Specialist Mathematics Units 3 & 4

Practice Examinations

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Contents

Exam 1 — Technology-free (Easy)

Reading time: 15 minutes. Writing time: 1 hour. Total 40 marks. Technology (calculators/notes) is NOT permitted. A formula sheet is provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Unless otherwise indicated, diagrams are not drawn to scale.

Section A — Short-answer

Question 1 (4 marks)

Prove by mathematical induction that $\sum_{r=1}^{n}(3r-1)=\frac{n(3n+1)}{2}$ for all integers $n\geq 1$. Let $P(n)$ be this statement.

a. Verify the base case $P(1)$. 1 mark

b. Write down the inductive hypothesis $P(k)$. 1 mark

c. Assuming $P(k)$, prove $P(k+1)$, and state the conclusion of the proof. 2 marks

Question 2 (5 marks)

Let $z=-5\sqrt{3}-5i$.

a. Find $|z|$ and $Arg(z)$. 2 marks

b. Hence write z in the form $r\,cis(\theta)$. 1 mark

c. Find z^2, giving your answer in the form $x+iy$. 2 marks

Question 3 (4 marks)

Consider the vectors $a=i+2j+3k$ and $b=3i+2j+k$.

a. Find $a\cdot b$. 1 mark

b. Find $|a|$ and $|b|$. 1 mark

c. Hence find $\cos(\theta)$, where θ is the angle between a and b. 2 marks

Question 4 (4 marks)

The line ℓ passes through the point $A(1,2,-1)$ and is parallel to the vector $d=2i+j-2k$.

a. Write down a vector equation of ℓ. 1 mark

b. Show that the point $P(5,4,-5)$ lies on ℓ, and state the corresponding value of the parameter. 2 marks

c. Find a unit vector in the direction of ℓ. 1 mark

Question 5 (4 marks)

a. Differentiate $y=\tan^{-1}(3x)$ with respect to x. 2 marks

b. The curve $x^2+y^2=25$ passes through the point $(3,4)$. Use implicit differentiation to find the value of $\frac{dy}{dx}$ at this point. 2 marks

Question 6 (5 marks)

a. Find $\int\frac{8}{\sqrt{16-x^2}}dx$. 2 marks

b. Evaluate $\int_{1}^{2} \frac{4}{2x-1}\,dx$, giving your answer as an exact value. 3 marks

Question 7 (4 marks)

Solve the differential equation $\frac{dy}{dx}=\frac{2x}{y}$ given that $y=3$ when $x=0$, expressing y in terms of x. 4 marks

Question 8 (4 marks)

A particle moves in a plane so that its position vector at time $t\geq 0$ seconds is $r(t)=4t\,i+(t^2-3)\,j$, where distances are measured in metres.

a. Find the velocity vector $\dot{r}(t)$. 1 mark

b. Find the speed of the particle when $t=2$. 2 marks

c. Find the acceleration vector $\ddot{r}(t)$. 1 mark

Question 9 (6 marks)

The volume X (in millilitres) dispensed into a bottle by a filling machine is normally distributed with mean μ and standard deviation $\sigma=2.5$. Bottles are filled independently.

a. Let $T=X_1+X_2+X_3$ be the total volume of three independently filled bottles. Find $E(T)$ in terms of μ, and find $Var(T)$. 2 marks

b. For a random sample of $n=25$ bottles, find the standard deviation of the sample mean $\overline{X}$. 1 mark

c. A particular sample of 25 bottles has a sample mean of $\bar{x}=199$. Taking $z=1.96$, calculate an approximate 95% confidence interval for μ. 3 marks

End of Exam 1

Exam 1 Formula Sheet

Formula sheet

This formula sheet is provided with every examination in this booklet.

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$
volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$
area of a triangle	$\frac{1}{2} b c \sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3} A h$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\,cis(\theta)$

modulus	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$
argument	$-\pi < Arg(z) \le \pi$
product	$z_1 z_2 = r_1 r_2\, cis(\theta_1 + \theta_2)$
quotient	$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$
de Moivre's theorem	$z^n = r^n\, cis(n\theta)$

Data analysis, probability and statistics

For independent random variables $X_1, X_2 \ldots X_n$:

$E(aX_1 + b) = aE(X_1) + b$

$E(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$

$Var(aX_1 + b) = a^2 Var(X_1)$

$Var(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$

For independent identically distributed variables $X_1, X_2 \ldots X_n$:

$E(X_1 + X_2 + \ldots + X_n) = n\mu\, Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$

mean of sample mean $\overline{X}$	$E(\overline{X})=\mu$
variance of sample mean $\overline{X}$	$Var(\overline{X})=\dfrac{\sigma^2}{n}$

Calculus — derivatives and integrals

$\dfrac{d}{dx}(x^n)=nx^{n-1}$	$\int x^n\,dx=\dfrac{1}{n+1}x^{n+1}+c,\ n\neq-1$
$\dfrac{d}{dx}(e^{ax})=ae^{ax}$	$\int e^{ax}\,dx=\dfrac{1}{a}e^{ax}+c$
$\dfrac{d}{dx}(\log_e(x))=\dfrac{1}{x}$	$\int \dfrac{1}{x}\,dx=\log_e\lvert x\rvert+c$
$\dfrac{d}{dx}(\sin(ax))=a\cos(ax)$	$\int \sin(ax)\,dx=-\dfrac{1}{a}\cos(ax)+c$
$\dfrac{d}{dx}(\cos(ax))=-a\sin(ax)$	$\int \cos(ax)\,dx=\dfrac{1}{a}\sin(ax)+c$
$\dfrac{d}{dx}(\tan(ax))=a\,sec^2(ax)$	$\int sec^2(ax)\,dx=\dfrac{1}{a}\tan(ax)+c$
$\dfrac{d}{dx}(\cot(ax))=-a\,cosec^2(ax)$	$\int cosec^2(ax)\,dx=-\dfrac{1}{a}\cot(ax)+c$
$\dfrac{d}{dx}(sec(ax))=a\,sec(ax)\tan(ax)$	$\int sec(ax)\tan(ax)\,dx=\dfrac{1}{a}sec(ax)+c$
$\dfrac{d}{dx}(cosec(ax))=-a\,cosec(ax)\cot(ax)$	$\int cosec(ax)\cot(ax)\,dx=-\dfrac{1}{a}cosec(ax)+c$
$\dfrac{d}{dx}(\sin^{-1}(ax))=\dfrac{a}{\sqrt{1-(ax)^2}}$	$\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\dfrac{x}{a}\right)+c,\ a>0$
$\dfrac{d}{dx}(\cos^{-1}(ax))=\dfrac{-a}{\sqrt{1-(ax)^2}}$	$\int \dfrac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\dfrac{x}{a}\right)+c,\ a>0$
$\dfrac{d}{dx}(\tan^{-1}(ax))=\dfrac{a}{1+(ax)^2}$	$\int \dfrac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\dfrac{x}{a}\right)+c$

$$\int (ax+b)^n\,dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq-1$$

$$\int \frac{1}{ax+b}\,dx=\frac{1}{a}\log_e\lvert ax+b\rvert+c$$

Calculus — rules and applications

product rule	$\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$
quotient rule	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$
chain rule	$\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$

integration by parts: $\int u\dfrac{dv}{dx}\,dx=uv-\int v\dfrac{du}{dx}\,dx$

Euler's method: if $\frac{dy}{dx}=f(x,y)$, $x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+h\times f(x_n,y_n)$

arc length (parametric) $$\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

surface area (Cartesian, about x-axis) $$\int_{x_1}^{x_2}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

surface area (Cartesian, about y-axis) $$\int_{y_1}^{y_2}2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

surface area (parametric, about x-axis) $$\int_{t_1}^{t_2}2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

surface area (parametric, about y-axis) $$\int_{t_1}^{t_2}2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Kinematics

acceleration: $a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant acceleration formulas:

$v=u+at$ $\qquad$ $s=ut+\frac{1}{2}at^2$

$v^2=u^2+2as$ $\qquad$ $s=\frac{1}{2}(u+v)t$

Vectors in two and three dimensions

position vector: $r(t)=x(t)i+y(t)j+z(t)k$

magnitude: $|r(t)|=\sqrt{x(t)^2+y(t)^2+z(t)^2}$

derivative with respect to t: $\dot{r}(t)=\frac{dr}{dt}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$

For $r_1=x_1i+y_1j+z_1k$ and $r_2=x_2i+y_2j+z_2k$:

vector scalar product: $r_1\cdot r_2=|r_1||r_2|\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

vector cross product: $r_1\times r_2=/\begin{matrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2\end{matrix}/=(y_1z_2-y_2z_1)i+(x_2z_1-x_1z_2)j+(x_1y_2-x_2y_1)k$

vector equation of a line: $r(t)=r_1+tr_2=(x_1+x_2t)i+(y_1+y_2t)j+(z_1+z_2t)k$

parametric equation of a line: $x(t)=x_1+x_2t,\ y(t)=y_1+y_2t,\ z(t)=z_1+z_2t$

vector equation of a plane: $r(s,t)=r_0+sr_1+tr_2=(x_0+x_1s+x_2t)i+(y_0+y_1s+y_2t)j+(z_0+z_1s+z_2t)k$

parametric equation of a plane: $x(s,t)=x_0+x_1s+x_2t,\ y(s,t)=y_0+y_1s+y_2t,\ z(s,t)=z_0+z_1s+z_2t$

Cartesian equation of a plane: $ax+by+cz=d$

Circular functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=sec^2(x)$

$\cot^2(x)+1=cosec^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$

$\sin^2(ax)=\dfrac{1}{2}(1-\cos(2ax))$

$\cos^2(ax)=\dfrac{1}{2}(1+\cos(2ax))$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

End of formula sheet

Exam 1 — Answers and solutions

Question 1.

a. For $P(1)$ the left-hand side is $3(1)-1=2$ and the right-hand side is $\frac{1(3(1)+1)}{2}=\frac{4}{2}=2$. The two sides are equal, so $P(1)$ is true.

b. Assume $P(k)$ is true for some integer $k\geq 1$; that is, $\sum_{r=1}^{k}(3r-1)=\frac{k(3k+1)}{2}$.

c. The $(k+1)$-th term is $3(k+1)-1=3k+2$. Adding it to both sides of the hypothesis,

$$\sum_{r=1}^{k+1}(3r-1)=\frac{k(3k+1)}{2}+(3k+2)=\frac{k(3k+1)+2(3k+2)}{2}=\frac{3k^2+7k+4}{2}=\frac{(k+1)(3k+4)}{2}.$$

Since $3(k+1)+1=3k+4$, this equals $\frac{(k+1)(3(k+1)+1)}{2}$, which is exactly $P(k+1)$. Hence $P(k)\Rightarrow P(k+1)$. As $P(1)$ is true and the implication holds for every $k\geq 1$, by the principle of mathematical induction the formula holds for all integers $n\geq 1$.

Question 2.

a. $|z|=\sqrt{(-5\sqrt{3})^2+(-5)^2}=\sqrt{75+25}=\sqrt{100}=10$. The point $(-5\sqrt{3},-5)$ lies in the third quadrant, and the reference angle is $\tan^{-1}\left(\frac{5}{5\sqrt{3}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$, so $Arg(z)=-\left(\pi-\frac{\pi}{6}\right)=-\frac{5\pi}{6}$.

b. $z=10\,cis\left(-\frac{5\pi}{6}\right)$.

c. $z^2=(-5\sqrt{3}-5i)^2=75+50\sqrt{3}i+25i^2=75+50\sqrt{3}i-25=50+50\sqrt{3}i$.

Question 3.

a. $a\cdot b=(1)(3)+(2)(2)+(3)(1)=3+4+3=10$.

b. $|a|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$ and $|b|=\sqrt{3^2+2^2+1^2}=\sqrt{14}$.

c. $\cos(\theta)=\frac{a\cdot b}{|a||b|}=\frac{10}{\sqrt{14}\times\sqrt{14}}=\frac{10}{14}=\frac{5}{7}$.

Question 4.

a. $r(t)=(i+2j-k)+t(2i+j-2k)$, that is $r(t)=(1+2t)i+(2+t)j+(-1-2t)k$.

b. Equating the i-component to that of P: $1+2t=5$, so $t=2$. With $t=2$ the j-component is $2+2=4$ and the k-component is $-1-2(2)=-5$, which match $P(5,4,-5)$. Hence P lies on ℓ, at $t=2$.

c. $|d|=\sqrt{2^2+1^2+(-2)^2}=\sqrt{9}=3$, so a unit vector in the direction of ℓ is $\frac{1}{3}(2i+j-2k)$.

Question 5.

a. Using $\frac{d}{dx}\tan^{-1}(ax)=\frac{a}{1+(ax)^2}$ with $a=3$,

$$\frac{dy}{dx}=\frac{3}{1+(3x)^2}=\frac{3}{1+9x^2}.$$

b. Differentiating $x^2+y^2=25$ implicitly gives $2x+2y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{x}{y}$. At the point $(3,4)$, $\frac{dy}{dx}=-\frac{3}{4}$.

Question 6.

a. This matches the standard form $\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c$ with $a=4$:

$$\int\frac{8}{\sqrt{16-x^2}}dx=8\sin^{-1}\left(\frac{x}{4}\right)+c.$$

b. Since $\int\frac{1}{2x-1}dx=\frac{1}{2}\log_e|2x-1|+c$,

$$\int_1^2\frac{4}{2x-1}dx=4\times\frac{1}{2}[\log_e|2x-1|]_1^2=2(\log_e 3-\log_e 1)=2\log_e 3.$$

Question 7.

Separating the variables in $\frac{dy}{dx}=\frac{2x}{y}$ gives $y\,dy=2x\,dx$. Integrating both sides,

$$\int y\,dy=\int 2x\,dx\Rightarrow\frac{y^2}{2}=x^2+c_1,$$

so $y^2=2x^2+c$ (where $c=2c_1$). Substituting $x=0$ and $y=3$ gives $9=c$. Hence $y^2=2x^2+9$, and since $y=3>0$ we take the positive root: $y=\sqrt{2x^2+9}$.

Question 8.

a. Differentiating each component with respect to t, $\dot{r}(t)=4i+2t\,j$.

b. At $t=2$, $\dot{r}(2)=4i+4j$, so the speed is $|\dot{r}(2)|=\sqrt{4^2+4^2}=\sqrt{32}=4\sqrt{2}$ m/s.

c. Differentiating the velocity, $\ddot{r}(t)=2j$ m/s^2 (constant).

Question 9.

a. By the linearity of expectation, $E(T)=E(X_1)+E(X_2)+E(X_3)=3\mu$. Because the bottles are filled independently, $Var(T)=Var(X_1)+Var(X_2)+Var(X_3)=3\sigma^2=3(2.5)^2=3\times 6.25=18.75$.

b. $sd(\overline{X})=\frac{\sigma}{\sqrt{n}}=\frac{2.5}{\sqrt{25}}=\frac{2.5}{5}=0.5$.

c. The margin of error is $z\times sd(\overline{X})=1.96\times 0.5=0.98$. The approximate 95% confidence interval for μ is

$$\bar{x}\pm 0.98=199\pm 0.98=(198.02,199.98).$$

Exam 2 — Technology-free (Medium)

Reading time: 15 minutes. Writing time: 1 hour. Total 40 marks. Technology (calculators/notes) is NOT permitted. A formula sheet is provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Unless otherwise indicated, diagrams are not drawn to scale.

Section A — Short-answer

Question 1 (4 marks)

Let $P(n)$ be the statement that

$$\sum_{r=1}^{n}(4r-3)=n(2n-1)$$

for all integers $n\geq 1$.

a. Verify the base case $P(1)$. 1 mark

b. Assuming that $P(k)$ is true for some integer $k\geq 1$, prove that $P(k+1)$ is true, and state the conclusion of the proof. 3 marks

Question 2 (5 marks)

Let $z=-3\sqrt{3}+3i$.

a. Find $|z|$ and $Arg(z)$, and hence write z in the form $r\,cis(\theta)$. 2 marks

b. Hence find z^3, giving your answer in Cartesian form $x+iy$. 2 marks

c. Hence write down all three solutions of $w^3=216i$, giving each in Cartesian form. 1 mark

Question 3 (4 marks)

Consider the vectors $a=2i+3j-6k$ and $b=i-2j+2k$.

a. Find $a\cdot b$, and hence find the cosine of the angle between a and b. 2 marks

b. Find $a\times b$. 2 marks

Question 4 (4 marks)

The line ℓ has vector equation $r(t)=(2+t)i+(-1+2t)j+(1-t)k$, where $t\in\mathbb{R}$. The plane Π has Cartesian equation $x-y+2z=2$.

a. Show that the point $A(2,-1,1)$ on ℓ does not lie in Π, and find the coordinates of the point where ℓ meets Π. 2 marks

b. Find the acute angle between the line ℓ and the plane Π. 2 marks

Question 5 (4 marks)

Let $f(x)=x\sin^{-1}(x)+\sqrt{1-x^2}$, for $-1<x<1$.

a. Show that $f'(x)=\sin^{-1}(x)$. 3 marks

b. Hence find the gradient of the graph of $y=f(x)$ at $x=\frac{1}{2}$, giving your answer in exact form. 1 mark

Question 6 (5 marks)

Consider the function $g(x)=\dfrac{4}{(x+1)(x+3)}$.

a. Express $g(x)$ in partial fractions. 2 marks

b. Hence evaluate $\displaystyle\int_0^1 \frac{4}{(x+1)(x+3)}\,dx$, giving your answer in the form $a\log_e(b)$. 3 marks

Question 7 (5 marks)

A curve $y=f(x)$ passes through the point $(0,1)$ and satisfies the differential equation $\dfrac{dy}{dx}=x^2y^2$.

a. Solve the differential equation to show that $y=\dfrac{3}{3-x^3}$. 3 marks

b. State the largest interval containing $x=0$ on which this solution is defined. 1 mark

c. Find the value of $\dfrac{dy}{dx}$ when $x=1$. 1 mark

Question 8 (4 marks)

The position of a particle at time $t\geq 0$ is given by $r(t)=(t-2)i+(t^2-4t+3)j$, where distances are measured in metres and t is in seconds.

a. Find the velocity of the particle, and show that its speed when $t=3$ is $\sqrt{5}\,\mathrm{m/s}$. 2 marks

b. Find the Cartesian equation of the path of the particle, stating any restriction on x. 2 marks

Question 9 (5 marks)

The mass, in grams, of a biscuit produced by a machine is a normally distributed random variable X with mean 25 and standard deviation 4. The masses of different biscuits are independent.

a. A packet contains 9 biscuits selected at random. Find the mean and standard deviation of the total mass T of the biscuits in the packet. 2 marks

b. A packet of 9 biscuits is called *oversize* if its total mass exceeds 243 grams. Given that $P(Z\leq 1.5)=0.93$, correct to two decimal places, find the probability that a randomly selected packet of 9 biscuits is oversize. 2 marks

c. Find the mean and standard deviation of the sample mean mass $\overline{X}$ of a random sample of 16 biscuits. 1 mark

End of Exam 2

Exam 2 Formula Sheet

Formula sheet

This formula sheet is provided with every examination in this booklet.

