? In one sentence, what does it say about the three triangles' angles?

Q3. A triangle similar to the 3–4–5 triangle has a scale factor of 4, with θ opposite the shortest side in both. a Write down the sides of the larger triangle. b Work out $\sin\theta$ in both triangles. c Explain in one sentence why the two values must be equal.

Q4. For some angle θ, $\sin\theta = 0.6$ and $\cos\theta = 0.8$. a Write down the altitude and the base of the right-angled triangle with hypotenuse 1. b Enlarge that triangle until the hypotenuse is 25. What are the altitude and the base now? c Check your answer to b with Pythagoras' theorem.

Q5. The line $y = 2x$ meets the positive direction of the horizontal axis at the angle θ. a Write down the gradient of the line. b Use the tangent–gradient connection to write down $\tan\theta$. c Check with the triangle formed by the origin, $(1,0)$ and $(1,2)$.

Q6. Vertical lines at $x=4$, $x=8$ and $x=12$ meet the line $y=\frac{3}{4}x$ at $(4,3)$, $(8,6)$ and $(12,9)$, making three nested right-angled triangles like the theory figure. a Use Pythagoras' theorem to find the three hypotenuses. b Work out rise over run for each triangle. c Which of the three ratios $\sin\theta$, $\cos\theta$, $\tan\theta$ is rise over run? What does its value tell you?

Unscaffolded

Q7. A right-angled triangle has sides 8, 15 and 17, with θ opposite the side of length 8. Write $\sin\theta$, $\cos\theta$ and $\tan\theta$ as fractions and as decimals to 4 decimal places.

Q8. A right-angled triangle has sides 7.5, 10 and 12.5, with θ opposite the 7.5. Work out the three ratios for θ and compare them with the 3–4–5 triangle. What one word describes the two triangles?

Q9. Two right-angled triangles have sides 9, 12, 15 and 4.5, 6, 7.5, with θ opposite the shortest side in both. Find the scale factor between them, work out $\cos\theta$ in each, and say why the values must agree.

Q10. A right-angled triangle has sides 5, 12 and 13, with θ opposite the 5. Write down $\sin\theta$ and $\cos\theta$, use them to give the altitude and base of the unit-hypotenuse triangle, and hence find the altitude and base of the similar triangle with hypotenuse 26.

Q11. A right-angled triangle has hypotenuse 1 and base 0.6, with θ between them. Use Pythagoras' theorem to find the altitude, then write $\sin\theta$, $\cos\theta$ and $\tan\theta$, giving $\tan\theta$ as a fraction in simplest form.

Q12. A line through the origin passes through $(6,8)$ and meets the positive direction of the horizontal axis at θ. Find the gradient of the line and hence $\tan\theta$ as a fraction and as a decimal to 4 decimal places.

Q13. A line through the origin has gradient $\frac{5}{12}$ and meets the positive direction of the horizontal axis at θ. Write down $\tan\theta$ as a decimal to 4 decimal places, and find the y-value of the point on the line with $x=12$.

Q14. Every side of a 3–4–5 right-angled triangle is doubled. Work out $\tan\theta$ before and after, and give the result as one general sentence about all similar triangles.

Q15. In a 3–4–5 right-angled triangle, let θ be opposite the 3 and φ opposite the 4. Work out $\tan\theta$ and $\tan\varphi$, find their product, and find $\theta+\varphi$. Explain the product in one sentence.

Q16. The lines $y=\frac{3}{4}x$ and $y=2x$ meet the positive direction of the horizontal axis at angles θ_1 and θ_2. Use the tangents to decide which angle is larger, given that $\theta_1\approx 36.9°$ and $\theta_2\approx 63.4°$. State the connection between the size of the tangent and the size of the angle.

Q17. A right-angled triangle is similar to the 6–8–10 triangle, with θ opposite the shortest side, and its hypotenuse is 25. Find the scale factor and hence the sides opposite and adjacent to θ. Check with Pythagoras' theorem.

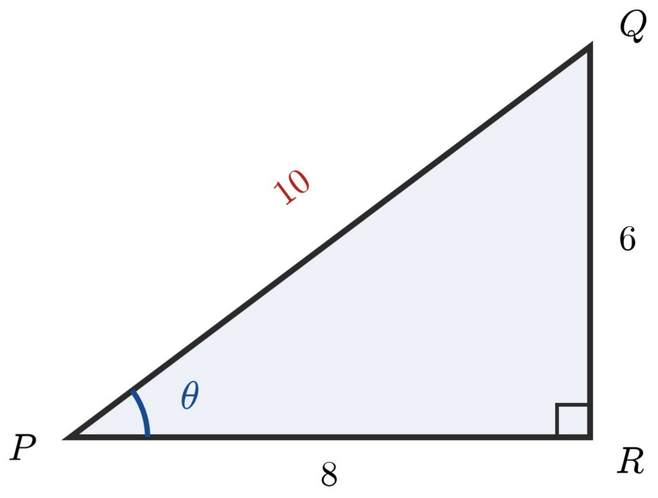
Q18. Mia claims that because the 6–8–10 triangle has a longer opposite side than the 3–4–5 triangle, its sine of θ (with θ opposite the shortest side) must be bigger. Use the two ratios to show that she is wrong, and state what the sine of an angle actually depends on.

Answers

Q1 a PQ (opposite the right angle, the longest side). b opposite $QR=6$; adjacent $PR=8$. c opposite $PR=8$; adjacent $QR=6$; the hypotenuse PQ never changes. **Q2** a B: $6,8,10$; C: $9,12,15$. b $\frac{3}{5}=\frac{6}{10}=\frac{9}{15}=0.6$ in all three. c The ratios are equal, so the triangles have the same angles — they are similar. **Q3** a $12,16,20$. b $\frac{3}{5}=\frac{12}{20}=0.6$ in both. c Similar triangles have equal ratios of corresponding sides, and $\sin\theta$ is such a ratio. **Q4** a altitude 0.6, base 0.8. b altitude 15, base 20. c $15^2+20^2=225+400=625=25^2$. **Q5** a 2. b $\tan\theta=2$. c opposite 2, adjacent 1: $\frac{2}{1}=2$. **Q6** a $5,10,15$. b $\frac{3}{4}$ in each. c $\tan\theta$; it is the same 0.75 in every triangle of the family. **Q7** $\sin\theta=\frac{8}{17}\approx 0.4706$; $\cos\theta=\frac{15}{17}\approx 0.8824$; $\tan\theta=\frac{8}{15}\approx 0.5333$. **Q8** $0.6,0.8,0.75$ — the same as the 3–4–5 triangle; similar. **Q9** Scale factor $\frac{1}{2}$; $\cos\theta=\frac{12}{15}=\frac{6}{7.5}=0.8$ in both; similar triangles, same angle, same ratio. **Q10** $\sin\theta=\frac{5}{13}$, $\cos\theta=\frac{12}{13}$; unit triangle: altitude $\frac{5}{13}$, base $\frac{12}{13}$; hypotenuse 26: altitude 10, base 24. **Q11** altitude 0.8; $\sin\theta=0.8$, $\cos\theta=0.6$, $\tan\theta=\frac{4}{3}$. **Q12** gradient $\frac{8}{6}=\frac{4}{3}$; $\tan\theta=\frac{4}{3}\approx 1.3333$. **Q13** 0.4167; $y=5$. **Q14** Before $\frac{3}{4}$, after $\frac{6}{8}=\frac{3}{4}$; doubling (or any enlargement) leaves $\tan\theta$ unchanged. **Q15** $\tan\theta=\frac{3}{4}$, $\tan\varphi=\frac{4}{3}$; product 1; $\theta+\varphi=90°$; the two acute angles of a right-angled triangle add to $90°$, so each is the other's opposite-over-adjacent turned upside down. **Q16** θ_2; $\tan\theta_2=2>\frac{3}{4}=\tan\theta_1$, and $63.4°>36.9°$; a larger tangent goes with a larger angle. **Q17** Scale factor 2.5; opposite 15, adjacent 20; $15^2+20^2=625=25^2$. **Q18** $\frac{3}{5}=\frac{6}{10}=0.6$: the sines are equal, so Mia is wrong; the sine depends on the angle, not on the size of the triangle.

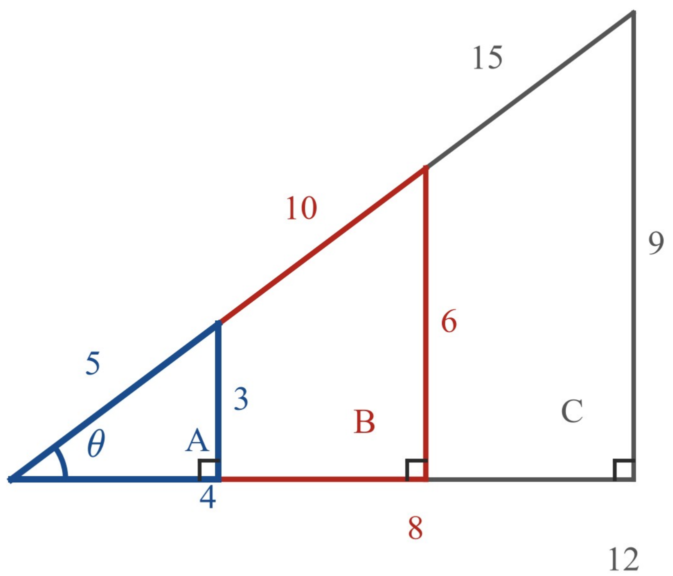
Fully-worked solutions

Q1. a The right angle is at R, so the hypotenuse is the side opposite R: $PQ=10$. b θ is at P: the side across from P is $QR=6$ (opposite); the side next to P that is not the hypotenuse is $PR=8$ (adjacent). c At Q: opposite is $PR=8$; adjacent is $QR=6$. The hypotenuse PQ keeps its name at both angles. ∴ Naming opposite and adjacent needs a marked angle; the hypotenuse is fixed.



Triangle PQR with the right angle at R: P at the left, R at the bottom right, Q at the top right, the base PR of 8, the altitude RQ of 6, the hypotenuse PQ of 10, and the angle theta marked at P

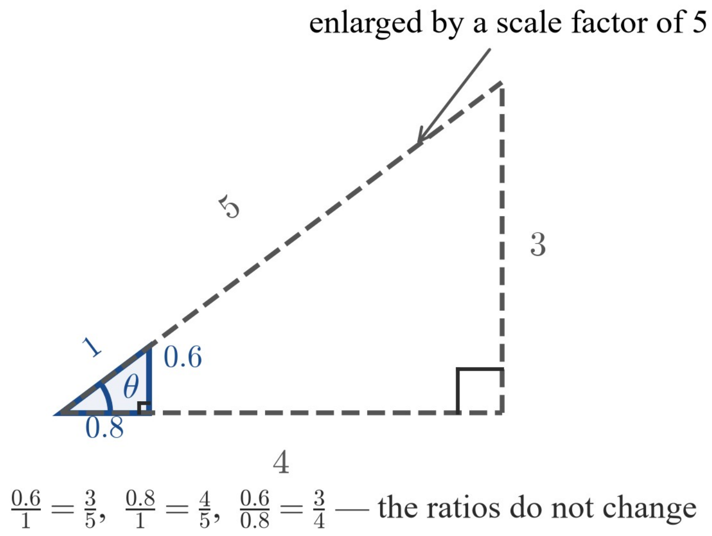
Q2. a B: opposite 6, adjacent 8, hypotenuse 10. C: opposite 9, adjacent 12, hypotenuse 15. b A: $\frac{3}{5}=0.6$; B: $\frac{6}{10}=0.6$; C: $\frac{9}{15}=0.6$. c The ratio opposite over hypotenuse is 0.6 in all three — the same ratio, so the same angle θ in each triangle: the triangles are similar. ∴ $\sin\theta=0.6$ for the whole nested family.



Three nested similar right-angled triangles A, B and C sharing the angle theta at the origin: A with sides 3, 4 and 5 in blue, B with sides 6, 8 and 10 in red, C with sides 9, 12 and 15 in grey, each with a small square at its right angle

Q3. a Scale factor 4: $3\times4=12$, $4\times4=16$, $5\times4=20$. b Small: $\sin\theta=\frac{3}{5}=0.6$. Large: $\sin\theta=\frac{12}{20}=0.6$. c $\sin\theta$ is a ratio of corresponding sides; similar triangles keep all such ratios equal. ∴ Both give $\sin\theta=0.6$ — the ratio is constant under enlargement.

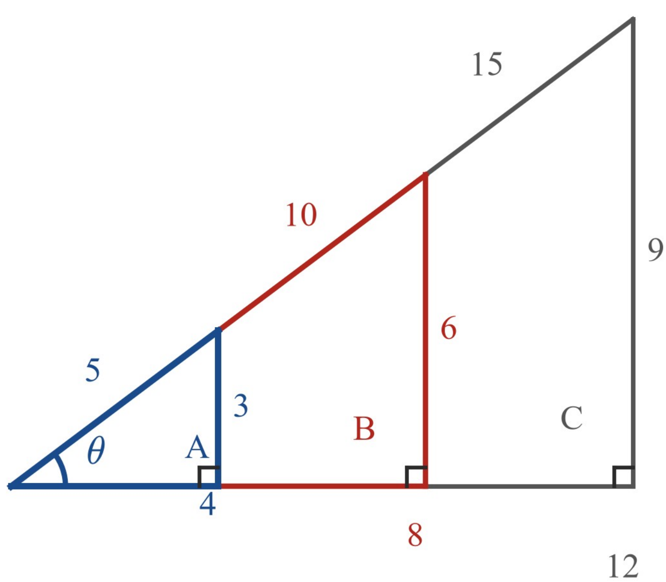
Q4. a With hypotenuse 1: altitude $=\sin\theta=0.6$, base $=\cos\theta=0.8$. b Scale factor from 1 to 25 is 25: altitude $=0.6\times25=15$, base $=0.8\times25=20$. c $15^2+20^2=225+400=625$ and $25^2=625$ — Pythagoras' theorem holds. ∴ Altitude 15, base 20; the enlarged triangle is similar, so its ratios are still 0.6 and 0.8.



The 0.6, 0.8, 1 triangle in solid blue nested inside its enlargement by a scale factor of 5, the 3, 4, 5 triangle in dashed grey, sharing the angle theta at the origin, with the equal ratios written below

Q5. a $y=2x$ has gradient 2. b The tangent of the angle a line makes with the positive horizontal direction is its gradient: $\tan\theta=2$. c Check: in the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$: opposite $=2$, adjacent $=1$, so $\tan\theta=\frac{2}{1}=2$. ∴ Both methods give $\tan\theta=2$.

Q6. a $\sqrt{4^2+3^2}=\sqrt{25}=5$; $\sqrt{8^2+6^2}=\sqrt{100}=10$; $\sqrt{12^2+9^2}=\sqrt{225}=15$. b $\frac{3}{4}$, $\frac{6}{8}=\frac{3}{4}$, $\frac{9}{12}=\frac{3}{4}$. c Rise over run is opposite over adjacent $=\tan\theta$; its value 0.75 is the same in all three triangles, and it is the gradient of $y=\frac{3}{4}x$. ∴ Hypotenuses $5, 10, 15$; $\tan\theta=0.75$ throughout the family.



Three nested similar right-angled triangles A, B and C sharing the angle theta at the origin: A with sides 3, 4 and 5 in blue, B with sides 6, 8 and 10 in red, C with sides 9, 12 and 15 in grey, each with a small square at its right angle

Q7. Hypotenuse $=17$ (opposite the right angle, since $8^2+15^2=64+225=289=17^2$). Opposite $=8$, adjacent $=15$. $\sin\theta=\frac{8}{17}\approx0.4706$; $\cos\theta=\frac{15}{17}\approx0.8824$; $\tan\theta=\frac{8}{15}\approx0.5333$. ∴ The three ratios are $\frac{8}{17}, \frac{15}{17}, \frac{8}{15}$.

Q8. $\sin\theta = \frac{7.5}{12.5} = 0.6$; $\cos\theta = \frac{10}{12.5} = 0.8$; $\tan\theta = \frac{7.5}{10} = 0.75$. These are exactly the ratios of the 3–4–5 triangle, because $\frac{7.5}{3} = \frac{10}{4} = \frac{12.5}{5} = 2.5$: one is an enlargement of the other. ∴ Same ratios — the triangles are **similar**.

Q9. Scale factor: $\frac{4.5}{9} = \frac{6}{12} = \frac{7.5}{15} = \frac{1}{2}$. $\cos\theta$ in the first: $\frac{12}{15} = 0.8$; in the second: $\frac{6}{7.5} = 0.8$. They agree because a scale factor makes similar triangles, and similar triangles give equal ratios for the same angle. ∴ Scale factor $\frac{1}{2}$; $\cos\theta = 0.8$ in both.

Q10. $\sin\theta = \frac{5}{13}$ and $\cos\theta = \frac{12}{13}$, so the unit-hypotenuse triangle has altitude $\frac{5}{13}$ and base $\frac{12}{13}$. Scale factor from 1 to 26 is 26: altitude $= \frac{5}{13} \times 26 = 10$, base $= \frac{12}{13} \times 26 = 24$. Check: $10^2 + 24^2 = 100 + 576 = 676 = 26^2$. ∴ Altitude 10, base 24.

Q11. $\text{altitude}^2 = 1^2 - 0.6^2 = 1 - 0.36 = 0.64$, so altitude $= 0.8$. $\sin\theta = \frac{0.8}{1} = 0.8$; $\cos\theta = \frac{0.6}{1} = 0.6$; $\tan\theta = \frac{0.8}{0.6} = \frac{4}{3}$. ∴ Altitude 0.8; the ratios are 0.8, 0.6, $\frac{4}{3}$.

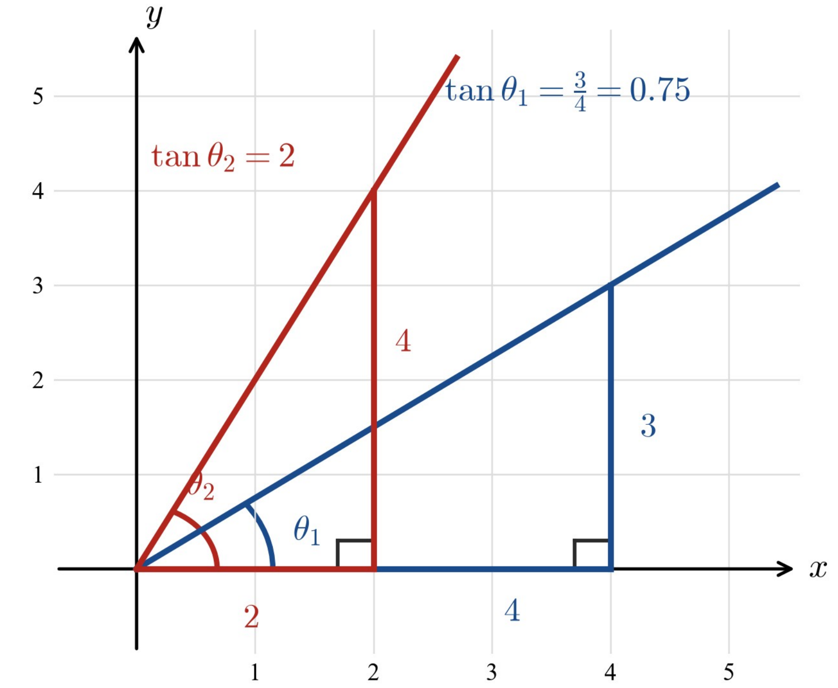
Q12. Gradient $= \frac{8-0}{6-0} = \frac{8}{6} = \frac{4}{3}$. The tangent of the angle the line makes with the positive horizontal direction is the gradient: $\tan\theta = \frac{4}{3} \approx 1.3333$. ∴ $\tan\theta = \frac{4}{3}$.

