Mathematics Level 8

Chapter 2 — Algebra: linear expressions, relations and algorithms

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Contents

Expanding, factorising, rearranging and simplifying linear expressions

Content description `VC2M8A01`: create, expand, factorise, rearrange and simplify linear expressions, applying the associative, commutative, identity, distributive and inverse properties

*A bracket is just a wrapped-up piece of mathematics. This subtopic teaches you the four moves that keep algebra short and useful: **simplifying** (collecting like terms), **expanding** (unwrapping the bracket), **factorising** (wrapping it back up), and **rearranging** (turning a formula around to feature a different letter). Every one of these moves is nothing new — it is the five familiar properties of numbers, doing their ordinary work with letters.*

Theory

The parts of a linear expression

Last year you used letters to stand for numbers (`VC2M7A01`). This subtopic works with **linear expressions** — expressions in which no letter is squared or cubed, and no letter sits under a fraction bar. Each letter stands on its own, multiplied by at most a number.

The parts of a linear expression

$$3x \quad + 2y \quad - 5$$

$3x$ — the x-term; its coefficient is 3

$2y$ — the y-term; its coefficient is 2

-5 — the constant term (no letter attached)

the three terms are separated by $+$ and $-$ signs

Not linear: $x^2 + 4$ — the letter is squared

Not linear: $\frac{6}{t}$ — the letter is under the fraction bar

The linear expression 3x + 2y − 5 with its three terms colour-coded — the x-term 3x in blue, the y-term 2y in red and the constant −5 in grey — and two non-examples, x squared plus 4 and 6 over t, marked as not linear

The + and − signs cut an expression into **terms**, and each sign belongs to the term after it:

- $3x$ is a **term**; the number 3 multiplying the letter is its **coefficient**;
- $2y$ is a term with coefficient 2;
- -5 is the **constant term** — a term with no letter, so its value never changes.

$3x+2y-5$ is linear. x^2+4 is not (the letter is squared), and $\frac{6}{t}$ is not (the letter is under the fraction bar).

The five properties of algebra

The content description for this subtopic names five properties. With numbers you have used them all since primary school; now they give you permission to make the same moves with letters.

Five properties of algebra

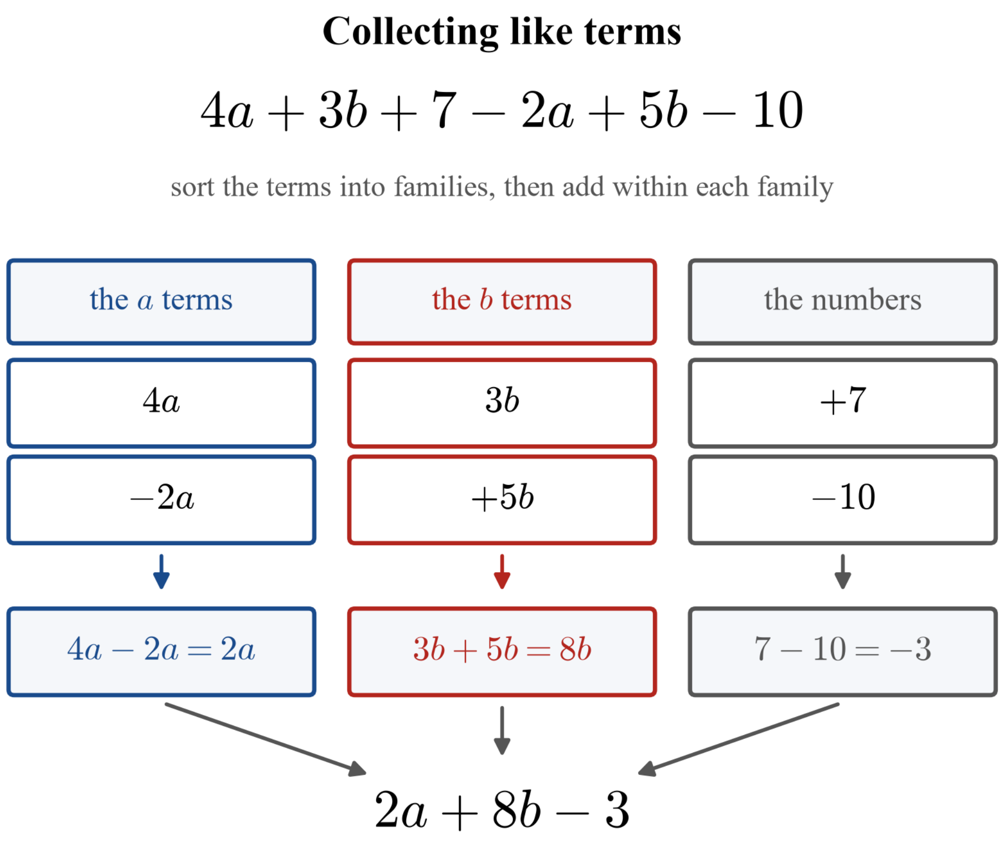
name	with numbers	with letters
Commutative order can swap	$3+5=5+3$	$a+b=b+a$
Associative grouping can move	$(2+3)+4=2+(3+4)$	$(a+b)+c=a+(b+c)$
Identity add 0, or multiply by 1, and nothing changes	$6+0=6, \quad 1\times 6=6$	$a+0=a, \quad 1\times a=a$
Inverse combine to make 0 (adding) or 1 (multiplying)	$5+(-5)=0, \quad 4\times\frac{1}{4}=1$	$a+(-a)=0$
Distributive multiply everything inside the bracket	$3(4+5)=12+15$	$a(b+c)=ab+ac$

A table of the five properties of algebra — commutative, associative, identity, inverse and distributive — each shown with numbers and with letters

- **Commutative** — order can swap: $a+b=b+a$. So $3x+5$ is the same as $5+3x$.
- **Associative** — grouping can move: $(a+b)+c=a+(b+c)$. When you add $4a+2a$ as $(4+2)a$, this is the property doing the work.
- **Identity** — adding 0 or multiplying by 1 changes nothing: $a+0=a$ and $1\times a=a$. That is why a on its own means $1a$.
- **Inverse** — a number combined with its opposite makes 0 (adding) or 1 (multiplying): $a+(-a)=0$. This is why $5a-5a=0$.
- **Distributive** — multiply everything inside the bracket: $a(b+c)=ab+ac$. This one is the engine of expanding and factorising.

Simplifying: collect the like terms

Like terms are terms with exactly the same letter part — $4a$ and $-2a$ are like terms; $4a$ and $4b$ are not. Like terms can be added or subtracted by adding or subtracting their coefficients, and that is what **simplifying** means.

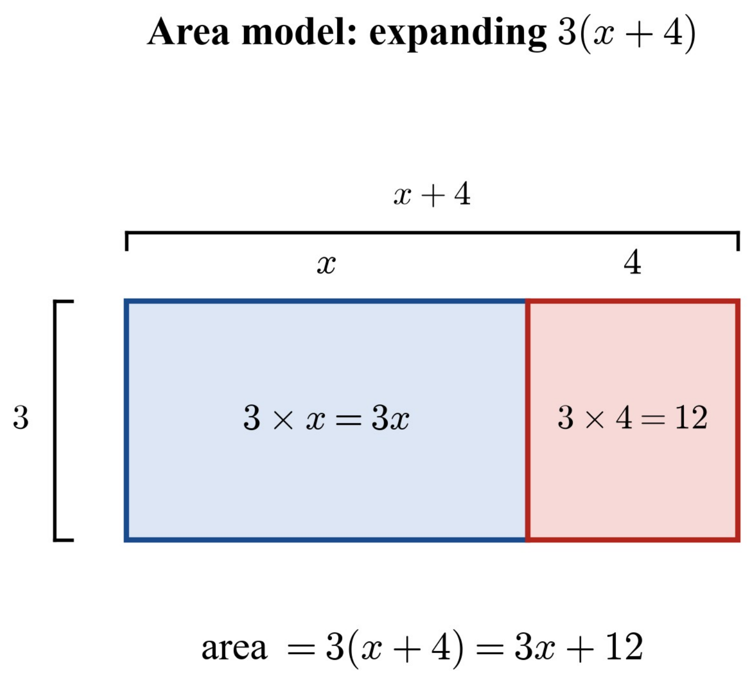


Flow diagram collecting like terms in 4a + 3b + 7 − 2a + 5b − 10: the a terms give 2a, the b terms give 8b and the numbers give −3, so the expression simplifies to 2a + 8b − 3

Sort the terms into families, keep each term's sign with it, and add within each family. The x terms, the y terms and the plain numbers never mix.

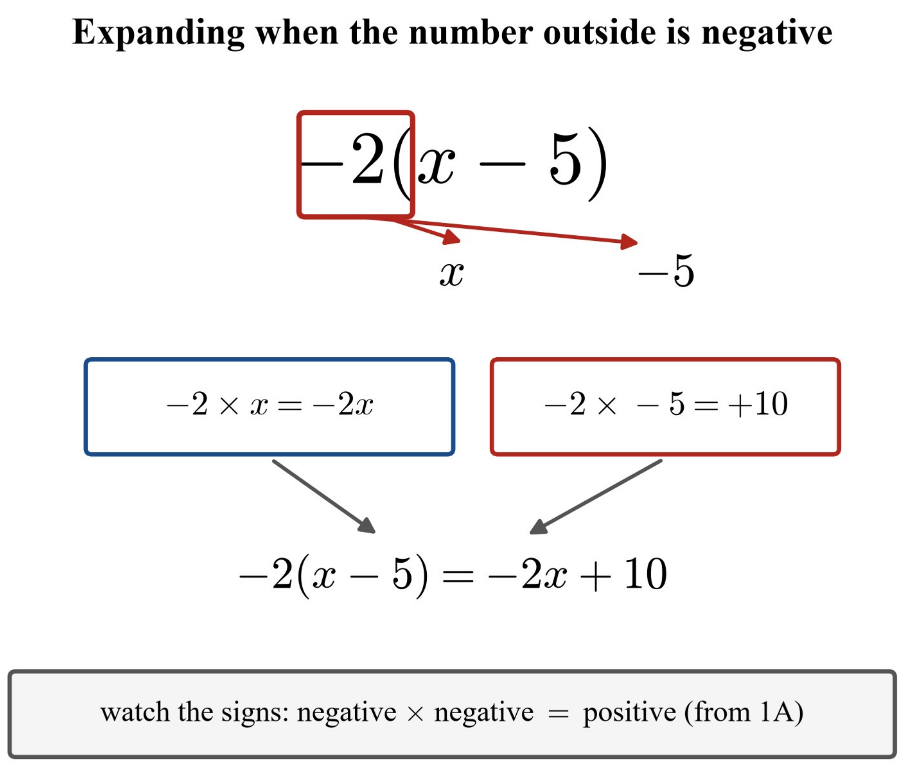
Expanding brackets

To **expand** means to multiply out the bracket, and the distributive property is the rule for it: the number outside multiplies every term inside. An area model shows why — the whole rectangle's area equals the sum of its pieces.



Area model for expanding 3(x + 4): a rectangle of height 3 split into a 3 by x piece and a 3 by 4 piece, showing 3(x + 4) = 3x + 12

When the number outside the bracket is negative, it still multiplies every term inside — and the sign rules from 1A apply to each product.



Arrows from the −2 outside the bracket in −2(x − 5) to each term inside, giving −2 times x equals −2x and −2 times −5 equals +10, so −2(x − 5) = −2x + 10

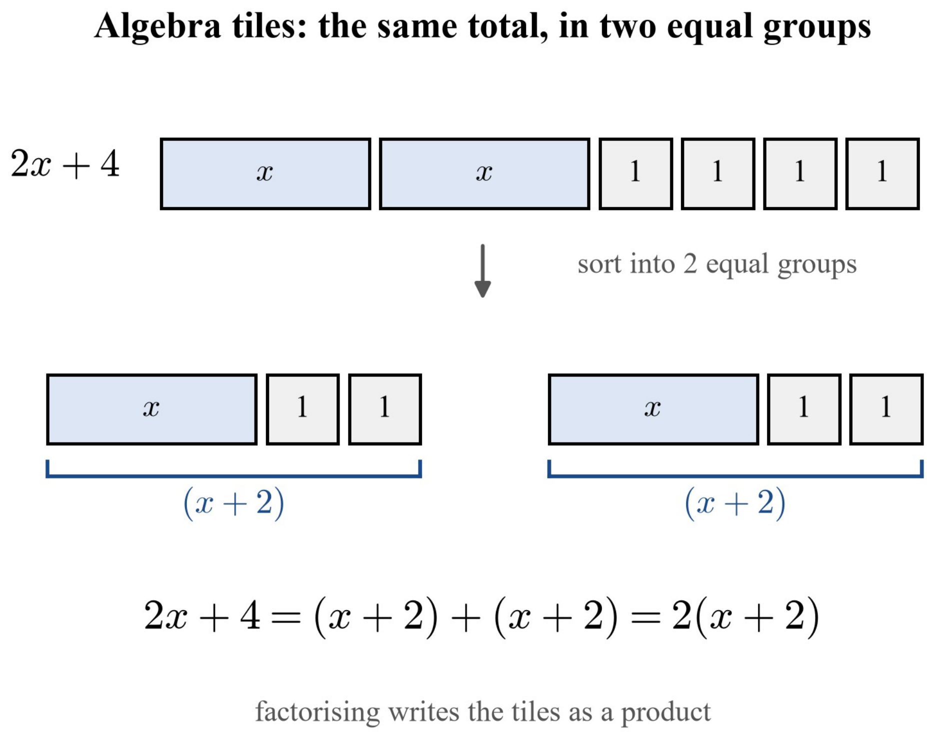
Expanding and then collecting like terms is how longer expressions are simplified. For example:

$$5(m+2n)+3m-4n=5m+10n+3m-4n=8m+6n$$

Factorising: put the bracket back

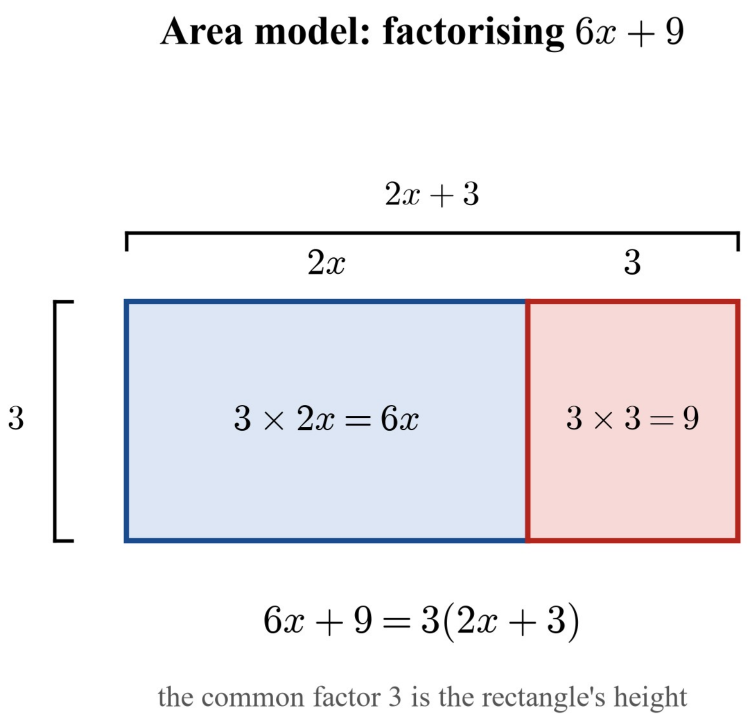
Factorising is the reverse of expanding: an expression is written as a product, with a **common factor** standing outside a bracket. Look for the largest number that divides every term.

Algebra tiles show what is happening. The tiles for $2x+4$ can be sorted into two equal groups, and equal groups are what multiplication means.



Algebra tiles showing 2x + 4 as two x-tiles and four unit tiles, sorted into two equal groups of (x + 2), so 2x + 4 = 2(x + 2)

The area model shows the same idea: the common factor is the rectangle's height, and the bracket is its width.

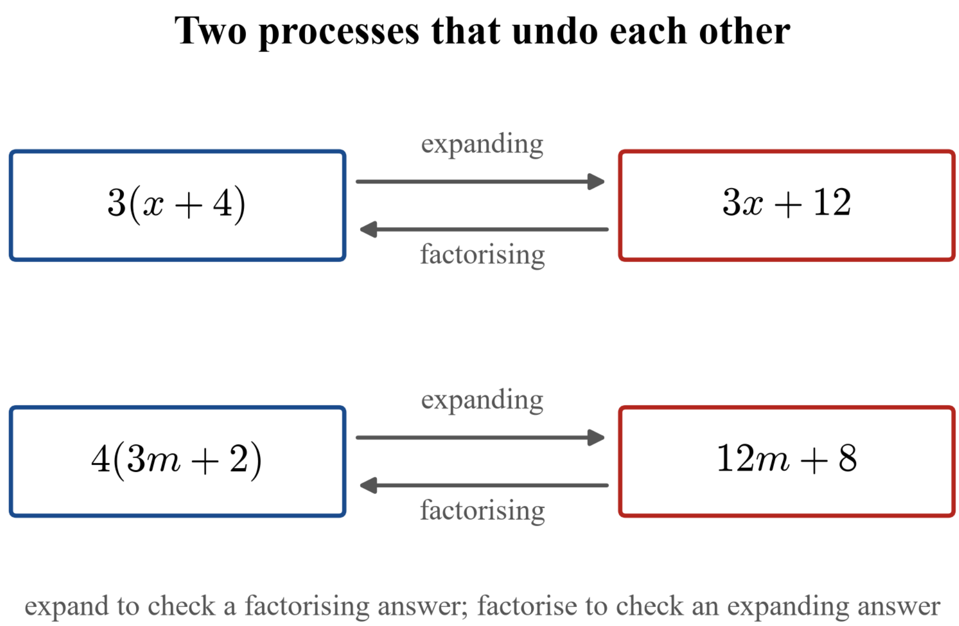


Area model for factorising 6x + 9: a rectangle of area 6x + 9 with height 3 and width 2x + 3, showing 6x + 9 = 3(2x + 3)

In $6x+9$, the largest number dividing both 6 and 9 is 3, so $6x+9=3(2x+3)$: divide each term by 3 to fill the bracket.

Expanding and factorising undo each other

The two processes run in opposite directions, so each one checks the other.

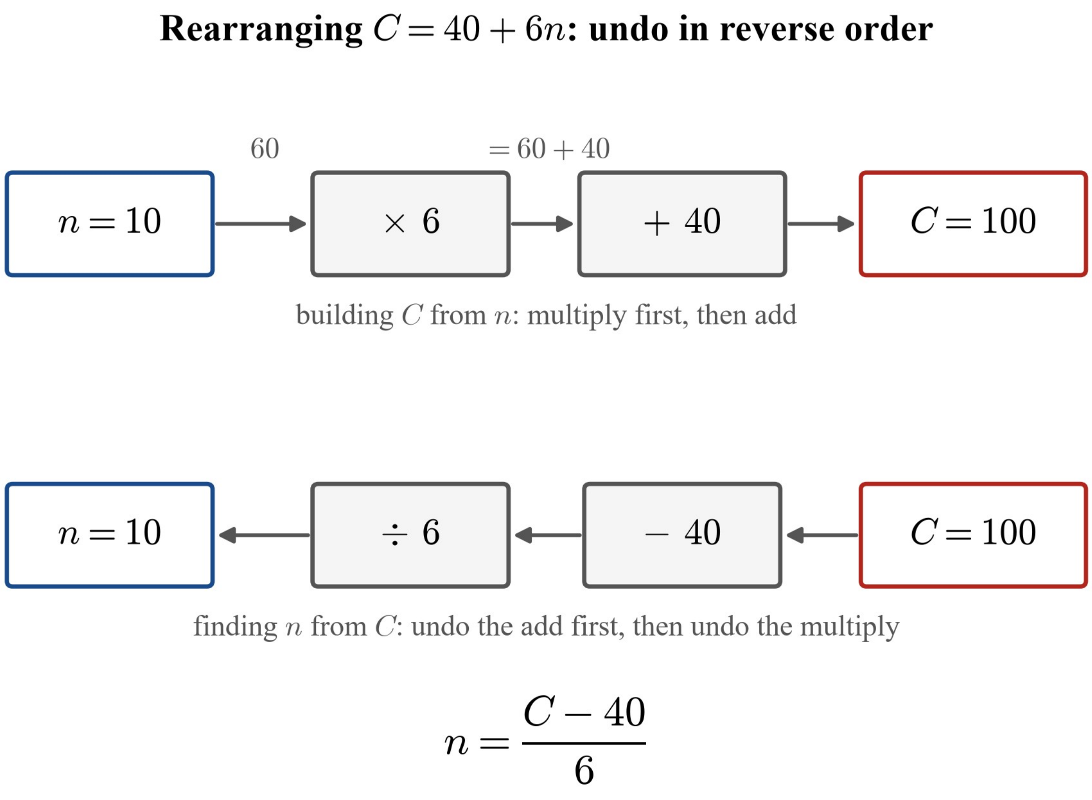


Paired cards with opposite arrows: 3(x + 4) expands to 3x + 12 and factorises back again, and 4(3m + 2) expands to 12m + 8 and factorises back again

After factorising, expand your answer in your head: $3(2x+3)=6x+9$ — back where you started, so the factorising was right.

Rearranging: undo in reverse order

A formula is a recipe that builds one quantity from another. To **rearrange** it — to make a different letter the subject — undo the recipe's steps in reverse order, using inverse operations (the inverse property at work).



Machine diagram for C = 40 + 6n: forwards, n = 10 is multiplied by 6 and then 40 is added to give C = 100; backwards, C = 100 has 40 subtracted and is then divided by 6 to return n = 10, so n = (C − 40) over 6

The forward recipe for $C = 40 + 6n$ is "multiply by 6, then add 40". To find n from C, undo those steps backwards: subtract 40 first, then divide by 6:

$$n = \frac{C - 40}{6}$$

Check with the numbers in the figure: $C = 100$ gives $n = \frac{100 - 40}{6} = \frac{60}{6} = 10$ — the machine returns to its start.

This subtopic in one picture

This subtopic

Simplify

collect the like terms

$$5a + 3a - a = 7a$$

Expand

multiply out the bracket

$$3(x + 4) = 3x + 12$$

Factorise

take out the common factor

$$6x + 9 = 3(2x + 3)$$

Rearrange

undo in reverse order

$$C = 40 + 6n \Rightarrow n = \frac{C - 40}{6}$$

Expanding and factorising undo each other — use one to check the other.

Four summary cards: Simplify collects the like terms, 5a + 3a − a = 7a; Expand multiplies out the bracket, 3(x + 4) = 3x + 12; Factorise takes out the common factor, 6x + 9 = 3(2x + 3); Rearrange undoes the operations in reverse order, C = 40 + 6n gives n = (C − 40) over 6

Words of this subtopic

Word	What it means here
expression	a collection of terms joined by + and − signs
term	a part of an expression separated by + or −; the sign belongs to the term after it
coefficient	the number multiplying the letter in a term
constant	a term with no letter; its value never changes
linear expression	an expression in which no letter is squared and no letter sits under a fraction bar
like terms	terms with exactly the same letter part
simplify	collect the like terms to write an expression more shortly
expand	multiply out the bracket
factorise	write as a product, with a common factor outside a bracket
common factor	a number that divides every term of the expression
rearrange	turn a formula around so that a different letter is the subject
property	a rule about numbers that is always true, and so also works with letters

Worked examples

Example 1 — with help

Simplify $7a+3-2a+5b-8-6b$.

Working	Comment
$7a-2a$	the a terms
$=5a$	$7-2=5$
$5b-6b$	the b terms
$=-b$	$5-6=-1$, and $-1b$ is written $-b$
$3-8=-5$	the constant terms

$\therefore 7a+3-2a+5b-8-6b=5a-b-5$.

Check by counting terms: six terms went in, three came out — one per family.

Example 2 — with help

Expand.

(a) $5(2x+3)$
(b) $-4(3-2y)$
(c) $3(a+4)+2(a-1)$

Working	Comment
(a) $5\times 2x+5\times 3=10x+15$	the 5 multiplies both terms inside.
(b) $-4\times 3-(-4)\times 2y=-12+8y$	negative outside: $-4\times -2y=+8y$.
(c) $3a+12+2a-2=5a+10$	expand each bracket, then collect like terms.

$\therefore$ (a) $10x+15$; (b) $-12+8y$; (c) $5a+10$.

Example 3 — with help

Factorise.

(a) $12m+8$
(b) $15a-10b$
(c) $9x+12y+6$

Working	Comment
(a) $12m+8=4(3m+2)$	4 is the largest number dividing 12 and 8.
(b) $15a-10b=5(3a-2b)$	5 divides 15 and 10.
(c) $9x+12y+6=3(3x+4y+2)$	3 divides all three terms; the last term becomes 2, not 0.

$\therefore$ (a) $4(3m+2)$; (b) $5(3a-2b)$; (c) $3(3x+4y+2)$. Check each by expanding: $4\times 3m+4\times 2=12m+8$ ✓.

Example 4 — on your own

Expand and simplify.

(a) $4(2x-5)-3(x-6)$
(b) $2(p+3q)-(p-4q)$

Expand each bracket, keeping the signs with their terms. In (b), the bracket with no number in front is being multiplied by -1.

(a) $4(2x-5)-3(x-6)=8x-20-3x+18=5x-2$.

(b) $2(p+3q)-(p-4q)=2p+6q-p+4q=p+10q$.

∴ (a) $5x-2$; (b) $p+10q$.

Example 5 — on your own

Rearrange each formula to make the letter in brackets the subject.

(a) $d=75t$ (t)
(b) $P=2l+2w$ (w)
(c) $C=3n-7$ (n)

Undo the operations in reverse order.

(a) t is multiplied by 75, so undo with division: $t=\dfrac{d}{75}$.

(b) w is multiplied by 2 and then $2l$ is added. Undo the add first: $P-2l=2w$, then undo the multiply: $w=\dfrac{P-2l}{2}$.

(c) n is multiplied by 3 and then 7 is subtracted. Undo the subtract first: $C+7=3n$, then divide: $n=\dfrac{C+7}{3}$.

∴ (a) $t=\dfrac{d}{75}$; (b) $w=\dfrac{P-2l}{2}$; (c) $n=\dfrac{C+7}{3}$.

Example 6 — on your own

A fruit shop sells gift bags. Each bag holds x apples and y oranges.

(a) Write an expression, with a bracket, for the number of pieces of fruit in 6 bags.

(b) Expand your expression.

(c) A seventh bag is different: it holds 2 of each fruit, plus 3 loose apples are added at the counter. Write an expression for the fruit in this order and expand it.

(d) When $x=4$ and $y=3$, check your expanded answer to (c) against the bracketed one (substitution, from Level 7).

(e) Six bags each holding $(x+y)$ pieces: $6(x+y)$.

(f) $6(x+y)=6x+6y$.

(g) $2(x+y)+3=2x+2y+3$.

(h) Bracketed: $2(4+3)+3=2\times 7+3=17$. Expanded: $2\times 4+2\times 3+3=8+6+3=17$. Same total — the expansion checks out.

∴ (a) $6(x+y)$; (b) $6x+6y$; (c) $2x+2y+3$; (d) both forms give 17 pieces of fruit.

Practice questions

Work these questions by hand. No calculator is needed in this subtopic — the skill is the algebra, and the numbers are chosen to be kind.

With help

Q1. Which expressions are linear? For the ones that are not, say why.

a. $3x+7$
b. x^2+4
c. $5-2y$
d. $\frac{6}{t}$

Q2. Look at the expression $8m-5+3n$.

a. List its three terms.
b. What is the coefficient of m?
c. What is the constant term?

Q3. Each line shows one of the five properties at work. Name it.

a. $3+5=5+3$
b. $3(x+4)=3x+12$
c. $6+0=6$
d. $5+(-5)=0$
e. $(2+3)+4=2+(3+4)$

Q4. Simplify.

a. $3x+2x+3x$
b. $6a+2b+4a$
c. $4m+9+5m+6$

Q5. Simplify.

a. $2y+7+3y-5$
b. $6p-9+2p-4$
c. $5a-3b-3a+b+4$

Q6. Expand.

a. $4(x+5)$
b. $3(2a-7)$
c. $6(3-y)$
d. $-2(m+8)$

Q7. Match each expression with its factorised form.

Expressions: $5x+15$ · $8a-12$ · $10+2y$ · $9m-3n$

Factorised: $4(2a-3)$ · $2(y+5)$ · $5(x+3)$ · $3(3m-n)$

Q8. Factorise.

a. $6x+10$
b. $9a-12$
c. $14m+21n$

On your own

Q9. Expand and simplify.

a. $2(x+y)+2x+4y$
b. $2a+3b-5a+b-3$

c. $6n-3(m+n)-2$

Q10. Expand and simplify.

a. $3(x+4)+2x$
b. $4(2a+1)+a-2$
c. $2(5x-3)+2x+1$

Q11. Expand, watching the signs.

a. $-3(x+4)$
b. $-5(2-y)$
c. $2(a+4)-6$
d. $-4(x-2)+4x+2$

Q12. Factorise.

a. $18x+24$
b. $15a-25b$
c. $12m+8n-4$
d. $7x+14y+21$

Q13. Each line of working contains one error. Find it and fix it.

a. $4a+3a=7a^2$
b. $2(x+5)=2x+5$
c. $5x-(x-3)=4x-3$

Q14. Rearrange to make the letter in brackets the subject.

a. $C=8n$ (n)
b. $A=b+12$ (b)
c. $A=3w+5$ (w)

Q15. Rearrange to make the letter in brackets the subject.

a. $C=60+25n$ (n)
b. $A=4w-9$ (w)
c. $V=500-20t$ (t)
d. $P=2a+2b$ (a)

Maths in real life

Q16. A stationery shop packs boxes for schools. Each box holds p pencils and r rulers, and the shop fills 5 boxes. The shop also adds 8 loose rulers to the order.

a. Write an expression, with a bracket, for the total number of items.
b. Expand your expression.
c. Write an expression for the total number of items including the loose rulers.
d. Find the total when $p=12$ and $r=2$.

Q17. A rectangular garden bed is w metres wide and 3 metres longer than it is wide.

a. Write a rule for the perimeter P metres of the bed, and simplify it.
b. Find the perimeter when $w=4$.
c. The perimeter is 26 m. Use your rule to find w.

Q18. A cinema sells family tickets. A family group costs $2a+3c$ dollars, where a is the adult price and c is the child price. A school holiday group books 3 families.

a. Write an expression, with a bracket, for the total cost, and expand it.
b. Find the total cost when $a=12$ and $c=5$.

Q19. An isosceles triangle has two equal sides of length x cm and a base of b cm.

a. Write a rule for the perimeter P cm.
b. Find P when $x=9$ and $b=5$.
c. When $P=31$ and $b=7$, rearrange your rule (or work backwards) to find x.