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta-\sin(\theta))$
volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$
area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
sine rule	$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$
cosine rule	$c^2=a^2+b^2-2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z=x+iy=r(\cos(\theta)+i\sin(\theta))=r\,cis(\theta)$

modulus	$\lvert z\rvert=\sqrt{x^2+y^2}=r$
argument	$-\pi<Arg(z)\le\pi$
product	$z_1z_2=r_1r_2\,cis(\theta_1+\theta_2)$
quotient	$\frac{z_1}{z_2}=\frac{r_1}{r_2}cis(\theta_1-\theta_2)$
de Moivre's theorem	$z^n=r^n\,cis(n\theta)$

Data analysis, probability and statistics

For independent random variables $X_1, X_2 \ldots X_n$:

$E(aX_1+b)=aE(X_1)+b$

$E(a_1X_1+a_2X_2+\ldots+a_nX_n)=a_1E(X_1)+a_2E(X_2)+\ldots+a_nE(X_n)$

$Var(aX_1+b)=a^2Var(X_1)$

$Var(a_1X_1+a_2X_2+\ldots+a_nX_n)=a_1^2Var(X_1)+a_2^2Var(X_2)+\ldots+a_n^2Var(X_n)$

For independent identically distributed variables $X_1, X_2 \ldots X_n$:

$E(X_1+X_2+\ldots+X_n)=n\mu\,Var(X_1+X_2+\ldots+X_n)=n\sigma^2$

approximate confidence interval for μ	$\left(\bar{x}-z\frac{s}{\sqrt{n}},\bar{x}+z\frac{s}{\sqrt{n}}\right)$

mean of sample mean $\overline{X}$ $E(\overline{X})=\mu$

variance of sample mean $\overline{X}$ $Var(\overline{X})=\frac{\sigma^2}{n}$

Calculus — derivatives and integrals

$\frac{d}{dx}(x^n)=nx^{n-1}$ $\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,n\neq-1$

$\frac{d}{dx}(e^{ax})=ae^{ax}$ $\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$

$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$ $\int\frac{1}{x}\,dx=\log_e|x|+c$

$\frac{d}{dx}(\sin(ax))=a\cos(ax)$ $\int\sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$

$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$ $\int\cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$

$\frac{d}{dx}(\tan(ax))=a\,sec^2(ax)$ $\int sec^2(ax)\,dx=\frac{1}{a}\tan(ax)+c$

$\frac{d}{dx}(\cot(ax))=-a\,cosec^2(ax)$ $\int cosec^2(ax)\,dx=-\frac{1}{a}\cot(ax)+c$

$\frac{d}{dx}(sec(ax))=a\,sec(ax)\tan(ax)$ $\int sec(ax)\tan(ax)\,dx=\frac{1}{a}sec(ax)+c$

$\frac{d}{dx}(cosec(ax))=-a\,cosec(ax)\cot(ax)$ $\int cosec(ax)\cot(ax)\,dx=-\frac{1}{a}cosec(ax)+c$

$\frac{d}{dx}(\sin^{-1}(ax))=\frac{a}{\sqrt{1-(ax)^2}}$ $\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c,a>0$

$\frac{d}{dx}(\cos^{-1}(ax))=\frac{-a}{\sqrt{1-(ax)^2}}$ $\int\frac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+c,a>0$

$\frac{d}{dx}(\tan^{-1}(ax))=\frac{a}{1+(ax)^2}$ $\int\frac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\frac{x}{a}\right)+c$

$\int(ax+b)^n\,dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,n\neq-1$

$\int\frac{1}{ax+b}\,dx=\frac{1}{a}\log_e|ax+b|+c$

Calculus — rules and applications

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

integration by parts: $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$

Euler's method: if $\frac{dy}{dx}=f(x,y)$, $x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+h\times f(x_n,y_n)$

arc length (parametric) $\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (Cartesian, about x-axis) $\int_{x_1}^{x_2}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

surface area (Cartesian, about y-axis) $\int_{y_1}^{y_2}2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

surface area (parametric, about x-axis) $\int_{t_1}^{t_2}2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (parametric, about y-axis) $\int_{t_1}^{t_2}2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration: $a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant acceleration formulas:

$v=u+at$ $\qquad s=ut+\frac{1}{2}at^2$

$v^2=u^2+2as$ $\qquad s=\frac{1}{2}(u+v)t$

Vectors in two and three dimensions

position vector: $r(t)=x(t)i+y(t)j+z(t)k$

magnitude: $|r(t)|=\sqrt{x(t)^2+y(t)^2+z(t)^2}$

derivative with respect to t: $\dot{r}(t)=\frac{dr}{dt}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$

For $r_1=x_1i+y_1j+z_1k$ and $r_2=x_2i+y_2j+z_2k$:

vector scalar product: $r_1\cdot r_2=|r_1||r_2|\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

vector cross product: $r_1\times r_2=\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=(y_1z_2-y_2z_1)i+(x_2z_1-x_1z_2)j+(x_1y_2-x_2y_1)k$

vector equation of a line: $r(t)=r_1+tr_2=(x_1+x_2t)i+(y_1+y_2t)j+(z_1+z_2t)k$

parametric equation of a line: $x(t)=x_1+x_2t,\ y(t)=y_1+y_2t,\ z(t)=z_1+z_2t$

vector equation of a plane: $r(s,t)=r_0+sr_1+tr_2=(x_0+x_1s+x_2t)i+(y_0+y_1s+y_2t)j+(z_0+z_1s+z_2t)k$

parametric equation of a plane: $x(s,t)=x_0+x_1s+x_2t,\ y(s,t)=y_0+y_1s+y_2t,\ z(s,t)=z_0+z_1s+z_2t$

Cartesian equation of a plane: $ax+by+cz=d$

Circular functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=sec^2(x)$

$\cot^2(x)+1=cosec^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$

$\sin^2(ax)=\dfrac{1}{2}(1-\cos(2ax))$

$\cos^2(ax)=\dfrac{1}{2}(1+\cos(2ax))$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

End of formula sheet

Exam 2 — Answers and solutions

Question 1.

a. For $n=1$, the left-hand side is $4(1)-3=1$ and the right-hand side is $1(2(1)-1)=1\times1=1$. The two sides are equal, so $P(1)$ is true.

b. Assume $P(k)$ is true for some integer $k\ge1$; that is,

$$\sum_{r=1}^{k}(4r-3)=k(2k-1).$$

Adding the $(k+1)$-th term, whose value is $4(k+1)-3=4k+1$,

$$\sum_{r=1}^{k+1}(4r-3)=k(2k-1)+(4k+1)=2k^2-k+4k+1=2k^2+3k+1=(k+1)(2k+1).$$

Since $(k+1)(2(k+1)-1)=(k+1)(2k+1)$, this is exactly $P(k+1)$, so $P(k)\Rightarrow P(k+1)$. As $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for every integer $k\ge1$, by the principle of mathematical induction $P(n)$ holds for all integers $n\ge1$.

Question 2.

a. $|z|=\sqrt{(-3\sqrt{3})^2+3^2}=\sqrt{27+9}=\sqrt{36}=6$. The point $(-3\sqrt{3},3)$ lies in the second quadrant with reference angle $\tan^{-1}\left(\frac{3}{3\sqrt{3}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$, so $Arg(z)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$. Hence $z=6\,cis\left(\frac{5\pi}{6}\right)$.

b. By de Moivre's theorem,

$$z^3=6^3\,cis\left(3\times\frac{5\pi}{6}\right)=216\,cis\left(\frac{5\pi}{2}\right)=216\,cis\left(\frac{\pi}{2}\right)=216\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)=216i,$$

using $\frac{5\pi}{2}-2\pi=\frac{\pi}{2}$. So $z^3=216i$ (that is, $0+216i$).

c. Part **b** shows that $z=-3\sqrt{3}+3i$ is one solution of $w^3=216i$. The three cube roots all have modulus $216^{1/3}=6$, with arguments $\frac{5\pi}{6}$, $\frac{5\pi}{6}-\frac{2\pi}{3}=\frac{\pi}{6}$ and $\frac{5\pi}{6}+\frac{2\pi}{3}=\frac{3\pi}{2}$, equivalently $-\frac{\pi}{2}$, that is $6\,cis\left(\frac{5\pi}{6}\right)$, $6\,cis\left(\frac{\pi}{6}\right)$ and $6\,cis\left(-\frac{\pi}{2}\right)$, so

$$w=-3\sqrt{3}+3i,\ w=3\sqrt{3}+3i,\ w=-6i.$$

Question 3.

a. $a\cdot b=(2)(1)+(3)(-2)+(-6)(2)=2-6-12=-16$. Also $|a|=\sqrt{2^2+3^2+(-6)^2}=\sqrt{49}=7$ and $|b|=\sqrt{1^2+(-2)^2+2^2}=\sqrt{9}=3$. Hence

$$\cos\theta=\frac{a\cdot b}{|a||b|}=\frac{-16}{7\times3}=-\frac{16}{21}.$$

b. Expanding the determinant,

$$a\times b=\begin{vmatrix} i & j & k \\ 2 & 3 & -6 \\ 1 & -2 & 2\end{vmatrix}=((3)(2)-(-6)(-2))i-((2)(2)-(-6)(1))j+((2)(-2)-(3)(1))k.$$

This gives $a\times b=(6-12)i-(4+6)j+(-4-3)k=-6i-10j-7k$.

Question 4.

a. At $A(2,-1,1)$, $x-y+2z=2-(-1)+2(1)=5\neq 2$, so A does not lie in Π. Substituting the line into the plane equation,

$$(2+t)-(-1+2t)+2(1-t)=2+t+1-2t+2-2t=5-3t.$$

Setting $5-3t=2$ gives $t=1$, so ℓ meets Π at $(2+1,-1+2,1-1)=(3,1,0)$. (Check: $3-1+2(0)=2$.)

b. The direction of ℓ is $d=i+2j-k$ with $|d|=\sqrt{6}$, and a normal to Π is $n=i-j+2k$ with $|n|=\sqrt{6}$. Then $d\cdot n=(1)(1)+(2)(-1)+(-1)(2)=-3$. If α is the acute angle between the line and the plane, then

$$\sin\alpha=\frac{|d\cdot n|}{|d||n|}=\frac{3}{\sqrt{6}\times\sqrt{6}}=\frac{3}{6}=\frac{1}{2},$$

so $\alpha=\frac{\pi}{6}$ (that is, $30°$).

Question 5.

a. Differentiate term by term. For $x\sin^{-1}(x)$, the product rule gives

$$\frac{d}{dx}\left(x\sin^{-1}(x)\right)=\sin^{-1}(x)+x\cdot\frac{1}{\sqrt{1-x^2}}.$$

For $\sqrt{1-x^2}=(1-x^2)^{1/2}$, the chain rule gives $\frac{d}{dx}(1-x^2)^{1/2}=\frac{1}{2}(1-x^2)^{-1/2}(-2x)=\frac{-x}{\sqrt{1-x^2}}$. Adding these,

$$f'(x)=\sin^{-1}(x)+\frac{x}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}=\sin^{-1}(x).$$

b. $f'\left(\frac{1}{2}\right)=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$.

Question 6.

a. Write $\frac{4}{(x+1)(x+3)}=\frac{A}{x+1}+\frac{B}{x+3}$, so $4=A(x+3)+B(x+1)$. Setting $x=-1$ gives $4=2A$, so $A=2$; setting $x=-3$ gives $4=-2B$, so $B=-2$. Hence

$$\frac{4}{(x+1)(x+3)}=\frac{2}{x+1}-\frac{2}{x+3}.$$

b. Using part **a**,

$$\int_0^1\frac{4}{(x+1)(x+3)}\,dx=\int_0^1\left(\frac{2}{x+1}-\frac{2}{x+3}\right)dx=[2\log_e|x+1|-2\log_e|x+3|]_0^1.$$

Evaluating at the terminals,

$$=(2\log_e 2-2\log_e 4)-(2\log_e 1-2\log_e 3)=2\log_e 2-2\log_e 4+2\log_e 3=2\log_e\left(\frac{2\times 3}{4}\right)=2\log_e\left(\frac{3}{2}\right).$$

Question 7.

a. The equation is separable. For $y\neq 0$, divide by y^2 and integrate each side:

$$\int \frac{1}{y^2} dy = \int x^2 dx \Rightarrow -\frac{1}{y} = \frac{x^3}{3} + c.$$

The curve passes through $(0, 1)$, so $-\frac{1}{1} = 0 + c$, giving $c = -1$. Then $-\frac{1}{y} = \frac{x^3}{3} - 1 = \frac{x^3 - 3}{3}$, so $\frac{1}{y} = \frac{3 - x^3}{3}$ and therefore $y = \frac{3}{3 - x^3}$.

b. The solution is undefined only where the denominator vanishes, that is at $x = \sqrt[3]{3}$. The largest interval containing $x = 0$ on which the solution is defined is therefore $x < \sqrt[3]{3}$, that is $(-\infty, \sqrt[3]{3})$.

c. From the differential equation, $\frac{dy}{dx} = x^2 y^2$. At $x = 1$, $y = \frac{3}{3-1} = \frac{3}{2}$, so $\frac{dy}{dx} = (1)^2 \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

Question 8.

a. Differentiating with respect to t, the velocity is $\dot{r}(t) = i + (2t - 4) j$. At $t = 3$, $\dot{r}(3) = i + 2j$, so the speed is $|\dot{r}(3)| = \sqrt{1^2 + 2^2} = \sqrt{5}$ m/s.

b. With $x = t - 2$, we have $t = x + 2$. Substituting into $y = t^2 - 4t + 3$,

$$y = (x+2)^2 - 4(x+2) + 3 = x^2 + 4x + 4 - 4x - 8 + 3 = x^2 - 1.$$

Since $t \geq 0$, $x = t - 2 \geq -2$, so the particle traverses only the part of the parabola from $(-2, 3)$ onwards. The Cartesian equation of the path is $y = x^2 - 1$ for $x \geq -2$.

Question 9.

a. Let $T = X_1 + X_2 + \ldots + X_9$. Then $E(T) = 9 \times 25 = 225$ grams. By independence, $Var(T) = 9 \times 4^2 = 9 \times 16 = 144$, so the standard deviation of T is $\sqrt{144} = 12$ grams.

b. Since T is normally distributed with mean 225 and standard deviation 12, standardising gives

$$P(T > 243) = P\left(Z > \frac{243 - 225}{12}\right) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93 = 0.07.$$

c. For a sample of size $n = 16$, $E(\overline{X}) = 25$ grams and $Var(\overline{X}) = \frac{\sigma^2}{n} = \frac{16}{16} = 1$, so the standard deviation of $\overline{X}$ is $\sqrt{1} = 1$ gram.

Exam 3 — Technology-free (Hard)

Reading time: 15 minutes. Writing time: 1 hour. Total 40 marks. Technology (calculators/notes) is NOT permitted. A formula sheet is provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Unless otherwise indicated, diagrams are not drawn to scale.

Section A — Short-answer

Question 1 (4 marks)

Prove by mathematical induction that $\sum_{r=1}^{n} r \cdot 2^{r-1} = (n-1)2^n + 1$ for all integers $n \geq 1$. Let $P(n)$ be this statement.

a. Verify the base case $P(1)$. 1 mark

b. Write down the inductive hypothesis $P(k)$. 1 mark

c. Assuming $P(k)$, prove $P(k+1)$, and state the conclusion of the proof. 2 marks

Question 2 (3 marks)

The algorithm below reads a positive integer n.

```
input n
a ← 1
b ← 1
for i from 3 to n do
    c ← a + b
    a ← b
    b ← c
end for
output b
```

a. Complete a trace table showing the values of a, b and c after each pass of the loop for $n = 6$, and state the output. 2 marks

b. State, with a reason, what the output of the algorithm represents for a general integer $n \geq 2$. 1 mark

Question 3 (5 marks)

Consider the polynomial $P(z) = z^3 + z + 10$, where $z \in \mathbb{C}$.

a. Show that $z = 1 + 2i$ is a solution of $P(z) = 0$. 2 marks

b. Write down a second solution of $P(z) = 0$, giving a reason, and hence find the third solution. 2 marks

c. Hence express $P(z)$ as a product of a real linear factor and a real quadratic factor. 1 mark

Question 4 (5 marks)

The points $A(1, 0, 2)$, $B(3, 1, 1)$ and $C(2, 2, 3)$ have position vectors measured from the origin.

a. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. 2 marks

b. Hence find the Cartesian equation of the plane through A, B and C. 2 marks

c. Find the exact area of triangle ABC. 1 mark

Question 5 (4 marks)

Let $y = \tan^{-1}\left(\frac{x}{2}\right)$.

a. Find $\frac{dy}{dx}$. 1 mark

b. Show that $\frac{d^2y}{dx^2} = \frac{-4x}{\left(4+x^2\right)^2}$, and hence find the coordinates of the point of inflection of the curve. 3 marks

Question 6 (4 marks)

a. Using partial fractions, find $\int \frac{5}{(x-1)(x+4)}\,dx$. 2 marks

b. Find $\int x\cos(2x)\,dx$. 2 marks

Question 7 (4 marks)

A particle moves in a straight line. At time t seconds its displacement from a fixed origin O is x metres, and its acceleration is given by $a = -9x$. When the particle is at $x = 2$ it has velocity $v = 6$ m/s.

a. Show that $v^2 = 9\left(8 - x^2\right)$. 2 marks

b. Find the amplitude of the motion. 1 mark

c. Find the maximum speed of the particle, and the position at which it occurs. 1 mark

Question 8 (3 marks)

A particle moves so that its position vector at time $t \geq 0$ seconds is $r(t) = 3\cos(t)i + 3\sin(t)j + 4tk$, where distances are measured in metres.

a. Find the velocity $\dot{r}(t)$, and show that the speed of the particle is constant, stating its value. 2 marks

b. Show that the acceleration $\ddot{r}(t)$ is perpendicular to the velocity for all t. 1 mark

Question 9 (4 marks)

The time taken by machine A to complete a task is a normally distributed random variable X with mean 30 minutes and standard deviation 4 minutes. The time taken by machine B is a normally distributed random variable Y with mean 20 minutes and standard deviation 3 minutes. The two machines operate independently.

a. Find the mean and the standard deviation of the total time $X+Y$. 2 marks

b. Find the mean and the variance of the difference $X-Y$. 2 marks

Question 10 (4 marks)

The reaction times of a large population are normally distributed with a known standard deviation of $\sigma = 0.4$ seconds. A random sample of $n = 64$ people has a mean reaction time of $\bar{x} = 1.30$ seconds.

a. Taking $z = 1.96$, calculate an approximate 95% confidence interval for the population mean reaction time μ. 2 marks

b. A researcher tests $H_0: \mu = 1.35$ against $H_1: \mu \neq 1.35$ at the 5% level of significance. Calculate the value of the test statistic z, and state the conclusion of the test. 2 marks

End of Exam 3

Exam 3 Formula Sheet

Formula sheet

This formula sheet is provided with every examination in this booklet.