Q13. \$θ = \$ gradient $= \frac{5}{12} \approx 0.4167$. At $x = 12$: $y = \frac{5}{12} \times 12 = 5$ — the triangle with run 12 has rise 5, giving $\tan\theta = \frac{5}{12}$ as required. ∴ $\tan\theta \approx 0.4167$; the point is $(12, 5)$.

Q14. Before: $\tan\theta = \frac{3}{4}$. After doubling: sides $6, 8, 10$, so $\tan\theta = \frac{6}{8} = \frac{3}{4}$. The value is unchanged, because doubling every side is an enlargement, and enlargements keep the ratios of corresponding sides equal. ∴ $\tan\theta$ is the same before and after: the ratios of a right-angled triangle depend on its angles, not on its size.

Q15. $\tan\theta = \frac{3}{4}$ (opposite 3, adjacent 4); $\tan\varphi = \frac{4}{3}$ (opposite 4, adjacent 3). Product: $\frac{3}{4} \times \frac{4}{3} = 1$. $\theta + \varphi = 180° - 90° = 90°$, since the three angles of the triangle add to $180°$ and the right angle takes $90°$. ∴ $\tan\theta \times \tan\varphi = 1$: the side that is opposite θ is adjacent to φ, so each tangent is the reciprocal of the other.

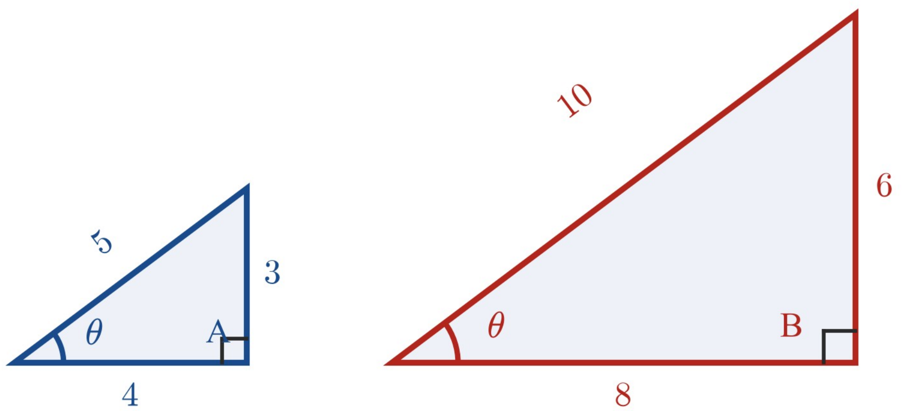
Q16. $\tan\theta_1 = \frac{3}{4} = 0.75$ and $\tan\theta_2 = 2$. $2 > 0.75$, and in fact $\theta_2 \approx 63.4° > \theta_1 \approx 36.9°$. A steeper line makes a larger angle with the horizontal axis and has a larger gradient, so a larger tangent goes with a larger angle. ∴ θ_2 is the larger angle; the size of the tangent and the size of the angle grow together.



Two lines through the origin on one set of axes: y equals three-quarters x in blue making the angle theta 1 with run 4 and rise 3, and y equals 2x in red making the larger angle theta 2 with run 2 and rise 4; tan theta 1 equals 0.75 and tan theta 2 equals 2

Q17. Scale factor: $\frac{25}{10} = 2.5$. Opposite: $6 \times 2.5 = 15$; adjacent: $8 \times 2.5 = 20$. Check: $15^2 + 20^2 = 225 + 400 = 625 = 25^2$. $\therefore$ Opposite 15, adjacent 20.

Q18. In the 3–4–5 triangle: $\sin\theta = \frac{3}{5} = 0.6$. In the 6–8–10 triangle: $\sin\theta = \frac{6}{10} = 0.6$. The opposite side did double, but so did the hypotenuse, so the ratio — the sine — is unchanged. $\therefore$ Mia is wrong: the sine of θ depends on the angle θ, not on how long the sides are.



Triangle A with sides 3, 4 and 5 in blue beside triangle B with sides 6, 8 and 10 in red, both with the angle theta opposite the shortest side and a small square at the right angle

The enlargement transformation: what is preserved and what changes

4B · Chapter 4 — Space: similarity, the trigonometric ratios and geometric algorithms · Level 9

Content description `VC2M9SP02` (word for word): *apply the enlargement transformation to shapes and objects using dynamic geometry software as appropriate; identify and explain, using language of similarity, ratio and scale, aspects that remain the same and those that change*

Achievement standard anchor (AS 14): *Students apply the enlargement transformation to images of shapes and objects, and interpret results.*

Elaboration ids for linkage: `VC2M9SP02.E01`, `VC2M9SP02.E02`, `VC2M9SP02.E03`

Prerequisite pointer: none within this book — the Level 8 idea of similar shapes and scale factors is rehearsed here as a tool, not re-taught. This section's scale-factor work is used again in 5C and 5E.

DIGITAL TOOL — one investigation in this section works well in dynamic geometry software, but the software is permitted, not required. Your teacher will nominate the tool. The investigation says exactly what to draw and what you should see, and the figures show the expected output, so the same investigation can be followed with ruler and grid paper if the tool is not in front of you.

Note on figures: every triangle, rectangle and cube in this section is drawn from its own lengths, and every marked length from its own value (R7); every context is a generic one, so no real data ships and no source line is owed (R1, R2).

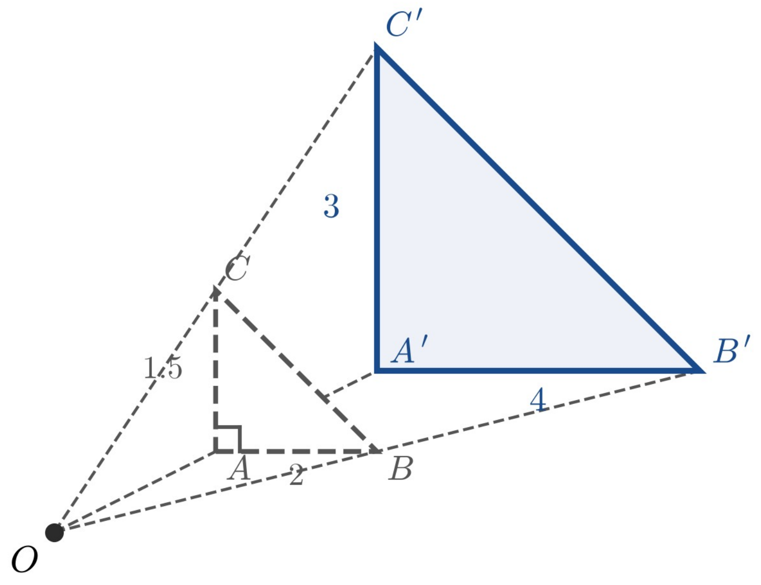
Theory

The enlargement transformation

In everyday speech, to *enlarge* something means to make it bigger. In this section the word is used more widely, exactly as the curriculum uses it: an **enlargement** is a transformation that multiplies every length in a shape by the same number k, called the **scale factor**, measured out from a fixed point called the **centre of enlargement**. An enlargement with $k > 1$ does make the shape bigger; an enlargement with $0 < k < 1$ makes it smaller — a smaller image is still an enlargement, and is also called a **reduction**. The starting shape is the **object**; the shape the enlargement produces is the **image**.

Each vertex of the image lies on the ray from the centre through the matching vertex of the object, at k times the distance from the centre. In the figure, the centre is O and $k = 2$: the object's sides of 2 and 1.5 become 4 and 3, and the right angle at A is still a right angle at A'.

enlargement from centre O, scale factor $k = 2$: every length doubles



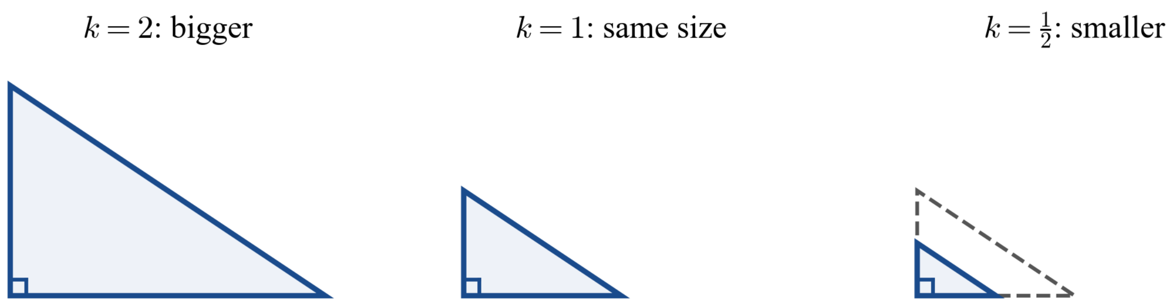
A dashed grey triangle with vertices A, B, C enlarged from centre O by scale factor 2 to a solid blue triangle with vertices A-prime, B-prime, C-prime; dashed rays from O pass through each matching pair of vertices, the sides 2 and 1.5 become 4 and 3, and right-angle marks at A and A-prime show the angle unchanged

The scale factor decides what the image looks like:

- $k > 1$: the image is bigger than the object;
- $k = 1$: the image is the same size as the object (a **congruent** copy);
- $0 < k < 1$: the image is smaller than the object — a reduction.

In all three cases the angles of the triangle are untouched.

the same triangle enlarged by three scale factors — angles never change



The same right-angled triangle in three panels: enlarged by k = 2 and bigger, by k = 1 and the same size, and by k = one-half and smaller, with a right-angle mark in each panel to show the angles never change

What is preserved and what changes

Under an enlargement with scale factor k:

- **preserved** (the same): every angle; the shape of the object, so the image is **similar** to the object; the ratio of any two lengths inside the shape; and parallel lines, which map to parallel lines;
- **changed**: every length is multiplied by k, so the perimeter is multiplied by k; the area is multiplied by k^2; and for a solid object, the volume is multiplied by k^3.

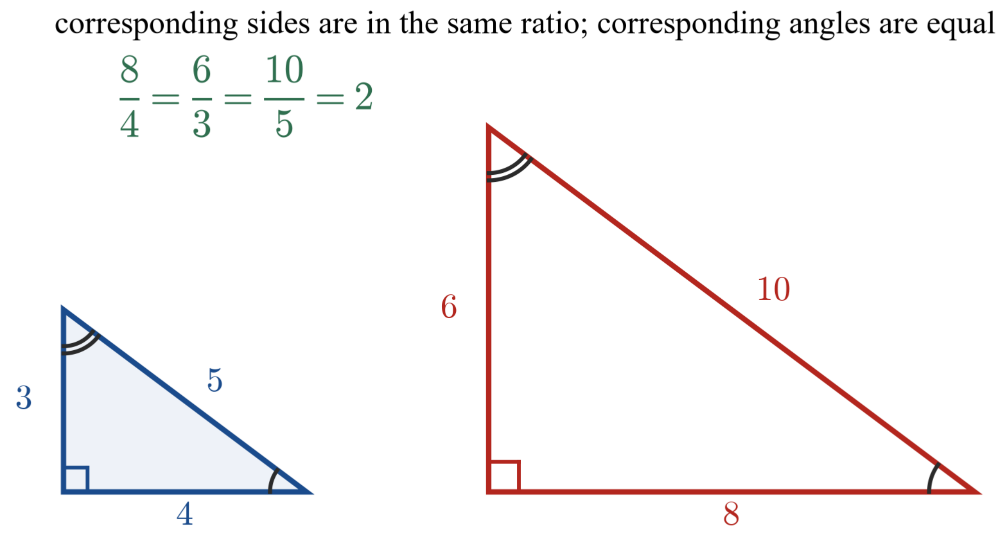
That last line — lengths $\times k$, areas $\times k^2$, volumes $\times k^3$ — is the load-bearing result of this section. The rest of the theory builds up to it.

Corresponding sides and angles (E01)

Matching parts of the object and the image are called **corresponding** parts. In similar shapes, corresponding angles are equal and corresponding sides are in the same ratio — that ratio is the scale factor. The figure shows a 3-4-5 triangle and its image under $k=2$, a 6-8-10 triangle: the matching angles carry the same marks, and

$$\frac{8}{4}=\frac{6}{3}=\frac{10}{5}=2.$$

Comparing the ratio of corresponding sides is how you check that two shapes are similar, and how you recover k when it is not given.

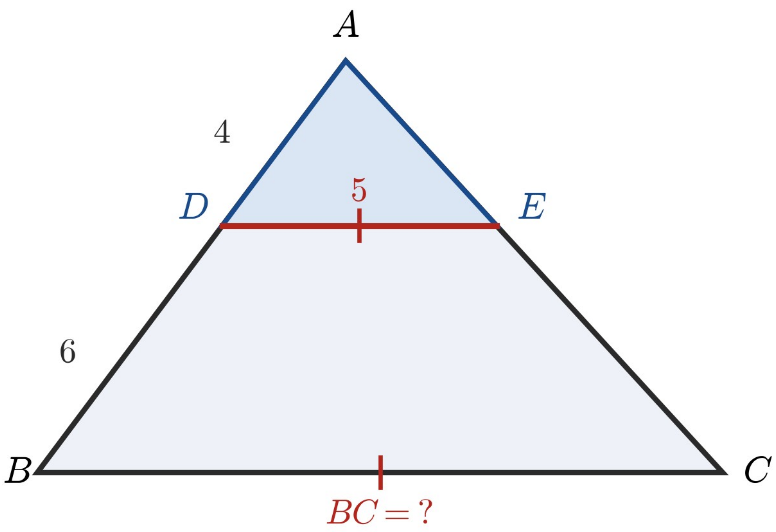


A blue 3-4-5 right-angled triangle beside a red 6-8-10 right-angled triangle with matching angle marks, and the equal ratios 8 over 4 = 6 over 3 = 10 over 5 = 2 written above

Using similarity to find unknown lengths (E02)

One fact does most of the work in enlargement problems: *a line drawn parallel to one side of a triangle cuts off a smaller triangle similar to the whole.* In the figure, DE is parallel to BC, so triangle ABC is an enlargement of triangle ADE, with scale factor $\frac{AB}{AD}$. Any pair of corresponding sides gives that same ratio, so $\frac{AD}{AB}=\frac{DE}{BC}$, and an unknown length falls out of a proportion of corresponding sides.

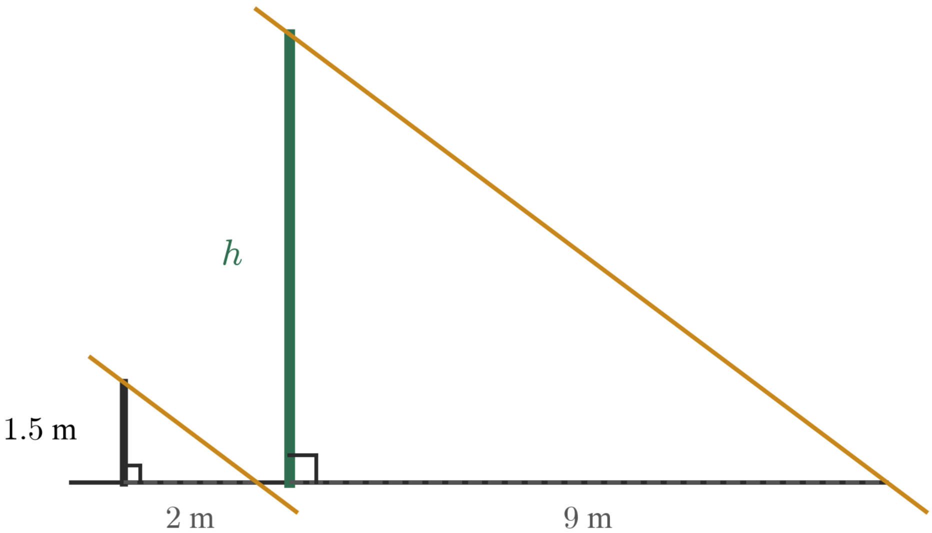
DE is parallel to BC, so triangle ADE is similar to triangle ABC



Triangle ABC with D on AB and E on AC, the segment DE in red and parallel to BC, labelled AD = 4, DB = 6, DE = 5 and BC = ?, with parallel marks on DE and BC

The same idea works outdoors. A vertical post and a vertical tree both stand at right angles to level ground, and the sun's rays are parallel, so the ray makes the same angle with the ground in both triangles. Two pairs of equal angles make the post-and-shadow triangle similar to the tree-and-shadow triangle — which is why a shadow can measure a height.

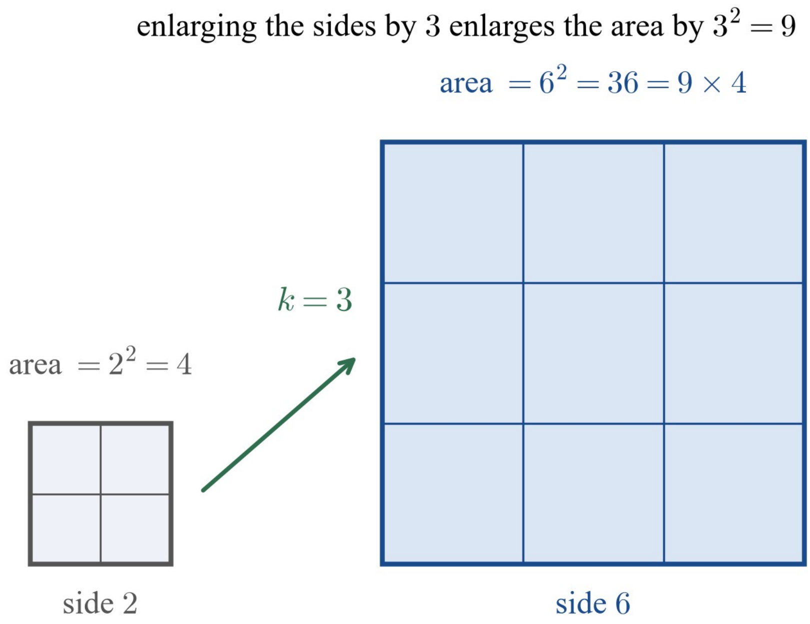
the sun's rays are parallel, so the two triangles are similar



A 1.5 m post with a 2 m shadow and a tree of height h with a 9 m shadow on the same level ground, with parallel amber sun rays forming two similar right-angled triangles

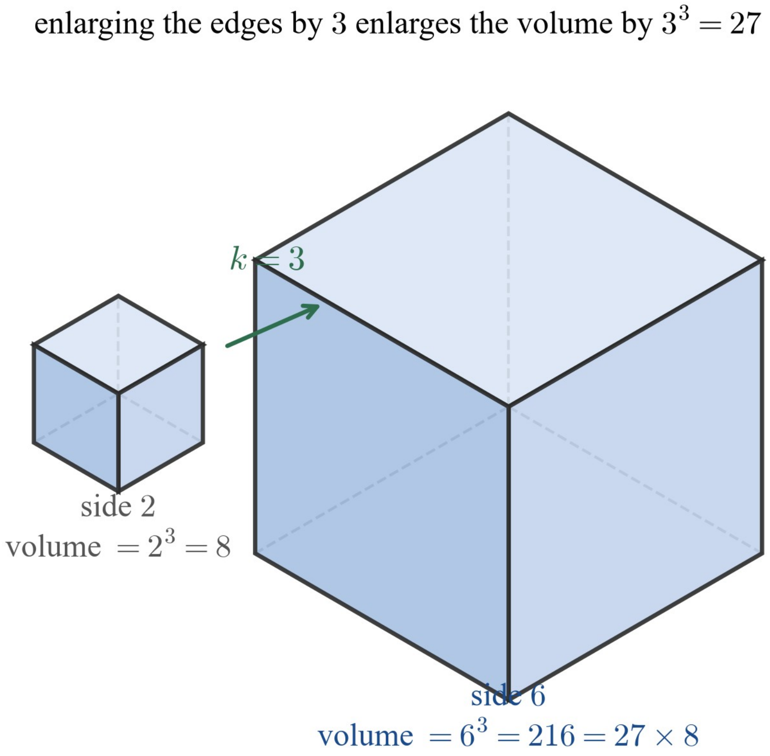
The pattern for area and volume (E03)

Take the curriculum's own example. A square of side 2 enlarged by $k=3$ has side 6; its area goes from $2^2=4$ to $6^2=36$, and $36=9\times4$. The figure shows why the factor is 9: the big square is exactly a 3×3 array of copies of the small one. In general, side s becomes side ks, and the area goes from s^2 to $(ks)^2=k^2s^2$ — multiplied by k^2.



