# Specialist Mathematics Units 3 & 4

## Chapter 9 — Kinematics



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### Contents

### Theory

**Kinematics** is the study of motion. This chapter treats **rectilinear motion**: the motion of a single particle along a straight line, described entirely by calculus. Choose a fixed **origin** $O$ on the line and a **positive direction**. The particle's **position** at time $t$ is the signed distance from $O$, written $x(t)$ (some texts use $s(t)$); a positive value lies on the positive side of $O$, a negative value on the other. Time is measured in seconds, position in metres unless stated otherwise.

**Position and displacement.**

The **displacement** of the particle between times $t_1$ and $t_2$ is the change in position,

$$\text{displacement} = x(t_2) - x(t_1),$$

a signed quantity: its sign records the net direction of travel and its size the net distance moved. Displacement is not the same as the total distance travelled, a point developed below.

**Velocity.**

The **velocity** $v$ is the instantaneous rate of change of position with respect to time,

$$v = \frac{dx}{dt} = \dot{x}.$$

The overdot is Newton's notation for differentiation with respect to time. Velocity is a signed quantity: $v > 0$ means the particle is moving in the positive direction, $v < 0$ in the negative direction, and $v = 0$ means the particle is **instantaneously at rest**. At an instant where $v = 0$ the position graph has a stationary point, so a particle that reverses direction does so exactly where $v = 0$ — these are the **turning points** of the motion.

**Acceleration.**

The **acceleration** $a$ is the instantaneous rate of change of velocity with respect to time,

$$a = \frac{dv}{dt} = \dot{v} = \ddot{x} = \frac{d^2x}{dt^2}.$$

So acceleration is the second derivative of position. A positive acceleration increases the velocity (making it more positive, or a negative velocity less negative); a negative acceleration decreases it. Acceleration is measured in metres per second per second, written $\mathrm{m/s^2}$.

**Finding velocity and acceleration by differentiation.**

Given the position $x(t)$, differentiate once for the velocity and again for the acceleration:

$$x(t) \xrightarrow{\frac{d}{dt}} v(t) = \dot{x} \xrightarrow{\frac{d}{dt}} a(t) = \ddot{x}.$$

Each step uses the ordinary rules of differentiation.

**Recovering velocity and position by antidifferentiation.**

The process reverses. If the acceleration is known, antidifferentiate to recover the velocity, and antidifferentiate again for the position:

$$v = \int a\,dt, x = \int v\,dt.$$

Each antidifferentiation introduces an arbitrary constant, so two pieces of information — the **initial conditions**, usually the velocity and position at $t=0$ — are needed to pin the constants down. For example, if $a=6t$ and the particle starts ($t=0$) at $x=5$ with velocity $v=2$, then

$$v=\int 6t\,dt=3t^2+c,$$

and $v=2$ at $t=0$ gives $c=2$, so $v=3t^2+2$; then

$$x=\int\left(3t^2+2\right)dt=t^3+2t+d,$$

and $x=5$ at $t=0$ gives $d=5$, so $x=t^3+2t+5$.

**Speed.**

**Speed** is the magnitude of the velocity,

$$\text{speed}=|v|,$$

an unsigned quantity that measures how fast the particle moves regardless of direction. A particle with velocity $-6$ m/s has speed $6$ m/s.

**Displacement and distance travelled.**

The **distance travelled** over an interval is the total length of the path, counting every part as positive whether the particle moves forwards or backwards. When the particle keeps one direction throughout, distance travelled equals the size of the displacement. When it reverses, the two must be found separately. The method is to **split the interval at every time where** $v=0$ (each a possible change of direction), find the displacement over each piece, and **add the magnitudes**:

$$\text{distance travelled}=\sum\left|x\left(\text{end of piece}\right)-x\left(\text{start of piece}\right)\right|.$$

The displacement over the whole interval is still just $x\left(t_2\right)-x\left(t_1\right)$, found in one step.

**Average velocity and average speed.**

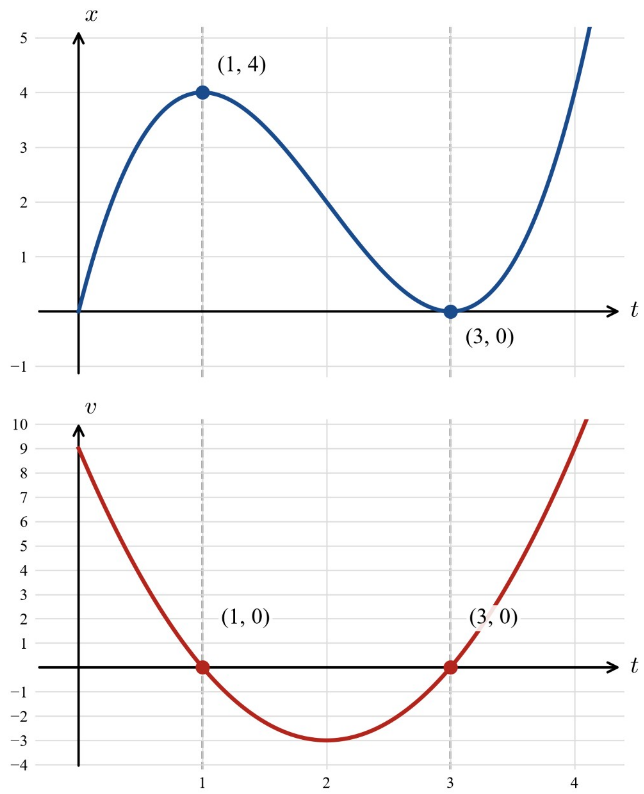
Over an interval from $t_1$ to $t_2$,

$$\text{average velocity}=\frac{\text{displacement}}{\text{time taken}}=\frac{x\left(t_2\right)-x\left(t_1\right)}{t_2-t_1},\text{average speed}=\frac{\text{distance travelled}}{\text{time taken}}.$$

Average velocity is signed and uses displacement; average speed is unsigned and uses the total distance. They agree only when the particle never reverses. (These averages differ from the *instantaneous* velocity $v$ and speed $|v|$, which are found by differentiating.)

**Position–time and velocity–time graphs.**

Two graphs describe a motion. On the **position–time** graph the height is $x$; its gradient at any instant is the velocity $\dot{x}$, so the curve rises where $v>0$, falls where $v<0$, and is stationary where $v=0$. On the **velocity–time** graph the height is $v$; its gradient is the acceleration $\dot{v}$, and the graph crosses the $t$-axis exactly at the instants the particle is at rest. The figure shows one motion, $x=t^3-6t^2+9t$, with its velocity $v=3t^2-12t+9=3\left(t-1\right)\left(t-3\right)$ beneath: the position graph turns at $t=1$ and $t=3$, precisely where the velocity graph meets the axis.




Between $t=1$ and $t=3$ the velocity is negative, so the particle moves back in the negative direction before advancing again — the reason distance travelled must be summed piece by piece.

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## Worked examples

### *Example 1 — scaffolded*

A particle moves in a straight line with position $x=t^3-9t^2+15t+4$ metres at time $t$ seconds, $t\geq 0$. Find its velocity and acceleration, evaluate both when $t=2$, and find when the particle is instantaneously at rest.

| Working | Comment |
|---|---|
| $v=\dot{x}=3t^2-18t+15$. | Velocity is the first derivative of position. |
| $a=\ddot{x}=6t-18$. | Acceleration is the derivative of velocity. |
| At $t=2$: $v=3(4)-18(2)+15=-9$ m/s. | Substitute $t=2$ into $v$; the sign shows motion in the negative direction. |
| At $t=2$: $a=6(2)-18=-6\text{ m/s}^2$. | Substitute $t=2$ into $a$. |
| At rest when $v=0$: $3t^2-18t+15=3(t-1)(t-5)=0$, so $t=1$ or $t=5$. | Factorise; the particle is instantaneously at rest at these two times. |

### *Example 2 — scaffolded*

A particle moves with acceleration $a=6t-4\text{ m/s}^2$. When $t=0$ its velocity is 1 m/s and its position is 2 m. Find the velocity and position as functions of $t$.

| Working | Comment |
|---|---|
| $v=\int a\,dt=\int(6t-4)\,dt=3t^2-4t+c$. | Antidifferentiate the acceleration; keep the constant $c$. |
| $v=1$ at $t=0$: $1=0-0+c$, so $c=1$. Hence $v=3t^2-4t+1$. | Apply the initial velocity to fix $c$. |
| $x=\int v\,dt=\int(3t^2-4t+1)\,dt=t^3-2t^2+t+d$. | Antidifferentiate the velocity; keep the constant $d$. |
| $x=2$ at $t=0$: $2=0-0+0+d$, so $d=2$. Hence | Apply the initial position to fix $d$. |

| Working | Comment |
|---|---|
| $x=t^3-2t^2+t+2$. | |

### *Example 3 — unscaffolded*

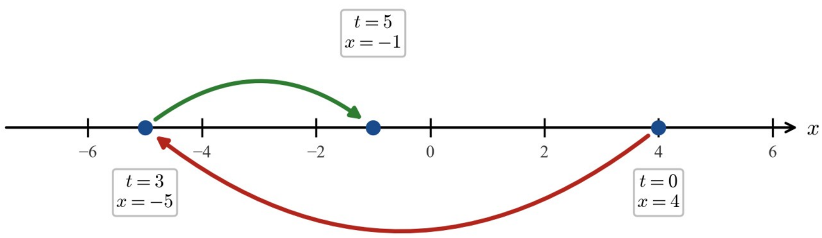
A particle moves along a line with velocity $v=2t-6$ m/s, $t\ge 0$, and is at $x=4$ m when $t=0$. Find the displacement and the distance travelled during the first 5 seconds.

The particle is at rest when $v=0$, that is $2t-6=0$, so $t=3$; this is where it may change direction. Antidifferentiating the velocity,

$$x=\int v\,dt=\int(2t-6)\,dt=t^2-6t+c,$$

and $x=4$ at $t=0$ gives $c=4$, so $x=t^2-6t+4$. Evaluating the position at the relevant times, $x(0)=4$, $x(3)=9-18+4=-5$ and $x(5)=25-30+4=-1$.

The **displacement** over the first 5 seconds is $x(5)-x(0)=-1-4=-5$ m (a net 5 m in the negative direction). For the **distance travelled**, split at $t=3$: from $t=0$ to $t=3$ the particle moves from 4 to $-5$, a distance $|-5-4|=9$ m; from $t=3$ to $t=5$ it moves from $-5$ to $-1$, a distance $|-1-(-5)|=4$ m. The number line shows the two legs.




$\therefore$ the displacement is $-5$ m and the distance travelled is $9+4=13$ m.

### *Example 4 — unscaffolded*

A particle moves with position $x=t^3-3t^2$ metres, $t\ge 0$. Find the times at which it is at rest, and its average velocity and average speed over the first 3 seconds.

Differentiating, $v=\dot{x}=3t^2-6t=3t(t-2)$, which is zero at $t=0$ and $t=2$; the particle starts from rest and is again at rest at $t=2$. The positions at the key times are $x(0)=0$, $x(2)=8-12=-4$ and $x(3)=27-27=0$.

Over the first 3 seconds the displacement is $x(3)-x(0)=0-0=0$ m, so the

$$\text{average velocity}=\frac{0}{3}=0\,\text{m/s}.$$

For the distance travelled, split at $t=2$: from 0 to $-4$ is 4 m, then from $-4$ back to 0 is 4 m, a total of 8 m. Hence

$$\text{average speed}=\frac{8}{3}\,\text{m/s}.$$

$\therefore$ the particle is at rest at $t=0$ and $t=2$ s; its average velocity is 0 but its average speed is $\frac{8}{3}$ m/s — the two differ because the particle returns to its start.