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta-\sin(\theta))$
volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$
area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
sine rule	$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$
cosine rule	$c^2=a^2+b^2-2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z=x+iy=r(\cos(\theta)+i\sin(\theta))=r\,cis(\theta)$

modulus	$\lvert z\rvert=\sqrt{x^2+y^2}=r$
argument	$-\pi<Arg(z)\le\pi$
product	$z_1z_2=r_1r_2\,cis(\theta_1+\theta_2)$
quotient	$\frac{z_1}{z_2}=\frac{r_1}{r_2}cis(\theta_1-\theta_2)$
de Moivre's theorem	$z^n=r^n\,cis(n\theta)$

Data analysis, probability and statistics

For independent random variables $X_1, X_2 \ldots X_n$:

$E(aX_1+b)=aE(X_1)+b$

$E(a_1X_1+a_2X_2+\ldots+a_nX_n)=a_1E(X_1)+a_2E(X_2)+\ldots+a_nE(X_n)$

$Var(aX_1+b)=a^2Var(X_1)$

$Var(a_1X_1+a_2X_2+\ldots+a_nX_n)=a_1^2Var(X_1)+a_2^2Var(X_2)+\ldots+a_n^2Var(X_n)$

For independent identically distributed variables $X_1, X_2 \ldots X_n$:

$E(X_1+X_2+\ldots+X_n)=n\mu\,Var(X_1+X_2+\ldots+X_n)=n\sigma^2$

approximate confidence interval for μ	$\left(\bar{x}-z\frac{s}{\sqrt{n}},\bar{x}+z\frac{s}{\sqrt{n}}\right)$

mean of sample mean $\overline{X}$ — $E(\overline{X})=\mu$

variance of sample mean $\overline{X}$ — $Var(\overline{X})=\frac{\sigma^2}{n}$

Calculus — derivatives and integrals

$$\frac{d}{dx}(x^n)=nx^{n-1} \qquad \int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$$

$$\frac{d}{dx}(e^{ax})=ae^{ax} \qquad \int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$$

$$\frac{d}{dx}(\log_e(x))=\frac{1}{x} \qquad \int \frac{1}{x}\,dx=\log_e|x|+c$$

$$\frac{d}{dx}(\sin(ax))=a\cos(ax) \qquad \int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$$

$$\frac{d}{dx}(\cos(ax))=-a\sin(ax) \qquad \int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$$

$$\frac{d}{dx}(\tan(ax))=a\,sec^2(ax) \qquad \int sec^2(ax)\,dx=\frac{1}{a}\tan(ax)+c$$

$$\frac{d}{dx}(\cot(ax))=-a\,cosec^2(ax) \qquad \int cosec^2(ax)\,dx=-\frac{1}{a}\cot(ax)+c$$

$$\frac{d}{dx}(sec(ax))=a\,sec(ax)\tan(ax) \qquad \int sec(ax)\tan(ax)\,dx=\frac{1}{a}sec(ax)+c$$

$$\frac{d}{dx}(cosec(ax))=-a\,cosec(ax)\cot(ax) \qquad \int cosec(ax)\cot(ax)\,dx=-\frac{1}{a}cosec(ax)+c$$

$$\frac{d}{dx}(\sin^{-1}(ax))=\frac{a}{\sqrt{1-(ax)^2}} \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c,\ a>0$$

$$\frac{d}{dx}(\cos^{-1}(ax))=\frac{-a}{\sqrt{1-(ax)^2}} \qquad \int \frac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+c,\ a>0$$

$$\frac{d}{dx}(\tan^{-1}(ax))=\frac{a}{1+(ax)^2} \qquad \int \frac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\int (ax+b)^n\,dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$$

$$\int \frac{1}{ax+b}\,dx=\frac{1}{a}\log_e|ax+b|+c$$

Calculus — rules and applications

product rule — $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule — $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule — $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

integration by parts: $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$

Euler's method: if $\frac{dy}{dx}=f(x,y)$, $x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+h\times f(x_n,y_n)$

arc length (parametric) $\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (Cartesian, about x-axis) $\int_{x_1}^{x_2}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

surface area (Cartesian, about y-axis) $\int_{y_1}^{y_2}2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

surface area (parametric, about x-axis) $\int_{t_1}^{t_2}2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (parametric, about y-axis) $\int_{t_1}^{t_2}2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration: $a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant acceleration formulas:

$v=u+at$ $\qquad s=ut+\frac{1}{2}at^2$

$v^2=u^2+2as$ $\qquad s=\frac{1}{2}(u+v)t$

Vectors in two and three dimensions

position vector: $r(t)=x(t)i+y(t)j+z(t)k$

magnitude: $|r(t)|=\sqrt{x(t)^2+y(t)^2+z(t)^2}$

derivative with respect to t: $\dot{r}(t)=\frac{dr}{dt}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$

For $r_1=x_1i+y_1j+z_1k$ and $r_2=x_2i+y_2j+z_2k$:

vector scalar product: $r_1\cdot r_2=|r_1||r_2|\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

vector cross product: $r_1\times r_2=\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=(y_1z_2-y_2z_1)i+(x_2z_1-x_1z_2)j+(x_1y_2-x_2y_1)k$

vector equation of a line: $r(t)=r_1+tr_2=(x_1+x_2t)i+(y_1+y_2t)j+(z_1+z_2t)k$

parametric equation of a line: $x(t)=x_1+x_2t,\ y(t)=y_1+y_2t,\ z(t)=z_1+z_2t$

vector equation of a plane: $r(s,t)=r_0+sr_1+tr_2=(x_0+x_1s+x_2t)i+(y_0+y_1s+y_2t)j+(z_0+z_1s+z_2t)k$

parametric equation of a plane: $x(s,t)=x_0+x_1s+x_2t,\ y(s,t)=y_0+y_1s+y_2t,\ z(s,t)=z_0+z_1s+z_2t$

Cartesian equation of a plane: $ax+by+cz=d$

Circular functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=sec^2(x)$

$\cot^2(x)+1=cosec^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$

$\sin^2(ax)=\dfrac{1}{2}(1-\cos(2ax))$

$\cos^2(ax)=\dfrac{1}{2}(1+\cos(2ax))$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

End of formula sheet

Exam 3 — Answers and solutions

Question 1.

a. For $P(1)$ the left-hand side is $\sum_{r=1}^{1} r \cdot 2^{r-1} = 1 \cdot 2^0 = 1$, and the right-hand side is $(1-1)2^1 + 1 = 0 + 1 = 1$. The two sides are equal, so $P(1)$ is true.

b. Assume $P(k)$ is true for some integer $k \ge 1$; that is, $\sum_{r=1}^{k} r \cdot 2^{r-1} = (k-1)2^k + 1$.

c. The $(k+1)$-th term is $(k+1)2^{(k+1)-1} = (k+1)2^k$. Adding it to both sides of the hypothesis,

$$\sum_{r=1}^{k+1} r \cdot 2^{r-1} = (k-1)2^k + 1 + (k+1)2^k = 2^k[(k-1)+(k+1)] + 1 = 2^k(2k) + 1 = k \cdot 2^{k+1} + 1.$$

Since $((k+1)-1)2^{k+1} + 1 = k \cdot 2^{k+1} + 1$, this is exactly $P(k+1)$. Hence $P(k) \Rightarrow P(k+1)$. As $P(1)$ is true and the implication holds for every $k \ge 1$, by the principle of mathematical induction the formula holds for all integers $n \ge 1$.

Question 2.

a. The variables start at $a=1$, $b=1$. Each pass computes $c = a+b$, then $a \leftarrow b$ and $b \leftarrow c$:

i	a	b	c
start	1	1	—
3	1	2	2
4	2	3	3
5	3	5	5
6	5	8	8

After the pass with $i=6$ the loop stops, and the output is $b=8$.

b. The successive values of b are $1, 1, 2, 3, 5, 8, \ldots$, in which each term is the sum of the two before it. This is the Fibonacci sequence (with the first two terms equal to 1), so for a general integer $n \ge 2$ the algorithm outputs the n-th Fibonacci number. (For $n=6$ this is 8, agreeing with part **a**.)

Question 3.

a. First $(1+2i)^2 = 1 + 4i + 4i^2 = 1 + 4i - 4 = -3 + 4i$, so

$$(1+2i)^3 = (1+2i)(-3+4i) = -3 + 4i - 6i + 8i^2 = -3 - 2i - 8 = -11 - 2i.$$

Therefore $P(1+2i) = (-11-2i) + (1+2i) + 10 = 0$, so $z = 1+2i$ is a solution.

b. Because P has real coefficients, its non-real roots occur in conjugate pairs, so $z = 1-2i$ is also a solution. Since $P(z) = z^3 + z + 10$ has no z^2 term, the sum of the three roots is 0. Hence the third root is $-[(1+2i)+(1-2i)] = -2$. (Check: $P(-2) = -8 - 2 + 10 = 0$.)

c. The conjugate pair gives the real quadratic factor

$$(z-(1+2i))(z-(1-2i)) = z^2 - 2z + (1^2 + 2^2) = z^2 - 2z + 5,$$

and the real root $z = -2$ gives the linear factor $(z+2)$. Therefore $P(z) = (z+2)(z^2 - 2z + 5)$.

Question 4.

$\overrightarrow{AB} = B - A = (2, 1, -1)$ and $\overrightarrow{AC} = C - A = (1, 2, 1)$.

a. Expanding the determinant,

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}=(1\cdot 1-(-1)\cdot 2)i-(2\cdot 1-(-1)\cdot 1)j+(2\cdot 2-1\cdot 1)k=3i-3j+3k.$$

b. The cross product is normal to the plane, and is parallel to $(1,-1,1)$. Using the point $A(1,0,2)$,

$$1(x-1)-1(y-0)+1(z-2)=0\Rightarrow x-y+z=3.$$

(Check: B gives $3-1+1=3$ and C gives $2-2+3=3$.)

c. The area of the triangle is half the magnitude of the cross product:

$$\text{area}=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{3^2+(-3)^2+3^2}=\frac{1}{2}\sqrt{27}=\frac{3\sqrt{3}}{2}.$$

Question 5.

a. With $u=\frac{x}{2}$, using $\frac{d}{dx}\tan^{-1}(u)=\frac{1}{1+u^2}\cdot\frac{du}{dx}$,

$$\frac{dy}{dx}=\frac{1/2}{1+(x/2)^2}=\frac{1/2}{\frac{4+x^2}{4}}=\frac{2}{4+x^2}.$$

b. Writing $\frac{dy}{dx}=2(4+x^2)^{-1}$ and differentiating with the chain rule,

$$\frac{d^2y}{dx^2}=2\cdot(-1)(4+x^2)^{-2}\cdot 2x=\frac{-4x}{(4+x^2)^2}.$$

This is zero only when $x=0$. For $x<0$ it is positive (the curve is concave up) and for $x>0$ it is negative (concave down), so the concavity changes at $x=0$ and there is a point of inflection there. Since $y=\tan^{-1}(0)=0$, the point of inflection is $(0,0)$.

Question 6.

a. Write $\frac{5}{(x-1)(x+4)}=\frac{A}{x-1}+\frac{B}{x+4}$, so that $5=A(x+4)+B(x-1)$. Putting $x=1$ gives $5=5A$, so $A=1$; putting $x=-4$ gives $5=-5B$, so $B=-1$. Hence

$$\int\frac{5}{(x-1)(x+4)}dx=\int\left(\frac{1}{x-1}-\frac{1}{x+4}\right)dx=\log_e|x-1|-\log_e|x+4|+c.$$

b. Integrating by parts with $u=x$ and $\frac{dv}{dx}=\cos(2x)$, so $\frac{du}{dx}=1$ and $v=\frac{1}{2}\sin(2x)$,

$$\int x\cos(2x)\,dx=\frac{1}{2}x\sin(2x)-\int\frac{1}{2}\sin(2x)\,dx=\frac{1}{2}x\sin(2x)+\frac{1}{4}\cos(2x)+c.$$

Question 7.

a. Using $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-9x$ and integrating with respect to x,

$$\frac{1}{2}v^2=\int-9x\,dx=-\frac{9}{2}x^2+C.$$

When $x=2$, $v=6$, so $\frac{1}{2}(36)=18=-\frac{9}{2}(4)+C=-18+C$, giving $C=36$. Then $\frac{1}{2}v^2=36-\frac{9}{2}x^2$, so $v^2=72-9x^2=9(8-x^2)$.

b. The particle is momentarily at rest ($v=0$) at the extremes of the motion: $9(8-x^2)=0$ gives $x^2=8$, so $x=\pm 2\sqrt{2}$. The amplitude is $2\sqrt{2}$ m.

c. Since $v^2=72-9x^2$ is greatest when x^2 is least, the maximum speed occurs at $x=0$, where $v^2=72$. The maximum speed is $\sqrt{72}=6\sqrt{2}$ m/s, at the origin.

Question 8.

a. Differentiating each component with respect to t,

$$\dot{r}(t)=-3\sin(t)i+3\cos(t)j+4k.$$

The speed is

$$|\dot{r}(t)|=\sqrt{9\sin^2(t)+9\cos^2(t)+16}=\sqrt{9+16}=\sqrt{25}=5,$$

which is independent of t, so the speed is constant at 5 m/s.

b. Differentiating again, $\ddot{r}(t)=-3\cos(t)i-3\sin(t)j$. Then

$$\ddot{r}(t)\cdot\dot{r}(t)=(-3\cos t)(-3\sin t)+(-3\sin t)(3\cos t)+(0)(4)=9\sin t\cos t-9\sin t\cos t=0$$

for all t. Hence the acceleration is always perpendicular to the velocity, consistent with the speed being constant.

Question 9.

a. By linearity of expectation, $E(X+Y)=E(X)+E(Y)=30+20=50$ minutes. Because X and Y are independent, the variances add:

$$Var(X+Y)=Var(X)+Var(Y)=4^2+3^2=16+9=25,$$

so the standard deviation of $X+Y$ is $\sqrt{25}=5$ minutes.

b. The mean is $E(X-Y)=30-20=10$ minutes. For independent variables the variances still add (the coefficient -1 is squared):

$$Var(X-Y)=Var(X)+(-1)^2Var(Y)=16+9=25.$$

Question 10.

The standard deviation of the sample mean is $\frac{\sigma}{\sqrt{n}}=\frac{0.4}{\sqrt{64}}=\frac{0.4}{8}=0.05$.

a. The margin of error is $z\times\frac{\sigma}{\sqrt{n}}=1.96\times 0.05=0.098$. The approximate 95% confidence interval for μ is

$$\bar{x}\pm 0.098=1.30\pm 0.098=(1.202, 1.398).$$

b. The test statistic is

$$z=\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}=\frac{1.30-1.35}{0.05}=\frac{-0.05}{0.05}=-1.$$

Since $|-1|=1<1.96$, the statistic does not lie in the rejection region, so H_0 is not rejected at the 5% level. There is insufficient evidence to conclude that the mean reaction time differs from 1.35 seconds.

Exam 4 — Technology-active (Easy)

Reading time: 15 minutes. Writing time: 2 hours. Section A: 20 multiple-choice questions (20 marks). Section B: extended-response (60 marks). Total 80 marks. One bound reference and one approved CAS technology are permitted. A formula sheet is provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Unless otherwise indicated, diagrams are not drawn to scale.

Section A — Multiple-choice

Answer all questions. Each correct answer is worth 1 mark; no mark is deducted for an incorrect answer. Choose the response that is best.

Question 1

Which of the following is a counterexample to the statement "for all real numbers x, $x^2 > x$"?

