Specialist Mathematics Units 3 & 4

Chapter 10 — Vector functions and vector calculus

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Contents

Theory

Vector functions.

A rule that assigns a vector to each value of a real variable t is called a **vector function**. In two dimensions it is written

$$r(t) = x(t)i + y(t)j,$$

where $x(t)$ and $y(t)$ are ordinary real functions of t called the **components**, and i, j are the unit vectors along the x- and y-axes. In three dimensions there is a third component, $r(t) = x(t)i + y(t)j + z(t)k$. The variable t is the **parameter**.

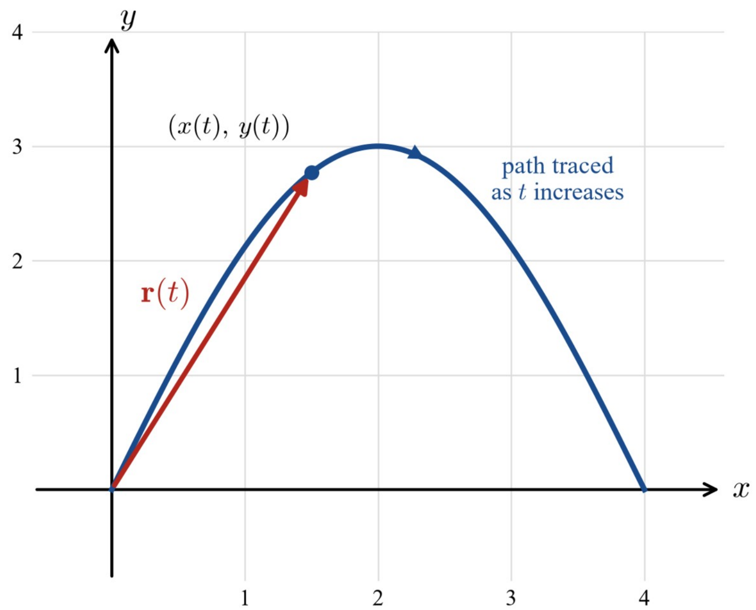
The **domain** of r is the set of values of t for which every component is defined; unless a smaller domain is stated we take the largest possible set of real numbers t.

The path of a vector function.

Draw $r(t)$ as a **position vector** — an arrow from the origin O to the point $(x(t), y(t))$. As t varies the tip of this arrow moves and traces a curve; this curve is the **path** (or **trajectory**) of the vector function. The point on the path is $(x(t), y(t))$, so the two component functions

$$x = x(t), y = y(t)$$

are exactly the **parametric equations** of the curve. The direction in which the point moves as t **increases** is the **direction of travel**, and we always mark it on a sketch with an arrow.



From parametric to Cartesian form.

A path described parametrically can usually also be described by a single **Cartesian equation** relating x and y directly. We obtain it by **eliminating the parameter** t: either make t the subject of one equation and substitute into the other, or combine the two equations using an identity. The result identifies the curve — a line, a parabola, a circle, an ellipse or a hyperbola.

Circles.

The most important standard path uses cos and sin. For a constant $a > 0$,

$$r(t)=a\cos t\, i+a\sin t\, j,\, 0\le t<2\pi,$$

has components $x=a\cos t$ and $y=a\sin t$. Squaring and adding, then using the Pythagorean identity $\cos^2 t+\sin^2 t=1$,

$$x^2+y^2=a^2\cos^2 t+a^2\sin^2 t=a^2(\cos^2 t+\sin^2 t)=a^2.$$

So the path is the **circle** of radius a centred at the origin. At $t=0$ the point is $(a,0)$ and at $t=\frac{\pi}{2}$ it is $(0,a)$, so the point travels **anticlockwise**. Replacing $\sin t$ by $-\sin t$ (equivalently t by $-t$) reverses this to **clockwise**. Adding constants inside the components shifts the centre: $r(t)=(h+a\cos t)i+(k+a\sin t)j$ traces $(x-h)^2+(y-k)^2=a^2$.

Ellipses.

If the two coefficients differ, say $a\ne b$ (both positive), then

$$r(t)=a\cos t\, i+b\sin t\, j,\, 0\le t<2\pi,$$

gives $x=a\cos t$ and $y=b\sin t$, so $\cos t=\frac{x}{a}$ and $\sin t=\frac{y}{b}$. The same identity now gives

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2 t+\sin^2 t=1,$$

an **ellipse** with x-intercepts $(\pm a,0)$ and y-intercepts $(0,\pm b)$. A circle is the special case $a=b$.

Hyperbolas.

For a hyperbola we use the identity $sec^2 t-\tan^2 t=1$ (obtained by dividing $\cos^2 t+\sin^2 t=1$ by $\cos^2 t$). With

$$r(t)=a\, sec\, t\, i+b\tan t\, j,$$

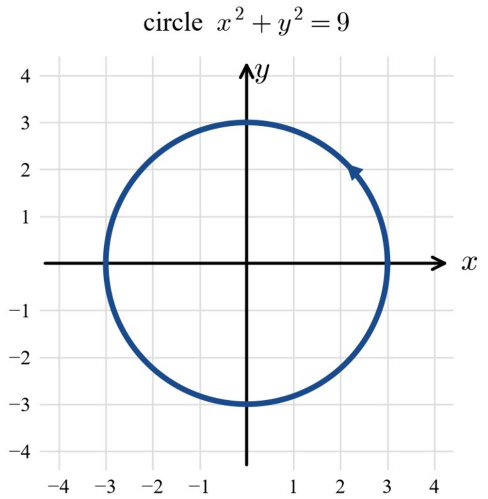
we have $sec\, t=\frac{x}{a}$ and $\tan t=\frac{y}{b}$, so

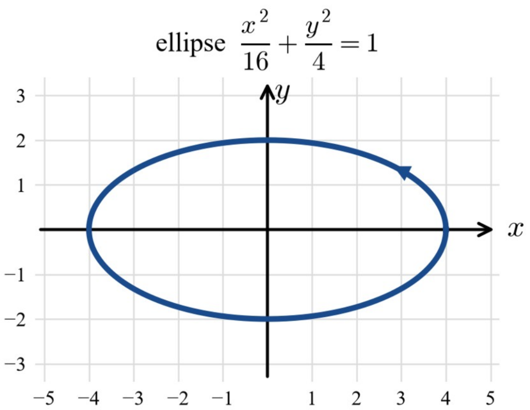
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=sec^2 t-\tan^2 t=1,$$

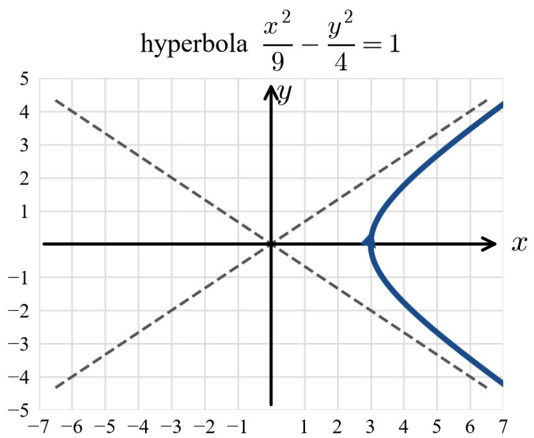
a **hyperbola** with vertices $(\pm a,0)$ and asymptotes $y=\pm\frac{b}{a}x$. On the domain $-\frac{\pi}{2}<t<\frac{\pi}{2}$ we have $\cos t>0$, so $x=a\, sec\, t\ge a$ and only the **right branch** ($x\ge a$) is traced. Interchanging the components, $r(t)=b\tan t\, i+a\, sec\, t\, j$, gives the vertically-opening hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

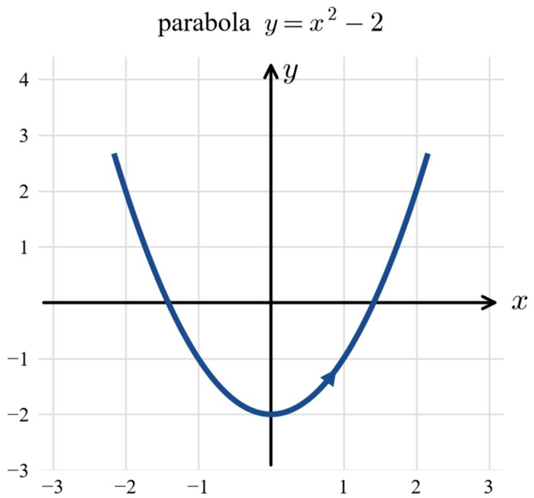
Lines and parabolas.

Not every path is a conic. If both components are linear in t, the path is a **straight line**; if one component is t (or linear in t) and the other is quadratic, the path is a **parabola**. Here we eliminate t directly. For example, $x=t$, $y=t^2$ gives $y=x^2$; and $x=1+2t$, $y=3-t$ gives $t=3-y$, hence $x=1+2(3-y)=7-2y$, the line $x+2y=7$.









Restricted domains: arcs and branches.

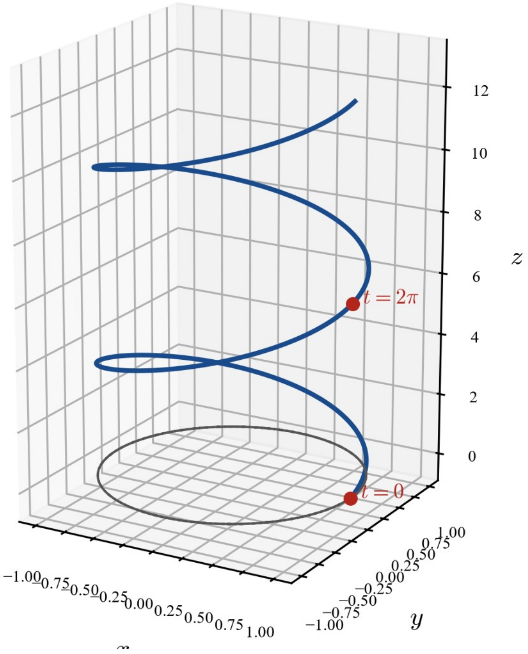
When the parameter is restricted to part of its natural domain, only **part** of the full curve is traced. The Cartesian equation of the whole curve is unchanged, but x and y are now limited to a range and the path becomes an **arc** (part of a circle or ellipse) or a single **branch**. To find where an arc starts and ends, substitute the endpoint values of t. For instance $r(t)=2\cos t\, i+2\sin t\, j$ on $0\le t\le \pi$ traces only the **upper half** of $x^2+y^2=4$ (where $y\ge 0$), from $(2,0)$ to $(-2,0)$.

A path in three dimensions.

A vector function with three components traces a curve in space. A key example is the **circular helix**

$$r(t)=\cos t\, i+\sin t\, j+t\, k,\, t\ge 0.$$

Its first two components satisfy $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the point always lies above the **unit circle** in the $x\,y$-plane; meanwhile $z=t$ increases steadily, so the point winds around and rises, like a point on a spiral staircase. One complete turn (t increasing by 2π) raises it by 2π in the z-direction.



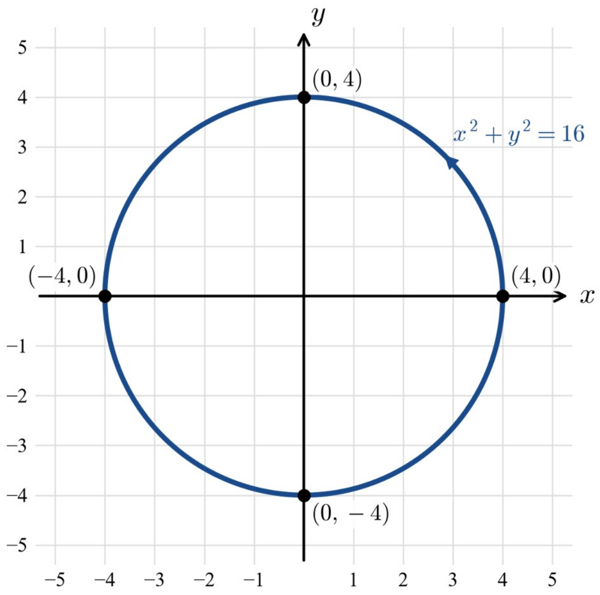
To sketch or identify the path of a vector function: read off the parametric equations $x=x(t)$, $y=y(t)$; eliminate t to get the Cartesian equation and name the curve; plot a few points from convenient values of t; and mark the direction of travel and any endpoints coming from the domain.

Worked examples

Example 1 — scaffolded

The vector function $r(t)=4\cos t\, i+4\sin t\, j$, $0\le t<2\pi$, traces a path. Find its Cartesian equation, describe the curve, and state the direction of travel.

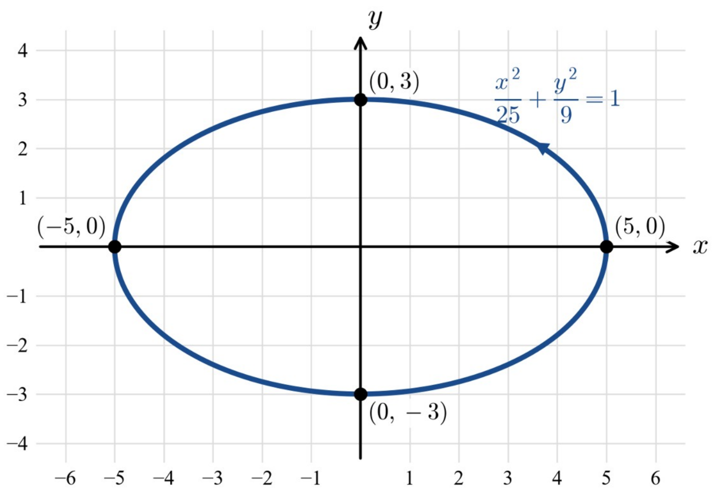
Working	Comment
$x=4\cos t,\ y=4\sin t$.	Read the parametric equations off the components.
$x^2+y^2=16\cos^2 t+16\sin^2 t$.	Square each component and add.
$=16(\cos^2 t+\sin^2 t)=16$.	Use $\cos^2 t+\sin^2 t=1$ to remove t.
Path: $x^2+y^2=16$, a circle centred at O with radius 4.	Compare with $x^2+y^2=a^2$; here $a=4$.
$t=0\to(4,0)$; $t=\frac{\pi}{2}\to(0,4)$: anticlockwise.	Test two values of t to fix the direction of travel.



Example 2 — scaffolded

Find the Cartesian equation of the path of $r(t)=5\cos t\, i+3\sin t\, j$, $0\le t<2\pi$, name the curve, and state its axis intercepts.

Working	Comment
$x=5\cos t,\ y=3\sin t$.	The components give the parametric equations.
$\cos t=\frac{x}{5},\ \sin t=\frac{y}{3}$.	Make the trigonometric functions the subject.
$\frac{x^2}{25}+\frac{y^2}{9}=\cos^2 t+\sin^2 t=1$.	Substitute into $\cos^2 t+\sin^2 t=1$.
Path: $\frac{x^2}{25}+\frac{y^2}{9}=1$, an ellipse.	Standard form with $a=5$ and $b=3$.
Intercepts $(\pm5,0)$ and $(0,\pm3)$; anticlockwise.	$t=0\rightarrow(5,0)$, $t=\frac{\pi}{2}\rightarrow(0,3)$.



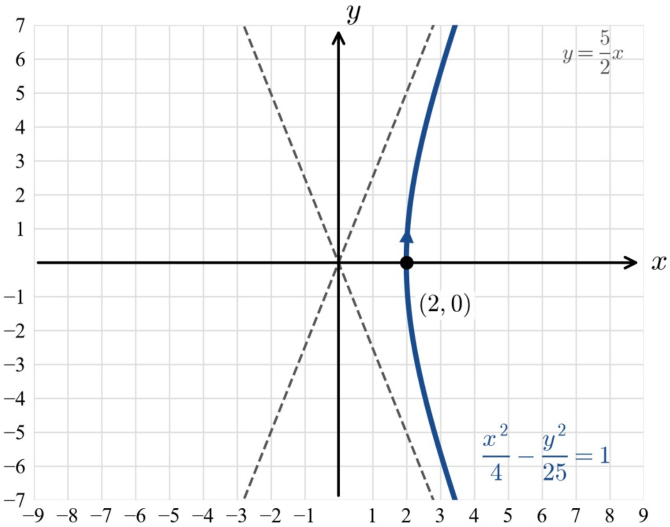
Example 3 — unscaffolded

The path of a point is $r(t)=2\sec t\, i+5\tan t\, j$ for $-\frac{\pi}{2}<t<\frac{\pi}{2}$. Find the Cartesian equation, name the curve, state which part of it is traced and in which direction, and give the asymptotes.

The components give $x=2\sec t$ and $y=5\tan t$, so $\sec t=\frac{x}{2}$ and $\tan t=\frac{y}{5}$. Using the identity $\sec^2 t-\tan^2 t=1$,

$$\frac{x^2}{4}-\frac{y^2}{25}=\sec^2 t-\tan^2 t=1.$$

This is a hyperbola with vertices $(\pm 2,0)$ and asymptotes $y=\pm\frac{5}{2}x$. On $-\frac{\pi}{2}<t<\frac{\pi}{2}$ we have $\cos t>0$, so $x=2\sec t\geq 2$ and only the right branch is traced. At $t=0$ the point is at the vertex $(2,0)$; as t increases towards $\frac{\pi}{2}$, $y=5\tan t\rightarrow+\infty$, while as t decreases towards $-\frac{\pi}{2}$, $y\rightarrow-\infty$. So as t increases the point travels up the right branch.



∴ the path is the right branch ($x\geq 2$) of $\frac{x^2}{4}-\frac{y^2}{25}=1$, traced upward as t increases.