## Practice questions

### *Scaffolded*

**Q1.** A particle has position $x = 2t^3 - 9t^2 + 12t$ metres, $t \ge 0$. (a) Find the velocity $v$. (b) Find the acceleration $a$. (c) State the velocity and acceleration at $t = 0$. (d) Find the times at which the particle is instantaneously at rest.

**Q2.** A particle moves with acceleration $a = 6t + 2\,\mathrm{m/s^2}$; when $t = 0$ its velocity is $-4$ m/s and its position is 1 m. (a) Antidifferentiate to find $v$ in terms of $t$ and $c$. (b) Use the initial velocity to find $c$. (c) Antidifferentiate to find $x$ in terms of $t$ and $d$. (d) Use the initial position to find $d$.

**Q3.** A stone is thrown vertically and moves with acceleration $a = -10\,\mathrm{m/s^2}$ (taking up as positive). At $t = 0$ it leaves the origin with velocity $20$ m/s. (a) Find the velocity $v$ at time $t$. (b) Find the position $x$ at time $t$. (c) Find the time at which the stone is momentarily at rest. (d) Find its greatest height.

**Q4.** A particle has velocity $v = 3t^2 - 12$ m/s, $t \ge 0$, and is at $x = 2$ m when $t = 0$. (a) Find the time at which the particle is at rest. (b) Find the position $x$ in terms of $t$. (c) Find the displacement during the first $4$ seconds. (d) Find the distance travelled during the first $4$ seconds.

**Q5.** A particle has position $x = t^2 - 4t + 7$ metres, $t \ge 0$. (a) Find the velocity $v$. (b) Find the speed at $t = 1$. (c) Find the values of $t$ for which the particle moves in the negative direction. (d) Find the average velocity over the first $5$ seconds.

**Q6.** A particle has position $x = t^3 - 6t^2$ metres, $t \ge 0$. (a) Find the times at which the particle is at rest. (b) Find the displacement during the first $5$ seconds. (c) Find the distance travelled during the first $5$ seconds. (d) Find the average velocity and the average speed over the first $5$ seconds.

### *Unscaffolded*

**Q7.** A particle has position $x = t^3 - 5t^2 + 3t - 1$ metres, $t \ge 0$. Find its velocity and acceleration, and evaluate each when $t = 2$.

**Q8.** A particle has position $x = t^3 - 9t^2 + 24t$ metres, $t \ge 0$. Find the times at which it is instantaneously at rest and its position at each of those times.

**Q9.** A particle moves with acceleration $a = 12t - 6\,\mathrm{m/s^2}$; when $t = 0$ its velocity is $5$ m/s and its position is $-2$ m. Find the velocity and position as functions of $t$.

**Q10.** A ball is thrown upward from ground level with acceleration $a = -10\,\mathrm{m/s^2}$ and initial velocity $25$ m/s (up positive), starting at $x = 0$. Find its velocity and position at time $t$, its greatest height, and the time at which it returns to the ground.

**Q11.** A particle moves with velocity $v = 3t^2 - 24t + 36$ m/s, $t \ge 0$, starting at the origin. Find the displacement and the distance travelled during the first $6$ seconds.

**Q12.** A particle moves with velocity $v = 2t - 8$ m/s, $t \ge 0$, and is at $x = 3$ m when $t = 0$. Find the displacement and the distance travelled during the first $6$ seconds.

**Q13.** A particle has position $x = t^2 - 12t + 20$ metres, $t \ge 0$. Find its velocity, its speed at $t = 1$, and the values of $t$ for which it moves in the positive direction.

**Q14.** A particle has position $x = 2t^3 - 15t^2 + 24t$ metres, $t \ge 0$. Find its average velocity and its average speed over the first $5$ seconds.

**Q15.** A particle moves with velocity $v = 3t^2 - 4t + 2$ m/s, and when $t = 1$ its position is $x = 4$ m. Find its position in terms of $t$, and its acceleration when $t = 2$.

**Q16.** A particle moves with velocity $v = 3t^2 - 6t - 9$ m/s, $t \geq 0$, starting at the origin. (a) Find the time at which it is at rest. (b) Find the displacement during the first 4 seconds. (c) Find the distance travelled during the first 4 seconds, and illustrate the motion on a number line.

**Q17.** A particle has position $x = t^3 - 6t^2 + 5t + 1$ metres, $t \geq 0$. Find its velocity and acceleration, the minimum velocity and the time at which it occurs, and the particle's position at that time.

**Q18.** A particle moves with velocity $v = 12 - 3t^2$ m/s, $t \geq 0$, starting at the origin. (a) Find the time at which it is at rest. (b) Find the displacement during the first 3 seconds. (c) Find the distance travelled during the first 3 seconds. (d) Find the average velocity and the average speed over the first 3 seconds.

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### Answers

**Q1.** (a) $v=6t^2-18t+12$; (b) $a=12t-18$; (c) $v=12$ m/s, $a=-18\,\text{m/s}^2$; (d) $t=1$ s and $t=2$ s. **Q2.** (a) $v=3t^2+2t+c$; (b) $c=-4$, so $v=3t^2+2t-4$; (c) $x=t^3+t^2-4t+d$; (d) $d=1$, so $x=t^3+t^2-4t+1$. **Q3.** (a) $v=20-10t$; (b) $x=20t-5t^2$; (c) $t=2$ s; (d) 20 m. **Q4.** (a) $t=2$ s; (b) $x=t^3-12t+2$; (c) 16 m; (d) 48 m. **Q5.** (a) $v=2t-4$; (b) 2 m/s; (c) $0\le t<2$; (d) 1 m/s. **Q6.** (a) $t=0$ and $t=4$ s; (b) $-25$ m; (c) 39 m; (d) average velocity $-5$ m/s, average speed 7.8 m/s. **Q7.** $v=3t^2-10t+3$, $a=6t-10$; at $t=2$, $v=-5$ m/s and $a=2\,\text{m/s}^2$. **Q8.** $t=2$ s (at $x=20$ m) and $t=4$ s (at $x=16$ m). **Q9.** $v=6t^2-6t+5$, $x=2t^3-3t^2+5t-2$. **Q10.** $v=25-10t$, $x=25t-5t^2$; greatest height 31.25 m at $t=2.5$ s; returns to the ground at $t=5$ s. **Q11.** Displacement 0 m; distance travelled 64 m. **Q12.** Displacement $-12$ m; distance travelled 20 m. **Q13.** $v=2t-12$; speed at $t=1$ is 10 m/s; positive direction for $t>6$. **Q14.** Average velocity $-1$ m/s; average speed 9.8 m/s. **Q15.** $x=t^3-2t^2+2t+3$; $a=8\,\text{m/s}^2$ at $t=2$. **Q16.** (a) $t=3$ s; (b) $-20$ m; (c) 34 m. **Q17.** $v=3t^2-12t+5$, $a=6t-12$; minimum velocity $-7$ m/s at $t=2$ s; position $x=-5$ m. **Q18.** (a) $t=2$ s; (b) 9 m; (c) 23 m; (d) average velocity 3 m/s, average speed $\frac{23}{3}$ m/s.

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### Fully-worked solutions

**Q1.** Differentiating $x=2t^3-9t^2+12t$ gives the velocity $v=\dot{x}=6t^2-18t+12$, and differentiating again gives the acceleration $a=\ddot{x}=12t-18$. At $t=0$, $v=12$ m/s and $a=-18\,\text{m/s}^2$. The particle is at rest when $v=0$: $6t^2-18t+12=6(t^2-3t+2)=6(t-1)(t-2)=0$, so $t=1$ s or $t=2$ s.

**Q2.** Antidifferentiating the acceleration,

$$v=\int a\,dt=\int(6t+2)\,dt=3t^2+2t+c.$$

Since $v=-4$ at $t=0$, $-4=c$, so $v=3t^2+2t-4$. Antidifferentiating again,

$$x=\int v\,dt=\int(3t^2+2t-4)\,dt=t^3+t^2-4t+d.$$

Since $x=1$ at $t=0$, $1=d$, so $x=t^3+t^2-4t+1$.

**Q3.** Antidifferentiating the constant acceleration,

$$v=\int a\,dt=\int(-10)\,dt=-10t+c.$$

Since $v=20$ at $t=0$, $c=20$, so $v=20-10t$. Antidifferentiating again,

$$x=\int v\,dt=\int(20-10t)\,dt=20t-5t^2+d,$$

and $x=0$ at $t=0$ gives $d=0$, so $x=20t-5t^2$. The stone is momentarily at rest at the top of its flight, where $v=0$: $20-10t=0$, so $t=2$ s. Its greatest height is then $x(2)=40-20=20$ m.

**Q4.** The particle is at rest when $v=0$: $3t^2-12=0$ gives $t^2=4$, so $t=2$ s (taking $t\ge0$). Antidifferentiating,

$$x=\int v\,dt=\int(3t^2-12)\,dt=t^3-12t+c,$$

and $x=2$ at $t=0$ gives $c=2$, so $x=t^3-12t+2$. The relevant positions are $x(0)=2$, $x(2)=8-24+2=-14$ and $x(4)=64-48+2=18$. The displacement over the first 4 seconds is $x(4)-x(0)=18-2=16$ m. For the distance travelled, split at $t=2$: from 2 to $-14$ is $|-14-2|=16$ m, then from $-14$ to 18 is $|18-(-14)|=32$ m, a total of $16+32=48$ m.

**Q5.** Differentiating $x=t^2-4t+7$ gives $v=2t-4$. At $t=1$, $v=2-4=-2$, so the speed is $|-2|=2$ m/s. The particle moves in the negative direction while $v<0$, that is $2t-4<0$, giving $0\le t<2$. The average velocity over the first 5 seconds uses the displacement: $x(0)=7$ and $x(5)=25-20+7=12$, so

$$\text{average velocity} = \frac{x(5) - x(0)}{5} = \frac{12 - 7}{5} = 1\ \text{m/s}.$$

**Q6.** Differentiating $x = t^3 - 6t^2$ gives $v = 3t^2 - 12t = 3t(t - 4)$, which is zero at $t = 0$ and $t = 4$ s. The positions are $x(0) = 0$, $x(4) = 64 - 96 = -32$ and $x(5) = 125 - 150 = -25$. The displacement over the first 5 seconds is $x(5) - x(0) = -25$ m. For the distance, split at $t = 4$: from 0 to $-32$ is 32 m, then from $-32$ to $-25$ is $|-25 - (-32)| = 7$ m, a total of 39 m. Hence the average velocity is $\frac{-25}{5} = -5$ m/s and the average speed is $\frac{39}{5} = 7.8$ m/s.

**Q7.** Differentiating $x = t^3 - 5t^2 + 3t - 1$ gives $v = \dot{x} = 3t^2 - 10t + 3$, and differentiating again gives $a = \ddot{x} = 6t - 10$. At $t = 2$, $v = 3(4) - 10(2) + 3 = 12 - 20 + 3 = -5$ m/s and $a = 6(2) - 10 = 2\ \text{m/s}^2$.

**Q8.** Differentiating $x = t^3 - 9t^2 + 24t$ gives $v = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8) = 3(t - 2)(t - 4)$, which is zero at $t = 2$ s and $t = 4$ s. The positions are $x(2) = 8 - 36 + 48 = 20$ m and $x(4) = 64 - 144 + 96 = 16$ m.

**Q9.** Antidifferentiating the acceleration,

$$v = \int a\,dt = \int (12t - 6)\,dt = 6t^2 - 6t + c,$$

and $v = 5$ at $t = 0$ gives $c = 5$, so $v = 6t^2 - 6t + 5$. Antidifferentiating again,

$$x = \int v\,dt = \int (6t^2 - 6t + 5)\,dt = 2t^3 - 3t^2 + 5t + d,$$

and $x = -2$ at $t = 0$ gives $d = -2$, so $x = 2t^3 - 3t^2 + 5t - 2$.