A. $x = 2$

B. $x = -1$

C. $x = \frac{1}{2}$

D. $x = 3$

Question 2

The negation of the statement "$\forall x \in \mathbb{R}, x^2 \geq 0$" is

A. $\exists x \in \mathbb{R}, x^2 < 0$

B. $\forall x \in \mathbb{R}, x^2 < 0$

C. $\exists x \in \mathbb{R}, x^2 \geq 0$

D. $\forall x \in \mathbb{R}, x^2 \leq 0$

Question 3

The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

A. $\frac{\pi}{6}$

B. $\frac{\pi}{2}$

C. $\frac{2\pi}{3}$

D. $\frac{\pi}{3}$

Question 4

If $\cos\theta = \frac{3}{5}$ and θ is acute, then $\sin(2\theta)$ equals

A. $\frac{7}{25}$

B. $\frac{24}{25}$

C. $\frac{12}{25}$

D. $-\frac{24}{25}$

Question 5

For $z = -\sqrt{3} - i$, the value of $Arg(z)$ is

A. $\frac{5\pi}{6}$

B. $-\frac{\pi}{6}$

C. $\frac{7\pi}{6}$

D. $-\frac{5\pi}{6}$

Question 6

If $z_1 = 2\,cis\frac{\pi}{4}$ and $z_2 = 3\,cis\frac{\pi}{6}$, then $z_1 z_2$ equals

A. $6\,cis\frac{5\pi}{12}$

B. $5\,cis\frac{5\pi}{12}$

C. $6\,cis\frac{\pi}{12}$

D. $6\,cis\frac{5\pi}{24}$

Question 7

Using De Moivre's theorem, $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^3$ equals

A. $\frac{\sqrt{3}}{2}+\frac{1}{2}i$

B. i

C. $-i$

D. 1

Question 8

If $y=\tan^{-1}(4x)$, then $\frac{dy}{dx}$ equals

A. $\frac{1}{1+16x^2}$

B. $\frac{4}{1+4x^2}$

C. $\frac{4}{1+16x^2}$

D. $\frac{4}{\sqrt{1-16x^2}}$

Question 9

The x-coordinate of the point of inflection of the curve $y=x^3-6x^2+5$ is

A. $x=2$

B. $x=0$

C. $x=4$

D. $x=-2$

Question 10

$\int\frac{1}{\sqrt{25-x^2}}dx$ equals

A. $\frac{1}{5}\sin^{-1}\left(\frac{x}{5}\right)+c$

B. $\cos^{-1}\left(\frac{x}{5}\right)+c$

C. $\tan^{-1}\left(\frac{x}{5}\right)+c$

D. $\sin^{-1}\left(\frac{x}{5}\right)+c$

Question 11

$\int_1^2 \frac{1}{x} dx$ equals

A. $\frac{1}{2}$

B. $\log_e 2$

C. $\log_e \frac{1}{2}$

D. 1

Question 12

The region bounded by $y=\sqrt{x}$, the x-axis and the line $x=4$ is rotated about the x-axis. The volume of the solid of revolution is

A. 4π

B. $\frac{64\pi}{3}$

C. 8π

D. 16π

Question 13

For the differential equation $\frac{dy}{dx}=2x-y$ with $y(1)=3$, one step of Euler's method with step size $h=0.1$ gives $y(1.1)$ approximately equal to

A. 2.8

B. 3.0

C. 3.1

D. 2.9

Question 14

A particle moves in a straight line with velocity $v=5t^2+2t$ (in m/s) at time t seconds. Its acceleration at $t=3$ seconds is

A. $10\,\text{m/s}^2$

B. $32\,\text{m/s}^2$

C. $30\,\text{m/s}^2$

D. $51\,\text{m/s}^2$

Question 15

The scalar product $(3i+2j-k)\cdot(i-4j+2k)$ equals

A. -7

B. -3

C. 9

D. 13

Question 16

A unit vector in the direction of $a=3i-4j$ is

A. $3i-4j$

B. $\frac{1}{7}(3i-4j)$

C. $\frac{4}{5}i-\frac{3}{5}j$

D. $\frac{3}{5}i-\frac{4}{5}j$

Question 17

A vector normal to the plane $2x-y+3z=6$ is

A. $2i+j+3k$

B. $2i-j+3k$

C. $i-j+k$

D. $6i$

Question 18

A particle has position vector $r(t)=(t^2+3t)i+(4-t^3)j$. Its velocity at $t=1$ is

A. $5i-3j$

B. $4i+3j$

C. $2i-3j$

D. $(2t+3)i-3t^2j$

Question 19

X and Y are independent random variables with $Var(X)=4$ and $Var(Y)=9$. Then $Var(X-Y)$ equals

A. 13

B. $\sqrt{13}$

C. 5

D. 25

Question 20

A random sample of size $n=25$ is taken from a normal population with standard deviation $\sigma=10$. The standard deviation of the sample mean $\overline{X}$ is

A. 10

B. $\sqrt{10}$

C. 2

D. 0.4

End of Section A

Section B — Extended-response

Answer all questions in the spaces provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (10 marks)

Let $z=1+\sqrt{3}i$.

a. Express z in the polar form $r\, cis\, \theta$, where $r>0$ and $-\pi<\theta\leq\pi$. 3 marks

b. Using De Moivre's theorem, show that $z^3=-8$. 3 marks

c. The equation $w^3=-8$ therefore has z as one solution. Find all three solutions of $w^3=-8$, giving each in the form $x+i\,y$, and plot them on an Argand diagram. 4 marks

Question 2 (10 marks)

Consider the vectors $a=2i+2j-k$ and $b=i-2j+2k$.

a. Find $|a|$ and $|b|$. 2 marks

b. Find $a \cdot b$, and hence find the angle between a and b, correct to the nearest degree. 3 marks

c. Find the scalar resolute (scalar projection) of a in the direction of b. 2 marks

d. Find $a\times b$, and hence write down a Cartesian equation of the plane through the origin that contains both a and b. 3 marks

Question 3 (11 marks)

The region R is bounded by the curve $y=\sin(x)$ and the x-axis for $0\leq x\leq\pi$.

a. Find $\frac{dy}{dx}$, and hence the gradient of the curve at the point where $x=\frac{\pi}{3}$. 2 marks

b. Find the exact area of the region R. 3 marks

c. The region R is rotated about the x-axis. Using the identity $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$, find the exact volume of the solid of revolution formed. 4 marks

d. State the maximum value of $y=\sin(x)$ on $0\leq x\leq\pi$, and the value of x at which it occurs. 2 marks

Question 4 (10 marks)

The temperature $T\,°\mathrm{C}$ of a liquid, t minutes after it begins to cool, satisfies the differential equation

$$\frac{dT}{dt} = -\frac{1}{5}(T-20), T(0)=80.$$

a. Show that $T = 20 + 60e^{-t/5}$ satisfies both the differential equation and the initial condition. 3 marks

b. Find the temperature of the liquid after 5 minutes, correct to one decimal place. 2 marks

A particle moves in a straight line. Its acceleration is $a = 6t - 4$ (in $\mathrm{m/s^2}$) at time t seconds. Initially the particle has velocity 1 m/s and is at the origin.

c. Find the velocity v of the particle in terms of t. 2 marks

d. Find the displacement x of the particle in terms of t, and hence its position when $t = 2$ seconds. 3 marks

Question 5 (9 marks)

a. Prove by mathematical induction that, for all integers $n \geq 1$,

$$\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}.$$

Set out your proof by verifying the base case, stating the inductive hypothesis, completing the inductive step, and stating the conclusion. 5 marks

Consider the algorithm below, in which n is a positive integer.

```
input n
s ← 0
d ← 1
for k from 1 to n do
    s ← s + d
    d ← d + 2
end for
output s
```

b. Complete a trace table with columns k, s and d recording their values after each pass of the loop, for $n = 5$, and state the output. 2 marks

c. Name the sequence of values taken by s, and write down, with a reason, the value the algorithm outputs for a general positive integer n. 2 marks

Question 6 (10 marks)

The mass, in grams, of an apple grown at an orchard is normally distributed with mean μ grams and known standard deviation $\sigma = 12$ grams. A random sample of 36 apples is taken and found to have a sample mean mass of $\bar{x} = 205$ grams.

a. Find an approximate 95% confidence interval for μ, giving the endpoints correct to one decimal place. (Use $z = 1.96$.) 3 marks

b. Find the standard deviation of the sample mean $\overline{X}$ if the sample size were increased to 144, and state the effect this has on the width of the 95% confidence interval. 2 marks

c. It is claimed that $\mu = 200$ grams. To test this claim against the alternative $\mu \neq 200$ at the 5% level of significance, calculate the value of the test statistic z based on $\bar{x} = 205$, and state your conclusion. 3 marks

d. Find the p-value for the test in part **c.**, correct to four decimal places, and confirm your conclusion. 2 marks

End of Exam 4

Exam 4 Formula Sheet

Formula sheet

This formula sheet is provided with every examination in this booklet.

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$
volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$
area of a triangle	$\frac{1}{2} b c \sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3} A h$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\,cis(\theta)$

modulus	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$
argument	$-\pi < Arg(z) \leq \pi$
product	$z_1 z_2 = r_1 r_2\, cis(\theta_1 + \theta_2)$
quotient	$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$
de Moivre's theorem	$z^n = r^n cis(n\theta)$

Data analysis, probability and statistics

For independent random variables $X_1, X_2 \ldots X_n$:

$E(aX_1 + b) = aE(X_1) + b$

$E(a_1 X_1 + a_2 X_2 + \ldots + a_n X_n) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$

$Var(aX_1 + b) = a^2 Var(X_1)$

$Var(a_1 X_1 + a_2 X_2 + \ldots + a_n X_n) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$

For independent identically distributed variables $X_1, X_2 \ldots X_n$:

$E(X_1 + X_2 + \ldots + X_n) = n\mu\ Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$

mean of sample mean $\overline{X}$ $\quad E(\overline{X})=\mu$

variance of sample mean $\overline{X}$ $\quad Var(\overline{X})=\frac{\sigma^2}{n}$

Calculus — derivatives and integrals

$$\frac{d}{dx}(x^n)=nx^{n-1} \qquad \int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$$

$$\frac{d}{dx}(e^{ax})=ae^{ax} \qquad \int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$$

$$\frac{d}{dx}(\log_e(x))=\frac{1}{x} \qquad \int \frac{1}{x}\,dx=\log_e|x|+c$$

$$\frac{d}{dx}(\sin(ax))=a\cos(ax) \qquad \int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$$

$$\frac{d}{dx}(\cos(ax))=-a\sin(ax) \qquad \int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$$

$$\frac{d}{dx}(\tan(ax))=a\,sec^2(ax) \qquad \int sec^2(ax)\,dx=\frac{1}{a}\tan(ax)+c$$

$$\frac{d}{dx}(\cot(ax))=-a\,cosec^2(ax) \qquad \int cosec^2(ax)\,dx=-\frac{1}{a}\cot(ax)+c$$

$$\frac{d}{dx}(sec(ax))=a\,sec(ax)\tan(ax) \qquad \int sec(ax)\tan(ax)\,dx=\frac{1}{a}sec(ax)+c$$

$$\frac{d}{dx}(cosec(ax))=-a\,cosec(ax)\cot(ax) \qquad \int cosec(ax)\cot(ax)\,dx=-\frac{1}{a}cosec(ax)+c$$

$$\frac{d}{dx}(\sin^{-1}(ax))=\frac{a}{\sqrt{1-(ax)^2}} \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c,\ a>0$$

$$\frac{d}{dx}(\cos^{-1}(ax))=\frac{-a}{\sqrt{1-(ax)^2}} \qquad \int \frac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+c,\ a>0$$

$$\frac{d}{dx}(\tan^{-1}(ax))=\frac{a}{1+(ax)^2} \qquad \int \frac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\int (ax+b)^n\,dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$$

$$\int \frac{1}{ax+b}\,dx=\frac{1}{a}\log_e|ax+b|+c$$

Calculus — rules and applications

product rule $\quad \frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\quad \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\quad \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

integration by parts: $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$

Euler's method: if $\frac{dy}{dx}=f(x,y)$, $x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+h\times f(x_n,y_n)$

arc length (parametric) $\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (Cartesian, about x-axis) $\int_{x_1}^{x_2}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

surface area (Cartesian, about y-axis) $\int_{y_1}^{y_2}2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

surface area (parametric, about x-axis) $\int_{t_1}^{t_2}2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (parametric, about y-axis) $\int_{t_1}^{t_2}2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration: $a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant acceleration formulas:

$v=u+at$ $\qquad s=ut+\frac{1}{2}at^2$

$v^2=u^2+2as$ $\qquad s=\frac{1}{2}(u+v)t$

Vectors in two and three dimensions

position vector: $r(t)=x(t)i+y(t)j+z(t)k$

magnitude: $|r(t)|=\sqrt{x(t)^2+y(t)^2+z(t)^2}$

derivative with respect to t: $\dot{r}(t)=\frac{dr}{dt}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$

For $r_1=x_1i+y_1j+z_1k$ and $r_2=x_2i+y_2j+z_2k$:

vector scalar product: $r_1\cdot r_2=|r_1||r_2|\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

vector cross product: $r_1\times r_2=\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=(y_1z_2-y_2z_1)i+(x_2z_1-x_1z_2)j+(x_1y_2-x_2y_1)k$

vector equation of a line: $r(t)=r_1+tr_2=(x_1+x_2t)i+(y_1+y_2t)j+(z_1+z_2t)k$

parametric equation of a line: $x(t)=x_1+x_2t,\ y(t)=y_1+y_2t,\ z(t)=z_1+z_2t$

vector equation of a plane: $r(s,t)=r_0+sr_1+tr_2=(x_0+x_1s+x_2t)i+(y_0+y_1s+y_2t)j+(z_0+z_1s+z_2t)k$

parametric equation of a plane: $x(s,t)=x_0+x_1s+x_2t,\ y(s,t)=y_0+y_1s+y_2t,\ z(s,t)=z_0+z_1s+z_2t$

Cartesian equation of a plane: $ax+by+cz=d$

Circular functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=sec^2(x)$

$\cot^2(x)+1=cosec^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$

$\sin^2(ax)=\dfrac{1}{2}(1-\cos(2ax))$

$\cos^2(ax)=\dfrac{1}{2}(1+\cos(2ax))$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

End of formula sheet

Exam 4 — Answers and solutions

Section A — Multiple-choice answer key

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Answer	C	A	D	B	D	A	B	C	A	D	B	C	D	B	A	D	B	A	A	C

Reasons.

1. **C** — $x=1/2$ gives $1/4>1/2$, which is false; the other values all satisfy $x^2>x$.
2. **A** — negating $\forall$ gives $\exists$, and the negation of $x^2\geq 0$ is $x^2<0$.
3. **D** — $\sin(\pi/3)=\sqrt{3}/2$ and $\pi/3$ lies in the range $[-\pi/2,\pi/2]$ of $\sin^{-1}$.
4. **B** — $\sin\theta=4/5$, so $\sin(2\theta)=2\sin\theta\cos\theta=2\cdot 4/5\cdot 3/5=24/25$.
5. **D** — z lies in the third quadrant with reference angle $\tan^{-1}(1/\sqrt{3})=\pi/6$, so $Arg(z)=-(\pi-\pi/6)=-5\pi/6$.
6. **A** — moduli multiply ($2\times 3=6$) and arguments add ($\pi/4+\pi/6=5\pi/12$).
7. **B** — by De Moivre's theorem the result is $cis(3\times\pi/6)=cis\,\pi/2=i$.
8. **C** — chain rule: $\dfrac{d}{dx}\tan^{-1}(4x)=\dfrac{4}{1+(4x)^2}=\dfrac{4}{1+16x^2}$.
9. **A** — $y''=6x-12=0$ at $x=2$, where y'' changes sign.
10. **D** — standard form $\int\dfrac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\dfrac{x}{a}\right)+c$ with $a=5$.
11. **B** — $[\log_e|x|]_1^2=\log_e 2-\log_e 1=\log_e 2$.
12. **C** — $V=\pi\int_0^4(\sqrt{x})^2dx=\pi\int_0^4 x\,dx=\pi\cdot 16/2=8\pi$.
13. **D** — $y_1=y_0+hf(x_0,y_0)=3+0.1(2(1)-3)=3-0.1=2.9$.
14. **B** — $a=\dfrac{dv}{dt}=10t+2$; at $t=3$, $a=32\,\text{m/s}^2$.
15. **A** — $3(1)+2(-4)+(-1)(2)=3-8-2=-7$.
16. **D** — $|a|=\sqrt{9+16}=5$, so the unit vector is $1/5(3i-4j)=3/5i-4/5j$.
17. **B** — the coefficients of x, y, z give the normal $2i-j+3k$.
18. **A** — $\dot{r}(t)=(2t+3)i-3t^2j$; at $t=1$ this is $5i-3j$.
19. **A** — for independent variables $Var(X-Y)=Var(X)+Var(Y)=4+9=13$.
20. **C** — $sd(\overline{X})=\dfrac{\sigma}{\sqrt{n}}=\dfrac{10}{\sqrt{25}}=2$.

Section B — worked solutions

Question 1.

a. The modulus is $r=|z|=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{1+3}=2$. Since $z=1+\sqrt{3}i$ lies in the first quadrant, $Arg(z)=\tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right)=\dfrac{\pi}{3}$. Hence

$$z=2\,cis\frac{\pi}{3}.$$

b. By De Moivre's theorem,

$$z^3 = 2^3 cis\left(3 \times \frac{\pi}{3}\right) = 8\,cis(\pi) = 8(\cos(\pi) + i\sin(\pi)) = 8(-1+0i) = -8.$$

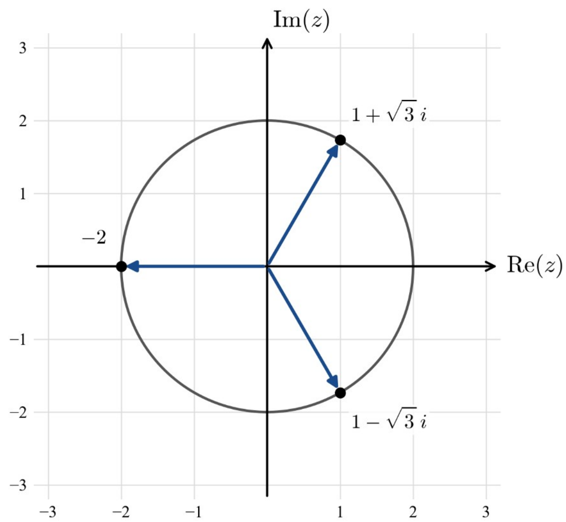
c. Write $-8 = 8\,cis(\pi)$. The solutions of $w^3 = -8$ have modulus $8^{1/3} = 2$ and arguments $\dfrac{\pi + 2k\pi}{3}$ for $k = 0, 1, 2$:

$$w = 2\,cis\frac{\pi}{3}, 2\,cis(\pi), 2\,cis\left(-\frac{\pi}{3}\right).$$

Converting to Cartesian form,

$$w = 1 + \sqrt{3}i, -2, 1 - \sqrt{3}i.$$

The three solutions lie on a circle of radius 2, equally spaced $120°$ apart.