Q20. Make your own. Write a linear expression with two letters and a constant, then write it three ways: as it stands, expanded after putting a number outside a bracket you create, and factorised. Swap with a partner and check each other's expanding and factorising undo one another.

Answers

Q1. a. Linear. b. Not linear — the letter is squared. c. Linear. d. Not linear — the letter is under the fraction bar.

Q2. a. $8m$, -5, $3n$. b. 8. c. -5.

Q3. a. Commutative. b. Distributive. c. Identity. d. Inverse. e. Associative.

Q4. a. $8x$. b. $10a+2b$. c. $9m+15$.

Q5. a. $5y+2$. b. $8p-13$. c. $2a-2b+4$.

Q6. a. $4x+20$. b. $6a-21$. c. $18-6y$. d. $-2m-16$.

Q7. $5x+15 \rightarrow 5(x+3)$; $8a-12 \rightarrow 4(2a-3)$; $10+2y \rightarrow 2(y+5)$; $9m-3n \rightarrow 3(3m-n)$.

Q8. a. $2(3x+5)$. b. $3(3a-4)$. c. $7(2m+3n)$.

Q9. a. $4x+6y$. b. $-3a+4b-3$. c. $-3m+3n-2$.

Q10. a. $5x+12$. b. $9a+2$. c. $12x-5$.

Q11. a. $-3x-12$. b. $-10+5y$. c. $2a+2$. d. 10.

Q12. a. $6(3x+4)$. b. $5(3a-5b)$. c. $4(3m+2n-1)$. d. $7(x+2y+3)$.

Q13. a. $4a+3a=7a$ (like terms add their coefficients; the letter is unchanged). b. $2(x+5)=2x+10$ (the 2 multiplies both terms). c. $5x-(x-3)=4x+3$ (subtracting the bracket subtracts -3, which adds 3).

Q14. a. $n=\frac{C}{8}$. b. $b=A-12$. c. $w=\frac{A-5}{3}$.

Q15. a. $n=\frac{C-60}{25}$. b. $w=\frac{A+9}{4}$. c. $t=\frac{500-V}{20}$. d. $a=\frac{P-2b}{2}$.

Q16. a. $5(p+r)$. b. $5p+5r$. c. $5p+5r+8$. d. 78 items.

Q17. a. $P=4w+6$. b. 22 m. c. $w=5$ m.

Q18. a. $3(2a+3c)=6a+9c$. b. $117.

Q19. a. $P=2x+b$. b. 23 cm. c. $x=12$ cm.

Q20. Answers will vary. Check: the expanded and factorised forms undo each other.

Fully-worked solutions

Q1. A linear expression has no squared letters and no letter under a fraction bar.

a. $3x+7$: the letter stands on its own — **linear**.
b. x^2+4: the letter is squared — **not linear**.
c. $5-2y$: the letter stands on its own — **linear**.
d. $\frac{6}{t}$: the letter is under the fraction bar — **not linear**.

Q2. a. The signs cut the expression into **$8m$, -5 and $3n$** — each sign belongs to the term after it.

b. The coefficient of m is **8**.

c. The constant term is -5 — the term with no letter.

Q3. a. Order swaps — **commutative**. b. The 3 multiplies everything inside — **distributive**. c. Adding 0 changes nothing — **identity**. d. A number plus its opposite makes 0 — **inverse**. e. The grouping moves — **associative**.

Q4. a. \$3x + 2x + 3x = (3 + 2 + 3)x = \$ $8x$.

b. \$6a + 4a + 2b = \$ $10a+2b$ — the a terms combine; the b term stands alone.

c. \$4m + 5m + 9 + 6 = \$ $9m+15$.

Q5. a. \$2y + 3y + 7 - 5 = \$ $5y+2$.

b. \$6p + 2p - 9 - 4 = \$ $8p-13$.

c. \$5a - 3a - 3b + b + 4 = \$ $2a-2b+4$.

Q6. a. \$4 x + 4 = \$ $4x+20$.

b. \$3 2a - 3 = \$ $6a-21$.

c. \$6 - 6 y = \$ $18-6y$.

d. \$-2 m - 2 = \$ $-2m-16$.

Q7. $5x+15$: common factor 5 → $5(x+3)$. $8a-12$: common factor 4 → $4(2a-3)$. $10+2y$: common factor 2 → $2(5+y)=2(y+5)$. $9m-3n$: common factor 3 → $3(3m-n)$.

∴ $5x+15 \rightarrow 5(x+3)$; $8a-12 \rightarrow 4(2a-3)$; $10+2y \rightarrow 2(y+5)$; $9m-3n \rightarrow 3(3m-n)$.

Q8. a. \$6x + 10 = \$ $2(3x+5)$.

b. \$9a - 12 = \$ $3(3a-4)$.

c. \$14m + 21n = \$ $7(2m+3n)$.

Q9. a. \$2x + 2y + 2x + 4y = \$ $4x+6y$.

b. \$2a - 5a + 3b + b - 3 = \$ $-3a+4b-3$.

c. \$6n - 3m - 3n - 2 = \$ $-3m+3n-2$.

Q10. a. \$3x + 12 + 2x = \$ $5x+12$.

b. \$8a + 4 + a - 2 = \$ $9a+2$.

c. \$10x - 6 + 2x + 1 = \$ $12x-5$.

Q11. a. \$-3 x - 3 = \$ $-3x-12$.

b. \$-5 - (-5) y = -10 + 5y = \$ $-10+5y$.

c. \$2a + 8 - 6 = \$ $2a+2$.

d. \$-4x + 8 + 4x + 2 = \$ 10 — the x terms cancel ($-4x+4x=0$, inverse property at work).

Q12. a. \$18x + 24 = \$ $6(3x+4)$ — 6 divides 18 and 24.

b. \$15a - 25b = \$ $5(3a-5b)$.

c. \$12m + 8n - 4 = \$ $4(3m+2n-1)$ — the constant term becomes 1, not 0.

d. \$7x + 14y + 21 = \$ $7(x+2y+3)$.

Q13. a. $4a+3a=7a$, not $7a^2$: like terms add their coefficients and keep the letter as it is.

b. $2(x+5)=2x+10$: the 2 must multiply the 5 as well.

c. $5x-(x-3)=5x-x+3=4x+3$: subtracting the bracket flips both signs inside, so -3 becomes $+3$.

Q14. a. n is multiplied by 8; undo with division: $n=\frac{C}{8}$.

b. 12 is added to b; undo with subtraction: $b=A-12$.

c. w is multiplied by 3, then 5 is added: undo the add, then the multiply: $w=\frac{A-5}{3}$.

Q15. a. $C-60=25n$, so $n=\frac{C-60}{25}$.

b. $A+9=4w$, so $w=\frac{A+9}{4}$.

c. $V-500=-20t$, so $t=\frac{V-500}{-20}$, usually written as $t=\frac{500-V}{20}$.

d. $P-2b=2a$, so $a=\frac{P-2b}{2}$.

Q16. a. Five boxes each holding $(p+r)$ items: $5(p+r)$.

b. \$5(p + r) = \$ $5p+5r$.

c. $5p+5r+8$.

d. $5\times12+5\times2+8=60+10+8=78$: **78 items**. Check with the bracket: $5(12+2)+8=5\times14+8=78$ ✓.

Q17. a. The length is $w+3$, so \$P = 2(w + (w + 3)) = 2(2w + 3) = \$ $4w+6$.

b. $P=4\times4+6=16+6=22$: **22 m**.

c. $26=4w+6$, so $4w=20$ and $w=5$: **5 m**.

Q18. a. \$3(2a + 3c) = \$ $6a+9c$.

b. $6\times12+9\times5=72+45=117$: **\$117**. Check with the bracket: $3(2\times12+3\times5)=3(24+15)=3\times39=117$ ✓.

Q19. a. Two equal sides and a base: $P=2x+b$.

b. $P=2\times9+5=18+5=23$: **23 cm**.

c. $31=2x+7$, so $2x=24$ and $x=12$: **12 cm**.

Q20. Answers will vary. Example: start with $4m+12$. Factorise: $4(m+3)$. Check by expanding: $4(m+3)=4m+12$ ✓ — the two processes undo each other.

Graphing linear relations; solving linear equations and one-variable inequalities

> **Content description `VC2M8A02`:** graph linear relations on the Cartesian plane using digital tools where appropriate; solve linear equations and one-variable inequalities using graphical and algebraic techniques; verify solutions by substitution

A table of numbers that grows by the same amount every step, a straight line on a grid, and an equation that undoes itself to hand back one number: these are three faces of the same idea. In this subtopic you will graph linear relations, read equations and inequalities straight off their graphs, solve them by undoing operations in reverse order — and check every answer by substitution, every time.

Theory

Recall: plotting points and checking solutions

In Year 7 you built tables of values and plotted points in all four quadrants of the Cartesian plane (`VC2M7A05`), and you checked solutions of equations by substituting them back in (`VC2M7A03`). Both habits are used all through this subtopic. As a warm-up: does the point $(2,5)$ sit on the graph of $y=2x+1$? Substitute $x=2$: $2\times2+1=5$, which matches the y-coordinate — it does.

Recognising a linear relation: first differences

A **relation** is a rule connecting two quantities. A relation is **linear** when equal steps in x produce the same step in y every time. That step in y is called the **first difference**.

$y=3x+2$: every step adds 3

x	0	1	2	3	4
y	2	5	8	11	14

3 3 3 3

$y=x^2$: the steps keep changing

x	1	2	3	4	5
y	1	4	9	16	25

3 5 7 9

Constant first differences → linear. Changing first differences → not linear.

Two tables of values with arrows between successive y-values: for y = 3x + 2 every first difference is 3, so the relation is linear; for y = x squared the differences 3, 5, 7, 9 keep changing, so it is not linear.

In the left table, x goes up by 1 each time and y goes up by 3 each time: the first differences are constant, and the relation $y=3x+2$ is linear. In the right table the differences keep changing $(3,5,7,9)$, so $y=x^2$ is not linear.

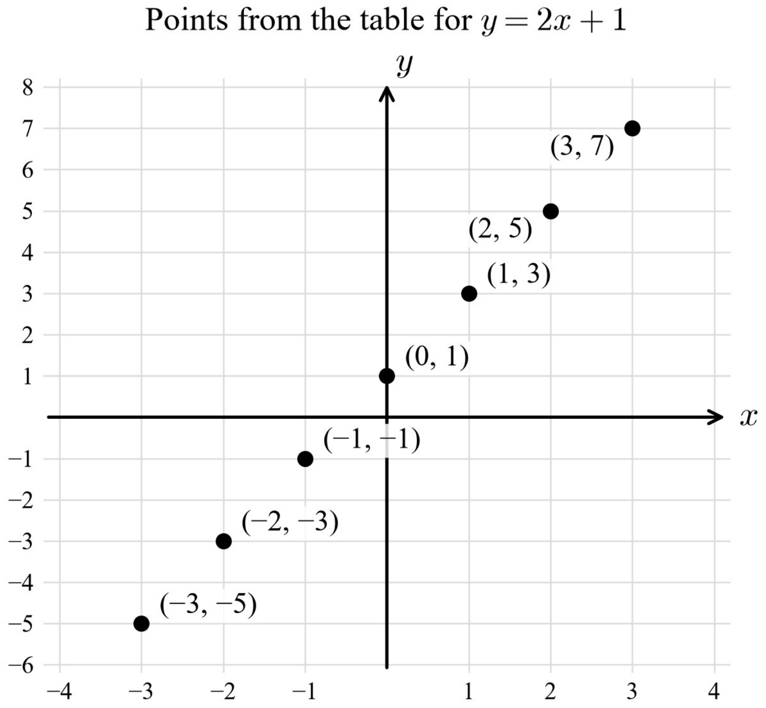
Constant first differences (for equal steps in x) → linear. Changing first differences → not linear.

Graphing a linear relation from a table of values

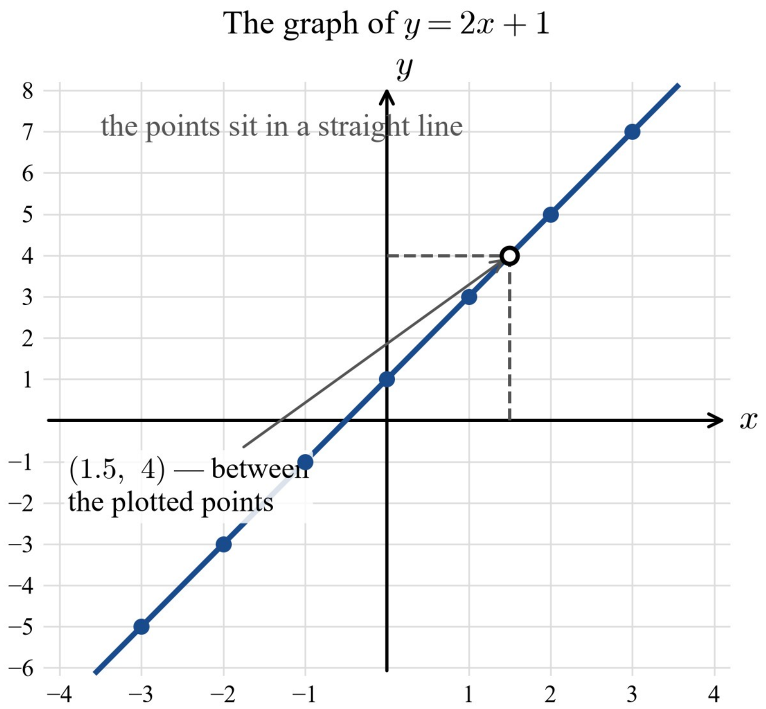
To graph a linear relation:

1. Choose some x-values and build a **table of values** by substituting into the rule.
2. Plot the points on the Cartesian plane.
3. The first differences are constant, so the points sit in a straight line — join them with a ruler.

The next two figures do this for $y=2x+1$.



The points from the table for y = 2x + 1, from (−3, −5) to (3, 7), plotted on the Cartesian plane.

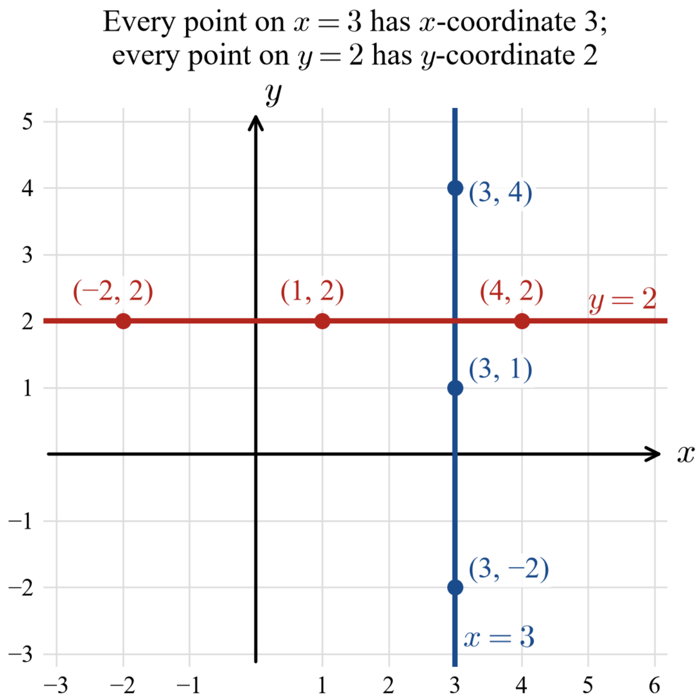


The plotted points joined with a straight line: the graph of y = 2x + 1, with the in-between point (1.5, 4) marked.

The line runs on in both directions past the plotted points, and points **between** the table entries sit on the line too: at $x=1.5$ the rule gives $y=2\times1.5+1=4$, and $(1.5, 4)$ is on the line. Graphing software or a graphics calculator will draw the line for you once you enter the rule — but plot a few points by hand first, so that you can tell when the screen is wrong.

Horizontal and vertical lines

Two special linear relations are worth knowing on sight.



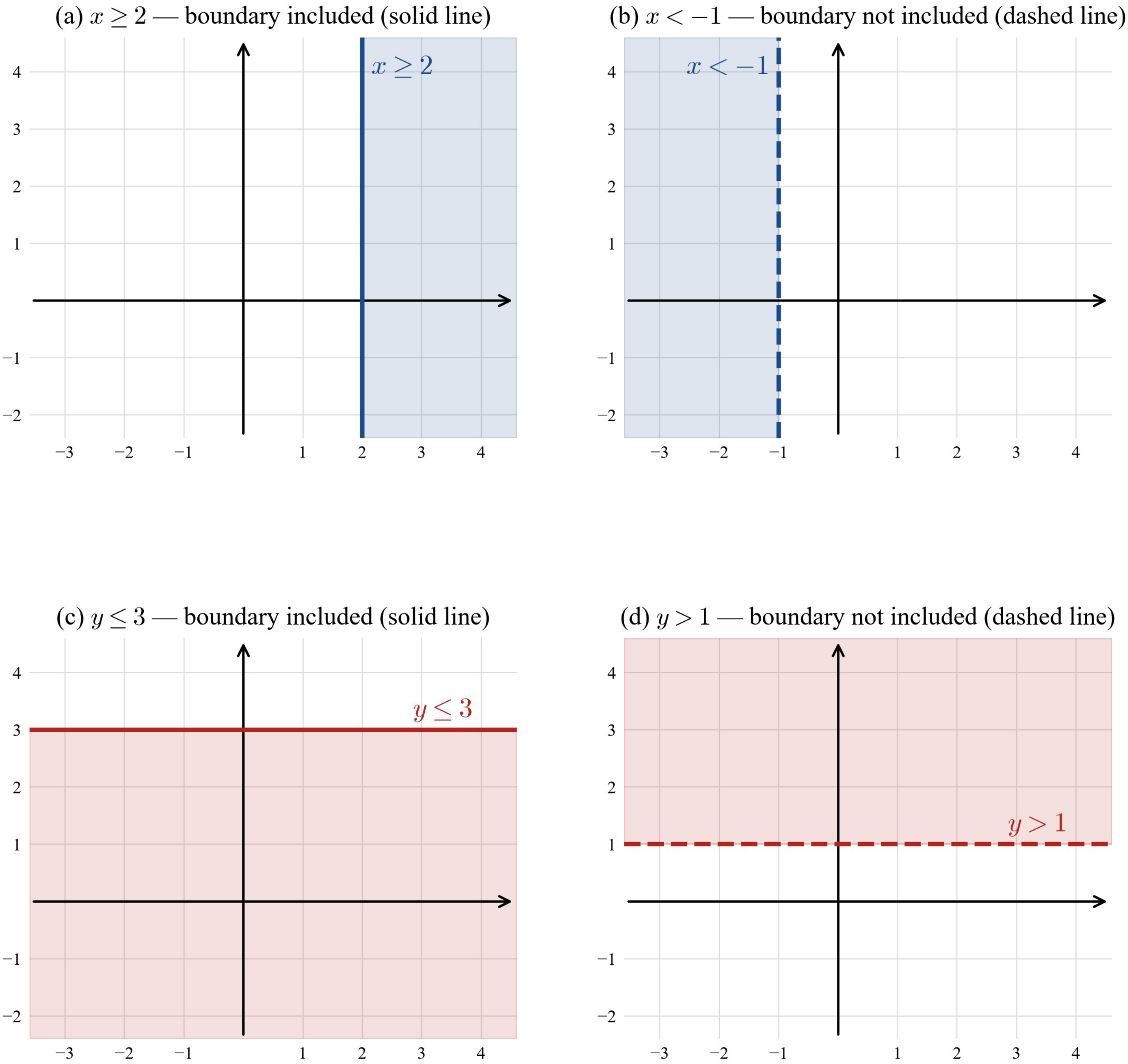
The vertical line x = 3 and the horizontal line y = 2 on the Cartesian plane, with sample points labelled on each.

Every point on the vertical line has x-coordinate 3, whatever y is, so the line is $x=3$. Every point on the horizontal line has y-coordinate 2, whatever x is, so the line is $y=2$. In general:

- $x=a$ is the **vertical** line through a on the x-axis.
- $y=a$ is the **horizontal** line through a on the y-axis.

Inequalities on the Cartesian plane

An **inequality** such as $x \geq 2$ is true for many points at once, so its graph is a whole region of the plane, not just a line.

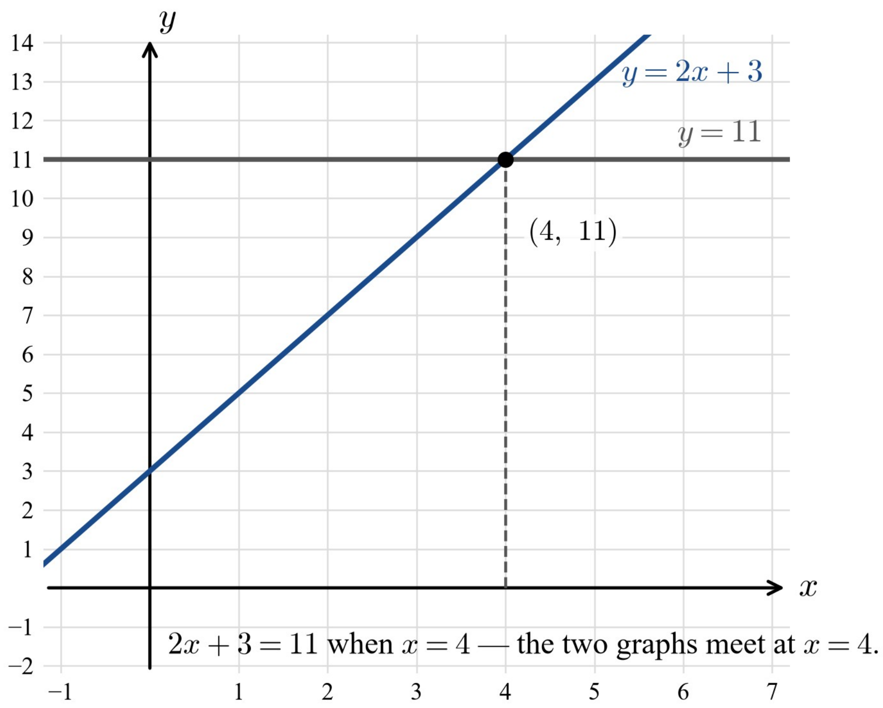


Four panels shading the regions $x \geq 2$ (solid boundary, shaded right), $x < -1$ (dashed boundary, shaded left), $y \leq 3$ (solid boundary, shaded below) and $y > 1$ (dashed boundary, shaded above).

The line itself is the **boundary** of the region. With ≥ or ≤ the boundary is part of the solution and is drawn **solid**; with < or > the boundary is not part of the solution and is drawn **dashed**. A test point tells you which side to shade: for $x \geq 2$, try $(3, 0)$ — there $x = 3 \geq 2$ is true, so the side containing $(3, 0)$ is shaded.

Reading an equation off a graph

An equation such as $2x + 3 = 11$ can be read off a graph: draw $y = 2x + 3$ and $y = 11$, and look at where they meet.

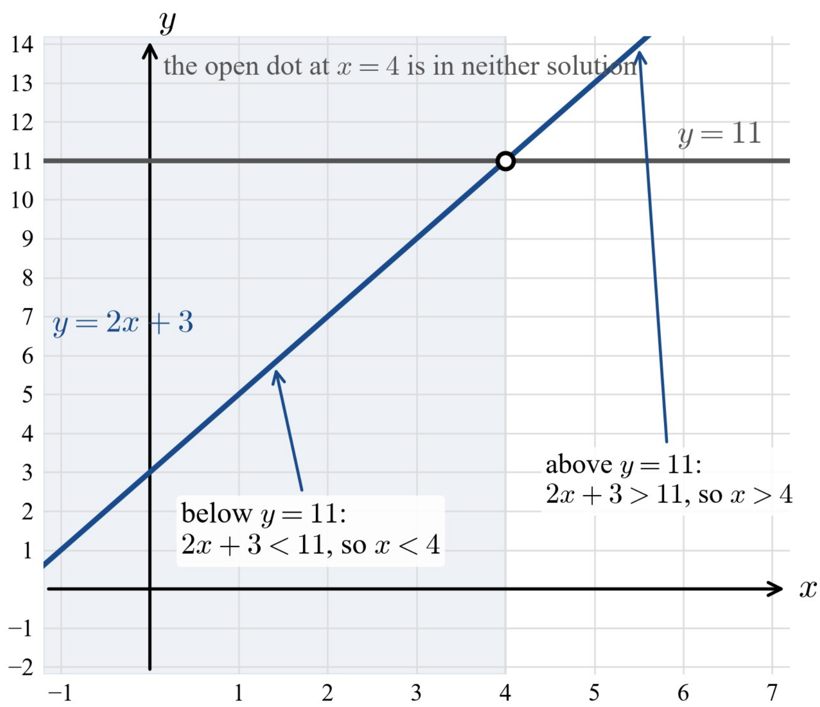


The graphs of y = 2x + 3 and y = 11 meeting at the point (4, 11), showing that 2x + 3 = 11 when x = 4.

The graphs meet at $(4, 11)$, so $2x+3=11$ has the solution $x=4$. Check by substitution: $2\times4+3=11$. ✓

Reading an inequality off a graph

The same picture answers the inequalities as well. The line $y=2x+3$ is **below** $y=11$ exactly where $2x+3<11$, and **above** $y=11$ exactly where $2x+3>11$.

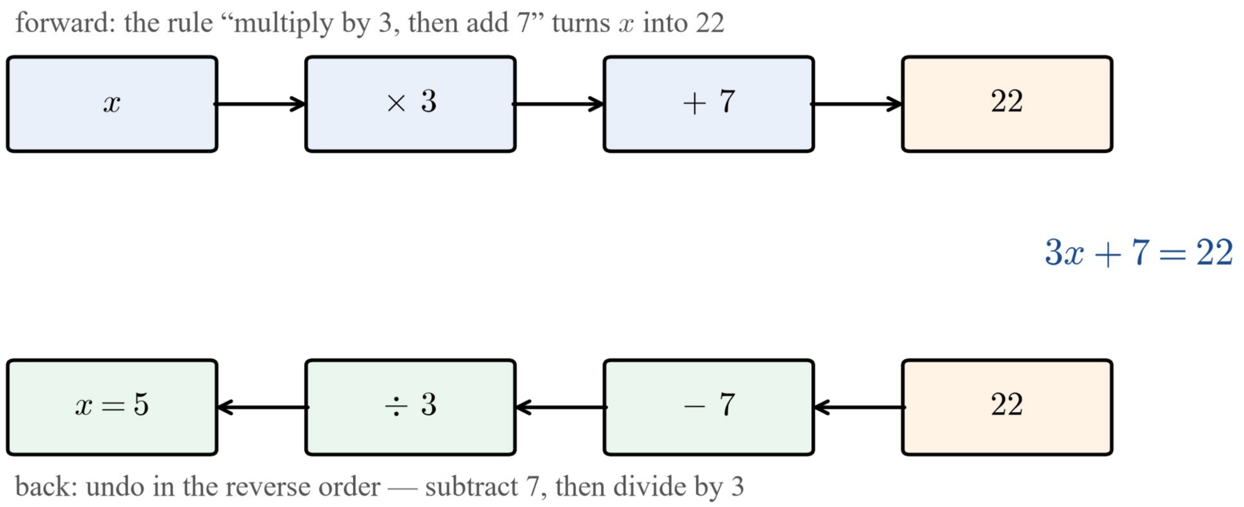


The graph of y = 2x + 3 against the line y = 11: below y = 11 the relation gives 2x + 3 < 11, so x < 4; above it, 2x + 3 > 11, so x > 4; the open dot at x = 4 belongs to neither solution.

Reading off the graph: $2x+3<11$ when $x<4$, and $2x+3>11$ when $x>4$. At $x=4$ the two sides are equal, so $x=4$ belongs to **neither** inequality solution — that is why the dot is open there.

Solving equations by inverse operations

A graph shows a solution at a glance; algebra finds it exactly. The equation $3x+7=22$ says: start with x, multiply by 3, then add 7, and the result is 22. To find x, **undo** each operation **in the reverse order**, doing the same to both sides.



Flow diagram for 3x + 7 = 22: the forward row x, times 3, add 7, 22; the return row 22, subtract 7, divide by 3, x = 5.

Working	Comment
$3x+7=22$	the equation as given
$3x=22-7=15$	undo the $+7$ first: subtract 7 from both sides
$x=\dfrac{15}{3}=5$	then undo the $\times 3$: divide both sides by 3
check: $3\times 5+7=15+7=22$ ✓	substitute back — the left side returns the right side

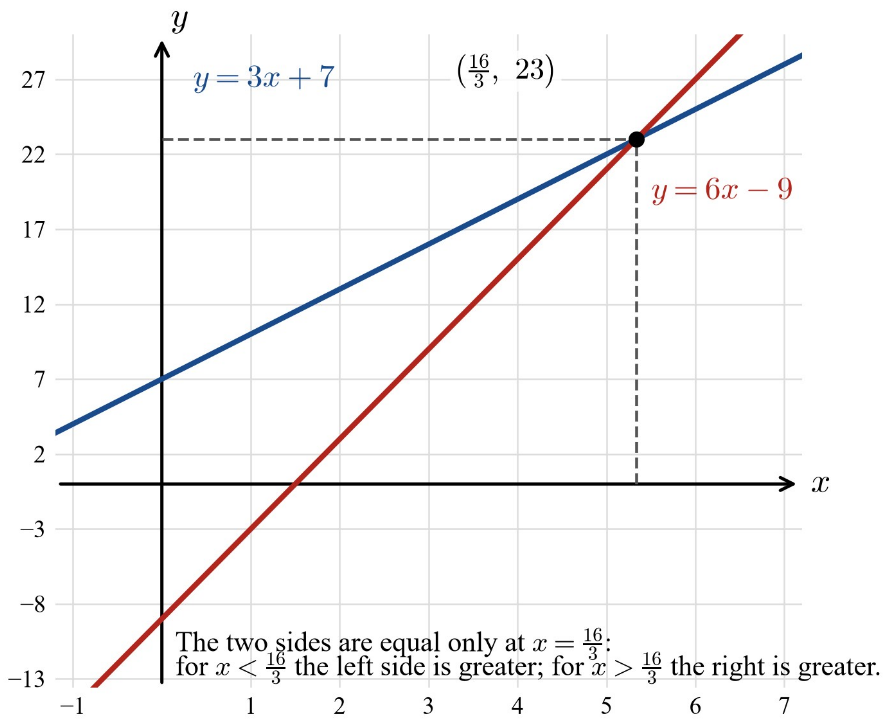
The check is not optional in this subtopic: the content description says *verify solutions by substitution*, so every equation solved here ends with one.

The variable on both sides

When the variable sits on both sides, as in $3x+7=6x-9$, first gather the x-terms on one side. Subtract $3x$ from both sides: $7=3x-9$. Now add 9 to both sides: $16=3x$, so $x=\dfrac{16}{3}$. The answer is a fraction — solutions in this subtopic are allowed to be fractions, not just whole numbers.