**Q10.** Antidifferentiating $a = -10$,

$$v = \int a\,dt = \int (-10)\,dt = -10t + c,$$

with $v = 25$ at $t = 0$ giving $c = 25$, so $v = 25 - 10t$. Antidifferentiating again,

$$x = \int v\,dt = \int (25 - 10t)\,dt = 25t - 5t^2 + d,$$

with $x = 0$ at $t = 0$ giving $d = 0$, so $x = 25t - 5t^2$. The ball reaches its greatest height where $v = 0$: $25 - 10t = 0$, so $t = 2.5$ s, and the height is $x(2.5) = 62.5 - 31.25 = 31.25$ m. It returns to the ground where $x = 0$ again: $25t - 5t^2 = 5t(5 - t) = 0$, so $t = 5$ s.

**Q11.** The velocity $v = 3t^2 - 24t + 36 = 3(t^2 - 8t + 12) = 3(t - 2)(t - 6)$ is zero at $t = 2$ and $t = 6$ s. Antidifferentiating,

$$x = \int v\,dt = \int (3t^2 - 24t + 36)\,dt = t^3 - 12t^2 + 36t + c,$$

and $x = 0$ at $t = 0$ gives $c = 0$, so $x = t^3 - 12t^2 + 36t$. The positions are $x(0) = 0$, $x(2) = 8 - 48 + 72 = 32$ and $x(6) = 216 - 432 + 216 = 0$. The displacement over the first 6 seconds is $x(6) - x(0) = 0$ m. For the distance, split at $t = 2$: from 0 to 32 is 32 m, then from 32 back to 0 is 32 m, a total of 64 m.

**Q12.** The velocity $v = 2t - 8$ is zero at $t = 4$ s. Antidifferentiating,

$$x = \int v\,dt = \int (2t - 8)\,dt = t^2 - 8t + c,$$

and $x = 3$ at $t = 0$ gives $c = 3$, so $x = t^2 - 8t + 3$. The positions are $x(0) = 3$, $x(4) = 16 - 32 + 3 = -13$ and $x(6) = 36 - 48 + 3 = -9$. The displacement over the first 6 seconds is $x(6) - x(0) = -9 - 3 = -12$ m. For the distance, split at $t = 4$: from 3 to $-13$ is 16 m, then from $-13$ to $-9$ is 4 m, a total of 20 m.

**Q13.** Differentiating $x=t^2-12t+20$ gives $v=2t-12$. At $t=1$, $v=2-12=-10$, so the speed is $|-10|=10$ m/s. The particle moves in the positive direction while $v>0$, that is $2t-12>0$, giving $t>6$.

**Q14.** Differentiating $x=2t^3-15t^2+24t$ gives $v=6t^2-30t+24=6(t-1)(t-4)$, which is zero at $t=1$ s and $t=4$ s. The positions are $x(0)=0$, $x(1)=2-15+24=11$, $x(4)=128-240+96=-16$ and $x(5)=250-375+120=-5$. The displacement over the first 5 seconds is $x(5)-x(0)=-5$ m, so the average velocity is $\frac{-5}{5}=-1$ m/s. For the distance, split at $t=1$ and $t=4$: from 0 to 11 is 11 m, from 11 to $-16$ is $|-16-11|=27$ m, then from $-16$ to $-5$ is $|-5-(-16)|=11$ m, a total of 49 m; the average speed is $\frac{49}{5}=9.8$ m/s.

**Q15.** Antidifferentiating the velocity,

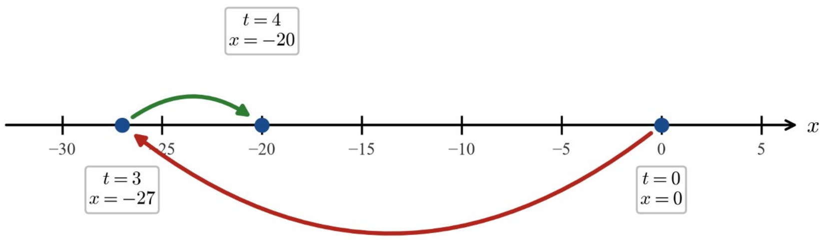
$$x=\int v\,dt=\int(3t^2-4t+2)\,dt=t^3-2t^2+2t+c.$$

Since $x=4$ at $t=1$, $4=1-2+2+c=1+c$, so $c=3$ and $x=t^3-2t^2+2t+3$. The acceleration is $a=\dot{v}=6t-4$, so at $t=2$, $a=12-4=8\,\text{m/s}^2$.

**Q16.** The velocity $v=3t^2-6t-9=3(t^2-2t-3)=3(t-3)(t+1)$ is zero at $t=3$ s (rejecting $t=-1$). Antidifferentiating,

$$x=\int v\,dt=\int(3t^2-6t-9)\,dt=t^3-3t^2-9t+c,$$

and $x=0$ at $t=0$ gives $c=0$, so $x=t^3-3t^2-9t$. The positions are $x(0)=0$, $x(3)=27-27-27=-27$ and $x(4)=64-48-36=-20$. The displacement over the first 4 seconds is $x(4)-x(0)=-20$ m. For the distance, split at $t=3$: from 0 to $-27$ is 27 m, then from $-27$ to $-20$ is 7 m, a total of 34 m, as the number line shows.




**Q17.** Differentiating $x=t^3-6t^2+5t+1$ gives $v=3t^2-12t+5$ and $a=6t-12$. The velocity is a parabola opening upward, so it is least where its derivative (the acceleration) is zero: $6t-12=0$, giving $t=2$ s. The minimum velocity is $v(2)=3(4)-12(2)+5=12-24+5=-7$ m/s, and the particle's position then is $x(2)=8-24+10+1=-5$ m.

**Q18.** The velocity $v=12-3t^2=3(4-t^2)=3(2-t)(2+t)$ is zero at $t=2$ s (taking $t\geq 0$). Antidifferentiating,

$$x=\int v\,dt=\int(12-3t^2)\,dt=12t-t^3+c,$$

and $x=0$ at $t=0$ gives $c=0$, so $x=12t-t^3$. The positions are $x(0)=0$, $x(2)=24-8=16$ and $x(3)=36-27=9$. The displacement over the first 3 seconds is $x(3)-x(0)=9$ m. For the distance, split at $t=2$: from 0 to 16 is 16 m, then from 16 to 9 is 7 m, a total of 23 m. Hence the average velocity is $\frac{9}{3}=3$ m/s and the average speed is $\frac{23}{3}$ m/s.

## Theory

For a particle moving along a straight line, its **position** $x$, **velocity** $v=\frac{dx}{dt}=\dot{x}$ and **acceleration** $a=\frac{dv}{dt}=\ddot{x}$ are linked by calculus: velocity is the derivative of position, and acceleration is the derivative of velocity. Reversing these relationships by **antidifferentiation** recovers velocity from acceleration, and position from velocity. This section studies the important **special case** in which the acceleration is **constant** — the same fixed number at every instant. When that is so, antidifferentiation can be carried out once and for all, producing the standard **constant-acceleration formulae**. Deriving them here **by calculus**, rather than simply quoting them, also makes clear exactly when they may be used.

**The constant-acceleration model.**

Suppose a particle moves in a straight line with constant acceleration $a$. Write

| symbol | meaning |
|---|---|
| $u$ | the velocity at the start, when $t=0$ (the **initial velocity**) |
| $x_0$ | the position at the start, when $t=0$ |
| $v$ | the velocity at a later time $t$ |
| $x$ | the position at a later time $t$ |
| $s=x-x_0$ | the **displacement** from the starting point |

The whole model rests on one statement: $a$ is a **constant**, not depending on $t$, on $v$ or on $x$. Everything below is a consequence of antidifferentiating that constant.

**Velocity from acceleration: $v=u+at$.**

Acceleration is the rate of change of velocity, so $\frac{dv}{dt}=a$. Because $a$ is constant, antidifferentiate with respect to $t$:

$$v=\int a\,dt=at+c_1.$$

The constant $c_1$ is fixed by the initial velocity. At $t=0$ the velocity is $u$, so $u=a(0)+c_1$, giving $c_1=u$. Hence

$$v=u+at.$$

This is a linear function of time: the velocity changes by the same amount $a$ each second.

**Position from velocity: $x=x_0+ut+1/2\,at^2$.**

Velocity is the rate of change of position, so $\frac{dx}{dt}=v=u+at$. Antidifferentiate again with respect to $t$:

$$x=\int(u+at)\,dt=ut+1/2\,at^2+c_2.$$

At $t=0$ the position is $x_0$, so $x_0=c_2$. Hence

$$x=x_0+ut+1/2\,at^2.$$

Measuring position from the start, so that $s=x-x_0$ is the displacement, this reads

$$s=ut+1/2\,at^2.$$

**The velocity–position formula: $v^2=u^2+2a(x-x_0)$.**

The two results above both involve $t$. To connect velocity directly to position, use the chain-rule form of acceleration,

$$a = v\frac{dv}{dx} = \frac{d}{dx}\left(1/2\,v^2\right).$$

Since $a$ is constant, antidifferentiate this with respect to $x$:

$$1/2\,v^2 = \int a\,dx = ax + c_3.$$

At the start, $x = x_0$ and $v = u$, so $1/2\,u^2 = a x_0 + c_3$, which gives $c_3 = 1/2\,u^2 - a x_0$. Substituting back,

$$1/2\,v^2 = ax + 1/2\,u^2 - a x_0 = 1/2\,u^2 + a\left(x - x_0\right),$$

and multiplying through by $2$,

$$v^2 = u^2 + 2a\left(x - x_0\right), \text{or } v^2 = u^2 + 2as.$$

This formula contains no $t$, so it is the one to use when time is neither given nor wanted.

**Summary of the constant-acceleration formulae.**

For motion in a straight line with constant acceleration $a$, initial velocity $u$, and displacement $s = x - x_0$:

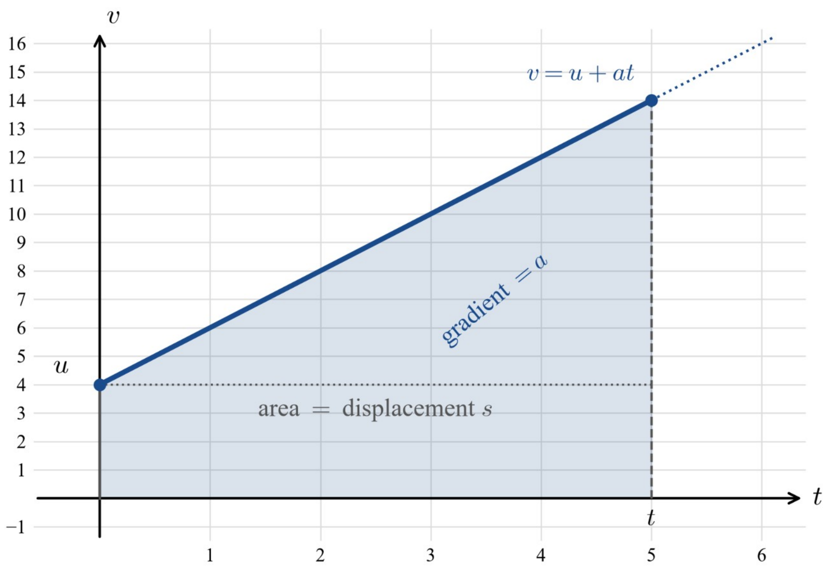
$$v = u + at,\ s = ut + 1/2\,at^2,\ v^2 = u^2 + 2as.$$

Each is exact; each follows from antidifferentiating the constant $a$. A fourth relation, $s = 1/2\left(u + v\right)t$, comes from the graph below.