Question 2.

a. $|a| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$ and $|b| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$.

b. $a \cdot b = (2)(1) + (2)(-2) + (-1)(2) = 2 - 4 - 2 = -4$. Then

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{-4}{3 \times 3} = -\frac{4}{9},$$

so $\theta = \cos^{-1}\left(-\dfrac{4}{9}\right) \approx 116°$ (to the nearest degree).

c. The scalar resolute of a in the direction of b is

$$\frac{a \cdot b}{|b|} = \frac{-4}{3} = -\frac{4}{3}.$$

d. The cross product is

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 1 & -2 & 2 \end{vmatrix} = (2(2) - (-1)(-2))i - (2(2) - (-1)(1))j + (2(-2) - 2(1))k = 2i - 5j - 6k.$$

This vector is normal to the plane, so a Cartesian equation of the plane through the origin containing a and b is

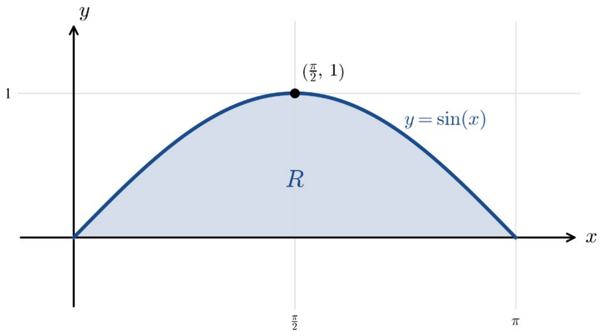
$$2x - 5y - 6z = 0.$$

Question 3.

a. $\frac{dy}{dx}=\cos(x)$. At $x=\frac{\pi}{3}$, the gradient is $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$.

b. The area of R is

$$\int_0^{\pi}\sin(x)\,dx=[-\cos(x)]_0^{\pi}=-\cos(\pi)-(-\cos(0))=-(-1)+1=2.$$



c. Rotating R about the x-axis gives

$$V=\pi\int_0^{\pi}\sin^2(x)\,dx=\pi\int_0^{\pi}\frac{1}{2}(1-\cos(2x))\,dx=\frac{\pi}{2}\left[x-\frac{1}{2}\sin(2x)\right]_0^{\pi}=\frac{\pi}{2}(\pi-0)=\frac{\pi^2}{2}.$$

d. On $0\le x\le\pi$, the maximum value of $y=\sin(x)$ is 1, occurring at $x=\frac{\pi}{2}$.

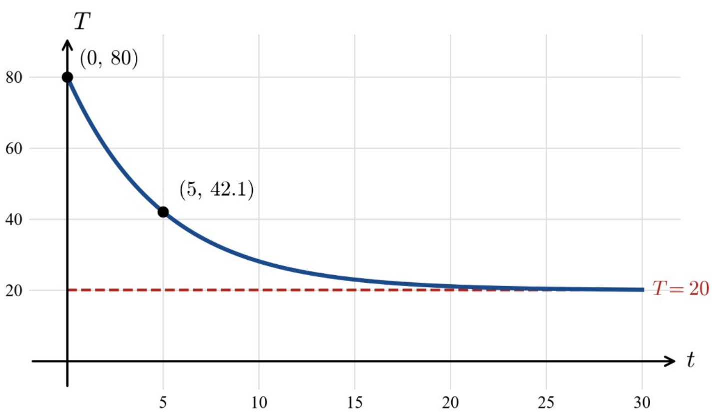
Question 4.

a. Differentiating $T=20+60e^{-t/5}$ gives

$$\frac{dT}{dt}=60\times\left(-\frac{1}{5}\right)e^{-t/5}=-12e^{-t/5}.$$

Also $-\frac{1}{5}(T-20)=-\frac{1}{5}\left(60e^{-t/5}\right)=-12e^{-t/5}$, so the differential equation is satisfied. At $t=0$, $T=20+60e^{0}=20+60=80$, so the initial condition holds.

b. $T(5)=20+60e^{-5/5}=20+60e^{-1}=20+60(0.367879\ldots)=42.07\ldots\approx 42.1\,°\text{C}$.



c. Integrating the acceleration, $v=\int(6t-4)\,dt=3t^2-4t+C$. Since $v=1$ when $t=0$, $C=1$, so

$$v=3t^2-4t+1.$$

d. Integrating the velocity, $x=\int\left(3t^2-4t+1\right)dt=t^3-2t^2+t+D$. Since $x=0$ when $t=0$, $D=0$, so

$$x=t^3-2t^2+t.$$

When $t=2$, $x=8-8+2=2$, so the particle is 2 m from the origin (on the positive side).

Question 5.

a. Let $P(n)$ be the statement $\sum_{r=1}^{n} r(r+1)=\frac{n(n+1)(n+2)}{3}$.

Base case. For $n=1$, the left-hand side is $1(1+1)=2$ and the right-hand side is $\frac{1\times2\times3}{3}=2$. The sides agree, so $P(1)$ is true.

Inductive hypothesis. Assume $P(k)$ is true for some integer $k\geq1$; that is,

$$\sum_{r=1}^{k} r(r+1)=\frac{k(k+1)(k+2)}{3}.$$

Inductive step. Adding the $(k+1)$-th term $(k+1)(k+2)$ to both sides,

$$\sum_{r=1}^{k+1} r(r+1)=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)=(k+1)(k+2)\left(\frac{k}{3}+1\right)=\frac{(k+1)(k+2)(k+3)}{3}.$$

This is $P(k+1)$, so $P(k)\Rightarrow P(k+1)$.

Conclusion. Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for every $k\geq1$, by the principle of mathematical induction $P(n)$ is true for all integers $n\geq1$.

b. Starting from $s=0$, $d=1$, each pass adds the current value of d to s and then increases d by 2:

k	s after pass	d after pass
1	1	3
2	4	5
3	9	7
4	16	9
5	25	11

After the fifth pass $s=25$, so the output is 25.

c. The successive values of s are $1,4,9,16,25$ — the perfect squares. The variable d runs through the odd numbers $1,3,5,\ldots,2n-1$, so after n passes s is the sum of the first n odd numbers. That sum is n^2, so for a general positive integer n the algorithm outputs n^2.

Question 6.

a. The standard deviation of the sample mean is $\frac{\sigma}{\sqrt{n}}=\frac{12}{\sqrt{36}}=2$. The margin of error is $z\times\frac{\sigma}{\sqrt{n}}=1.96\times2=3.92$, so an approximate 95% confidence interval for μ is

$$(205-3.92,205+3.92)=(201.08,208.92)\approx(201.1,208.9).$$

b. For $n=144$, $sd(\overline{X})=\frac{12}{\sqrt{144}}=1$ gram, half the value of 2 grams used in part **a.** (the standard deviation is halved when the sample size is quadrupled). The margin of error becomes $1.96\times1=1.96$, so the confidence interval is half as wide as the one found in part **a.**

c. Under $H_0 : \mu = 200$ the test statistic is

$$z = \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}} = \frac{205 - 200}{2} = 2.5.$$

For a two-tailed test at the 5% level the critical value is 1.96. Since $|2.5| > 1.96$, we reject H_0: there is evidence at the 5% level that the mean mass differs from 200 grams.

d. The p-value is

$$p = 2(1 - \Phi(2.5)) = 2(1 - 0.99379) = 2(0.00621) = 0.0124.$$

Since $0.0124 < 0.05$, we reject H_0, confirming the conclusion of part **c.**

Exam 5 — Technology-active (Medium)

Reading time: 15 minutes. Writing time: 2 hours. Section A: 20 multiple-choice questions (20 marks). Section B: extended-response (60 marks). Total 80 marks. One bound reference and one approved CAS technology are permitted. A formula sheet is provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Unless otherwise indicated, diagrams are not drawn to scale.

Section A — Multiple-choice

Answer all questions. Each correct answer is worth 1 mark; no mark is deducted for an incorrect answer. Choose the response that is best.

Question 1

Consider the following algorithm, in which s is an integer variable.

```
s ← 0
for i from 1 to 5 do
    s ← s + i × i
end for
print s
```

The value printed is

A. 15

B. 30

C. 55

D. 25

Question 2

Which one of the following expressions is divisible by 6 for **every** positive integer n?

A. n^2+n

B. n^3-n

C. n^2-n

D. n^3+n

Question 3

The graph of $y=\dfrac{3x+1}{x-2}$ has asymptotes

A. $x=2, y=3$

B. $x=-2, y=3$

C. $x=2, y=1$

D. $x=3, y=2$

Question 4

The non-vertical asymptote of the graph of $y=\dfrac{x^2+1}{x-1}$ is

A. $y=x+2$

B. $y=x$

C. $y=x+1$

D. $y=x-1$

Question 5

If $z=3\,cis\left(\dfrac{\pi}{4}\right)$, then z^4 equals

A. 81

B. -81

C. $81i$

D. $-81i$

Question 6

If $z=3-3i$, then $Arg(z)$ equals

A. $\dfrac{\pi}{4}$

B. $\dfrac{3\pi}{4}$

C. $-\dfrac{3\pi}{4}$

D. $-\dfrac{\pi}{4}$

Question 7

The set of points z in the complex plane for which $|z-(1+i)|=|z-(3-i)|$ is a straight line with equation

A. $y=x+2$

B. $y=-x+2$

C. $y=-x-2$

D. $y=x-2$

Question 8

The angle between the vectors $a = i + 2j + 2k$ and $b = 2i - 2j + k$ is

A. 45°

B. 60°

C. 90°

D. 0°

Question 9

The area of the triangle with sides represented by $a = 2i + j - k$ and $b = i + 3j + k$ is

A. $5\sqrt{2}$

B. $\frac{25}{2}$

C. 5

D. $\frac{5\sqrt{2}}{2}$

Question 10

The distance from the point $(4, -2, 1)$ to the plane $x - 2y + 2z = 3$ is

A. $\frac{13}{3}$

B. $\frac{7}{9}$

C. $\frac{7}{3}$

D. 7

Question 11

A particle has position vector $r(t) = t^3 i + 4t\, j$, where $t \geq 0$. Its speed at $t = 1$ is

A. 7

B. 5

C. $\sqrt{7}$

D. 25

Question 12

$\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$ equals

A. $-\frac{1}{\sqrt{4-x^2}}$

B. $\frac{1}{\sqrt{4-x^2}}$

C. $-\frac{1}{2\sqrt{1-x^2}}$

D. $-\frac{2}{\sqrt{4-x^2}}$

Question 13

$\int \frac{1}{x^2+9}\,dx$ equals

A. $3\tan^{-1}\left(\frac{x}{3}\right)+c$

B. $\tan^{-1}\left(\frac{x}{3}\right)+c$

C. $\frac{1}{3}\tan^{-1}(3x)+c$

D. $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$

Question 14

$\int_0^{\pi/2} \sin^2(x)\,dx$ equals

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. $\frac{1}{2}$

D. π

Question 15

The solution of the differential equation $\frac{dy}{dx} = xy$ for which $y = 2$ when $x = 0$ is

A. $y = 2e^{x}$

B. $y = 2e^{x^2/2}$

C. $y = e^{x^2/2} + 1$

D. $y = 2e^{x^2}$

Question 16

A particle moves in a straight line with velocity $v = 3t^2 - 12t + 9$ (in m/s), where $t \geq 0$ is in seconds. The particle is momentarily at rest when

A. $t = 1$ or $t = 3$

B. $t = 2$

C. $t = 3$ only

D. $t = 1$ only

Question 17

Euler's method with step size $h = 0.2$ is used to solve $\frac{dy}{dx} = x^2 - y$ with $y = 2$ when $x = 1$. The estimate of y when $x = 1.2$ is

A. 2.2

B. 1.6

C. 1.8

D. 2.0

Question 18

The independent normal random variables $X \sim N(50, 8^2)$ and $Y \sim N(30, 15^2)$. The standard deviation of $X + Y$ is

A. 23

B. $\sqrt{23}$

C. 289

D. 17

Question 19

A random sample of 36 observations is taken from a population that is $N(80, 15^2)$. The probability that the sample mean $\acute{X}$ exceeds 85 is closest to

A. 0.1587

B. 0.0228

C. 0.3446

D. 0.0013

Question 20

A population has known standard deviation $\sigma = 20$. For a random sample of size $n = 16$, the margin of error of a 95% confidence interval for the population mean is closest to

A. 9.80

B. 1.96

C. 5.00

D. 19.60

End of Section A

Section B — Extended-response

Answer all questions. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Total 60 marks.

Question 1 (10 marks)

Let $z = \sqrt{3} + i$.

a. Express z in polar form $r\,cis(\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. 2 marks

b. Hence find z^6, giving your answer in the form $a + b\,i$. 2 marks

c. Find the three cube roots of z, giving each in polar form $r\,cis(\theta)$ with $-\pi < \theta \le \pi$. 3 marks

d. Plot the three cube roots from part **c.** on an Argand diagram, and describe the polygon whose vertices are these three points. 3 marks

Question 2 (11 marks)

Two straight lines ℓ_1 and ℓ_2 have vector equations

$$r_1(s) = (2i + 3k) + s\,u, r_2(t) = (-i - j) + t\,v,$$

where $u = i + j - k$, $v = 2i + j + k$ and $s, t \in \mathbb{R}$.

a. Show that ℓ_1 and ℓ_2 intersect, and find the coordinates of their point of intersection P. 3 marks

b. Find the acute angle between ℓ_1 and ℓ_2, correct to the nearest degree. 2 marks

c. Find $u \times v$, and hence show that the plane Π containing both lines has Cartesian equation $2x - 3y - z = 1$. 3 marks

d. A third line m has vector equation $r(\lambda) = (4j + k) + \lambda(2i - 3j - k)$. Explain why m is perpendicular to Π, and find the coordinates of the point at which m meets Π. 3 marks

Question 3 (10 marks)

A quantity $y > 0$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{y}{x(x+1)}, x > 0,$$

with $y = 1$ when $x = 1$.

a. Express $\frac{1}{x(x+1)}$ in partial fractions. 2 marks

b. Hence solve the differential equation, showing that $y = \frac{2x}{x+1}$. 4 marks

c. State the value that y approaches as $x \to \infty$. 2 marks

d. Sketch the graph of $y = \frac{2x}{x+1}$ for $x > 0$, labelling the point $(1, 1)$ and the horizontal asymptote. 2 marks

Question 4 (10 marks)

A particle moves in a straight line. Its acceleration is $a = -9x$ (in $\mathrm{m/s^2}$), where x metres is its displacement from a fixed point O. When the particle is at O its velocity is $6\,\mathrm{m/s}$ in the positive direction.

a. Using $a = \frac{d}{dx}\left(1/2\, v^2\right)$, show that $v^2 = 36 - 9x^2$. 3 marks

b. Find the amplitude of the motion. 2 marks

c. Find the exact speed of the particle when $x = 1$. 2 marks

d. The displacement is also given by $x(t) = 2\sin(3t)$, where $t \geq 0$ is in seconds and $x(0) = 0$. Find the first time $t > 0$ at which the particle is momentarily at rest. 3 marks

Question 5 (10 marks)

A machine fills bags with coffee. The mass, X grams, of a bag is normally distributed with mean 250 and standard deviation 4; bags are filled independently.

a. A carton contains 12 bags. Let $T = X_1 + X_2 + \cdots + X_{12}$ be the total mass of the bags in a carton. Find $E(T)$ and $sd(T)$, giving the standard deviation in exact form. 2 marks

b. Find $P(T < 2980)$, correct to four decimal places. 2 marks

c. A quality inspector weighs a random sample of 25 bags and finds a sample mean of $\acute{x} = 248.6$ g. Taking $\sigma = 4$ as known, find a 95% confidence interval for the population mean fill μ, with endpoints correct to two decimal places. 3 marks

d. The manufacturer claims that $\mu = 250$ g. Using the sample in part **c.**, carry out a two-tailed test at the 5% level of significance. Write down the null and alternative hypotheses, evaluate the test statistic, give the p-value correct to four decimal places, and say what the inspector should conclude about the claim. 3 marks

Question 6 (9 marks)

Consider the statement: for every positive integer n, the number $7^n - 1$ is divisible by 6.

a. Verify the statement for $n = 1$ and for $n = 2$. 2 marks

b. Prove the statement by mathematical induction. 5 marks

c. Hence find the remainder when 7^{100} is divided by 6. 2 marks

End of Exam 5

Exam 5 Formula Sheet

Formula sheet

This formula sheet is provided with every examination in this booklet.

Mensuration

area of a circle segment	$\frac{r^2}{2}\left(\theta - \sin(\theta)\right)$
volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$
area of a triangle	$\frac{1}{2} b c \sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3} A h$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r\left(\cos(\theta) + i\sin(\theta)\right) = r\,cis(\theta)$

modulus	$\lvert z \rvert = \sqrt{x^2 + y^2} = r$
argument	$-\pi < Arg(z) \le \pi$
product	$z_1 z_2 = r_1 r_2\, cis(\theta_1 + \theta_2)$
quotient	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, cis(\theta_1 - \theta_2)$
de Moivre's theorem	$z^n = r^n\, cis(n\theta)$

Data analysis, probability and statistics

For independent random variables $X_1, X_2 \ldots X_n$:

$E(aX_1 + b) = aE(X_1) + b$

$E(a_1 X_1 + a_2 X_2 + \ldots + a_n X_n) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$

$Var(aX_1 + b) = a^2 Var(X_1)$

$Var(a_1 X_1 + a_2 X_2 + \ldots + a_n X_n) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$

For independent identically distributed variables $X_1, X_2 \ldots X_n$:

$E(X_1 + X_2 + \ldots + X_n) = n\mu \; Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$

mean of sample mean $\overline{X}$ $E(\overline{X})=\mu$

variance of sample mean $\overline{X}$ $Var(\overline{X})=\frac{\sigma^2}{n}$

Calculus — derivatives and integrals

$\frac{d}{dx}(x^n)=nx^{n-1}$ $\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,n\neq-1$

$\frac{d}{dx}(e^{ax})=ae^{ax}$ $\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$

$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$ $\int\frac{1}{x}\,dx=\log_e|x|+c$

$\frac{d}{dx}(\sin(ax))=a\cos(ax)$ $\int\sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$

$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$ $\int\cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$

$\frac{d}{dx}(\tan(ax))=a\,sec^2(ax)$ $\int sec^2(ax)\,dx=\frac{1}{a}\tan(ax)+c$

$\frac{d}{dx}(\cot(ax))=-a\,cosec^2(ax)$ $\int cosec^2(ax)\,dx=-\frac{1}{a}\cot(ax)+c$

$\frac{d}{dx}(sec(ax))=a\,sec(ax)\tan(ax)$ $\int sec(ax)\tan(ax)\,dx=\frac{1}{a}sec(ax)+c$

$\frac{d}{dx}(cosec(ax))=-a\,cosec(ax)\cot(ax)$ $\int cosec(ax)\cot(ax)\,dx=-\frac{1}{a}cosec(ax)+c$

$\frac{d}{dx}(\sin^{-1}(ax))=\frac{a}{\sqrt{1-(ax)^2}}$ $\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c,a>0$

$\frac{d}{dx}(\cos^{-1}(ax))=\frac{-a}{\sqrt{1-(ax)^2}}$ $\int\frac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+c,a>0$

$\frac{d}{dx}(\tan^{-1}(ax))=\frac{a}{1+(ax)^2}$ $\int\frac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\frac{x}{a}\right)+c$