Check: left side $3\times\dfrac{16}{3}+7=16+7=23$; right side $6\times\dfrac{16}{3}-9=32-9=23$. Both sides return 23. ✓

The graph shows what the solution means. Think of the two sides as two rules, $y=3x+7$ and $y=6x-9$:



The lines y = 3x + 7 and y = 6x − 9 meeting at (16/3, 23): the two sides of the equation 3x + 7 = 6x − 9 are equal only at x = 16/3.

The two sides are equal exactly where the lines meet, at $x = \dfrac{16}{3}$. This is one equation being tested at each value of x — not two equations solved together, which is a later topic.

Solving one-variable inequalities by algebra

The same undoing works for inequalities; the only new idea is that the answer is a range of values instead of a single number. This year we solve inequalities in which the number multiplying x is positive, and then every undoing step keeps the inequality sign pointing the same way.

Solve $5x - 4 < 21$: add 4 to both sides, $5x < 25$; divide both sides by 5, $x < 5$. The solution is every number less than 5.

Check two things — the **boundary** and a **test value**. At $x = 5$ (the boundary), $5 \times 5 - 4 = 21$: the sides are equal, so $x = 5$ itself is not part of a "<" solution. Try $x = 4$ (inside the range): $5 \times 4 - 4 = 16 < 21$ ✓. The graph agrees: $y = 5x - 4$ is below $y = 21$ exactly where $x < 5$.

Words of this subtopic

Word	What it means here
relation	a rule connecting two quantities, like $y = 2x + 1$
linear relation	a relation whose y-values change by the same amount for equal steps in x; its graph is a straight line
table of values	a table of chosen x-values with the matching y-values
first difference	the change in y from one table entry to the next
equation	a statement that two expressions are equal; solving it finds the variable's value
solution	a value of the variable that makes the statement true
inequality	a statement using <, >, ≤ or ≥; its solution is usually a range of values
boundary	the line that edges an inequality region; solid when it is part of the solution, dashed when it is not

Word	What it means here
inverse operations	operation pairs that undo each other: add/subtract, multiply/divide
verify by substitution	put the answer back into the original statement and check that it holds

Worked examples

Example 1 — with help

(a) Build a table of values for $y = 4x - 1$ using $x = 0, 1, 2, 3, 4, 5$.
(b) Find the first differences. What do they tell you?
(c) Plot the points and join them. Read y at $x = 2.5$ from your line, then check with the rule.

Working	Comment
$x=0$: $y=-1$; $x=1$: $y=3$; $x=2$: $y=7$; $x=3$: $y=11$; $x=4$: $y=15$; $x=5$: $y=19$	substitute each x into the rule
differences: $4, 4, 4, 4, 4$	constant first difference → the relation is linear
read from the line: $y=9$ at $x=2.5$; rule: $4 \times 2.5 - 1 = 10 - 1 = 9$ ✓	the line gives values between the table entries

∴ (a) $y = -1, 3, 7, 11, 15, 19$; (b) the first difference is 4, constant, so the relation is linear; (c) $y = 9$ at $x = 2.5$, and the rule agrees.

Example 2 — with help

Draw the lines $x = -2$ and $y = 3$ on one set of axes. Say which of the points $(-2, 5)$, $(4, 3)$, $(-2, 3)$ and $(3, -2)$ sit on each line.

Working	Comment
$x=-2$ is vertical: every point on it has first coordinate -2.	$(-2, 5)$ and $(-2, 3)$ sit on it
$y=3$ is horizontal: every point on it has second coordinate 3.	$(4, 3)$ and $(-2, 3)$ sit on it
$(-2, 3)$ has first coordinate -2 **and** second coordinate 3.	it sits on both lines — where they cross
$(3, -2)$ has first coordinate 3 and second coordinate -2.	it sits on neither line

∴ on $x=-2$: $(-2, 5)$ and $(-2, 3)$; on $y=3$: $(4, 3)$ and $(-2, 3)$. The point $(-2, 3)$ is on both lines; $(3, -2)$ is on neither.

Example 3 — with help

Use the graphs of $y = 2x + 3$ and $y = 11$ above to read off the solutions of:

(a) $2x + 3 = 11$
(b) $2x + 3 < 11$
(c) $2x + 3 > 11$

Then verify (a) by substitution.

Working	Comment
(a) the lines meet at $(4, 11)$: $x = 4$	equal where the graphs meet
(b) the line is below $y = 11$ to the left of $x = 4$: $x < 4$	below ↔ left side smaller

Working	Comment
(c) the line is above $y=11$ to the right of $x=4$: $x>4$	above ↔ left side greater
check (a): $2\times4+3=11$ ✓	substitute back

∴ (a) $x=4$; (b) $x<4$; (c) $x>4$.

Example 4 — on your own

Solve $3x+7=22$ using inverse operations, and check your answer.

Undo in reverse order: subtract 7, then divide by 3.

$3x+7=22$ $3x=22-7=15$ (subtract 7 from both sides) $x=\frac{15}{3}=5$ (divide both sides by 3) check: $3\times5+7=15+7=22$ ✓

∴ $x=5$.

Example 5 — on your own

Solve $3x+7=6x-9$, verify by substitution, and say what the solution shows on the graph of the two lines.

Gather the x-terms: subtract $3x$ from both sides, $7=3x-9$. Add 9 to both sides, $16=3x$. Divide by 3, $x=\frac{16}{3}$.

check: left side $3\times\frac{16}{3}+7=23$; right side $6\times\frac{16}{3}-9=23$ ✓

On the graph, $x=\frac{16}{3}$ is where the lines $y=3x+7$ and $y=6x-9$ meet — the only place the two sides are equal.

∴ $x=\frac{16}{3}$ (both sides equal 23).

Example 6 — on your own

Solve $5x-4<21$. Check with a test value, and say where the solution appears on a graph.

Add 4 to both sides: $5x<25$. Divide both sides by 5: $x<5$.

check: boundary $x=5$ gives $5\times5-4=21$, equal and therefore not part of a "<" solution; test value $x=4$ gives $5\times4-4=16<21$ ✓

On a graph, $y=5x-4$ is below $y=21$ exactly where $x<5$ — the same answer read off the picture.

∴ $x<5$.

Practice questions

Work these questions by hand. The numbers are chosen so that no calculator is needed, and every equation solution should be checked by substitution — that is part of the skill.

With help

Q1. Find the first differences and say whether each table shows a linear relation.

x	0	1	2	3	4
y	2	5	8	11	14

x	1	2	3	4	5

x	0	1	2	3	4
y	1	4	9	16	25

Q2. Build a table of values for $y=3x-2$ using $x=-1,0,1,2,3$.

Q3. Plot your points from Q2 on a grid and join them. Read y at $x=1.5$ from the line, then check with the rule.

Q4. Draw the lines $y=3$, $x=-2$, $y=-1$ and $x=4$ on one set of axes.

Q5. For each inequality, say whether the boundary line is solid or dashed, and which side of it is shaded.

a. $x\leq 2$
b. $y>3$
c. $x\geq -1$
d. $y<0$

Q6. The graphs of $y=3x-2$ and $y=10$ meet at $(4,10)$. Read off:

a. the solution of $3x-2=10$, and verify it by substitution
b. the solution of $3x-2<10$
c. the solution of $3x-2>10$

Q7. Solve and check:

a. $x+6=15$
b. $4x=28$
c. $2x+5=17$
d. $5x-3=22$

Q8. Solve and check:

a. $5x+4=2x+19$
b. $7x-2=3x+14$

On your own

Q9. Build a table of values for $y=7-2x$ using $x=0,1,2,3,4$. Find the first differences. Is the relation linear?

Q10. A table has a constant first difference of 4. Copy and complete it, then write the rule connecting x and y.

x	0	1	2	3	4
y	5	9	▢	17	▢

Q11. Plot $y=-x+4$ for $x=-2,-1,0,1,2,3,4$ and join the points. Read the value of x where $y=2$, and check it by substitution.

Q12. Say which of the points $(0,-2)$, $(3,-2)$, $(3,0)$, $(-1,-2)$ and $(4,2)$ sit on the line $y=-2$, which sit on $x=3$, which sit on both, and which sit on neither.

Q13. Sketch each region on its own axes, showing a solid or dashed boundary:

a. $y\geq -2$
b. $x>3$
c. $x\geq 0$
d. $y\leq 5$

Q14. The graphs of $y=4x+1$ and $y=21$ meet at $(5,21)$. Read off the solution of $4x+1=21$ (and verify it by substitution), and the solution of $4x+1<21$.

Q15. Solve and check:

a. $2x+9=6$
b. $5x-1=3x+9$
c. $8x-5=3x+10$
d. $9-2x=x+3$

Q16. Solve each inequality. Check the boundary and one test value in each case.

a. $x+4<10$
b. $3x\geq 21$
c. $2x-5>7$
d. $5x+2\leq 27$

Q17. Solve:

a. $4x+3<2x+15$
b. $6x-1\leq 4x+9$

Q18. A seedling is 8 cm tall and grows 3 cm each week: $h=8+3w$, with h the height in cm after w weeks.

a. Build a table of values for $w=0,1,2,3,4$.
b. Plot the points and join them.
c. Read from your graph the number of weeks when $h=17$.
d. Solve $8+3w=26$ and check your answer.
e. Solve $8+3w>29$ and say what your answer means for the seedling.

Maths in real life

Q19. A phone leaves the charger at 62% battery and loses 4% for every hour of use: $B=62-4t$, with B the battery percent after t hours.

a. Explain why the relation is linear.
b. Copy and complete the table.

t (hours)	0	2	4	6
B (percent)	▢	▢	▢	▢

c. Plot the points and join them.
d. Read t when $B=30$ from your graph, then solve $62-4t=30$ and check.

Q20. In a fictional grid town, the tram route runs along the line $x=2$ and the river runs along the line $y=-1$.

a. Draw both lines on a grid.
b. The station, the dock, the bridge and the shop sit at $(2,4)$, $(5,-1)$, $(2,-1)$ and $(-1,2)$ in some order. Match each place to its point.

Q21. A car park charges a \$3 entry fee plus \$5 per hour: $C=3+5n$, with C the cost in dollars for n hours.

a. Solve $3+5n=28$ and check your answer.
b. Read from the graph of $C=3+5n$ where $3+5n>28$.
c. What does your answer to (b) mean for a driver?

Q22. Make up your own equation with solution $x=4$, and your own inequality with solution $x<3$. Swap with a partner, solve each other's, and check by substitution.

Answers

Q1. a. Differences $3, 3, 3, 3$ — constant, so linear. b. Differences $3, 5, 7, 9$ — changing, so not linear.

Q2. $y = -5, -2, 1, 4, 7$.

Q3. $y = 2.5$; check $3 \times 1.5 - 2 = 4.5 - 2 = 2.5$ ✓

Q4. $y = 3$ and $y = -1$ are horizontal lines; $x = -2$ and $x = 4$ are vertical lines.

Q5. a. solid, the left side. b. dashed, above. c. solid, the right side. d. dashed, below.

Q6. a. $x = 4$ (check $3 \times 4 - 2 = 10$ ✓). b. $x < 4$. c. $x > 4$.

Q7. a. $x = 9$. b. $x = 7$. c. $x = 6$. d. $x = 5$.

Q8. a. $x = 5$. b. $x = 4$.

Q9. $y = 7, 5, 3, 1, -1$; differences $-2, -2, -2, -2$ — constant, so linear.

Q10. 13 and 21; $y = 5 + 4x$.

Q11. $x = 2$; check $-2 + 4 = 2$ ✓

Q12. On $y = -2$: $(0, -2)$, $(3, -2)$, $(-1, -2)$. On $x = 3$: $(3, -2)$, $(3, 0)$. On both: $(3, -2)$. On neither: $(4, 2)$.

Q13. a. solid, above. b. dashed, the right side. c. solid, the right side. d. solid, below.

Q14. $x = 5$ (check $4 \times 5 + 1 = 21$ ✓); $x < 5$.

Q15. a. $x = -1.5$. b. $x = 5$. c. $x = 3$. d. $x = 2$.

Q16. a. $x < 6$. b. $x \geq 7$. c. $x > 6$. d. $x \leq 5$.

Q17. a. $x < 6$. b. $x \leq 5$.

Q18. a. $h = 8, 11, 14, 17, 20$. c. $w = 3$. d. $w = 6$. e. $w > 7$ — after 7 weeks the seedling is taller than 29 cm.

Q19. a. The battery loses the same amount (4%) each hour, so the first differences are constant. b. $B = 62, 54, 46, 38$. d. $t = 8$.

Q20. station $(2, 4)$; dock $(5, -1)$; bridge $(2, -1)$; shop $(-1, 2)$.

Q21. a. $n = 5$. b. $n > 5$. c. Parking for longer than 5 hours costs more than \$28.

Q22. Answers will vary. Samples: $2x + 1 = 9$ has solution $x = 4$; $2x < 6$ has solution $x < 3$.

Fully-worked solutions

Q1. Subtract each y-value from the next.

a. $5 - 2 = 3$, $8 - 5 = 3$, $11 - 8 = 3$, $14 - 11 = 3$: the first differences are constant, so the relation is **linear**.
b. $4 - 1 = 3$, $9 - 4 = 5$, $16 - 9 = 7$, $25 - 16 = 9$: the differences change, so the relation is **not linear**.

Q2. Substitute each x into $y = 3x - 2$: $x = -1$: $y = -3 - 2 = -5$; $x = 0$: $y = -2$; $x = 1$: $y = 1$; $x = 2$: $y = 4$; $x = 3$: $y = 7$.

x	−1	0	1	2	3
y	−5	−2	1	4	7

Q3. The points sit in a straight line; joining them and reading across at $x = 1.5$ gives $y = 2.5$. Check: $3 \times 1.5 - 2 = 4.5 - 2 = 2.5$ ✓

Q4. $y=3$ and $y=-1$ are horizontal lines through 3 and -1 on the y-axis; $x=-2$ and $x=4$ are vertical lines through -2 and 4 on the x-axis.

Q5. ≥ and ≤ include the boundary (solid); < and > do not (dashed). a. $x\le 2$: solid line $x=2$, shaded to the left. b. $y>3$: dashed line $y=3$, shaded above. c. $x\ge -1$: solid line $x=-1$, shaded to the right. d. $y<0$: dashed line $y=0$ (the x-axis), shaded below.

Q6. a. The graphs meet at $x=4$, so $3x-2=10$ has solution $x=4$. Check: $3\times 4-2=12-2=10$ ✓ b. The line $y=3x-2$ is below $y=10$ where $x<4$. c. It is above $y=10$ where $x>4$.

Q7. a. $x=15-6=9$; check $9+6=15$ ✓ b. $x=28\div 4=7$; check $4\times 7=28$ ✓ c. $2x=17-5=12$, $x=6$; check $2\times 6+5=17$ ✓ d. $5x=22+3=25$, $x=5$; check $5\times 5-3=22$ ✓

Q8. a. $5x+4=2x+19$: subtract $2x$, $3x+4=19$; subtract 4, $3x=15$; $x=5$. Check: left $5\times 5+4=29$; right $2\times 5+19=29$ ✓ b. $7x-2=3x+14$: subtract $3x$, $4x-2=14$; add 2, $4x=16$; $x=4$. Check: left $7\times 4-2=26$; right $3\times 4+14=26$ ✓

Q9. $x=0$: $y=7$; $x=1$: $y=5$; $x=2$: $y=3$; $x=3$: $y=1$; $x=4$: $y=-1$. The first differences are -2 every time — constant, so the relation is linear (a line running downhill).

Q10. Constant difference 4: $9+4=13$ and $17+4=21$. The y-value starts at 5 when $x=0$ and grows by 4 for each step in x: $y=5+4x$. Check at $x=3$: $5+12=17$ ✓

Q11. The table: $(-2,6)$,$(-1,5)$,$(0,4)$,$(1,3)$,$(2,2)$,$(3,1)$,$(4,0)$. The points sit in a straight line; reading along it, $y=2$ at $x=2$. Check: $-2+4=2$ ✓

Q12. A point sits on $y=-2$ when its second coordinate is -2: $(0,-2)$, $(3,-2)$, $(-1,-2)$. It sits on $x=3$ when its first coordinate is 3: $(3,-2)$, $(3,0)$. The point $(3,-2)$ sits on both; $(4,2)$ sits on neither.

Q13. a. solid line $y=-2$, shaded above. b. dashed line $x=3$, shaded to the right. c. solid line $x=0$ (the y-axis), shaded to the right. d. solid line $y=5$, shaded below.

Q14. The graphs meet at $x=5$: $4x+1=21$ has solution $x=5$. Check: $4\times 5+1=21$ ✓ The line $y=4x+1$ is below $y=21$ where $x<5$, so $4x+1<21$ has solution $x<5$.

Q15. a. $2x=6-9=-3$, $x=\dfrac{-3}{2}=-1.5$. Check: $2\times(-1.5)+9=-3+9=6$ ✓ b. $5x-1=3x+9$: subtract $3x$, $2x-1=9$; add 1, $2x=10$; $x=5$. Check: left 24; right 24 ✓ c. $8x-5=3x+10$: subtract $3x$, $5x-5=10$; add 5, $5x=15$; $x=3$. Check: left 19; right 19 ✓ d. $9-2x=x+3$: add $2x$, $9=3x+3$; subtract 3, $6=3x$; $x=2$. Check: left $9-4=5$; right $2+3=5$ ✓

Q16. a. $x<10-4=6$. Boundary: $6+4=10$, not <10 ✓; test $x=0$: $4<10$ ✓ b. $x\ge 21\div 3=7$. Boundary: $3\times 7=21\ge 21$ ✓; test $x=8$: $24\ge 21$ ✓ c. $2x>7+5=12$, $x>6$. Boundary: $12-5=7$, not >7 ✓; test $x=10$: $15>7$ ✓ d. $5x\le 27-2=25$, $x\le 5$. Boundary: $25+2=27\le 27$ ✓; test $x=0$: $2\le 27$ ✓

Q17. a. $4x+3<2x+15$: subtract $2x$, $2x+3<15$; subtract 3, $2x<12$; $x<6$. Test $x=0$: $3<15$ ✓ b. $6x-1\le 4x+9$: subtract $4x$, $2x-1\le 9$; add 1, $2x\le 10$; $x\le 5$. Test $x=5$: $29\le 29$ ✓

Q18. a. $w=0$: $h=8$; $w=1$: $h=11$; $w=2$: $h=14$; $w=3$: $h=17$; $w=4$: $h=20$. b. The points sit in a straight line; join them. c. $h=17$ at $w=3$. d. $3w=26-8=18$, $w=6$; check $8+18=26$ ✓ e. $3w>29-8=21$, $w>7$: from week 7 on, the seedling is taller than 29 cm. Test $w=8$: $8+24=32>29$ ✓

Q19. a. Each hour of use removes the same 4%, so the first differences are constant (-4 per hour): the relation is linear. b. $t=0$: $B=62$; $t=2$: $B=54$; $t=4$: $B=46$; $t=6$: $B=38$. c. The points sit in a straight line; join them. d. Reading the line, $B=30$ at $t=8$. Algebra: $4t=62-30=32$, $t=8$. Check: $62-4\times 8=62-32=30$ ✓

Q20. a. $x=2$ vertical, $y=-1$ horizontal. b. The station is on the tram route only: $(2,4)$. The dock is on the river only: $(5,-1)$. The bridge is where the tram crosses the river, on both lines: $(2,-1)$. The shop is on neither: $(-1,2)$.

Q21. a. $5n = 28 - 3 = 25$, $n = 5$. Check: $3 + 5 \times 5 = 3 + 25 = 28$ ✓ b. The line $C = 3 + 5n$ is above $C = 28$ where $n > 5$. c. A driver who stays more than 5 hours pays more than $28.

Q22. Answers will vary. Check that the equation really returns $x = 4$ (for example $2x + 1 = 9$: $2 \times 4 + 1 = 9$ ✓) and the inequality really gives $x < 3$ (for example $2x < 6$: dividing by 2 gives $x < 3$ ✓).

Modelling with linear relations, including profit and loss

> **Content description `VC2M8A03`:** use mathematical modelling to solve applied problems involving linear relations, including financial contexts involving profit and loss; formulate problems with linear functions, and choose a representation; interpret and communicate solutions in terms of the context, and review the appropriateness of the model

A taxi fare, a leaking tank, a class fundraiser and a Saturday job all change in the same pattern: a starting amount, with the same amount added (or lost) at each step. By the end of this subtopic you will be able to turn situations like these into a linear rule, solve the questions they raise, and — just as importantly — say when the rule stops being a good description of real life.

Theory

The modelling cycle

Last year you read relationships from graphs built from real data (`VC2M7A04`) and described what you saw. This subtopic turns that around: instead of reading a situation **from** a graph, you build the mathematics **from** the situation. That is **mathematical modelling**, and it runs in a cycle of five steps. Every worked example and every full modelling question in this subtopic follows them.

1. **Formulate.** Decide what the question is really asking. Choose letters to stand for the quantities (for example, C for total cost, n for number of hours) and write the relationship between them.
2. **Choose a representation.** A table of values, a rule, or a graph — pick the one that suits the question. You can move between them.
3. **Solve.** Do the mathematics: substitute into the rule, read from the table, or work backwards from an output to its input.
4. **Interpret and communicate.** State the answer **in terms of the context**, with the right units. "t = 4" is not an answer; "the job took 4 hours" is.
5. **Review.** Ask whether the answer and the model itself make sense. Does the size of the answer fit the situation? For which inputs does the rule actually apply?

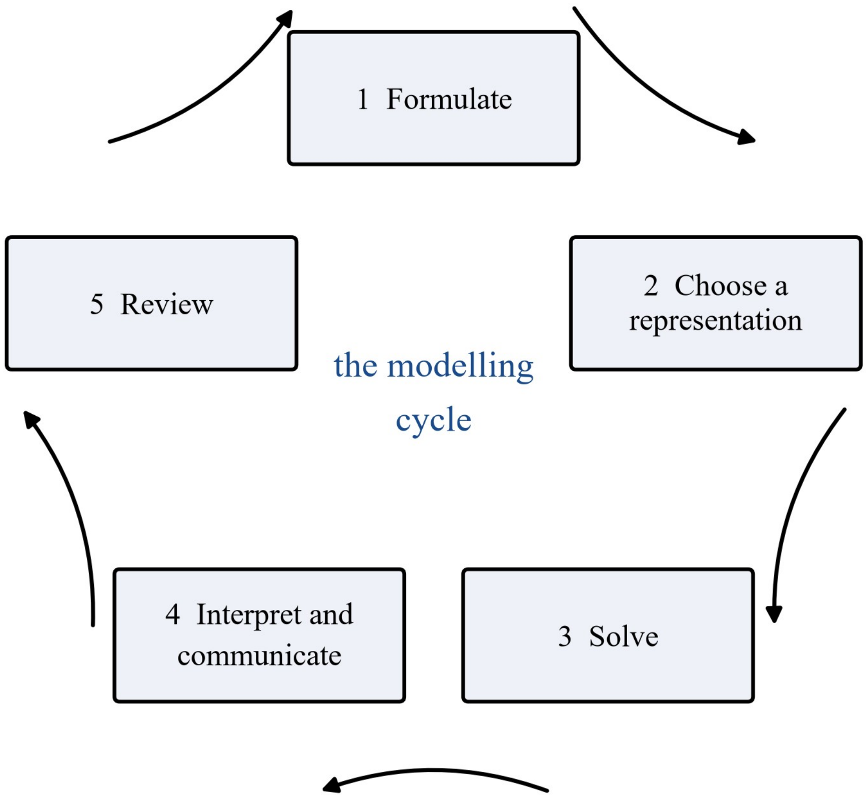


Diagram of the modelling cycle: five boxes in a ring joined by arrows going around — 1 formulate, 2 choose a representation, 3 solve, 4 interpret and communicate, 5 review.

A solution that stops at step 3 is only half done. The review step is part of the mathematics here, not a decoration — models are approximations of real life, and saying where a model works and where it breaks down is a skill this subtopic builds.

Initial value and constant rate of change

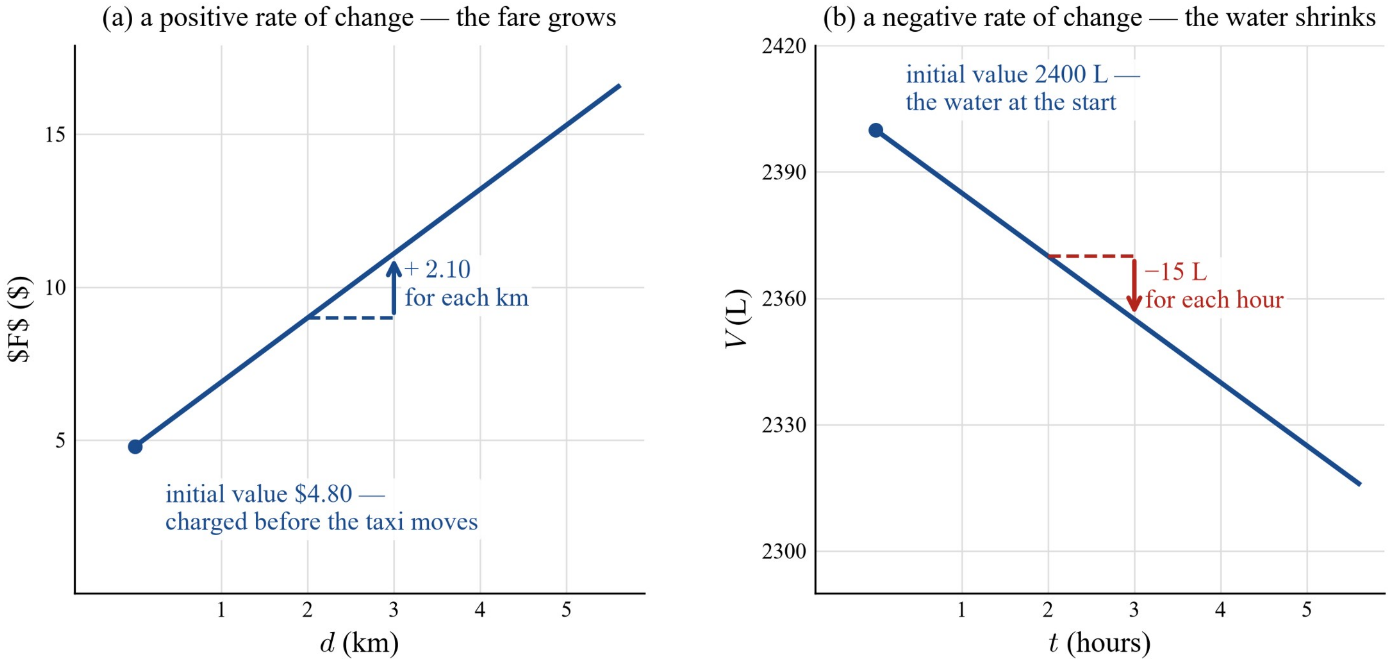
A relation is **linear** when the output changes by the **same amount** each time the input goes up by 1. In a table of values, the differences between successive outputs are constant —you met this test in 2B. Two numbers describe every linear relation completely:

- the **initial value** — the output when the input is 0; the starting amount;
- the **constant rate of change** — the amount added to (or subtracted from) the output for each increase of 1 in the input.

Read these two numbers **in the context**:

Situation	Initial value	Constant rate of change
A taxi fare with a flag fall, then a charge per km	the flag fall — charged before the taxi moves	the charge per km
A tradesperson’s bill with a call-out fee, then a charge per hour	the call-out fee — charged even for 0 hours of work	the charge per hour
Water leaking from a full tank	the amount of water at the start	the amount lost each hour (a negative rate)
Pay at an hourly rate	$0 — no hours worked, no pay	the hourly rate

If the rate of change is positive the quantity **grows**; if it is negative the quantity **shrinks**. A leaking tank loses 15 L each hour, so its rate of change is −15 L per hour.



Two graphs. Left: a taxi fare that starts at $4.80 and grows by $2.10 for each kilometre, with the initial value and one step of $2.10 marked. Right: a tank that starts at 2400 litres and loses 15 litres each hour, with the initial value and one step down of 15 litres marked.

Writing a rule

Use letters that remind you of the quantities, and write:

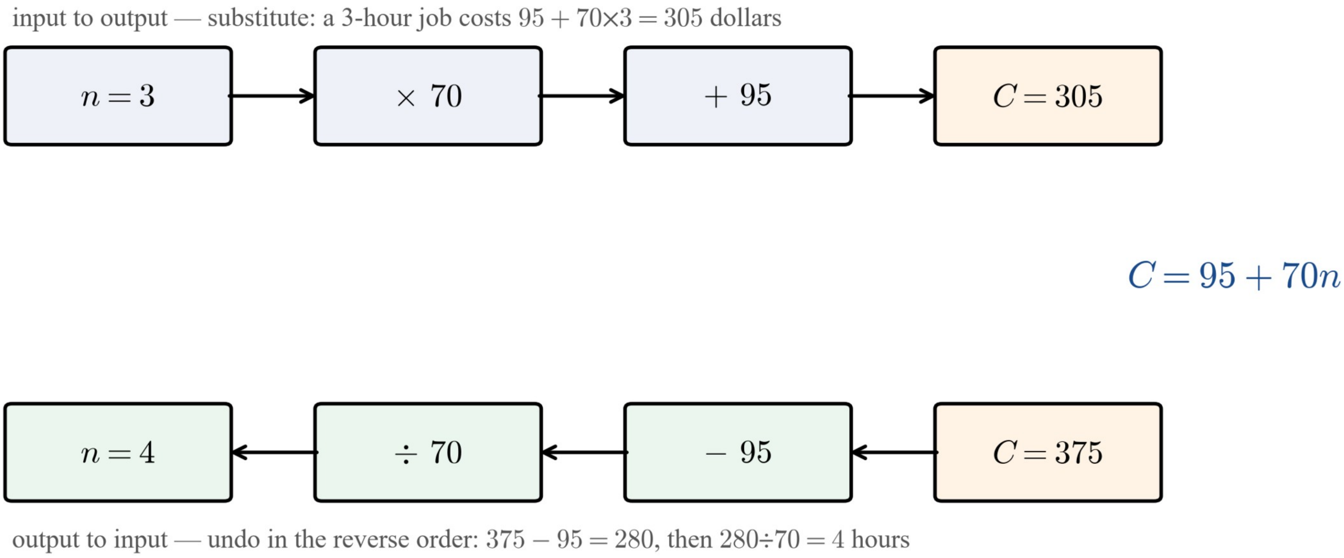
$$\text{output} = \text{initial value} + \text{rate of change} \times \text{input}$$

A plumber charges a $95 call-out fee and then $70 for each hour of work. With C for the total cost in dollars and n for the number of hours worked:

$$C = 95 + 70n$$

With this rule you can work in both directions:

- **Input to output** — substitute. A 3-hour job costs $C = 95 + 70 \times 3 = 95 + 210 = 305$, so $305.
- **Output to input** — undo the steps. A bill of $375 comes from $375 - 95 = 280$ dollars of hourly work, and $280 \div 70 = 4$, so the job took 4 hours. (2B takes this further as solving equations.)