**The velocity–time graph.**

Because $v = u + at$ is linear in $t$, the **velocity–time graph** of constant-acceleration motion is a **straight line**. Its **gradient** is $\frac{dv}{dt} = a$, the acceleration; a constant acceleration shows up as a constant gradient. The **signed area** between the graph and the time axis is the **displacement**, since antidifferentiating $v$ recovers $x$; area lying below the axis, where $v < 0$, counts as negative. The **total** area, with every part counted as positive, is instead the **distance travelled**, and the two agree only when $v$ keeps the one sign. In the figure below $u$ and $v$ are both positive, so the shaded region is a trapezium with parallel sides $u$ and $v$ and width $t$, and its area gives

$$s = 1/2\left(u + v\right)t.$$




**Vertical motion under gravity.**

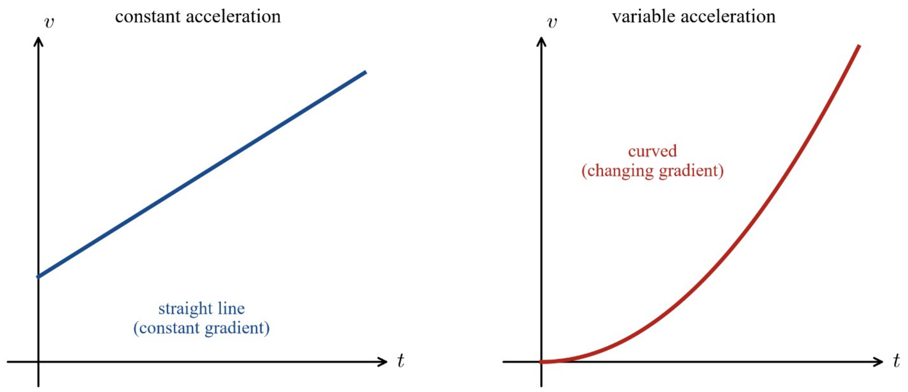
Near the Earth's surface a freely moving body — one acted on by gravity alone, with air resistance ignored — has a **constant** downward acceleration of magnitude $g \approx 9.8\,\text{m/s}^2$. Motion under gravity is therefore a case of constant acceleration, and the three formulae apply. A **sign convention** must be chosen first. Taking **up as positive**, the acceleration points downward, so

$$a = -g = -9.8\,\text{m/s}^2,$$

whatever the direction of motion — on the way up, at the highest point, and on the way down. (If instead **down** is taken as positive, then $a = +9.8\,\text{m/s}^2$.) At the highest point of a throw the velocity is momentarily $v = 0$, which is often the key to finding the maximum height.

**Recognising constant versus variable acceleration.**

The formulae above hold **only** when $a$ is constant. Acceleration is variable — and the formulae do **not** apply — whenever $a$ depends on the time $t$, the velocity $v$, or the position $x$. A quick test is the velocity–time graph: it is a straight line exactly when the acceleration is constant, and curved when the acceleration varies.




For example, a particle with $v = 5 - 3t$ has $a = \dfrac{dv}{dt} = -3$, a constant, so the formulae apply. But a particle with $v = t^2$ has $a = \dfrac{dv}{dt} = 2t$, which changes with time, so the constant-acceleration formulae must **not** be used; such motion is handled by antidifferentiating the actual expression for the acceleration, as developed elsewhere in this chapter.

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## Worked examples

### *Example 1 — scaffolded*

A train starts from rest and accelerates uniformly at $0.6\,\text{m/s}^2$ along a straight track. Find its velocity after 20 seconds, and the distance it travels in that time.

| Working | Comment |
|---|---|
| Constant acceleration, so the formulae apply. Here $u = 0$, $a = 0.6\,\text{m/s}^2$, $t = 20$. | "Starts from rest" means the initial velocity is $u = 0$. |
| $v = u + at = 0 + 0.6 \times 20 = 12$. | Use $v = u + at$ for the velocity after 20 s. |
| So the velocity after 20 s is 12 m/s. | State the result with its unit. |
| $s = ut + 1/2\,at^2 = 0 \times 20 + 1/2 \times 0.6 \times 20^2 = 1/2 \times 0.6 \times 4$ . | Use $s = ut + 1/2\,at^2$ for the distance. |
| The train travels 120 m. (Check: $v^2 = u^2 + 2as = 0 + 2 \times 0.6 \times 120 = 144 = 12^2$.) | The velocity–position formula confirms the answer. |

### *Example 2 — scaffolded*

A ball is thrown vertically upward from ground level with a speed of 19.6 m/s. Taking up as positive and $g=9.8\,\mathrm{m/s^2}$, find the time it takes to reach its highest point, and the maximum height it reaches.

| Working | Comment |
|---|---|
| Up is positive, so $a=-g=-9.8\,\mathrm{m/s^2}$; also $u=19.6$. | Under gravity the acceleration is constant and points down. |
| At the highest point the velocity is $v=0$. | The ball is momentarily at rest at the top. |
| $v=u+at \Rightarrow 0=19.6-9.8t \Rightarrow t=\dfrac{19.6}{9.8}=2$. | Solve $v=u+at$ for the time to the top. |
| The highest point is reached after 2 s. | State the time. |
| $s=ut+1/2\,at^2=19.6\times 2+1/2(-9.8)(2^2)=39.2-19.$ . | Substitute into $s=ut+1/2\,at^2$. |
| The maximum height is 19.6 m. (Check: $v^2=u^2+2as$ gives $0=19.6^2-2\times 9.8\times 19.6$.) | The velocity–position formula agrees. |

### *Example 3 — unscaffolded*

A car travelling at 25 m/s brakes with **constant** deceleration and comes to rest after 62.5 m. Find the magnitude of the deceleration and the time taken to stop.

The braking is at constant acceleration, so the formulae apply. Take the direction of travel as positive: then $u=25$, the final velocity $v=0$, and the displacement $s=62.5$. Since time is not involved, use the velocity–position formula:

$$v^2=u^2+2as \Rightarrow 0=25^2+2a(62.5) \Rightarrow 125a=-625 \Rightarrow a=-5.$$

The negative sign shows the acceleration opposes the motion, as expected for braking. For the time, use $v=u+at$:

$$0=25+(-5)t \Rightarrow t=5.$$

∴ the deceleration has magnitude $5\,\mathrm{m/s^2}$ and the car stops after 5 s.

### *Example 4 — unscaffolded*

A particle moves in a straight line with constant acceleration $a=-2\,\mathrm{m/s^2}$. At $t=0$ it is at the origin ($x_0=0$) with velocity 8 m/s. By antidifferentiation, find $v$ and $x$ as functions of $t$. Hence find the particle's velocity and position when $t=6$, and the total distance it travels in the first 6 seconds.

The acceleration is constant, so antidifferentiate $\dfrac{dv}{dt}=-2$ with respect to $t$:

$$v=\int(-2)\,dt=-2t+c_1,$$

and $v=8$ at $t=0$ gives $c_1=8$, so $v=8-2t$. Antidifferentiating once more, using $\dfrac{dx}{dt}=v$,

$$x=\int(8-2t)\,dt=8t-t^2+c_2,$$

and $x=0$ at $t=0$ gives $c_2=0$, so $x=8t-t^2$. At $t=6$,

$$v=8-2(6)=-4,\ x=8(6)-6^2=48-36=12.$$

The velocity is negative, so the particle is now moving in the negative direction: it must have turned around. It is momentarily at rest when $v=8-2t=0$, that is at $t=4$, where $x=8(4)-4^2=16$. From $t=0$ to $t=4$ it moves from 0 to 16 (a distance of 16 m); from $t=4$ to $t=6$ it moves back from 16 to 12 (a further 4 m). Displacement and distance differ because of the reversal.

$\therefore v=8-2t$ and $x=8t-t^2$; at $t=6$ the velocity is $-4$ m/s and the position is 12 m, while the total distance travelled is $16+4=20$ m.

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## Practice questions

### *Scaffolded*

**Q1.** A cyclist accelerates uniformly from 4 m/s to 10 m/s in 3 seconds along a straight path. (a) Find the acceleration $a$ using $v=u+at$. (b) Find the distance travelled using $s=ut+1/2at^2$. (c) Check the distance using $v^2=u^2+2as$.

**Q2.** A stone is dropped from rest from the top of a cliff and hits the ground 3 seconds later. Take down as positive and $g=9.8\,\text{m/s}^2$. (a) State the values of $u$ and $a$. (b) Find the speed of the stone on impact using $v=u+at$. (c) Find the height of the cliff using $s=ut+1/2at^2$.

**Q3.** A car travelling at 30 m/s brakes with constant deceleration $4\,\text{m/s}^2$ until it stops. (a) State $u$, $v$ and $a$, being careful with the sign of $a$. (b) Find the time to stop using $v=u+at$. (c) Find the stopping distance using $v^2=u^2+2as$
.

**Q4.** A particle moves in a straight line with constant acceleration $a=4\,\text{m/s}^2$. At $t=0$ it is at $x_0=2$ m with velocity $u=3$ m/s. (a) By antidifferentiating $\frac{dv}{dt}=4$, find $v$ as a function of $t$. (b) By antidifferentiating $\frac{dx}{dt}=v$, find $x$ as a function of $t$. (c) Find the position of the particle when $t=2$.

**Q5.** A ball is thrown vertically upward at 14 m/s from a point 1.5 m above the ground. Take up as positive and $g=9.8\,\text{m/s}^2$. (a) State the values of $u$ and $a$. (b) Using $v^2=u^2+2as$ with $v=0$ at the top, find the maximum height above the point of projection. (c) State the greatest height reached above the ground.

**Q6.** For each particle moving in a straight line, decide whether the constant-acceleration formulae apply, giving a reason. (a) The acceleration is $a=6\,\text{m/s}^2$ at every instant. (b) The velocity is $v=5-3t$; find $a=\frac{dv}{dt}$ and decide. (c) The position is $x=t^3$; find $a=\frac{d^2x}{dt^2}$ and decide.

### *Unscaffolded*

**Q7.** A car accelerates uniformly from rest to 24 m/s in 8 seconds. Find the acceleration and the distance travelled.

**Q8.** A train moving at 20 m/s decelerates uniformly at $0.5\,\text{m/s}^2$ until it stops. Find the time taken and the distance travelled.

**Q9.** A stone is dropped from rest down a well and takes 2 seconds to reach the water. Taking down as positive with $g=9.8\,\text{m/s}^2$, find the depth of the well and the speed of the stone when it reaches the water.

**Q10.** A ball is thrown vertically upward with speed 29.4 m/s. Taking up as positive with $g=9.8\,\text{m/s}^2$, find (a) the time to reach the highest point, and (b) the maximum height reached.

**Q11.** A particle moves with constant acceleration. At $t=0$ it passes the origin with velocity 12 m/s, and 4 seconds later its velocity is 4 m/s. Find (a) the acceleration, and (b) the displacement during those 4 seconds.

**Q12.** A particle moves in a straight line with constant acceleration $-3\,\text{m/s}^2$ and has velocity 15 m/s at $t=0$. By antidifferentiation find $v(t)$, and, taking $x=0$ at $t=0$, find $x(t)$. Hence find when the particle is momentarily at rest and its position then.

**Q13.** A car accelerating uniformly from 8 m/s covers 200 m in 10 seconds. Find the acceleration and the final velocity.

**Q14.** A ball is thrown vertically upward from ground level and returns to the ground 5 seconds later. Taking up as positive with $g = 9.8\,\mathrm{m/s^2}$, find its initial speed and the maximum height it reaches.

**Q15.** An object is projected vertically downward from the top of a tower with initial speed 5 m/s. Taking down as positive with $g = 9.8\,\mathrm{m/s^2}$, find its velocity and the distance it has fallen after 3 seconds.

**Q16.** A particle moving with constant acceleration travels 30 m while its velocity increases from 4 m/s to 16 m/s. Find (a) the acceleration, and (b) the time taken.

**Q17.** Explain, in terms of the velocity–time graph, why a particle's acceleration is constant if and only if its velocity–time graph is a straight line. State what the gradient of the graph and the area between the graph and the time axis each represent.