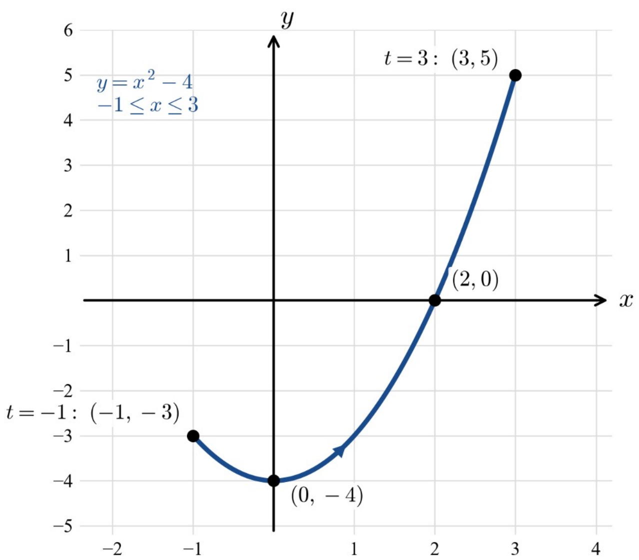
Example 4 — unscaffolded

For $r(t)=t\,i+(t^2-4)\,j$ with $-1\leq t\leq 3$, find the Cartesian equation, describe the path (including its endpoints and direction of travel), and find where it meets the coordinate axes.

Since the i-component is t, we have $x=t$; substituting directly,

$$y=t^2-4=x^2-4.$$

The path lies on the parabola $y=x^2-4$, which has vertex $(0,-4)$. Because $-1\leq t\leq 3$ and $x=t$, only the arc with $-1\leq x\leq 3$ is traced. Its endpoints are at $t=-1$, giving $(-1,-3)$, and $t=3$, giving $(3,5)$; as t increases, $x=t$ increases, so the point travels from left to right. The path crosses the y-axis at $t=0$, giving $(0,-4)$, and crosses the x-axis where $y=0$, i.e. $t^2=4$ so $t=\pm 2$ — but only $t=2$ lies in the domain, giving $(2,0)$.



$\therefore$ the path is the arc of $y=x^2-4$ from $(-1,-3)$ to $(3,5)$ (with $-1\le x\le 3$), travelled left to right.

Practice questions

Scaffolded

Q1. The vector function $r(t)=6\cos t\, i+6\sin t\, j,\ 0\le t<2\pi$, traces a path. (a) Write down the parametric equations for x and y. (b) Find x^2+y^2 and simplify using $\cos^2 t+\sin^2 t=1$. (c) State the Cartesian equation and name the curve. (d) Find the points at $t=0$ and $t=\frac{\pi}{2}$, and hence the direction of travel.

Q2. Consider $r(t)=2\cos t\, i+5\sin t\, j,\ 0\le t<2\pi$. (a) Write $\cos t$ and $\sin t$ in terms of x and y. (b) Use $\cos^2 t+\sin^2 t=1$ to eliminate t. (c) State the Cartesian equation and name the curve. (d) Write down the x- and y-intercepts.

Q3. Consider $r(t)=4\sec t\, i+3\tan t\, j,\ -\frac{\pi}{2}<t<\frac{\pi}{2}$. (a) Write $\sec t$ and $\tan t$ in terms of x and y. (b) Use $\sec^2 t-\tan^2 t=1$ to eliminate t. (c) State the Cartesian equation and the equations of the asymptotes. (d) Explain why only the branch with $x\ge 4$ is traced.

Q4. A point moves with $r(t)=(2+3t)i+(1-t)j$. (a) Write down x and y in terms of t. (b) Make t the subject of $x=2+3t$. (c) Substitute into $y=1-t$ to find the Cartesian equation. (d) State the gradient of the line and the point given by $t=0$.

Q5. Consider $r(t)=t\, i+(t^2+1)j,\ t\ge 0$. (a) Explain why $x=t$. (b) Substitute to find y in terms of x. (c) State the Cartesian equation and name the curve. (d) The domain is $t\ge 0$. Which half of the curve is traced, and where does it start?

Q6. Consider $r(t)=3\cos t\, i+3\sin t\, j,\ 0\le t\le\frac{\pi}{2}$. (a) Find the Cartesian equation of the whole curve. (b) Find the points at $t=0$ and $t=\frac{\pi}{2}$. (c) Describe the arc that is actually traced. (d) State the direction of travel.

Unscaffolded

Q7. Find the Cartesian equation and describe the path of $r(t)=7\cos t\, i+7\sin t\, j,\ 0\le t<2\pi$, stating the direction of travel.

Q8. Show that $r(t)=3\cos t\, i+8\sin t\, j$ traces an ellipse, and state its axis intercepts.

Q9. Find the Cartesian equation of the path of $r(t)=5\,sec\,t\,i+2\tan t\,j$ for $-\frac{\pi}{2}<t<\frac{\pi}{2}$, and give the equations of its asymptotes.

Q10. The path of a point is $r(t)=(4-t)i+(2t+1)j$. Find its Cartesian equation and the gradient of the line.

Q11. Find the Cartesian equation of the path of $r(t)=(t+2)i+t^2 j$, and state the coordinates of its vertex.

Q12. Show that $r(t)=(2+4\cos t)i+(-1+4\sin t)j$ traces a circle, and state its centre and radius.

Q13. The vector function $r(t)=6\cos t\,i+4\sin t\,j$ is restricted to $0\le t\le\pi$. Find the Cartesian equation of the full curve, and describe the arc that is traced, including its endpoints.

Q14. Consider $r(t)=5\cos t\,i+5\sin t\,j$ and $s(t)=5\cos t\,i-5\sin t\,j$ on $0\le t<2\pi$. Show that they have the same Cartesian equation but opposite directions of travel.

Q15. Find the Cartesian equation of the path of $r(t)=3\tan t\,i+5\,sec\,t\,j$ for $-\frac{\pi}{2}<t<\frac{\pi}{2}$, name the curve, and state which branch is traced.

Q16. A point moves so that $r(t)=\cos t\,i+\cos 2t\,j$, $0\le t<2\pi$. Using $\cos 2t=2\cos^2 t-1$, show that the path lies on a parabola, and state the range of x.

Q17. For $r(t)=2t\,i+(t^2-4)j$, find the Cartesian equation of the path and the coordinates of the points where it crosses the coordinate axes.

Q18. A particle moves along the helix $r(t)=3\cos t\,i+3\sin t\,j+2t\,k$, $t\ge 0$. (a) Show that its shadow in the $x\,y$-plane moves on a circle, and state the radius of that circle. (b) Find its position when it first returns to a point directly above its starting point.

Answers

Q1. $x^2+y^2=36$; a circle centred at O with radius 6; anticlockwise. **Q2.** $\frac{x^2}{4}+\frac{y^2}{25}=1$; an ellipse; intercepts $(\pm 2,0)$ and $(0,\pm 5)$. **Q3.** $\frac{x^2}{16}-\frac{y^2}{9}=1$; asymptotes $y=\pm\frac{3}{4}x$; right branch $x\geq 4$. **Q4.** $x+3y=5$ (that is, $y=\frac{5-x}{3}$); gradient $-\frac{1}{3}$; through $(2,1)$. **Q5.** $y=x^2+1$; a parabola; the right half $x\geq 0$, starting at the vertex $(0,1)$. **Q6.** $x^2+y^2=9$; a quarter circle in the first quadrant from $(3,0)$ to $(0,3)$, anticlockwise. **Q7.** $x^2+y^2=49$; a circle centred at O with radius 7; anticlockwise. **Q8.** $\frac{x^2}{9}+\frac{y^2}{64}=1$; intercepts $(\pm 3,0)$ and $(0,\pm 8)$. **Q9.** $\frac{x^2}{25}-\frac{y^2}{4}=1$; asymptotes $y=\pm\frac{2}{5}x$ (right branch $x\geq 5$). **Q10.** $2x+y=9$ (that is, $y=9-2x$); gradient -2. **Q11.** $y=(x-2)^2$; vertex $(2,0)$. **Q12.** $(x-2)^2+(y+1)^2=16$; centre $(2,-1)$, radius 4. **Q13.** $\frac{x^2}{36}+\frac{y^2}{16}=1$; the upper half ($y\geq 0$) from $(6,0)$ through $(0,4)$ to $(-6,0)$, anticlockwise. **Q14.** Both give $x^2+y^2=25$; r is anticlockwise and s is clockwise. **Q15.** $\frac{y^2}{25}-\frac{x^2}{9}=1$; a hyperbola; the upper branch $y\geq 5$. **Q16.** $y=2x^2-1$, with $-1\leq x\leq 1$. **Q17.** $y=\frac{x^2}{4}-4$; crosses the x-axis at $(\pm 4,0)$ and the y-axis at $(0,-4)$. **Q18.** (a) $x^2+y^2=9(\cos^2 t+\sin^2 t)=9$, a circle of radius 3; (b) $(3,0,4\pi)$, reached at $t=2\pi$.

Fully-worked solutions

Q1. The components give $x=6\cos t$ and $y=6\sin t$. Then $x^2+y^2=36\cos^2 t+36\sin^2 t=36(\cos^2 t+\sin^2 t)=36$, so the path is $x^2+y^2=36$, a circle of radius 6 centred at the origin. At $t=0$ the point is $(6,0)$ and at $t=\frac{\pi}{2}$ it is $(0,6)$, so the point travels anticlockwise.

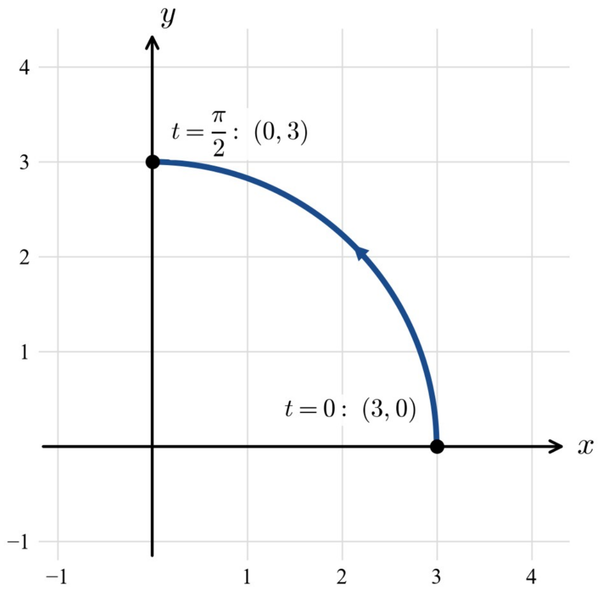
Q2. From $x=2\cos t$ and $y=5\sin t$ we get $\cos t=\frac{x}{2}$ and $\sin t=\frac{y}{5}$. Substituting into $\cos^2 t+\sin^2 t=1$ gives $\frac{x^2}{4}+\frac{y^2}{25}=1$, an ellipse. Setting $y=0$ gives $x=\pm 2$, and $x=0$ gives $y=\pm 5$, so the intercepts are $(\pm 2,0)$ and $(0,\pm 5)$.

Q3. From $x=4\sec t$ and $y=3\tan t$ we have $\sec t=\frac{x}{4}$ and $\tan t=\frac{y}{3}$. Then $\sec^2 t-\tan^2 t=1$ gives $\frac{x^2}{16}-\frac{y^2}{9}=1$, a hyperbola with $a=4$, $b=3$ and asymptotes $y=\pm\frac{3}{4}x$. For $-\frac{\pi}{2}<t<\frac{\pi}{2}$, $\cos t>0$, so $x=4\sec t=\frac{4}{\cos t}\geq 4$; only the right branch ($x\geq 4$) is traced.

Q4. Here $x=2+3t$ and $y=1-t$. Making t the subject of the first, $t=\frac{x-2}{3}$. Substituting, $y=1-\frac{x-2}{3}=\frac{3-(x-2)}{3}=\frac{5-x}{3}$, that is $x+3y=5$. This is a straight line of gradient $-\frac{1}{3}$; at $t=0$ it passes through $(2,1)$.

Q5. Because the i-component is t, we have $x=t$, and then $y=t^2+1=x^2+1$. The path lies on the parabola $y=x^2+1$ with vertex $(0,1)$. Since $t\geq 0$ and $x=t$, only the part with $x\geq 0$ is traced — the right half of the parabola — starting at the vertex $(0,1)$ (when $t=0$) and rising to the right.

Q6. With $x=3\cos t$ and $y=3\sin t$, $x^2+y^2=9(\cos^2 t+\sin^2 t)=9$, so the whole curve is the circle $x^2+y^2=9$. On the restricted domain, $t=0$ gives $(3,0)$ and $t=\frac{\pi}{2}$ gives $(0,3)$; as t runs from 0 to $\frac{\pi}{2}$ both $x\geq 0$ and $y\geq 0$, so only the quarter of the circle in the first quadrant is traced, anticlockwise from $(3,0)$ to $(0,3)$.



Q7. $x=7\cos t$ and $y=7\sin t$, so $x^2+y^2=49(\cos^2 t+\sin^2 t)=49$: a circle centred at the origin with radius 7. At $t=0$ the point is $(7,0)$ and at $t=\frac{\pi}{2}$ it is $(0,7)$, so the travel is anticlockwise.

Q8. From $\cos t=\frac{x}{3}$ and $\sin t=\frac{y}{8}$, $\frac{x^2}{9}+\frac{y^2}{64}=\cos^2 t+\sin^2 t=1$, an ellipse. Its intercepts are $(\pm 3,0)$ and $(0,\pm 8)$. (Because the larger denominator is under y^2, the major axis lies along the y-axis.)

Q9. From $sec\,t=\frac{x}{5}$ and $\tan t=\frac{y}{2}$, $\frac{x^2}{25}-\frac{y^2}{4}=sec^2 t-\tan^2 t=1$, a hyperbola with $a=5$, $b=2$ and asymptotes $y=\pm\frac{2}{5}x$. On $-\frac{\pi}{2}<t<\frac{\pi}{2}$ only the right branch ($x\geq 5$) is traced.

Q10. $x=4-t$ and $y=2t+1$. From the first, $t=4-x$; substituting, $y=2(4-x)+1=9-2x$, that is $2x+y=9$, a straight line of gradient -2.

Q11. $x=t+2$ gives $t=x-2$, and $y=t^2=(x-2)^2$. The path is the parabola $y=(x-2)^2$, which opens upward with vertex $(2,0)$.

Q12. $x=2+4\cos t$ and $y=-1+4\sin t$, so $\cos t=\frac{x-2}{4}$ and $\sin t=\frac{y+1}{4}$. Then $(x-2)^2+(y+1)^2=16(\cos^2 t+\sin^2 t)=16$, a circle with centre $(2,-1)$ and radius 4.

Q13. From $\cos t=\frac{x}{6}$ and $\sin t=\frac{y}{4}$, $\frac{x^2}{36}+\frac{y^2}{16}=1$, an ellipse. For $0\leq t\leq\pi$, $\sin t\geq 0$, so $y=4\sin t\geq 0$: only the upper half is traced. It runs from $t=0$ at $(6,0)$, through $t=\frac{\pi}{2}$ at $(0,4)$, to $t=\pi$ at $(-6,0)$, anticlockwise.

Q14. For r: $x=5\cos t$, $y=5\sin t$, so $x^2+y^2=25$. For s: $x=5\cos t$, $y=-5\sin t$, so again $x^2+y^2=25\cos^2 t+25\sin^2 t=25$. Both trace the circle of radius 5 centred at O. However, at $t=\frac{\pi}{2}$, r is at $(0,5)$

(anticlockwise from $(5,0)$) while s is at $(0,-5)$ (clockwise). So the paths coincide but the directions of travel are opposite.

Q15. $x=3\tan t$ and $y=5\sec t$, so $\tan t=\frac{x}{3}$ and $\sec t=\frac{y}{5}$. Then $\frac{y^2}{25}-\frac{x^2}{9}=\sec^2 t-\tan^2 t=1$, a hyperbola opening vertically. For $-\frac{\pi}{2}<t<\frac{\pi}{2}$, $\cos t>0$, so $y=5\sec t\geq 5$: the upper branch is traced, with vertex $(0,5)$.

Q16. $x=\cos t$ and $y=\cos 2t=2\cos^2 t-1=2x^2-1$, so the path lies on the parabola $y=2x^2-1$. Since $x=\cos t$ and $0\leq t<2\pi$, the value of x ranges over all of $[-1,1]$; hence the path is the arc of the parabola with $-1\leq x\leq 1$, from $(-1,1)$ through the vertex $(0,-1)$ to $(1,1)$.

Q17. $x=2t$ gives $t=\frac{x}{2}$, so $y=t^2-4=\frac{x^2}{4}-4$. It crosses the y-axis where $x=0$: $y=-4$, that is $(0,-4)$. It crosses the x-axis where $y=0$: $\frac{x^2}{4}=4$, so $x^2=16$ and $x=\pm 4$, giving $(\pm 4,0)$.

Q18. (a) The i- and j-components of the position are $x=3\cos t$ and $y=3\sin t$, so the shadow in the $x\,y$-plane (ignoring z) satisfies $x^2+y^2=9\cos^2 t+9\sin^2 t=9(\cos^2 t+\sin^2 t)=9$: it moves on the circle of radius 3 centred at the origin. (b) At $t=0$ the particle is at $(3,0,0)$. It is directly above this point again when $x=3$ and $y=0$, i.e. $\cos t=1$ and $\sin t=0$; the first positive solution is $t=2\pi$. Then $z=2t=4\pi$, so the position is $(3,0,4\pi)$.

Theory

A position vector that changes with time.

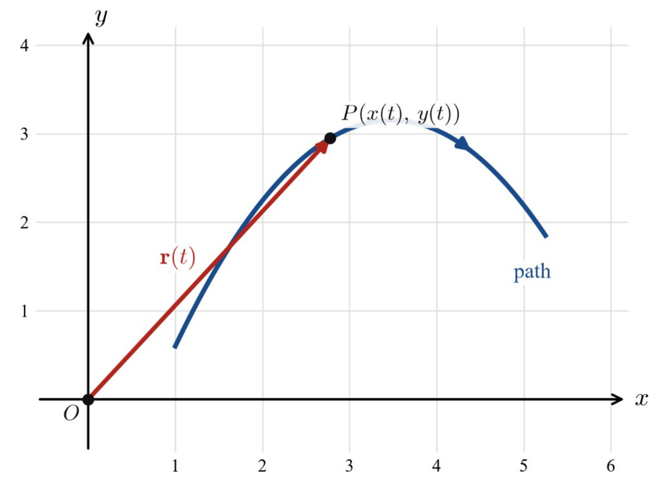
When a particle moves in a plane, its location at each instant can be recorded by the **position vector** drawn from a fixed origin O to the particle. As the particle moves this vector changes, so its components are functions of the time t. We write

$$r(t) = x(t)i + y(t)j,$$

where $x(t)$ and $y(t)$ are the coordinates of the particle at time t, and i, j are the standard unit vectors along the axes. A rule of this kind is called a **vector function** of t. Unless stated otherwise the time satisfies $t \geq 0$.