A 2 by 2 square divided into four unit squares enlarged by k = 3 to a 6 by 6 square divided into a 3 by 3 array of copies of the small square, labelled area = 4 and area = 36 = 9 times 4

For a solid, one more direction joins in. A cube of side 2 enlarged by $k = 3$ has side 6; its volume goes from $2^3 = 8$ to $6^3 = 216$, and $216 = 27 \times 8$. In general the volume goes from s^3 to $(k\,s)^3 = k^3 s^3$ — multiplied by k^3. The big cube is a $3 \times 3 \times 3$ stack of copies of the small one: 27 of them.



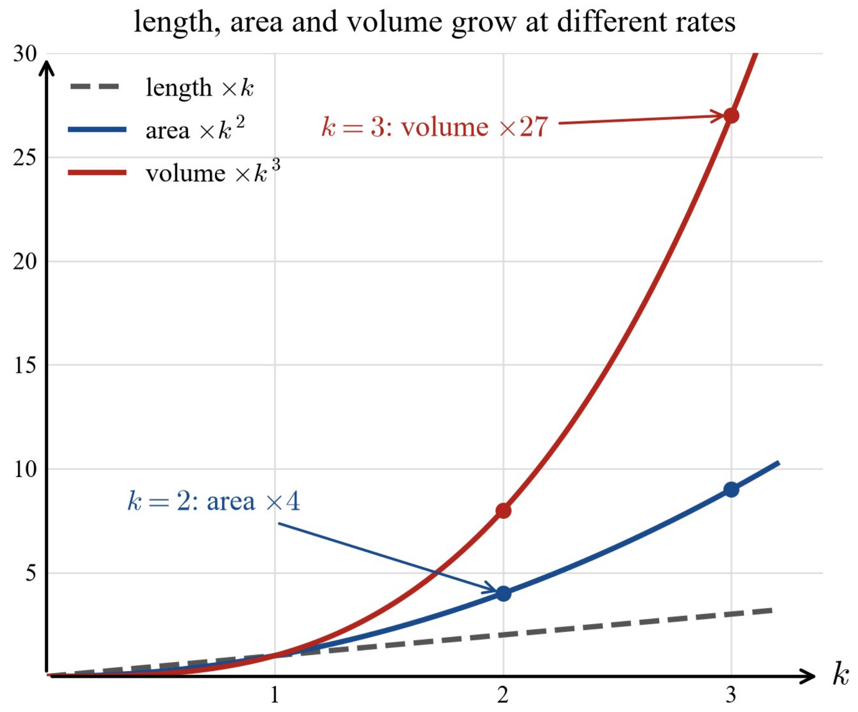
A small cube of side 2 labelled volume = 8 beside a large cube of side 6 labelled volume = 216 = 27 times 8, with an arrow labelled k = 3 between them

The pattern holds for any scale factor, whole or not:

scale factor k	lengths $\times k$	areas $\times k^2$	volumes $\times k^3$
2	2	4	8

scale factor k	lengths $\times k$	areas $\times k^2$	volumes $\times k^3$
3	3	9	27
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

The three rules grow at very different speeds. At $k=2$ the volume is already 8 times the original while the lengths have only doubled; at $k=3$ the area is 9 times and the volume 27 times the original.



Three curves on axes of scale factor k against growth factor: the dashed grey line of length times k, the blue curve of area times k-squared through (2, 4) and (3, 9), and the steeper red curve of volume times k-cubed through (2, 8) and (3, 27)

Surface area against volume

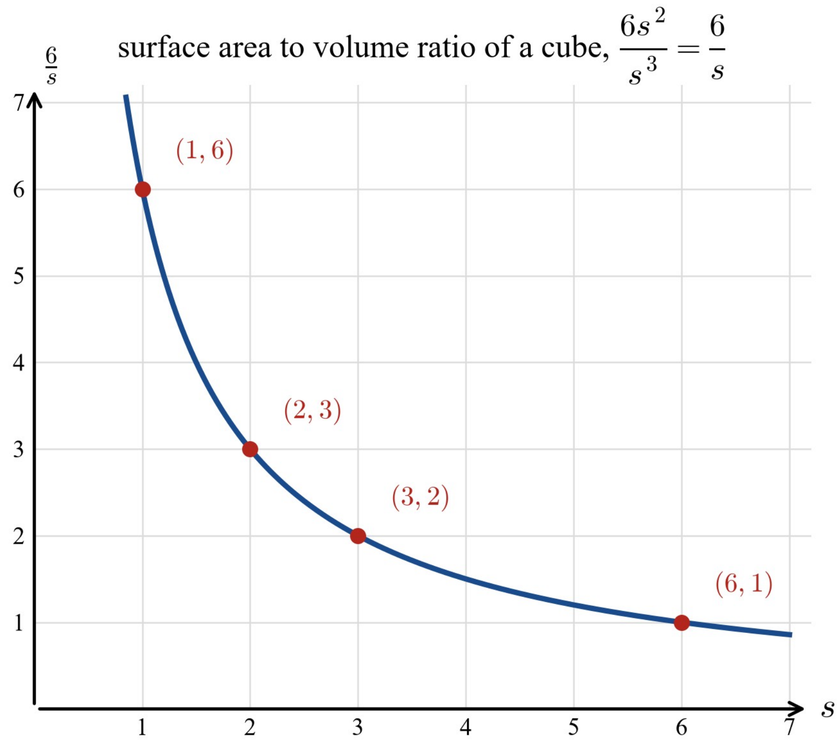
A cube of side s has surface area $6s^2$ (six faces, each s by s) and volume s^3. The ratio of surface area to volume is

$$\frac{6s^2}{s^3}=\frac{6}{s},$$

so the bigger the cube, the smaller the ratio:

side s (cm)	surface area (cm²)	volume (cm³)	ratio $\frac{6}{s}$
1	6	1	6
2	24	8	3
3	54	27	2
6	216	216	1

This is the k^2-against-k^3 pattern in disguise: as the cube grows, volume (a k^3 quantity) outruns surface area (a k^2 quantity). It explains a fact of biology without any biology: a cell takes in what it needs through its surface, so a cell that grew too large would have too little surface for its volume — which is why cells stay small.



The curve of 6 over s for s from 1 to 7, with the points (1, 6), (2, 3), (3, 2) and (6, 1) marked, falling as s grows

Investigation — see it happen (dynamic geometry software permitted, not required)

With the tool: draw any triangle; enlarge it by $k=2$ about any centre; then measure the three sides and three angles of both triangles. On paper: draw the triangle on grid paper, count squares for the sides, and redraw it with every length doubled.

What you should see: each side of the image is exactly twice the matching side — three ratios, all equal to 2 — while each angle of the image equals the matching angle of the object. Lengths doubled, angles untouched: that single experiment is the whole section in one picture.

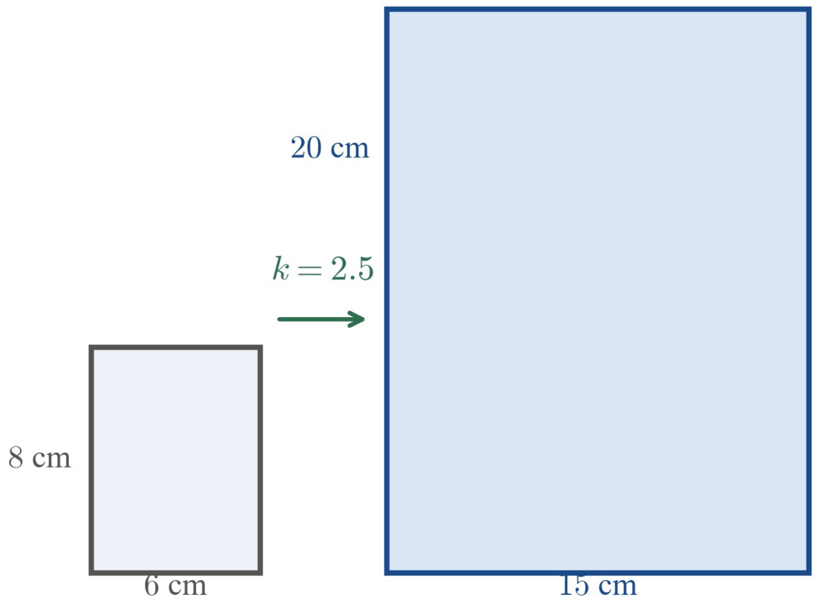
Worked examples

Example 1 — scaffolded

A photo is 6 cm wide and 8 cm high. It is enlarged so that the width becomes 15 cm. Find the scale factor, the height of the enlarged photo, and the factor by which the area grows.

Working	Comment
$k=\frac{15}{6}=\frac{5}{2}=2.5$	Scale factor = image length ÷ object length, using the widths.
height $=8\times2.5=20$ cm	Every length is multiplied by k.
Check: $\frac{20}{8}=2.5$	The heights give the same ratio as the widths — the shapes are similar.
areas: $6\times8=48$ cm² → $15\times20=300$ cm², and $300\div48=6.25=2.5^2$	Area is multiplied by k^2, here $2.5^2=6.25$.

a photo enlarged from 6 cm by 8 cm to 15 cm by 20 cm



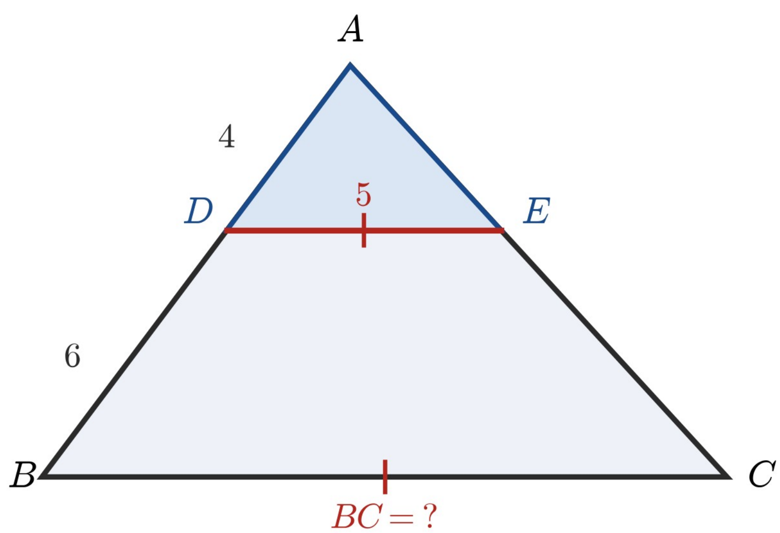
A grey 6 cm by 8 cm rectangle beside a blue 15 cm by 20 cm rectangle, with an arrow between them labelled k = 2.5

Example 2 — scaffolded

In triangle ABC, D lies on AB and E lies on AC, with DE parallel to BC. Given $AD=4$, $DB=6$ and $DE=5$, find BC.

Working	Comment
$AB=AD+DB=4+6=10$	The whole side of the big triangle.
$k=\frac{AB}{AD}=\frac{10}{4}=2.5$	$DE \parallel BC$, so triangle ABC is an enlargement of triangle ADE.
$BC=k\times DE=2.5\times5=12.5$	Corresponding sides are multiplied by k.
Check: $\frac{12.5}{5}=2.5$	The same ratio as $\frac{AB}{AD}$, as similarity requires.

DE is parallel to BC, so triangle ADE is similar to triangle ABC

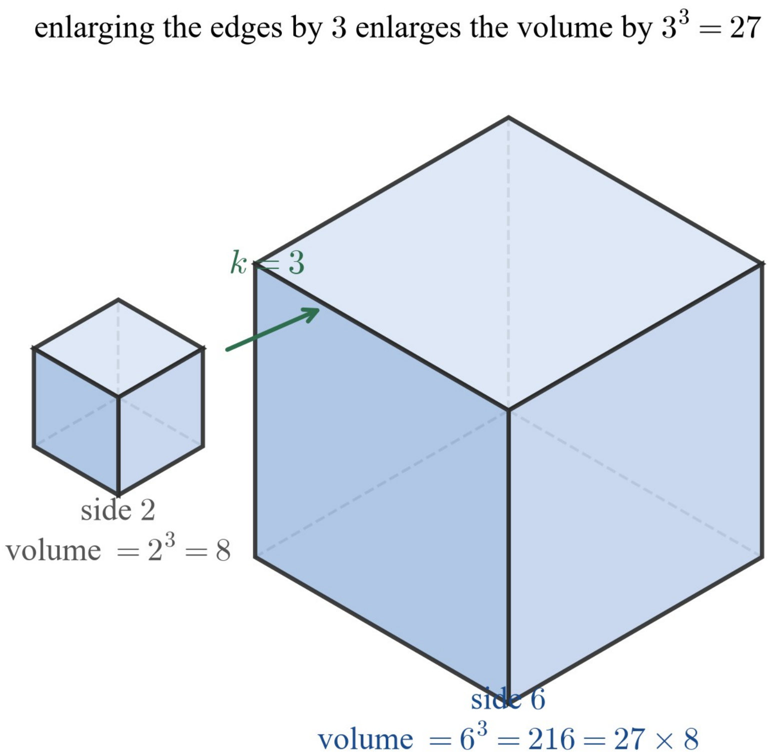


Triangle ABC with D on AB and E on AC, DE in red and parallel to BC, labelled AD = 4, DB = 6, DE = 5 and BC = ?

Example 3 — unscaffolded

A cube has side 2 cm. It is enlarged by scale factor 3. Find the side, the surface area and the volume of the image, and the factor by which the surface area and the volume grow.

Side of the image: $2 \times 3 = 6$ cm. Surface area of the object: $6 \times 2^2 = 24$ cm²; of the image: $6 \times 6^2 = 216$ cm². Factor for the surface area: $216 \div 24 = 9 = 3^2$. Volume of the object: $2^3 = 8$ cm³; of the image: $6^3 = 216$ cm³. Factor for the volume: $216 \div 8 = 27 = 3^3$. ∴ The image cube has side 6 cm, surface area 9 times the original, and volume 27 times the original — the factors k^2 and k^3 for $k = 3$.



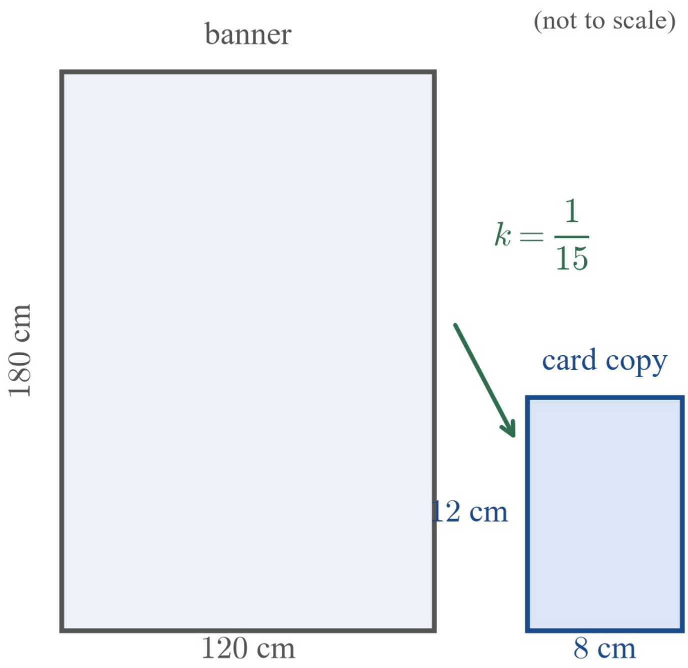
A small cube of side 2 labelled volume = 8 beside a large cube of side 6 labelled volume = 216 = 27 times 8, with an arrow labelled k = 3

Example 4 — unscaffolded

A banner measures 120 cm by 180 cm. A card copy of the banner is 8 cm wide. Find the scale factor of the enlargement, the height of the card copy, and the factor relating the two areas.

Scale factor: $k = \frac{8}{120} = \frac{1}{15}$ — a reduction, since $0 < k < 1$. Height of the card copy: $180 \times \frac{1}{15} = 12$ cm. Areas: banner $120 \times 180 = 21600$ cm²; card copy $8 \times 12 = 96$ cm². Factor: $96 \div 21600 = \frac{1}{225} = \left(\frac{1}{15}\right)^2$. ∴ The card copy is 8 cm by 12 cm, and its area is $\frac{1}{225}$ of the banner's — a reduction obeys the same k^2 rule as an enlargement.

the 120 cm by 180 cm banner reduced to 8 cm by 12 cm



A large grey rectangle labelled 120 cm by 180 cm (the banner) with an arrow labelled k = one-fifteenth to a small blue rectangle labelled 8 cm by 12 cm (the card copy), marked not to scale

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Practice questions

Scaffolded

Q1. A triangle has sides 3 cm, 5 cm and 7 cm. It is enlarged by scale factor 4.

a. Find the three sides of the image.
b. What happens to the angles of the triangle?

Q2. An enlargement takes a side of 8 cm to a side of 20 cm.

a. Find the scale factor.
b. Find the image of a side of 13 cm.
c. The object has perimeter 32 cm. Find the perimeter of the image.

Q3. A rectangle 10 cm by 6 cm is enlarged by scale factor $\frac{1}{2}$.

a. Find the measurements of the image.
b. Find the area of the object and of the image.
c. Find the factor relating the two areas, and compare it with $\frac{1}{2}$.