$\int(ax+b)^n\,dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,n\neq-1$

$\int\frac{1}{ax+b}\,dx=\frac{1}{a}\log_e|ax+b|+c$

Calculus — rules and applications

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

integration by parts: $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$

Euler's method: if $\frac{dy}{dx}=f(x,y)$, $x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+h\times f(x_n,y_n)$

arc length (parametric) $\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (Cartesian, about x-axis) $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

surface area (Cartesian, about y-axis) $\int_{y_1}^{y_2} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

surface area (parametric, about x-axis) $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

surface area (parametric, about y-axis) $\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration: $a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant acceleration formulas:

$v=u+at$ $\qquad s=ut+\frac{1}{2}at^2$

$v^2=u^2+2as$ $\qquad s=\frac{1}{2}(u+v)t$

Vectors in two and three dimensions

position vector: $r(t)=x(t)i+y(t)j+z(t)k$

magnitude: $|r(t)|=\sqrt{x(t)^2+y(t)^2+z(t)^2}$

derivative with respect to t: $\dot{r}(t)=\frac{dr}{dt}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$

For $r_1=x_1i+y_1j+z_1k$ and $r_2=x_2i+y_2j+z_2k$:

vector scalar product: $r_1\cdot r_2=|r_1||r_2|\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

vector cross product: $r_1\times r_2=\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2\end{vmatrix}=(y_1z_2-y_2z_1)i+(x_2z_1-x_1z_2)j+(x_1y_2-x_2y_1)k$

vector equation of a line: $r(t)=r_1+tr_2=(x_1+x_2t)i+(y_1+y_2t)j+(z_1+z_2t)k$

parametric equation of a line: $x(t)=x_1+x_2t,\ y(t)=y_1+y_2t,\ z(t)=z_1+z_2t$

vector equation of a plane: $r(s,t)=r_0+sr_1+tr_2=(x_0+x_1s+x_2t)i+(y_0+y_1s+y_2t)j+(z_0+z_1s+z_2t)k$

parametric equation of a plane: $x(s,t)=x_0+x_1s+x_2t,\ y(s,t)=y_0+y_1s+y_2t,\ z(s,t)=z_0+z_1s+z_2t$

Cartesian equation of a plane: $ax+by+cz=d$

Circular functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=sec^2(x)$

$\cot^2(x)+1=cosec^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$

$\sin^2(ax)=\dfrac{1}{2}(1-\cos(2ax))$

$\cos^2(ax)=\dfrac{1}{2}(1+\cos(2ax))$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

End of formula sheet

Exam 5 — Answers and solutions

Section A — Multiple-choice answer key

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans	C	B	A	C	B	D	D	C	D	C	B	A	D	A	B	A	C	D	B	A

Section A — reasons

1. **C.** $s=1^2+2^2+3^2+4^2+5^2=1+4+9+16+25=55$.
2. **B.** $n^3-n=(n-1)n(n+1)$ is a product of three consecutive integers, so it is divisible by both 2 and 3, hence by 6. Each other option has a counterexample: n^2+n and n^3+n both equal 2 at $n=1$, and n^2-n equals 2 at $n=2$.
3. **A.** Vertical asymptote where $x-2=0$, so $x=2$; as $x\to\infty$, $y\to\frac{3x}{x}=3$, so $y=3$.
4. **C.** Dividing, $\frac{x^2+1}{x-1}=x+1+\frac{2}{x-1}$, so the oblique asymptote is $y=x+1$.
5. **B.** $z^4=3^4 cis(4\cdot\pi/4)=81\,cis(\pi)=-81$.
6. **D.** $3-3i$ lies in the fourth quadrant with $\tan\theta=\frac{-3}{3}=-1$, so $Arg(z)=-\frac{\pi}{4}$.
7. **D.** The locus is the perpendicular bisector of $(1,1)$ and $(3,-1)$. Equating $(x-1)^2+(y-1)^2=(x-3)^2+(y+1)^2$ gives $x-y-2=0$, i.e. $y=x-2$.
8. **C.** $a\cdot b=2-4+2=0$, so the vectors are perpendicular: the angle is 90°.
9. **D.** $a\times b=4i-3j+5k$, $|a\times b|=\sqrt{50}=5\sqrt{2}$; area $=1/2|a\times b|=\frac{5\sqrt{2}}{2}$.
10. **C.** Distance $=\frac{|1(4)-2(-2)+2(1)-3|}{\sqrt{1^2+(-2)^2+2^2}}=\frac{|7|}{3}=\frac{7}{3}$.
11. **B.** $\dot{r}(t)=3t^2 i+4j$; at $t=1$, $\dot{r}=3i+4j$ and speed $=\sqrt{3^2+4^2}=5$.
12. **A.** $\frac{d}{dx}\cos^{-1}\left(\frac{x}{2}\right)=\frac{-1/2}{\sqrt{1-(x/2)^2}}=\frac{-1/2}{1/2\sqrt{4-x^2}}=-\frac{1}{\sqrt{4-x^2}}$.
13. **D.** $\int\frac{1}{x^2+3^2}dx=\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$.
14. **A.** Using $\sin^2 x=1/2(1-\cos 2x)$, $\int_0^{\pi/2}\sin^2 x\,dx=[x/2-\sin 2x/4]_0^{\pi/2}=\frac{\pi}{4}$.
15. **B.** Separating, $\int\frac{1}{y}dy=\int x\,dx$ gives $\log_e y=x^2/2+c$; with $y(0)=2$, $y=2e^{x^2/2}$.
16. **A.** $v=3(t^2-4t+3)=3(t-1)(t-3)=0$ at $t=1$ and $t=3$.
17. **C.** $y_1=y_0+hf(x_0,y_0)=2+0.2(1^2-2)=2-0.2=1.8$.
18. **D.** $Var(X+Y)=8^2+15^2=64+225=289$, so $sd(X+Y)=\sqrt{289}=17$.
19. **B.** $\acute{X}\sim N(80,15^2/36)$, $sd(\acute{X})=2.5$; $z=\frac{85-80}{2.5}=2$, so $P(\acute{X}>85)=P(Z>2)\approx 0.0228$.
20. **A.** Margin $=z^*\frac{\sigma}{\sqrt{n}}=1.96\times\frac{20}{\sqrt{16}}=1.96\times 5=9.80$.

Section B — worked solutions

Question 1.

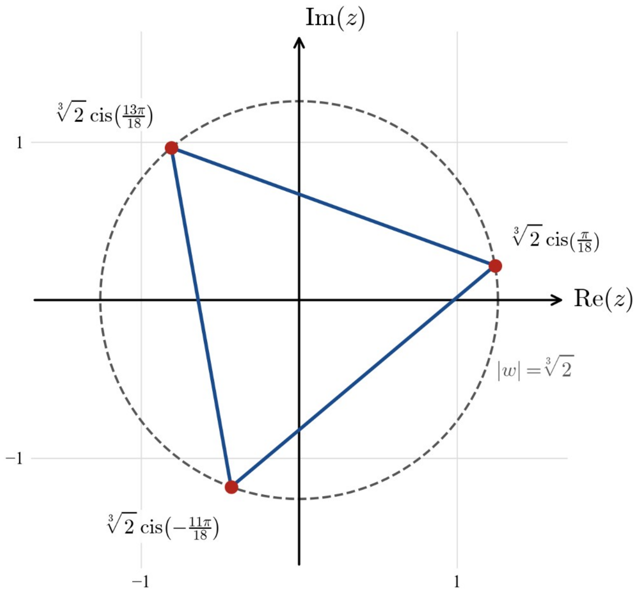
a. $r=|z|=\sqrt{(\sqrt{3})^2+1^2}=\sqrt{4}=2$. Since z is in the first quadrant with $\tan\theta=\frac{1}{\sqrt{3}}$, $\theta=\frac{\pi}{6}$. Thus $z=2\,cis\left(\frac{\pi}{6}\right)$.

b. By de Moivre's theorem, $z^6=2^6\,cis\left(6\cdot\frac{\pi}{6}\right)=64\,cis(\pi)=-64$. So $z^6=-64+0i=-64$.

c. The cube roots have modulus $2^{1/3}=\sqrt[3]{2}$ and arguments $\frac{\pi/6+2k\pi}{3}$ for $k=0,1,2$, that is $\frac{\pi}{18}+\frac{2k\pi}{3}$. Reducing to $(-\pi,\pi]$:

$$\sqrt[3]{2}\,cis\left(\frac{\pi}{18}\right),\sqrt[3]{2}\,cis\left(\frac{13\pi}{18}\right),\sqrt[3]{2}\,cis\left(-\frac{11\pi}{18}\right).$$

d. The three points all lie on the circle of radius $\sqrt[3]{2}\approx1.26$, with arguments 10°, 130° and -110° — equally spaced 120° apart. They are therefore the vertices of an **equilateral triangle** centred at the origin.



Question 2.

a. A point of ℓ_1 has coordinates $(2+s,s,3-s)$ and a point of ℓ_2 has coordinates $(-1+2t,-1+t,t)$. Equating the x- and y-components,

$$2+s=-1+2t,s=-1+t.$$

Substituting the second equation into the first gives $2+(t-1)=-1+2t$, so $t=2$ and $s=1$. These values also satisfy the z-equation, since $3-s=2$ and $t=2$. The lines therefore intersect, at $P(3,1,2)$.

b. $u\cdot v=(1)(2)+(1)(1)+(-1)(1)=2$, with $|u|=\sqrt{3}$ and $|v|=\sqrt{6}$, so

$$\cos\theta=\frac{u\cdot v}{|u||v|}=\frac{2}{\sqrt{3}\cdot\sqrt{6}}=\frac{2}{3\sqrt{2}}=\frac{\sqrt{2}}{3}\approx0.4714,$$

giving $\theta=\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)\approx62^\circ$ (to the nearest degree).

c.

$$u\times v=/\begin{matrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 1 & 1\end{matrix}/=(1\cdot1-(-1)\cdot1)i-(1\cdot1-(-1)\cdot2)j+(1\cdot1-1\cdot2)k=2i-3j-k.$$

Both lines lie in Π, so $n=2i-3j-k$ is normal to Π. Using the point $P(3,1,2)$ from part **a.**,

$$2(x-3)-3(y-1)-(z-2)=0 \Rightarrow 2x-3y-z=1,$$

as required. (Checking the base points: $2(2)-3(0)-3=1$ and $2(-1)-3(-1)-0=1$.)

d. The direction of m is $2i-3j-k$, which is exactly the normal n of Π; a line parallel to the normal of a plane is perpendicular to that plane. A point of m has coordinates $(2\lambda,4-3\lambda,1-\lambda)$, and substituting into $2x-3y-z=1$ gives

$$4\lambda-3(4-3\lambda)-(1-\lambda)=14\lambda-13=1,$$

so $\lambda=1$ and m meets Π at $(2,1,0)$.

Question 3.

a. Writing $\dfrac{1}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}$ gives $1=A(x+1)+Bx$. Setting $x=0$: $A=1$; setting $x=-1$: $B=-1$. Hence

$$\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}.$$

b. Separating the variables and integrating,

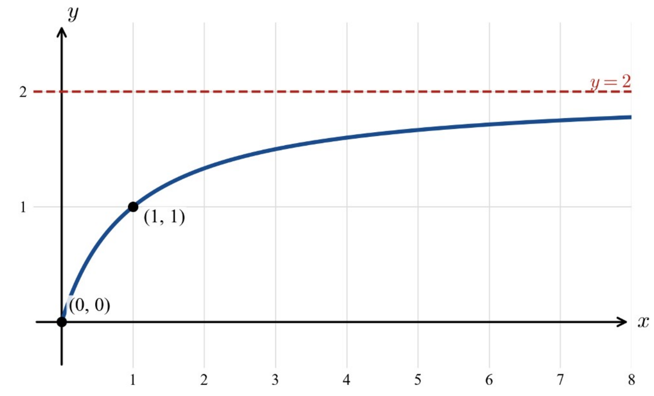
$$\int\frac{1}{y}dy=\int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx \Rightarrow \log_e y=\log_e x-\log_e(x+1)+c.$$

Thus $\log_e y=\log_e\left(\dfrac{x}{x+1}\right)+c$, so $y=A\dfrac{x}{x+1}$ where $A=e^c$. Using $y=1$ when $x=1$: $1=A\cdot\dfrac{1}{2}$, so $A=2$ and

$$y=\frac{2x}{x+1}.$$

c. As $x\to\infty$, $\dfrac{2x}{x+1}=\dfrac{2}{1+1/x}\to 2$. The value approached is $y=2$.

d. The curve passes through the origin, increases for $x>0$, passes through $(1,1)$, and approaches the horizontal asymptote $y=2$ from below.



Question 4.

a. Using $a=\dfrac{d}{dx}(1/2v^2)=-9x$ and integrating with respect to x,

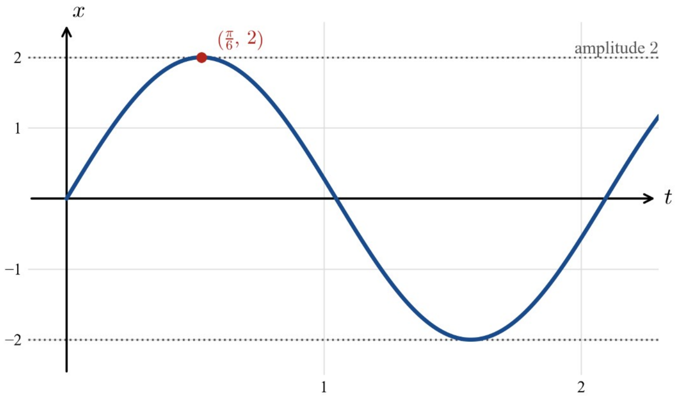
$$1/2v^2=\int(-9x)dx=-\frac{9x^2}{2}+C.$$

When $x=0$, $v=6$, so $1/2(36)=C$, giving $C=18$. Hence $1/2\,v^2=18-\dfrac{9x^2}{2}$, that is $v^2=36-9x^2$.

b. At the extremes of the motion $v=0$: $36-9x^2=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$. The amplitude is $2\,\text{m}$.

c. When $x=1$, $v^2=36-9(1)^2=27$, so the speed is $|v|=\sqrt{27}=3\sqrt{3}\,\text{m/s}$.

d. With $x(t)=2\sin(3t)$, the velocity is $\dot{x}(t)=6\cos(3t)$. The particle is at rest when $\cos(3t)=0$, i.e. $3t=\dfrac{\pi}{2},\dfrac{3\pi}{2},\ldots$ The first positive solution is $3t=\dfrac{\pi}{2}$, so $t=\dfrac{\pi}{6}\,\text{s}$.



Question 5.

a. The bags are independent, so means and variances add:

$$E(T)=12\times 250=3000\,\text{g},\, Var(T)=12\times 4^2=192,\, sd(T)=\sqrt{192}=8\sqrt{3}\,\text{g}.$$

b. $T\sim N(3000,192)$. Standardising, $z=\dfrac{2980-3000}{\sqrt{192}}=\dfrac{-20}{8\sqrt{3}}\approx -1.4434$, so

$$P(T<2980)=P(Z<-1.4434)\approx 0.0745.$$

c. With $\sigma=4$, $n=25$ and $z^*=1.9600$ for 95% confidence, the standard error is $\dfrac{\sigma}{\sqrt{n}}=\dfrac{4}{5}=0.8$ and the margin of error is $z^*\dfrac{\sigma}{\sqrt{n}}=1.9600\times 0.8=1.568$. The confidence interval is

$$\acute{x}\pm 1.568=248.6\pm 1.568=(247.03,250.17)\,\text{g}.$$

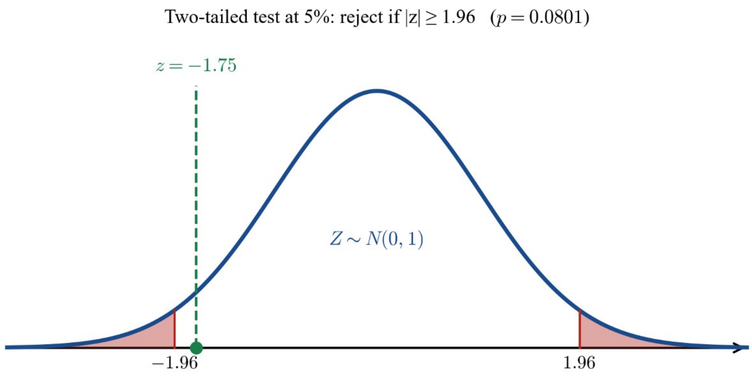
d. The claim names no direction, so the test is two-tailed:

$$H_0:\mu=250,\, H_1:\mu\neq 250.$$

The standard error is $\dfrac{\sigma}{\sqrt{n}}=\dfrac{4}{5}=0.8$, so $z=\dfrac{248.6-250}{0.8}=-1.75$. The two-tailed p-value is

$$p=2P(Z<-1.75)=2\times 0.04006=0.0801.$$

Since $0.0801>0.05$, we do **not** reject H_0: at the 5% level there is insufficient evidence that the mean fill differs from 250 g, so the sample does not contradict the manufacturer's claim. (Equivalently, 250 lies inside the confidence interval found in part **c.**)



Question 6.

a. $n=1$: $7^1-1=6=6\times1$, divisible by 6. $n=2$: $7^2-1=48=6\times8$, divisible by 6. The statement holds for $n=1$ and $n=2$.

b. Claim: 7^n-1 is divisible by 6 for every positive integer n.

Base case: for $n=1$, $7^1-1=6=6\times1$, which is divisible by 6.

Inductive step: assume the statement is true for some positive integer $n=k$, that is $7^k-1=6m$ for some integer m (so $7^k=6m+1$). Then

$$7^{k+1}-1=7\cdot7^k-1=7(6m+1)-1=42m+7-1=42m+6=6(7m+1).$$

Since $7m+1$ is an integer, $7^{k+1}-1$ is divisible by 6. Thus the statement holds for $n=k+1$.

By the principle of mathematical induction, 7^n-1 is divisible by 6 for every positive integer n.

c. Taking $n=100$, part **b.** shows $7^{100}-1$ is divisible by 6, so $7^{100}=6q+1$ for some integer q. Hence the remainder when 7^{100} is divided by 6 is 1.

Exam 6 — Technology-active (Hard)

Reading time: 15 minutes. Writing time: 2 hours. Section A: 20 multiple-choice questions (20 marks). Section B: extended-response (60 marks). Total 80 marks. One bound reference and one approved CAS technology are permitted. A formula sheet is provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Unless otherwise indicated, diagrams are not drawn to scale.

Section A — Multiple-choice

Answer all questions. Each correct answer is worth 1 mark; no mark is deducted for an incorrect answer. Choose the response that is best.