Locating the particle.

To find where the particle is at a particular instant, substitute that value of t into the components. At time t the particle is at the point $(x(t), y(t))$; for example the starting point is $r(0)$. The same idea in reverse answers *when* a particle reaches a given point: set $x(t)$ and $y(t)$ equal to the coordinates of the point and solve — the value of t must satisfy **both** component equations.



The path of the particle.

As t increases the point $(x(t), y(t))$ traces out a curve called the **path** (or **trajectory**) of the particle. The vector function gives the path in **parametric form**, with t the parameter. Eliminating t between $x = x(t)$ and $y = y(t)$ produces the **Cartesian equation** of the path, relating x and y directly. Because t is restricted, the path is in general only *part* of the curve with that equation, so the range of x (or of y) actually traced should be stated as well. The standard parametric curves — circles, ellipses and hyperbolas, and parabolas — and how to sketch them were developed in an earlier section; here the emphasis is on reading a path off a position function and on comparing the motion of two particles. (The path records only the set of points visited; how *fast* the particle moves along it, through the velocity $\dot{r} = v$, is taken up in a later section.)

Distance from the origin and from a fixed point.

The distance of the particle from the origin at time t is the magnitude of its position vector,

$$|r(t)| = \sqrt{x(t)^2 + y(t)^2}.$$

More generally, the distance from a **fixed point** D with position vector d is the magnitude of the vector from D to the particle,

$$|r(t)-d|=\sqrt{(x(t)-d_1)^2+(y(t)-d_2)^2}.$$

Each of these is a function of t. Its **least value** — the closest the particle ever comes to that point — is found most easily by working with the *square* of the distance, which avoids the surd. Because a distance is never negative, it is smallest at exactly the same time as its square is smallest, so we minimise

$$f(t)=|r(t)-d|^2$$

by differentiating and solving $f'(t)=0$, then take the square root at the end. When both components are linear in t the function f is a **quadratic** in t, and completing the square gives the same answer without calculus; when a component is non-linear — a parabolic path, for instance — f has higher degree and the calculus method is the one to use.

Two particles: do they meet, or do their paths merely cross?

Suppose two particles have position vectors $r_1(t)$ and $r_2(t)$. Two quite different questions can be asked.

Do the particles meet (collide)? They meet only if they are at the **same place at the same time**. So we set the position vectors equal at the *same* value of t,

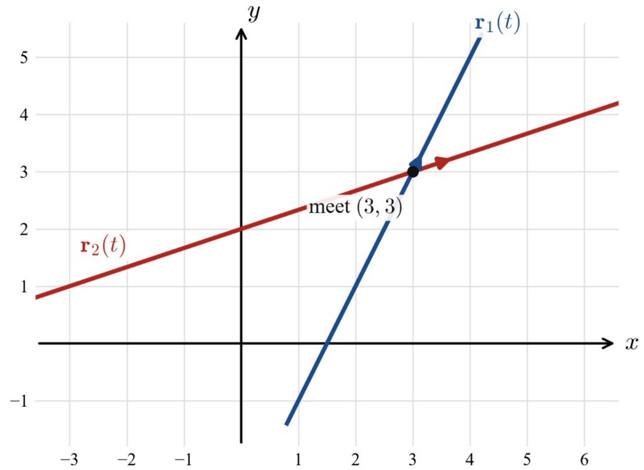
$$r_1(t)=r_2(t),$$

which gives one equation from the i-components and one from the j-components. The particles collide only if a **single** value of t satisfies **both** equations; that common t gives the time and place of the collision. If no value of t satisfies both equations, the particles never meet.

Do their paths cross? Two paths cross wherever they share a common point, even if the particles pass through it at **different** times. To test this we allow the parameters to be different — say s for the first particle and t for the second — and solve

$$r_1(s)=r_2(t).$$

A solution (s,t) with **both** parameters in the allowed domain gives a crossing point of the paths; a solution that needs a value of s or t outside the domain must be rejected, because neither particle ever visits the point it gives. The particles actually collide at a crossing only in the special case $s=t$. So meeting is a stronger condition than the paths crossing.



The distance between the two particles, and their closest approach.

The vector from the first particle to the second is $r_2(t)-r_1(t)$, so the distance between them at time t is

$$D(t)=|r_2(t)-r_1(t)|.$$

This is a function of t in its own right. The **closest approach** is its minimum value. As before we minimise the square $D(t)^2$ — a quadratic in t whenever both motions are linear in t: differentiate, solve $\frac{d}{dt}\left(D(t)^2\right)=0$ for the time of closest approach, and substitute back to get the minimum distance. If that minimum distance is 0 the particles actually collide; if it is positive they never touch, and it measures how near they come.

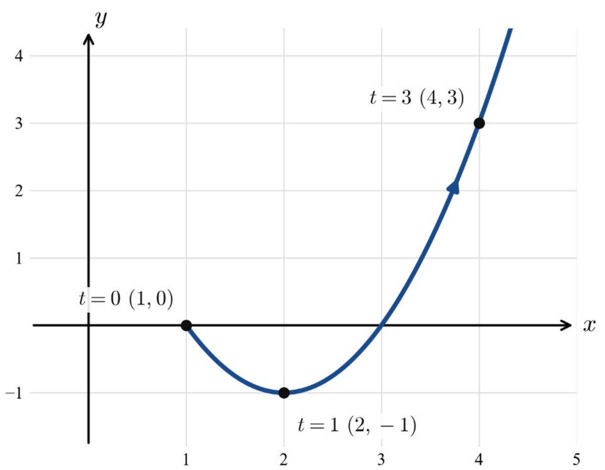
Worked examples

Example 1 — scaffolded

A particle moves so that its position vector at time $t\geq 0$ is $r(t)=(t+1)i+(t^2-2t)j$. Find its position at $t=0$, $t=1$ and $t=3$, and find the Cartesian equation of its path.

Working	Comment
$r(0)=(0+1)i+(0-0)j=i$, the point $(1,0)$.	Substitute $t=0$ into each component.
$r(1)=2i+(1-2)j=2i-j$, the point $(2,-1)$.	Substitute $t=1$.
$r(3)=4i+(9-6)j=4i+3j$, the point $(4,3)$.	Substitute $t=3$.
$x=t+1\Rightarrow t=x-1$.	Make t the subject of the simpler component.
$y=t^2-2t=(x-1)^2-2(x-1)=x^2-4x+3$.	Substitute into y and expand to eliminate t.

The path is the parabola $y=x^2-4x+3$, drawn for $x\geq 1$ since $t\geq 0$. The particle starts at $(1,0)$ and moves along the curve in the direction of increasing t.



Example 2 — scaffolded

A particle has position vector $r(t)=(t+3)i+(2t-4)j$, $t\geq 0$. Find its distance from the origin as a function of t, its distance from the fixed point $A(2,4)$ when $t=2$, and the time at which it is closest to the origin.

Working	Comment
$\lvert r(t)\rvert^2=(t+3)^2+(2t-4)^2=5t^2-10t+25$.	Square the components and add; expand.
$\lvert r(t)\rvert=\sqrt{5t^2-10t+25}$; at $t=0$, $\lvert r(0)\rvert=\sqrt{25}=5$.	Distance from O; the particle starts 5 units from O.
$r(2)=5i+0j$, so $\lvert r(2)-a\rvert=\sqrt{(5-2)^2+(0-4)^2}=\sqrt{25}=5$.	Distance from $A(2,4)$: subtract $a=2i+4j$ first.
Let $f(t)=5t^2-10t+25$; then $f'(t)=10t-10=0$ gives $t=1$.	Minimise the *square* of the distance; solve $f'(t)=0$.
$\lvert r(1)\rvert=\sqrt{5-10+25}=\sqrt{20}=2\sqrt{5}$, at the point $(4,-2)$.	Least distance from O ($f''(t)=10>0$ confirms a minimum).

Example 3 — unscaffolded

Two particles have position vectors $r_1(t)=(t-1)i+(t+2)j$ and $r_2(t)=(3+2t)i+(6-t)j$, $t\geq 0$. Determine whether the particles collide, and whether their paths cross.

For a collision the particles must be at the same place at the same time, so set $r_1(t)=r_2(t)$ and equate components:

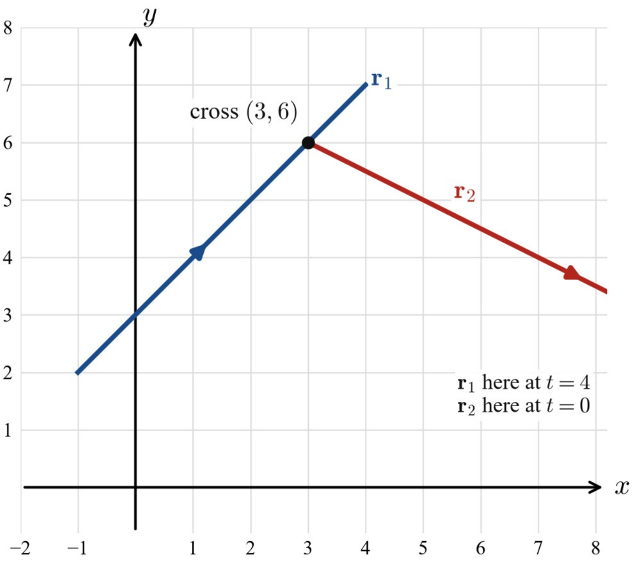
$$t-1=3+2t \Rightarrow t=-4, t+2=6-t \Rightarrow t=2.$$

The two components demand different values of t, and no single time satisfies both, so the particles **do not collide**.

To test whether the paths cross, allow different times s and t and solve $r_1(s)=r_2(t)$:

$$s-1=3+2t, s+2=6-t.$$

From the second equation $s=4-t$; substituting into the first gives $(4-t)-1=3+2t$, that is $3-t=3+2t$, so $t=0$ and $s=4$. The paths therefore cross: $r_2(0)=3i+6j$ and $r_1(4)=3i+6j$ are the same point $(3,6)$. The first particle reaches it at $t=4$ and the second at $t=0$, so they pass through the crossing point at different times.



$\therefore$ the particles do not collide, but their paths cross at $(3,6)$.

Example 4 — unscaffolded

Two particles have position vectors $r_1(t)=(3t-6)i+(2t+1)j$ and $r_2(t)=ti+(t-1)j$, $t\geq 0$. Find the least distance between the particles and the time at which it occurs.

The vector from the second particle to the first is

$$r_1(t)-r_2(t)=(3t-6-t)i+(2t+1-(t-1))j=(2t-6)i+(t+2)j.$$

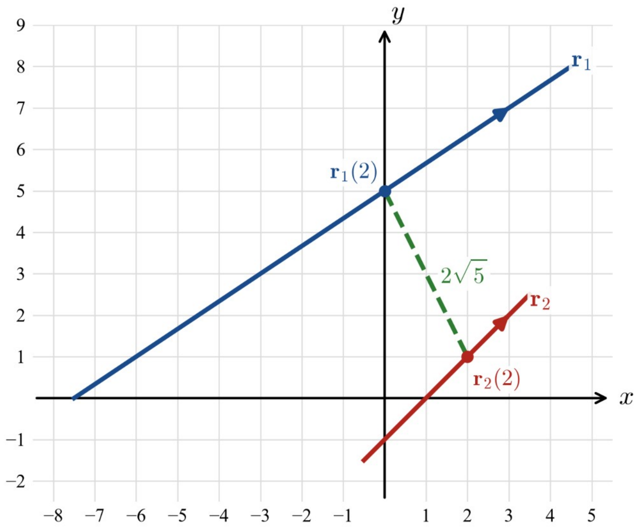
The square of the distance between the particles is

$$D(t)^2=(2t-6)^2+(t+2)^2=5t^2-20t+40.$$

Minimising this quadratic, $\frac{d}{dt}\left(D(t)^2\right)=10t-20=0$ gives $t=2$, and the second derivative $10>0$ confirms a minimum. The least distance is therefore

$$D(2)=\sqrt{5(2)^2-20(2)+40}=\sqrt{20}=2\sqrt{5}.$$

At that instant the particles are at $r_1(2)=0i+5j=(0,5)$ and $r_2(2)=2i+j=(2,1)$, which are indeed $2\sqrt{5}$ apart. As the minimum distance is not zero, the particles never collide.



$\therefore$ the closest approach is $2\sqrt{5}$, occurring at $t = 2$.

Practice questions

Scaffolded

Q1. A particle has position vector $r(t) = (2t-1)i + (t+3)j$, $t \geq 0$. (a) Find its position when $t = 0$. (b) Find its position when $t = 2$. (c) By making t the subject of $x = 2t-1$, find the Cartesian equation of the path. (d) State what kind of curve the path is.

Q2. A particle has position vector $r(t) = (3t+1)i + (4t-7)j$, $t \geq 0$. (a) Show that $|r(t)|^2 = 25t^2 - 50t + 50$. (b) Find the distance of the particle from the origin when $t = 3$. (c) Minimise $|r(t)|^2$ to find the time when the particle is closest to the origin. (d) State the least distance from the origin.

Q3. Two particles have position vectors $r_1(t) = (2t+1)i + (t-2)j$ and $r_2(t) = (t+3)i + (3t-6)j$. (a) Set the i-components equal and solve for t. (b) Set the j-components equal and solve for t. (c) Explain why the particles collide, stating the time. (d) State the point at which they meet.

Q4. Two particles have position vectors $r_1(t) = ti + (t+1)j$ and $r_2(t) = 2ti + (7-t)j$. (a) By solving $r_1(t) = r_2(t)$, show that the particles do not collide. (b) To test the paths, solve $r_1(s) = r_2(t)$ for s and t. (c) State the crossing point of the two paths. (d) State the time at which each particle passes through the crossing point.

Q5. Two particles have position vectors $r_1(t) = (2t+1)i + (3t-8)j$ and $r_2(t) = ti + tj$. (a) Find $r_1(t) - r_2(t)$. (b) Show that the square of the distance between the particles is $5t^2 - 30t + 65$. (c) Find the time of closest approach. (d) Find the least distance between the particles.

Q6. A particle has position vector $r(t) = (t+1)i + (2t-1)j$, and A is the fixed point with position vector $a = i + 4j$. (a) Find the vector $r(t) - a$. (b) Show that $|r(t) - a|^2 = 5t^2 - 20t + 25$. (c) Find the time at which the particle is closest to A. (d) Find the least distance from A.

Unscaffolded

Q7. A particle has position vector $r(t) = (3-t)i + t^2 j$, $t \geq 0$. Find its position at $t = 0$, $t = 1$ and $t = 2$, and show that the Cartesian equation of its path is $y = (x-3)^2$.

Q8. A particle has position vector $r(t) = (t+1)i + (t-5)j$, $t \geq 0$. Find the least distance of the particle from the origin and the time at which it occurs.

Q9. Two particles have position vectors $r_1(t) = (4t-5)i + (t+2)j$ and $r_2(t) = (t+4)i + (2t-1)j$. Show that the particles collide, and state the time and place of the collision.

Q10. Two particles have position vectors $r_1(t)=(t+2)i+2t\,j$ and $r_2(t)=t\,i+(5-t)\,j$. Determine whether the particles meet, and whether their paths cross; if the paths cross, state the point.

Q11. Two particles have position vectors $r_1(t)=(2t-1)i+(2t-5)\,j$ and $r_2(t)=t\,i+t\,j$. Find the least distance between the particles and the time at which it occurs.

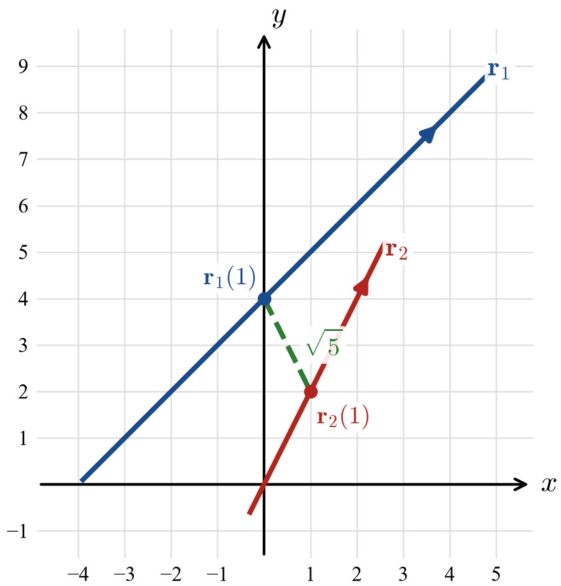
Q12. A particle has position vector $r(t)=3t\,i+(t+5)\,j$. Find the least distance of the particle from the fixed point $(4,3)$.

Q13. Two particles have position vectors $r_1(t)=(t+1)i+2t\,j$ and $r_2(t)=(3t-1)i+(t+2)\,j$. (a) Find the distance $D(t)$ between the particles as a function of t. (b) Find the distance between them when $t=2$. (c) Determine whether the particles collide.

Q14. A particle moves in three dimensions with position vector $r(t)=t\,i+(2t-1)\,j+(t+2)k$, $t\geq 0$. (a) Find the position of the particle at $t=1$ and at $t=2$. (b) Find the distance of the particle from the origin as a function of t. (c) Hence find the least distance of the particle from the origin.

Q15. Two particles have position vectors $r_1(t)=t\,i+t^2\,j$ and $r_2(t)=2t\,i+(2t+2)\,j$, $t\geq 0$. By solving $r_1(s)=r_2(t)$, show that the paths cross at exactly one point and state it, explaining why the second solution of your equation must be rejected. Determine whether the particles collide.

Q16. Two particles have position vectors $r_1(t)=(3t-3)i+(3t+1)\,j$ and $r_2(t)=t\,i+2t\,j$. Find the least distance between the particles and the time at which it occurs.



Q17. Two drones fly in a horizontal plane with position vectors (in metres, with t in seconds) $r_1(t)=(3t-1)i+(2t+1)\,j$ and $r_2(t)=(t+7)i+(4t-7)\,j$. Show that the drones collide, and find when and where.

Q18. Two particles have position vectors $r_1(t)=(2t+3)i+(3t-1)\,j$ and $r_2(t)=(t+4)i+(2t+8)\,j$, $t\geq 0$. (a) Show that the particles never collide. (b) Find the distance between them as a function of t. (c) Find the time of closest approach and the least distance between the particles.