Flow diagram for C = 95 + 70n. Top row, input to output: n = 3, times 70, add 95, C = 305. Bottom row, output to input: C = 375, subtract 95, divide by 70, n = 4.

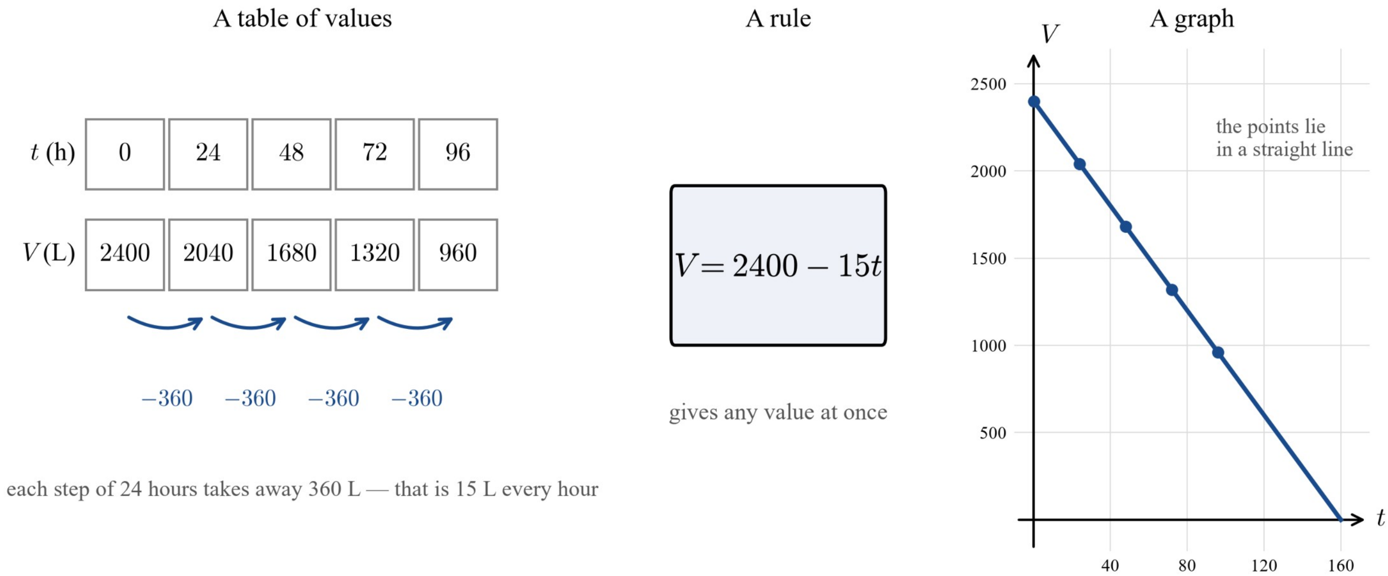
Choosing a representation

The same situation can be shown three ways. A tank starts with 2400 L and loses 15 L each hour:

- **A table** lists particular values and shows the constant difference:

t (hours)	0	24	48	72	96
V (litres)	2400	2040	1680	1320	960

- **A rule** gives any value at once: $V = 2400 - 15t$.
- **A graph** of the table's points gives the whole picture; because the relation is linear, the points lie on a straight line, and you can read values between the entries you plotted (2B covers the graphing in full).



The same leaking tank shown three ways: a table of values for t = 0, 24, 48, 72 and 96 hours with the same drop of 360 litres between entries, the rule V = 2400 − 15t, and a graph whose five points sit in a straight line.

Choose the representation that fits the question. A table is good for spotting the pattern; a rule is good for exact values far apart; a graph is good for seeing the overall behaviour.

Profit and loss

Financial contexts give the most useful linear models of all. For any money-making venture:

- the **costs** are what you spend — often a fixed amount (materials, a stall fee) plus an amount per item;
- the **income** is what you receive — the price times the number of items sold;
- the **profit** is income minus costs.

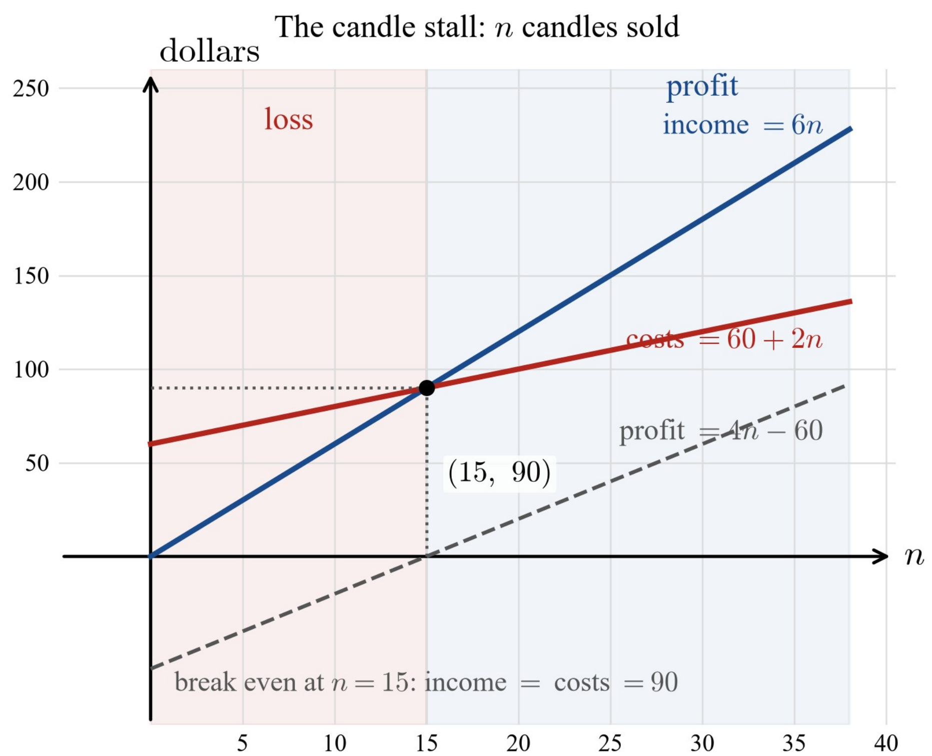
$$\text{profit} = \text{income} - \text{costs}$$

If the profit is positive, you are ahead. If the profit is **negative**, the venture is making a **loss** — negative numbers have a real meaning here, just as they do in 1A. The **break-even point** is where profit is exactly 0: income equals costs, and every item sold after that point makes money.

Candles for a school fete cost \$2 each to make, the stall costs a fixed \$60, and each candle sells for 6. With n candles sold:

$$\text{income} = 6n, \text{costs} = 60 + 2n, \text{profit} = 6n - (60 + 2n) = 4n - 60$$

Each candle sold adds \$4 to the profit, and the fixed \$60 must be covered first: $60 \div 4 = 15$, so the stall breaks even at 15 candles. Fewer than 15 — a loss; more than 15 — a profit.

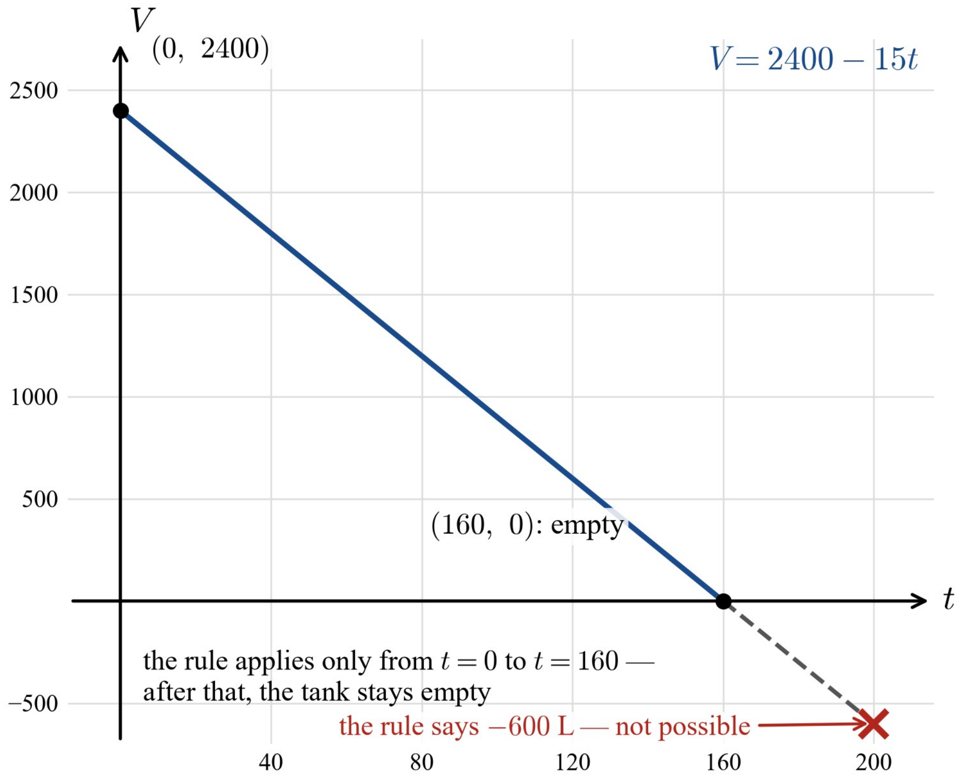


Graphs of income = 6n, costs = 60 + 2n and profit = 4n − 60 for the candle stall. The income and costs lines meet at n = 15, the break-even point, with the loss side shaded red and the profit side shaded blue.

Reviewing the model — when a rule stops making sense

A rule does not know what it is modelling. Part of modelling is working out which inputs the rule can sensibly take:

- **The empty tank.** For $V = 2400 - 15t$, substituting $t = 200$ gives $V = 2400 - 3000 = -600$. Negative litres make no sense. The tank is empty after $2400 \div 15 = 160$ hours, and once it is empty, no more water can flow from it — it stays empty. The rule applies only for t from 0 to 160.
- **Whole numbers.** You can sell 15 candles but not 15.5 candles. If a break-even calculation gives 42.5, then it takes 43 sales to make a profit.
- **Constant rates in real life.** Real cars slow for lights, real leaks widen, real jobs attract weekend rates. A constant rate of change is often a good description for a while, and then it is not.



Graph of V = 2400 − 15t from t = 0 to t = 160, where the line reaches 0 and the tank is empty. Past t = 160 a dashed part of the line keeps falling to −600 at t = 200, marked with a cross because negative litres are not possible.

Before you accept any modelled answer, ask three questions:

1. Does the answer fit the context — the right size and the right units?
2. Does the input make sense — not before the start, not past the end, and in the right kind of numbers (whole, or decimal, or time)?
3. For how long, or for which values, does the constant-rate assumption hold?

Words of this subtopic

Word	What it means here
model	a mathematical version of a real situation — a rule, table or graph that stands in for it
linear relation	a relation whose output changes by the same amount each time the input goes up by 1
initial value	the output when the input is 0; the starting amount
constant rate of change	the amount added to (or subtracted from) the output for each increase of 1 in the input
rule	a formula, like $C = 95 + 70n$, giving the output for any input
costs	what a venture spends: fixed amounts plus amounts per item
income	what a venture receives: price times number of items sold
profit	income minus costs; negative profit is a loss
break-even point	the input where profit is exactly 0
flag fall	the fixed charge a taxi makes at the start of a trip, before any distance
call-out fee	the fixed charge a tradesperson makes just to attend, before any work time

Worked examples

Example 1 — with help

A plumber charges a $95 call-out fee plus $70 per hour of work.

(a) Write a rule for the total cost C of a job that takes n hours.
(b) Make a table of costs for jobs from 0 to 5 hours.
(c) What does a 3-hour job cost?
(d) A customer is billed $375. How long did the job take?
(e) What do the numbers 95 and 70 mean in this context?

Working	Comment
$C=95+70n$	Start with the call-out fee; add $70 for each hour.
Table: $n=0,1,2,3,4,5$ give $C=95,165,235,305,375,445$.	Each extra hour adds $70 — the differences are constant, so the relation is linear.
(c) $C=95+70\times3=95+210=305$	Substitute the input.
(d) $375-95=280$, then $280\div70=4$	Undo the rule: remove the call-out fee, then divide by the hourly rate.
(e) 95 is charged even for 0 hours; 70 is added for each hour worked.	Read both numbers in context.

∴ (a) $C=95+70n$; (c) a 3-hour job costs $305; (d) the job took 4 hours. Review: the rule assumes work is billed at the same rate for every hour — a reasonable description for a standard job, but it would need adjusting for overtime or weekend rates.

Example 2 — with help

During a dry spell in January, a farm tank loses water through a crack. It starts with 2400 L and leaks 15 L each hour. Let V be the volume of water left after t hours.

(a) Write a rule for V.
(b) When is the tank empty?
(c) What does the rule give at $t=200$? Why is that answer unsuitable?

Working	Comment
$V=2400-15t$	Start at 2400 L; subtract 15 L for each hour.
$2400\div15=160$	The tank empties after 160 hours.
$V=2400-15\times200=2400-3000=-600$	The rule gives −600 L at $t=200$.

∴ (a) $V=2400-15t$; (b) the tank is empty after 160 hours (about 6 days and 16 hours); (c) −600 L is not a possible volume — negative litres do not exist. Once the tank is empty, no more water can flow from it, so it stays empty. The rule applies only for t from 0 to 160.

Notice the two numbers in context: 2400 is the starting volume, and 15 is the amount lost each hour — a rate of change of −15 L per hour.

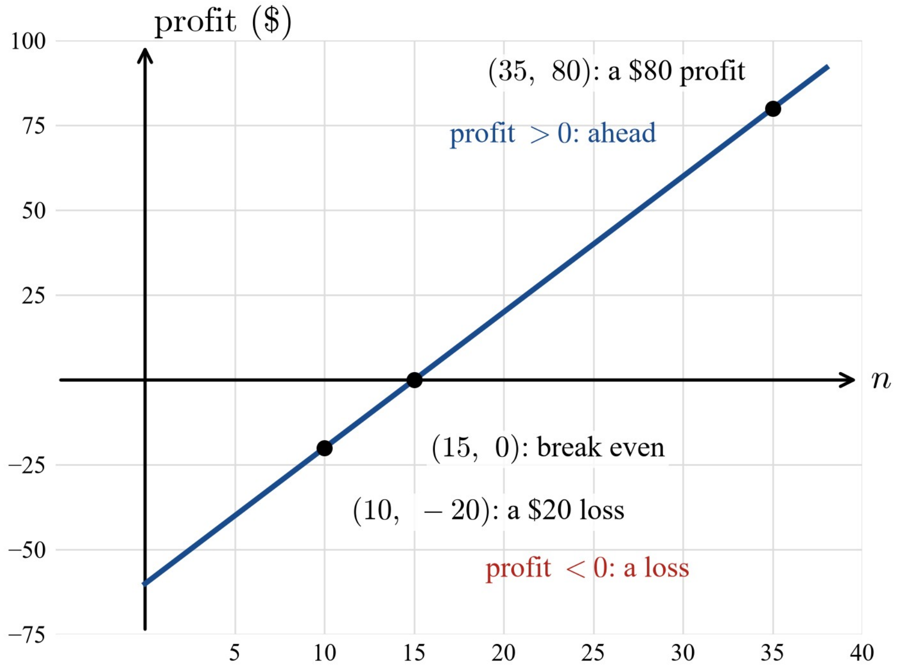
Example 3 — with help

A Year 8 class makes candles for the school fete. Materials cost $2 per candle and the stall fee is a fixed $60. The candles sell for $6 each. Let n be the number of candles sold.

(a) Write rules for the income and the costs.
(b) Write a rule for the profit.
(c) Find the profit when 10, 15 and 35 candles are sold.
(d) How many candles must be sold to break even?

Working	Comment
income $= 6n$; costs $= 60 + 2n$	Price × number sold; fixed fee plus \$2 per candle.
profit $= 6n - (60 + 2n) = 4n - 60$	Profit = income − costs.
$n = 10$: $4 \times 10 - 60 = -20$	A profit of −\$20 is a \$20 loss.
$n = 15$: $4 \times 15 - 60 = 0$	Neither profit nor loss.
$n = 35$: $4 \times 35 - 60 = 80$	A profit of \$80.
$60 \div 4 = 15$	The fixed \$60 needs 15 candles at \$4 each to cover it.

∴ (b) profit $= 4n - 60$; (c) 10 candles → a \$20 loss, 15 candles → break even, 35 candles → a \$80 profit; (d) the break-even point is 15 candles. Each candle sold after the 15th adds \$4 to the profit.



Graph of profit = 4n − 60: at n = 10 the profit is −20 (a \$20 loss), at n = 15 it is 0 (break even), and at n = 35 it is 80 (a \$80 profit). Below n = 15 the profit is negative; above it, positive.

Example 4 — on your own

A taxi company charges a flag fall of \$4.80 and then \$2.10 per kilometre.

(a) Write a rule for the fare F of a trip of d km.
(b) Find the fare for a 12 km trip.
(c) A passenger pays \$38.40. How far did the taxi travel?
(d) What do the two numbers in the rule mean in this context?

The flag fall is the initial value — charged before the taxi moves — and \$2.10 is added for each kilometre:

$$F = 4.80 + 2.10d$$

(b) $F = 4.80 + 2.10 \times 12 = 4.80 + 25.20 = 30.00$.

(c) Work backwards: $38.40 - 4.80 = 33.60$ comes from the distance charge, and $33.60 \div 2.10 = 16$.

∴ (a) $F = 4.80 + 2.10d$; (b) the 12 km trip costs \$30; (c) the taxi travelled 16 km; (d) \$4.80 is the flag fall, \$2.10 is the charge per km. Review: this is a model of one flat rate. Real fare structures can add booking fees and higher rates at night, so the rule is a guide for ordinary trips, not a promise.

Example 5 — on your own

A car leaves a town and travels along a country road at a constant speed of 90 km/h.

(a) Write a rule for the distance d km travelled after t hours.
(b) How far has the car travelled after 2.5 hours?
(c) How long does it take to travel 315 km?
(d) Is this rule a good model for any real trip? Give your reasons.

The car covers 90 km in each hour, so the rate of change is 90 km per hour, and the journey starts at distance 0:

$$d = 90t$$

(b) $d = 90 \times 2.5 = 225$.

(c) Work backwards: $315 \div 90 = 3.5$.

∴ (a) $d = 90t$; (b) the car has travelled 225 km; (c) the trip takes 3.5 hours. (d) The rule is a good model only while the speed really is constant. A real car slows for corners, traffic and stops, so $d = 90t$ describes the trip well as an average picture, but not minute by minute. (You will meet distance–time modelling again, as a rate, in 4C.)

Example 6 — on your own

From 1 July 2026, the national minimum wage in Australia is $26.44 per hour.

Source: Fair Work Ombudsman, *Minimum wages*. Retrieved 19 August 2026.

Let P be the pay, in dollars, for h hours of work at this rate.

(a) Write a rule for P.
(b) Copy and complete the table.

h (hours)	0	2	4	6	8
P (dollars)	▢	▢	▢	▢	▢

(c) How many hours of work earn $396.60?
(d) What does each number in the rule mean here? What is different about this initial value compared with the taxi and the plumber?

The pay starts at $0 — no hours, no pay — and grows by $26.44 for each hour worked:

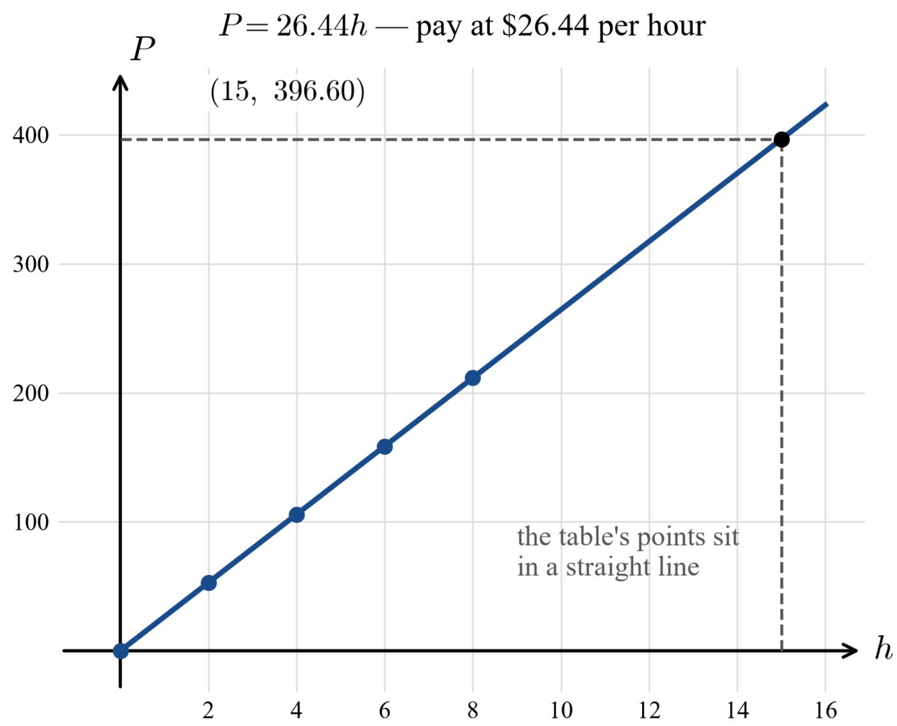
$$P = 26.44h$$

(b) Each entry is 26.44 times the hours:

h (hours)	0	2	4	6	8
P (dollars)	0	52.88	105.76	158.64	211.52

(c) $396.60 \div 26.44 = 15$.

∴ (a) $P = 26.44h$; (c) 15 hours of work earn $396.60; (d) there is no starting payment — the initial value is 0 — and $26.44 is the amount added for each hour worked. The taxi and the plumber had a fee charged before any distance or time; pay at an hourly rate starts from nothing. Plotting the table's points on a grid shows them in a straight line — a graph of the rule lets you read pay for any number of hours between the entries. Review: the model assumes the same rate for every hour. It does not include weekend or late rates, overtime, or tax — so it suits ordinary hours at the base rate, and not much else.



Graph of P = 26.44h: the points for 0, 2, 4, 6 and 8 hours sit in a straight line, and dashed guides from 15 hours read off a pay of \$396.60.

Practice questions

Work these questions by hand. The numbers are chosen so that no calculator is needed — the skill here is the modelling, not the arithmetic.

With help

Q1. Each rule models a situation. State the initial value and the rate of change, and say what each one stands for.

a. $C = 40 + 30n$, the cost in dollars of a handyman's visit of n hours
b. $V = 500 - 8t$, the volume in litres of a cattle trough after t days, as water is used
c. $h = 12 + 3w$, the height in cm of a seedling w weeks after planting
d. $P = 5n - 80$, the profit in dollars from selling n badges

Q2. The handyman from Q1a charges $C = 40 + 30n$.

a. Copy and complete the table.

n (hours)	0	1	2	3	4	5
C (dollars)	▢	▢	▢	▢	▢	▢

b. What is the difference between each C value and the one before it? Why is it the same every time?
c. Plot the points on a grid. What do you notice about them?

Q3. Write a rule for each situation. Choose your own letters and say what they stand for.

a. A bucket starts with 12 L of water. Water is added at 5 L per minute.
b. A candle is 80 mm tall and burns down at 6 mm per hour.
c. A student earns \$15 for each hour of weekend work.

Q4. A tank holds 900 L and drains at 12 L per minute: $V = 900 - 12t$, with V in litres and t in minutes.

a. Find V when $t = 30$.

b. Find t when $V=420$.
c. When is the tank empty?
d. For how long does the tank hold more than 300 L?

Q5. A candle is 18 cm tall and burns down at 1.5 cm per hour.

a. Write a rule for its height h cm after t hours.
b. How tall is it after 4 hours?
c. When has it burned away completely?
d. For which values of t does your rule make sense? Explain.

Q6. A cake stall sells cakes for $5 each. The stall costs a fixed $40 to set up, and each cake costs $1 to make. Let n be the number of cakes sold.

a. Write a rule for the income.
b. Write a rule for the costs.
c. Write a rule for the profit.
d. How many cakes must be sold to break even?
e. Find the profit when 25 cakes are sold.

Q7. Which tables show a linear relation? For the ones that do, give the rate of change.

t	0	1	2	3
d	0	60	120	180

n	1	2	3	4
A	2	4	8	16

t	0	1	2	3
V	100	92	84	76

Q8. Match each situation with its rule.

i. A candle is 20 cm tall and burns down at 2 cm per hour.
ii. A $50 gift card is spent at $8 per week.
iii. A bean plant is 5 cm tall and grows 1.5 cm each week.

Rules: $h=5+1.5w$ · $G=50-8w$ · $h=20-2t$

On your own

Q9. An electrician charges an $80 call-out fee and then $65 per hour.

a. Write a rule for the cost C of a job taking n hours.
b. Make a table of costs for 0 to 4 hours.
c. Find the cost of a job that takes 2.5 hours.
d. A job costs $405. How long did it take?
e. What does the 80 stand for in this context?

Q10. A tank on a farm holds 3200 L. During a dry February it leaks 25 L per hour.

a. Write a rule for the volume V left after t hours.
b. How much water is left after 24 hours?
c. When will the tank be empty?
d. Substituting $t=200$ into your rule gives $V=-1800$. Explain why this cannot be the amount of water left, and say what actually happens to the tank.
e. After how many hours is less than 1000 L left in the tank?

Q11. A cyclist rides along a flat road at a constant 24 km/h.

a. Write a rule for the distance d km ridden in t hours.
b. How far does the cyclist ride in 3.5 hours?
c. How long does it take to ride 108 km?
d. Would $d=24t$ be a good model for a real 4-hour ride? Give your reasons.

Q12. A taxi charges a flag fall of $3.60 and then $1.80 per km.

a. Write a rule for the fare F of a trip of d km.
b. Find the fare for a 9 km trip.
c. A fare comes to $28.80. How far was the trip?
d. A booking fee of $2 is added to every trip. Write the new rule. Which number in the rule changes, and which stays the same?

Q13. Maya has a weekend job paid at $31.50 per hour.

a. Write a rule for her pay P for h hours of work.
b. Find her pay for 4 hours and for 7 hours.
c. One weekend she earns $472.50. How many hours did she work?
d. Draw a graph of P against h for h from 0 to 8 on grid paper. Use your graph to estimate her pay for 5 hours, then check the estimate by calculation.

Q14. The Year 8 formal is held in the school hall. The DJ and the hall together cost a fixed $340, and each student pays $8 to get in.

a. Write rules for the income and the profit when n students attend.
b. Find the profit when 30 students attend.
c. How many students must attend for the formal to make a profit? (Be careful: you cannot have half a student.)
d. Find the profit when 60 students attend.

Q15. After coming out of the oven, a roast rests on the kitchen bench. For the first while, its temperature falls by about 2 °C each minute. Its temperature when it leaves the oven is 74 °C, and the room is 20 °C.

a. Write a rule for the temperature T °C after m minutes of resting.
b. What is the temperature after 10 minutes?
c. How long does the rule predict it takes to reach 20 °C?
d. Give two reasons why this linear rule is only a rough guide.

Q16. A table of values for a linear relation is shown.

t	0	1	2	3	4
C	12	19	26	33	40

a. Find the rate of change.
b. State the initial value.
c. Write the rule for C in terms of t.
d. Use the rule to find C when $t=10$.

Q17. A gym offers two plans. Plan A has no joining fee and costs $12 per visit. Plan B costs a $60 joining fee and then $6 per visit. Let n be the number of visits.

a. Write a rule for the total cost of each plan.
b. Copy and complete the table.

n (visits)	2	4	6	8	10	12	14
Plan A	▢	▢	▢	▢	▢	▢	▢

n (visits)	2	4	6	8	10	12	14
(dollars)							
Plan B (dollars)	▢	▢	▢	▢	▢	▢	▢

c. For how many visits do the two plans cost the same? What is that cost?
d. Which plan is cheaper for 6 visits? Which is cheaper for 15 visits?
e. In words, explain why the cheaper plan depends on how often you visit.

Q18. Full modelling cycle. The school garden club sells trays of seedlings. The club spends a fixed $25 on materials, plus $3 to raise each tray. Each tray sells for $8.

a. **Formulate:** choose letters for the quantities, and write rules for the costs, the income and the profit when n trays are sold.
b. **Choose a representation and solve:** how many trays must the club sell to make a profit of at least $100?
c. **Interpret and communicate:** state your answer in terms of the seedlings, in a full sentence.
d. **Review:** give two reasons why the real profit might differ from your model.

Maths in real life

Q19. From 1 July 2026, the national minimum wage in Australia is $26.44 per hour, and the weekly rate for a standard working week is $1004.90.

Source: Fair Work Ombudsman, *Minimum wages*. Retrieved 19 August 2026.

a. Write a rule for the pay P for h hours of work at the hourly rate.
b. Find the pay for 6, 12, 20 and 30 hours of work.
c. How many hours of work earn $661?
d. At the hourly rate, 38 hours of work gives $38 \times 26.44 = 1004.72$ — but the official weekly rate above is 1004.90. Suggest why the two amounts are not exactly the same.

Q20. A car leaves a town and travels at a steady 50 km/h.

a. Write a rule for the distance d km from the town after t hours.
b. How far is the car from the town after 12 minutes? (Give t in hours.)
c. When is the car 35 km from the town? Give your answer in minutes.
d. Give one reason why the rule might not match a real trip out of a town.

Q21. In July, steady rain fills a garden water tank. The tank starts with 120 L, and the rain adds 40 L each hour. The tank's capacity is 500 L.

a. Write a rule for the volume V in the tank after t hours of rain.
b. Find the volume after 6 hours.
c. When does the tank reach its capacity?
d. What happens after that, and for which values of t does your rule describe the tank?

Q22. Make your own. Write a profit-and-loss situation for a stall or a fundraiser: choose a fixed cost, a cost per item and a selling price. Write the rules for costs, income and profit, find the break-even point, and write one sentence reviewing your own model. Swap with a partner and check each other's working.

Answers

Q1. a. Initial value 40 (the call-out charge); rate 30 ($ per hour). b. Initial value 500 (the starting volume); rate −8 (litres used per day). c. Initial value 12 (height at planting); rate 3 (cm grown per week). d. Initial value −80 (the 80 spent before any badge is sold); rate 5(added per badge).

Q2. a. $C = 40, 70, 100, 130, 160, 190$. b. 30; each extra hour adds $30. c. The points lie on a straight line.

Q3. a. $V = 12 + 5m$ (V litres after m minutes). b. $h = 80 - 6t$ (h mm after t hours). c. $P = 15h$ (P dollars for h hours).

Q4. a. 540 L. b. 40 min. c. 75 min. d. 50 min.