Q4. A triangle has angles 35°, 65° and 80°, and sides 4 cm, 6 cm and 9 cm. It is enlarged by scale factor 3.

a. Write down the angles of the image.
b. Find the sides of the image.
c. Write the ratios of corresponding sides as three equal fractions.

Q5. A square of side 5 cm is enlarged by scale factor 2.

a. Find the area of the object and of the image.
b. Find the factor relating the two areas.

c. Compare that factor with the scale factor, and state the general rule.

Q6. A box measures 2 cm by 3 cm by 4 cm. It is enlarged by scale factor 3.

a. Find the volume of the object and of the image.
b. Find the factor relating the two volumes.
c. Compare that factor with the scale factor, and state the general rule.

Unscaffolded

Q7. A photo 10 cm wide and 15 cm high is enlarged so that its width becomes 25 cm. Find the height of the enlarged photo.

Q8. A triangle with sides 6 cm, 8 cm and 10 cm is enlarged so that its shortest side becomes 15 cm. Find the other two sides of the image, and show that the image is still right-angled.

Q9. In triangle ABC, D lies on AB and E lies on AC, with DE parallel to BC. Given $AD=3$, $DB=9$ and $DE=4$, find BC.

Q10. The image of an enlargement with scale factor 5 has sides 45 cm and 30 cm, perimeter 150 cm and area 400 cm². Find the matching sides, the perimeter and the area of the object.

Q11. A circle of radius 7 cm is enlarged by scale factor 2. Find the radius of the image and the exact areas of both circles, in terms of π. By what factor did the area grow?

Q12. A rectangle 8 cm by 12 cm is enlarged by scale factor $\frac{3}{2}$. Find the measurements of the image and the factor relating the areas.

Q13. A model of a water tank is made at a scale of 1:20. The model measures 6 cm by 4 cm by 5 cm.

a. Find the measurements of the real tank, in centimetres.
b. Find the volume of the real tank in litres. (1 L = 1000 cm³)
c. Write the ratio of model volume to real volume as $1:n$.

Q14. "Doubling the side of a square doubles its area." Use a square of side 3 cm to show that this claim is wrong, and state the correct factor.

Q15. A rectangular design 2 m by 1.5 m is painted on a wall with its width enlarged to 6 m.

a. Find the scale factor and the height of the wall painting.
b. Find the area of the design and of the wall painting.
c. The design used 1 L of paint. At the same rate of coverage, how many litres does the wall painting use?

Q16. A vertical post 1.5 m tall casts a shadow 2 m long on level ground. At the same time, a tree casts a shadow 9 m long on the same ground. Explain why the post-and-shadow triangle is similar to the tree-and-shadow triangle, and hence find the height of the tree.

Q17. Four cubes have sides 1 cm, 2 cm, 3 cm and 6 cm.

a. For each cube, find the surface area and the volume.
b. For each cube, find the single number surface area ÷ volume.
c. Describe what happens to this number as the cube grows.
d. A cell takes in what it needs through its surface. In one sentence, use the pattern from part c to say why cells stay small.

Q18. Mia prints a photo 4 cm by 6 cm at 150% of its size and claims the print has 1.5 times the area of the original. Find the measurements of the print, the true area factor, and explain why Mia's claim is wrong.

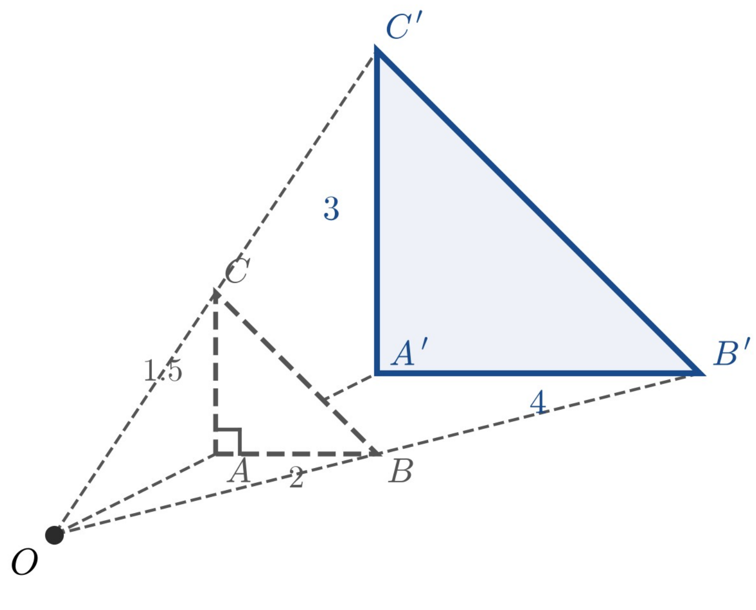
Answers

Q1 a 12 cm, 20 cm, 28 cm; b nothing — every angle is preserved. **Q2** a 2.5; b 32.5 cm; c 80 cm. **Q3** a 5 cm by 3 cm; b 60 cm² and 15 cm²; c $15 \div 60 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$. **Q4** a 35°, 65°, 80°; b 12 cm, 18 cm, 27 cm; c $\frac{12}{4} = \frac{18}{6} = \frac{27}{9} = 3$. **Q5** a 25 cm² and 100 cm²; b 4; c $4 = 2^2$ — area is multiplied by k^2. **Q6** a 24 cm³ and 648 cm³; b 27; c $27 = 3^3$ — volume is multiplied by k^3. **Q7** 37.5 cm. **Q8** $k = 2.5$; 20 cm and 25 cm; $15^2 + 20^2 = 625 = 25^2$, so the image is right-angled. **Q9** 16. **Q10** sides 9 cm and 6 cm; perimeter 30 cm; area 16 cm². **Q11** 14 cm; 49π cm² and 196π cm²; factor 4. **Q12** 12 cm by 18 cm; $\frac{9}{4}$ $(= 2.25)$. **Q13** a 120 cm by 80 cm by 100 cm; b 960 000 cm³ = 960 L; c $1:8000$. **Q14** side $3 \to 6$ cm; area $9 \to 36$ cm²; the factor is $4 = 2^2$, not 2. **Q15** a $k = 3$, height 4.5 m; b 3 m² and 27 m²; c 9 L. **Q16** 6.75 m. **Q17** a 6, 24, 54, 216 cm² and 1, 8, 27, 216 cm³; b 6, 3, 2, 1; c it falls as the cube grows; d a large cell has too little surface for its volume, so cells stay small. **Q18** 6 cm by 9 cm; $24 \to 54$ cm²; factor $\frac{9}{4} = 2.25$, not 1.5 — area grows by k^2, not k.

Fully-worked solutions

Q1. a Each side is multiplied by 4: $3 \times 4 = 12$, $5 \times 4 = 20$, $7 \times 4 = 28$. b Enlargements preserve angles: each angle of the image equals the matching angle of the object. ∴ The image has sides 12 cm, 20 cm, 28 cm and exactly the same angles as the object.

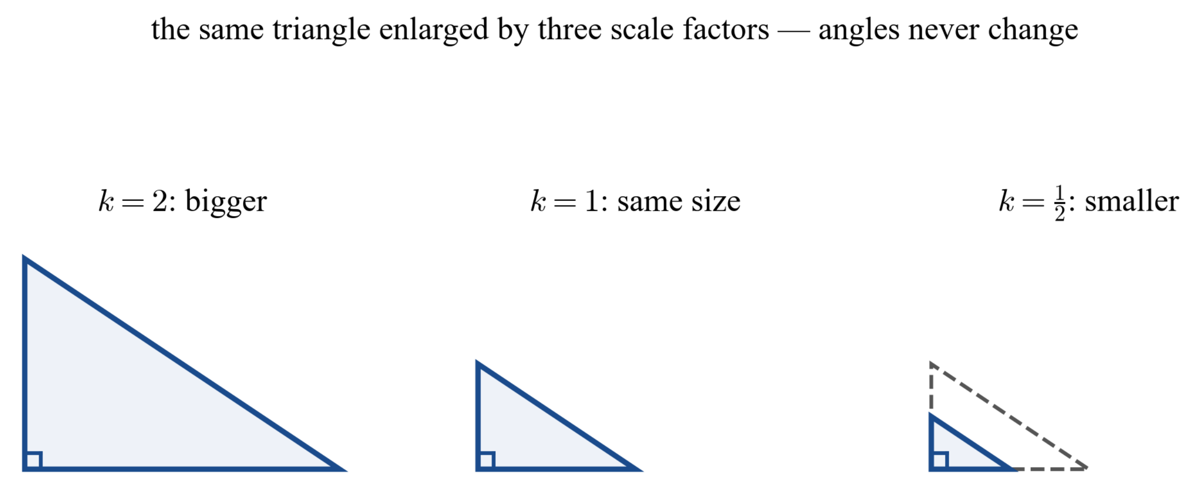
enlargement from centre O, scale factor $k = 2$: every length doubles



A dashed grey triangle enlarged from centre O by scale factor 2 to a solid blue triangle, every length doubled and every angle unchanged

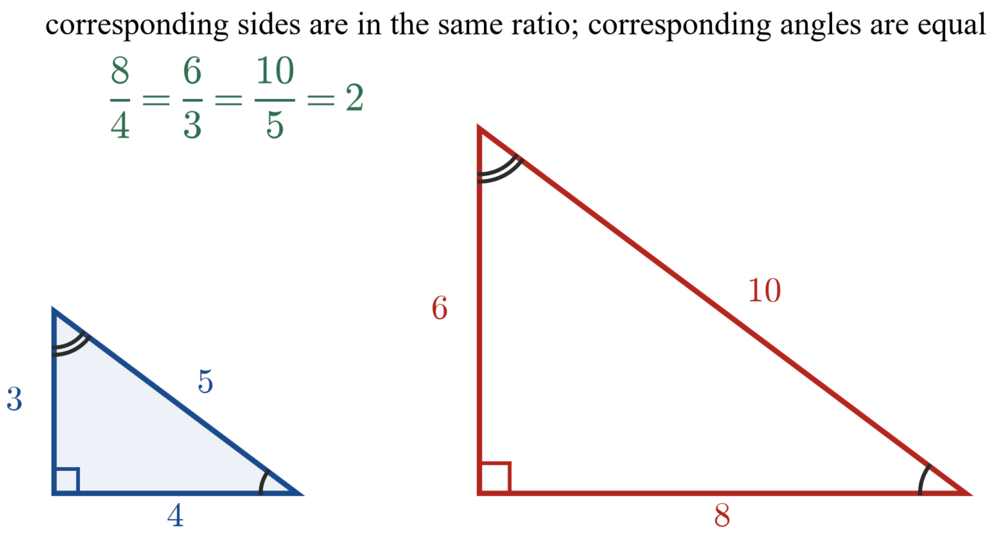
Q2. a $k = 20 \div 8 = 2.5$. b $13 \times 2.5 = 32.5$ cm. c Perimeter is a length, so it is multiplied by k: $32 \times 2.5 = 80$ cm. ∴ $k = 2.5$; the 13 cm side maps to 32.5 cm and the perimeter to 80 cm.

Q3. a $10 \times \frac{1}{2} = 5$ cm and $6 \times \frac{1}{2} = 3$ cm. b Object: $10 \times 6 = 60$ cm²; image: $5 \times 3 = 15$ cm². c $15 \div 60 = \frac{1}{4}$, and $\frac{1}{4} = \left(\frac{1}{2}\right)^2 = k^2$. ∴ The image is 5 cm by 3 cm, and the area is multiplied by $\frac{1}{4}$, the square of the scale factor.



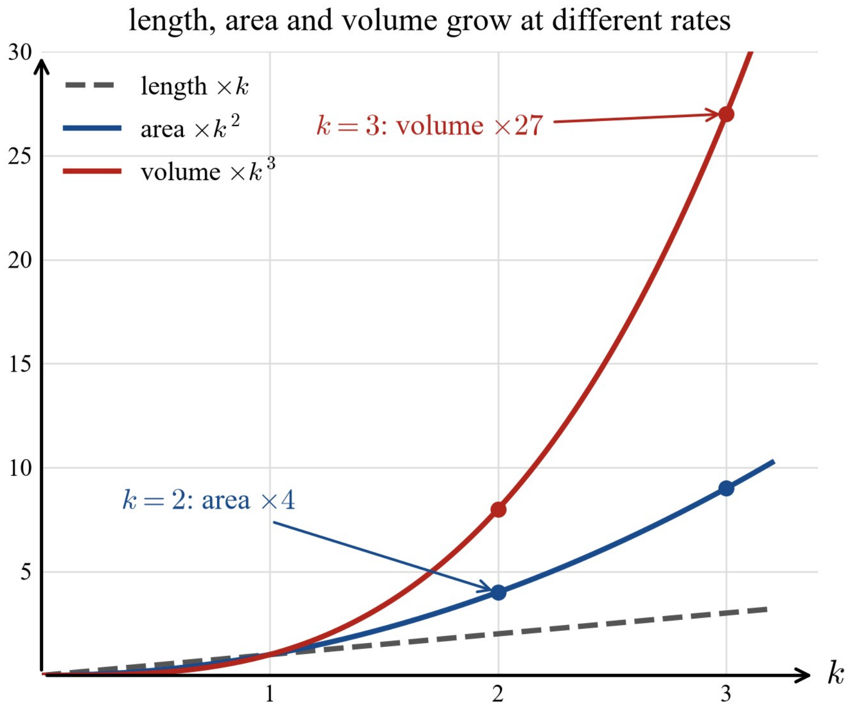
The same right-angled triangle enlarged by k = 2 (bigger), k = 1 (same) and k = one-half (smaller); the third panel shows this question's reduction

Q4. a Angles are preserved: 35°, 65°, 80°. b $4 \times 3 = 12$, $6 \times 3 = 18$, $9 \times 3 = 27$. c $\frac{12}{4} = \frac{18}{6} = \frac{27}{9} = 3$ — every pair of corresponding sides gives k. ∴ The image has the same three angles and sides 12 cm, 18 cm, 27 cm, the ratios of corresponding sides all equal to 3.



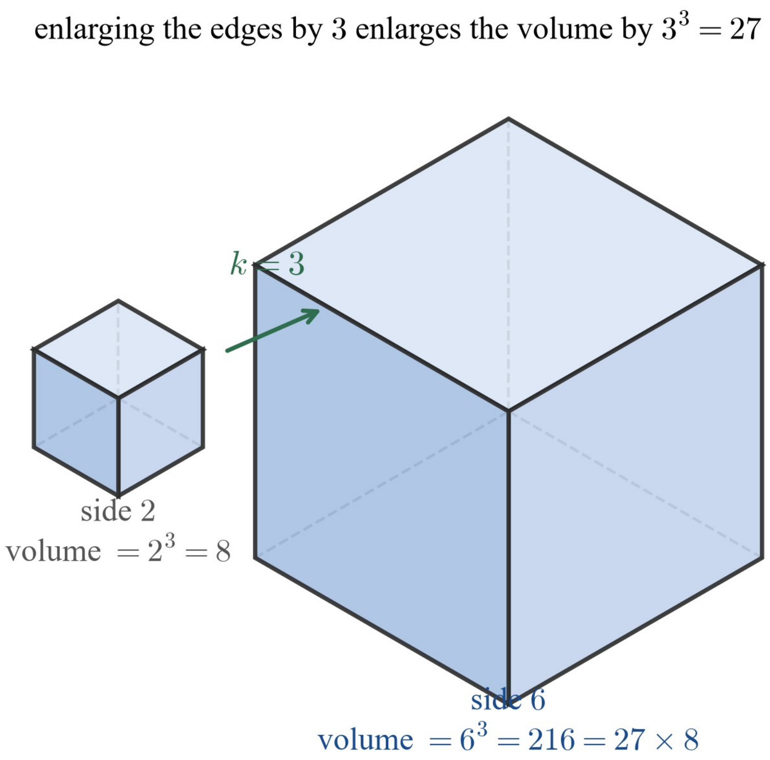
A blue 3-4-5 triangle beside a red 6-8-10 triangle with matching angle marks; the same equal-ratio check as part c with different values

Q5. a Object: $5^2 = 25$ cm²; image side $5 \times 2 = 10$ cm, area $10^2 = 100$ cm². b $100 \div 25 = 4$. c $4 = 2^2$: the area factor is the square of the scale factor — area is multiplied by k^2. ∴ Areas 25 cm² and 100 cm²; the area grows by $4 = k^2$, not by 2.



Three growth curves against scale factor k; the point (2, 4) on the area curve shows this question's factor 4 at k = 2

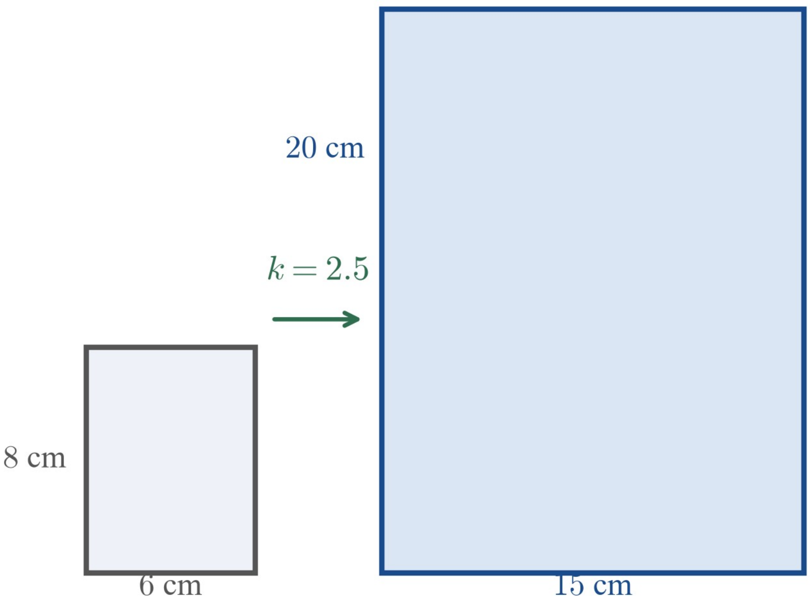
Q6. a Object: $2 \times 3 \times 4 = 24$ cm³; image sides 6, 9, 12 cm, volume $6 \times 9 \times 12 = 648$ cm³. b $648 \div 24 = 27$. c $27 = 3^3$: the volume factor is the cube of the scale factor — volume is multiplied by k^3. ∴ Volumes 24 cm³ and 648 cm³; the volume grows by $27 = k^3$.