Answers

Q1. (a) $(-1,3)$; (b) $(3,5)$; (c) $y=\frac{x+7}{2}$; (d) a straight line — only the part with $x\geq -1$ is traced, the ray from $(-1,3)$. **Q2.** (a) shown; (b) $5\sqrt{5}$; (c) $t=1$; (d) 5. **Q3.** (a) $t=2$; (b) $t=2$; (c) both components give $t=2$, so they are at the same point at that time; (d) $(5,0)$. **Q4.** (a) the components give $t=0$ and $t=3$, no common t; (b) $s=4$, $t=2$; (c) $(4,5)$; (d) particle 1 at $t=4$, particle 2 at $t=2$. **Q5.** (a) $(t+1)i+(2t-8)j$; (b) shown; (c) $t=3$; (d) $2\sqrt{5}$. **Q6.** (a) $ti+(2t-5)j$; (b) shown; (c) $t=2$; (d) $\sqrt{5}$. **Q7.** $(3,0)$, $(2,1)$, $(1,4)$; path $y=(x-3)^2$, for $x\leq 3$ only. **Q8.** least distance $3\sqrt{2}$ at $t=2$. **Q9.** collide at $t=3$ at the point $(7,5)$. **Q10.** they do not meet; the paths cross at $(3,2)$. **Q11.** least distance $2\sqrt{2}$ at $t=3$. **Q12.** $\sqrt{10}$ (at $t=1$). **Q13.** (a) $D(t)=\sqrt{5t^2-12t+8}$; (b) 2; (c) no common time, so they do not collide. **Q14.** (a) $(1,1,3)$ and $(2,3,4)$; (b) $|r(t)|=\sqrt{6t^2+5}$; (c) $\sqrt{5}$ at $t=0$. **Q15.** the paths cross at exactly one point, $(2,4)$ (the other root, $t=-1/2$ with $s=-1$, is rejected since $t\geq 0$); the particles do not collide. **Q16.** least distance $\sqrt{5}$ at $t=1$. **Q17.** collide after 4 seconds at the point $(11,9)$. **Q18.** (a) shown; (b) $D(t)=\sqrt{2t^2-20t+82}$; (c) $t=5$, least distance $4\sqrt{2}$.

Fully-worked solutions

Q1. Substituting, $r(0)=-i+3j$, the point $(-1,3)$, and $r(2)=3i+5j$, the point $(3,5)$. From $x=2t-1$, $t=\frac{x+1}{2}$, so

$$y=t+3=\frac{x+1}{2}+3=\frac{x+7}{2}.$$

Since y is a linear function of x, the path is a straight line, of gradient $1/2$; and since $t\geq 0$ gives $x=2t-1\geq -1$, only the part with $x\geq -1$ is traced — the ray starting at $(-1,3)$.

Q2. (a) $|r(t)|^2=(3t+1)^2+(4t-7)^2=9t^2+6t+1+16t^2-56t+49=25t^2-50t+50$. (b) At $t=3$, $|r(3)|^2=25(9)-150+50=125$, so $|r(3)|=\sqrt{125}=5\sqrt{5}$. (c) Let $f(t)=25t^2-50t+50$; then $f'(t)=50t-50=0$ gives $t=1$, and $f''(t)=50>0$, so this is the minimum. (d) $|r(1)|=\sqrt{25-50+50}=\sqrt{25}=5$, at the point $(4,-3)$.

Q3. Setting the i-components equal, $2t+1=t+3$, so $t=2$. Setting the j-components equal, $t-2=3t-6$, so $2t=4$ and $t=2$. Both components give the same time $t=2$, so the particles are at the same point at the same instant and therefore collide. At $t=2$, $r_1(2)=5i+0j$, so they meet at $(5,0)$.

Q4. (a) Solving $r_1(t)=r_2(t)$: from the i-components $t=2t$, so $t=0$; from the j-components $t+1=7-t$, so $t=3$. The two times differ, so the particles do not collide. (b) Allowing different parameters, $r_1(s)=r_2(t)$ gives $s=2t$ and $s+1=7-t$. Substituting, $2t+1=7-t$, so $3t=6$, $t=2$ and $s=4$. (c) The crossing point is $r_1(4)=4i+5j$, that is $(4,5)$. (d) Particle 1 is there at $t=4$ and particle 2 at $t=2$; since these differ, the paths cross without the particles meeting.

Q5. (a) $r_1(t)-r_2(t)=(2t+1-t)i+(3t-8-t)j=(t+1)i+(2t-8)j$. (b) The square of the distance is $(t+1)^2+(2t-8)^2=t^2+2t+1+4t^2-32t+64=5t^2-30t+65$. (c) Differentiating, $10t-30=0$ gives $t=3$ (and the second derivative $10>0$ confirms a minimum). (d) The least distance is $\sqrt{5(9)-30(3)+65}=\sqrt{45-90+65}=\sqrt{20}=2\sqrt{5}$.

Q6. (a) $r(t)-a=(t+1-1)i+(2t-1-4)j=ti+(2t-5)j$. (b) $|r(t)-a|^2=t^2+(2t-5)^2=t^2+4t^2-20t+25=5t^2-20t+25$. (c) Differentiating, $10t-20=0$ gives $t=2$. (d) The least distance is $\sqrt{5(4)-20(2)+25}=\sqrt{20-40+25}=\sqrt{5}$.

Q7. Substituting, $r(0)=3i$, the point $(3,0)$; $r(1)=2i+j$, the point $(2,1)$; and $r(2)=i+4j$, the point $(1,4)$. From $x=3-t$, $t=3-x$, so $y=t^2=(3-x)^2=(x-3)^2$. Since $t\geq 0$ gives $x=3-t\leq 3$, the path is not the whole parabola but only the part of $y=(x-3)^2$ with $x\leq 3$ — the left arm, traced from the vertex $(3,0)$ upwards.

Q8. $|r(t)|^2=(t+1)^2+(t-5)^2=t^2+2t+1+t^2-10t+25=2t^2-8t+26$. Differentiating, $4t-8=0$ gives $t=2$ (second derivative $4>0$, a minimum). Then $|r(2)|=\sqrt{2(4)-16+26}=\sqrt{18}=3\sqrt{2}$, so the least distance from the origin is $3\sqrt{2}$, at $t=2$ (the point $(3,-3)$).

Q9. Setting $r_1(t)=r_2(t)$: the i-components give $4t-5=t+4$, so $3t=9$ and $t=3$; the j-components give $t+2=2t-1$, so $t=3$. Both give $t=3$, so the particles collide at that time. Then $r_1(3)=7i+5j$, so the collision is at $(7,5)$.

Q10. For a collision, $r_1(t)=r_2(t)$ requires $t+2=t$ from the i-components, which is impossible; the particles never meet. For the paths, solve $r_1(s)=r_2(t)$: $s+2=t$ and $2s=5-t$. Substituting $t=s+2$ into the second equation, $2s=5-(s+2)=3-s$, so $3s=3$, giving $s=1$ and $t=3$. The crossing point is $r_1(1)=3i+2j$, that is $(3,2)$. So the paths cross at $(3,2)$ although the particles do not meet.

Q11. $r_1(t)-r_2(t)=(2t-1-t)i+(2t-5-t)j=(t-1)i+(t-5)j$. The square of the distance is $(t-1)^2+(t-5)^2=t^2-2t+1+t^2-10t+25=2t^2-12t+26$. Differentiating, $4t-12=0$ gives $t=3$ (a minimum, since $4>0$). The least distance is $\sqrt{2(9)-36+26}=\sqrt{8}=2\sqrt{2}$, occurring at $t=3$.

Q12. With $d=4i+3j$, $r(t)-d=(3t-4)i+(t+5-3)j=(3t-4)i+(t+2)j$. The square of the distance is $(3t-4)^2+(t+2)^2=9t^2-24t+16+t^2+4t+4=10t^2-20t+20$. Differentiating, $20t-20=0$ gives $t=1$ (a minimum, since $20>0$), and the least distance is $\sqrt{10-20+20}=\sqrt{10}$; at that instant the particle is at $(3,6)$.

Q13. (a) $r_1(t)-r_2(t)=(t+1-(3t-1))i+(2t-(t+2))j=(2-2t)i+(t-2)j$, so

$$D(t)=\sqrt{(2-2t)^2+(t-2)^2}=\sqrt{4t^2-8t+4+t^2-4t+4}=\sqrt{5t^2-12t+8}.$$

(b) $D(2)=\sqrt{5(4)-24+8}=\sqrt{4}=2$. (c) A collision needs $r_1(t)=r_2(t)$: the i-components give $t+1=3t-1$, so $t=1$, while the j-components give $2t=t+2$, so $t=2$. No single time works, so the particles do not collide (consistent with $D(t)>0$ for all t).

Q14. (a) $r(1)=i+j+3k$, the point $(1,1,3)$; $r(2)=2i+3j+4k$, the point $(2,3,4)$. (b) $|r(t)|^2=t^2+(2t-1)^2+(t+2)^2=t^2+4t^2-4t+1+t^2+4t+4=6t^2+5$, so $|r(t)|=\sqrt{6t^2+5}$. (c) Since $6t^2+5$ is smallest when $t=0$ (for $t\geq 0$), the least distance from the origin is $\sqrt{5}$, at $t=0$.

Q15. Solving $r_1(s)=r_2(t)$: the i-components give $s=2t$, and the j-components give $s^2=2t+2$. Substituting $s=2t$,

$$(2t)^2=2t+2\Rightarrow 4t^2-2t-2=0\Rightarrow 2t^2-t-1=0\Rightarrow(2t+1)(t-1)=0,$$

so $t=1$ or $t=-1/2$. The root $t=1$ gives $s=2$; both parameters are then non-negative, as required, and the point is $r_2(1)=2i+4j=(2,4)$, which is also $r_1(2)$. The root $t=-1/2$ needs $s=-1$, and both of these values are negative, so this solution must be rejected: the point it gives, $(-1,1)$, lies on neither path, since particle 1 traces only the half-parabola $y=x^2$ with $x\geq 0$, and particle 2 only the ray $y=x+2$ with $x\geq 0$ from $(0,2)$. So the paths cross at exactly one point, $(2,4)$. For a collision we would need $r_1(t)=r_2(t)$; the i-components give $t=2t$, so $t=0$, but then $r_1(0)=(0,0)$ while $r_2(0)=(0,2)$, which differ. Hence the particles do not collide.

Q16. $r_1(t)-r_2(t)=(3t-3-t)i+(3t+1-2t)j=(2t-3)i+(t+1)j$. The square of the distance is $(2t-3)^2+(t+1)^2=4t^2-12t+9+t^2+2t+1=5t^2-10t+10$. Differentiating, $10t-10=0$ gives $t=1$ (a minimum). The least distance is $\sqrt{5-10+10}=\sqrt{5}$, occurring at $t=1$; then the particles are at $r_1(1)=(0,4)$ and $r_2(1)=(1,2)$, which are $\sqrt{5}$ apart.

Q17. Setting $r_1(t)=r_2(t)$: the i-components give $3t-1=t+7$, so $2t=8$ and $t=4$; the j-components give $2t+1=4t-7$, so $8=2t$ and $t=4$. Both components give $t=4$, so the drones are at the same point at the same time and collide. At $t=4$, $r_1(4)=11i+9j$, so the collision occurs after 4 seconds at the point $(11,9)$, measured in metres from the origin.

Q18. (a) A collision needs $r_1(t)=r_2(t)$: the i-components give $2t+3=t+4$, so $t=1$, while the j-components give $3t-1=2t+8$, so $t=9$. No single time satisfies both, so the particles never collide. (b) $r_1(t)-r_2(t)=((2t+3)-(t+4))i+((3t-1)-(2t+8))j=(t-1)i+(t-9)j$, so

$$D(t)=\sqrt{(t-1)^2+(t-9)^2}=\sqrt{t^2-2t+1+t^2-18t+81}=\sqrt{2t^2-20t+82}.$$

(c) Minimising $2t^2-20t+82$, the derivative $4t-20=0$ gives $t=5$ (a minimum, since $4>0$). The least distance is $\sqrt{2(25)-100+82}=\sqrt{32}=4\sqrt{2}$; at that instant the particles are at $r_1(5)=(13,14)$ and $r_2(5)=(9,18)$.

Theory

A **vector function** (or vector-valued function)

$$r(t)=x(t)i+y(t)j+z(t)k$$

assigns a vector to each value of the parameter t, usually **time**. As t varies, the point with position vector $r(t)$ traces a **path** in the plane or in space. This section develops the calculus of such functions: how to **differentiate** $r(t)$ to measure its rate of change, and how to **antidifferentiate** to recover $r(t)$ from a known rate of change. Throughout, i, j and k are the fixed standard unit vectors, which do **not** change with t.

Differentiating component-wise.

The derivative of $r(t)$ with respect to t is found by differentiating each component separately:

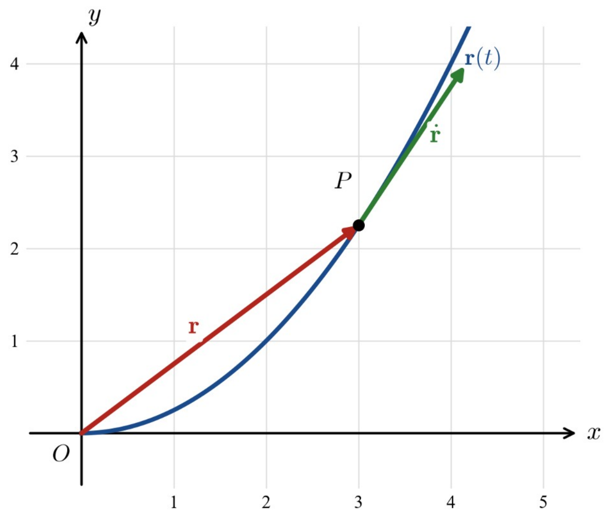
$$\dot{r}(t)=\frac{dr}{dt}=\dot{x}i+\dot{y}j+\dot{z}k.$$

This works because i, j, k are constant, so only the scalar coefficients $x(t)$, $y(t)$, $z(t)$ are differentiated. The overdot is Newton's notation for differentiation with respect to time. Geometrically, $\dot{r}(t)$ is a vector **tangent to the path** at the point $r(t)$, pointing in the direction of increasing t.

When $r(t)$ is the position of a moving particle, its derivative is the **velocity** and the derivative of the velocity is the **acceleration**:

$$v=\dot{r}=\dot{x}i+\dot{y}j+\dot{z}k,a=\ddot{r}=\ddot{x}i+\ddot{y}j+\ddot{z}k.$$

The **speed** is the magnitude of the velocity, $|v|=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$, a scalar. The figure shows a path with the position vector r to a point P and the velocity $\dot{r}$ drawn along the tangent there.



Rules of differentiation.

Because differentiation acts component-wise, the familiar product rules carry over to vector functions. For vector functions $r(t)$ and $s(t)$, a scalar function $f(t)$ and a constant k:

- **Linearity:** $\frac{d}{dt}(r+s)=\dot{r}+\dot{s}$ and $\frac{d}{dt}(kr)=k\dot{r}$.

- **Scalar function times a vector:** $\frac{d}{dt}(f(t)r(t))=f'(t)r(t)+f(t)\dot{r}(t)$.
- **Dot product:** $\frac{d}{dt}(r\cdot s)=\dot{r}\cdot s+r\cdot\dot{s}$.

Each has the same shape as the ordinary product rule. For example, with $f(t)=t^2$ and $r(t)=\cos t\, i+\sin t\, j$,

$$\frac{d}{dt}(t^2 r)=2t(\cos t\, i+\sin t\, j)+t^2(-\sin t\, i+\cos t\, j)=(2t\cos t-t^2\sin t)i+(2t\sin t+t^2\cos t)j.$$

A vector of constant length.

Applying the dot-product rule to $r\cdot r$ gives a useful identity:

$$\frac{d}{dt}(r\cdot r)=\dot{r}\cdot r+r\cdot\dot{r}=2r\cdot\dot{r}.$$

Since $r\cdot r=|r|^2$, if the **length** $|r|$ is **constant** then $r\cdot r$ is constant, its derivative is zero, and so

$$r\cdot\dot{r}=0,$$

which means r is **perpendicular** to $\dot{r}$ (whenever $\dot{r}\neq 0$). This is exactly the situation of a particle moving on a **circle** centred at the origin: its distance from O never changes, so the velocity is always at right angles to the position vector — the velocity runs along the tangent while the position vector is a radius.

Antidifferentiating component-wise.

Antidifferentiation reverses the process and is also carried out component-wise. If $r(t)=x(t)i+y(t)j+z(t)k$, then

$$\int r(t)\,dt=\left(\int x\,dt\right)i+\left(\int y\,dt\right)j+\left(\int z\,dt\right)k+c,$$

where the arbitrary constant is now a **constant vector** $c=c_1 i+c_2 j+c_3 k$ — one arbitrary constant per component, gathered into a single vector.

Recovering a vector function from its derivative.

Antidifferentiation together with an **initial condition** pins down the constant vector. Given the velocity $\dot{r}(t)$ and the position r at one instant, antidifferentiate and substitute the known value to find c. For instance, if $\dot{r}=2i+6t\,j$ and $r(0)=i-j$, then

$$r(t)=\int(2i+6t\,j)\,dt=2t\,i+3t^2 j+c,$$

and putting $t=0$ gives $r(0)=c=i-j$, so $r(t)=(2t+1)i+(3t^2-1)j$.

If instead the **acceleration** $\ddot{r}(t)$ is given, antidifferentiate **once** for the velocity (introducing a constant vector fixed by the initial velocity), then **again** for the position (introducing a second constant vector fixed by the initial position). Two vector initial conditions are needed, one for each antidifferentiation.

Worked examples

Example 1 — scaffolded

A particle has position vector $r(t)=(t^2+1)i+(t^3-4t)j$. Find the velocity $\dot{r}$ and acceleration $\ddot{r}$, and find the velocity, acceleration and speed when $t=2$.

Working	Comment
$\dot{r}=2t\,i+(3t^2-4)j$.	Differentiate each component of r.

Working	Comment
$\ddot{r}=2i+6tj$.	Differentiate each component of $\dot{r}$.
At $t=2$: $\dot{r}=4i+8j$.	Substitute $t=2$ into $\dot{r}$.
At $t=2$: $\ddot{r}=2i+12j$.	Substitute $t=2$ into $\ddot{r}$.
Speed $=\lvert\dot{r}(2)\rvert=\sqrt{4^2+8^2}=\sqrt{80}=4\sqrt{5}$.	Speed is the magnitude of the velocity.