Q5. a. $h = 18 - 1.5t$. b. 12 cm. c. 12 hours. d. From 0 to 12 hours — after that the candle has burned away and the rule would give negative heights.

Q6. a. income $= 5n$. b. costs $= 40 + n$. c. profit $= 4n - 40$. d. 10 cakes. e. $60.

Q7. a. Linear; rate 60. b. Not linear (differences 2, 4, 8). c. Linear; rate −8.

Q8. i → $h = 20 - 2t$; ii → $G = 50 - 8w$; iii → $h = 5 + 1.5w$.

Q9. a. $C = 80 + 65n$. b. $C = 80, 145, 210, 275, 340$. c. $242.50. d. 5 hours. e. The call-out fee, charged even for 0 hours of work.

Q10. a. $V = 3200 - 25t$. b. 2600 L. c. 128 hours. d. A volume cannot be negative; the tank is empty after 128 hours and stays empty. e. After 88 hours.

Q11. a. $d = 24t$. b. 84 km. c. 4.5 hours. d. Sample: not for the whole ride — a real ride has hills, lights and rests, so the speed is not really constant.

Q12. a. $F = 3.60 + 1.80d$. b. $19.80. c. 14 km. d. $F = 5.60 + 1.80d$; the initial value changes (3.60 → 5.60) and the rate per km stays 1.80.

Q13. a. $P = 31.50h$. b. $126 and $220.50. c. 15 hours. d. Estimate about $157.50; check: $31.50 \times 5 = 157.50$.

Q14. a. income $= 8n$; profit $= 8n - 340$. b. −$100, a loss of $100. c. 43 students (break-even is 42.5). d. $140.

Q15. a. $T = 74 - 2m$. b. 54 °C. c. 27 minutes. d. Sample: cooling slows as the roast gets closer to room temperature, and it cannot cool below 20 °C.

Q16. a. 7. b. 12. c. $C = 12 + 7t$. d. 82.

Q17. a. Plan A: $A = 12n$; Plan B: $B = 60 + 6n$. b. A: 24, 48, 72, 96, 120, 144, 168; B: 72, 84, 96, 108, 120, 132, 144. c. 10 visits; $120. d. Plan A for 6 visits ($72 against $96); Plan B for 15 visits ($150 against $180). e. Sample: Plan A costs more per visit but has no joining fee, so it suits few visits; Plan B's $60 fee is only worth paying if you visit often.

Q18. a. Sample letters: costs $= 25 + 3n$, income $= 8n$, profit $= 5n - 25$. b. 25 trays. c. Sample: "The club must sell at least 25 trays of seedlings to make a profit of $100." d. Sample: some trays may not sell; raising costs may be more (or less) than $3 per tray.

Q19. a. $P = 26.44h$. b. $158.64, $317.28, $528.80, $793.20. c. 25 hours. d. Sample: the hourly and weekly rates are set separately and rounded, so hourly rate × 38 is close to, but not exactly, the weekly rate.

Q20. a. $d = 50t$. b. 10 km. c. 42 min. d. Sample: the car must slow for lights, traffic or turns.

Q21. a. $V = 120 + 40t$. b. 360 L. c. 9.5 hours. d. Once full, the tank overflows and stays at 500 L; the rule applies for t from 0 to 9.5.

Q22. Answers will vary. Check: profit = income − costs, the break-even point makes the profit 0, and the review says something real about where the model could fail.

Fully-worked solutions

Q1. Read the two numbers in the rule: the number on its own is the initial value (the output when the input is 0), and the number multiplying the letter is the rate of change.

a. $C = 40 + 30n$: initial value **40** — the call-out charge, paid even for a 0-hour job; rate of change **30** — dollars added for each hour of work.
b. $V = 500 - 8t$: initial value **500** — the starting volume in litres; rate of change **−8** — 8 litres used each day.
c. $h = 12 + 3w$: initial value **12** — the height in cm at planting (week 0); rate of change **3** — cm grown each week.
d. $P = 5n - 80$: initial value **−80** — $80 spent before any badge is sold; rate of change **5** — dollars added for each badge sold.

∴ a. 40 and 30; b. 500 and −8; c. 12 and 3; d. −80 and 5, with the meanings above.

Q2. a. Substitute each value of n into $C = 40 + 30n$:

$n = 0$: $C = 40$; $n = 1$: $C = 40 + 30 = 70$; $n = 2$: $C = 40 + 60 = 100$; $n = 3$: $C = 130$; $n = 4$: $C = 160$; $n = 5$: $C = 190$.

n (hours)	0	1	2	3	4	5
C (dollars)	40	70	100	130	160	190

b. $70 - 40 = 30$, $100 - 70 = 30$, and so on: the difference is **30** each time, because each extra hour adds $30 to the cost — that constant difference is what makes the relation linear.

c. Plotting $(0, 40)$, $(1, 70)$, $(2, 100)$, $(3, 130)$, $(4, 160)$, $(5, 190)$ on a grid, the points lie on a straight line.

Q3. a. Start at 12 L, add 5 L each minute. With V in litres and m in minutes:

$$V = 12 + 5m$$

b. Start at 80 mm, lose 6 mm each hour. With h in mm and t in hours:

$$h = 80 - 6t$$

c. Start at $0, add $15 each hour. With P in dollars and h in hours:

$$P = 15h$$

Q4. a. $V = 900 - 12 \times 30 = 900 - 360 = 540$: **540 L**.

b. Work backwards from 420: $900 - 420 = 480$ litres have drained, and $480 \div 12 = 40$: **40 minutes**.

c. Empty means all 900 L have drained: $900 \div 12 = 75$: **75 minutes**.

d. More than 300 L means less than $900 - 300 = 600$ L has drained: $600 \div 12 = 50$: the tank holds more than 300 L for the **first 50 minutes**.

Q5. a. Start at 18 cm, lose 1.5 cm each hour: $h = 18 - 1.5t$.

b. $h = 18 - 1.5 \times 4 = 18 - 6 = 12$: **12 cm**.

c. Burned away means height 0: $18 \div 1.5 = 12$: **12 hours**.

d. The rule makes sense from $t = 0$ to $t = 12$. After 12 hours the candle has gone, and the rule would give negative heights (at $t = 13$, $h = 18 - 19.5 = -1.5$), which are not possible.

Q6. a. income $= 5n$ ($5 per cake times n cakes).

b. costs $= 40 + n$ (fixed $40, plus $1 per cake).

c. profit = income − costs $= 5n - (40 + n) = 4n - 40$.

d. Break even means profit 0: $4n = 40$, so $n = 10$: **10 cakes**.

e. $4 \times 25 - 40 = 100 - 40 = 60$: a profit of **$60**.

Q7. Find the differences between successive outputs.

a. $60 - 0 = 60$, $120 - 60 = 60$, $180 - 120 = 60$: constant, so **linear**, rate of change **60**.

b. $4 - 2 = 2$, $8 - 4 = 4$, $16 - 8 = 8$: the differences grow, so **not linear**.

c. $92 - 100 = -8$, $84 - 92 = -8$, $76 - 84 = -8$: constant, so **linear**, rate of change **−8**.

Q8. Sort by initial value and rate of change.

i. Starts at 20 cm, loses 2 cm per hour → $h = 20 - 2t$.
ii. Starts at $50, loses $8 per week → $G = 50 - 8w$.
iii. Starts at 5 cm, gains 1.5 cm per week → $h = 5 + 1.5w$.

∴ i → $h = 20 - 2t$; ii → $G = 50 - 8w$; iii → $h = 5 + 1.5w$.

Q9. a. $C = 80 + 65n$.

b. $n = 0$: $C = 80$; $n = 1$: $C = 145$; $n = 2$: $C = 210$; $n = 3$: $C = 275$; $n = 4$: $C = 340$.

c. $C = 80 + 65 \times 2.5 = 80 + 162.50 = 242.50$: **$242.50**.

d. $405 - 80 = 325$ comes from the hourly work, and $325 \div 65 = 5$: **5 hours**.

e. 80 is the **call-out fee** — charged even when 0 hours of work are done.

Q10. a. $V = 3200 - 25t$.

b. $V = 3200 - 25 \times 24 = 3200 - 600 = 2600$: **2600 L**.

c. $3200 \div 25 = 128$: **128 hours**.

d. A volume cannot be negative: −1800 L is not a possible amount of water. The rule only applies until the tank empties at 128 hours; after that, the tank is empty and stays empty.

e. Less than 1000 L means more than $3200 - 1000 = 2200$ L have leaked: $2200 \div 25 = 88$, so **after 88 hours** less than 1000 L remains.

Q11. a. $d = 24t$.

b. $d = 24 \times 3.5 = 84$: **84 km**.

c. $108 \div 24 = 4.5$: **4.5 hours**.

d. Not for the whole ride. A real cyclist slows for hills, lights and rests, so the speed is only about 24 km/h on average; $d = 24t$ is a useful summary of the ride, not an exact minute-by-minute account.

Q12. a. $F = 3.60 + 1.80d$.

b. $F = 3.60 + 1.80 \times 9 = 3.60 + 16.20 = 19.80$: **$19.80**.

c. $28.80 - 3.60 = 25.20$ comes from the distance charge, and $25.20 \div 1.80 = 14$: **14 km**.

d. The $2 adds to the starting charge: $F = 5.60 + 1.80d$. The **initial value changes** (from 3.60 to 5.60); the rate of $1.80 per km stays the same.

Q13. a. $P = 31.50h$.

b. $31.50 \times 4 = 126$: **$126**. $31.50 \times 7 = 220.50$: **$220.50**.

c. $472.50 \div 31.50 = 15$: **15 hours**.

d. Plot $(0,0)$, $(2,63)$, $(4,126)$, $(6,189)$, $(8,252)$ and join them. Reading from the line at $h=5$ gives about 157.50. Check: $31.50\times 5=157.50$ ✓.

Q14. a. income $=8n$; costs $=340$; profit $=8n-340$.

b. $8\times 30-340=240-340=-100$: a **loss of $100**.

c. Break even: $8n=340$, so $n=42.5$. Since you cannot have half a student, it takes **43 students** to make a profit (42 students would still be a $4 loss).

d. $8\times 60-340=480-340=140$: a profit of **$140**.

Q15. a. $T=74-2m$.

b. $T=74-2\times 10=54$: **54 °C**.

c. Reaching 20 °C means a drop of $74-20=54$ degrees, and $54\div 2=27$: **27 minutes**.

d. Sample reasons: cooling is not really at a constant rate — it slows as the roast gets closer to room temperature; and the roast cannot cool below the room's 20 °C, so the rule must stop making sense before long.

Q16. a. $19-12=7$, $26-19=7$: the rate of change is **7**.

b. At $t=0$, $C=12$: the initial value is **12**.

c. $C=12+7t$.

d. $C=12+7\times 10=82$: **82**.

Q17. a. Plan A: $A=12n$; Plan B: $B=60+6n$.

n (visits)	2	4	6	8	10	12	14
Plan A (dollars)	24	48	72	96	120	144	168
Plan B (dollars)	72	84	96	108	120	132	144

c. The table shows both plans cost **$120 at 10 visits**. Check: Plan A $12\times 10=120$; Plan B $60+6\times 10=120$ ✓.

d. At 6 visits: Plan A $72, Plan B $96 — **Plan A is cheaper**. At 15 visits: Plan A $12\times 15=180$, Plan B $60+6\times 15=150$ — **Plan B is cheaper**.

e. Plan A costs more per visit but has no joining fee, so it costs less when you visit rarely. Plan B's $60 joining fee is only worth paying when you visit often enough for its lower per-visit price to pay the fee back — from 10 visits onwards.

Q18. a. **Formulate.** Let n be the number of trays sold. costs $=25+3n$ · income $=8n$ · profit $=8n-(25+3n)=5n-25$.

b. **Choose a representation and solve.** The rule is fastest: profit at least $100 means $5n-25\geq 100$, so $5n\geq 125$ and $n\geq 25$. (Check with a table: at $n=24$, profit $=5\times 24-25=95$, not enough; at $n=25$, profit $=100$ ✓.)

c. **Interpret and communicate.** The garden club must sell at least 25 trays of seedlings to make a profit of $100.

d. **Review.** Sample: some trays may not sell at all; the real cost of raising a tray may be more or less than $3 once materials, seeds and weather are considered. The model assumes every tray raised is sold at $8, which may not hold.

Q19. a. $P=26.44h$.

b. $26.44 \times 6 = 158.64$; $26.44 \times 12 = 317.28$; $26.44 \times 20 = 528.80$; $26.44 \times 30 = 793.20$. Pay: **\$158.64, \$317.28, \$528.80, \$793.20**.

c. $661 \div 26.44 = 25$: **25 hours**.

d. The hourly rate and the weekly rate are set separately and each is rounded to a convenient amount, so $38 \times 26.44 = 1004.72$ lands close to, but not exactly on, the weekly rate of \$1004.90. The model $P = 26.44h$ uses the hourly rate only — a reminder to check which rate applies before modelling pay.

Q20. a. $d = 50t$.

b. 12 minutes $= \frac{12}{60} = 0.2$ hours; $d = 50 \times 0.2 = 10$: **10 km**.

c. $35 \div 50 = 0.7$ hours; $0.7 \times 60 = 42$: **42 minutes**.

d. Sample: leaving a town there are lights, roundabouts and slower zones, so the car cannot hold a perfectly steady 50 km/h the whole way.

Q21. a. $V = 120 + 40t$.

b. $V = 120 + 40 \times 6 = 120 + 240 = 360$: **360 L**.

c. Reaching 500 L means a gain of $500 - 120 = 380$ L, and $380 \div 40 = 9.5$: **9.5 hours**.

d. After 9.5 hours the tank is full; any further rain overflows, and the volume stays at 500 L rather than climbing. The rule describes the tank only for t from 0 to 9.5.

Q22. Answers will vary. Example: a sausage sizzle where the hire of the barbecue costs a fixed \$30, each sausage costs \$1, and each sausage sells for \$3.50. costs $= 30 + n$, income $= 3.50n$, profit $= 2.50n - 30$. Break-even: $30 \div 2.50 = 12$ sausages. Sample review: the model assumes every sausage cooked is sold, and that exactly \$30 and \$1 are the true costs — wind, rain and hungry teachers could all change the real result.

Experimenting with linear functions: making and testing conjectures

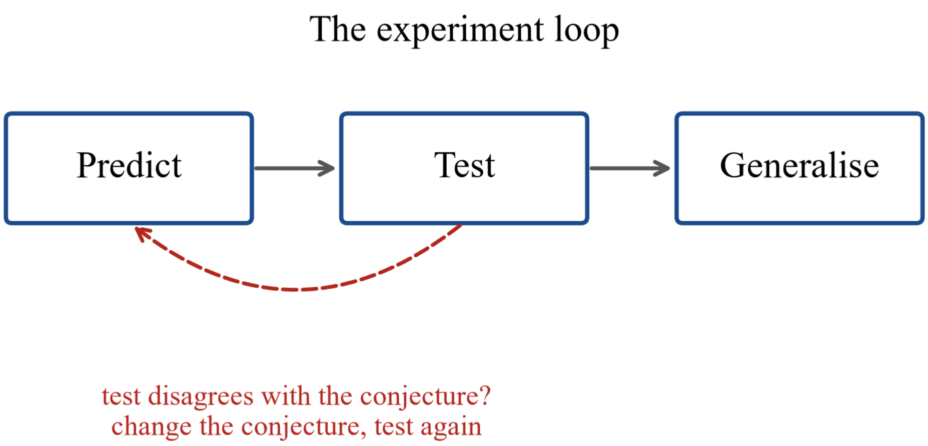
Content description `VC2M8A05`: experiment with linear functions and relations using digital tools, making and testing conjectures and generalising emerging patterns

Mathematics grows by noticing patterns and then daring to say they always hold. In this subtopic you will use graphing software as a testing bench: you predict what a line will do, check it on screen, and — when every test agrees — state the pattern as a rule. A conjecture that survives your best attempts to break it becomes a generalisation.

Theory

The experiment loop

In 2C you built linear rules **from** situations. Here you work the other way: you start from the rule itself, $y = ax + b$, and ask what the numbers a and b **do** to the graph. The method is the one scientists use — a loop of three moves.



Flowchart of the experiment loop: Predict leads to Test, Test leads to Generalise, and a dashed return arrow from Test to Predict is labelled "test disagrees with the conjecture? change the conjecture, test again".

- **Predict.** State what you think will happen, clearly enough that a graph could prove you wrong. That statement is a **conjecture**.
- **Test.** Put the conjecture in front of the evidence. With graphing software you can type in five rules in ten seconds — so test widely, and test nasty cases on purpose.
- **Generalise.** When every test agrees, state the pattern that covers them all. When a test disagrees, the conjecture is broken: fix it and test again.

One passing test proves almost nothing; a conjecture is only as strong as the tests it has survived. So half the skill of this subtopic is choosing tests that **could** break your idea — different signs, different sizes, and the awkward case where a number is 0.

One rule, two numbers to vary

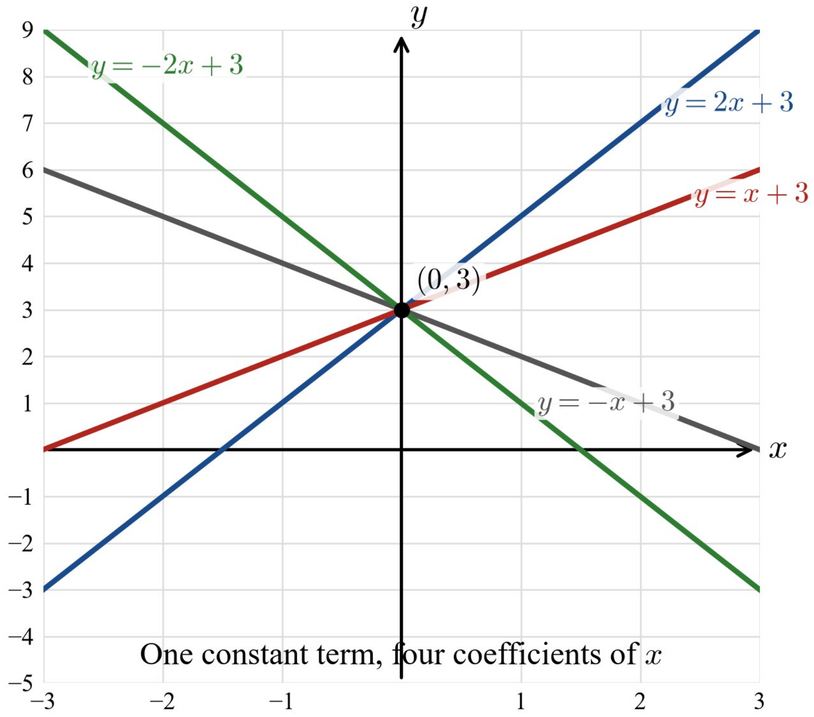
Every rule in this subtopic has the shape

$$y = ax + b$$

with two numbers you are free to change:

- a, the **coefficient of** x — the number multiplying x;
- b, the **constant term** — the number standing on its own.

We call a and b the **parameters** of the rule: numbers you vary, one at a time, to see what changes in the graph. Keeping b fixed and letting a run through $2, 1, -1, -2$ gives a whole **family of lines**:

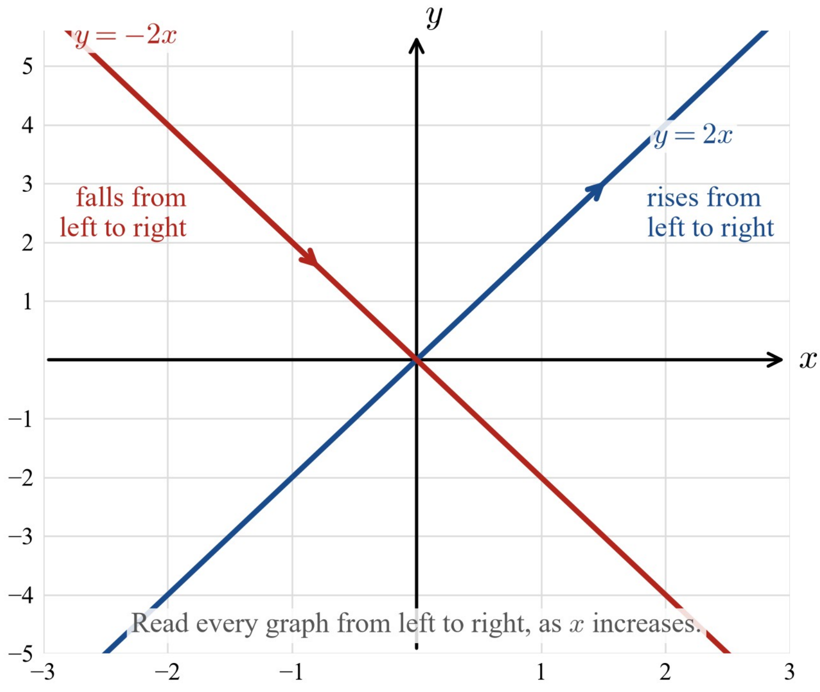


Four lines y = 2x + 3, y = x + 3, y = −x + 3 and y = −2x + 3 drawn on one grid, all crossing the y-axis at the point (0, 3).

All four lines cross the y-axis at the same point, $(0, 3)$ — yet they head off in different directions at different steepnesses. So the picture already suggests a division of labour: a controls direction and steepness, and b controls where the line crosses the y-axis. The rest of the theory tests those two conjectures.

What the coefficient of x does

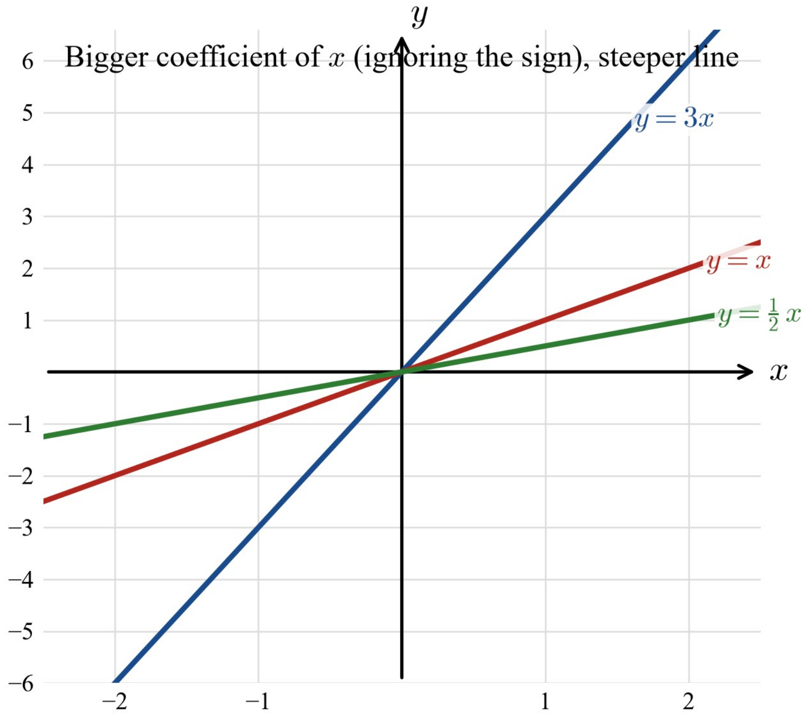
Direction. Type $y = 2x$ and $y = -2x$ into your graphing tool and read the screen from left to right, the way x increases:



The line y = 2x rising from left to right beside the line y = −2x falling from left to right, with arrows showing the direction of each line.

The first conjecture is almost visible on screen: *when the coefficient of x is positive the line rises from left to right, and when it is negative the line falls.*

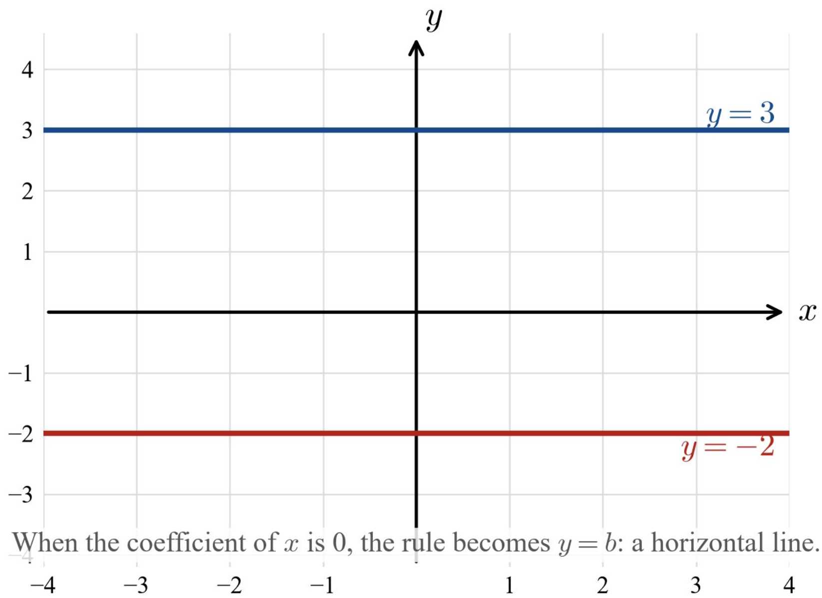
Steepness. Now keep the sign and change the size. Type $y=3x$, $y=x$ and $y=\frac{1}{2}x$:



The lines y = 3x, y = x and y = ½x on one grid, showing that a bigger coefficient of x (ignoring the sign) gives a steeper line.

A second conjecture: *the bigger the coefficient of x (ignoring its sign), the steeper the line.* So $y=-3x$ is just as steep as $y=3x$ — it falls instead of rising, but the steepness matches.

The edge case $a=0$. A conjecture about “the coefficient of x” should survive the case where that coefficient is 0. Type $y=3$ and $y=-2$ (that is, $y=0x+3$ and $y=0x-2$):

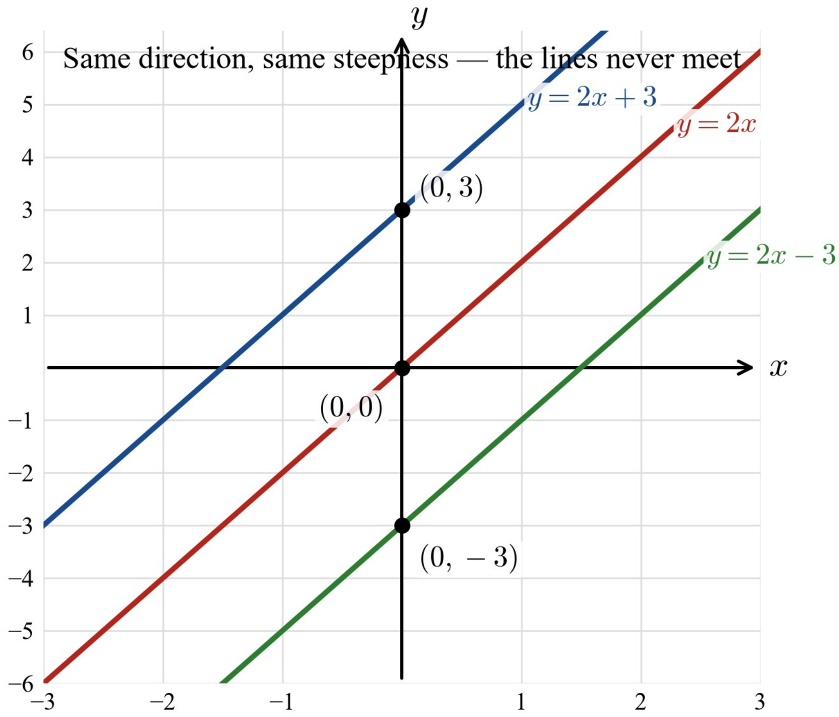


The horizontal lines y = 3 and y = −2 on one grid, showing that when the coefficient of x is 0 the rule becomes y = b, a horizontal line.

With $a=0$ the x-term contributes nothing, the output is b for **every** input, and the graph is a horizontal line at height b. Any generalisation about direction must say what happens at $a=0$ — a horizontal line neither rises nor falls.

What the constant term does

Keep a fixed and let b run. Type $y = 2x + 3$, $y = 2x$ and $y = 2x - 3$:



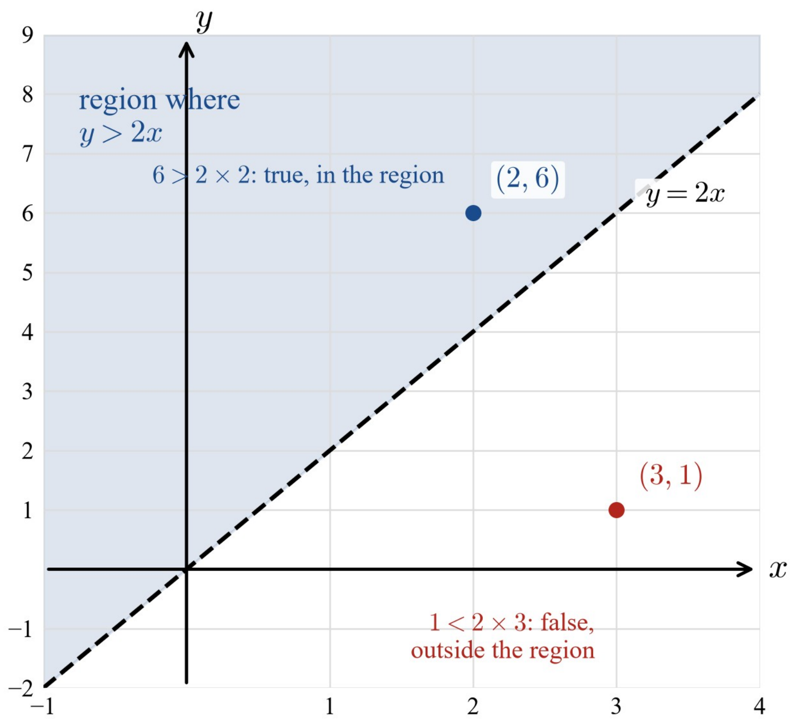
The three parallel lines y = 2x + 3, y = 2x and y = 2x − 3, crossing the y-axis at (0, 3), (0, 0) and (0, −3).

The lines have the same direction and the same steepness — they never meet. Lines that never meet are **parallel**. The only thing that changes is where each line crosses the y-axis, and the pattern is exact: $y = ax + b$ crosses the y-axis at $(0, b)$.

This is the initial value you met in 2C. Substitute $x = 0$ into $y = ax + b$ and the ax term dies: $y = a \times 0 + b = b$. So the constant term is the output when the input is 0 — on the graph, the crossing point on the y-axis.