A small cube of side 2 beside a large cube of side 6 with an arrow labelled k = 3; the same k-cubed volume pattern as this question

Q7. $k = 25 \div 10 = 2.5$. height $= 15 \times 2.5 = 37.5$ cm. Check: $\dfrac{37.5}{15} = 2.5$, the same ratio as the widths. ∴ The enlarged photo is 37.5 cm high.

a photo enlarged from 6 cm by 8 cm to 15 cm by 20 cm

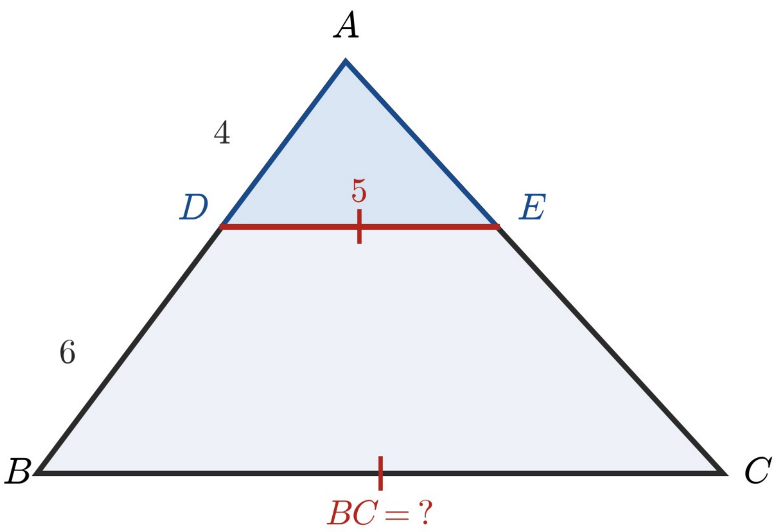


A grey 6 cm by 8 cm rectangle enlarged to a blue 15 cm by 20 cm rectangle; the same width-to-height ratio method as this question

Q8. $k=15\div6=2.5$. The other sides: $8\times2.5=20$ cm and $10\times2.5=25$ cm. Right angle: $15^2+20^2=225+400=625=25^2$, so the image satisfies the converse of Pythagoras' theorem, exactly as the object does ($6^2+8^2=10^2$). ∴ The image has sides 15 cm, 20 cm, 25 cm and is still right-angled — enlargements preserve angles.

Q9. $AB=AD+DB=3+9=12$. $DE\parallel BC$, so triangle ABC is an enlargement of triangle ADE with $k=\frac{AB}{AD}=\frac{12}{3}=4$. $BC=k\times DE=4\times4=16$. Check: $\frac{16}{4}=4=\frac{12}{3}$. ∴ $BC=16$.

DE is parallel to BC, so triangle ADE is similar to triangle ABC



Triangle ABC with D on AB and E on AC, DE in red and parallel to BC; the same similar-triangle set-up as this question

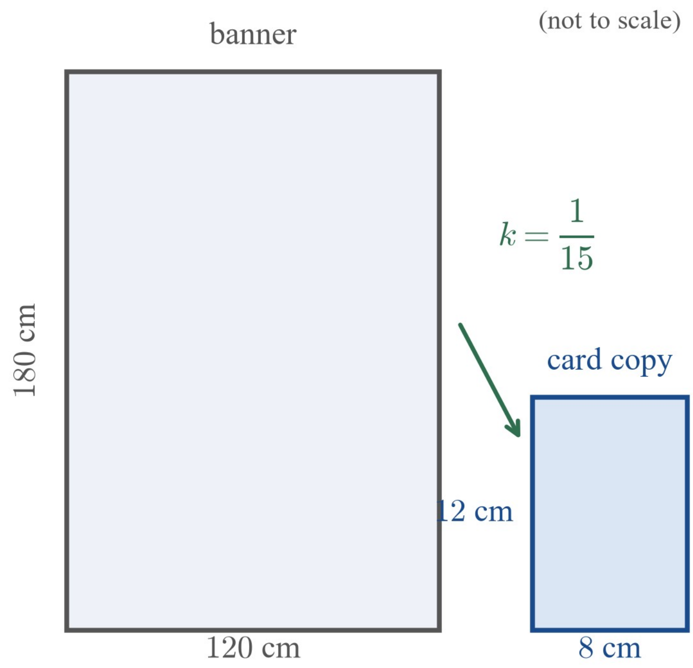
Q10. Object sides: $45\div5=9$ cm and $30\div5=6$ cm. Perimeter is a length: $150\div5=30$ cm. Area: divide by $k^2=25$: $400\div25=16$ cm². ∴ The object has sides 9 cm and 6 cm, perimeter 30 cm and area 16 cm².

Q11. Image radius: $7 \times 2 = 14$ cm. Object area: $\pi \times 7^2 = 49\pi$ cm²; image area: $\pi \times 14^2 = 196\pi$ cm². Factor: $196\pi \div 49\pi = 4 = 2^2$. ∴ Radius 14 cm; exact areas 49π cm² and 196π cm²; the area grew by $4 = k^2$.

Q12. Image: $8 \times \frac{3}{2} = 12$ cm and $12 \times \frac{3}{2} = 18$ cm. Areas: $8 \times 12 = 96$ cm² and $12 \times 18 = 216$ cm²; factor $216 \div 96 = \frac{9}{4}$. And $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ — the k^2 rule holds for a fraction greater than 1. ∴ The image is 12 cm by 18 cm and the areas are related by $\frac{9}{4}$.

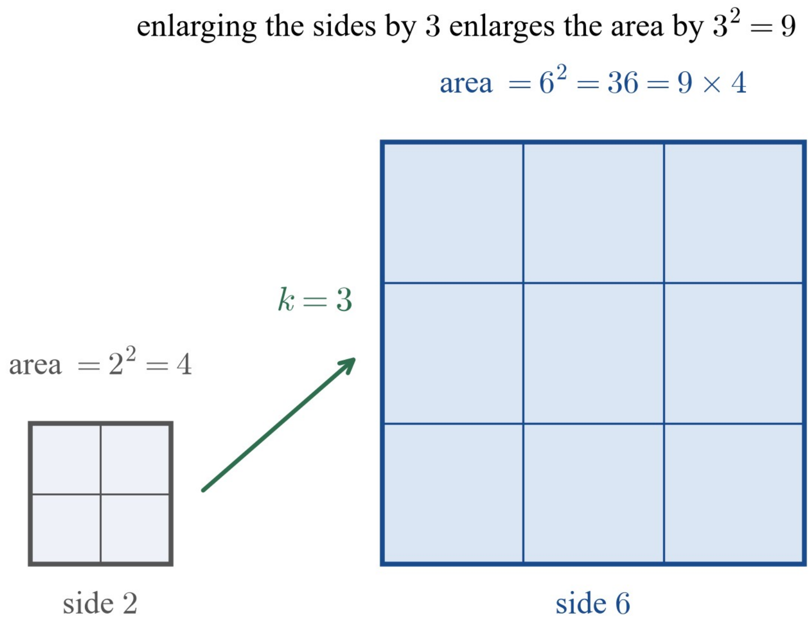
Q13. a The model is a reduction of the real tank with $k = \frac{1}{20}$, so the real tank is 20 times the model: $6 \times 20 = 120$ cm, $4 \times 20 = 80$ cm, $5 \times 20 = 100$ cm. b Real volume: $120 \times 80 \times 100 = 960\,000$ cm³ $= 960$ L. c Model volume: $6 \times 4 \times 5 = 120$ cm³; ratio $120 : 960\,000 = 1 : 8000$, and $8000 = 20^3$. ∴ The real tank is 120 cm by 80 cm by 100 cm, holds 960 L, and the volumes are in the ratio $1 : 8000 = 1 : 20^3$.

the 120 cm by 180 cm banner reduced to 8 cm by 12 cm



A large banner rectangle reduced by k = one-fifteenth to a small card copy; the same reduction idea as a 1:20 model

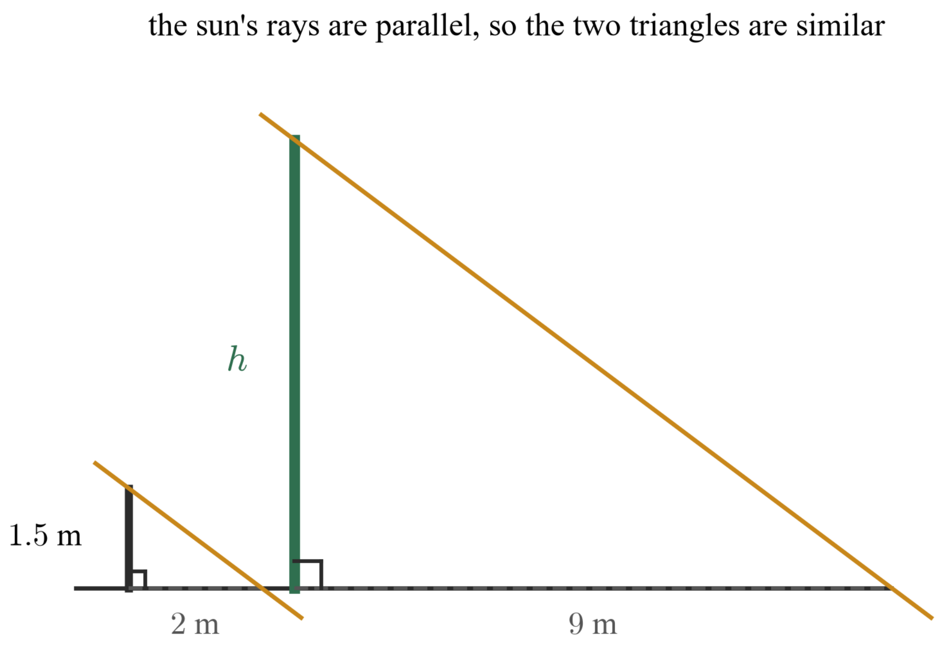
Q14. Object: side 3 cm, area $3^2 = 9$ cm². Image: side $3 \times 2 = 6$ cm, area $6^2 = 36$ cm². Factor: $36 \div 9 = 4$, not 2. ∴ Doubling the side multiplies the area by $4 = 2^2$ — the claim is wrong; areas follow k^2.



A 2 by 2 square enlarged by k = 3 to a 6 by 6 square made of nine copies; the same area-times-k-squared pattern the 3 cm square shows with k = 2

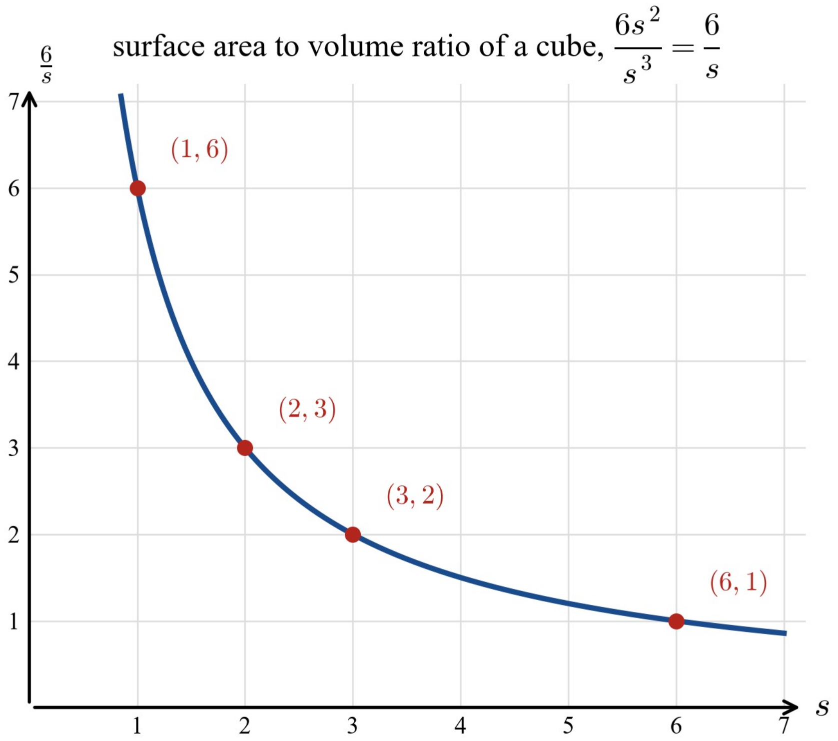
Q15. a $k = 6 \div 2 = 3$; height $= 1.5 \times 3 = 4.5$ m. b Design: $2 \times 1.5 = 3$ m²; wall: $6 \times 4.5 = 27$ m². c Area factor $27 \div 3 = 9 = 3^2$, so paint $= 1 \times 9 = 9$ L. $\therefore$ $k = 3$, the wall painting is 6 m by 4.5 m with area 27 m², and it uses 9 L of paint.

Q16. Both the post and the tree stand at right angles to the level ground, so both triangles have a right angle. The sun's rays are parallel, so the ray makes the same angle with the ground in both triangles; two pairs of equal angles make the triangles similar. Ratio of corresponding sides: $\frac{h}{1.5} = \frac{9}{2}$, so $h = 1.5 \times \frac{9}{2} = \frac{27}{4} = 6.75$ m. Check: $\frac{6.75}{9} = 0.75 = \frac{1.5}{2}$ — the same ratio of height to shadow in both triangles. $\therefore$ The tree is 6.75 m tall.



A 1.5 m post with a 2 m shadow and a tree of height h with a 9 m shadow, the parallel sun rays making two similar right-angled triangles

Q17. a Surface areas $6s^2$: 6, 24, 54, 216 cm². Volumes s^3: 1, 8, 27, 216 cm³. b $6 \div 1 = 6$; $24 \div 8 = 3$; $54 \div 27 = 2$; $216 \div 216 = 1$. c The number falls — 6, then 3, then 2, then 1 — as the cube grows; it is $\frac{6}{s}$. d A larger cell has less surface for each cubic unit of volume, so it cannot take in enough through its surface — which is why cells stay small. ∴ The surface-area-to-volume ratio falls from 6 to 1 as the side grows from 1 cm to 6 cm.



The falling curve of 6 over s with the points (1, 6), (2, 3), (3, 2) and (6, 1) marked — the ratios computed in part b

Q18. 150 % means $k = \frac{3}{2}$; the print is $4 \times \frac{3}{2} = 6$ cm by $6 \times \frac{3}{2} = 9$ cm. Areas: original $4 \times 6 = 24$ cm²; print $6 \times 9 = 54$ cm². Factor: $54 \div 24 = \frac{9}{4} = 2.25$, and $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$. ∴ The print has 2.25 times the area, not 1.5 times — Mia multiplied by k where area must be multiplied by k^2.

Designing, testing and refining algorithms from geometric constructions and theorems

4C · Chapter 4 — Space: similarity, the trigonometric ratios and geometric algorithms · Level 9

> **Content description `VC2M9SP03` (word for word):** *design, test and refine algorithms involving a sequence of steps and decisions based on geometric constructions and theorems; discuss and evaluate refinements*
>
> **Achievement standard anchor (AS 15):** *They design, use and test algorithms based on geometric constructions or theorems.*
>
> **Elaboration ids for linkage:** `VC2M9SP03.E01`, `VC2M9SP03.E02`, `VC2M9SP03.E03`
>
> **Prerequisite pointer:** none within this book. This subtopic stands on assumed Level 8 knowledge — Pythagoras' theorem (`VC2M8M06`) and the angle facts of a triangle — which is rehearsed below as a tool, not re-taught.

Theory

What an algorithm is

An **algorithm** is a finite set of instructions that finishes a task. The instructions are a sequence of **steps** and **decisions**: a step tells you what to do, and a decision asks a yes-or-no question and sends you down one path or the other.

In this subtopic, algorithms are written in one of two ways:

1. **Numbered steps** — a list you follow in order, one line at a time.
2. **A flow chart** — a diagram of the same steps, built from standard shapes:
   - a rounded box marks the **start** or the **end**;
   - a rectangle is a **step** — an input, a calculation, or an output;
   - a diamond is a **decision** with a yes-or-no answer;
   - arrows show the order the steps run in.

The same algorithm can be written both ways. A flow chart is often easier to follow, and numbered steps are often easier to write; both are used here.

The cycle: design, test, refine, evaluate

This subtopic runs one cycle over and over.

- **Design.** Write a first version of the algorithm. It does not have to be perfect — it has to be clear enough to follow.
- **Test.** Run the algorithm on **test cases**. Each test case records three things: the **input**, the **expected output**, and the **actual output**. Always include cases that should **pass** and cases that should **fail**. A test set that only checks easy, successful cases proves very little. Cases that sit on the **boundary** — the edge between a pass and a fail — are the most useful of all.
- **Refine.** When a test case fails, find the step at fault and fix the algorithm. Write the refinement down, then re-run *every* test case, not just the one you fixed. A fix that breaks a case that used to pass is a new flaw, not a fix.
- **Evaluate.** Judge the finished algorithm. What does it always get right? What can it never check? Could someone else follow your steps and get the same result?

The theorems our algorithms use

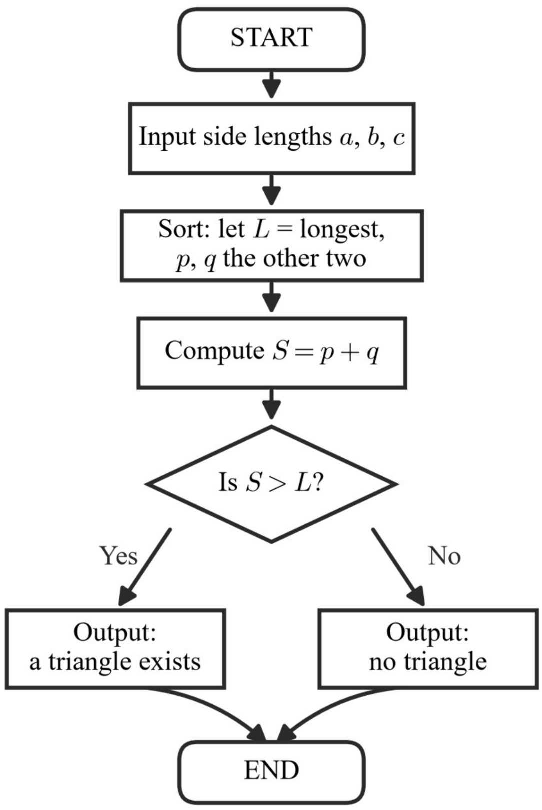
A **theorem** is a mathematical statement that is known to be always true. At Level 9 you *use* theorems as reliable tools; proving them belongs to Level 10. This subtopic uses three facts you already know.

1. **Pythagoras' theorem.** In a right-angled triangle, the square on the **hypotenuse** — the longest side, opposite the right angle — equals the sum of the squares on the other two sides. The **converse** runs the test backwards: if the square on the longest side of a triangle equals the sum of the squares on the other two sides, then the angle opposite the longest side is a right angle.
2. **The triangle inequality.** Three lengths form a triangle when the sum of the two shorter lengths is greater than the longest length. If the sum equals the longest length, the three sides sit flat in a straight line and no triangle exists. If the sum is less, the two short sides can never reach each other.
3. **The angle sum of a triangle.** The three angles of any triangle add to 180°.

Algorithm 1 — Is it a triangle?

1. Input three side lengths a, b and c.
2. Sort them: call the longest L, and call the other two p and q.
3. Compute $S = p + q$.
4. Decision: is $S > L$? If yes, output "a triangle exists". If no, output "no triangle".

The same algorithm as a flow chart:

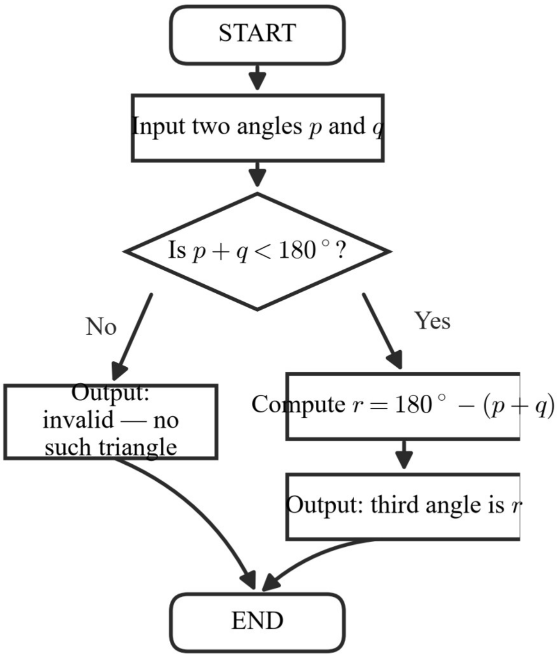


Algorithm 2 — Is it right-angled?