Question 1

The exact value of $\cos\left(\frac{5\pi}{12}\right)$ is

A. $\frac{\sqrt{6}+\sqrt{2}}{4}$

B. $\frac{\sqrt{2}-\sqrt{6}}{4}$

C. $\frac{\sqrt{6}-\sqrt{2}}{4}$

D. $\frac{\sqrt{6}-\sqrt{2}}{2}$

Question 2

Consider the statement: “for every real number x, if $x>2$ then $x^2>4$.” The negation of this statement is

A. for every real number x, $x>2$ and $x^2\le 4$

B. there exists a real number x such that $x>2$ and $x^2\le 4$

C. there exists a real number x such that $x>2$ and $x^2>4$

D. for every real number x, if $x\le 2$ then $x^2\le 4$

Question 3

$\tan^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right)$ equals

A. $\frac{5\pi}{6}$

B. $\frac{\pi}{6}$

C. $-\frac{5\pi}{6}$

D. $-\frac{\pi}{6}$

Question 4

If $z = \dfrac{2}{1+\sqrt{3}i}$, then z equals

A. $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

B. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

C. $1 - \sqrt{3}i$

D. $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Question 5

One of the solutions of the equation $z^3 = -8i$ is

A. $1 - \sqrt{3}i$

B. $\sqrt{3} + i$

C. $\sqrt{3} - i$

D. $-2i$

Question 6

The region of the complex plane defined by $|z - i| \le |z + i|$ is

A. $\{z : \text{Im}(z) \le 0\}$

B. $\{z : \text{Im}(z) \ge 0\}$

C. $\{z : \text{Re}(z) \ge 0\}$

D. $\{z : \text{Re}(z) \le 0\}$

Question 7

The scalar resolute of $a = 2i - j + 2k$ in the direction of $b = 3i + 4k$ is

A. $\frac{14}{25}$

B. $\frac{14}{9}$

C. $\frac{14}{3}$

D. $\frac{14}{5}$

Question 8

The points A and B have position vectors $i+2j-k$ and $3i-2j+3k$ respectively. A unit vector in the direction of $\overrightarrow{AB}$ is

A. $\frac{1}{3}(i-2j+2k)$

B. $\frac{1}{6}(i-2j+2k)$

C. $\frac{1}{3}(2i-4j+4k)$

D. $\frac{1}{3}(-i+2j-2k)$

Question 9

The distance from the point $P(3,1,-2)$ to the plane $x+2y-2z=4$ is

A. 3

B. $\frac{5}{\sqrt{3}}$

C. $\frac{5}{3}$

D. $\frac{5}{9}$

Question 10

For $x>0$, the graph of $y=\frac{x^2+4}{x}$ has a

A. local maximum at $(2,4)$

B. local minimum at $(2,4)$

C. local minimum at $(4,5)$

D. local maximum at $(-2,-4)$

Question 11

The curve $x^2 + xy + y^2 = 7$ passes through the point $(1, 2)$. At this point $\frac{dy}{dx}$ equals

A. $-\frac{4}{5}$

B. $-\frac{5}{4}$

C. $\frac{4}{5}$

D. $-\frac{1}{2}$

Question 12

A spherical balloon is inflated so that its volume increases at a constant rate of $20\,\text{cm}^3/\text{s}$. When the radius is $5\,\text{cm}$, the rate at which the radius is increasing, in cm/s, is

A. $\frac{4}{5\pi}$

B. $\frac{1}{25\pi}$

C. $\frac{1}{100\pi}$

D. $\frac{1}{5\pi}$

Question 13

$\int \frac{1}{\sqrt{25-4x^2}}\,dx$ equals

A. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right)+c$

B. $\sin^{-1}\left(\frac{2x}{5}\right)+c$

C. $\frac{1}{2}\sin^{-1}\left(\frac{x}{5}\right)+c$

D. $\frac{1}{5}\sin^{-1}\left(\frac{2x}{5}\right)+c$

Question 14

$\int_3^4 \frac{1}{x^2 - x - 2} dx$ equals

A. $\log_e\left(\frac{8}{5}\right)$

B. $\frac{1}{3}\log_e\left(\frac{2}{5}\right)$

C. $\frac{1}{3}\log_e\left(\frac{8}{5}\right)$

D. $\frac{1}{3}\log_e\left(\frac{5}{8}\right)$

Question 15

The length of the arc of the curve $y = \frac{2}{3}(x-1)^{3/2}$ from $x = 1$ to $x = 4$ is

A. $\frac{7}{3}$

B. $\frac{14}{3}$

C. $\frac{16}{3}$

D. $\frac{14\pi}{3}$

Question 16

Euler's method with step size $h = 0.2$ is used to solve $\frac{dy}{dx} = 2x - y$ with $y = 3$ when $x = 1$. The estimate of y when $x = 1.4$ is

A. 2.8

B. 2.6

C. 3.2

D. 2.72

Question 17

A particle moving in a straight line has velocity v m/s and satisfies $v\frac{dv}{dx}=3x^2$, where x is its displacement in metres. Given $v=2$ when $x=0$, the speed of the particle when $x=2$ is

A. 10

B. $\sqrt{10}$

C. $2\sqrt{6}$

D. $2\sqrt{5}$

Question 18

A particle moves so that its position vector at time t is $r(t)=2\cos(t)i+3\sin(t)j$. The speed of the particle at the instant it crosses the positive y-axis is

A. 3

B. 2

C. $\sqrt{13}$

D. $\sqrt{5}$

Question 19

X and Y are independent random variables with $E(X)=8$, $Var(X)=5$, $E(Y)=3$ and $Var(Y)=2$. If $W=2X-3Y+4$, then $E(W)$ and $Var(W)$ are respectively

A. 7 and 38

B. 11 and 16

C. 11 and 38

D. 11 and 42

Question 20

A 95% confidence interval for the mean μ of a normal population with known standard deviation σ is constructed from a random sample of size 36. Using the same population and the same level of confidence, the sample size required to halve the width of the interval is

A. 144

B. 72

C. 18

D. 576

End of Section A

Section B — Extended-response

Answer all questions. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Total 60 marks.

Question 1 (10 marks)

Let $u=-1+i$.

a. Find $|u|$ and $Arg(u)$, and hence express u in the polar form $r\,cis(\theta)$ with $-\pi<\theta\leq\pi$. 2 marks

b. Prove by mathematical induction that $(r\,cis(\theta))^n=r^n\,cis(n\theta)$ for every positive integer n. Set out your proof by verifying the base case, stating the inductive hypothesis, completing the inductive step, and stating the conclusion. 4 marks

c. Hence show that $u^8=16$. 1 mark

d. Find all four solutions of $z^4=-4$, giving each in the Cartesian form $x+i\,y$. Plot the four solutions on an Argand diagram and state the geometric shape whose vertices they form. 3 marks

Question 2 (11 marks)

Relative to an origin O, the points A, B and C have coordinates $A(2,1,0)$, $B(4,3,2)$ and $C(1,4,3)$.

a. Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. 2 marks

b. Find $\overrightarrow{AB}\times\overrightarrow{AC}$, and hence the exact area of triangle ABC. 3 marks

c. Show that the Cartesian equation of the plane Π through A, B and C is $y-z=1$. 2 marks

d. The line ℓ has vector equation $r(t)=(3i+k)+t(i+j+2k)$. Find the coordinates of the point at which ℓ meets the plane Π. 2 marks

e. Find the acute angle between the line ℓ and the plane Π, correct to the nearest degree. 2 marks

Question 3 (10 marks)

Consider the quotient function $f(x)=\dfrac{x}{e^x}$, for $x\geq 0$.

a. Find $f'(x)$, and show that the graph of $y=f(x)$ has a local maximum at $(1,e^{-1})$. 2 marks

b. Find the coordinates of the point of inflection of the graph, and state the equation of the horizontal asymptote of the graph, justifying your answer by considering the behaviour of $f(x)$ as $x\to\infty$. 3 marks

c. Find the exact area A of the region enclosed by the graph of $y=f(x)$, the x-axis and the line $x=1$. 2 marks

d. The region described in part **c.** is rotated about the x-axis. Find the exact volume of the solid of revolution formed, and give its value correct to three decimal places. 3 marks

Question 4 (10 marks)

An object is taken from an oven and left to cool in a room at a constant temperature of $20\,°\mathrm{C}$. Its temperature $T\,°\mathrm{C}$, t minutes after it is removed, satisfies

$$\frac{dT}{dt}=-k(T-20),k>0,$$

with $T=90$ when $t=0$ and $T=60$ when $t=10$.

a. Verify that $T=20+70e^{-kt}$ satisfies the differential equation and the condition $T(0)=90$. 2 marks

b. Using $T(10)=60$, show that $k=\dfrac{1}{10}\log_e\left(\dfrac{7}{4}\right)$. 3 marks

c. Find the temperature of the object 20 minutes after it is removed, correct to the nearest degree. 2 marks

d. Find, correct to the nearest minute, the time at which the temperature reaches $30\,°\mathrm{C}$. State the temperature the object approaches in the long run, and explain why this temperature is never actually attained. 3 marks

Question 5 (10 marks)

A projectile is launched from the origin. Its position vector at time t seconds is

$$r(t)=15t\,i+\left(20t-5t^2\right)j,$$

where displacements are in metres, i is a unit vector horizontally and j is a unit vector vertically upward.

a. Find the velocity vector $\dot{r}(t)$, and hence the initial speed of the projectile and its angle of projection above the horizontal, correct to the nearest degree. 3 marks

b. Find the time at which the projectile reaches its greatest height, and find that greatest height. 2 marks

c. Find the time of flight and the horizontal range of the projectile. 2 marks

d. Find the speed of the projectile at the instant it strikes the ground, and the angle below the horizontal at which it strikes, correct to the nearest degree. 3 marks

Question 6 (9 marks)

A machine produces metal rods whose length $X\,\mathrm{mm}$ is normally distributed with unknown mean μ and known standard deviation $\sigma=4\,\mathrm{mm}$. A random sample of 16 rods has sample mean $\acute{x}=502\,\mathrm{mm}$.

a. Find a $95\,\%$ confidence interval for μ, giving the endpoints correct to two decimal places. 2 marks

b. The target length is $500\,\mathrm{mm}$. Test, at the $5\,\%$ level of significance, whether the mean length differs from $500\,\mathrm{mm}$. State the hypotheses, the value of the test statistic, the p-value (correct to four decimal places), and your conclusion in context. 3 marks

c. Two rods are chosen independently from the production line. Find the standard deviation of the difference in their lengths, giving your answer in exact form. 1 mark

d. For the two-tailed test in part **b.**, suppose the true mean length is in fact $\mu=503\,\mathrm{mm}$. Find the probability of a Type II error (failing to reject H_0), correct to three decimal places. 3 marks

End of Exam 6

Exam 6 Formula Sheet

Formula sheet

This formula sheet is provided with every examination in this booklet.

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$
volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$
area of a triangle	$\frac{1}{2} b c \sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3} A h$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\,cis(\theta)$

modulus	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$
argument	$-\pi < Arg(z) \le \pi$
product	$z_1 z_2 = r_1 r_2\,cis(\theta_1 + \theta_2)$
quotient	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2)$
de Moivre's theorem	$z^n = r^n\,cis(n\theta)$

Data analysis, probability and statistics

For independent random variables $X_1, X_2 \ldots X_n$:

$E(aX_1 + b) = aE(X_1) + b$

$E(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$

$Var(aX_1 + b) = a^2 Var(X_1)$

$Var(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$

For independent identically distributed variables $X_1, X_2 \ldots X_n$:

$E(X_1 + X_2 + \ldots + X_n) = n\mu\; Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$

mean of sample mean $\overline{X}$ $E(\overline{X})=\mu$

variance of sample mean $\overline{X}$ $Var(\overline{X})=\frac{\sigma^2}{n}$

Calculus — derivatives and integrals

$$\frac{d}{dx}(x^n)=nx^{n-1} \qquad \int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,\, n\neq -1$$

$$\frac{d}{dx}(e^{ax})=ae^{ax} \qquad \int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$$

$$\frac{d}{dx}(\log_e(x))=\frac{1}{x} \qquad \int \frac{1}{x}\,dx=\log_e|x|+c$$

$$\frac{d}{dx}(\sin(ax))=a\cos(ax) \qquad \int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$$

$$\frac{d}{dx}(\cos(ax))=-a\sin(ax) \qquad \int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$$

$$\frac{d}{dx}(\tan(ax))=a\,sec^2(ax) \qquad \int sec^2(ax)\,dx=\frac{1}{a}\tan(ax)+c$$

$$\frac{d}{dx}(\cot(ax))=-a\,cosec^2(ax) \qquad \int cosec^2(ax)\,dx=-\frac{1}{a}\cot(ax)+c$$

$$\frac{d}{dx}(sec(ax))=a\,sec(ax)\tan(ax) \qquad \int sec(ax)\tan(ax)\,dx=\frac{1}{a}sec(ax)+c$$

$$\frac{d}{dx}(cosec(ax))=-a\,cosec(ax)\cot(ax) \qquad \int cosec(ax)\cot(ax)\,dx=-\frac{1}{a}cosec(ax)+c$$

$$\frac{d}{dx}(\sin^{-1}(ax))=\frac{a}{\sqrt{1-(ax)^2}} \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c,\, a>0$$

$$\frac{d}{dx}(\cos^{-1}(ax))=\frac{-a}{\sqrt{1-(ax)^2}} \qquad \int \frac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+c,\, a>0$$

$$\frac{d}{dx}(\tan^{-1}(ax))=\frac{a}{1+(ax)^2} \qquad \int \frac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\int (ax+b)^n\,dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\, n\neq -1$$

$$\int \frac{1}{ax+b}\,dx=\frac{1}{a}\log_e|ax+b|+c$$

Calculus — rules and applications

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

integration by parts: $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$

Euler's method: if $\frac{dy}{dx}=f(x,y)$, $x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+h\times f(x_n,y_n)$

arc length (parametric)	$\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$
surface area (Cartesian, about x-axis)	$\int_{x_1}^{x_2}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
surface area (Cartesian, about y-axis)	$\int_{y_1}^{y_2}2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$
surface area (parametric, about x-axis)	$\int_{t_1}^{t_2}2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$
surface area (parametric, about y-axis)	$\int_{t_1}^{t_2}2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration: $a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant acceleration formulas:

$v=u+at$	$s=ut+\frac{1}{2}at^2$
$v^2=u^2+2as$	$s=\frac{1}{2}(u+v)t$

Vectors in two and three dimensions

position vector: $r(t)=x(t)i+y(t)j+z(t)k$

magnitude: $|r(t)|=\sqrt{x(t)^2+y(t)^2+z(t)^2}$

derivative with respect to t: $\dot{r}(t)=\frac{dr}{dt}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$

For $r_1=x_1i+y_1j+z_1k$ and $r_2=x_2i+y_2j+z_2k$:

vector scalar product: $r_1\cdot r_2=|r_1||r_2|\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

vector cross product: $r_1\times r_2=\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2\end{vmatrix}=(y_1z_2-y_2z_1)i+(x_2z_1-x_1z_2)j+(x_1y_2-x_2y_1)k$

vector equation of a line: $r(t)=r_1+tr_2=(x_1+x_2t)i+(y_1+y_2t)j+(z_1+z_2t)k$

parametric equation of a line: $x(t)=x_1+x_2t,\ y(t)=y_1+y_2t,\ z(t)=z_1+z_2t$

vector equation of a plane: $r(s,t)=r_0+sr_1+tr_2=(x_0+x_1s+x_2t)i+(y_0+y_1s+y_2t)j+(z_0+z_1s+z_2t)k$

parametric equation of a plane: $x(s,t)=x_0+x_1s+x_2t,\ y(s,t)=y_0+y_1s+y_2t,\ z(s,t)=z_0+z_1s+z_2t$

Cartesian equation of a plane: $ax+by+cz=d$

Circular functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=sec^2(x)$

$\cot^2(x)+1=cosec^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$

$\sin^2(ax)=\dfrac{1}{2}(1-\cos(2ax))$

$\cos^2(ax)=\dfrac{1}{2}(1+\cos(2ax))$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

End of formula sheet

Exam 6 — Answers and solutions

Section A — Multiple-choice answer key

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans	C	B	D	A	C	B	D	A	C	B	A	D	A	C	B	D	D	B	C	A

Section A — reasons

1. **C.** $\frac{5\pi}{12}=\frac{\pi}{4}+\frac{\pi}{6}$, so $\cos\left(\frac{5\pi}{12}\right)=\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$.
2. **B.** The negation of $\forall x(P\Rightarrow Q)$ is $\exists x(P\wedge\neg Q)$. Here P is $x>2$ and Q is $x^2>4$, so the negation is: there exists a real x with $x>2$ and $x^2\leq 4$.
3. **D.** $\tan\left(\frac{5\pi}{6}\right)=-\frac{1}{\sqrt{3}}$; since the range of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$.
4. **A.** $z=\frac{2}{1+\sqrt{3}i}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i}=\frac{2(1-\sqrt{3}i)}{1+3}=\frac{1-\sqrt{3}i}{2}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$.
5. **C.** $-8i=8\,cis\left(-\frac{\pi}{2}\right)$, so the cube roots have modulus 2 and one has argument $-\frac{\pi}{6}$: $2\,cis\left(-\frac{\pi}{6}\right)=\sqrt{3}-i$. (Indeed $(\sqrt{3}-i)^3=-8i$.)
6. **B.** $|z-i|\leq|z+i|$ means z is at least as close to i as to $-i$. Squaring, $|z+i|^2-|z-i|^2=4\,\mathrm{Im}(z)\geq 0$, so $\mathrm{Im}(z)\geq 0$.
7. **D.** Scalar resolute $=\frac{a\cdot b}{|b|}=\frac{(2)(3)+(-1)(0)+(2)(4)}{\sqrt{3^2+4^2}}=\frac{14}{5}$.
8. **A.** $\overrightarrow{AB}=(3-1)i+(-2-2)j+(3-(-1))k=2i-4j+4k$, with $|\overrightarrow{AB}|=\sqrt{4+16+16}=6$; the unit vector is $\frac{1}{6}(2i-4j+4k)=\frac{1}{3}(i-2j+2k)$.
9. **C.** Distance $=\frac{|(3)+2(1)-2(-2)-4|}{\sqrt{1^2+2^2+(-2)^2}}=\frac{|3+2+4-4|}{3}=\frac{5}{3}$.
10. **B.** $\frac{x^2+4}{x}=x+\frac{4}{x}$, so $\frac{dy}{dx}=1-\frac{4}{x^2}=0$ gives $x=2$ for $x>0$, where $y=4$. Since $\frac{d^2y}{dx^2}=\frac{8}{x^3}>0$ for $x>0$, the point $(2,4)$ is a local minimum. (The point $(4,5)$ lies on the graph but is not stationary, and $(-2,-4)$ is outside the domain.)
11. **A.** Differentiating implicitly, $2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{2x+y}{x+2y}$. At $(1,2)$ this is $-\frac{2+2}{1+4}=-\frac{4}{5}$.
12. **D.** $V=\frac{4}{3}\pi r^3$, so $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$. Then $\frac{dr}{dt}=\frac{20}{4\pi(5)^2}=\frac{20}{100\pi}=\frac{1}{5\pi}$.
13. **A.** Since $25-4x^2=25-(2x)^2$, substitute $u=2x$: $\int\frac{1}{\sqrt{25-4x^2}}dx=\frac{1}{2}\int\frac{1}{\sqrt{25-u^2}}du=\frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right)+c$.
14. **C.** $x^2-x-2=(x-2)(x+1)$ and $\frac{1}{(x-2)(x+1)}=\frac{1}{3}\left(\frac{1}{x-2}-\frac{1}{x+1}\right)$, so $\int_3^4\frac{1}{x^2-x-2}dx=\frac{1}{3}[\log_e|x-2|-\log_e|x+1|]_3^4=\frac{1}{3}\log_e\left(\frac{8}{5}\right)$.
15. **B.** $y'=(x-1)^{1/2}$, so $1+(y')^2=x$ and the arc length is $\int_1^4\sqrt{x}\,dx=\left[\frac{2}{3}x^{3/2}\right]_1^4=\frac{2}{3}(8-1)=\frac{14}{3}$.