Example 2 — scaffolded

A particle has velocity $\dot{r}(t)=(4t-1)i+3t^2j$ and is at the point with position vector $r(0)=2i-j$ when $t=0$. Find its position vector $r(t)$.

Working	Comment
$r(t)=\int\left((4t-1)i+3t^2j\right)dt=(2t^2-t)i+t^3j+c$.	Antidifferentiate each component; add a constant vector c.
At $t=0$: $r(0)=c=2i-j$.	Every t-term vanishes at $t=0$, leaving c.
$r(t)=(2t^2-t+2)i+(t^3-1)j$.	Substitute c back into the antiderivative.

Example 3 — unscaffolded

A particle moves so that its position vector is $r(t)=2\cos 3t\,i+2\sin 3t\,j$. Show that $\lvert r\rvert$ is constant, and hence that the velocity $\dot{r}$ is always perpendicular to r.

The magnitude of the position vector is

$$\lvert r\rvert=\sqrt{(2\cos 3t)^2+(2\sin 3t)^2}=\sqrt{4\cos^2 3t+4\sin^2 3t}=\sqrt{4(\cos^2 3t+\sin^2 3t)}=\sqrt{4}=2,$$

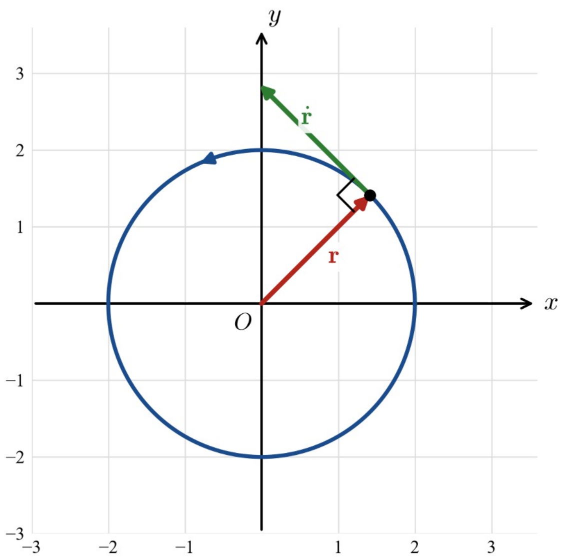
using $\cos^2\theta+\sin^2\theta=1$. So the particle stays a fixed distance 2 from the origin, and its path is a circle of radius 2. Differentiating component-wise,

$$\dot{r}=-6\sin 3t\,i+6\cos 3t\,j.$$

The dot product of the position and velocity is

$$r\cdot\dot{r}=(2\cos 3t)(-6\sin 3t)+(2\sin 3t)(6\cos 3t)=-12\sin 3t\cos 3t+12\sin 3t\cos 3t=0.$$

Since $r\cdot\dot{r}=0$ and neither vector is the zero vector, r and $\dot{r}$ are perpendicular — as expected for motion on a circle centred at O, where the velocity is tangent to the circle and the position vector is a radius. The figure shows the position vector and the tangent velocity at one instant.



$\therefore \lvert r\rvert=2$ is constant and $r\cdot\dot{r}=0$, so $\dot{r}\perp r$ at every instant.

Example 4 — unscaffolded

A particle has acceleration $\ddot{r}(t)=6t\,i-2\,j$. When $t=0$ its velocity is $\dot{r}(0)=i+4\,j$ and its position vector is $r(0)=-i$. Find $r(t)$.

Antidifferentiate the acceleration to find the velocity, keeping a constant vector c_1:

$$\dot{r}(t)=\int(6t\,i-2\,j)\,dt=3t^2 i-2t\,j+c_1.$$

Setting $t=0$, $\dot{r}(0)=c_1=i+4\,j$, so $\dot{r}(t)=(3t^2+1)i+(4-2t)\,j$. Antidifferentiate once more for the position, with a second constant vector c_2:

$$r(t)=\int\big((3t^2+1)i+(4-2t)\,j\big)\,dt=(t^3+t)i+(4t-t^2)\,j+c_2.$$

Setting $t=0$, $r(0)=c_2=-i$.

$\therefore r(t)=(t^3+t-1)i+(4t-t^2)\,j.$

Practice questions

Scaffolded

Q1. A particle has position vector $r(t)=(2t^2-t)i+(t^3+2)\,j$. (a) Find the velocity $\dot{r}$. (b) Find the acceleration $\ddot{r}$. (c) Find the velocity when $t=1$. (d) Find the speed when $t=1$.

Q2. A particle has velocity $\dot{r}(t)=(2t+3)i+(6t^2-2)\,j$ and $r(0)=i+4\,j$. (a) Antidifferentiate each component, writing the constant of integration as a vector c. (b) Use $r(0)=i+4\,j$ to find c. (c) State $r(t)$. (d) Find the position vector when $t=1$.

Q3. A particle has acceleration $\ddot{r}(t)=2i+12t\,j$, with $\dot{r}(0)=3i-j$ and $r(0)=j$. (a) Antidifferentiate $\ddot{r}$ to find $\dot{r}$ in terms of t and a constant vector c_1. (b) Use the initial velocity to find c_1, and state $\dot{r}(t)$. (c) Antidifferentiate $\dot{r}$ to find r in terms of t and c_2. (d) Use the initial position to find c_2, and state $r(t)$.

Q4. Let $f(t)=t^2+1$ and $r(t)=3t\,i-t^2\,j$. Use the product rule $\frac{d}{dt}(fr)=f'r+f\dot{r}$ to differentiate $f(t)r(t)$. (a) Find $f'(t)$ and $\dot{r}(t)$. (b) Find $f'(t)r(t)$. (c) Find $f(t)\dot{r}(t)$. (d) Add the results to find $\frac{d}{dt}\big(f(t)r(t)\big)$.

Q5. A particle has position vector $r(t)=5\cos 2t\,i+5\sin 2t\,j$. (a) Find $|r|$ and state whether it is constant. (b) Find the velocity $\dot{r}$. (c) Find $r\cdot\dot{r}$. (d) Hence explain why $\dot{r}$ is perpendicular to r.

Q6. A particle moves in space with position vector $r(t)=t^2 i+2t\,j+t^3 k$. (a) Find the velocity $\dot{r}$. (b) Find the acceleration $\ddot{r}$. (c) Find the velocity when $t=1$. (d) Find the speed when $t=1$.

Unscaffolded

Q7. A particle has position vector $r(t)=(t^3-2t)i+4t^2\,j$. Find $\dot{r}$ and $\ddot{r}$, and evaluate each when $t=1$.

Q8. A particle has position vector $r(t)=3\cos t\,i+3\sin t\,j$. Find its velocity, and show that its speed is constant.

Q9. A particle has velocity $\dot{r}(t)=6t^2 i+(2t-3)\,j$ and $r(0)=i-2\,j$. Find $r(t)$.

Q10. A particle has acceleration $\ddot{r}(t)=4i-6t\,j$, with $\dot{r}(0)=-2i+j$ and $r(0)=3i$. Find $r(t)$.

Q11. Let $f(t)=\sin t$ and $r(t)=t\,i+t^2\,j$. Find $\frac{d}{dt}\big(f(t)r(t)\big)$.

Q12. For $r(t)=ti+t^2 j$ and $s(t)=2ti+3t^2 j$, verify that $\frac{d}{dt}(r\cdot s)=\dot{r}\cdot s+r\cdot\dot{s}$, and evaluate it when $t=1$.

Q13. A particle has position vector $r(t)=4\cos\frac{t}{2}i+4\sin\frac{t}{2}j$. Show that $|r|$ is constant, and hence that $\dot{r}$ is perpendicular to r.

Q14. A particle has position vector $r(t)=2ti+(t^2-9)j$. Find the value of $t>0$ at which the velocity is perpendicular to the position vector.

Q15. A particle moves in space with velocity $\dot{r}(t)=2i+6tj+3t^2k$ and $r(0)=i-j+2k$. Find $r(t)$.

Q16. A particle has velocity $\dot{r}(t)=3t^2i+(2t+1)j$ and is at the point with position vector $2i+3j$ when $t=1$. Find $r(t)$.

Q17. For $r(t)=(t+1)i+2tj$, find $\frac{d}{dt}(r\cdot r)$ directly, and verify that it equals $2r\cdot\dot{r}$.

Q18. A projectile has acceleration $\ddot{r}(t)=-10j$, with $\dot{r}(0)=3i+20j$ and $r(0)=0$. (a) Find $r(t)$. (b) Find the time at which the j-component of the velocity is zero (the highest point), and the position vector at that time.

Answers

Q1. (a) $\dot{r}=(4t-1)i+3t^2 j$; (b) $\ddot{r}=4i+6t\,j$; (c) $\dot{r}(1)=3i+3j$; (d) speed $=3\sqrt{2}$. **Q2.** (a) $(t^2+3t)i+(2t^3-2t)j+c$; (b) $c=i+4j$; (c) $r(t)=(t^2+3t+1)i+(2t^3-2t+4)j$; (d) $r(1)=5i+4j$. **Q3.** (a) $\dot{r}=2t\,i+6t^2 j+c_1$; (b) $c_1=3i-j$, so $\dot{r}=(2t+3)i+(6t^2-1)j$; (c) $r=(t^2+3t)i+(2t^3-t)j+c_2$; (d) $c_2=j$, so $r(t)=(t^2+3t)i+(2t^3-t+1)j$. **Q4.** (a) $f'(t)=2t$, $\dot{r}=3i-2t\,j$; (b) $6t^2 i-2t^3 j$; (c) $(3t^2+3)i-(2t^3+2t)j$; (d) $(9t^2+3)i-(4t^3+2t)j$. **Q5.** (a) $|r|=5$, constant; (b) $\dot{r}=-10\sin 2t\,i+10\cos 2t\,j$; (c) $r\cdot\dot{r}=0$; (d) the dot product is zero, so the vectors are perpendicular. **Q6.** (a) $\dot{r}=2t\,i+2j+3t^2 k$; (b) $\ddot{r}=2i+6t\,k$; (c) $\dot{r}(1)=2i+2j+3k$; (d) speed $=\sqrt{17}$. **Q7.** $\dot{r}=(3t^2-2)i+8t\,j$, $\ddot{r}=6t\,i+8j$; at $t=1$, $\dot{r}=i+8j$ and $\ddot{r}=6i+8j$. **Q8.** $\dot{r}=-3\sin t\,i+3\cos t\,j$; $|\dot{r}|=3$, which is constant. **Q9.** $r(t)=(2t^3+1)i+(t^2-3t-2)j$. **Q10.** $r(t)=(2t^2-2t+3)i+(t-t^3)j$. **Q11.** $(t\cos t+\sin t)i+(t^2\cos t+2t\sin t)j$. **Q12.** $\frac{d}{dt}(r\cdot s)=12t^3+4t$; at $t=1$ it equals 16. **Q13.** $|r|=4$ (constant), and $r\cdot\dot{r}=0$, so $\dot{r}\perp r$. **Q14.** $t=\sqrt{7}$. **Q15.** $r(t)=(2t+1)i+(3t^2-1)j+(t^3+2)k$. **Q16.** $r(t)=(t^3+1)i+(t^2+t+1)j$. **Q17.** $\frac{d}{dt}(r\cdot r)=10t+2$, and $2r\cdot\dot{r}=10t+2$; they agree. **Q18.** (a) $r(t)=3t\,i+(20t-5t^2)j$; (b) $t=2$, at position $6i+20j$.

Fully-worked solutions

Q1. Differentiating $r=(2t^2-t)i+(t^3+2)j$ component-wise gives $\dot{r}=(4t-1)i+3t^2 j$, and differentiating again gives $\ddot{r}=4i+6t\,j$. At $t=1$, $\dot{r}=(4-1)i+3j=3i+3j$, so the speed is $|\dot{r}(1)|=\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$.

Q2. Antidifferentiating each component,

$$r(t)=\int\big((2t+3)i+(6t^2-2)j\big)dt=(t^2+3t)i+(2t^3-2t)j+c.$$

At $t=0$ the t-terms vanish, so $r(0)=c=i+4j$. Hence $r(t)=(t^2+3t+1)i+(2t^3-2t+4)j$, and at $t=1$, $r(1)=(1+3+1)i+(2-2+4)j=5i+4j$.

Q3. Antidifferentiating the acceleration,

$$\dot{r}(t)=\int(2i+12t\,j)dt=2t\,i+6t^2 j+c_1.$$

Since $\dot{r}(0)=c_1=3i-j$, the velocity is $\dot{r}(t)=(2t+3)i+(6t^2-1)j$. Antidifferentiating again,

$$r(t)=\int\big((2t+3)i+(6t^2-1)j\big)dt=(t^2+3t)i+(2t^3-t)j+c_2.$$

Since $r(0)=c_2=j$, we get $r(t)=(t^2+3t)i+(2t^3-t+1)j$.

Q4. Here $f'=2t$ and $\dot{r}=3i-2t\,j$. The two pieces of the product rule are

$$f'r=2t(3t\,i-t^2 j)=6t^2 i-2t^3 j,\; f\dot{r}=(t^2+1)(3i-2t\,j)=(3t^2+3)i-(2t^3+2t)j.$$

Adding,

$$\frac{d}{dt}(fr)=f'r+f\dot{r}=(9t^2+3)i-(4t^3+2t)j.$$

(As a check, $fr=(3t^3+3t)i-(t^4+t^2)j$, whose derivative is the same.)

Q5. The magnitude is $|r|=\sqrt{(5\cos 2t)^2+(5\sin 2t)^2}=\sqrt{25(\cos^2 2t+\sin^2 2t)}=5$, which is constant. Differentiating, $\dot{r}=-10\sin 2t\,i+10\cos 2t\,j$, and so

$$r\cdot\dot{r}=(5\cos 2t)(-10\sin 2t)+(5\sin 2t)(10\cos 2t)=-50\sin 2t\cos 2t+50\sin 2t\cos 2t=0.$$

Because the dot product is zero (and neither vector is 0), $\dot{r}$ is perpendicular to r; this is the general fact that a vector of constant length is perpendicular to its derivative.

Q6. Differentiating $r=t^2 i+2t\,j+t^3 k$ component-wise gives $\dot{r}=2t\,i+2\,j+3t^2 k$, and again $\ddot{r}=2\,i+6t\,k$. At $t=1$, $\dot{r}=2\,i+2\,j+3\,k$, so the speed is $|\dot{r}(1)|=\sqrt{2^2+2^2+3^2}=\sqrt{17}$.

Q7. Differentiating $r=(t^3-2t)i+4t^2 j$ gives $\dot{r}=(3t^2-2)i+8t\,j$, and again $\ddot{r}=6t\,i+8\,j$. At $t=1$, $\dot{r}=(3-2)i+8\,j=i+8\,j$ and $\ddot{r}=6\,i+8\,j$.

Q8. Differentiating $r=3\cos t\,i+3\sin t\,j$ gives $\dot{r}=-3\sin t\,i+3\cos t\,j$. Its magnitude is

$$|\dot{r}|=\sqrt{(-3\sin t)^2+(3\cos t)^2}=\sqrt{9\sin^2 t+9\cos^2 t}=\sqrt{9(\sin^2 t+\cos^2 t)}=\sqrt{9}=3,$$

which does not depend on t, so the speed is constant at 3.

Q9. Antidifferentiating,

$$r(t)=\int\left(6t^2 i+(2t-3)\,j\right)dt=2t^3 i+(t^2-3t)\,j+c.$$

Since $r(0)=c=i-2\,j$, we get $r(t)=(2t^3+1)i+(t^2-3t-2)\,j$.

Q10. Antidifferentiating the acceleration,

$$\dot{r}(t)=\int(4\,i-6t\,j)\,dt=4t\,i-3t^2 j+c_1,$$

and $\dot{r}(0)=c_1=-2\,i+j$ gives $\dot{r}(t)=(4t-2)i+(1-3t^2)\,j$. Antidifferentiating again,

$$r(t)=\int\left((4t-2)i+(1-3t^2)\,j\right)dt=(2t^2-2t)i+(t-t^3)\,j+c_2,$$

and $r(0)=c_2=3\,i$ gives $r(t)=(2t^2-2t+3)i+(t-t^3)\,j$.

Q11. With $f=\sin t$, $f'=\cos t$, and $\dot{r}=i+2t\,j$, the product rule gives

$$\frac{d}{dt}(f\,r)=f'\,r+f\,\dot{r}=\cos t\,(t\,i+t^2 j)+\sin t\,(i+2t\,j)=(t\cos t+\sin t)\,i+(t^2\cos t+2t\sin t)\,j.$$

Q12. Directly, $r\cdot s=(t)(2t)+(t^2)(3t^2)=2t^2+3t^4$, so $\frac{d}{dt}(r\cdot s)=4t+12t^3$. By the rule, with $\dot{r}=i+2t\,j$ and $\dot{s}=2\,i+6t\,j$,

$$\dot{r}\cdot s+r\cdot\dot{s}=\left[(1)(2t)+(2t)(3t^2)\right]+\left[(t)(2)+(t^2)(6t)\right]=(2t+6t^3)+(2t+6t^3)=4t+12t^3,$$

which matches. At $t=1$ the value is $4+12=16$.

Q13. The magnitude is $|r|=\sqrt{(4\cos\frac{t}{2})^2+(4\sin\frac{t}{2})^2}=\sqrt{16\left(\cos^2\frac{t}{2}+\sin^2\frac{t}{2}\right)}=4$, which is constant. Differentiating, $\dot{r}=-2\sin\frac{t}{2}\,i+2\cos\frac{t}{2}\,j$, so

$$r\cdot\dot{r}=(4\cos t/2)(-2\sin t/2)+(4\sin t/2)(2\cos t/2)=-8\sin t/2\cos t/2+8\sin t/2\cos t/2=0.$$

Since the dot product is zero, $\dot{r}$ is perpendicular to r.

Q14. The velocity is $\dot{r}=2\,i+2t\,j$. The velocity is perpendicular to the position vector when their dot product is zero:

$$r\cdot\dot{r}=(2t)(2)+(t^2-9)(2t)=4t+2t^3-18t=2t^3-14t=2t(t^2-7).$$

Setting this to zero gives $t=0$ or $t^2=7$. Taking $t>0$, $t=\sqrt{7}$.