Beyond equations: inequalities and their regions

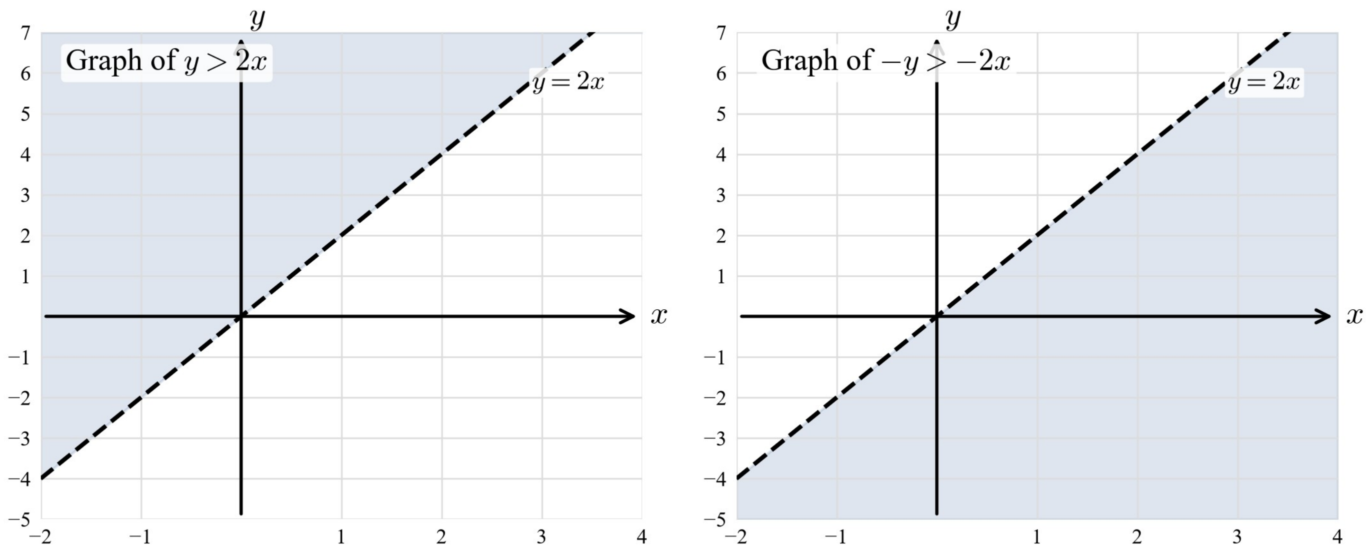
An equation like $y = 2x$ picks out a line. An **inequality** like $y > 2x$ picks out a **region** — a whole side of the plane. The line $y = 2x$ itself is the **boundary line** between the side where $y > 2x$ is true and the side where it is false, and a graphing tool draws it dashed to show that points on the line are not included. Testing a point tells you which side is which:



The dashed boundary line y = 2x with the region where y > 2x shaded above it; the point (2, 6) lies inside the shaded region and the point (3, 1) lies outside it.

Substituting is the hand-check the software is making: at $(2,6)$, $6>2\times 2$ is true, so $(2,6)$ sits in the region; at $(3,1)$, $1>2\times 3$ is false, so $(3,1)$ sits outside it.

Now the experiment the elaboration of this CD points at. Type $y>2x$ and $-y>-2x$ side by side. The two rules look like the same statement with both sides' signs changed — a student might conjecture they shade the same side. The screens disagree:



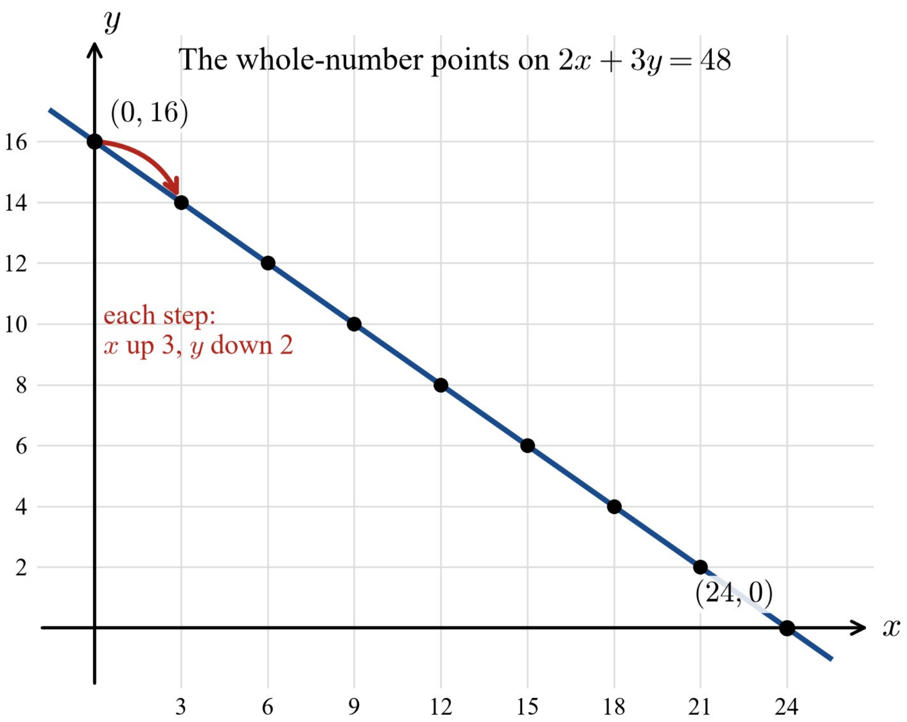
Both rules use " > ", but the shaded side is different — dividing both sides by −1 reversed the inequality.

Two panels side by side: the graph of y > 2x shaded above the dashed line y = 2x, and the graph of −y > −2x shaded below the same dashed line.

Both rules use the symbol >, yet the shaded side is different. Changing the sign of both sides of an inequality reverses it — $-y>-2x$ says the same thing as $y<2x$ — so the region flips. A conjecture about inequalities must track the signs on **both** sides, not just the symbol in the middle.

Whole-number solutions

Some questions only care about whole-number answers — you can sell 15 candles but not 15.5. Take the equation $2x+3y=48$ and ask for solutions where x and y are both whole numbers. A graphing tool draws the line; the whole-number solutions are the points on it with whole-number coordinates:



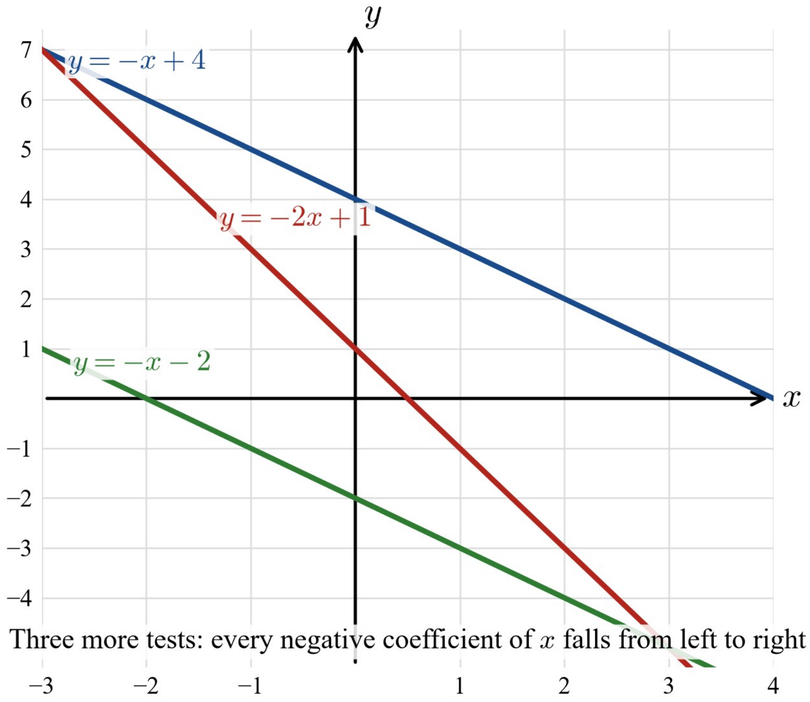
The line 2x + 3y = 48 with its nine whole-number points marked, from (0, 16) to (24, 0), and an arrow from (0, 16) to (3, 14) labelled "each step: x up 3, y down 2".

Starting at $(0,16)$, the next whole-number point is $(3,14)$, then $(6,12)$, and so on to $(24,0)$ — nine solutions, each step taking x up 3 and y down 2. The step is not a coincidence: raising x by 3 adds $2\times3=6$ to the left side, and dropping y by 2 removes $3\times2=6$, so the total stays 48. The pattern in the steps is itself a conjecture worth generalising — Q20 picks this up.

Good tests: how to break a conjecture

The loop only works if your tests are honest attempts to break the idea. Three habits make tests stronger:

- **Vary the sign.** A conjecture about coefficients of x tested only on positive numbers has never really been tested. The three lines below are stronger evidence for "negative coefficients fall" than any number of positive examples.



The three falling lines y = −x + 4, y = −2x + 1 and y = −x − 2 on one grid — further tests of the conjecture that a negative coefficient of x falls from left to right.

- **Test zero.** The case $a = 0$ broke into every direction conjecture above; a generalisation that forgets it is incomplete.
- **Test a broken conjecture on purpose.** Conjecture: *if b is positive, the line rises.* One type-in breaks it: $y = -x + 5$ has $b = 5$, positive, and falls. A single breaking test outweighs a hundred confirming ones — the conjecture must be rewritten (direction depends on a, not on b) and the loop runs again.

Words of this subtopic

Word	What it means here
conjecture	a statement you think is true, written clearly enough that a test can prove it wrong
test	a check of a conjecture — graphing rules in software, or substituting a point by hand
generalisation	the pattern every test agrees on; a conjecture that has survived testing
parameter	a number in a rule that you choose to vary to see what changes (a and b in $y = ax + b$)
coefficient of x	the number multiplying x — the a in $y = ax + b$
constant term	the number standing on its own — the b in $y = ax + b$
family of lines	the set of lines you get by varying the parameters of one rule-shape
parallel	said of lines that never meet; same direction and steepness
boundary line	the line separating the side where an inequality is true from the side where it is false
region	the side of the boundary line where an inequality holds; the shaded part of the graph
whole-number solution	a pair of whole numbers that makes an equation in x and y true

Worked examples

Example 1 — with help

A student writes a conjecture after seeing $y = 2x$ rise and $y = -2x$ fall:

> "If the coefficient of x is negative, the line falls from left to right — whatever the constant term."

Run the experiment loop on this conjecture.

Working	Comment
Predict. $y = -x$, $y = -2x$ fall; the constant term should not matter, so $y = -x + 4$, $y = -2x + 1$, $y = -x - 2$ should fall too.	The conjecture makes two claims — about the sign and about the constant term — so test both.
Test. Graph all five rules. Every line falls from left to right.	The constants 4, 1, -2 cover a positive and a negative constant term; the figure in "Good tests" shows three of them.
Test harder. Add $y = -0.001x + 100$: a huge constant term, a tiny negative coefficient. It still falls — slowly, but it falls.	A breaking test would have one line rising. None does.
Generalise. A negative coefficient of x gives a falling line, whatever the constant term; the constant term moves the line up or down without turning it.	The surviving conjecture becomes a generalisation.

∴ The conjecture stands: sign of a controls direction, b only moves the line. Notice the shape of the answer — a stated conjecture, tests that could have broken it, and a generalisation.

Example 2 — with help

Conjecture: *the line $y = ax + b$ crosses the y-axis at $(0, b)$.*

Working	Comment
Predict. $y = 2x + 3$ crosses at $(0, 3)$; $y = 2x$ at $(0, 0)$; $y = 2x - 3$ at $(0, -3)$.	Read straight off the constant term.
Test. Graph the three rules and read the crossing points. All three agree with the prediction.	Same a, different b: the lines are parallel, only the crossing moves.
Test the edge. $y = 0x + 3$, that is $y = 3$: a horizontal line crossing at $(0, 3)$. Agrees.	The case $a = 0$ must work too, and it does.
Check by hand. Substitute $x = 0$: $y = a \times 0 + b = b$, for **every** a and b.	One substitution covers infinitely many graphs — the algebra explains the pattern.

∴ Generalisation: $y = ax + b$ crosses the y-axis at $(0, b)$, always. In the language of 2C, b is the initial value — the output when the input is 0.

Example 3 — with help

Find every whole-number solution of $2x + 3y = 48$, and describe the pattern in them.

Working	Comment
Start at $x = 0$: $3y = 48$, so $y = 16$. First solution $(0, 16)$.	A solution with $x = 0$ is the easiest to find.
Step: x up 3 adds 6 to $2x$; y down 2 removes 6 from $3y$; total stays 48.	The step $(+3, -2)$ keeps the equation true.
Solutions: $(0,16)$, $(3,14)$, $(6,12)$, $(9,10)$, $(12,8)$, $(15,6)$, $(18,4)$, $(21,2)$, $(24,0)$.	Stop at $(24, 0)$: the next step would make y negative.
Hand-check one in the middle:	The software draws the line; one substitution confirms

Working	Comment
$2 \times 12 + 3 \times 8 = 24 + 24 = 48$. ✓	you read it right.

∴ Nine whole-number solutions, each step x up 3, y down 2. Conjecture generalised: on $2x + 3y = 48$ the whole-number points come at even steps, because the step is built to add and remove the same amount.

Example 4 — on your own

Experiment with the pair $y > 2x$ and $y < 2x$. Predict how their regions are related, test with the points $(2, 6)$ and $(3, 1)$, and generalise.

At $(2, 6)$: $6 > 2 \times 2$ is true and $6 < 2 \times 2$ is false. At $(3, 1)$: $1 > 2 \times 3$ is false and $1 < 2 \times 3$ is true. Each point satisfies exactly one of the two inequalities.

∴ The two regions are opposite sides of the same boundary line $y = 2x$: $y > 2x$ shades the side **above** the line, $y < 2x$ the side **below**. Generalisation: swapping > for < swaps the shaded side.

Example 5 — on your own

A student conjectures that $y > 2x$ and $-y > -2x$ shade the **same** region, “because both statements use >”. Test the conjecture with the points $(2, 6)$ and $(3, 1)$, then generalise.

At $(2, 6)$: $y > 2x$ gives $6 > 4$, true; $-y > -2x$ gives $-6 > -4$, **false**. At $(3, 1)$: $y > 2x$ gives $1 > 6$, false; $-y > -2x$ gives $-1 > -6$, **true**. The two rules disagree at every point tested — the conjecture is broken.

∴ $-y > -2x$ shades the opposite side to $y > 2x$; it is the same region as $y < 2x$. Generalisation: changing the sign of both sides of an inequality reverses it, so the region flips. The symbol in the middle never tells the whole story on its own.

Example 6 — on your own

Run a full experiment on the family $y = ax + 5$ with $a = 4, 1, 0, -1, -4$.

Conjecture (stated before graphing): all five lines cross the y-axis at $(0, 5)$; the lines with $a = 4, 1$ rise, the lines with $a = -1, -4$ fall, $a = 0$ gives a horizontal line; and $a = 4$ is steeper than $a = 1$, as -4 is steeper than -1.

Test. Graphing the five rules: every line passes through $(0, 5)$; the directions and the steepness ordering are exactly as predicted; $y = 5$ (the case $a = 0$) is horizontal through $(0, 5)$.

∴ **Generalisation.** Varying a turns the line about the fixed point $(0, 5)$ without moving it: the sign of a sets the direction, the size of a (ignoring the sign) sets the steepness, $a = 0$ gives the horizontal line $y = 5$, and the constant term keeps the crossing point at $(0, 5)$ throughout. One point, a whole family of lines turning about it.

Practice questions

Keep a graphing tool open (graphing software or a spreadsheet graph) for every question in this subtopic. The habit is always the same: predict first, then type it in — the software checks your prediction, but the predicting is the mathematics.

With help

Q1. Predict whether each line rises or falls from left to right, then check with your graphing tool.

a. $y = 5x$
b. $y = -5x$
c. $y = 0.5x$
d. $y = -0.5x$

State the conjecture your four tests support.

Q2. In each pair, predict which line is steeper, then check.

a. $y=7x$ and $y=3x$
b. $y=-7x$ and $y=-3x$
c. $y=2x$ and $y=0.5x$

State the conjecture your tests support about steepness and the coefficient of x.

Q3. Predict where each line crosses the y-axis, then check.

a. $y=3x+4$
b. $y=3x-4$
c. $y=3x$
d. $y=-x+1$

State the conjecture your tests support about the constant term.

Q4. A student conjectures: "Changing the constant term **turns** the line." Test the conjecture with $y=2x$, $y=2x+3$ and $y=2x-3$. Does it stand or break? Rewrite it as a correct generalisation.

Q5. Predict the shape of the graphs of $y=0x+3$ and $y=0x-2$, then type them in. Generalise: what does $y=ax+b$ look like when $a=0$?

Q6. Find every whole-number solution of $x+2y=12$, starting at $(0,6)$. Describe the step from one solution to the next, and check two of your solutions by substitution.

Q7. Decide by substitution whether each point is in the region $y>2x$, then confirm with your graphing tool.

a. $(2,6)$
b. $(3,1)$
c. $(0,1)$
d. $(-1,-3)$

Q8. Conjecture: "If the constant term is positive, the line rises." Find one rule that breaks it (type in a few until one does), and rewrite the conjecture correctly.

On your own

Q9. Without graphing, arrange these lines from steepest to flattest: $y=-8x$, $y=5x$, $y=-x$, $y=0.5x$. Then check with your tool.

Q10. Test the conjecture "$y=ax+b$ crosses the y-axis at $(0,b)$" on the rules $y=4x+2$, $y=-x-6$, $y=3x$ and $y=7$. Predict each crossing point first. Does the conjecture stand?

Q11. Without graphing, sort these four lines into rising and falling, then list them from steepest to flattest: $y=-6x$, $y=x$, $y=-x$, $y=9x$. Check with your tool.

Q12. Find every whole-number solution of $2x+5y=30$. Describe the step, and explain why that step keeps the equation true.

Q13. Find every whole-number solution of $5x+4y=100$. Describe the step.

Q14. The region $y>2x$ is shaded above its boundary line. Predict which side of the line $y=2x$ the region $y<2x$ takes, and test your prediction with the point $(3,1)$.

Q15. Conjecture: "$y>3x$ and $-y>-3x$ shade the same side of the line $y=3x$." Test it with the point $(1,5)$. Does the conjecture stand? Generalise.

Q16. Conjecture: “Two lines with the same coefficient of x must be the same line.” Find two rules that break it, and say what the two lines look like side by side.

Q17. Conjecture: “If the constant term is negative, the line crosses the y-axis below the origin.” Test it on $y=x-3$, $y=4x-1$ and $y=-2x-7$, and include one edge case of your own choosing. Does the conjecture stand?

Q18. Conjecture: “If a rule has no x-term, its graph is horizontal.” Test it on $y=7$, $y=-1$ and $y=0$. Generalise, and connect your answer to the case $a=0$.

Full experiments

These questions are full experiments: a stated conjecture, tests, and a generalisation, in that order. Write all three.

Q19. A phone company offers two plans. Plan A costs a fixed \$20 per month plus \$5 per gigabyte; Plan B has no fixed charge and costs \$9 per gigabyte. Let w be the gigabytes used in a month, so Plan A costs $A=20+5w$ dollars and Plan B costs $B=9w$ dollars.

a. Conjecture: “Plan B is always cheaper.” Build a table of both costs for $w=0,1,2,\ldots,8$ (a spreadsheet does this in seconds) and test the conjecture against it.
b. For which w do the plans cost the same? What is that cost?
c. Generalise: for which usage is Plan B cheaper, and for which is Plan A cheaper?
d. Review: give one reason why a real phone bill might not match either rule.

Q20. Look back at the whole-number solutions in this subtopic:

Equation	Step from one whole-number solution to the next
$2x+3y=48$	x up 3, y down 2
$x+2y=12$	x up 2, y down 1
$2x+5y=30$	x up 5, y down 2

a. Conjecture a connection between the steps and the coefficients of the equation.
b. Test your conjecture on a new equation, $4x+3y=36$: find its whole-number solutions and check the step.
c. Generalise. Explain **why** the step works, in terms of what each coefficient adds or removes.

Q21. Conjecture: “For any a and b, the region $y>ax+b$ is the side **above** the boundary line, and $y<ax+b$ is the side below.”

a. Test the conjecture on $y>2x$, $y>-2x$ and $y>2x+3$, using one test point per rule that lies above the boundary line. Show each substitution.
b. Test once more with a point **below** the line $y=2x+3$.
c. Does the conjecture stand? Generalise, and say why comparing the point’s y-value with the rule’s y-value makes “above means >” inevitable.

Answers

Q1. a. Rises. b. Falls. c. Rises. d. Falls. Conjecture: positive coefficient of x → rises; negative → falls.

Q2. a. $y=7x$. b. $y=-7x$. c. $y=2x$. Conjecture: bigger coefficient ignoring the sign → steeper line.

Q3. a. $(0,4)$. b. $(0,-4)$. c. $(0,0)$. d. $(0,1)$. Conjecture: the line crosses the y-axis at $(0,b)$.

Q4. Breaks. The three lines have the same direction and steepness (parallel); changing b moves the line up or down without turning it.

Q5. Horizontal lines at height 3 and -2. When $a=0$ the rule is $y=b$: a horizontal line at height b.

Q6. $(0,6)$, $(2,5)$, $(4,4)$, $(6,3)$, $(8,2)$, $(10,1)$, $(12,0)$; step x up 2, y down 1.

Q7. a. In. b. Out. c. In. d. Out.

Q8. Sample breaker: $y=-x+5$ (constant term $+5$, positive, but the line falls). Corrected: direction depends on the sign of the coefficient of x, not on the constant term.

Q9. $y=-8x$, $y=5x$, $y=-x$, $y=0.5x$.

Q10. Predictions $(0,2)$, $(0,-6)$, $(0,0)$, $(0,7)$; all four confirmed. The conjecture stands.

Q11. Rising: $y=9x$, $y=x$. Falling: $y=-6x$, $y=-x$. Steepest to flattest: $y=9x$, $y=-6x$, then $y=x$ and $y=-x$ (equal steepness).

Q12. $(0,6)$, $(5,4)$, $(10,2)$, $(15,0)$; step x up 5, y down 2.

Q13. $(0,25)$, $(4,20)$, $(8,15)$, $(12,10)$, $(16,5)$, $(20,0)$; step x up 4, y down 5.

Q14. The side below the line; $(3,1)$ gives $1<6$, true, and $(3,1)$ sits below the line.

Q15. Breaks. $(1,5)$: $5>3$ true but $-5>-3$ false — opposite sides. Generalisation: changing the sign of both sides reverses the inequality and flips the region.

Q16. Sample: $y=2x+1$ and $y=2x+4$ — different parallel lines, same coefficient of x.

Q17. Stands. All three cross below $(0,0)$; sample edge case $y=0x-2$ (that is $y=-2$) crosses at $(0,-2)$, below the origin.

Q18. Stands. $y=7$, $y=-1$, $y=0$ are all horizontal; they are the case $a=0$, $y=b$.

Q19. a. The conjecture breaks: at $w=8$, $A=60$ but $B=72$. b. $w=5$, cost \$45. c. Plan B cheaper for $w<5$; Plan A cheaper for $w>5$; equal at $w=5$. d. Sample: real plans charge in blocks of data, add flags or caps, or change price mid-contract.

Q20. a. Sample: the x-step is the coefficient of y, and the y-step is the coefficient of x downwards. b. $(0,12)$, $(3,8)$, $(6,4)$, $(9,0)$; step x up 3, y down 4 — as conjectured. c. On $ax+by=c$, stepping x up b adds ab and stepping y down a removes ab, so the total is unchanged.

Q21. a. $(0,1)$: $1>0$ ✓; $(0,1)$: $1>0$ ✓; $(0,4)$: $4>3$ ✓ — all above and true. b. $(0,1)$ below $y=2x+3$: $1>3$ false ✓. c. Stands. Above the line means the point's y is bigger than the rule's y at that x — which is exactly $y>ax+b$.

Fully-worked solutions

Q1. Read the sign of the coefficient of x; positive rises, negative falls, left to right.

a. $5>0$: **rises**. b. $-5<0$: **falls**. c. $0.5>0$: **rises**. d. $-0.5<0$: **falls**.

∴ Conjecture supported: the **sign** of the coefficient of x sets the direction — positive rises, negative falls.

Q2. Steepness follows the size of the coefficient, ignoring its sign.

a. $7>3$: $y=7x$ **steeper**. b. Ignoring signs, $7>3$: $y=-7x$ **steeper**.
b. $2>0.5$: $y=2x$ **steeper**.

∴ Conjecture supported: the bigger the coefficient of x ignoring the sign, the steeper the line.

Q3. Substitute $x=0$ in your head: the crossing point is $(0,b)$.

a. $b=4$: $(0,4)$. b. $b=-4$: $(0,-4)$. c. $b=0$: $(0,0)$.
b. $b=1$: $(0,1)$.

∴ Conjecture supported: $y=ax+b$ crosses the y-axis at $(0,b)$.

Q4. Graph $y=2x$, $y=2x+3$, $y=2x-3$: all three rise at the same steepness and never meet — they are parallel. Changing the constant term moved the lines up and down; it turned nothing.

∴ The conjecture **breaks**. Correct generalisation: changing b moves the line up or down without turning it; parallel lines differ only in b.

Q5. $y=0x+3$ is $y=3$: every input gives 3, a **horizontal line at height 3**. Likewise $y=0x-2$ is horizontal at height -2.

∴ When $a=0$ the rule becomes $y=b$, a horizontal line at height b — it neither rises nor falls.

Q6. Start at $(0,6)$. The step that keeps $x+2y$ at 12: x up 2 adds 2, y down 1 removes 2.

Solutions: $(0,6)$, $(2,5)$, $(4,4)$, $(6,3)$, $(8,2)$, $(10,1)$, $(12,0)$ — then y would go negative, so stop.

Checks: $(4,4)$: $4+2\times4=12$ ✓. $(10,1)$: $10+2\times1=12$ ✓.

Q7. Substitute into $y>2x$.

a. $(2,6)$: $6>4$ true — **in the region**.
b. $(3,1)$: $1>6$ false — **outside**.
c. $(0,1)$: $1>0$ true — **in the region**.
d. $(-1,-3)$: $-3>-2$ false — **outside**.

Q8. The conjecture says a positive b forces a rising line. Type in $y=-x+5$: $b=5$ is positive, yet the line falls. One breaking test is enough.

∴ Corrected conjecture: the direction a line rises or falls depends only on the **sign of the coefficient of** x; the constant term plays no part in it.

Q9. Steepness uses the size ignoring the sign: sizes are $8,5,1,0.5$.

∴ Steepest to flattest: $y=-8x$, then $y=5x$, then $y=-x$, then $y=0.5x$. The screen confirms it — ignoring the signs, 8 beats 5, 5 beats 1, 1 beats 0.5.

Q10. Predict from the constant term: $(0,2)$, $(0,-6)$, $(0,0)$, $(0,7)$ ($y=7$ is $0x+7$). Graphing confirms all four crossing points.

∴ The conjecture **stands**, including the edge case $a=0$.

Q11. Signs: -6 falls, 1 rises, -1 falls, 9 rises. Sizes: $9,6,1,1$.

∴ Rising: $y=9x$, $y=x$; falling: $y=-6x$, $y=-x$. Steepest to flattest: $y=9x$, $y=-6x$, then $y=x$ and $y=-x$ tied for flattest.

Q12. Start at $(0,6)$. Step: x up 5 adds $2\times5=10$; y down 2 removes $5\times2=10$; the total stays 30.

Solutions: $(0,6)$, $(5,4)$, $(10,2)$, $(15,0)$.

Q13. Start at $(0,25)$. Step: x up 4 adds $5\times4=20$; y down 5 removes $4\times5=20$; the total stays 100.

Solutions: $(0,25)$, $(4,20)$, $(8,15)$, $(12,10)$, $(16,5)$, $(20,0)$.

Q14. Predict: the opposite side to $y>2x$, i.e. below the line. Test $(3,1)$: $1<2\times3=6$ is true, and $(3,1)$ lies below the line — prediction confirmed.

$\therefore$ $y<2x$ shades the side below the boundary line.

Q15. Test $(1,5)$ on both rules. $y>3x$: $5>3$, true. $-y>-3x$: $-5>-3$, false. The point is in one region and not the other.

$\therefore$ The conjecture **breaks**: the regions are opposite sides. Generalisation: changing the sign of both sides of an inequality reverses it, so the shaded side flips.

Q16. Take $y=2x+1$ and $y=2x+4$. Same coefficient of x, different constant terms — and they are clearly two different lines, parallel, one above the other.

$\therefore$ The conjecture **breaks**. The coefficient of x fixes direction and steepness only; the constant term still chooses **which** of the parallel lines you get.

Q17. Predict crossings $(0,-3)$, $(0,-1)$, $(0,-7)$ — all below the origin. Graphing confirms. Edge case: $y=0x-2$, horizontal at height -2, crossing at $(0,-2)$ — still below.

$\therefore$ The conjecture **stands**: a negative constant term puts the crossing point below the origin, because the crossing point is $(0,b)$ itself.

Q18. $y=7$, $y=-1$, $y=0$ graph as horizontal lines at heights 7, -1, 0.

$\therefore$ The conjecture **stands**, and it is the case $a=0$: with no x-term the output never changes, so $y=b$ is horizontal at height b.

Q19. a. The table:

w	0	1	2	3	4	5	6	7	8
$A=20\cdot$	20	25	30	35	40	45	50	55	60
$B=9w$	0	9	18	27	36	45	54	63	72

Plan B is cheaper at $w=0,1,2,3,4$ — but at $w=6,7,8$ Plan A is cheaper. The conjecture "B is always cheaper" **breaks**.

b. The costs are equal at $w=5$, both \$45.

c. Generalisation: Plan B is cheaper below 5 gigabytes, Plan A cheaper above 5, and at exactly 5 they tie. The fixed \$20 makes Plan A dearer for light users and cheaper for heavy ones.

d. Sample: real bills charge per started block of data, add a connection flag, or cap the price at a bundle — so the per-gigabyte rule is a guide, not a promise.

Q20. a. Reading across the three equations: the x-step equals the coefficient of y (3, 2, 5), and the y-step equals the coefficient of x downwards (2, 1, 2). Conjecture: on $ax+by=c$, whole-number solutions step by x up b, y down a.

b. $4x+3y=36$: start at $(0,12)$; conjectured step x up 3, y down 4 gives $(3,8)$, $(6,4)$, $(9,0)$. Check $(6,4)$: $4\times6+3\times4=24+12=36$ ✓. The conjecture stands.

c. Generalisation: stepping x up b adds $a\times b$ to the left side, and stepping y down a removes $b\times a$ — the same amount — so the total never changes. That is why the coefficients of the equation **are** the steps.

Q21. a. Test points above each boundary line:

- $y>2x$ with $(0,1)$: $1>2\times0=0$ — true.
- $y>-2x$ with $(0,1)$: $1>-2\times0=0$ — true.
- $y>2x+3$ with $(0,4)$: $4>2\times0+3=3$ — true.