**Q18.** A stone is thrown vertically upward from the edge of a cliff with speed 14.7 m/s. It eventually lands at the base of the cliff, 19.6 m below the point of projection. Taking up as positive with $g = 9.8\,\mathrm{m/s^2}$, find (a) its velocity when it lands, and (b) the total time of flight.

### Answers

**Q1.** $a=2\,\mathrm{m/s^2}$; $s=21$ m; check $v^2=16+84=100=10^2$. **Q2.** $u=0$, $a=9.8\,\mathrm{m/s^2}$; impact speed 29.4 m/s; cliff height 44.1 m. **Q3.** $u=30$, $v=0$, $a=-4\,\mathrm{m/s^2}$; time 7.5 s; stopping distance 112.5 m. **Q4.** $v=3+4t$; $x=2+3t+2t^2$; at $t=2$, $x=16$ m. **Q5.** $u=14$, $a=-9.8\,\mathrm{m/s^2}$; maximum height above projection 10 m; greatest height above ground 11.5 m. **Q6.** (a) applies ($a$ constant); (b) $a=-3\,\mathrm{m/s^2}$, constant, so applies; (c) $a=6t$, depends on $t$, so does not apply. **Q7.** $a=3\,\mathrm{m/s^2}$; distance 96 m. **Q8.** time 40 s; distance 400 m. **Q9.** depth 19.6 m; speed 19.6 m/s. **Q10.** (a) 3 s; (b) 44.1 m. **Q11.** (a) $a=-2\,\mathrm{m/s^2}$; (b) displacement 32 m. **Q12.** $v=15-3t$; $x=15t-3/2t^2$; at rest at $t=5$ s, position 37.5 m. **Q13.** $a=2.4\,\mathrm{m/s^2}$; final velocity 32 m/s. **Q14.** initial speed 24.5 m/s; maximum height 30.625 m. **Q15.** velocity 34.4 m/s; distance 59.1 m. **Q16.** (a) $a=4\,\mathrm{m/s^2}$; (b) time 3 s. **Q17.** Gradient $=\dfrac{dv}{dt}=a$, so constant acceleration ⇔ constant gradient ⇔ straight line; the signed area between the graph and the time axis is the displacement (the total area is the distance travelled). **Q18.** (a) velocity $-24.5$ m/s (speed 24.5 m/s, moving down); (b) time of flight 4 s.

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### Fully-worked solutions

**Q1.** The acceleration is constant, so the formulae apply, with $u=4$, and $v=10$ when $t=3$. (a) $v=u+at$ gives $10=4+3a$, so $3a=6$ and $a=2\,\mathrm{m/s^2}$. (b) $s=ut+1/2\,at^2=4\times3+1/2\times2\times3^2=12+9=21$ m. (c) $v^2=u^2+2as=4^2+2\times2\times21=16+84=100=10^2$, which matches the final velocity of 10 m/s.

**Q2.** Down is positive, and the stone is dropped, so $u=0$ and $a=g=9.8\,\mathrm{m/s^2}$; here $t=3$. (b) $v=u+at=0+9.8\times3=29.4$, so the impact speed is 29.4 m/s. (c) $s=ut+1/2\,at^2=0+1/2\times9.8\times3^2=1/2\times9.8\times9=44.1$, so the cliff is $44.1$ m high.

**Q3.** Take the direction of travel as positive, so $u=30$ and $v=0$; the deceleration makes $a=-4\,\mathrm{m/s^2}$. (b) $v=u+at$ gives $0=30-4t$, so $t=7.5$ s. (c) $v^2=u^2+2as$ gives $0=30^2+2(-4)s=900-8s$, so $8s=900$ and $s=112.5$ m.

**Q4.** The acceleration is constant, so antidifferentiate. From $\dfrac{dv}{dt}=4$,

$$v=\int 4\,dt=4t+c_1,$$

and $v=3$ at $t=0$ gives $c_1=3$, so $v=3+4t$. Then from $\dfrac{dx}{dt}=3+4t$,

$$x=\int(3+4t)\,dt=3t+2t^2+c_2,$$

and $x=2$ at $t=0$ gives $c_2=2$, so $x=2+3t+2t^2$. At $t=2$, $x=2+3(2)+2(2^2)=2+6+8=16$ m.

**Q5.** Up is positive, so $u=14$ and $a=-9.8\,\mathrm{m/s^2}$. At the highest point $v=0$. (b) $v^2=u^2+2as$ gives $0=14^2+2(-9.8)s=196-19.6s$, so $19.6s=196$ and $s=10$ m above the point of projection. (c) The projection point is 1.5 m above the ground, so the greatest height above the ground is $10+1.5=11.5$ m.

**Q6.** The formulae apply exactly when the acceleration is a constant. (a) $a=6\,\mathrm{m/s^2}$ at every instant is constant, so the formulae apply. (b) $a=\dfrac{dv}{dt}=\dfrac{d}{dt}(5-3t)=-3\,\mathrm{m/s^2}$, a constant, so the formulae apply. (c) $\dfrac{dx}{dt}=3t^2$ and $a=\dfrac{d^2x}{dt^2}=6t$, which depends on $t$; the acceleration is variable, so the constant-acceleration formulae do **not** apply.

**Q7.** Constant acceleration with $u=0$, $v=24$, $t=8$. From $v=u+at$, $24=0+8a$, so $a=3\,\mathrm{m/s^2}$. The distance is $s=ut+1/2\,at^2=0+1/2\times3\times8^2=1/2\times3\times64=96$ m.

**Q8.** Take the direction of travel as positive, so $u=20$, $v=0$, and $a=-0.5\,\mathrm{m/s^2}$. From $v=u+at$, $0=20-0.5t$, so $t=40$ s. From $v^2=u^2+2as$, $0=20^2+2(-0.5)s=400-s$, so $s=400$ m.

**Q9.** Down is positive, dropped from rest, so $u=0$, $a=9.8\,\mathrm{m/s^2}$, $t=2$. The depth is $s=ut+1/2at^2=0+1/2\times9.8\times2^2=19.6$ m. The speed at the water is $v=u+at=0+9.8\times2=19.6$ m/s.

**Q10.** Up is positive, so $u=29.4$ and $a=-9.8\,\mathrm{m/s^2}$. At the top $v=0$. (a) $v=u+at$ gives $0=29.4-9.8t$, so $t=\frac{29.4}{9.8}=3$ s. (b) $s=ut+1/2at^2=29.4\times3+1/2(-9.8)(3^2)=88.2-44.1=44.1$ m. (Or $v^2=u^2+2as$ gives $0=29.4^2-19.6s$, so $s=44.1$ m.)

**Q11.** Constant acceleration with $u=12$, and $v=4$ when $t=4$. (a) $v=u+at$ gives $4=12+4a$, so $4a=-8$ and $a=-2\,\mathrm{m/s^2}$. (b) $s=ut+1/2at^2=12\times4+1/2(-2)(4^2)=48-16=32$ m. (Equivalently $s=1/2(u+v)t=1/2(12+4)(4)=32$ m.)

**Q12.** The acceleration is constant, so antidifferentiate $\frac{dv}{dt}=-3$:

$$v=\int(-3)\,dt=-3t+c_1,$$

and $v=15$ at $t=0$ gives $c_1=15$, so $v=15-3t$. Then from $\frac{dx}{dt}=15-3t$,

$$x=\int(15-3t)\,dt=15t-3/2t^2+c_2,$$

and $x=0$ at $t=0$ gives $c_2=0$, so $x=15t-3/2t^2$. The particle is at rest when $v=15-3t=0$, that is $t=5$ s, where $x=15(5)-3/2(5^2)=75-37.5=37.5$ m.

**Q13.** Constant acceleration with $u=8$, $s=200$, $t=10$. From $s=ut+1/2at^2$, $200=8\times10+1/2a\times10^2=80+50a$, so $50a=120$ and $a=2.4\,\mathrm{m/s^2}$. The final velocity is $v=u+at=8+2.4\times10=32$ m/s.

**Q14.** Up is positive, so $a=-9.8\,\mathrm{m/s^2}$. By symmetry the ball rises for half the total time, so it reaches the top after 2.5 s, where $v=0$. From $v=u+at$, $0=u-9.8\times2.5$, so $u=24.5$ m/s. The maximum height is $s=ut+1/2at^2=24.5\times2.5+1/2(-9.8)(2.5^2)=61.25-30.625=30.625$ m.

**Q15.** Down is positive, projected downward, so $u=5$, $a=9.8\,\mathrm{m/s^2}$, $t=3$. The velocity is $v=u+at=5+9.8\times3=34.4$ m/s. The distance fallen is $s=ut+1/2at^2=5\times3+1/2\times9.8\times3^2=15+44.1=59.1$ m.

**Q16.** Constant acceleration with $u=4$, $v=16$, $s=30$. (a) $v^2=u^2+2as$ gives $16^2=4^2+2a(30)$, so $256=16+60a$, giving $60a=240$ and $a=4\,\mathrm{m/s^2}$. (b) $v=u+at$ gives $16=4+4t$, so $4t=12$ and $t=3$ s.

**Q17.** On a velocity–time graph the gradient at any instant is $\frac{dv}{dt}$, which is the acceleration $a$. A straight line has a single constant gradient, so its acceleration is constant; conversely, if the acceleration is constant then the gradient never changes and the graph must be a straight line. Hence the acceleration is constant if and only if the velocity–time graph is a straight line. The signed area between the graph and the time axis is the displacement, because antidifferentiating the velocity recovers the position; area below the axis, where $v<0$, counts as negative. Counting every part as positive instead gives the total area, which is the distance travelled.

**Q18.** Up is positive, with the point of projection taken as $x=0$, so $u=14.7$ and $a=-9.8\,\mathrm{m/s^2}$. The stone lands 19.6 m below the projection point, so its displacement there is $s=-19.6$. (a) $v^2=u^2+2as=14.7^2+2(-9.8)(-19.6)=216.09+384.16=600.25$, so $v=\pm24.5$; the stone is moving downward as it lands, so $v=-24.5$ m/s (a speed of 24.5 m/s). (b) $v=u+at$ gives $-24.5=14.7-9.8t$, so $9.8t=39.2$ and $t=4$ s. (Check: $s=ut+1/2at^2=14.7\times4-4.9\times4^2=58.8-78.4=-19.6$, as required.)

## Theory

For a particle moving in a straight line, its **velocity** $v$ is the rate of change of position, $v=\frac{dx}{dt}$, and its **acceleration** is the rate of change of velocity, $a=\frac{dv}{dt}$. A **velocity–time graph** plots $v$ (in m/s) on the vertical axis against time $t$ (in seconds) on the horizontal axis. Read the right way, such a graph tells us the acceleration, the displacement and the distance travelled at a glance — this section develops those readings and then uses them to compute with the graph.

**The gradient of a velocity–time graph is the acceleration.**

Because $a=\frac{dv}{dt}$, the acceleration at any instant is the **gradient** of the $v$–$t$ graph at that instant. So

- a straight segment sloping **upward** has constant positive acceleration — the velocity is increasing at a constant rate;
- a **horizontal** segment has zero gradient, hence zero acceleration — the velocity is constant;
- a straight segment sloping **downward** has constant negative acceleration — the velocity is decreasing at a constant rate;
- a **curved** graph has a changing gradient, so the acceleration is changing; its value at a point is the slope of the tangent there.

For a straight segment the gradient is found as $\frac{\text{rise}}{\text{run}}$, exactly as for any line.