1. Input three side lengths.
2. Call the longest L; call the other two a and b.
3. Compute $a^2 + b^2$ and compute L^2.
4. Decision: is $a^2 + b^2 = L^2$? If yes, output "right-angled". If no, output "not right-angled".

Algorithm 3 — The third angle

1. Input two angles p and q of a triangle.
2. Decision: is $p+q<180°$? If no, output "invalid — no such triangle" and stop.
3. Compute $r=180°-(p+q)$.
4. Output "the third angle is r".



Algorithm 4 — A Pythagorean-triple generator

A **Pythagorean triple** is a set of three whole-number side lengths that passes the right angle test; $(3,4,5)$ is the smallest example. Choose whole numbers m and n with $m>n\geq 1$; the three lengths

$$m^2-n^2, 2mn, m^2+n^2$$

always pass the right angle test, and the hypotenuse is always m^2+n^2.

1. Input m and n.
2. Decision: is $m>n\geq 1$? If no, output "invalid input" and stop — otherwise m^2-n^2 is zero or negative, and a length cannot be either.
3. Compute m^2-n^2, $2mn$ and m^2+n^2.
4. Output the three numbers as a triple.

Construction algorithms

Equipment for this subtopic: ruler, compasses, protractor.

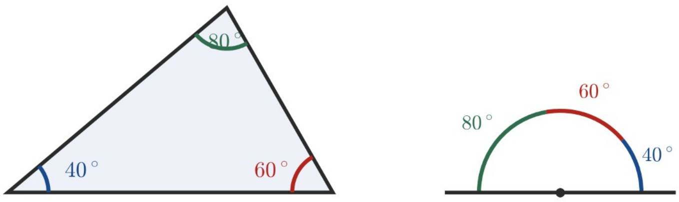
A geometric **construction** draws an exact figure using a ruler (straight lines only — never used to measure lengths into place) and compasses (arcs and circles). The protractor is used to *test* the finished construction, never to draw it. Three constructions appear in this subtopic.

- **A 60° angle.** Each angle of an equilateral triangle is $180° \div 3 = 60°$. Draw a base AB, set the compasses to the length of the base, and draw an arc from each end; where the arcs cross is the third corner C, and angle CAB is 60°.
- **The bisector of an angle.** To cut an angle exactly in half: from the vertex, draw an arc that crosses both arms (label the crossings P and Q); from P and from Q, draw two arcs with the **same** compass width so that they cross at R; draw the ray from the vertex through R.
- **The perpendicular bisector of a segment.** From each end of the segment, draw arcs with the same compass width, one pair above the segment and one pair below; join the two crossing points. The joining line cuts the segment in half at 90°.

Each construction is itself an algorithm, and each one is tested the same way: run it, then measure the result with a protractor or ruler and compare with the expected value.

Visual proof as an animation

An **animation** is a sequence of still pictures — **frames** — that shows a construction one step at a time. Planning the frames before drawing them (a frame-by-frame plan is called a **storyboard**) is a way of designing an algorithm, and an animation is tested like any other algorithm: run a case that should work and a case that should fail, and check every frame.



The three angles of a triangle …… fit exactly along a straight line: $40 + 60 + 80 = 180°$

The picture above is the final frame of a visual argument for the angle sum of a triangle: the three corners of a 40°–60°–80° triangle have been moved down onto a straight line, where they fill 180° exactly.

A note about proof

Testing an algorithm on many cases builds strong evidence, but checking cases — even very many cases — is not the same as a formal proof. A proof shows that a result *must* hold in every case, and deductive proof belongs to Level 10. In this subtopic, an algorithm that passes all of its test cases is trusted for the cases it was tested on, and you will be asked to say exactly what it has and has not shown.

Worked examples

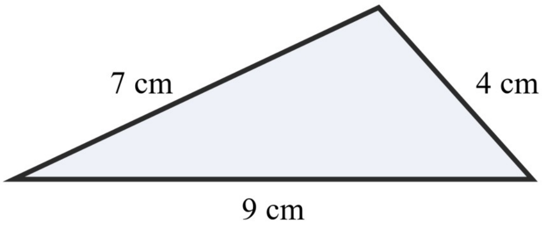
Example 1 — scaffolded

Do side lengths 4 cm, 9 cm and 7 cm form a triangle? Trace Algorithm 1.

Working	Comment
Input: $a=4$, $b=9$, $c=7$	Step 1: three side lengths.
$L=9$, $p=4$, $q=7$	Step 2: sort — 9 is the longest.
$S=4+7$	Step 3: add the two shorter sides.
$S=11$	One step at a time.
$11>9$, so $S>L$	Step 4: the decision is yes.
Output: a triangle exists	The two shorter sides are long enough to reach each other.

The figure shows the triangle; the check $4+7=11>9$ is exactly the decision the algorithm made.

two shorter sides: $4+7=11>9$ (triangle exists)



Example 2 — scaffolded

Trace Algorithm 1 for (a) 2 cm, 3 cm, 7 cm and (b) 1 cm, 2 cm, 3 cm.

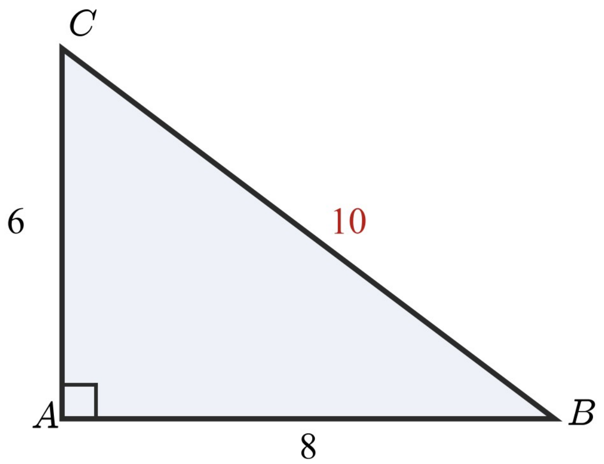
Working	Comment
(a) $L=7,\ p=2,\ q=3$	Step 2: sort.
$S=2+3=5$	Step 3: add the two shorter sides.
$5<7$, so S is not greater than L	Step 4: the decision is no.
Output: no triangle	The 2 cm and 3 cm sides can never reach each other. This is a **fail case** — the algorithm correctly says no.
(b) $L=3,\ p=1,\ q=2$	Sort.
$S=1+2=3$	Add the two shorter sides.
$S=L$, not greater	The decision asks whether $S>L$; $3>3$ is false.
Output: no triangle	The sides sit flat in a straight line. This **boundary case** sits exactly on the edge between a pass and a fail — the most useful kind of test case.

Example 3 — scaffolded

Is a triangle with sides 6 cm, 8 cm and 10 cm right-angled? Trace Algorithm 2.

Working	Comment
$L=10$	Step 2: the longest side is the only possible hypotenuse.
$a^2+b^2=6^2+8^2$	Step 3: square the two shorter sides.
$a^2+b^2=36+64$	One step at a time.
$a^2+b^2=100$	
$L^2=10^2$	Square the longest side.
$L^2=100$	
$100=100$, so $a^2+b^2=L^2$	Step 4: the decision is yes.
Output: right-angled	By the converse of Pythagoras' theorem, the angle opposite the 10 cm side is 90°.

$$6^2 + 8^2 = 36 + 64 = 100 = 10^2$$

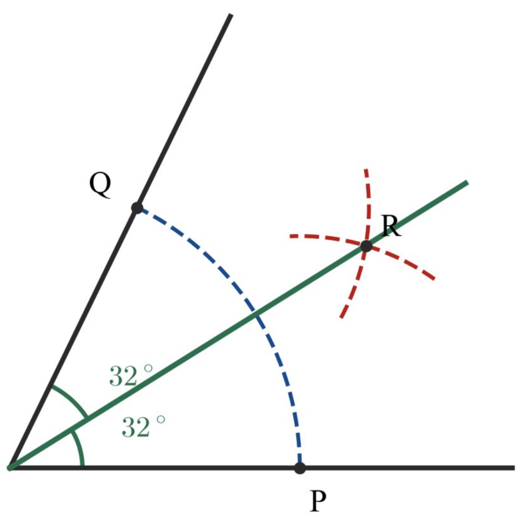


Example 4 — scaffolded

Write a construction algorithm that bisects a 64° angle, then test it.

Working	Comment
1. Draw an angle of 64° with vertex O	The angle to bisect. A protractor is used here only because the angle is *given*.
2. With centre O, draw an arc crossing both arms; label the crossings P and Q	Any compass width works; P and Q end up the same distance from O.
3. With centre P, draw an arc inside the angle	Keep the compass width the same for this step and the next.
4. With centre Q, draw an arc of the same width crossing the first arc; label the crossing R	R ends up the same distance from P as from Q.
5. Draw the ray from O through R	This ray is the bisector.
Test: measure both halves with a protractor	Expected: two equal angles of $64° \div 2 = 32°$.
$32° + 32° = 64°$	The readings are 32° and 32° — this test case **passes**.

Bisecting a $64°$ angle: each half is $32°$



A fail case to watch. If step 4 uses a *different* compass width from step 3, the arcs cross at a point that is not on the bisector. A protractor check then reads two unequal angles — say 40° on one side and 24° on the other, which add to 64° but are not equal. The construction fails.

Refinement (written): *use the same compass width for the arcs in steps 3 and 4.*

Example 5 — unscaffolded

Alex's first draft of a triangle test says: "Add the first two numbers you are given. If the sum is greater than the third number, the lengths form a triangle." Test the draft on (a) 3 cm, 4 cm, 5 cm and (b) 2 cm, 9 cm, 4 cm, then refine it.

(a) Input as given: $a=3$, $b=4$, $c=5$. $3+4=7$. $7>5$, so the draft says "triangle". Here that is correct — the draft passes this case.

(b) Input as given: $a=2$, $b=9$, $c=4$. $2+9=11$. $11>4$, so the draft says "triangle". But the two shorter sides are 2 and 4, and $2+4=6<9$ — no triangle exists. The draft has passed a case that should fail: its flaw is assuming the third number given is the longest side.

Refinement (written): *sort the three lengths first, then test whether the sum of the two shorter is greater than the longest.* This is exactly Algorithm 1.

Re-run every test case on the refined algorithm: $(3,4,5)$: sorted, $L=5$; $3+4=7>5$ — triangle, correct. $(2,9,4)$: sorted, $L=9$; $2+4=6$, and 6 is not greater than 9 — no triangle, correct. ∴ The refined algorithm passes both the case that should pass and the case that should fail.

Example 6 — unscaffolded

Use Algorithm 2 to decide whether each triangle is right-angled: (a) 5 cm, 12 cm, 13 cm; (b) 5 cm, 6 cm, 7 cm.

(a) $L=13$; the other sides are 5 and 12. $5^2+12^2=25+144$. $25+144=169$. $13^2=169$. $169=169$, so $a^2+b^2=L^2$. Output: right-angled — the angle opposite the 13 cm side is 90°.

(b) $L=7$; the other sides are 5 and 6. $5^2+6^2=25+36$. $25+36=61$. $7^2=49$. $61\neq 49$, so $a^2+b^2\neq L^2$. Output: not right-angled.

The order of the input does not matter — Algorithm 2 sorts the sides first, so $(13,5,12)$ gives the same output as $(5,12,13)$. Together, part (a) is a case that passes and part (b) is a case that fails, exactly as a good test set requires. ∴ $(5,12,13)$ is right-angled and $(5,6,7)$ is not.

Example 7 — unscaffolded

Use Algorithm 4 with $m=4$ and $n=1$ to generate a triple, and test it. Then explain why $m=4$, $n=2$ and $m=3$, $n=1$ produce nothing new, and refine the algorithm.

Input $m=4$, $n=1$; the decision holds since $4>1\geq 1$. $m^2-n^2=16-1=15$. $2mn=2\times 4\times 1=8$. $m^2+n^2=16+1=17$. The triple is $(15,8,17)$. Test with Algorithm 2: $15^2+8^2=225+64=289$, and $17^2=289$, so the triple passes.

Now $m=4$, $n=2$: the output is $(12,16,20)$. It passes the right angle test, but it is just $4\times(3,4,5)$ — a multiple of a triple we already have. And $m=3$, $n=1$ gives $(8,6,10)=2\times(3,4,5)$ — the same smaller triple again.

Refinement (written): *to produce a brand-new triple — one that is not a multiple of a smaller triple — add two conditions to step 2: m and n must have no common factor, and they must not both be odd.*

Check the refinement on $m=5$, $n=2$: no common factor, and one is even, so both conditions hold. The output is $(21,20,29)$, and $21^2+20^2=441+400=841=29^2$ — a new triple that passes the test. ∴ The generator works, and with the two added conditions it produces brand-new triples such as $(15,8,17)$ and $(21,20,29)$.

Example 8 — unscaffolded

Using a ruler and compasses only, construct an angle of 30°. Test your construction.

Step 1 — construct a 60° angle. Draw a base AB. Set the compasses to the length of AB. With centre A, draw an arc above the base. With centre B and the same width, draw an arc crossing the first arc at C. Draw the ray from A through C. Triangle ABC has three equal sides, so it is equilateral, and angle $CAB = 180° \div 3 = 60°$.

Step 2 — bisect angle CAB using the five steps of Example 4. The bisector cuts the 60° angle into two equal halves: $60° \div 2 = 30°$.

Test: measure the new angle with a protractor. Expected: 30°. Reading: 30°. The test case passes. ∴ A 60° construction followed by a bisection produces a 30° angle. (Bisecting again gives 15° — see Question 25.)

Practice questions

Equipment for the construction questions: ruler, compasses, protractor.

Scaffolded

Q1. Decide whether 5 cm, 6 cm and 9 cm form a triangle.

a. Which length is L? Which are p and q?
b. Compute $S = p + q$.
c. Is $S > L$? Write your conclusion.

Q2. Decide whether 3 cm, 5 cm and 8 cm form a triangle.

a. Sort the lengths and name L, p and q.
b. Compute $S = p + q$ and compare it with L.
c. What is special about this case? (Compare S and L very carefully.)

Q3. Two sides of a triangle are 7 cm and 10 cm. The third side is a whole number of centimetres.

a. The third side must be greater than $10 - 7$. Why?
b. The third side must be less than $7 + 10$. Why?
c. List all possible values of the third side. How many are there?

Q4. Trace the flow chart of Algorithm 1 for the input 4 cm, 4 cm, 4 cm. Write down L, p, q, S, the answer to the decision, and the output.

Q5. Is a triangle with sides 9 cm, 12 cm and 15 cm right-angled?

a. Which side is L?
b. Compute $a^2 + b^2$ for the two shorter sides.
c. Compute L^2, compare, and write your conclusion.

Q6. Use Algorithm 4 with $m = 3$ and $n = 2$.

a. Compute $m^2 - n^2$.
b. Compute $2mn$.
c. Compute $m^2 + n^2$ and write the triple.
d. Check your triple with Algorithm 2.

Q7. Two angles of a triangle are 62° and 48°.

a. Compute $p + q$.
b. Is $p + q < 180°$?
c. Use Algorithm 3 to find the third angle.

Q8. Two angles of a triangle are claimed to be 113° and 72°.

a. Compute $p+q$.
b. What does the decision in Algorithm 3 say?
c. Explain why that answer makes sense.

Q9. The five steps of the angle-bisector construction from Example 4 have been jumbled.

i. From Q, draw an arc of the same width crossing the first arc at R.
ii. Draw the ray from O through R.
iii. From O, draw an arc crossing both arms at P and Q.
iv. From P, draw an arc inside the angle.
v. Draw an angle with vertex O.
a. Write the steps in the correct order.
b. Apply the construction to a 70° angle. If the construction is correct, what should each half measure?

Q10. Copy and complete the 60° construction.

"Draw a base AB. Set the compasses to the length of ______. Draw an arc from ______, then an arc of the same width from B; label the crossing C. Triangle ABC is ______, so each of its angles is ______°."

Q11. Copy and complete the table using Algorithm 1.

Side lengths	S (sum of the two shorter)	L	Is $S>L$?	Triangle?
6, 8, 10	…	…	…	…
1, 1, 3	…	…	…	…
7, 7, 7	…	…	…	…

Q12. The triple $(8,15,17)$ passes the right angle test.

a. Verify this by calculation.
b. Doubling every side gives $(16,30,34)$. Verify that this triple also passes the test.
c. What do you notice about a multiple of a triple?

Unscaffolded

Q13. Which of these sets of lengths form a triangle? For each, show the comparison that decides it.

a. 4 cm, 5 cm, 8 cm
b. 4 cm, 5 cm, 9 cm
c. 2 cm, 2 cm, 2 cm
d. 6 cm, 9 cm, 16 cm

Q14. Two sides of a triangle are 12 cm and 5 cm, and the third side is a whole number of centimetres. Find the smallest and the largest possible length of the third side.

Q15. Two sides of a triangle are both 6 cm, and the third side x is a whole number of centimetres. Find every possible value of x.

Q16. Mia's triangle test checks only the first pair of sides it is given: "Is $a+b>c$? If yes, the lengths form a triangle."

a. Show that her test gives the right answer for 5 cm, 7 cm, 9 cm.
b. Show that her test gives the wrong answer for 4 cm, 9 cm, 3 cm.
c. Write the refinement that fixes her algorithm.
d. Re-run both cases on the refined algorithm.

Q17. Use Algorithm 2 to decide whether each triangle is right-angled.

a. 7 cm, 24 cm, 25 cm

b. 8 cm, 11 cm, 14 cm
c. 20 cm, 21 cm, 29 cm

Q18. Use Algorithm 4 with $m=5$ and $n=2$ to generate a triple, then verify it with Algorithm 2.

Q19. Use Algorithm 4 with $m=5$ and $n=3$.

a. What triple do you get?
b. Show that it is a multiple of a smaller triple.
c. Which of the two refinement conditions from Example 7 do 5 and 3 break?

Q20. The triple $(3,4,5)$ passes the right angle test.

a. Verify that $(12,16,20)=4\times(3,4,5)$ also passes.
b. Explain why multiplying every side of a right-angled triangle by the same whole number always gives another right-angled triangle. (What happens to each squared length?)

Q21. Design an algorithm that takes two angles of a triangle as input and either outputs the third angle or reports that no such triangle exists. Write it as numbered steps, including a decision. Then test it on

a. 60° and 70° — a case that should pass;
b. 100° and 90° — a case that should fail.

Q22. Using a ruler and compasses only, construct an angle of 120°. (Start from the 60° construction of Question 10, then extend one of the rays.) Test your construction with a protractor, and explain why it must work.