Q15. Antidifferentiating each component,

$$r(t)=\int(2i+6tj+3t^2k)\,dt=2ti+3t^2j+t^3k+c.$$

Since $r(0)=c=i-j+2k$, we get $r(t)=(2t+1)i+(3t^2-1)j+(t^3+2)k$.

Q16. Antidifferentiating,

$$r(t)=\int(3t^2i+(2t+1)j)\,dt=t^3i+(t^2+t)j+c.$$

The initial condition is given at $t=1$: $r(1)=i+2j+c=2i+3j$, so $c=i+j$. Hence $r(t)=(t^3+1)i+(t^2+t+1)j$.

Q17. Directly, $r\cdot r=(t+1)^2+(2t)^2=t^2+2t+1+4t^2=5t^2+2t+1$, so $\dfrac{d}{dt}(r\cdot r)=10t+2$. With $\dot{r}=i+2j$,

$$2r\cdot\dot{r}=2[(t+1)(1)+(2t)(2)]=2(t+1+4t)=2(5t+1)=10t+2,$$

which agrees with the direct calculation.

Q18. (a) Antidifferentiating the acceleration,

$$\dot{r}(t)=\int(-10j)\,dt=-10tj+c_1,$$

and $\dot{r}(0)=c_1=3i+20j$, so $\dot{r}(t)=3i+(20-10t)j$. Antidifferentiating again,

$$r(t)=\int(3i+(20-10t)j)\,dt=3ti+(20t-5t^2)j+c_2,$$

and $r(0)=c_2=0$, so $r(t)=3ti+(20t-5t^2)j$. (b) The j-component of the velocity is $20-10t$, which is zero when $t=2$. At that time $r(2)=6i+(40-20)j=6i+20j$.

Chapter 10 Vector functions and vector calculus — 10D Velocity and acceleration for motion in a plane and in space

Theory

When a particle moves in a plane or in space, its position is described by a **vector function of time**

$$r(t) = x(t)i + y(t)j \text{ (in a plane)}, r(t) = x(t)i + y(t)j + z(t)k \text{ (in space)},$$

where i, j, k are the standard unit vectors. As t varies, the tip of $r(t)$ traces the **path** of the particle. An earlier section showed how to **differentiate** and **antidifferentiate** a vector function component by component; this section applies that calculus to the motion itself — its velocity, its acceleration and its speed. Throughout, time is in seconds and distance in metres.

Velocity.

The **velocity** is the derivative of the position vector with respect to time,

$$v(t) = \dot{r} = \dot{x}i + \dot{y}j(+\dot{z}k),$$

found by differentiating each component. Geometrically the velocity vector is always **tangent to the path** and points in the **direction of motion** — the way the particle is heading at that instant.

Speed.

The **speed** is the magnitude of the velocity,

$$|v| = \sqrt{\dot{x}^2 + \dot{y}^2(+\dot{z}^2)},$$

an unsigned quantity measured in $\mathrm{m/s}$. Speed says how fast the particle moves; the velocity also records the direction.

Acceleration.

The **acceleration** is the derivative of the velocity, that is the second derivative of the position,

$$a(t) = \ddot{r} = \ddot{x}i + \ddot{y}j(+\ddot{z}k),$$

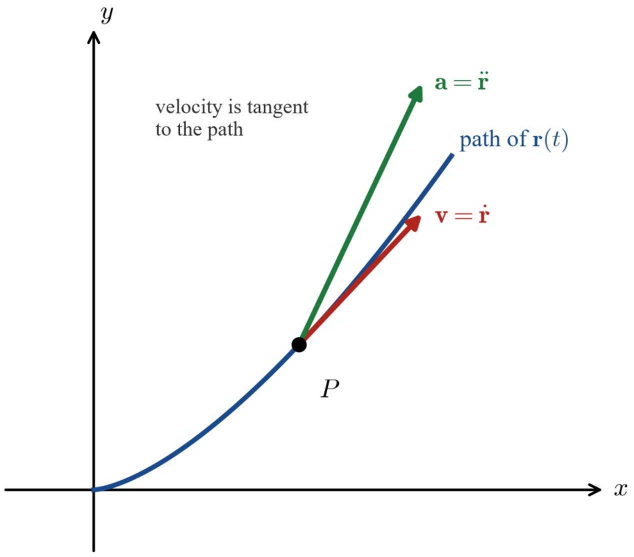
measured in $\mathrm{m/s}^2$. Unlike the velocity, the acceleration is generally **not** tangent to the path; it points towards the **concave** (inner) side of the curve, and its component across the path is what makes the direction of travel turn.

Direction of motion.

Since the velocity is tangent to the path, the **direction of motion** at any instant is the unit vector

$$\hat{v} = \frac{v}{|v|}.$$

The figure shows a curved path with the velocity drawn tangent at a point P and the acceleration drawn from P towards the concave side.



Recovering the motion by antidifferentiation.

The process reverses. If the acceleration is known, **antidifferentiate** to recover the velocity, and antidifferentiate again for the position:

$$v(t) = \int a(t)\,dt, r(t) = \int v(t)\,dt.$$

Each antidifferentiation is carried out component by component and introduces an arbitrary **constant vector**, so an **initial condition** is needed at each stage — usually the velocity and position at $t=0$ — to pin the constants down. For example, if $a(t) = 4i - 2j$ with $v(0) = -3i + 5j$ and $r(0) = 2i$, then

$$v(t) = \int (4i - 2j)\,dt = (4t + c_1)i + (-2t + c_2)j,$$

and $v(0) = c_1 i + c_2 j = -3i + 5j$ gives $c_1 = -3$, $c_2 = 5$, so $v(t) = (4t - 3)i + (5 - 2t)j$. Then

$$r(t) = \int ((4t - 3)i + (5 - 2t)j)\,dt = (2t^2 - 3t + d_1)i + (5t - t^2 + d_2)j,$$

and $r(0) = 2i$ gives $d_1 = 2$, $d_2 = 0$, so $r(t) = (2t^2 - 3t + 2)i + (5t - t^2)j$.

Projectile motion.

A **projectile** is a particle moving freely under gravity alone. Taking i horizontal and j vertically upward, the only acceleration is that due to gravity,

$$a = -gj,$$

where $g \approx 9.8\,\text{m/s}^2$ is the acceleration due to gravity, the same value used for motion under gravity earlier in the course. If the projectile leaves the origin with initial velocity $v(0) = ui + wj$, then antidifferentiating,

$$v(t) = \int (-gj)\,dt = ui + (w - gt)j, r(t) = \int (ui + (w - gt)j)\,dt = uti + \left(wt - 1/2gt^2\right)j.$$

So $x = ut$ and $y = wt - 1/2gt^2$; eliminating $t = \frac{x}{u}$ shows y is a quadratic in x, so the **path is a parabola**. The horizontal velocity u stays constant while the vertical velocity decreases at the constant rate g. The projectile reaches its **greatest height** when the vertical velocity is zero ($w - gt = 0$), and it returns to the launch level ($y = 0$) at the **time of flight** $t = \frac{2w}{g}$; the horizontal distance covered by then is the **range**.

Circular motion.

A particle with position

$$r(t) = R\cos(\omega t)i + R\sin(\omega t)j$$

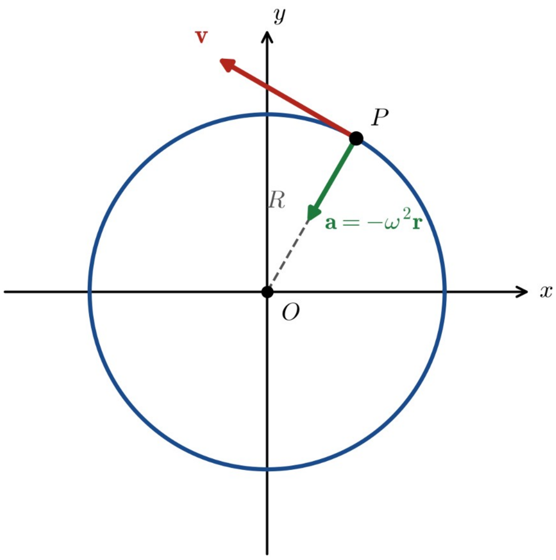
has $|r| = R$ for all t, so it moves on a **circle** of radius R centred at the origin, turning at the constant rate ω. Differentiating,

$$v(t) = -R\omega\sin(\omega t)i + R\omega\cos(\omega t)j, |v| = R\omega\,(\text{constant}),$$

and differentiating again,

$$a(t) = -R\omega^2\cos(\omega t)i - R\omega^2\sin(\omega t)j = -\omega^2 r.$$

Because $a = -\omega^2 r$ is a negative multiple of r, the acceleration points **opposite to the position vector — that is, from the particle straight towards the centre** O — with magnitude $|a| = R\omega^2$. A short calculation also gives $v \cdot r = 0$, confirming the velocity is perpendicular to the radius, hence tangent to the circle.



Worked examples

Example 1 — scaffolded

A particle moves in a plane with position vector $r(t) = (t^3 - 3t)i + 3t^2 j$ metres at time t seconds. Find its velocity and acceleration, and its speed and direction of motion when $t = 2$.

Working	Comment
$v = \dot{r} = (3t^2 - 3)i + 6t\,j$.	Differentiate each component of r.
$a = \ddot{r} = 6t\,i + 6\,j$.	Differentiate each component of v.
At $t = 2$: $v = (12 - 3)i + 12\,j = 9i + 12\,j$.	Substitute $t = 2$ into v.
At $t = 2$: $a = 12i + 6\,j$.	Substitute $t = 2$ into a.
Speed $= |v| = \sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15$ m/s.	Speed is the magnitude of the velocity.
Direction $= \dfrac{v}{|v|} = \dfrac{9i + 12\,j}{15} = \dfrac{3}{5}i + \dfrac{4}{5}j$.	Divide the velocity by the speed for the unit tangent.

Example 2 — scaffolded

A particle moves in space with position vector $r(t) = t^3 i + 6t\,j + 3t^2 k$ metres. Find its velocity, acceleration and speed when $t = 1$.

Working	Comment
$v=\dot{r}=3t^2 i+6j+6t\,k$.	Differentiate each of the three components.
$a=\ddot{r}=6t\,i+6k$.	The j-component of v is constant, so it differentiates to 0.
At $t=1$: $v=3i+6j+6k$.	Substitute $t=1$ into v.
At $t=1$: $a=6i+6k$.	Substitute $t=1$ into a.
Speed $=\sqrt{3^2+6^2+6^2}=\sqrt{9+36+36}=\sqrt{81}=9$ m/s.	In space the speed uses all three components.

Example 3 — unscaffolded

A particle moves in a plane with acceleration $a(t)=2i+6t\,j$. When $t=0$ its velocity is $i-2j$ and its position is $3i$. Find its velocity and position as functions of t.

Antidifferentiating the acceleration component by component and keeping a constant in each,

$$v(t)=\int(2i+6t\,j)\,dt=(2t+c_1)i+(3t^2+c_2)j.$$

The initial velocity $v(0)=c_1 i+c_2 j=i-2j$ gives $c_1=1$ and $c_2=-2$, so $v(t)=(2t+1)i+(3t^2-2)j$. Antidifferentiating again,

$$r(t)=\int((2t+1)i+(3t^2-2)j)\,dt=(t^2+t+d_1)i+(t^3-2t+d_2)j.$$

The initial position $r(0)=d_1 i+d_2 j=3i$ gives $d_1=3$ and $d_2=0$.

$\therefore v(t)=(2t+1)i+(3t^2-2)j$ and $r(t)=(t^2+t+3)i+(t^3-2t)j$.

Example 4 — unscaffolded

A ball is projected from the origin with a speed of 19.6 m/s at 30° above the horizontal. Taking i horizontal, j vertically upward and $g=9.8\,\text{m/s}^2$, find the velocity and position at time t, the time of flight, the range and the greatest height.

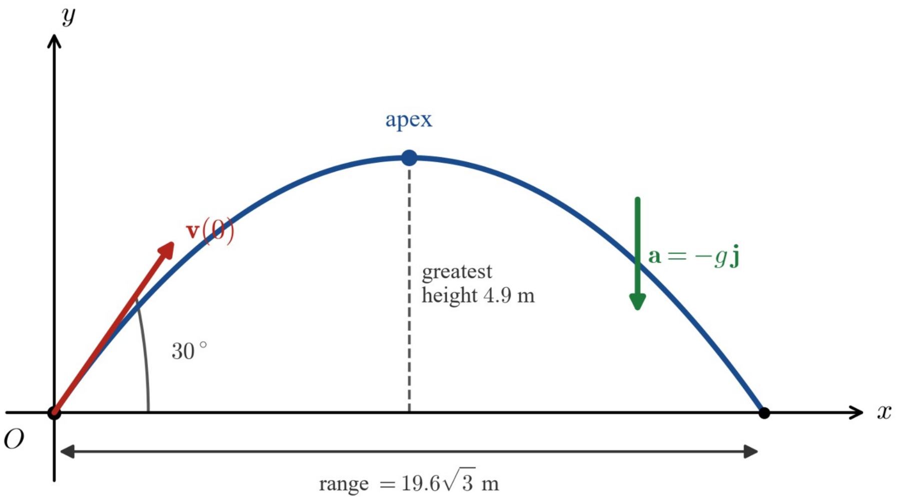
Resolving the launch velocity, $v(0)=19.6\cos 30^\circ i+19.6\sin 30^\circ j=9.8\sqrt{3}i+9.8j$. The acceleration is $a=-9.8j$, so

$$v(t)=\int(-9.8j)\,dt=9.8\sqrt{3}i+(9.8-9.8t)j,$$

using $v(0)=9.8\sqrt{3}i+9.8j$ to fix the constants. Then

$$r(t)=\int(9.8\sqrt{3}i+(9.8-9.8t)j)\,dt=9.8\sqrt{3}t\,i+(9.8t-4.9t^2)j,$$

since $r(0)=0$. The ball is back at launch level when the j-component is zero: $9.8t-4.9t^2=4.9t(2-t)=0$, so the **time of flight** is $t=2$ s. The **range** is the horizontal distance then, $x(2)=9.8\sqrt{3}\times 2=19.6\sqrt{3}$ m. The **greatest height** occurs where the vertical velocity is zero: $9.8-9.8t=0$, so $t=1$ s, and the height is $y(1)=9.8-4.9=4.9$ m.



∴ the time of flight is 2 s, the range is $19.6\sqrt{3}$ m and the greatest height is 4.9 m.

Practice questions

Scaffolded

Q1. A particle moves in a plane with $r(t)=(4t^2-t)i+2t^2 j$ metres. (a) Find the velocity v. (b) Find the acceleration a. (c) Find the speed when $t=2$. (d) Find the direction of motion when $t=2$.

Q2. A particle moves in space with $r(t)=t^2 i+3t\,j+3t^2 k$ metres. (a) Find the velocity v. (b) Find the acceleration a. (c) Find v and a when $t=1$. (d) Find the speed when $t=1$.

Q3. A particle moves in a plane with acceleration $a(t)=6t\,i+2j$. When $t=0$ its velocity is $-i+3j$ and its position is $2i-j$. (a) Antidifferentiate to find v in terms of t and two constants. (b) Use the initial velocity to find the constants, and state $v(t)$. (c) Antidifferentiate to find r in terms of t and two constants. (d) Use the initial position to find the constants, and state $r(t)$.

Q4. A particle moves in a plane with velocity $v(t)=(2t-4)i+3t^2 j$ m/s, and is at $r(0)=i+2j$ when $t=0$. (a) Antidifferentiate to find r in terms of t and two constants. (b) Use the initial position to find the constants. (c) State $r(t)$. (d) Find the position when $t=2$.

Q5. A stone is thrown from the origin with initial velocity $30i+19.6j$ m/s, with i horizontal, j vertically upward and $g=9.8\,\text{m/s}^2$. (a) Find the velocity $v(t)$. (b) Find the position $r(t)$. (c) Find the time of flight. (d) Find the range and the greatest height.

Q6. A particle moves with position vector $r(t)=3\cos(2t)i+3\sin(2t)j$ metres. (a) Show that $|r|=3$, so the particle moves on a circle of radius 3. (b) Find v and show that the speed $|v|=6$ is constant. (c) Find a and show that $a=-4r$. (d) Explain why the acceleration is directed towards the centre, and state its magnitude.

Unscaffolded

Q7. A particle moves in a plane with $r(t)=(t^3-4t)i+(t^2+2t)j$ metres. Find its velocity, acceleration, speed and direction of motion when $t=2$.

Q8. A particle moves in space with $r(t)=t^2 i+6t\,j+9t\,k$ metres. Find its velocity, acceleration and speed when $t=1$.

Q9. A particle moves in a plane with acceleration $a(t)=12t^2 i+6j$. When $t=0$ its velocity is $-i+2j$ and it is at the origin. Find its velocity and position as functions of t.

Q10. A particle moves in space with velocity $v(t) = 4t\,i + 3\,j + 6t^2 k$ m/s, and is at $r(0) = i + 2\,j - k$ when $t = 0$. Find $r(t)$ and the position when $t = 1$.

Q11. A ball is projected from the origin with a speed of 49 m/s at 30° above the horizontal ($g = 9.8\,\text{m/s}^2$). Find the time of flight, the range and the greatest height.

Q12. A ball is kicked from the origin with initial velocity $19.6\,i + 14.7\,j$ m/s ($g = 9.8\,\text{m/s}^2$, j upward). Find the time of flight, the range, the greatest height and the speed with which the ball strikes the ground.

Q13. A particle moves in a plane with $r(t) = (t^2 - 6t)\,i + t^2\,j$ metres. Find its speed as a function of t, and hence the minimum speed and the time at which it occurs.

Q14. A particle moves with position vector $r(t) = 5\cos(3t)\,i + 5\sin(3t)\,j$ metres. Show that it moves on a circle, find its (constant) speed, and show that its acceleration is directed towards the centre, stating the magnitude.

Q15. A particle moves in space with acceleration $a(t) = 6t\,i + 4\,j + 12t^2 k$. When $t = 0$ its velocity is $2i - 3\,j + 3k$ and its position is $-i + 2k$. Find its velocity and position as functions of t.

Q16. A particle moves in a plane with $r(t) = (t^3 - 12t)\,i + t^2\,j$ metres, $t \ge 0$. (a) Find the time at which the particle is moving parallel to j, and its speed then. (b) Find the time at which it is moving parallel to i, and its velocity then.