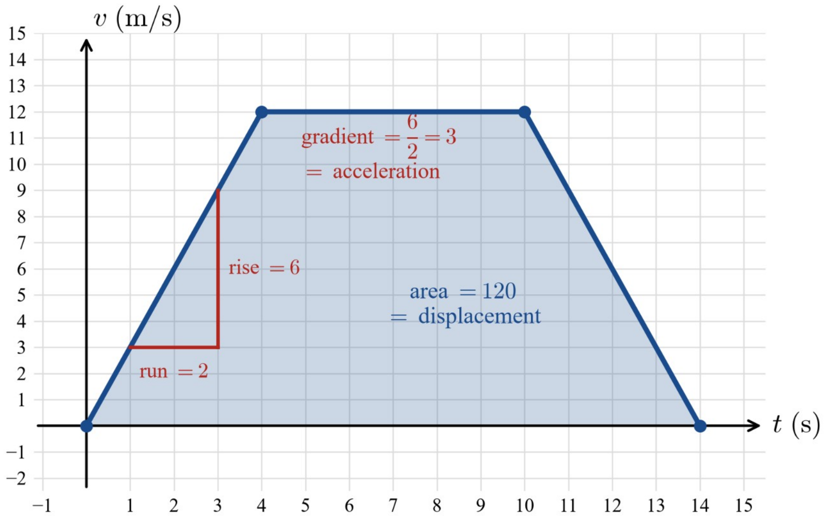
The sign of the acceleration tells you whether the **velocity** is increasing or decreasing, not whether the **speed** is: the particle speeds up when $a$ and $v$ have the **same** sign, and slows down when their signs are **opposite**. So an upward-sloping segment lying below the $t$-axis (where $v<0$) shows a particle slowing down as it moves in the negative direction, while a downward-sloping segment below the axis shows one speeding up in the negative direction.

**The signed area under a velocity–time graph is the displacement.**

Over the interval from $t=t_1$ to $t=t_2$ the change in position is

$$x(t_2)-x(t_1)=\int_{t_1}^{t_2} v\,dt,$$

which is the **signed area** between the graph and the $t$-axis. Area lying **above** the axis (where $v>0$, motion in the positive direction) counts as **positive**; area lying **below** the axis (where $v<0$, motion in the negative direction) counts as **negative**. The net result is the **displacement**. The graph below shows a piecewise-linear example: the shaded region has area $120$, and its first ramp has gradient $\frac{6}{2}=3$, so the displacement is $120$ m and the acceleration on that ramp is $3\,\text{m/s}^2$.




**The total area (magnitude) is the distance travelled.**

The **distance travelled** ignores direction: it adds up how far the particle moves whichever way it goes. On the graph it is the **total area** between the graph and the $t$-axis, with area below the axis counted as **positive**:

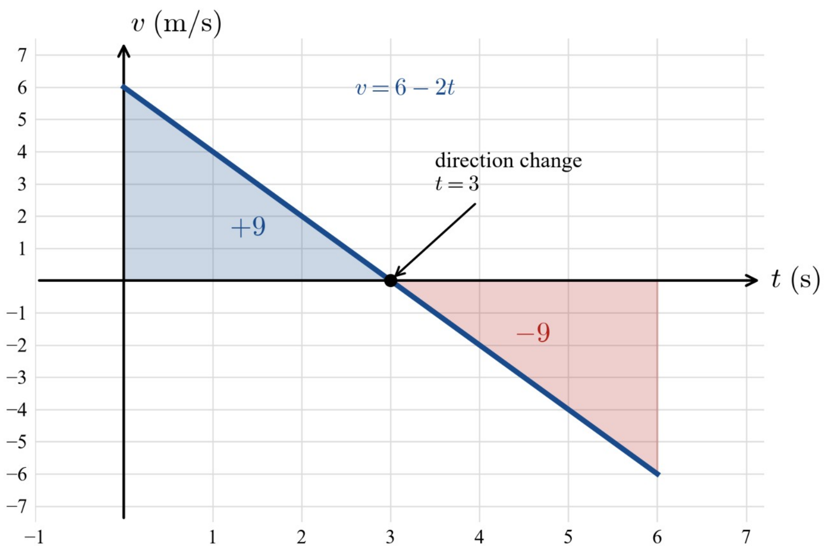
$$\text{distance} = \int_{t_1}^{t_2} |v|\, dt.$$

To evaluate this, split the motion at every time where $v = 0$ **and changes sign**, work out the area of each piece, and add the magnitudes. When the graph never dips below the axis the distance equals the displacement.

**Reading direction, rest and changing acceleration.**

- Where the graph **crosses** the $t$-axis (so $v$ changes sign) the particle is momentarily at rest and then **reverses direction**.
- Where the graph **lies on** the $t$-axis over an interval, $v = 0$ throughout, so the particle is **at rest** for that whole interval.
- A **straight** portion means constant acceleration; a **curved** portion means the acceleration is changing.

In the graph below the line $v = 6 - 2t$ crosses the axis at $t = 3$: the particle moves in the positive direction for $0 \le t < 3$ (area $+9$), is momentarily at rest at $t = 3$, then moves in the negative direction for $3 < t \le 6$ (area $-9$). Its displacement is $9 - 9 = 0$ (it returns to its starting point), while the distance travelled is $9 + 9 = 18$ m.




**Computing from a piecewise-linear graph.**

When the graph is made of straight segments, the region under it breaks into **triangles** and **trapezia**, whose areas are

$$\text{triangle} = 1/2(\text{base})(\text{height}), \text{trapezium} = 1/2(a+b)h,$$

where $a$ and $b$ are the two parallel sides and $h$ the perpendicular distance between them. Add the **signed** areas for the displacement, and the **magnitudes** for the distance.

**Computing from a curved graph.**

When the graph is a curve, use integration. The displacement is $\int v\,dt$ evaluated over the interval, and the distance is found by splitting at the zeros of $v$ and summing the magnitudes of the pieces, exactly as above.

**Average velocity and average speed.**

$$\text{average velocity} = \frac{\text{displacement}}{\text{time taken}}, \text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

These agree only when the particle never changes direction. Whenever there is a direction change the distance exceeds the magnitude of the displacement, so the average speed exceeds the magnitude of the average velocity.

**Sketching a velocity–time graph from a description.**

To sketch from a verbal description, translate each phase of the motion into a feature of the graph: *starting from rest* means the graph begins on the $t$-axis; *accelerating uniformly* means a straight segment with constant (nonzero) gradient; *constant velocity* means a horizontal segment; *decelerating to rest* means a straight segment returning to the axis; and *reversing direction* means the graph crosses below the axis.

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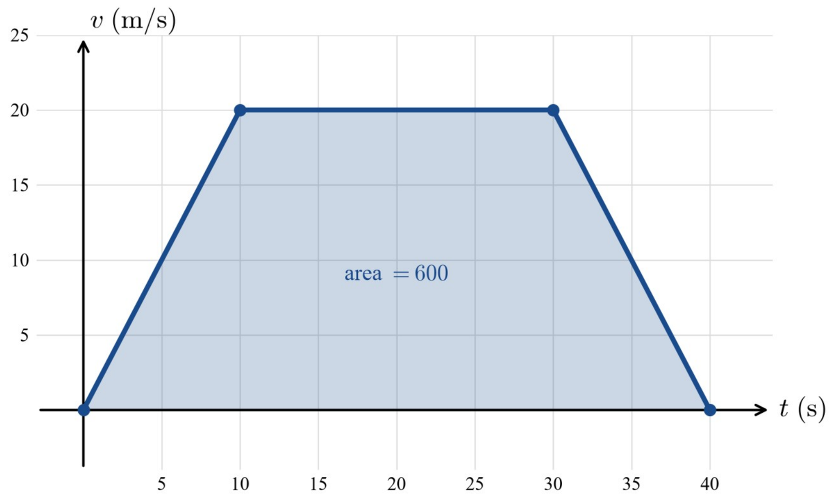
## Worked examples

### *Example 1 — scaffolded*

A vehicle starts from rest and accelerates uniformly to 20 m/s over 10 s. It then travels at 20 m/s for 20 s, and finally decelerates uniformly to rest over a further 10 s. Its velocity–time graph is shown.



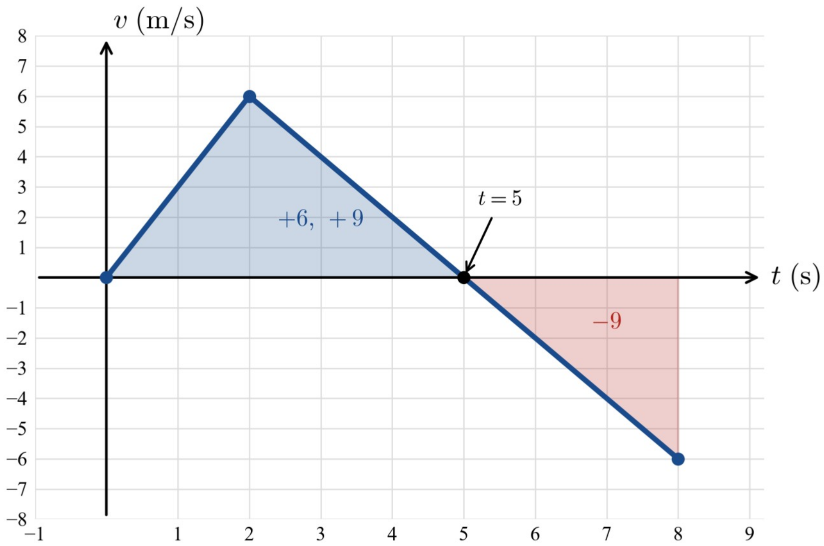

Find the acceleration in each phase, the displacement, the distance travelled, and the average velocity and average speed.

| Working | Comment |
|---|---|
| Phase 1 $(0 \le t \le 10)$: the graph rises from 0 to 20, so $a_1 = \dfrac{20-0}{10-0} = 2\,\text{m/s}^2$. | Acceleration is the gradient of the $v$–$t$ graph. |
| Phase 2 $(10 \le t \le 30)$: the graph is horizontal at $v=20$, so $a_2 = 0$. | A horizontal segment means constant velocity. |
| Phase 3 $(30 \le t \le 40)$: the graph falls from 20 to 0, so | A downward slope means $v$ is decreasing; here $v>0$, so |

| Working | Comment |
|---|---|
| $a_3=\dfrac{0-20}{40-30}=-2\,\text{m/s}^2$. | the vehicle is slowing down. |
| Displacement $=\frac{1}{2}(10)(20)+(20)(20)+\frac{1}{2}(10)(20)=100+400+10$ m. | Split the region into a triangle, a rectangle and a triangle; add the signed areas. |
| The graph never goes below the $t$-axis, so distance = displacement $=600$ m. | With no area below the axis, distance equals displacement. |
| Total time $=40$ s, so average velocity $=\dfrac{600}{40}=15$ m/s and average speed $=\dfrac{600}{40}=15$ m/s. | Average velocity uses displacement; average speed uses distance. |

### *Example 2 — scaffolded*

A particle moves in a straight line. It accelerates from rest, reaching $6$ m/s after $2$ s, then decelerates at a constant rate, so that its velocity–time graph continues as a single straight line to $-6$ m/s at $t=8$ s.




Find the acceleration in each phase, the time at which the particle changes direction, its displacement and the distance travelled, and its average velocity and average speed.

| Working | Comment |
|---|---|
| Phase 1 $(0\le t\le 2)$: gradient $=\dfrac{6-0}{2-0}=3\,\text{m/s}^2$. | A rising line: constant positive acceleration. |
| Phase 2 $(2\le t\le 8)$: gradient $=\dfrac{-6-6}{8-2}=-2\,\text{m/s}^2$. | One straight line from $(2,6)$ to $(8,-6)$. |
| On the second segment $v=6-2(t-2)=10-2t$, so $v=0$ at $t=5$ s. | The graph crosses the axis (sign change) here: the particle is momentarily at rest and reverses. |
| Displacement $=\frac{1}{2}(2)(6)+\frac{1}{2}(3)(6)-\frac{1}{2}(3)(6)=6+9-9=6$ m. | Areas above the axis are positive, below negative. |
| Distance $=6+9+9=24$ m. | Count the area below the axis as positive. |
| Total time $=8$ s, so average velocity $=\dfrac{6}{8}=\dfrac{3}{4}=0.75$ m/s | The direction change makes the two averages differ. |

and average speed $= \frac{24}{8} = 3$ m/s.