All three points sit above their line and satisfy the inequality.

b. $(0,1)$ lies below the line $y=2x+3$: $1>3$ is false — below means **not** $>$.

c. The conjecture **stands**. It must stand: a point above the boundary line has a y-coordinate bigger than the rule's output at that x, and "y bigger than $ax+b$" is literally the statement $y>ax+b$. The shading is the inequality, read as a picture.

Algorithms and testing procedures: identifying and correcting errors

Content description `VC2M8A04`: use algorithms and related testing procedures to identify and correct errors

A recipe, a bus timetable, the steps for long division — each one is a list of steps that works every time, as long as you follow it exactly. Mathematicians and programmers give lists like these a name, and they also have careful habits for checking them: trying them out on chosen numbers, watching where a run first goes wrong, and fixing the faulty step. This subtopic builds those habits.

Theory

What is an algorithm?

An **algorithm** is a set of **steps** that takes an **input**, does something to it, and produces an **output**. For a set of steps to count as an algorithm, three things must be true:

- the steps are in a **fixed order** — step 2 always comes after step 1;
- each step says **exactly** what to do, so two people following it get the same result;
- the steps **come to an end** — an algorithm finishes and hands over its output.

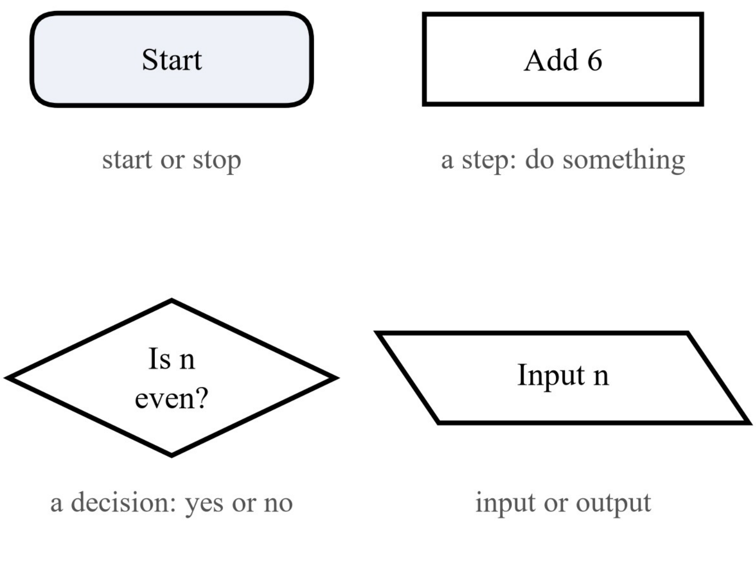
Here is an algorithm whose input is two numbers and whose output is their mean:

1. Input two numbers, a and b.
2. Add them: $s=a+b$.
3. Divide by 2: $m=s\div 2$.
4. Output m.

With input $a=4$, $b=10$: step 2 gives $s=14$, step 3 gives $m=7$, and the output is 7. With the same input, anyone following the steps gets 7 — that is the point of an algorithm. A step like “add a little” would not be allowed, because “a little” is not exact.

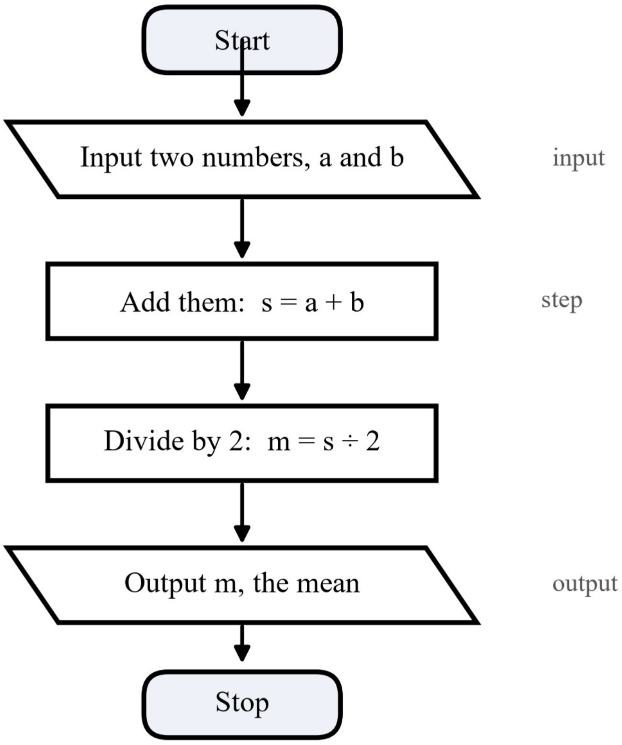
Flow charts

A **flow chart** draws an algorithm as boxes joined by arrows, so the order of the steps can be seen at a glance. Four box shapes are used, and each one means a different kind of step.



The four flow chart box shapes: a rounded box labelled Start meaning start or stop, a rectangle labelled Add 6 meaning a step that does something, a diamond labelled Is n even? meaning a decision with yes or no, and a slanted box labelled Input n meaning input or output

The mean algorithm drawn as a flow chart:



Flow chart for the mean of two numbers: Start, then input two numbers a and b, then the step add them s equals a plus b, then the step divide by 2 m equals s divided by 2, then output m the mean, then Stop

Rounded boxes mark the start and the stop, slanted boxes mark inputs and outputs, rectangles are steps, and diamonds are **decisions** — steps that ask a yes/no question, with one arrow leaving for yes and another for no. You will meet decisions in the divisibility flow chart below.

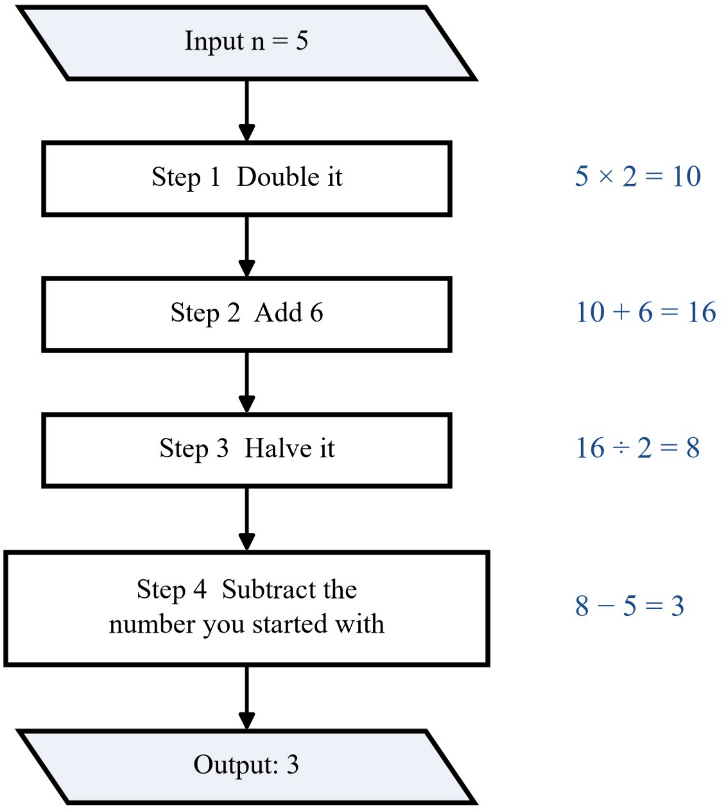
Tracing an algorithm

To **trace** an algorithm means to follow its steps with one particular input, writing down what happens at every step. A **trace table** keeps the record tidy: one row per step, with the result of that step beside it.

A number trick as an algorithm:

1. Input a number n.
2. Double it.
3. Add 6.
4. Halve it.
5. Subtract the number you started with.
6. Output the result.

Traced with input $n = 5$:



Trace of the number trick with input 5: doubling gives 5 times 2 equals 10, adding 6 gives 10 plus 6 equals 16, halving gives 16 divided by 2 equals 8, and subtracting the starting number gives 8 minus 5 equals 3, so the output is 3

Step	Done with $n=5$	Result
2 Double it	5×2	10
3 Add 6	$10+6$	16
4 Halve it	$16\div2$	8
5 Subtract the start	$8-5$	3

Output: 3. Try the trace with a different input and the output is 3 again — a trace can raise a good question (why always 3?), which the worked examples take up.

Tracing is also the main tool for **identifying errors**: when an output is wrong, the trace table shows the first step where the result stops being right, and that is the step to look at.

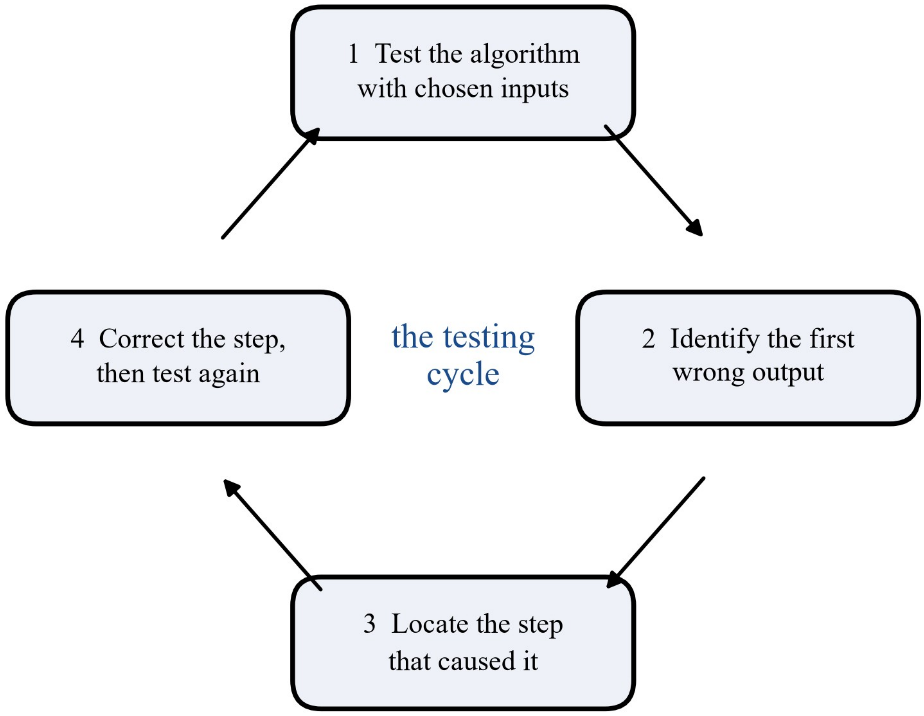
Testing procedures and test cases

An algorithm can look right and still be wrong. A **testing procedure** is the habit of checking an algorithm before trusting it:

- choose a **test case** — an input whose correct output you already know, so you can compare the algorithm's output with what it should be;
- run the algorithm on the test case and compare;
- if anything does not match, find the faulty step, **correct** it, and test again.

Good test cases include ordinary inputs and **boundary cases** — inputs at the edges, such as two equal numbers, 0, or a negative number, where rules are most likely to break. One careful run is never enough; a few well-chosen cases are.

When a test fails, the fixing runs in a cycle.



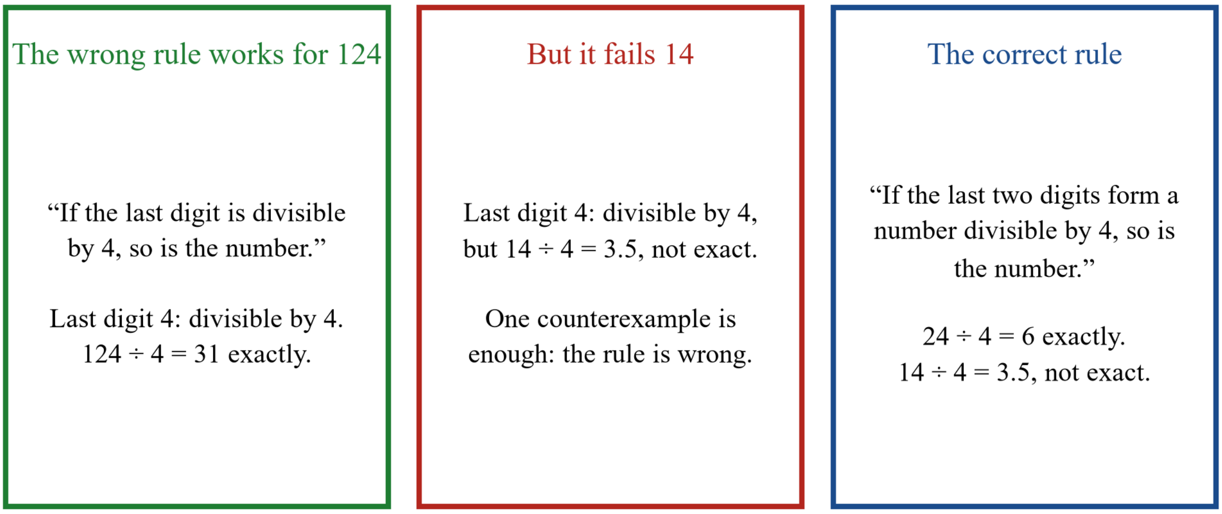
The testing cycle drawn as four boxes in a ring: 1 test the algorithm with chosen inputs, 2 identify the first wrong output, 3 locate the step that caused it, 4 correct the step then test again, with an arrow returning to step 1

Notice the last box: correcting a step is not the end. The corrected algorithm must be **tested again** — on the case that failed, and on a couple more besides.

Identifying errors: counterexamples

Sometimes tracing shows that every step was followed perfectly, yet the output is wrong. Then the fault is not in the following — it is in the **rule** the algorithm was built on. A rule that claims something is *always* true can be killed by a single example where it fails. That example is a **counterexample**, and one is enough.

A classmate offers this rule for testing whether a number is divisible by 4: “look at the last digit; if it is divisible by 4, so is the number.”

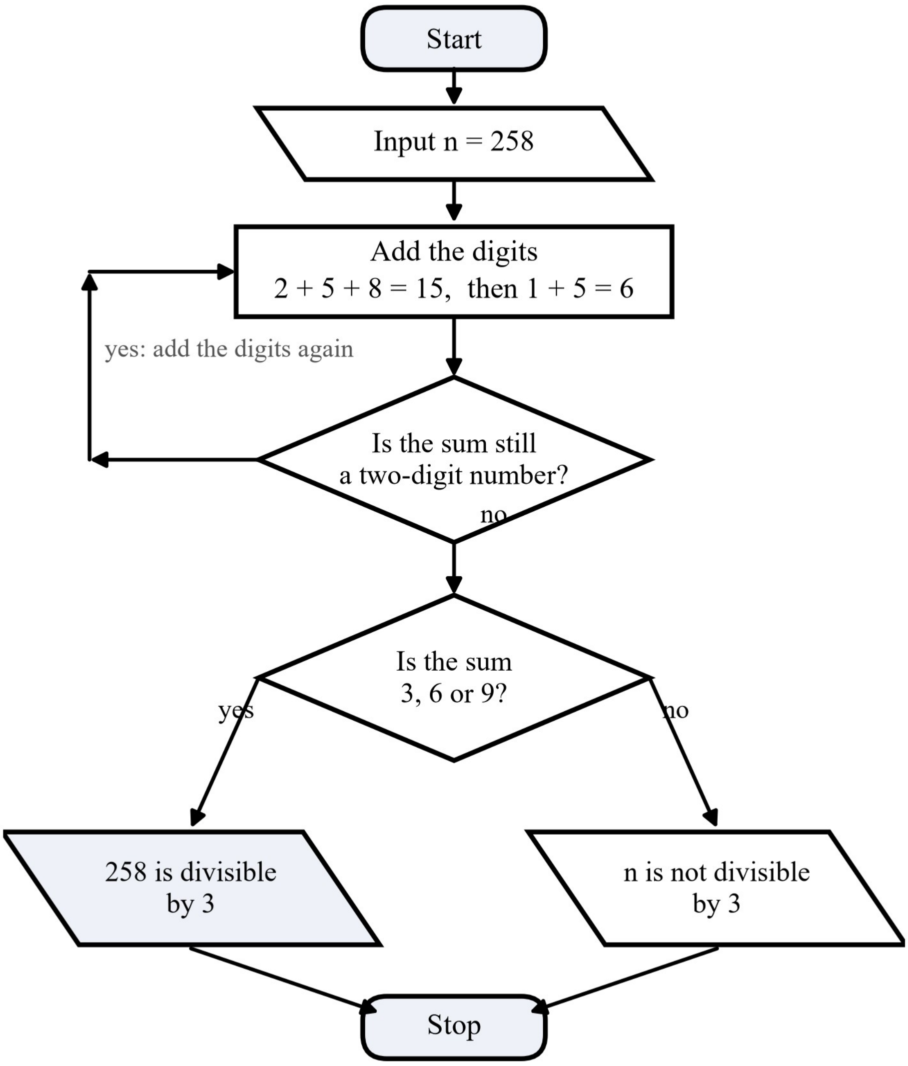


Three cards side by side. First card: the wrong rule works for 124 — if the last digit is divisible by 4 so is the number; last digit 4 is divisible by 4 and 124 divided by 4 is 31 exactly. Second card: but it fails 14 — last digit 4 is divisible by 4 but 14 divided by 4 is 3.5, not exact; one counterexample is enough, the rule is wrong. Third card: the correct rule — if the last two digits form a number divisible by 4 so is the number; 24 divided by 4 is 6 exactly, 14 divided by 4 is 3.5 not exact

The number 14 is a counterexample: its last digit (4) is divisible by 4, but $14 \div 4 = 3.5$, not exact. The rule is wrong, and no pile of passing cases like 124 can put that right. The correct rule looks at the **last two digits**: 24 is divisible by 4, so 124 is; 14 is not, so 14 is not.

An algorithm for divisibility by 3

Testing a number for divisibility is a natural home for algorithms. Here is one for divisibility by 3, drawn as a flow chart and traced with $n = 258$.



Flow chart for testing divisibility by 3, traced with 258: Start, input n equals 258, add the digits giving 2 plus 5 plus 8 equals 15 then 1 plus 5 equals 6, decision is the sum still a two-digit number with a yes arrow looping back to add the digits again, then decision is the sum 3, 6 or 9 with a yes arrow to the output 258 is divisible by 3 and a no arrow to the output n is not divisible by 3, both leading to Stop

Follow the arrows with 258: the digits add to $2+5+8=15$, which is still two digits, so they are added again: $1+5=6$. The sum 6 is one of 3, 6, 9, so 258 is divisible by 3 — and in fact $258 \div 3 = 86$ exactly. The loop ("add the digits again") is what a flow chart draws well and a single line of steps would hide.

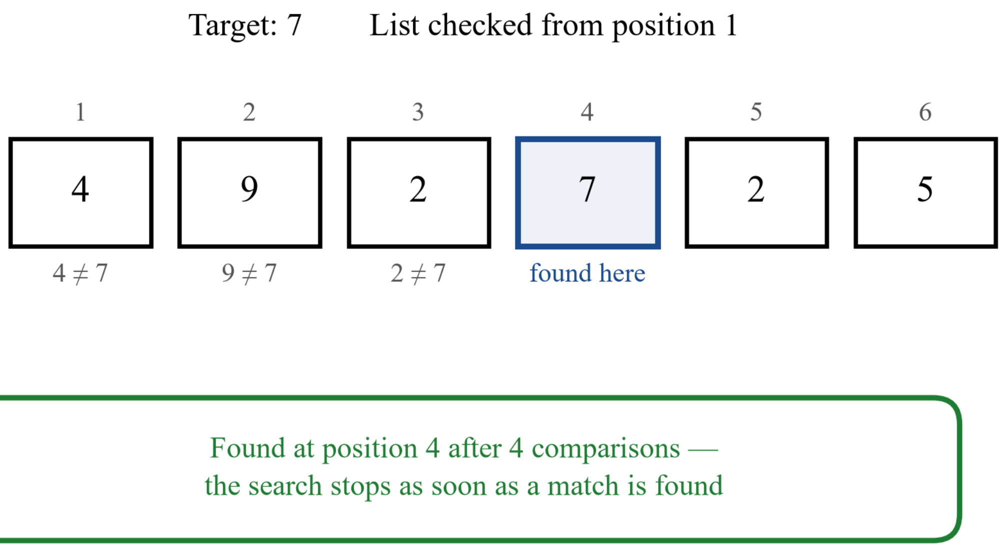
The same style of test exists for other divisors, and two are easy to state: a number is divisible by **2** when its last digit is 0, 2, 4, 6 or 8, and by **5** when its last digit is 0 or 5.

Search algorithms

A **search** looks through a list for a number called the **target**. The simplest search algorithm — a **linear search** — checks the list from the first position to the last:

1. Input a list and a target.
2. Start at position 1.
3. Compare the item with the target. If they match, output the position and stop.
4. If they do not match, move to the next position and go back to step 3.
5. If the list runs out, output "not in the list".

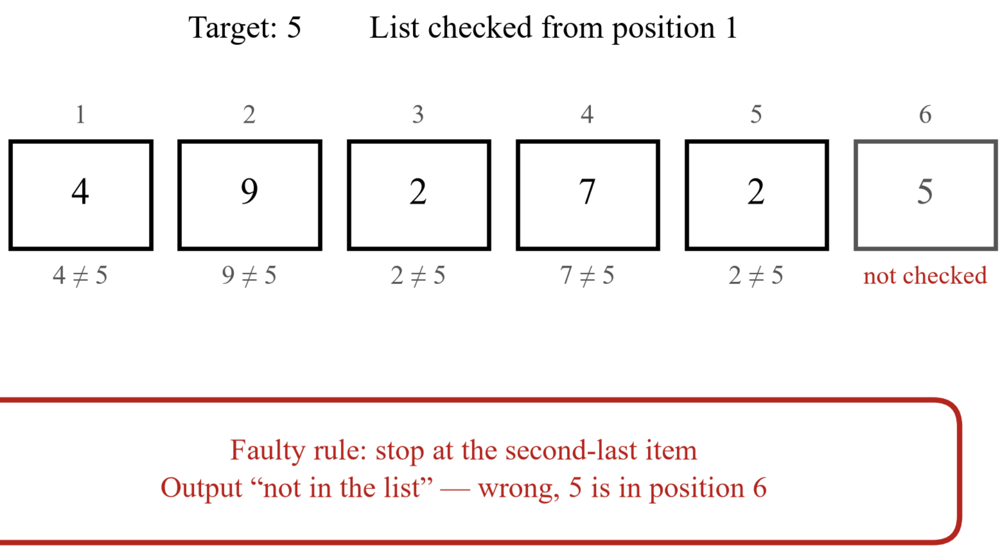
Traced on the list 4, 9, 2, 7, 2, 5 with target 7:



Linear search for 7 in the list 4, 9, 2, 7, 2, 5 checked from position 1: 4 is not 7, 9 is not 7, 2 is not 7, and 7 matches at position 4, so the target is found at position 4 after 4 comparisons and the search stops as soon as a match is found

Four comparisons find the target at position 4, and the search stops there — once a match is found, the work is done.

Searches are a classic place for errors to hide. Here is the same search with a faulty step 5: "stop at the second-last item and output 'not in the list'."



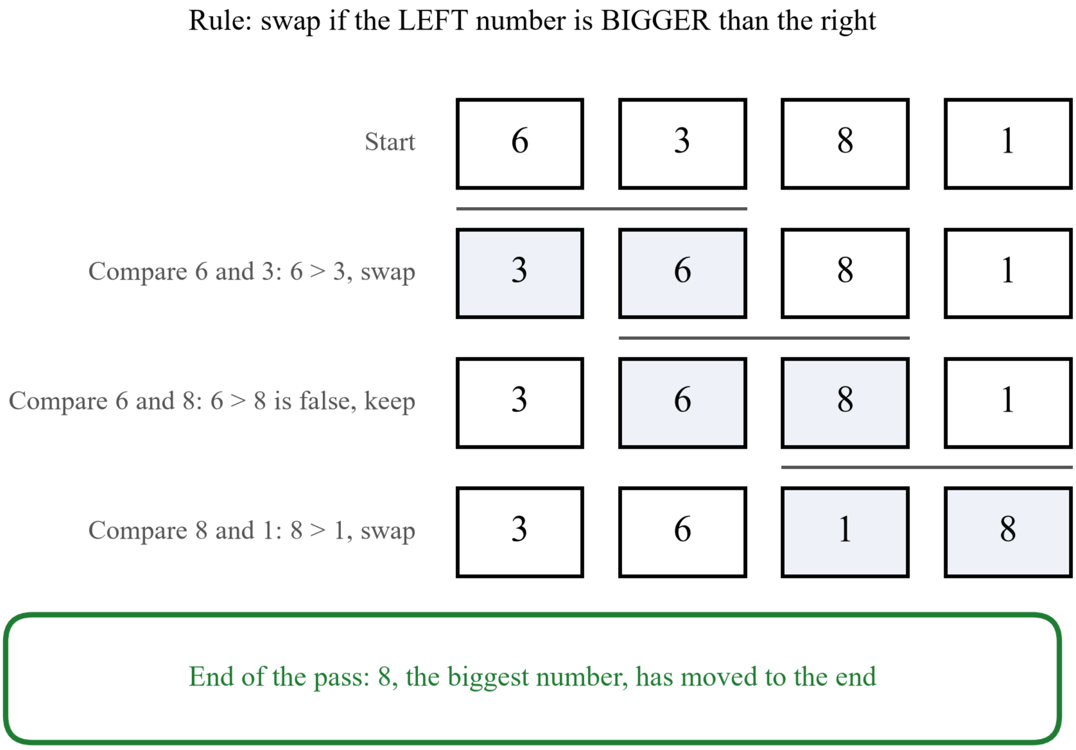
The same list with target 5 and a faulty rule that stops at the second-last item: 4, 9, 2, 7 and 2 are all compared and do not match, position 6 is never checked, and the search wrongly outputs not in the list when 5 is in position 6

The target 5 sits in position 6, which the faulty search never checks. A test case like this — target at the very end of the list — is exactly the boundary case a careful tester chooses.

Sorting algorithms

A **sort** puts a list in order, usually smallest first. One simple sorting method walks along the list comparing neighbouring numbers and swapping them when they are in the wrong order; one walk along the list is called a **pass**. The rule for a pass that orders a list smallest first is: *swap if the left number is bigger than the right.*

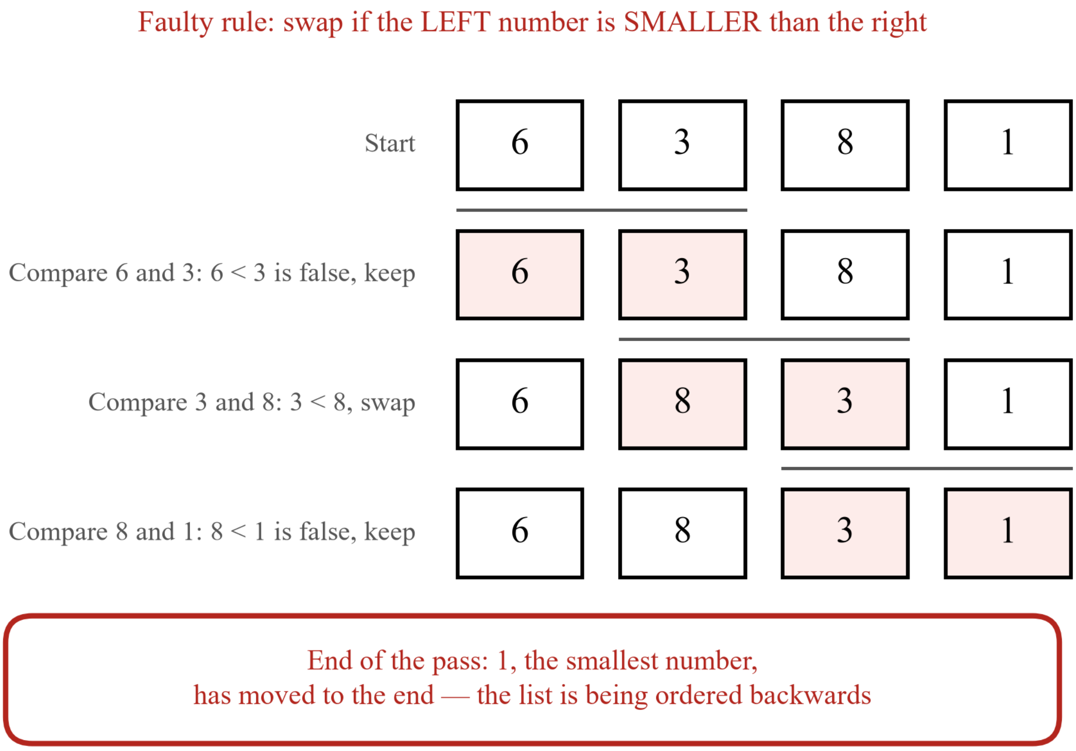
One pass on the list 6, 3, 8, 1:



One sorting pass on the list 6, 3, 8, 1 with the rule swap if the left number is bigger: comparing 6 and 3 swaps them to give 3, 6, 8, 1; comparing 6 and 8 keeps them; comparing 8 and 1 swaps them to give 3, 6, 1, 8; at the end of the pass 8, the biggest number, has moved to the end

After one pass the biggest number, 8, has moved to the end. Repeating passes keeps moving the next-biggest number into place, and when a whole pass finishes with no swaps, the list is sorted and the algorithm stops.

Now the same pass with a faulty rule: *swap if the left number is* ***smaller*** *than the right.*



One pass on the same list with the faulty rule swap if the left number is smaller: comparing 6 and 3 keeps them, comparing 3 and 8 swaps them to give 6, 8, 3, 1, comparing 8 and 1 keeps them, and at the end of the pass 1, the smallest number, has moved to the end — the list is being ordered backwards

Every step of the pass runs as written — and the list comes out ordered backwards, with the smallest number pushed to the end. Tracing with a short list is what makes an error like this visible.

Words of this subtopic

Word	Meaning here
algorithm	a set of steps in a fixed order, each exact, that takes an input to an output and ends
input / output	the number(s) an algorithm starts with / the result it produces
step	one instruction in an algorithm
decision	a step that asks a yes/no question, with an arrow for each answer
flow chart	a picture of an algorithm using the four box shapes
trace	to follow an algorithm with one input, recording every step's result
test case	an input whose correct output is known, used to check an algorithm
boundary case	a test at an edge — equal numbers, 0, the end of a list — where rules break first
counterexample	one example where a rule fails; enough to show the rule is wrong
search	an algorithm that looks through a list for a target
sort	an algorithm that puts a list in order
correct (debug)	to locate a faulty step, fix it, and test the algorithm again

Worked examples

Example 1 — with help

The number trick algorithm from the theory: input n; double it; add 6; halve it; subtract the number you started with; output the result.

(a) Trace the algorithm with $n=5$.
(b) Trace it with $n=12$.
(c) The output was 3 both times. Explain why the output is always 3.

Working	Comment
(a) $5\times 2=10$; $10+6=16$; $16\div 2=8$; $8-5=3$	One row of arithmetic per step — that is the trace.
(b) $12\times 2=24$; $24+6=30$; $30\div 2=15$; $15-12=3$	Same steps, different input, same output.
(c) $n\to 2n\to 2n+6\to n+3\to (n+3)-n=3$	Follow the steps with the letter n instead of a number.