Q17. A particle moves in a plane with acceleration $a(t) = -4i + (12t - 6)\,j$. When $t = 0$ its velocity is $4i - 2\,j$ and its position is $i + 3\,j$. Find its velocity and position as functions of t, and its speed when $t = 1$.

Q18. A particle moves with position vector $r(t) = 2\cos t\,i + 2\sin t\,j$ metres. Find r, v and a when $t = \frac{\pi}{6}$, show that the acceleration then points from the particle towards the centre O, and confirm the speed is constant.

Answers

Q1. (a) $v=(8t-1)i+4tj$; (b) $a=8i+4j$; (c) 17 m/s; (d) $\frac{15}{17}i+\frac{8}{17}j$. **Q2.** (a) $v=2ti+3j+6tk$; (b) $a=2i+6k$; (c) $v=2i+3j+6k$, $a=2i+6k$; (d) 7 m/s. **Q3.** (a) $v=(3t^2+c_1)i+(2t+c_2)j$; (b) $c_1=-1$, $c_2=3$, so $v=(3t^2-1)i+(2t+3)j$; (c) $r=(t^3-t+d_1)i+(t^2+3t+d_2)j$; (d) $d_1=2$, $d_2=-1$, so $r=(t^3-t+2)i+(t^2+3t-1)j$. **Q4.** (a) $r=(t^2-4t+c_1)i+(t^3+c_2)j$; (b) $c_1=1$, $c_2=2$; (c) $r=(t^2-4t+1)i+(t^3+2)j$; (d) $r(2)=-3i+10j$. **Q5.** (a) $v=30i+(19.6-9.8t)j$; (b) $r=30ti+(19.6t-4.9t^2)j$; (c) 4 s; (d) range 120 m, greatest height 19.6 m. **Q6.** (a) $|r|=\sqrt{9\cos^2(2t)+9\sin^2(2t)}=3$; (b) $v=-6\sin(2t)i+6\cos(2t)j$, $|v|=6$; (c) $a=-12\cos(2t)i-12\sin(2t)j=-4r$; (d) a is a negative multiple of r, so it points from the particle towards O; $|a|=12\,\text{m/s}^2$. **Q7.** $v=(3t^2-4)i+(2t+2)j$, $a=6ti+2j$; at $t=2$, $v=8i+6j$, $a=12i+2j$, speed 10 m/s, direction $\frac{4}{5}i+\frac{3}{5}j$. **Q8.** $v=2ti+6j+9k$, $a=2i$; at $t=1$, $v=2i+6j+9k$, speed 11 m/s. **Q9.** $v=(4t^3-1)i+(6t+2)j$; $r=(t^4-t)i+(3t^2+2t)j$. **Q10.** $r=(2t^2+1)i+(3t+2)j+(2t^3-1)k$; $r(1)=3i+5j+k$. **Q11.** Time of flight 5 s; range $122.5\sqrt{3}$ m; greatest height 30.625 m. **Q12.** Time of flight 3 s; range 58.8 m; greatest height 11.025 m; strikes the ground at 24.5 m/s. **Q13.** $|v|=\sqrt{8t^2-24t+36}$; minimum speed $3\sqrt{2}$ m/s at $t=\frac{3}{2}$ s. **Q14.** $|r|=5$ (circle of radius 5); $v=-15\sin(3t)i+15\cos(3t)j$, speed 15 m/s; $a=-9r$, directed towards the centre, $|a|=45\,\text{m/s}^2$. **Q15.** $v=(3t^2+2)i+(4t-3)j+(4t^3+3)k$; $r=(t^3+2t-1)i+(2t^2-3t)j+(t^4+3t+2)k$. **Q16.** (a) Parallel to j at $t=2$ s, speed 4 m/s; (b) parallel to i at $t=0$, $v=-12i$. **Q17.** $v=(4-4t)i+(6t^2-6t-2)j$; $r=(-2t^2+4t+1)i+(2t^3-3t^2-2t+3)j$; speed when $t=1$ is 2 m/s. **Q18.** $r=\sqrt{3}i+j$, $v=-i+\sqrt{3}j$, $a=-\sqrt{3}i-j=-r$ (towards O); $|v|=2$ m/s (constant).

Fully-worked solutions

Q1. Differentiating $r=(4t^2-t)i+2t^2j$ gives the velocity $v=(8t-1)i+4tj$, and differentiating again gives the acceleration $a=8i+4j$. At $t=2$, $v=15i+8j$, so the speed is $|v|=\sqrt{15^2+8^2}=\sqrt{225+64}=\sqrt{289}=17$ m/s and the direction of motion is $\frac{v}{|v|}=\frac{15i+8j}{17}=\frac{15}{17}i+\frac{8}{17}j$.

Q2. Differentiating $r=t^2i+3tj+3t^2k$ gives $v=2ti+3j+6tk$, and again $a=2i+6k$ (the constant j-component differentiates to 0). At $t=1$, $v=2i+3j+6k$ and $a=2i+6k$, so the speed is $|v|=\sqrt{2^2+3^2+6^2}=\sqrt{4+9+36}=\sqrt{49}=7$ m/s.

Q3. Antidifferentiating the acceleration,

$$v(t)=\int(6ti+2j)\,dt=(3t^2+c_1)i+(2t+c_2)j.$$

Since $v(0)=c_1i+c_2j=-i+3j$, we have $c_1=-1$ and $c_2=3$, so $v(t)=(3t^2-1)i+(2t+3)j$. Antidifferentiating again,

$$r(t)=\int((3t^2-1)i+(2t+3)j)\,dt=(t^3-t+d_1)i+(t^2+3t+d_2)j.$$

Since $r(0)=d_1i+d_2j=2i-j$, we have $d_1=2$ and $d_2=-1$, so $r(t)=(t^3-t+2)i+(t^2+3t-1)j$.

Q4. Antidifferentiating the velocity,

$$r(t)=\int((2t-4)i+3t^2j)\,dt=(t^2-4t+c_1)i+(t^3+c_2)j.$$

Since $r(0)=c_1 i+c_2 j=i+2j$, we have $c_1=1$ and $c_2=2$, so $r(t)=(t^2-4t+1)i+(t^3+2)j$. At $t=2$, $r(2)=(4-8+1)i+(8+2)j=-3i+10j$.

Q5. With $a=-9.8j$,

$$v(t)=\int(-9.8j)\,dt=30i+(19.6-9.8t)j,$$

using $v(0)=30i+19.6j$. Then

$$r(t)=\int(30i+(19.6-9.8t)j)\,dt=30ti+(19.6t-4.9t^2)j,$$

since $r(0)=0$. The stone returns to launch level when $19.6t-4.9t^2=4.9t(4-t)=0$, giving a time of flight of $t=4$ s; the range is $x(4)=30\times4=120$ m. The greatest height is where $19.6-9.8t=0$, i.e. $t=2$ s, and equals $y(2)=39.2-19.6=19.6$ m.

Q6. (a) $|r|=\sqrt{(3\cos 2t)^2+(3\sin 2t)^2}=\sqrt{9(\cos^2 2t+\sin^2 2t)}=\sqrt{9}=3$, a constant, so the particle stays 3 m from O — a circle of radius 3. (b) Differentiating, $v=-6\sin(2t)i+6\cos(2t)j$, so $|v|=\sqrt{36\sin^2 2t+36\cos^2 2t}=6$ m/s, constant. (c) Differentiating again, $a=-12\cos(2t)i-12\sin(2t)j=-4(3\cos(2t)i+3\sin(2t)j)=-4r$. (d) Because $a=-4r$ is a negative multiple of r, it points in the direction opposite to r — from the particle straight back towards the centre O — with magnitude $|a|=4|r|=12\,\text{m/s}^2$.

Q7. Differentiating $r=(t^3-4t)i+(t^2+2t)j$ gives $v=(3t^2-4)i+(2t+2)j$ and $a=6ti+2j$. At $t=2$, $v=(12-4)i+6j=8i+6j$ and $a=12i+2j$. The speed is $|v|=\sqrt{8^2+6^2}=\sqrt{100}=10$ m/s, and the direction of motion is $\dfrac{8i+6j}{10}=\dfrac{4}{5}i+\dfrac{3}{5}j$.

Q8. Differentiating $r=t^2 i+6tj+9tk$ gives $v=2ti+6j+9k$ and $a=2i$ (the j- and k-components of v are constant). At $t=1$, $v=2i+6j+9k$, so the speed is $|v|=\sqrt{2^2+6^2+9^2}=\sqrt{4+36+81}=\sqrt{121}=11$ m/s.

Q9. Antidifferentiating the acceleration,

$$v(t)=\int(12t^2 i+6j)\,dt=(4t^3+c_1)i+(6t+c_2)j.$$

Since $v(0)=c_1 i+c_2 j=-i+2j$, we have $c_1=-1$ and $c_2=2$, so $v(t)=(4t^3-1)i+(6t+2)j$. Antidifferentiating again,

$$r(t)=\int((4t^3-1)i+(6t+2)j)\,dt=(t^4-t+d_1)i+(3t^2+2t+d_2)j.$$

Since $r(0)=0$, both $d_1=0$ and $d_2=0$, so $r(t)=(t^4-t)i+(3t^2+2t)j$.

Q10. Antidifferentiating the velocity component by component,

$$r(t)=\int(4ti+3j+6t^2k)\,dt=(2t^2+c_1)i+(3t+c_2)j+(2t^3+c_3)k.$$

Since $r(0)=c_1 i+c_2 j+c_3 k=i+2j-k$, we have $c_1=1$, $c_2=2$ and $c_3=-1$, so $r(t)=(2t^2+1)i+(3t+2)j+(2t^3-1)k$. At $t=1$, $r(1)=3i+5j+k$.

Q11. The launch velocity is $v(0)=49\cos 30^\circ i+49\sin 30^\circ j=24.5\sqrt{3}i+24.5j$. With $a=-9.8j$,

$$v(t)=\int(-9.8j)\,dt=24.5\sqrt{3}i+(24.5-9.8t)j,\ r(t)=\int v(t)\,dt=24.5\sqrt{3}ti+(24.5t-4.9t^2)j.$$

The projectile lands when $24.5t-4.9t^2=4.9t(5-t)=0$, so the time of flight is $t=5$ s; the range is $x(5)=24.5\sqrt{3}\times5=122.5\sqrt{3}$ m. The greatest height is where $24.5-9.8t=0$, i.e. $t=2.5$ s, and equals $y(2.5)=61.25-30.625=30.625$ m.

Q12. With $v(0)=19.6i+14.7j$ and $a=-9.8j$,

$$v(t)=19.6i+(14.7-9.8t)j, r(t)=\int v(t)\,dt=19.6t\,i+(14.7t-4.9t^2)j.$$

The ball lands when $14.7t-4.9t^2=4.9t(3-t)=0$, so the time of flight is $t=3$ s and the range is $x(3)=19.6\times 3=58.8$ m. The greatest height is where $14.7-9.8t=0$, i.e. $t=1.5$ s, and equals $y(1.5)=22.05-11.025=11.025$ m. At landing, $v(3)=19.6i+(14.7-29.4)j=19.6i-14.7j$, so the ball strikes the ground at speed $|v(3)|=\sqrt{19.6^2+14.7^2}=\sqrt{600.25}=24.5$ m/s.

Q13. Differentiating $r=(t^2-6t)i+t^2j$ gives $v=(2t-6)i+2t\,j$, so

$$|v|=\sqrt{(2t-6)^2+(2t)^2}=\sqrt{4t^2-24t+36+4t^2}=\sqrt{8t^2-24t+36}.$$

The speed is least where the quadratic $8t^2-24t+36$ under the root is least, at $t=\dfrac{24}{2\times 8}=\dfrac{3}{2}$ s. There $8(3/2)^2-24(3/2)+36=18-36+36=18$, so the minimum speed is $\sqrt{18}=3\sqrt{2}$ m/s.

Q14. $|r|=\sqrt{25\cos^2(3t)+25\sin^2(3t)}=5$, a constant, so the path is a circle of radius 5. Differentiating, $v=-15\sin(3t)i+15\cos(3t)j$, so $|v|=\sqrt{225\sin^2(3t)+225\cos^2(3t)}=15$ m/s, constant. Differentiating again, $a=-45\cos(3t)i-45\sin(3t)j=-9(5\cos(3t)i+5\sin(3t)j)=-9r$. Since a is a negative multiple of r, it points from the particle towards the centre O, with magnitude $|a|=9|r|=45\,\text{m/s}^2$.

Q15. Antidifferentiating the acceleration,

$$v(t)=\int(6t\,i+4j+12t^2k)\,dt=(3t^2+c_1)i+(4t+c_2)j+(4t^3+c_3)k.$$

Since $v(0)=c_1i+c_2j+c_3k=2i-3j+3k$, we have $c_1=2$, $c_2=-3$, $c_3=3$, so $v(t)=(3t^2+2)i+(4t-3)j+(4t^3+3)k$. Antidifferentiating again,

$$r(t)=\int v(t)\,dt=(t^3+2t+d_1)i+(2t^2-3t+d_2)j+(t^4+3t+d_3)k.$$

Since $r(0)=-i+2k$, we have $d_1=-1$, $d_2=0$, $d_3=2$, so $r(t)=(t^3+2t-1)i+(2t^2-3t)j+(t^4+3t+2)k$.

Q16. Differentiating $r=(t^3-12t)i+t^2j$ gives $v=(3t^2-12)i+2t\,j$. (a) The particle moves parallel to j when the i-component of v is zero: $3t^2-12=0$, so $t=2$ s (taking $t\geq 0$). Then $v=0i+4j=4j$, a speed of 4 m/s. (b) It moves parallel to i when the j-component is zero: $2t=0$, so $t=0$, where $v=-12i$.

Q17. Antidifferentiating the acceleration,

$$v(t)=\int(-4i+(12t-6)j)\,dt=(-4t+c_1)i+(6t^2-6t+c_2)j.$$

Since $v(0)=c_1i+c_2j=4i-2j$, we have $c_1=4$ and $c_2=-2$, so $v(t)=(4-4t)i+(6t^2-6t-2)j$. Antidifferentiating again,

$$r(t)=\int v(t)\,dt=(-2t^2+4t+d_1)i+(2t^3-3t^2-2t+d_2)j.$$

Since $r(0)=i+3j$, we have $d_1=1$ and $d_2=3$, so $r(t)=(-2t^2+4t+1)i+(2t^3-3t^2-2t+3)j$. At $t=1$, $v(1)=0i+(6-6-2)j=-2j$, so the speed is $|v(1)|=2$ m/s.

Q18. At $t=\dfrac{\pi}{6}$, $r=2\cos\dfrac{\pi}{6}i+2\sin\dfrac{\pi}{6}j=\sqrt{3}i+j$. Differentiating, $v=-2\sin t\,i+2\cos t\,j$, so at $t=\dfrac{\pi}{6}$, $v=-2\sin\dfrac{\pi}{6}i+2\cos\dfrac{\pi}{6}j=-i+\sqrt{3}j$. Differentiating again, $a=-2\cos t\,i-2\sin t\,j$, so at $t=\dfrac{\pi}{6}$, $a=-\sqrt{3}i-j=-r$. Because $a=-r$, the acceleration points from the particle straight back towards the centre O. The speed is $|v|=\sqrt{(-2\sin t)^2+(2\cos t)^2}=\sqrt{4(\sin^2t+\cos^2t)}=2$ m/s for every t, so it is constant.

Topic 3 Test A — Technology-active

One bound reference and one approved technology (CAS calculator or software) are permitted. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (6 marks)

Consider the vectors $a=2i+3j+6k$ and $b=3i-4k$.

a. Find $|a|$ and the unit vector $\hat{a}$. 2 marks

b. Find $a \cdot b$ and hence the angle between a and b, to the nearest degree. 2 marks

c. Find the vector resolute of a parallel to b. 2 marks

Question 2 (7 marks)

The points $A(1,0,2)$, $B(3,2,1)$ and $C(2,-1,4)$ are the vertices of a triangle.

a. Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. 1 mark

b. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. 2 marks

c. Hence find the area of triangle ABC. 2 marks

d. Write down a vector equation of the line through A and B. 1 mark

e. Determine whether the point $D(7,6,-1)$ lies on the line through A and B. 1 mark

Question 3 (7 marks)

A plane Π has Cartesian equation $2x-y+2z=6$.

a. Find the perpendicular distance from the point $P(4,1,3)$ to Π. 2 marks

b. The line ℓ has vector equation $r=(i+2j+k)+t(i+2j+2k)$. Find the coordinates of the point where ℓ meets Π.
3 marks

c. Find, to the nearest degree, the acute angle between ℓ and Π. 2 marks

Question 4 (7 marks)

A particle moves in a plane so that its position at time t is $r(t)=3\cos t\, i+2\sin t\, j$, where $0 \le t \le 2\pi$.

a. Find the Cartesian equation of the path and identify the curve. 2 marks

b. Find $v(t)$ and the speed of the particle when $t=\frac{\pi}{3}$. 2 marks

c. Show that the acceleration satisfies $a=-r$. 1 mark

d. A second particle has path $s(t)=2\sec t\, i+3\tan t\, j$. Find the Cartesian equation of this path and identify the curve.
2 marks

Question 5 (7 marks)

Two particles move so that their positions at time t seconds (in metres) are

$$r_1(t)=(t+1)i+(t^2-2)j+2tk, r_2(t)=(3t-3)i+tj+(t+2)k.$$

a. Show that the two particles collide, and state the time and position of the collision. 4 marks

b. Find the velocity of each particle at the moment of collision, and hence the angle between their directions of motion, to the nearest degree. 3 marks

Question 6 (6 marks)

A ball is projected from the origin with initial velocity $20\,i+15\,j$ m/s, where i is horizontal and j is vertically upward. Take the acceleration due to gravity as $a=-10\,j$, so that $g=10\,\text{m/s}^2$.

a. Find the velocity $v(t)$ and the position $r(t)$. 2 marks

b. Find the time of flight and the range. 2 marks

c. Find the greatest height reached, and the speed of the ball at that instant. 2 marks

End of Topic 3 Test A

Test A — answers and solutions

Q1. (a) $|a|=\sqrt{2^2+3^2+6^2}=\sqrt{4+9+36}=\sqrt{49}=7$, so $\hat{a}=\frac{1}{7}(2i+3j+6k)$. (b) $a\cdot b=(2)(3)+(3)(0)+(6)(-4)=6-24=-18$. With $|b|=\sqrt{3^2+(-4)^2}=5$,

$$\cos\theta=\frac{a\cdot b}{|a||b|}=\frac{-18}{7\times5}=-\frac{18}{35},$$

so $\theta=\cos^{-1}\left(-\frac{18}{35}\right)\approx121°$. (c) The vector resolute of a parallel to b is

$$\left(\frac{a\cdot b}{|b|^2}\right)b=\frac{-18}{25}(3i-4k)=-\frac{54}{25}i+\frac{72}{25}k.$$

Q2. (a) $\overrightarrow{AB}=B-A=2i+2j-k$ and $\overrightarrow{AC}=C-A=i-j+2k$. (b)

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}=[(2)(2)-(-1)(-1)]i-[(2)(2)-(-1)(1)]j+[(2)(-1)-(2)(1)]k,$$

which gives $\overrightarrow{AB}\times\overrightarrow{AC}=3i-5j-4k$. (c) $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{3^2+(-5)^2+(-4)^2}=\sqrt{9+25+16}=\sqrt{50}=5\sqrt{2}$, so the area of the triangle is $\frac{1}{2}\times5\sqrt{2}=\frac{5\sqrt{2}}{2}$. (d) $r=(i+2k)+t(2i+2j-k)$, i.e. $r=(1+2t)i+2tj+(2-t)k$. (e) For $D(7,6,-1)$: $1+2t=7\Rightarrow t=3$; $2t=6\Rightarrow t=3$; $2-t=-1\Rightarrow t=3$. All three give $t=3$, so D lies on the line.