### *Example 3 — unscaffolded*

A particle moves in a straight line with velocity $v = t^2 - 4t + 3$ (m/s) for $0 \le t \le 4$, where $t$ is in seconds. Find the times at which it changes direction, its displacement over the interval, and the total distance it travels.

Factorising, $v = (t-1)(t-3)$, so $v = 0$ at $t = 1$ and $t = 3$. The velocity is positive on $0 \le t < 1$, negative on $1 < t < 3$, and positive again on $3 < t \le 4$; since $v$ changes sign at each zero, the particle **changes direction at $t = 1$ s and $t = 3$ s**.

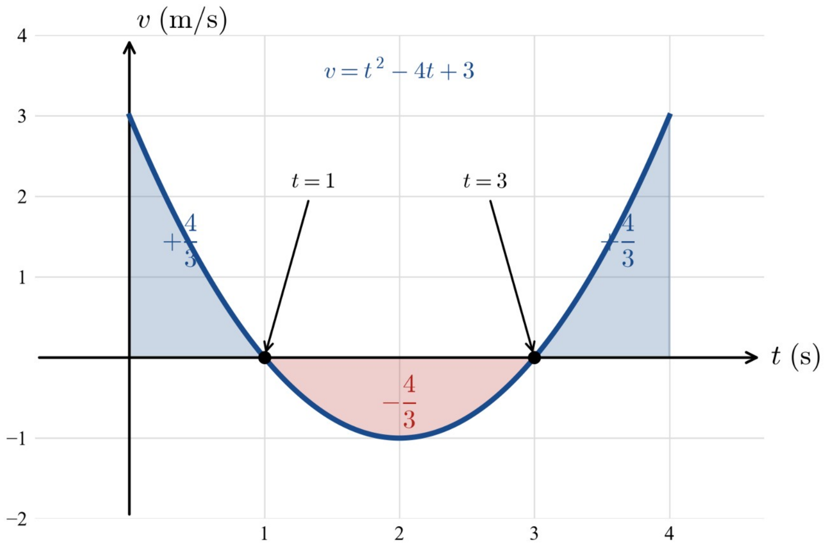
The displacement is the signed area, obtained by integrating $v$ over the whole interval:

$$\text{displacement} = \int_0^4 (t^2 - 4t + 3)\,dt = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^4 = \frac{64}{3} - 32 + 12 = \frac{4}{3}\,\text{m}.$$

For the distance, split at the direction changes $t = 1$ and $t = 3$ and integrate over each piece:

$$\int_0^1 (t^2 - 4t + 3)\,dt = \frac{4}{3}, \int_1^3 (t^2 - 4t + 3)\,dt = -\frac{4}{3}, \int_3^4 (t^2 - 4t + 3)\,dt = \frac{4}{3}.$$

Adding the magnitudes gives distance $= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ m. The three shaded lobes in the graph below have equal area $\frac{4}{3}$.

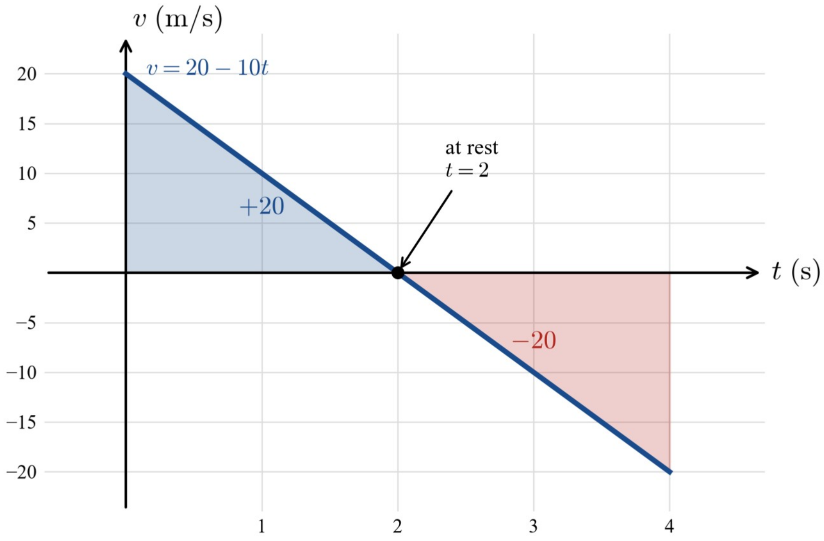



$\therefore$ the particle reverses at $t = 1$ s and $t = 3$ s; its displacement is $\frac{4}{3}$ m and it travels a distance of 4 m.

### *Example 4 — unscaffolded*

A particle moves along a straight line. It sets off with velocity 20 m/s and decelerates uniformly at $10\,\text{m/s}^2$. Sketch its velocity–time graph for the first 4 seconds, and find when it is momentarily at rest, its displacement, the farthest it gets from the start, and the total distance travelled.

Since the acceleration is a constant $-10\,\text{m/s}^2$, the velocity is $v = 20 - 10t$, a straight line starting at $(0, 20)$ with gradient $-10$. It meets the $t$-axis where $20 - 10t = 0$, i.e. at $t = 2$ s (momentarily **at rest**), and continues to $v = -20$ m/s at $t = 4$ s.




The displacement over the 4 seconds is

$$\text{displacement} = \int_0^4 (20 - 10t)\,dt = \left[20t - 5t^2\right]_0^4 = 80 - 80 = 0\text{ m},$$

so the particle returns to its starting point. Splitting at the direction change $t = 2$,

$$\int_0^2 (20 - 10t)\,dt = 20, \int_2^4 (20 - 10t)\,dt = -20,$$

so the particle travels 20 m out and 20 m back: the farthest it gets from the start is 20 m (reached at $t = 2$ s) and the total distance is $20 + 20 = 40$ m.

∴ it is momentarily at rest at $t = 2$ s, has displacement 0, reaches 20 m from the start, and travels 40 m in total.

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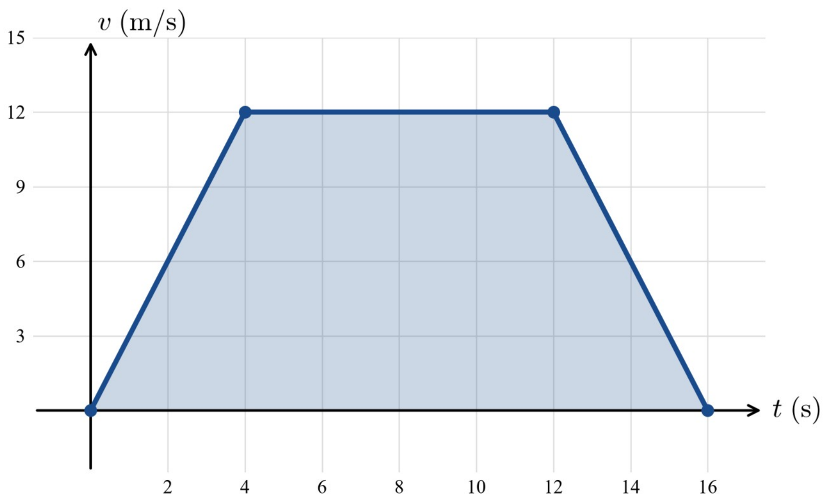
## Practice questions

### *Scaffolded*

**Q1.** The velocity–time graph shows a train's motion over 16 s.




(a) Find the acceleration in each of the three phases.
(b) Find the displacement (the area under the graph).
(c) Find the distance travelled.
(d) Find the average velocity.

**Q2.** A particle moves in a straight line; its velocity–time graph is the straight line segment from $(0,12)$ to $(5,-8)$ (with $t$ in seconds and $v$ in m/s). (a) Find the acceleration. (b) Find the time at which the particle changes direction. (c) Find the displacement over the $5$ s. (d) Find the distance travelled. (e) Find the average velocity and the average speed.

**Q3.** A particle has velocity $v=3t^2-12$ (m/s) for $0\le t\le 3$. (a) Find the time at which the velocity is zero. (b) Find the displacement. (c) Find the distance travelled. (d) Find the acceleration, and state its value at $t=2$.

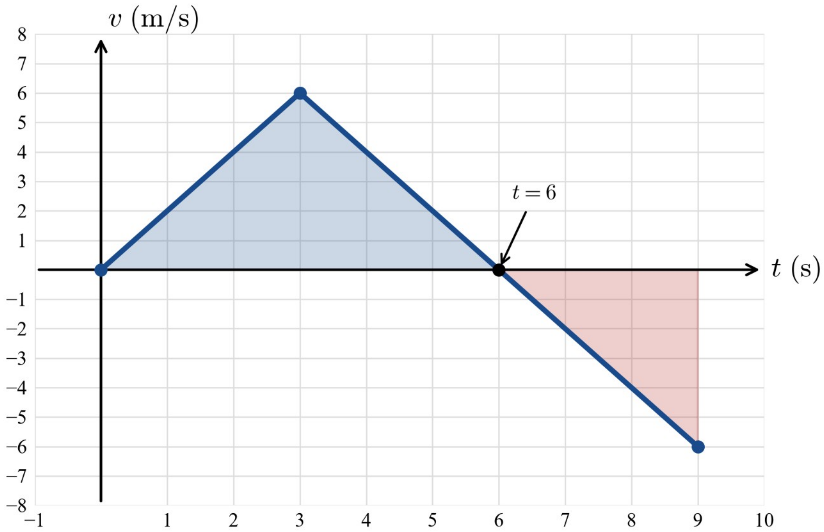
**Q4.** A particle has velocity $v=4t-t^2$ (m/s) for $0\le t\le 4$. (a) Show that $v\ge 0$ throughout the interval. (b) Find the displacement. (c) Hence state the distance travelled. (d) Find the average velocity.

**Q5.** A car's velocity–time graph is made of the straight segments joining $(0,0)$, $(2,8)$, $(6,8)$ and $(10,0)$ (with $t$ in s and $v$ in m/s). (a) Find the acceleration in each phase. (b) Find the displacement. (c) Find the average velocity.

**Q6.** A particle starts from rest and accelerates uniformly at $2\,\text{m/s}^2$ for $5$ s. It then travels at constant velocity for $10$ s, before decelerating uniformly at $5\,\text{m/s}^2$ until it stops. (a) Find its maximum velocity. (b) Find how long the deceleration takes. (c) Sketch the velocity–time graph. (d) Find the total displacement.

### *Unscaffolded*

**Q7.** The velocity–time graph of a particle is shown.




Find the acceleration in each phase, the time at which the particle changes direction, its displacement, the distance travelled, and its average velocity and average speed.

**Q8.** A particle moves with velocity $v=t^2-6t+8$ (m/s) for $0\le t\le 5$. Find the times at which it changes direction, its displacement, the distance travelled, and its acceleration as a function of $t$.

**Q9.** A particle has velocity $v=6-2t$ (m/s) for $0\le t\le 5$. Find its acceleration, the time at which it changes direction, its displacement, the distance travelled and its average speed.

**Q10.** A particle moves along a line with velocity $v=4t-8$ (m/s) for $0\le t\le 4$. Find the time at which it is at rest, its displacement over the interval and the distance travelled. What does the displacement tell you about where the particle finishes?

**Q11.** A ball moves in a straight line, starting with velocity $24$ m/s and decelerating uniformly at $8\,\text{m/s}^2$. (a) Sketch the velocity–time graph for the first $6$ s. (b) How far from the start is the ball when it is momentarily at rest? (c) What total distance has it travelled after $6$ s? (d) When does it return to its starting point?