Q23. An algorithm for the perpendicular bisector of a segment PQ says: "From P and from Q, draw arcs of any compass width, above and below PQ. Join the two crossing points."

a. For $PQ=8$ cm, show that the construction works with a width of 5 cm.
b. Show that it fails with a width of 3 cm. What goes wrong?
c. Write the refinement that fixes the algorithm.
d. Why must the width from P equal the width from Q?

Q24. Apply the angle-bisector algorithm of Example 4 to a straight angle of 180°. What angle do you get, and what is the special name for the line you have constructed?

Q25. In Example 8 you constructed 30°. Bisect that angle to construct 15°. What should each half measure, and what does your protractor read?

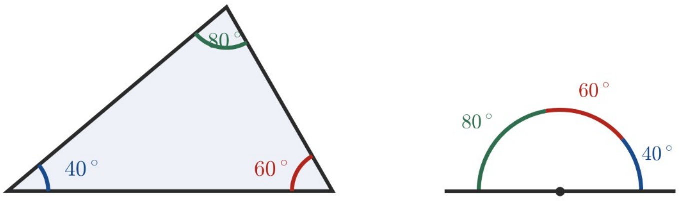
Q26. Trace Algorithm 2 for the sides given in the order 13 cm, 5 cm, 12 cm. What is the output? Explain why the order of the input cannot change the answer.

Q27. Jordan claims: "To test whether three lengths form a triangle, it is enough to add the two *shorter* sides and compare with the longest — the other two sums never need checking." Evaluate Jordan's claim using the lengths 4 cm, 7 cm and 9 cm. Is Jordan right? Give reasons.

Q28. Storyboard a six-frame animation of the perpendicular-bisector construction for a 6 cm segment PQ, using a compass width of 4 cm. State what is drawn in each frame, then test your storyboard.

a. Do the arcs meet? Show the calculation.
b. What happens if every frame is redrawn with $PQ=10$ cm but the width kept at 4 cm? What does this tell you about choosing the width?

Q29. The figure below is the final frame of an animation that argues for the angle sum of a triangle: the three corners of a 40°–60°–80° triangle have been moved onto a straight line.



The three angles of a triangle …… fit exactly along a straight line: $40+60+80=180°$

a. Write the storyboard of this animation as five frames.
b. Test the same animation on a triangle with angles 50°, 60° and 70°. What should the final frame show?
c. Evaluate: does this animation prove that the angles of *every* triangle add to 180°? What exactly has it shown?

Q30. A group designs an algorithm to list every right-angled triangle with whole-number sides and longest side at most 13 cm: “For each whole number L from 2 to 13, check whether L^2 can be split into the sum of two smaller squares.” Run the algorithm and list every triangle it finds. Organise your working in a table.

Q31. Copy and complete the table for Algorithm 4 with $n=1$.

m	m^2-n^2	$2mn$	m^2+n^2	Triple	Brand-new, or a multiple?
2	…	…	…	…	…
3	…	…	…	…	…
4	…	…	…	…	…
5	…	…	…	…	…

Q32. The three lengths n, $n+1$ and $n+2$ are given, where n is a whole number.

a. Test $n=1$: do the lengths form a triangle?
b. Test $n=2$ and $n=3$.
c. The sum of the two shorter sides is $n+(n+1)=2n+1$, and the longest side is $n+2$. For which values of n is $2n+1>n+2$? State the pattern.

Q33. Sam tests whether 4 cm, 5 cm and 6 cm form a right-angled triangle. He writes: “$4+5=9$, and $9>6$, so the triangle is right-angled.”

a. Sam’s answer is wrong in two ways. What are they?
b. Apply the correct algorithm to $(4,5,6)$ and give the correct output.

Q34. Linh’s 60° construction says: “Draw a base AB. Choose any compass width. Draw an arc from A, then an arc of the same width from B. Join the crossing point to A and to B.”

a. Test her construction with a width clearly shorter than AB — say $AB=6$ cm with a width of 4 cm. What do you notice about the angles at the base?
b. Explain why her algorithm does not always produce 60°.
c. Write the refinement that fixes it, and explain why the fixed version must give 60°.

Q35. A gardener makes triangular garden beds from three timbers of lengths a, b and c.

a. Design an algorithm that decides whether three timbers can form a triangular bed.
b. Give one test case that should pass and one that should fail, and run both through your algorithm: use 30 cm, 40 cm, 50 cm for the pass case and 10 cm, 30 cm, 60 cm for the fail case.
c. Evaluate your algorithm: what does it check, and what can it never check about a real garden bed?

Q36. Choose any one algorithm you designed in this subtopic (your own from Question 21, 22 or 30 is a good place to start). Invent two new test cases — one that should pass and one that should fail — and run them. Then evaluate your algorithm: write down one strength and one limit, and refine the algorithm if your test cases uncover a flaw.

Answers

Q1 Yes — $5+6=11>9$. **Q2** No — $3+5=8=L$; the sides sit flat. **Q3** The third side can be 4, 5, 6, …, 16 cm — 13 values. **Q4** $L=4$, $p=4$, $q=4$, $S=8$; $8>4$ — output: a triangle exists. **Q5** Yes — $9^2+12^2=225=15^2$. **Q6** $(5,12,13)$; $5^2+12^2=169=13^2$. **Q7** 70°. **Q8** Invalid — $113°+72°=185°\geq 180°$. **Q9** a v, iii, iv, i, ii; b 35°. **Q10** AB; A; equilateral; 60. **Q11** $(6,8,10)$: $S=14$, $L=10$, yes, triangle. $(1,1,3)$: $S=2$, $L=3$, no. $(7,7,7)$: $S=14$, $L=7$, yes, triangle. **Q12** a $8^2+15^2=289=17^2$; b $16^2+30^2=1156=34^2$; c a multiple of a triple is a triple. **Q13** a yes; b no; c yes; d no. **Q14** Smallest 8 cm, largest 16 cm. **Q15** $x=1,2,3,\ldots,11$ cm — 11 values. **Q16** a $5+7=12>9$, correct; b $4+9=13>3$ says "triangle", but $4+3=7<9$ — wrong; c sort first, then compare the two shorter sides with the longest; d both cases now come out correctly. **Q17** a yes; b no; c yes. **Q18** $(21,20,29)$; $21^2+20^2=841=29^2$. **Q19** a $(16,30,34)$; b $2\times(8,15,17)$; c 5 and 3 are both odd. **Q20** a $12^2+16^2=400=20^2$; b every squared length is multiplied by the same square number, so the equality still holds. **Q21** Steps: input p, q; if $p+q\geq 180°$, output "invalid"; otherwise output $180°-(p+q)$. a 50°; b invalid. **Q22** Construct 60° and extend the base past the vertex: $180°-60°=120°$; protractor reads 120°. **Q23** a $2\times5=10>8$ — arcs meet; b $2\times3=6<8$ — arcs never meet; c the width must be more than half of PQ; d each crossing point must be the same distance from P as from Q. **Q24** 90° — the line is perpendicular to the original ray. **Q25** 15°; the protractor reads 15°. **Q26** Right-angled; the algorithm sorts first, so order cannot matter. **Q27** Jordan is right — once the two shorter sides beat the longest, the two sums that include the longest side are automatically larger still. **Q28** Frames: 1 segment PQ; 2 arc from P above; 3 arc from P below; 4 arc from Q above, crossing at X; 5 arc from Q below, crossing at Y; 6 line XY meeting PQ at its midpoint. a $2\times4=8>6$ — they meet; b with $PQ=10$, $2\times4=8<10$ — the arcs fail to meet, so the width must be more than half the segment. **Q29** a See solution; b the three angles again fill the straight line: $50°+60°+70°=180°$; c it verifies the cases tested, it does not prove every case. **Q30** $(3,4,5)$, $(6,8,10)$, $(5,12,13)$. **Q31** $m=2$: $(3,4,5)$, brand-new; $m=3$: $(8,6,10)=2\times(3,4,5)$; $m=4$: $(15,8,17)$, brand-new; $m=5$: $(24,10,26)=2\times(5,12,13)$. **Q32** a no; b $n=2$ yes, $n=3$ yes; c every whole number $n\geq 2$. **Q33** a he ran the triangle test, not the right angle test, and "$9>6$" only shows the lengths form a triangle; b $4^2+5^2=41\neq 36=6^2$ — not right-angled (but it is a triangle). **Q34** a the base angles come out smaller than 60° (about 41° for $AB=6$ cm and width 4 cm) and change with the width; b equal angles need three equal sides, and her widths do not match the base; c set the width to the length of AB — then all three sides are equal, the triangle is equilateral, and each angle is 60°. **Q35** a sort the three lengths; if the sum of the two shorter is greater than the longest, output "bed possible", otherwise "no bed"; b $(30,40,50)$ passes since $30+40=70>50$; $(10,30,60)$ fails since $10+30=40<60$; c it checks only whether the bed can close — nothing about thickness of the timbers, the joins, straightness, or size. **Q36** Answers vary: any algorithm from this subtopic run correctly on one pass case and one fail case, with one strength, one limit, and a written refinement if a flaw was found.

Fully-worked solutions

Q1. $L=9$; $p=5$, $q=6$. $S=5+6=11$. $11>9$. ∴ The lengths form a triangle.

Q2. $L=8$; $p=3$, $q=5$. $S=3+5=8$. $S=L$, so S is not greater than L. ∴ No triangle — the sides sit flat in a straight line. This is a boundary case.

Q3. a If the third side were 3 cm or less, then third side + 7 cm ≤ 10 cm, and the two shorter sides could not reach past the 10 cm side. b If the third side were 17 cm or more, then $7+10=17$ cm would not be greater than it. c The third side x must satisfy $3<x<17$. Whole-number values: $x=4,5,6,\ldots,16$. ∴ There are 13 possible values.

Q4. Follow the flow chart of Algorithm 1 (shown in the theory). Input: 4, 4, 4. Sort: $L=4$, $p=4$, $q=4$. $S=p+q=4+4=8$. Decision: is $8>4$? Yes. ∴ Output: a triangle exists (an equilateral one).

Q5. a $L=15$. b $a^2+b^2=9^2+12^2=81+144=225$. c $L^2=15^2=225$. $225=225$. ∴ The triangle is right-angled.

Q6. a $m^2-n^2=9-4=5$. b $2mn=2\times3\times2=12$. c $m^2+n^2=9+4=13$; the triple is $(5,12,13)$. d $5^2+12^2=25+144=169$, and $13^2=169$. ∴ The triple passes the right angle test.

Q7. a $p+q=62°+48°=110°$. b $110°<180°$ — yes, continue. c $r=180°-110°=70°$. ∴ The third angle is 70°.

Q8. a $p+q=113°+72°=185°$. b $185°$ is not less than $180°$, so the decision says no. c Two angles alone already use up more than the 180° available, leaving nothing — in fact, a deficit — for a third angle. ∴ The input is invalid: no triangle has these two angles.

Q9. a First draw the angle, then the arc from O, then the arc from P, then the matching arc from Q, then the ray: **v, iii, iv, i, ii**. b Each half should measure $70°\div 2=35°$. ∴ Order v, iii, iv, i, ii; each half 35°.

Q10. "Draw a base AB. Set the compasses to the length of AB. Draw an arc from A, then an arc of the same width from B; label the crossing C. Triangle ABC is **equilateral**, so each of its angles is **60**°."

Q11. $(6,8,10)$: $S=6+8=14$, $L=10$; $14>10$ — yes, triangle. $(1,1,3)$: $S=1+1=2$, $L=3$; 2 is not greater than 3 — no. $(7,7,7)$: $S=7+7=14$, $L=7$; $14>7$ — yes, triangle.

Q12. a $8^2+15^2=64+225=289$, and $17^2=289$. b $16^2+30^2=256+900=1156$, and $34^2=1156$. c $(16,30,34)=2\times(8,15,17)$ — a multiple of a triple is itself a triple.

Q13. a $L=8$: $4+5=9>8$ — triangle. b $L=9$: $4+5=9=L$ — no triangle; the sides sit flat. c $L=2$: $2+2=4>2$ — triangle (equilateral). d $L=16$: $6+9=15<16$ — no triangle; the short sides cannot reach each other.

Q14. The third side x must satisfy $12-5<x<12+5$. That is, $7<x<17$. Whole numbers in this range run from 8 to 16. ∴ Smallest 8 cm, largest 16 cm.

Q15. The sides are 6, 6 and x. Check 1: $6+x>6$ needs $x>0$. Check 2: $6+6>x$ needs $x<12$. (These two checks are exactly the two-shortest-sides test: if $x\le 6$ then $L=6$ and the test becomes $x+6>6$; if $x>6$ then $L=x$ and the test becomes $6+6>x$.) ∴ $x=1,2,3,\ldots,11$ — 11 values.

Q16. a Input 5, 7, 9 as given: $5+7=12>9$ — Mia says "triangle", and $L=9$ with $5+7=12>9$ shows that is correct. b Input 4, 9, 3 as given: $4+9=13>3$ — Mia says "triangle". But sorted, $L=9$ and $4+3=7<9$, so no triangle exists. Her answer is wrong. c Refinement: *sort the three lengths first; then test whether the sum of the two shorter is greater than the longest.* d $(5,7,9)$: $5+7=12>9$ — triangle, correct. $(4,9,3)$ sorted is $3,4,9$: $3+4=7$ is not greater than 9 — no triangle, correct. ∴ The refined algorithm passes both cases.

Q17. a $L=25$: $7^2+24^2=49+576=625$, and $25^2=625$ — right-angled. b $L=14$: $8^2+11^2=64+121=185$, but $14^2=196$; $185\ne 196$ — not right-angled. c $L=29$: $20^2+21^2=400+441=841$, and $29^2=841$ — right-angled.

Q18. $m^2-n^2=25-4=21$. $2mn=2\times5\times2=20$. $m^2+n^2=25+4=29$. The triple is $(21,20,29)$. Check: $21^2+20^2=441+400=841$, and $29^2=841$. ∴ $(21,20,29)$ passes the right angle test.

Q19. a $m^2-n^2=25-9=16$; $2mn=2\times5\times3=30$; $m^2+n^2=25+9=34$. The triple is $(16,30,34)$. b $(16,30,34)=2\times(8,15,17)$ — a multiple of the triple $(8,15,17)$. c 5 and 3 are both odd — this breaks the condition "not both odd". (They do have no common factor, so that condition still holds.)

Q20. a $12^2+16^2=144+256=400$, and $20^2=400$ — passes. b Multiplying every side by k multiplies each squared length by k^2: $(ka)^2+(kb)^2=k^2(a^2+b^2)$. If $a^2+b^2=c^2$, then $k^2(a^2+b^2)=k^2c^2=(kc)^2$, so the equality still holds. ∴ Every whole-number multiple of a right-angled triangle is right-angled.

Q21. Design (numbered steps): 1. Input two angles p and q. 2. Decision: is $p+q<180°$? If no, output "invalid — no such triangle" and stop. 3. Compute $r=180°-(p+q)$. 4. Output "the third angle is r".

This is Algorithm 3; its flow chart appears in the theory. a $p+q=60°+70°=130°<180°$; $r=180°-130°=50°$. Pass case: correct. b $p+q=100°+90°=190°\ge 180°$; output "invalid". Fail case: correct. ∴ The algorithm passes one case that should pass and one that should fail.

Q22. Step 1: construct a 60° angle CAB by the method of Question 10 (base AB, arcs of width AB from A and B meeting at C, ray from A through C). Step 2: extend the base ray AB past A with the ruler. The extended ray and the ray AC now sit on a straight line through A. Angles on a straight line add to 180°, so the new angle is $180°-60°=120°$. Test: protractor reading 120° — expected 120°, actual 120°, pass. ∴ The construction works because it turns one known angle (60°) into its supplement.

Q23. a The arcs from P and Q meet when the two widths together reach further than PQ: width 5 cm gives $2\times5=10>8$, so the arcs cross above and below the segment — the construction works. b Width 3 cm gives $2\times3=6<8$; the arcs never meet, so there are no crossing points to join — the construction fails. c Refinement: *choose a compass width greater than half of* PQ. (At exactly half — 4 cm here — the arcs meet at a single point on PQ itself, which is useless, so the width must be strictly more than half.) d Each crossing point must be the same distance from P as from Q; unequal widths put the crossing points off that set of positions, and the join no longer cuts PQ in half at 90°.

Q24. A straight angle has two arms pointing in opposite directions from the vertex O. Step 2 of Example 4 draws an arc crossing both arms; the crossings P and Q sit on opposite sides of O, the same distance from it. Steps 3–5 then find a point R the same distance from P as from Q and draw the ray OR. R sits exactly halfway around from P to Q. ∴ Each half measures $180°\div2=90°$: the bisector of a straight angle is perpendicular to the original line.

Q25. Bisecting cuts the angle exactly in half. $30°\div2=15°$. Test: protractor reading 15° — expected 15°, actual 15°, pass. ∴ Each half measures 15°.

Q26. Input order: 13, 5, 12. Step 2 sorts first: $L=13$; the other sides are 5 and 12. $5^2+12^2=25+144=169$. $13^2=169$. $169=169$ — output: right-angled. ∴ The output is the same as for $(5,12,13)$, because sorting the sides is step 2 of the algorithm: whatever order the input arrives in, the decision is always made on the same sorted values.

Q27. Sort the lengths: $p=4$, $q=7$, $L=9$. The one check: $p+q=4+7=11>9$ — passes. Now the other two sums: $q+L=7+9=16>4=p$, and $p+L=4+9=13>7=q$. Both hold automatically: L is at least as long as each shorter side, so adding L to either shorter side must beat the remaining one. ∴ Jordan is right — once the two shorter sides together beat the longest side, the other two checks can never fail.

Q28. Storyboard: Frame 1: draw the segment $PQ=6$ cm. Frame 2: compasses at width 4 cm; draw an arc from P above PQ. Frame 3: same centre and width; draw an arc from P below PQ. Frame 4: same width; draw an arc from Q above PQ, crossing the first arc at X. Frame 5: same width; draw an arc from Q below PQ, crossing at Y. Frame 6: draw the line XY; it meets PQ at its midpoint at 90°.

a The arcs meet when $2\times\text{width}>PQ$: $2\times4=8>6$ — they meet. b With $PQ=10$ cm and the width still 4 cm: $2\times4=8<10$ — the arcs no longer meet, and frames 4–6 cannot be drawn at all. The lesson: the width must always be more than half the segment, so it must grow when the segment grows.

Q29. a Frame 1: draw a triangle with angles 40°, 60° and 80°, and a straight line below it. Frame 2: move the 40° corner down onto the line. Frame 3: move the 60° corner down, fitting it next to the 40° angle. Frame 4: move the 80° corner down, fitting it next to the 60° angle. Frame 5: the three angles sit side by side on the straight line, filling it exactly: $40°+60°+80°=180°$. b For 50°, 60° and 70° the final frame again shows the straight line filled: $50°+60°+70°=180°$. c No — the animation is a test of the cases it was run on, not a proof covering every triangle. It gives strong visual evidence that the angles of a triangle fill 180°, and checking a second triangle makes the evidence stronger, but a result about *every* triangle needs a proof, which belongs to Level 10.

Q30. Check each L from 2 to 13: can L^2 be written as the sum of two smaller squares?