∴ (a) and (b) the output is 3; (c) doubling, adding 6 and halving gives $n+3$, and subtracting n always leaves 3, whatever the input. Review: two traces show the pattern; the trace with the letter n explains it for every input at once.

Example 2 — with help

Use the divisibility-by-3 algorithm (flow chart above) on two numbers.

(a) Test 258.
(b) Test 350.
(c) Check both answers with a division.

Working	Comment
(a) $2+5+8=15$; $1+5=6$; 6 is one of 3, 6, 9	The sum is two digits once, so the digits are added

Working	Comment
	twice.
(b) $3+5+0=8$; 8 is not 3, 6 or 9	One digit already, so no second adding.
(c) $258\div3=86$ exactly; $350\div3=116$ remainder 2	The divisions confirm the algorithm.

∴ (a) 258 **is** divisible by 3; (b) 350 is **not**; (c) the divisions agree. Review: the algorithm and the division are two independent ways to the same answers — each one is a test case for the other.

Example 3 — with help

A linear search runs on the list 4, 9, 2, 7, 2, 5.

(a) Trace it with target 7. How many comparisons are made, and what is output?
(b) Trace it with target 2. Which position is output, even though 2 appears twice?

Working	Comment
(a) $4\neq7$, $9\neq7$, $2\neq7$, $7=7$	The search stops at the first match.
Output: position 4, after 4 comparisons	
(b) $4\neq2$, $9\neq2$, $2=2$	The match is the first 2, in position 3.
Output: position 3, after 3 comparisons	The second 2 (position 5) is never reached.

∴ (a) 4 comparisons, position 4; (b) position 3 — a linear search reports the **first** place the target appears. Review: the repeated 2 makes a good test case; it shows exactly what the algorithm does with repeats.

Example 4 — on your own

Algorithm B: input n; step 1 add 4; step 2 multiply by 2; step 3 subtract 2; output the result.

(a) Trace algorithm B with inputs 3 and 7.
(b) Write a rule for the output in terms of n.
(c) Choose a test case of your own and check the rule against the algorithm.

Input 3: $3+4=7$, $7\times2=14$, $14-2=12$. Input 7: $7+4=11$, $11\times2=22$, $22-2=20$.

(b) The steps do $(n+4)\times2-2=2n+8-2=2n+6$.

(c) Sample test case $n=0$: the algorithm gives $4\times2-2=6$, and the rule gives $2\times0+6=6$ — they match.

∴ (a) 3 → 12 and 7 → 20; (b) output $=2n+6$; (c) the test case agrees. Review: a rule found from two traces is a guess; the third case is what turns it into a tested claim.

Example 5 — on your own

One sorting pass, rule: *swap if the left number is bigger than the right.*

(a) Run one pass on the list 6, 3, 8, 1. Write the list after each comparison.

(b) What has happened to the biggest number by the end of the pass?

(c) Run one pass with the faulty rule *swap if the left number is smaller*. What goes wrong?

(d) Compare 6 and 3: $6>3$, swap → 3, 6, 8, 1. Compare 6 and 8: keep. Compare 8 and 1: swap → 3, 6, 1, 8.

(e) The biggest number, 8, has moved to the end — one pass pushes it all the way along.

(f) With the faulty rule: 6 and 3 keep; 3 and 8 swap → 6, 8, 3, 1; 8 and 1 keep. The pass ends at 6, 8, 3, 1 — the smallest number, 1, has moved to the end, so the pass is ordering the list backwards.

∴ (a) 3, 6, 8, 1 → 3, 6, 8, 1 → 3, 6, 1, 8; (b) 8 finishes at the end; (c) the faulty pass runs without stopping, yet puts the list in reverse order.

Example 6 — on your own

An algorithm should output the **biggest** number in a list:

1. Input a list. Let m be the first number.
2. For each next number x: if x is smaller than m, replace m with x.
3. Output m.

(a) Trace it on the list 3, 8, 2, 6. What is output?

(b) What should the output be?

(c) Correct step 2.

(d) Retest both the faulty and the corrected algorithm on the list 9, 1, 5, 7.

(e) $m = 3$; 8 is not smaller than 3; 2 is smaller, so $m = 2$; 6 is not smaller than 2. Output: 2.

(f) The biggest number is 8 — the algorithm should have output 8.

(g) Corrected step 2: if x is **bigger** than m, replace m with x.

(h) Corrected: $m = 9$, and no later number is bigger, so output 9. Faulty: 1 is smaller than 9, so $m = 1$; output 1. The corrected algorithm passes the retest; the faulty one fails it, which is what a testing procedure is for.

∴ (a) 2; (b) 8; (c) swap "smaller" for "bigger" in step 2; (d) corrected → 9, faulty → 1. Review: the faulty version is a perfectly followed algorithm built on a wrong step — tracing and retesting are what expose it.

Practice questions

Work every question by hand — no calculator is needed: the skill here is following, testing and fixing steps, not the arithmetic.

With help

Q1. Trace the number trick (double, add 6, halve, subtract the start) with input -4. What is the output?

Q2. Algorithm C: input n; step 1 multiply by 3; step 2 subtract 6; step 3 divide by 3; output the result.

a. Trace algorithm C with inputs 9 and 2.
b. Write a rule for the output in terms of n.

Q3. Use the divisibility-by-3 algorithm to test 417 and 224. State each sum of digits and the decision it leads to.

Q4. Copy the table. For each number, write yes or no in each column — does the number pass the test for divisibility by 2, by 5, and by 3?

Number	Divisible by 2?	Divisible by 5?	Divisible by 3?
370	▢	▢	▢
432	▢	▢	▢
585	▢	▢	▢

Q5. A classmate claims: "If a number ends in 3, then it is divisible by 3." Give one counterexample and show why it works.

Q6. A linear search runs on the list 4, 9, 2, 7, 2, 5 with target 5. How many comparisons are made, and what position is output?

Q7. A linear search runs on the list 8, 3, 5, 3, 9.

a. Target 3: how many comparisons, and which position?
b. Target 9: how many comparisons, and which position?
c. Target 4: how many comparisons, and what is output?

Q8. Run one sorting pass (swap if the left number is bigger) on the list 5, 2, 7, 4. Write the list after each comparison.

On your own

Q9. An algorithm: input n; if n is even, output $n \div 2$; if n is odd, output $3n+1$. Find the output for inputs 10 and 7.

Q10. A faulty rule for divisibility by 9: "add the digits; if the sum is 9, the number is divisible by 9."

a. Show that 99 is a counterexample to this rule.
b. Correct the rule.
c. Retest the corrected rule on 98, and check with a division.

Q11. The faulty divisibility-by-4 rule ("if the last digit is divisible by 4, so is the number").

a. Show that 14 is a counterexample. (Check 34 as well.)
b. The number 124 passes the faulty rule and really is divisible by 4. Why does 124 not rescue the rule?
c. State the correct rule and use it on 14 and 124.

Q12. A faulty linear search: "if an item is bigger than the target, stop and output 'not in the list'." It runs on the list 4, 7, 2, 6 with target 2.

a. What does the faulty search output?
b. What is the truth about 2 in this list?
c. The early stop would be safe on some lists. Which ones?
d. Correct the faulty step for use on any list.

Q13. Sort the list 4, 1, 6, 3 using passes of the rule *swap if the left number is bigger*. Write the list after each pass, and say how the algorithm knows when to stop.

Q14. Use the mean flow chart (input a and b; add them; divide by 2; output) with $a=9$ and $b=5$. Draw a trace table and give the output.

Q15. An algorithm should say which of two numbers is bigger:

1. Input a and b.
2. If a is bigger than b, output "a is bigger".
3. Otherwise, output "b is bigger".

a. Run it with input (3, 3). What does it say, and why is that wrong?
b. Correct the algorithm so it handles this boundary case.
c. Retest your corrected algorithm with (3, 3) and with (5, 2).

Q16. A rule is meant to calculate $3n+4$, but the algorithm was written as: step 1 add 4; step 2 multiply by 3.

a. Test it with $n=2$: what should the output be, and what is it?
b. Describe the error.
c. Correct the algorithm and retest with $n=5$.

Maths in real life

Q17. Barcodes carry a check digit so that misreads can be caught. For a 13-digit barcode, the first 12 digits are multiplied by 1 and 3 in turn (1, 3, 1, 3, …), the results are added, and the check digit is whatever brings the total up to the next multiple of 10. *(Source: GS1 General Specifications, the published barcode standard, retrieved 20 August 2026.)*

a. For the barcode 5 9 0 1 2 3 4 1 2 3 4 5 | 7, show that the check digit 7 is correct.
b. A misprint changes the 12th digit from 5 to 4, so the scanner reads 5 9 0 1 2 3 4 1 2 3 4 4 | 7. Show that the check now fails.
c. A misread changes the first digit from 5 to 6. Show that this too is caught.

Q18. The calendar's leap-year rule is an algorithm: input a year; if it is not divisible by 4, it is a common year; if it is not divisible by 100, it is a leap year; if it is divisible by 400, it is a leap year; otherwise it is a common year. Run the algorithm on the years 2028, 1900, 2000 and 2100, and state the decision made at each step.

Q19. A teacher records seven spelling marks out of 15: 12, 9, 14, 10, 11, 9, 15. To find the middle mark, she first lines the marks up from smallest to largest.

a. Run sorting passes on the marks (swap if the left number is bigger) and write the list after each pass until it is sorted.
b. The middle value of the sorted list is the **median** (you will meet it again in 5C). What is the median mark?

Q20. Design your own algorithm: test whether a number is divisible by 6, using the tests for 2 and for 3. Write it as numbered steps or a flow chart. Swap algorithms with a partner and test each other's on 42, 24, 26 and 21 — if your outputs differ, trace both algorithms to find the faulty step.

Answers

Q1. Output 3.

Q2. a. Input 9 → 7; input 2 → 0. b. Output $= n - 2$.

Q3. 417: sum 12, then 3 → divisible by 3. 224: sum 8 → not divisible by 3.

Q4. 370: yes, yes, no. 432: yes, no, yes. 585: no, yes, yes.

Q5. Sample: 13 ends in 3 but $13 \div 3$ is not exact. (23, 43 also work.)

Q6. 6 comparisons; position 6.

Q7. a. 2 comparisons; position 2. b. 5 comparisons; position 5. c. 5 comparisons; "not in the list".

Q8. $2, 5, 7, 4 \to 2, 5, 7, 4 \to 2, 5, 4, 7$.

Q9. 10 → 5; 7 → 22.

Q10. a. $9 + 9 = 18$, not 9, so the rule says no — but $99 \div 9 = 11$ exactly. b. Add the digits again until one digit is left; if it is 9, the number is divisible by 9. c. 98: $9 + 8 = 17$, $1 + 7 = 8$ → not divisible; $98 \div 9$ is not exact.

Q11. a. $14 \div 4 = 3.5$ and $34 \div 4 = 8.5$, not exact, though both end in 4. b. A rule is a claim about *every* number; one counterexample disproves it, and passing cases never prove it. c. Last **two** digits: 14 → no; 124 → 24 is divisible by 4, so yes.

Q12. a. "Not in the list" (it stops at the first item, 4). b. 2 is in the list, at position 3. c. Lists already sorted smallest first. d. Compare every item with the target before deciding.

Q13. Pass 1: 1, 4, 3, 6. Pass 2: 1, 3, 4, 6 (sorted). Stop when a whole pass makes no swaps.

Q14. $s = 14$, $m = 7$; output 7.

Q15. a. It says "b is bigger" — wrong, they are equal. b. Add: if $a = b$, output "the numbers are equal". c. (3, 3) → equal; (5, 2) → 5 is bigger.

Q16. a. Should be 10; is 18. b. The steps are in the wrong order — it calculates $3(n + 4)$. c. Multiply by 3 first, then add 4; $n = 5$ gives 19.

Q17. a. Weighted sum 83; next multiple of 10 is 90; $90 - 83 = 7$. b. Sum 80 → check digit would be 0, not 7 — caught. c. Sum 84 → check digit would be 6, not 7 — caught.

Q18. 2028: leap. 1900: common. 2000: leap. 2100: common.

Q19. a. Pass 1: 9, 12, 10, 11, 9, 14, 15; pass 2: 9, 10, 11, 9, 12, 14, 15; pass 3: 9, 10, 9, 11, 12, 14, 15; pass 4: 9, 9, 10, 11, 12, 14, 15; pass 5 makes no swaps. b. 11.

Q20. Sample: step 1, if the number is not even, output "no"; step 2, add the digits until one digit is left; step 3, if that digit is 3, 6 or 9, output "yes", otherwise "no". Check: 42 → yes, 24 → yes, 26 → no, 21 → no.

Fully-worked solutions

Q1. $-4 \times 2 = -8$; $-8 + 6 = -2$; $-2 \div 2 = -1$; $-1 - (-4) = -1 + 4 = 3$.

∴ The output is 3 — the trick gives 3 for negative inputs too.

Q2. a. Input 9: $9 \times 3 = 27$; $27 - 6 = 21$; $21 \div 3 = 7$. Input 2: $2 \times 3 = 6$; $6 - 6 = 0$; $0 \div 3 = 0$.

b. The steps give $(3n - 6) \div 3 = n - 2$. Check with the traces: $9 - 2 = 7$ and $2 - 2 = 0$.

∴ Output $= n - 2$.

Q3. 417: $4+1+7=12$, still two digits, so $1+2=3$; 3 is one of 3, 6, 9, so 417 is divisible by 3 (check: $417 \div 3 = 139$). 224: $2+2+4=8$; 8 is not 3, 6 or 9, so 224 is not divisible by 3 (check: $224 \div 3 = 74$ remainder 2).

Q4. Tests: last digit even (2); last digit 0 or 5 (5); digits add to 3, 6 or 9 after re-adding (3).

Number	By 2	By 5	By 3
370	yes (ends in 0)	yes (ends in 0)	no ($3+7+0=10 \rightarrow 1$)
432	yes (ends in 2)	no	yes ($4+3+2=9$)
585	no (ends in 5)	yes	yes ($5+8+5=18 \rightarrow 9$)

Q5. Take 13. It ends in 3, but $13 \div 3 = 4$ remainder 1 — not exact. One counterexample is enough, so the claim is wrong. (23 and 43 are counterexamples too; 33 is not, since $33 \div 3 = 11$.)

Q6. $4 \neq 5$, $9 \neq 5$, $2 \neq 5$, $7 \neq 5$, $2 \neq 5$, $5 = 5$: six comparisons, and the output is position 6. This is the boundary case from the faulty-search figure — here with the correct step 5, the last position really is checked.

Q7. a. $8 \neq 3$, $3 = 3$: 2 comparisons, position 2 (the first 3). b. $8 \neq 9$, $3 \neq 9$, $5 \neq 9$, $3 \neq 9$, $9 = 9$: 5 comparisons, position 5. c. All five items are compared and none matches: 5 comparisons, “not in the list”.

Q8. Compare 5 and 2: $5 > 2$, swap → 2, 5, 7, 4. Compare 5 and 7: keep. Compare 7 and 4: $7 > 4$, swap → 2, 5, 4, 7. End of the pass: 2, 5, 4, 7, with the biggest number, 7, at the end.

Q9. 10 is even: $10 \div 2 = 5$. 7 is odd: $3 \times 7 + 1 = 22$.

Q10. a. For 99 the rule adds $9+9=18$; 18 is not 9, so the rule says “not divisible by 9”. But $99 \div 9 = 11$ exactly — 99 is a counterexample. b. Corrected: add the digits; if the sum is still more than one digit, add again; the number is divisible by 9 when the final one-digit sum is 9. c. 98: $9+8=17$, $1+7=8$ → not divisible by 9; check $98 \div 9 = 10$ remainder 8. Retest 99 with the corrected rule: $18 \rightarrow 9$ → divisible, as the division showed.

Q11. a. 14 ends in 4, and 4 is divisible by 4, so the rule says yes — but $14 \div 4 = 3.5$. 34 fails the same way: $34 \div 4 = 8.5$. b. The rule claims to work for every number. 14 shows that claim is false, and a true example like 124 cannot undo a false one. c. Correct rule: the number formed by the last two digits must be divisible by 4. For 14: 14 is not, so no. For 124: 24 is ($24 \div 4 = 6$), so yes.

Q12. a. The very first item, 4, is bigger than the target 2, so the faulty search stops at once and outputs “not in the list”. b. False — 2 is in the list, at position 3. c. On a list already sorted smallest first: once an item is bigger than the target, every later item is bigger too, so the target cannot be further on. d. Corrected step: check every item; only after the last item has been compared may the algorithm output “not in the list”.

Q13. Pass 1: 4 and 1 swap → 1, 4, 6, 3; 4 and 6 keep; 6 and 3 swap → 1, 4, 3, 6. Pass 2: 1 and 4 keep; 4 and 3 swap → 1, 3, 4, 6; 4 and 6 keep → 1, 3, 4, 6. Pass 3 makes no swaps, so the algorithm stops; the list is sorted. The stop test — a whole pass with no swaps — is part of the algorithm, not an afterthought.

Q14.

Step	Done	Result
Add	$9+5$	$s=14$
Divide by 2	$14 \div 2$	$m=7$

∴ Output 7 — the mean of 9 and 5.

Q15. a. Step 2 is false (3 is not bigger than 3), so step 3 runs and says “b is bigger” — but the numbers are equal. Equal inputs are the boundary case the algorithm forgot. b. Insert after step 2: if $a=b$, output “the numbers are equal”; otherwise step 3 as before. c. (3, 3): “the numbers are equal” — right. (5, 2): step 2 is true, “5 is bigger” — right.

Q16. a. Intended: $3 \times 2 + 4 = 10$. As written: $2 + 4 = 6$, $6 \times 3 = 18$. The test case catches the error: $18 \neq 10$. b. The two steps are in the wrong order; the algorithm calculates $3(n + 4)$, not $3n + 4$. c. Corrected: step 1 multiply by 3; step 2 add 4. Retest with $n = 5$: $5 \times 3 = 15$, $15 + 4 = 19$, and $3 \times 5 + 4 = 19$ — the corrected algorithm now matches the rule.

Q17. a. Multiply the first 12 digits by 1, 3, 1, 3, … in turn: $5 + 27 + 0 + 3 + 2 + 9 + 4 + 3 + 2 + 9 + 4 + 15 = 83$. The next multiple of 10 is 90, and $90 - 83 = 7$ — the printed check digit, so the barcode passes. b. With the 12th digit read as 4: the sum becomes $83 - 15 + 12 = 80$; the check digit would have to be 0. The printed 7 does not match, so the scanner rejects the read. c. With the first digit read as 6: the sum becomes $83 - 5 + 6 = 84$; the check digit would have to be 6, not 7 — this misread is caught too. That is the testing procedure doing its job: a wrong read almost never passes the check.

Q18. 2028: $2028 \div 4 = 507$ exactly, and 2028 is not divisible by 100 → leap year. 1900: divisible by 4 and by 100, but $1900 \div 400$ is not exact → common year. 2000: divisible by 400 ($2000 \div 400 = 5$) → leap year. 2100: divisible by 4 and 100, but not by 400 → common year. The centuries 1900 and 2100 are the boundary cases — the "divisible by 400" decision exists exactly for them.

Q19. a. Pass 1: 12 and 9 swap → 9, 12, 14, 10, 11, 9, 15; 14 and 10 swap; 14 and 11 swap; 14 and 9 swap → end of pass 1: 9, 12, 10, 11, 9, 14, 15. Pass 2: 12 and 10 swap; 12 and 11 swap; 12 and 9 swap → 9, 10, 11, 9, 12, 14, 15. Pass 3: 11 and 9 swap → 9, 10, 9, 11, 12, 14, 15. Pass 4: 10 and 9 swap → 9, 9, 10, 11, 12, 14, 15. Pass 5 makes no swaps → stop. b. Seven marks, so the median is the 4th: 11.

Q20. Sample algorithm: 1. Input n. 2. If n is not even, output "not divisible by 6" and stop. 3. Add the digits of n until one digit is left. 4. If that digit is 3, 6 or 9, output "divisible by 6"; otherwise output "not divisible by 6". Testing: 42 → even, sum 6 → yes ($42 \div 6 = 7$). 24 → even, sum 6 → yes. 26 → even, but $2 + 6 = 8$ → no ($26 \div 6$ not exact). 21 → not even → no, even though 21 passes the 3-test. A number must pass **both** the 2-test and the 3-test; passing one is not enough.

Test

Mathematics Level 8 — Chapter test T2

Chapter 2 · Algebra: linear expressions, relations and algorithms

- **Time:** 30 minutes — 5 minutes reading time, then 25 minutes working.
- **Marks:** 25. Answer all five questions; each is worth 5 marks.
- **Equipment:** a ruler; graphing or spreadsheet software for Question 4.
- **Calculators:** a scientific calculator is permitted for Questions 2 and 4 only, and is optional there. Questions 1, 3 and 5 are **calculator-free**.
- **Instructions:** show your working — working earns marks even when a final number slips. Where a question says *check* or *verify*, substitute your answer back into the original statement. Write your generalisations in Question 4 in full sentences.

Question 1 (2A) — 5 marks · calculator-free

(a) Simplify $6x+4-2x+7y-9-5y$. **[1]**

(b) Expand $-3(2x-5)$. **[1]**

(c) Factorise $18a+12b-6$. **[1]**

(d) Rearrange $C=45+15n$ to make n the subject. **[1]**

(e) Expand and simplify $3(x+4)-2(x-1)$. **[1]**

Question 2 (2B) — 5 marks

(a) The table shows a relation between x and y.

x	0	1	2	3	4
y	-1	4	9	14	19

Find the first differences. Is the relation linear? **[1]**

(b) Solve $4x+5=29$. Check your answer by substitution. **[1]**

(c) Solve $3x+7=5x-9$. Check your answer by substitution. **[1]**

(d) Solve $4x-3<25$. **[1]**

(e) The graph of $y=3x-2$ and the horizontal line $y=13$ meet at $(5,13)$. Read off:

i. the solution of $3x-2=13$;
ii. the solution of $3x-2>13$.

[1]

Question 3 (2C) — 5 marks · calculator-free

The Year 8 fete runs a hot-chip stall. Hiring the stall costs a fixed \$70. It costs \$2 to make each serve of chips, and each serve sells for \$6.

Let n be the number of serves sold.

(a) Write a rule for the income, and a rule for the costs. **[1]**

(b) Show that the profit can be written as $4n-70$. **[1]**

(c) Find the profit when 25 serves are sold. **[1]**

(d) What is the least number of serves that must be sold for the stall to make a profit? Explain your answer briefly. **[1]**

(e) Give one reason why the actual profit on the day might differ from this model. **[1]**

Question 4 (2D) — 5 marks

Use graphing software, or a spreadsheet graph, for this question.

A student graphs the family $y=ax+4$, letting a take the values $3, 1, 0, -1, -3$.

(a) Before graphing, predict the direction of each of the five lines: which rise from left to right, which fall, and which are horizontal? **[1]**

(b) Graph the five rules. What do all five lines have in common? **[1]**

(c) State the generalisation your graphs support about a and the direction of a line. **[1]**

(d) Which line is steeper: the one with $a=3$ or the one with $a=1$? Which is steeper, $a=-3$ or $a=-1$? Generalise about steepness. **[1]**

(e) Another student claims: "If the constant term is positive, the line rises." Use one of your five lines to show that this claim is false. **[1]**

Question 5 (2E) — 5 marks · calculator-free

An algorithm is meant to calculate $5n+2$. It was written as:

1. Input a number n.
2. Add 2.
3. Multiply by 5.
4. Output the result.

(a) Test the algorithm with $n=3$. What output does it give, and what should the output have been? **[1]**

(b) Describe the error in the algorithm. **[1]**

(c) Correct the algorithm. **[1]**

(d) Retest your corrected algorithm with $n=4$, and check the result against $5n+2$. **[1]**

(e) A classmate claims: "Every number that ends in 6 is divisible by 6." Give one counterexample — a number that breaks the claim — and show that it does. **[1]**

End of test — total 25 marks

Solutions

Question 1

(a) Collect the like terms, keeping each term's sign with it:

$6x-2x=4x \cdot 7y-5y=2y \cdot 4-9=-5$

$\therefore 6x+4-2x+7y-9-5y=4x+2y-5$.

(b) The -3 multiplies every term inside the bracket, and the sign rules apply to each product: $-3\times 2x=-6x$ and $-3\times -5=+15$.

$\therefore -3(2x-5)=-6x+15$.

(c) The largest number dividing 18, 12 and 6 is 6. Divide each term by 6 to fill the bracket: $18a\div 6=3a$, $12b\div 6=2b$, $-6\div 6=-1$ — the last term becomes -1, not 0.

$\therefore 18a+12b-6=6(3a+2b-1)$.

Check by expanding: $6\times 3a+6\times 2b-6\times 1=18a+12b-6$ ✓

(d) Undo the steps in reverse order. n is multiplied by 15 and then 45 is added, so undo the add first, then the multiply:

$C-45=15n$, so $\therefore n=\dfrac{C-45}{15}$.

(e) Expand each bracket, watching the signs — the -2 multiplies both terms in the second bracket: $3(x+4)=3x+12$ and $-2(x-1)=-2x+2$. Then collect the like terms:

$3x+12-2x+2=x+14$.

Question 2

(a) Subtract each y-value from the next: $4-(-1)=5$, $9-4=5$, $14-9=5$, $19-14=5$. The first differences are constant (5), so the relation is **linear**.

(b) Undo in reverse order: subtract 5 from both sides, $4x=24$; divide both sides by 4, $x=6$.

Check: $4\times 6+5=24+5=29$ ✓ $\therefore x=6$.

(c) Gather the x-terms on one side. Subtract $3x$ from both sides: $7=2x-9$. Add 9 to both sides: $16=2x$. Divide by 2: $x=8$.

Check: left side $3\times 8+7=31$; right side $5\times 8-9=31$. Both sides return 31 ✓ $\therefore x=8$.

(d) Add 3 to both sides: $4x<28$. Divide both sides by 4: $x<7$.

Check: the boundary $x=7$ gives $4\times 7-3=25$, equal and therefore not part of a "<" solution; test value $x=0$ gives $-3<25$ ✓ $\therefore x<7$.

(e) The two sides are equal exactly where the graphs meet, and the left side is greater exactly where the line $y=3x-2$ sits above $y=13$.

i. The graphs meet at $(5,13)$, so $3x-2=13$ has solution $x=5$.
ii. The line is above $y=13$ to the right of $x=5$, so $3x-2>13$ has solution $x>5$.

Question 3

(a) Income is the price times the number sold; costs are the fixed hire plus the amount per serve.

∴ **income** $=6n$; **costs** $=70+2n$.

(b) Profit = income − costs. The bracket matters — the whole cost is subtracted:

profit $=6n-(70+2n)=6n-70-2n=4n-70$, as required.

(c) Substitute $n=25$: $4\times25-70=100-70=30$.

∴ The stall makes a **profit of $30** when 25 serves are sold.

(d) Break even means profit 0: $4n-70=0$, so $4n=70$ and $n=70\div4=17.5$. A half serve cannot be sold. Check the whole numbers each side: $4\times17-70=-2$ (a loss of $2), and $4\times18-70=2$ (a profit of $2).

∴ The stall first makes a profit at **18 serves**.

(e) Any one sensible review point earns the mark. Sample: the model assumes every serve made is sold at $6 — on a cold day some serves may not sell; or the real cost per serve may rise if the price of potatoes or oil changes; or the stall may sell out before meeting all demand. The model is a guide built on constant numbers, not a promise.

Question 4

(a) Read the sign of the coefficient of x: positive rises, negative falls, zero is horizontal.

∴ $a=3$ and $a=1$: **rise**; $a=0$: **horizontal**; $a=-1$ and $a=-3$: **fall**.

(b) All five lines pass through the point $(0,4)$ — they cross the y-axis at the same point, because the constant term is 4 in every rule.

(c) Sample generalisation: **the sign of the coefficient of x sets the direction of the line — positive coefficients rise from left to right, negative coefficients fall, and a coefficient of 0 gives a horizontal line.**

(d) $a=3$ is steeper than $a=1$, and $a=-3$ is steeper than $a=-1$. Generalisation: **the bigger the coefficient of x, ignoring its sign, the steeper the line.**

(e) The line $y=-x+4$ (the case $a=-1$) has constant term 4, which is positive — yet the line falls from left to right. One breaking test is enough, so the claim is false: the direction depends on the sign of the coefficient of x, not on the constant term. (The line $y=-3x+4$ breaks the claim as well.)

Question 5

(a) Run the algorithm as written with $n=3$: step 2, $3+2=5$; step 3, $5\times5=25$. It outputs **25**. The intended calculation is $5\times3+2=17$, so the output **should have been 17**. Since $25\neq17$, the test fails.

(b) The steps are in the wrong order. Adding 2 first means the 5 multiplies the whole $n+2$: the algorithm calculates $(n+2)\times5=5n+10$, not $5n+2$ — the 2 is multiplied by 5 when it should not be.

(c) Swap steps 2 and 3:

1. Input a number n.
2. Multiply by 5.
3. Add 2.
4. Output the result.

(d) Corrected algorithm with $n=4$: step 2, $4\times5=20$; step 3, $20+2=22$. Check against the rule: $5\times4+2=22$ ✓ — the corrected algorithm now matches $5n+2$, and the retest passes.

(e) Take 16. It ends in 6, so it satisfies the claim's condition — but $16\div6$ is not exact (the answer is between 2 and 3), so 16 is **not** divisible by 6. One counterexample is enough, so the claim is false. (26 and 46 are counterexamples too; 36 is not, since $36\div6=6$.)

Answer key

Q1. a $4x+2y-5$ · b $-6x+15$ · c $6(3a+2b-1)$ · d $n=\dfrac{C-45}{15}$ · e $x+14$

Q2. a first differences $5,5,5,5$ — constant, so linear · b $x=6$ · c $x=8$ · d $x<7$ · e i $x=5$, ii $x>5$

Q3. a income $=6n$, costs $=70+2n$ · b profit $=6n-(70+2n)=4n-70$ · c $30 profit · d 18 serves (break-even is 17.5, and 17 serves is still a $2 loss) · e any one sensible review point (e.g. some serves may not sell)

Q4. a $a=3,1$ rise; $a=0$ horizontal; $a=-1,-3$ fall · b all pass through $(0,4)$ · c the sign of a sets the direction · d $a=3$ steeper than $a=1$; $a=-3$ steeper than $a=-1$ — bigger coefficient ignoring the sign, steeper line · e $y=-x+4$ falls despite the positive constant term 4

Q5. a outputs 25; should be 17 · b the steps are in the wrong order — it calculates $(n+2)\times5$, so the 2 is also multiplied by 5 · c multiply by 5 first, then add 2 · d $4\times5=20$, $20+2=22$; check $5\times4+2=22$ ✓ · e sample: 16 ends in 6 but $16\div6$ is not exact