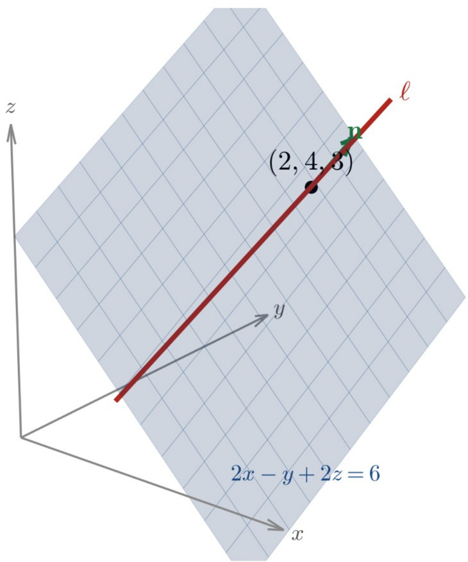
Q3. (a) The plane has normal $n=2i-j+2k$ with $|n|=\sqrt{4+1+4}=3$. The distance from $P(4,1,3)$ is

$$\frac{|2(4)-1+2(3)-6|}{3}=\frac{|8-1+6-6|}{3}=\frac{7}{3}.$$

(b) Substituting $x=1+t$, $y=2+2t$, $z=1+2t$ into $2x-y+2z=6$:

$$2(1+t)-(2+2t)+2(1+2t)=2+4t=6\Rightarrow t=1.$$

Then the point is $(1+1,2+2,1+2)=(2,4,3)$.



(c) With direction $d=i+2j+2k$ ($|d|=3$), the angle α between ℓ and Π satisfies

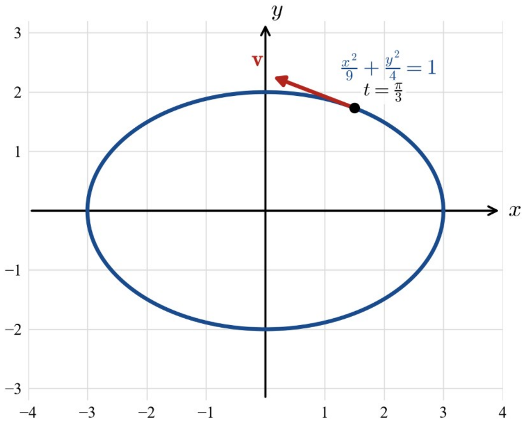
$$\sin\alpha=\frac{|n\cdot d|}{|n||d|}=\frac{|2-2+4|}{3\times3}=\frac{4}{9},$$

so $\alpha = \sin^{-1}\left(\frac{4}{9}\right) \approx 26°$.

Q4. (a) From $x = 3\cos t$ and $y = 2\sin t$, $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$. Using $\cos^2 t + \sin^2 t = 1$,

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

which is an ellipse.



(b) $v(t) = \dot{r} = -3\sin t\, i + 2\cos t\, j$. At $t = \frac{\pi}{3}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$, so $v = -\frac{3\sqrt{3}}{2} i + j$ and the speed is

$$|v| = \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + 1^2} = \sqrt{\frac{27}{4} + 1} = \sqrt{\frac{31}{4}} = \frac{\sqrt{31}}{2}.$$

(c) $a = \ddot{r} = -3\cos t\, i - 2\sin t\, j = -(3\cos t\, i + 2\sin t\, j) = -r$.

(d) From $x = 2\sec t$ and $y = 3\tan t$, $\sec t = \frac{x}{2}$ and $\tan t = \frac{y}{3}$. Using $\sec^2 t - \tan^2 t = 1$,

$$\frac{x^2}{4} - \frac{y^2}{9} = 1,$$

which is a hyperbola.

Q5. (a) For a collision the particles must share the same position at the same time. Equating components: i: $t+1 = 3t-3 \Rightarrow t = 2$; k: $2t = t+2 \Rightarrow t = 2$; j: $t^2 - 2 = t \Rightarrow t^2 - t - 2 = (t-2)(t+1) = 0 \Rightarrow t = 2$ or $t = -1$. The value $t = 2$ satisfies all three equations, so the particles collide at $t = 2$ s. The position is $r_1(2) = 3i + 2j + 4k$ (and $r_2(2) = 3i + 2j + 4k$ agrees), i.e. the point $(3,2,4)$. (b) Differentiating, $v_1 = i + 2t\, j + 2k$ and $v_2 = 3i + j + k$. At $t = 2$, $v_1 = i + 4j + 2k$ and $v_2 = 3i + j + k$. Then

$$\cos\theta = \frac{v_1 \cdot v_2}{|v_1||v_2|} = \frac{3+4+2}{\sqrt{21}\sqrt{11}} = \frac{9}{\sqrt{231}},$$

so $\theta = \cos^{-1}\left(\frac{9}{\sqrt{231}}\right) \approx 54°$.

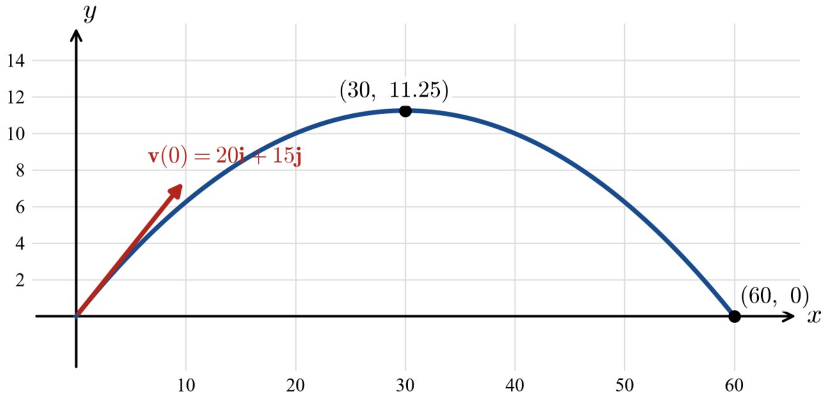
Q6. (a) Antidifferentiating $a = -10j$ and using $v(0) = 20i + 15j$,

$$v(t) = \int(-10j)\,dt = 20i + (15 - 10t)j.$$

Antidifferentiating again, with $r(0) = 0$,

$$r(t) = \int(20i + (15-10t)j)\,dt = 20t\, i + (15t - 5t^2)j.$$

(b) The ball returns to launch level when $15t-5t^2=5t(3-t)=0$, so the time of flight is $t=3$ s. The range is $x(3)=20\times3=60$ m.



(c) The greatest height occurs when the vertical velocity is zero: $15-10t=0\Rightarrow t=\frac{3}{2}$ s, giving height $y\left(\frac{3}{2}\right)=15\times\frac{3}{2}-5\times\frac{9}{4}=22.5-11.25=11.25$ m. At the top the velocity is horizontal, $v\left(\frac{3}{2}\right)=20i$, so the speed is 20 m/s.

Topic 3 Test B — Technology-free

Technology-free. No calculator. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

Consider the vectors $a=2i+2j+k$ and $b=3i-4k$.

a. Find $|a|$ and the unit vector $\hat{a}$. 2 marks

b. Find $a \cdot b$ and hence the scalar resolute of b in the direction of a. 2 marks

c. Find the vector resolute of b in the direction of a. 2 marks

d. Find the value of m for which $mi+4j-2k$ is perpendicular to a. 1 mark

Question 2 (6 marks)

Let $a=i+2j+2k$ and $b=2i-2j+k$.

a. Find $a \times b$. 2 marks

b. Hence find $|a \times b|$ and the area of the triangle having a and b as two of its sides. 2 marks

c. Find a unit vector perpendicular to both a and b. 2 marks

Question 3 (7 marks)

a. Show that the points $A(1,0,-1)$, $B(3,2,1)$ and $C(6,5,4)$ are collinear, and find the ratio $AB:BC$. 3 marks

b. Given $u=i+2j$, $v=2i-j$ and $w=4i+3j$, find scalars α and β such that $w=\alpha u+\beta v$. 2 marks

c. In triangle OAB, $\overrightarrow{OA}=a$, $\overrightarrow{OB}=b$ and M is the midpoint of AB. Prove that $\overrightarrow{OM}=\frac{1}{2}(a+b)$. 2 marks

Question 4 (7 marks)

Two lines are given by

$$\ell_1 : r=(i+j+k)+s(2i-j+k), \ell_2 : r=(5i+2j)+t(i+j-k).$$

a. Show that ℓ_1 and ℓ_2 intersect, and find the point of intersection. 4 marks

b. Find the angle between ℓ_1 and ℓ_2. 1 mark

c. A third line is $\ell_3 : r=(i+k)+u(2i-j+k)$. Show that ℓ_3 is parallel to ℓ_1 but distinct from it, so the two lines never meet. 2 marks

Question 5 (7 marks)

A particle moves in a plane with acceleration $a(t)=2i+6tj$. When $t=0$ its velocity is $i+j$ and its position is $3i-2j$.

a. Find the velocity $v(t)$. 2 marks

b. Find the position $r(t)$. 2 marks

c. Find the speed of the particle when $t=1$. 2 marks

d. Find the times at which the particle is moving parallel to the vector $i+j$. 1 mark

Question 6 (6 marks)

A particle moves so that its position at time t is $r(t) = 4\cos(2t)i + 4\sin(2t)j$ metres.

a. Show that the particle moves on a circle, stating its radius. 1 mark

b. Find $v(t)$ and show that the speed is constant, stating its value. 2 marks

c. Find $a(t)$ and show that $a = -4r$. 2 marks

d. State the position and velocity of the particle when $t = \frac{\pi}{4}$. 1 mark

End of Topic 3 Test B

Test B — answers and solutions

Q1. (a) $|a|=\sqrt{2^2+2^2+1^2}=\sqrt{9}=3$, so $\hat{a}=\frac{1}{3}(2i+2j+k)$. (b) $a\cdot b=(2)(3)+(2)(0)+(1)(-4)=6-4=2$. The scalar resolute of b in the direction of a is $\frac{a\cdot b}{|a|}=\frac{2}{3}$. (c) The vector resolute of b in the direction of a is

$$\left(\frac{a\cdot b}{|a|^2}\right)a=\frac{2}{9}(2i+2j+k)=\frac{4}{9}i+\frac{4}{9}j+\frac{2}{9}k.$$

(d) Perpendicular means the dot product is zero: $(m)(2)+(4)(2)+(-2)(1)=2m+8-2=2m+6=0$, so $m=-3$.

Q2. (a)

$$a\times b=\begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{vmatrix}=[(2)(1)-(2)(-2)]i-[(1)(1)-(2)(2)]j+[(1)(-2)-(2)(2)]k,$$

which gives $a\times b=6i+3j-6k$. (b) $|a\times b|=\sqrt{6^2+3^2+(-6)^2}=\sqrt{36+9+36}=\sqrt{81}=9$, so the triangle has area $\frac{1}{2}\times 9=\frac{9}{2}$. (c) Dividing by the magnitude, a unit vector perpendicular to both is

$$\pm\frac{1}{9}(6i+3j-6k)=\pm\frac{1}{3}(2i+j-2k).$$

Q3. (a) $\overrightarrow{AB}=B-A=2i+2j+2k$ and $\overrightarrow{BC}=C-B=3i+3j+3k$. Since $\overrightarrow{BC}=\frac{3}{2}\overrightarrow{AB}$, the vectors are parallel and share the point B, so A, B, C are collinear. As $|\overrightarrow{AB}|=2\sqrt{3}$ and $|\overrightarrow{BC}|=3\sqrt{3}$, the ratio is $AB:BC=2:3$. (b) $w=\alpha u+\beta v$ gives $\alpha+2\beta=4$ and $2\alpha-\beta=3$. From the second equation $\beta=2\alpha-3$; substituting, $\alpha+2(2\alpha-3)=4\Rightarrow 5\alpha=10\Rightarrow\alpha=2$, and then $\beta=1$. So $w=2u+v$. (c) Since M is the midpoint of AB, $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}=\frac{1}{2}(b-a)$. Then

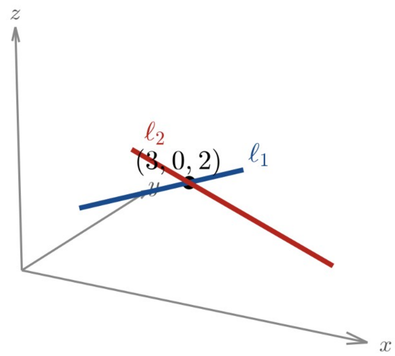
$$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=a+\frac{1}{2}(b-a)=\frac{1}{2}a+\frac{1}{2}b=\frac{1}{2}(a+b),$$

as required.

Q4. (a) Equating the two position vectors component by component:

$$1+2s=5+t,\ 1-s=2+t,\ 1+s=-t.$$

From the first equation $t=2s-4$. Substituting into the third, $1+s=-(2s-4)=-2s+4$, so $3s=3$ and $s=1$, giving $t=-2$. Checking the second equation: $1-1=0$ and $2+(-2)=0$, which agrees. Hence the lines intersect, at $r=(i+j+k)+(2i-j+k)=3i+2k$, i.e. the point $(3,0,2)$.



(b) The direction vectors are $d_1 = 2i - j + k$ and $d_2 = i + j - k$. Since $d_1 \cdot d_2 = (2)(1) + (-1)(1) + (1)(-1) = 0$, the lines are perpendicular, so the angle between them is $90°$.

(c) The direction of ℓ_3 is $2i - j + k = d_1$, so ℓ_3 is parallel to ℓ_1. If the point $(1, 0, 1)$ of ℓ_3 lay on ℓ_1 then $1 + 2s = 1$ (giving $s = 0$) and $1 - s = 0$ (giving $s = 1$) simultaneously, which is impossible. So $(1, 0, 1)$ is not on ℓ_1; the lines are parallel and distinct, and therefore never meet.

Q5. (a) Antidifferentiating the acceleration,

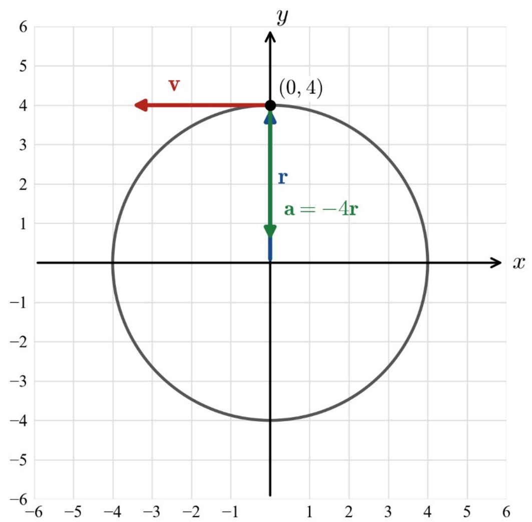
$$v(t) = \int (2i + 6t\,j)\,dt = (2t + c_1)i + (3t^2 + c_2)j.$$

Since $v(0) = c_1 i + c_2 j = i + j$, $c_1 = 1$ and $c_2 = 1$, so $v(t) = (2t+1)i + (3t^2+1)j$. (b) Antidifferentiating the velocity,

$$r(t) = \int \left((2t+1)i + (3t^2+1)j\right)dt = (t^2 + t + d_1)i + (t^3 + t + d_2)j.$$

Since $r(0) = d_1 i + d_2 j = 3i - 2j$, $d_1 = 3$ and $d_2 = -2$, so $r(t) = (t^2 + t + 3)i + (t^3 + t - 2)j$. (c) $v(1) = 3i + 4j$, so the speed is $|v(1)| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. (d) The velocity is parallel to $i + j$ when its i- and j-components are equal: $2t + 1 = 3t^2 + 1 \Rightarrow 3t^2 - 2t = 0 \Rightarrow t(3t - 2) = 0$, so $t = 0$ or $t = \frac{2}{3}$.

Q6. (a) $|r| = \sqrt{(4\cos 2t)^2 + (4\sin 2t)^2} = \sqrt{16(\cos^2 2t + \sin^2 2t)} = 4$, a constant, so the particle moves on a circle of radius 4. (b) $v(t) = \dot{r} = -8\sin(2t)i + 8\cos(2t)j$, so $|v| = \sqrt{64\sin^2 2t + 64\cos^2 2t} = 8$ m/s, which is constant. (c) $a(t) = \ddot{r} = -16\cos(2t)i - 16\sin(2t)j = -4(4\cos(2t)i + 4\sin(2t)j) = -4r$. (Being a negative multiple of r, the acceleration is directed from the particle toward the centre.) (d) At $t = \frac{\pi}{4}$, $2t = \frac{\pi}{2}$, so $r = 4\cos\frac{\pi}{2}i + 4\sin\frac{\pi}{2}j = 4j$ (the point $(0, 4)$) and $v = -8\sin\frac{\pi}{2}i + 8\cos\frac{\pi}{2}j = -8i$.



y
x
v
(0, 4)
r
a = −4r