L	L^2	Split into two squares?	Triangle
5	25	$9+16=3^2+4^2$	$(3,4,5)$
10	100	$36+64=6^2+8^2$	$(6,8,10)$
13	169	$25+144=5^2+12^2$	$(5,12,13)$

For every other L from 2 to 13, no split exists (for example $11^2=121$ is not $1+120$, $4+117$, $9+112$, … — none of the pairs are both squares). ∴ The algorithm finds exactly three triangles: $(3,4,5)$, $(6,8,10)$ and $(5,12,13)$.

Q31. With $n=1$: $m^2-n^2=m^2-1$, $2mn=2m$, $m^2+n^2=m^2+1$.

m	m^2-n^2	$2mn$	m^2+n^2	Triple	Brand-new, or a multiple?
2	3	4	5	$(3,4,5)$	brand-new
3	8	6	10	$(8,6,10)$	$2\times(3,4,5)$
4	15	8	17	$(15,8,17)$	brand-new
5	24	10	26	$(24,10,26)$	$2\times(5,12,13)$

Every odd m gives m and $n=1$ both odd, so the refinement conditions of Example 7 predict a multiple — and they are right in both rows.

Q32. a $n=1$: sides 1, 2, 3. $1+2=3=L$ — the sides sit flat; no triangle. b $n=2$: sides 2, 3, 4. $2+3=5>4$ — triangle. $n=3$: sides 3, 4, 5. $3+4=7>5$ — triangle. c $2n+1>n+2$ simplifies to $n>1$. ∴ The lengths form a triangle for every whole number $n\geq 2$, and never for $n=1$.

Q33. a First, Sam ran the triangle test (sums of sides) when the question asked for the right angle test (sums of squares). Second, "$9>6$" only shows the three lengths form *a* triangle — it says nothing about a right angle. b $L=6$; the other sides are 4 and 5. $4^2+5^2=16+25=41$. $6^2=36$. $41\neq 36$. ∴ Output: not right-angled. (The lengths do form a triangle, since $4+5=9>6$ — that is the only thing Sam actually showed.)

Q34. a With $AB=6$ cm and a width of 4 cm, the arcs still meet (since $4+4=8>6$), but the triangle has sides 6, 4, 4. The base angles come out about 41° — smaller than 60° — and a different width gives a different angle. b Her algorithm makes two sides equal to the chosen width, but the third side stays the length of AB. Equal angles need three equal sides; since the width and the base do not match, the angles are not 60° in general. c Refinement: *set the compass width to the length of* AB. Then all three sides equal AB, the triangle is equilateral, and each angle is $180°\div 3=60°$ — guaranteed, whatever the length of the base.

Q35. a Design: 1. Input the three timber lengths. 2. Sort them; call the longest L and the other two p and q. 3. Compute $S=p+q$. 4. Decision: is $S>L$? If yes, output "triangular bed possible". If no, output "no bed — the timbers cannot close up".

b Pass case: 30 cm, 40 cm, 50 cm. $L=50$; $S=30+40=70>50$ — bed possible, correct. Fail case: 10 cm, 30 cm, 60 cm. $L=60$; $S=10+30=40<60$ — no bed, correct. c The algorithm checks one thing only: whether the three timbers can meet to enclose a space. It can never check anything physical — the thickness of the timbers, how the joins are fastened, whether the sides are straight, whether the bed is a useful size, or how it is measured out on the ground. Evaluating an algorithm always includes naming what it leaves out.

Q36. Marking guide — a full answer contains all four parts: 1. A named algorithm from this subtopic (for example the two-angle algorithm of Question 21, the 120° construction of Question 22, or the listing algorithm of Question 30). 2. Two new test cases, clearly labelled as a pass case and a fail case, each run step by step with the input, the expected output and the actual output recorded. 3. An evaluation naming one strength (what the algorithm always gets right) and one limit (what it cannot check). 4. A written refinement if any test case uncovered a flaw, followed by a re-run of every test case.

Test

T4 · Chapter 4 — Space: similarity, the trigonometric ratios and geometric algorithms · Level 9

This is a school-designed paper. It is not a VCAA instrument. Achievement in the Victorian Curriculum is reported against the Level 9 achievement standard, not against a mark on this paper.

Time	27 minutes	5 minutes reading time, then 22 minutes writing time (1 minute per mark).
Total marks	22	Part A: 10 marks · Part B: 12 marks
Equipment — Part A	pencil, ruler, protractor, pair of compasses	Part A is technology-free: no calculator and no digital tool of any kind.
Equipment — Part B	pencil, ruler, protractor, pair of compasses, a scientific calculator, and dynamic geometry software as nominated by your teacher	Part B is technology-active. A CAS (computer algebra system) calculator is not permitted.

Coverage. This test scores the three subtopics of Chapter 4. It also draws on skills from the chapters before it: Question 2 uses gradient from 2B, and Questions 5 and 8 use Pythagoras' theorem (Level 8 prior knowledge) as a tool.

Subtopic	CD	Marks
4A — Similar right-angled triangles and the constancy of the sine, cosine and tangent ratios	`VC2M9SP01`	6
4B — The enlargement transformation: what is preserved and what changes	`VC2M9SP02`	6
4C — Designing, testing and refining algorithms from geometric constructions and theorems	`VC2M9SP03`	10

Instructions.

- Answer all questions. Show your working clearly.
- In parts that ask for an explanation, a correct number with no reason does not score full marks.
- Give exact values (fractions) unless a question asks for rounding, and state the rounding you use.

Part A — Technology-free (10 marks)

Question 1 (2 marks)

Triangle A is a right-angled triangle with sides 7 cm, 24 cm and 25 cm. Triangle B is its enlargement by scale factor 2, with sides 14 cm, 48 cm and 50 cm. In both triangles, θ is the angle opposite the shortest side.

a. Write $\dfrac{\text{opposite}}{\text{hypotenuse}}$ for triangle A and for triangle B, simplifying each fraction. [1]

b. Explain in one sentence why the two fractions must be equal. [1]

Question 2 (2 marks)

A straight line through the origin passes through the point $(12, 5)$. It meets the positive direction of the horizontal axis at the angle θ.

a. Find the gradient of the line. [1]

b. Hence write down the exact value of $\tan\theta$. [1]

Question 3 (1 mark)

Which of the following is preserved under every enlargement?

A. the sizes of the angles of the shape

B. the lengths of the sides of the shape

C. the area of the shape

D. the volume of the object

Question 4 (3 marks)

A rectangle 4 cm wide and 7 cm long is enlarged by scale factor 3.

a. Write down the measurements of the image. [1]

b. Find the area of the object and the area of the image. [1]

c. Explain why the area changes by the factor it does. [1]

Question 5 (2 marks)

The following algorithm decides whether a triangle is right-angled.

1. Input three side lengths.
2. Call the longest side L, and call the other two sides a and b.
3. Compute $a^2 + b^2$ and compute L^2.
4. Decision: is $a^2 + b^2 = L^2$? If yes, output "right-angled". If no, output "not right-angled".

Trace the algorithm for the input 9 cm, 40 cm and 41 cm. Show the value found at each step, and write the output. [2]

Part B — Technology-active (12 marks)

Use the dynamic geometry software nominated by your teacher, or a scientific calculator, as each question directs. A CAS (computer algebra system) calculator is not permitted.

Question 6 (2 marks)

a. In the dynamic geometry software, construct a right-angled triangle with one acute angle of $35°$. Measure the side lengths correct to 1 decimal place, and compute opposite ÷ hypotenuse correct to 2 decimal places. Then construct a second, larger right-angled triangle with the same $35°$ angle and compute the same ratio. Record both ratios. You may check your ratios against the sine key on your calculator. [1]

b. What do you notice about the two ratios? Answer in one sentence. [1]

Question 7 (2 marks)

A triangle has base 8 cm and height 3 cm.

a. Find the area of the triangle. [1]

b. The triangle is enlarged by scale factor 2.5. Use the dynamic geometry software to verify the area of the image: construct the object, enlarge it, and measure the base and the height of the image. Then explain the factor by which the area grows. [1]

Question 8 (8 marks)

A community sports club in Victoria is repainting a full-size netball court. A netball court is a rectangle 30.5 *m long and* 15.25 *m wide. The club checks the shape of the court with a* 35 *m tape measure.*

Source: World Netball, “Rules of Netball”, 2024 edition. Retrieved 23 August 2026. Court dimensions as at August 2026.

a. Use Pythagoras’ theorem to find the length of a diagonal of the court, correct to 1 decimal place. [1]

b. The club wants an algorithm that decides whether the marked court is a rectangle. Design the algorithm as numbered steps. Its input is the measured length, the measured width and the two measured diagonals; its output is either “rectangle” or “not a rectangle”. [2]

c. Name the theorem your algorithm in part b uses. [1]

d. Test your algorithm on these two sets of measurements, and state the output in each case.

   Case 1 — length 30.5 m, width 15.25 m, diagonals 34.1 m and 34.1 m.

   Case 2 — length 30.5 m, width 15.25 m, diagonals 33.8 m and 34.4 m. [2]

e. A club member says: “You do not need Pythagoras — just measure the two diagonals, and if they are equal, the court is a rectangle.” Explain why equal diagonals alone do not guarantee that the court is a rectangle. [1]

f. The measurements in part d are rounded to the nearest 10 cm. Suggest one refinement to your algorithm in part b that allows for the fact that real measurements are approximate. [1]

Solutions

Note on figures: the figures in these worked solutions are labelled sketches; every length shown is the value stated in the question, but a sketch is not drawn to scale.

Answer key

Q	Part	Answer	Marks
1	a	$\frac{7}{25}$ in both triangles ($\frac{14}{50}$ simplifies to $\frac{7}{25}$)	1
1	b	Similar triangles have equal ratios of corresponding sides	1
2	a	$\frac{5}{12}$	1
2	b	$\tan\theta=\frac{5}{12}$	1
3	—	A	1
4	a	12 cm by 21 cm	1
4	b	28 cm² and 252 cm²	1
4	c	Area is multiplied by $k^2=9$	1
5	—	$L=41$; $9^2+40^2=1681=41^2$; output "right-angled"	2
6	a	Both ratios ≈ 0.57 (accept 0.55–0.59 from consistent measurements)	1
6	b	The ratios are equal — the ratio belongs to the angle, not the size	1
7	a	12 cm²	1
7	b	Image area 75 cm²; area grows by $k^2=2.5^2=6.25$	1
8	a	34.1 m	1
8	b	See worked solution	2
8	c	Pythagoras' theorem (its converse)	1
8	d	Case 1: "rectangle"; Case 2: "not a rectangle"	2
8	e	A shape that is not a rectangle can have equal diagonals	1
8	f	Accept diagonals within a small margin of the expected value	1

Q	Part	Answer	Marks
		Total	**22**

Fully-worked solutions

Question 1.

a In triangle A, θ is opposite the shortest side, so opposite $=7$ and hypotenuse $=25$:

$$\frac{\text{opposite}}{\text{hypotenuse}}=\frac{7}{25}$$

In triangle B: opposite $=14$ and hypotenuse $=50$:

$$\frac{\text{opposite}}{\text{hypotenuse}}=\frac{14}{50}=\frac{7}{25}$$

b Triangle B is an enlargement of triangle A, so the two triangles are similar. Similar triangles have equal ratios of corresponding sides, and the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$ is such a ratio: it belongs to the angle θ, not to the size of the triangle.

$\therefore$ Both fractions are $\dfrac{7}{25}$ — the ratio is constant under enlargement.

```
      C                          C'
      |\                         |\
      | \                        | \
    7 |  \ 25                 14 |  \ 50
      |   \                      |   \
      | θ__\                     | θ__\
      A  24  B                   A  48  B

      triangle A                 triangle B
```

Question 2.

a Gradient = rise ÷ run. From the origin to $(12,5)$: rise $=5$, run $=12$.

$$\text{gradient}=\frac{5}{12}$$

b The tangent of the angle a line makes with the positive direction of the horizontal axis is the gradient of the line. In the right-angled triangle with vertices $(0,0)$, $(12,0)$ and $(12,5)$, opposite $=5$ and adjacent $=12$, so

$$\tan\theta=\frac{\text{opposite}}{\text{adjacent}}=\frac{5}{12}$$

$\therefore \tan\theta=\dfrac{5}{12}$.

```
 y
 |                           • (12, 5)
 |                         / |
 |                       /   | rise = 5
 |                     /     |
 |                   /       |
 |                 /  θ      |
 O_______________/___________|______ x
                run = 12
```

Question 3.

A. An enlargement multiplies every length by the scale factor k: side lengths are multiplied by k, areas by k^2 and volumes by k^3 — so B, C and D all change. Every angle is untouched: the image is similar to the object.

∴ The answer is **A**.

Question 4.

a Every length is multiplied by $k=3$:

width: $4 \times 3 = 12$ cm; length: $7 \times 3 = 21$ cm.

b Object: $4 \times 7 = 28$ cm².

Image: $12 \times 21 = 252$ cm².

c An enlargement multiplies every length by $k=3$. An area is a length times a length, so it is multiplied by $k \times k = k^2 = 9$. Check: $28 \times 9 = 252$. ✓

∴ The image is 12 cm by 21 cm; the areas are 28 cm² and 252 cm², and the area grows by the factor $9 = k^2$.

```
 object                           image
 ________
|        | 4 cm                  |←——— 21 cm ———→|
|________|                       __________________
  7 cm                           |                |
                                 |                |
                                 |                | 12 cm
                                 |                |
                                 |                |
                                 |________________|
```

Question 5.

Step 1: input 9, 40, 41.

Step 2: the longest side is $L=41$; the other two sides are $a=9$ and $b=40$.

Step 3: $a^2+b^2=9^2+40^2=81+1600=1681$; $L^2=41^2=1681$.

Step 4: $1681=1681$, so $a^2+b^2=L^2$ — the decision is yes.

∴ Output: **right-angled**. By the converse of Pythagoras' theorem, the angle opposite the 41 cm side is $90°$.

```
     |\
     | \
  40 |  \ 41
     |   \
     |____\
       9
```

Question 6.

a One valid run (measurements will vary slightly with the tool):

First triangle: hypotenuse 10.0 cm, opposite 5.7 cm.

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{5.7}{10} = 0.57$$

Second, larger triangle: hypotenuse 15.0 cm, opposite 8.6 cm.

$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{8.6}{15} \approx 0.57$

Check: $\sin 35° \approx 0.5736$, which rounds to 0.57.

b Both ratios are the same: the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ depends only on the angle ($35°$), not on the size of the triangle. That is the constancy of the sine ratio.

∴ Ratios ≈ 0.57 in both triangles; same angle, same ratio.

```
         |\                          |\
         | \                         | \
  5.7    |  \ 10.0           8.6     |  \ 15.0
         |   \                       |   \
         |35°_\                      |35°_ \
```

Question 7.

a area $= \frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} \times 8 \times 3 = 12$ cm².

b Construct the triangle (base 8 cm, height 3 cm), enlarge it by $k = 2.5$, and measure the image: base $8 \times 2.5 = 20$ cm, height $3 \times 2.5 = 7.5$ cm.

area of the image $= \frac{1}{2} \times 20 \times 7.5 = 75$ cm².

The area grows from 12 cm² to 75 cm², a factor of $75 \div 12 = 6.25$. Both the base and the height are multiplied by k, so the area is multiplied by $k \times k = k^2 = 2.5^2 = 6.25$.

∴ Area of the object 12 cm²; area of the image 75 cm² — the factor is k^2.

```
        /\                             /\
       /  \                           /  \
    3 /    \                     7.5 /    \
     /______\                       /      \
        8                          /________\
                                       20
```

Question 8.

a A diagonal cuts the court into two right-angled triangles with base 30.5 m and altitude 15.25 m. By Pythagoras' theorem,

$d^2 = 30.5^2 + 15.25^2$

$d^2 = 930.25 + 232.5625 = 1162.8125$

$d = \sqrt{1162.8125} \approx 34.1000$

∴ The diagonal is 34.1 m, correct to 1 decimal place.

b One correct algorithm:

1. Input the measured length l, the measured width w, and the two measured diagonals d_1 and d_2.
2. Compute $d = \sqrt{l^2 + w^2}$, and round it to the same number of decimal places as the measurements.
3. Decision: is $d_1 = d$ and $d_2 = d$?
4. If yes, output “rectangle”. If no, output “not a rectangle”.

(An algorithm that compares the squares — $d_1^2 = l^2 + w^2$ and $d_2^2 = l^2 + w^2$ — is equally valid.)

c Pythagoras' theorem — more precisely, its converse: if the square on the longest side of a triangle equals the sum of the squares on the other two sides, the angle between the two shorter sides is a right angle. Each half of the court must pass that test.

d Expected diagonal, from part a: $d = 34.1$ m.

Case 1: both diagonals read 34.1 m, so $d_1 = d$ and $d_2 = d$. Output: **rectangle**. This is the case that should pass.

Case 2: $d_1 = 33.8$ m and $d_2 = 34.4$ m; neither equals 34.1 m. Output: **not a rectangle**. This is the case that should fail: the corners have shifted, and the court is a slanted parallelogram rather than a rectangle.

e A shape that is not a rectangle can still have equal diagonals. For example, an isosceles trapezium — a trapezium whose two non-parallel sides are equal in length — has diagonals of equal length, but its angles are not right angles. A court whose corners had drifted into such a shape would pass the member's equal-diagonal check, so that check alone is not enough.

f Allow a margin: accept a diagonal as equal to the expected value when it lies within a small distance of it — say within 2 cm — instead of demanding an exact match, because tape-measure readings are approximate. Any sensible refinement that allows a small margin earns the mark.

∴ Diagonal 34.1 m; the algorithm passes the rectangle case and rejects the slanted case, and the refinement allows for measurement.

```
P ________________________________ Q
| \                                |
|    \                             | 15.25 m
|       \                          |
|__________\_______________________|
S            30.5 m                R

The diagonal joins opposite corners, P to R.
```

Marking guide

Marks map to the achievement standard as corrected in §4 of the series plan (D7). Reasoning is marked separately from the answer: `VC2M9SP02` requires students to *identify and explain*, so the reason in Question 4c and the explanation in Question 7b each carry their own mark, and a correct number with no reason does not score full marks.

Questions	CD	Achievement-standard language (corrected)
Q1, Q2, Q6 (6 marks)	`VC2M9SP01`	*They solve problems involving ratio, similarity and scale in two-dimensional situations.* (AS 9) and *Students apply Pythagoras' theorem and use trigonometric ratios to solve problems involving right-angled triangles.* (AS 11 — the restored sentence, D7)
Q3, Q4, Q7 (6 marks)	`VC2M9SP02`	*Students apply the enlargement transformation to images of shapes and objects, and interpret results.* (AS 14)
Q5, Q8 (10 marks)	`VC2M9SP03`	*They design, use and test algorithms based on geometric constructions or theorems.* (AS 15)

Question-by-question allocation: Q1 a 1 + b 1 · Q2 a 1 + b 1 · Q3 1 · Q4 a 1 + b 1 + c 1 (reason) · Q5 1 + 1 · Q6 a 1 + b 1 · Q7 a 1 + b 1 (explanation) · Q8 a 1 + b 2 + c 1 + d 2 + e 1 (reason) + f 1. **Total: 22 marks.**