16. **D.** Step 1: $y \approx 3 + 0.2(2(1) - 3) = 3 - 0.2 = 2.8$ at $x = 1.2$. Step 2: $y \approx 2.8 + 0.2(2(1.2) - 2.8) = 2.8 - 0.08 = 2.72$ at $x = 1.4$.

17. **D.** From $v\frac{dv}{dx} = 3x^2$, $\frac{1}{2}v^2 = x^3 + C$; with $v = 2$ at $x = 0$, $C = 2$, so $v^2 = 2x^3 + 4$. At $x = 2$, $v^2 = 20$ and the speed is $\sqrt{20} = 2\sqrt{5}$.

18. **B.** The particle is on the positive y-axis when $2\cos(t) = 0$ and $3\sin(t) > 0$, i.e. $t = \frac{\pi}{2}$. Then $\dot{r}(t) = -2\sin(t)i + 3\cos(t)j$ gives $\dot{r}\left(\frac{\pi}{2}\right) = -2i$, of speed 2.

19. **C.** $E(W) = 2E(X) - 3E(Y) + 4 = 16 - 9 + 4 = 11$. For independent X and Y the added constant does not affect the spread and each coefficient is squared, so $Var(W) = 2^2 Var(X) + (-3)^2 Var(Y) = 4(5) + 9(2) = 38$.

20. **A.** The width of the interval is $2z^* \frac{\sigma}{\sqrt{n}}$, which is inversely proportional to $\sqrt{n}$. Halving the width therefore requires $\sqrt{n}$ to be doubled, that is n to be multiplied by 4: $n = 4 \times 36 = 144$.

Section B — worked solutions

Question 1.

a. $|u| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. The point $-1 + i$ lies in the second quadrant with reference angle $\frac{\pi}{4}$, so $Arg(u) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. Hence

$$u = \sqrt{2}\,cis\left(\frac{3\pi}{4}\right).$$

b. Let $P(n)$ be the statement $(r\,cis(\theta))^n = r^n cis(n\theta)$, where n is a positive integer.

Base case. For $n = 1$ the left-hand side is $(r\,cis(\theta))^1 = r\,cis(\theta)$ and the right-hand side is $r^1 cis(1 \times \theta) = r\,cis(\theta)$, so $P(1)$ is true.

Inductive hypothesis. Assume $P(k)$ is true for some positive integer k, that is

$$(r\,cis(\theta))^k = r^k cis(k\theta).$$

Inductive step. Using the hypothesis and then the product rule $z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$,

$$(r\,cis(\theta))^{k+1} = (r\,cis(\theta))^k \times r\,cis(\theta) = r^k cis(k\theta) \times r\,cis(\theta) = r^{k+1} cis(k\theta + \theta) = r^{k+1} cis((k+1)\theta),$$

which is the statement $P(k+1)$. Hence $P(k)$ true implies $P(k+1)$ true.

Conclusion. $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for every positive integer k, so by the principle of mathematical induction $P(n)$ is true for all positive integers n.

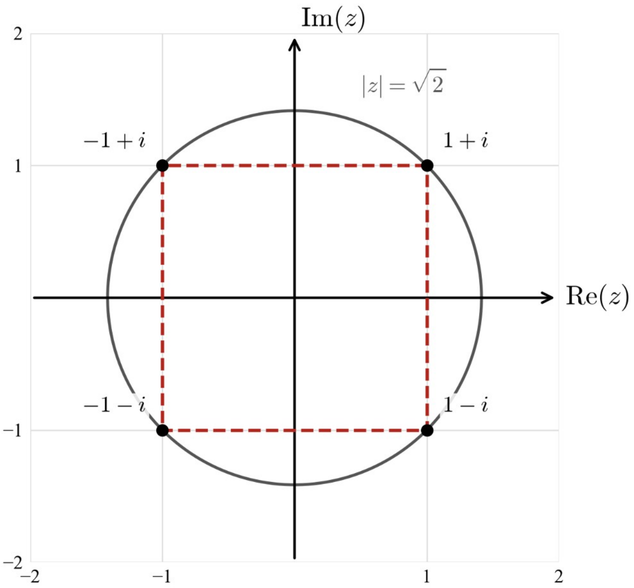
c. With $r = \sqrt{2}$, $\theta = \frac{3\pi}{4}$ and $n = 8$, part **b.** gives

$$u^8 = (\sqrt{2})^8 cis\left(8 \times \frac{3\pi}{4}\right) = 2^4 cis(6\pi) = 16\,cis(0) = 16.$$

d. Write $-4=4\,cis(\pi)$. The solutions of $z^4=-4$ have modulus $4^{1/4}=\sqrt{2}$ and arguments $\dfrac{\pi+2k\pi}{4}$ for $k=0,1,2,3$, that is $\dfrac{\pi}{4},\dfrac{3\pi}{4},-\dfrac{3\pi}{4},-\dfrac{\pi}{4}$. Converting to Cartesian form,

$$z=1+i,-1+i,-1-i,1-i.$$

The four points lie on the circle $|z|=\sqrt{2}$, equally spaced $90°$ apart, so they are the vertices of a **square** centred at the origin.



Question 2.

a. $\overrightarrow{AB}=B-A=(2,2,2)$, i.e. $2i+2j+2k$; $\overrightarrow{AC}=C-A=(-1,3,3)$, i.e. $-i+3j+3k$.

b.

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ 2 & 2 & 2 \\ -1 & 3 & 3 \end{vmatrix}=(2\cdot 3-2\cdot 3)i-(2\cdot 3-2\cdot(-1))j+(2\cdot 3-2\cdot(-1))k=0i-8j+8k.$$

Then $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{0^2+(-8)^2+8^2}=\sqrt{128}=8\sqrt{2}$, so the area of triangle ABC is $\dfrac{1}{2}(8\sqrt{2})=4\sqrt{2}$ square units.

c. The cross product $(0,-8,8)$ is parallel to $n=(0,1,-1)$, which is normal to Π. Using $A(2,1,0)$, $n\cdot\overrightarrow{OA}=0(2)+1(1)-1(0)=1$, so the plane is

$$0x+y-z=1,\text{i.e. }y-z=1.$$

(Check: B gives $3-2=1$ and C gives $4-3=1$.)

d. On ℓ, $r(t)=(3+t,t,1+2t)$. Substituting into $y-z=1$: $t-(1+2t)=1$, so $-t-1=1$, giving $t=-2$. The point of intersection is

$$(3+(-2),-2,1+2(-2))=(1,-2,-3).$$

e. The direction of ℓ is $d=(1,1,2)$. With α the angle between ℓ and Π,

$$\sin\alpha=\frac{|d\cdot n|}{|d||n|}=\frac{|0+1-2|}{\sqrt{6}\cdot\sqrt{2}}=\frac{1}{2\sqrt{3}}\approx 0.2887,$$

so $\alpha=\sin^{-1}(0.2887)\approx 17°$ (to the nearest degree).

Question 3.

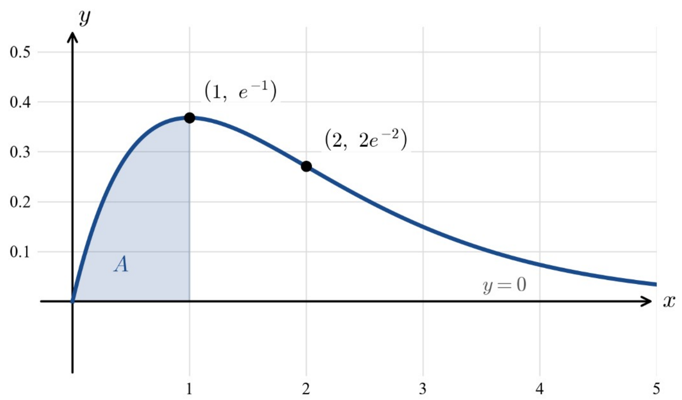
a. Writing $f(x)=\frac{x}{e^x}=xe^{-x}$, the quotient rule (or the product rule) gives

$$f'(x)=\frac{e^x(1)-xe^x}{(e^x)^2}=\frac{1-x}{e^x}=e^{-x}(1-x).$$

Setting $f'(x)=0$ gives $x=1$ (since $e^{-x}\neq 0$). The second derivative is $f''(x)=e^{-x}(x-2)$; at $x=1$, $f''(1)=-e^{-1}<0$, so the stationary point is a local maximum. Its value is $f(1)=1\cdot e^{-1}=e^{-1}$, giving the point $(1,e^{-1})$.

b. Points of inflection occur where $f''(x)=e^{-x}(x-2)=0$, i.e. $x=2$, and f'' changes sign there (negative for $0\le x<2$, positive for $x>2$). Since $f(2)=2e^{-2}$, the point of inflection is $(2,2e^{-2})$.

As $x\to\infty$ the denominator e^x grows exponentially while the numerator x grows only linearly, so $f(x)=\frac{x}{e^x}\to 0^+$. The graph therefore has the horizontal asymptote $y=0$, which it approaches from above.



c. The curve lies above the x-axis for $0<x\le 1$, so

$$A=\int_0^1 xe^{-x}\,dx=[-xe^{-x}]_0^1+\int_0^1 e^{-x}\,dx=-e^{-1}+[-e^{-x}]_0^1=-e^{-1}+(1-e^{-1})=1-\frac{2}{e}.$$

d. Rotating about the x-axis,

$$V=\pi\int_0^1 (xe^{-x})^2\,dx=\pi\int_0^1 x^2e^{-2x}\,dx=\pi\left(\frac{1}{4}-\frac{5}{4e^2}\right)\approx 0.254.$$

Question 4.

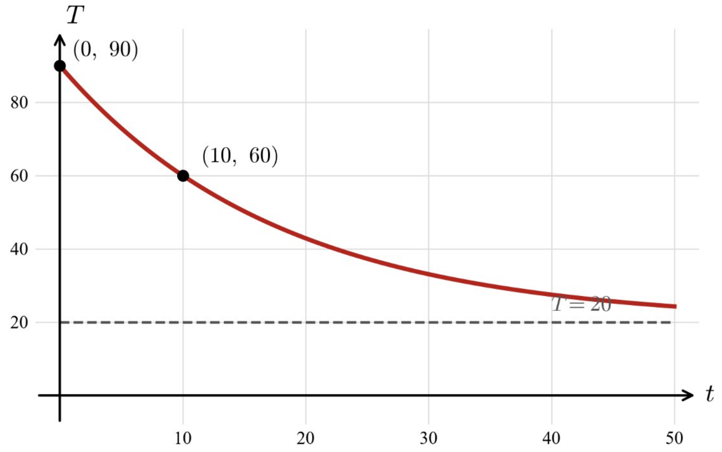
a. Differentiating $T=20+70e^{-kt}$ gives $\frac{dT}{dt}=-70ke^{-kt}$. Also $-k(T-20)=-k(70e^{-kt})=-70ke^{-kt}$, so the differential equation is satisfied. At $t=0$, $T=20+70e^0=20+70=90$, so $T(0)=90$ as required.

b. Setting $T(10)=60$: $20+70e^{-10k}=60$, so $70e^{-10k}=40$, giving $e^{-10k}=\frac{4}{7}$. Then $-10k=\log_e\left(\frac{4}{7}\right)=-\log_e\left(\frac{7}{4}\right)$, so

$$k=\frac{1}{10}\log_e\left(\frac{7}{4}\right).$$

c. Since $e^{-10k}=\frac{4}{7}$, we have $e^{-20k}=\left(\frac{4}{7}\right)^2=\frac{16}{49}$, so

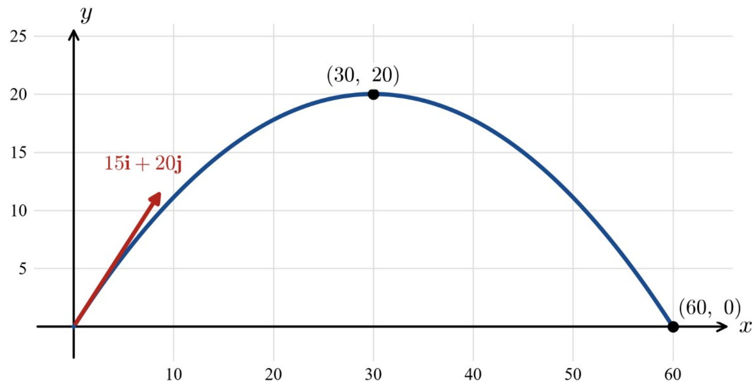
$$T(20)=20+70\cdot\frac{16}{49}=20+\frac{160}{7}\approx 42.86\,°\mathrm{C}\approx 43\,°\mathrm{C}.$$



d. Setting $T=30$: $20+70e^{-kt}=30$, so $e^{-kt}=\frac{1}{7}$ and $t=\frac{\log_e 7}{k}=\frac{10\log_e 7}{\log_e(7/4)}\approx 34.8\approx 35$ minutes. As $t\to\infty$, $70e^{-kt}\to 0$, so T approaches $20\,°\mathrm{C}$ (the room temperature). Because $70e^{-kt}>0$ for every finite t, we always have $T>20$, so the object never actually reaches $20\,°\mathrm{C}$.

Question 5.

a. $\dot{r}(t)=15i+(20-10t)j$. At $t=0$, $\dot{r}(0)=15i+20j$, so the initial speed is $\sqrt{15^2+20^2}=\sqrt{625}=25\,\mathrm{m/s}$. The angle of projection above the horizontal is $\tan^{-1}\left(\frac{20}{15}\right)\approx 53\,°$.



b. The greatest height occurs when the vertical velocity is zero: $20-10t=0$, so $t=2\,\mathrm{s}$. The height is then

$$y=20(2)-5(2)^2=40-20=20\,\mathrm{m}.$$

c. The projectile returns to ground level when $20t-5t^2=0$, i.e. $5t(4-t)=0$, giving $t=4\,\mathrm{s}$ (as $t=0$ is the launch). The horizontal range is $x=15(4)=60\,\mathrm{m}$.

d. At $t=4$, $\dot{r}(4)=15i+(20-40)j=15i-20j$, so the landing speed is $\sqrt{15^2+(-20)^2}=25\,\mathrm{m/s}$. The velocity makes an angle $\tan^{-1}\left(\frac{20}{15}\right)\approx 53\,°$ below the horizontal.

Question 6.

a. The standard error is $\frac{\sigma}{\sqrt{n}}=\frac{4}{\sqrt{16}}=1$. With $z^*=1.9600$ for $95\,\%$ confidence, the margin of error is $1.9600\times 1=1.9600$, so the confidence interval is

$$\acute{x} \pm 1.9600 = 502 \pm 1.9600 = (500.04\,, 503.96)\,\text{mm}.$$

b. The claim names no direction, so the test is two-tailed:

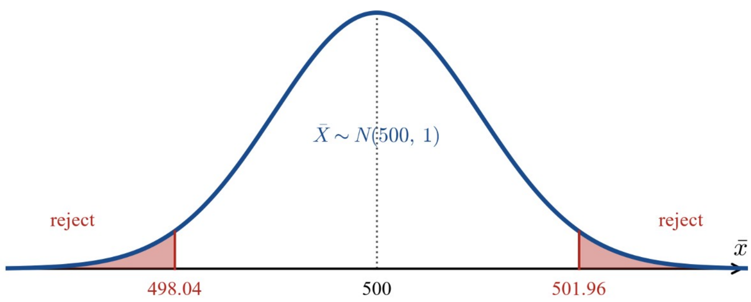
$$H_0 : \mu = 500\,, H_1 : \mu \neq 500.$$

The test statistic is $z = \dfrac{502 - 500}{1} = 2.0$. The two-tailed p-value is

$$p = 2P(Z > 2.0) = 2 \times 0.02275 = 0.0455.$$

Since $0.0455 < 0.05$, we reject H_0: at the $5\,\%$ level there is evidence that the mean rod length differs from $500\,\text{mm}$. (Equivalently, 500 lies outside the confidence interval found in part **a.**)

Two-tailed test at 5%: reject H_0 if $\bar{x} \leq 498.04$ or $\bar{x} \geq 501.96$



c. For two independent rods X_1 and X_2, $Var(X_1 - X_2) = 4^2 + 4^2 = 32$, so

$$sd(X_1 - X_2) = \sqrt{32} = 4\sqrt{2}\,\text{mm}.$$

d. For the two-tailed test at the $5\,\%$ level, H_0 is not rejected when $\acute{x}$ lies between $500 - 1.9600 = 498.04$ and $500 + 1.9600 = 501.96$. With standard error 1 and true mean $\mu = 503$,

$$\beta = P(498.04 < \acute{X} < 501.96) = \Phi\left(\frac{501.96 - 503}{1}\right) - \Phi\left(\frac{498.04 - 503}{1}\right) = \Phi(-1.04) - \Phi(-4.96) \approx 0.149.$$

The probability of a Type II error is approximately 0.149.