**Q12.** A runner's velocity–time graph joins the points $(0,0)$, $(4,8)$, $(8,0)$, $(12,-8)$ and $(16,0)$ (with $t$ in s and $v$ in m/s). (a) Find the displacement. (b) Find the distance travelled. (c) Find the average velocity and the average speed, and explain why they differ.

**Q13.** A particle has velocity $v=3t^2-6t$ (m/s) for $0\le t\le 3$. Find the times at which it is at rest, its displacement, the distance travelled and its acceleration.

**Q14.** A particle's velocity–time graph is the straight line through $(0,2)$ and $(5,12)$ (with $t$ in s and $v$ in m/s). Find its acceleration, the displacement over the $5$ s, and the average velocity.

**Q15.** A particle's velocity–time graph joins $(0,0)$, $(3,6)$, $(6,0)$, $(9,0)$ and $(12,-6)$ (with $t$ in s and $v$ in m/s). (a) For what values of $t$ is the particle at rest? (b) Find the displacement. (c) Find the distance travelled.

**Q16.** A particle has velocity $v=4-t^2$ (m/s) for $0\le t\le 3$. Find the time at which it changes direction, its displacement, the distance travelled and its acceleration.

**Q17.** The velocity–time graph of a particle in rectilinear motion is a smooth curve. State what each of the following tells you about the motion: (a) the graph crosses the $t$-axis; (b) the graph is horizontal; (c) the graph has a steep positive gradient; (d) the area between the graph and the $t$-axis. Then, for the particle with velocity $v=t^2-2t-3$ (m/s) on $0\le t\le 4$, find the displacement and the distance travelled.

**Q18.** A particle moves in a straight line with velocity $v=t^2-5t+4$ (m/s) for $0\le t\le 5$. (a) Find the times at which it changes direction. (b) Find its displacement. (c) Find the total distance travelled. (d) Find the acceleration at the instant the velocity is a minimum.

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### Answers

**Q1.** (a) 3, 0, $-3\,\mathrm{m/s^2}$; (b) 144 m; (c) 144 m; (d) 9 m/s. **Q2.** (a) $-4\,\mathrm{m/s^2}$; (b) $t=3$ s; (c) 10 m; (d) 26 m; (e) average velocity 2 m/s, average speed 5.2 m/s. **Q3.** (a) $t=2$ s; (b) $-9$ m; (c) 23 m; (d) $a=6t$, so $a=12\,\mathrm{m/s^2}$ at $t=2$. **Q4.** (a) $v=t(4-t)\ge 0$ for $0\le t\le 4$; (b) $\frac{32}{3}$ m; (c) $\frac{32}{3}$ m; (d) $\frac{8}{3}$ m/s. **Q5.** (a) $4$, $0$, $-2\,\mathrm{m/s^2}$; (b) 56 m; (c) 5.6 m/s. **Q6.** (a) 10 m/s; (b) 2 s; (c) segments $(0,0)$–$(5,10)$–$(15,10)$–$(17,0)$; (d) 135 m. **Q7.** accelerations 2 and $-2\,\mathrm{m/s^2}$; direction change $t=6$ s; displacement 9 m; distance 27 m; average velocity 1 m/s, average speed 3 m/s. **Q8.** direction changes at $t=2$ s and $t=4$ s; displacement $\frac{20}{3}$ m; distance $\frac{28}{3}$ m; $a=2t-6$. **Q9.** $a=-2\,\mathrm{m/s^2}$; direction change $t=3$ s; displacement 5 m; distance 13 m; average speed 2.6 m/s. **Q10.** at rest at $t=2$ s; displacement $0$; distance 16 m; the particle finishes exactly where it started. **Q11.** (a) straight line from $(0,24)$ to $(6,-24)$, crossing the axis at $t=3$; (b) 36 m; (c) 72 m; (d) $t=6$ s. **Q12.** (a) 0; (b) 64 m; (c) average velocity $0$, average speed 4 m/s; they differ because the runner reverses direction, so the return cancels the outward displacement but adds to the distance. **Q13.** at rest at $t=0$ and $t=2$ s; displacement $0$; distance $8$ m; $a=6t-6$. **Q14.** $a=2\,\mathrm{m/s^2}$; displacement 35 m; average velocity 7 m/s. **Q15.** (a) $6\le t\le 9$ s (and momentarily at $t=0$); (b) 9 m; (c) 27 m. **Q16.** direction change $t=2$ s; displacement 3 m; distance $\frac{23}{3}$ m; $a=-2t$. **Q17.** (a) the particle is momentarily at rest and reverses direction; (b) the acceleration is zero; (c) the acceleration is large and positive; (d) the signed area is the displacement, the total area the distance. For $v=t^2-2t-3$: displacement $-\frac{20}{3}$ m, distance $\frac{34}{3}$ m. **Q18.** (a) $t=1$ s and $t=4$ s; (b) $-\frac{5}{6}$ m; (c) $\frac{49}{6}$ m; (d) $0\,\mathrm{m/s^2}$ (at $t=\frac{5}{2}$ s).

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### Fully-worked solutions

**Q1.** Acceleration is the gradient of each segment. Phase 1 $(0\to 4)$: $a=\frac{12-0}{4-0}=3\,\mathrm{m/s^2}$; phase 2 $(4\to 12)$ is horizontal, $a=0$; phase 3 $(12\to 16)$: $a=\frac{0-12}{16-12}=-3\,\mathrm{m/s^2}$. The region under the graph is a triangle, a rectangle and a triangle, so the displacement is

$$1/2(4)(12)+(8)(12)+1/2(4)(12)=24+96+24=144\,\mathrm{m}.$$

The graph stays on or above the axis, so the distance also equals $144$ m. With total time $16$ s, the average velocity is $\frac{144}{16}=9$ m/s.

**Q2.** The line from $(0,12)$ to $(5,-8)$ has gradient $a=\frac{-8-12}{5-0}=-4\,\mathrm{m/s^2}$, so $v=12-4t$. It crosses the axis where $12-4t=0$, i.e. $t=3$ s (the direction change). The displacement is the signed area,

$$\int_0^5 (12-4t)\,dt=\left[12t-2t^2\right]_0^5=60-50=10\,\mathrm{m}.$$

Splitting at $t=3$: the triangle above the axis has area $1/2(3)(12)=18$ and the triangle below has area $1/2(2)(8)=8$, so the distance is $18+8=26$ m. Over $5$ s the average velocity is $\frac{10}{5}=2$ m/s and the average speed is $\frac{26}{5}=5.2$ m/s.

**Q3.** Here $v=3t^2-12=3(t^2-4)=3(t-2)(t+2)$, so on $0\le t\le 3$ the velocity is zero at $t=2$ s, negative for $0\le t<2$ and positive for $2<t\le 3$. The displacement is

$$\int_0^3 \left(3t^2-12\right)dt=\left[t^3-12t\right]_0^3=27-36=-9\,\text{m}.$$

Splitting at $t=2$,

$$\int_0^2 \left(3t^2-12\right)dt=-16,\ \int_2^3 \left(3t^2-12\right)dt=7,$$

so the distance is $16+7=23$ m. The acceleration is $a=\frac{dv}{dt}=6t$, which at $t=2$ gives $a=12\,\text{m/s}^2$.

**Q4.** Factorising, $v=4t-t^2=t(4-t)$. For $0\le t\le 4$ both $t\ge 0$ and $4-t\ge 0$, so $v\ge 0$ throughout: the particle never reverses. The displacement is

$$\int_0^4 \left(4t-t^2\right)dt=\left[2t^2-\frac{t^3}{3}\right]_0^4=32-\frac{64}{3}=\frac{32}{3}\,\text{m}.$$

Because $v\ge 0$, the distance equals the displacement, $\frac{32}{3}$ m. The average velocity is $\frac{1}{4}\cdot\frac{32}{3}=\frac{8}{3}$ m/s.

**Q5.** The gradients are: phase 1 $(0\to 2)$, $a=\frac{8-0}{2-0}=4\,\text{m/s}^2$; phase 2 $(2\to 6)$ horizontal, $a=0$; phase 3 $(6\to 10)$, $a=\frac{0-8}{10-6}=-2\,\text{m/s}^2$. The region is a triangle, a rectangle and a triangle, so the displacement is

$$1/2(2)(8)+(4)(8)+1/2(4)(8)=8+32+16=56\,\text{m}.$$

All of it is above the axis, so with total time 10 s the average velocity is $\frac{56}{10}=5.6$ m/s.

**Q6.** Accelerating at $2\,\text{m/s}^2$ from rest for 5 s gives a maximum velocity of $2\times 5=10$ m/s. Decelerating at $5\,\text{m/s}^2$ from 10 m/s to rest takes $\frac{10}{5}=2$ s. The velocity–time graph therefore consists of the segments joining $(0,0)$, $(5,10)$, $(15,10)$ and $(17,0)$: a ramp up, a horizontal stretch, then a steeper ramp down. The displacement is the area,

$$1/2(5)(10)+(10)(10)+1/2(2)(10)=25+100+10=135\,\text{m}.$$

**Q7.** The graph rises from $(0,0)$ to $(3,6)$ then falls straight to $(9,-6)$. Gradients: phase 1, $a=\frac{6-0}{3-0}=2\,\text{m/s}^2$; phase 2, $a=\frac{-6-6}{9-3}=-2\,\text{m/s}^2$. On the second segment $v=6-2(t-3)=12-2t$, which is zero at $t=6$ s — the direction change. The signed area is

$$1/2(3)(6)+1/2(3)(6)-1/2(3)(6)=9+9-9=9\,\text{m}\,(\text{displacement}),$$

and the total area is $9+9+9=27$ m (distance). Over 9 s the average velocity is $\frac{9}{9}=1$ m/s and the average speed is $\frac{27}{9}=3$ m/s.

**Q8.** Here $v=t^2-6t+8=(t-2)(t-4)$, so on $0\le t\le 5$ the velocity is zero at $t=2$ s and $t=4$ s, and it changes sign at each — the **direction changes at $t=2$ s and $t=4$ s**. The displacement is

$$\int_0^5 \left(t^2-6t+8\right)dt=\left[\frac{t^3}{3}-3t^2+8t\right]_0^5=\frac{125}{3}-75+40=\frac{20}{3}\,\text{m}.$$

Splitting at $t=2$ and $t=4$,

$$\int_0^2 (t^2-6t+8)\,dt=\frac{20}{3},\int_2^4 (t^2-6t+8)\,dt=-\frac{4}{3},\int_4^5 (t^2-6t+8)\,dt=\frac{4}{3},$$

so the distance is $\frac{20}{3}+\frac{4}{3}+\frac{4}{3}=\frac{28}{3}$ m. The acceleration is $a=\frac{dv}{dt}=2t-6$.

**Q9.** The acceleration is $a=\frac{dv}{dt}=-2\,\text{m/s}^2$. The velocity $v=6-2t$ is zero at $t=3$ s (the direction change). The displacement is

$$\int_0^5 (6-2t)\,dt=\left[6t-t^2\right]_0^5=30-25=5\,\text{m}.$$

Splitting at $t=3$,

$$\int_0^3 (6-2t)\,dt=9,\int_3^5 (6-2t)\,dt=-4,$$

so the distance is $9+4=13$ m. Over 5 s the average speed is $\frac{13}{5}=2.6$ m/s.

**Q10.** The velocity $v=4t-8$ is zero at $t=2$ s, so the particle is at rest then. The displacement is

$$\int_0^4 (4t-8)\,dt=\left[2t^2-8t\right]_0^4=32-32=0\,\text{m}.$$

Splitting at $t=2$,

$$\int_0^2 (4t-8)\,dt=-8,\int_2^4 (4t-8)\,dt=8,$$

so the distance is $8+8=16$ m. Since the displacement is $0$, the particle finishes exactly where it started.

**Q11.** With constant acceleration $-8\,\text{m/s}^2$ the velocity is $v=24-8t$: a straight line from $(0,24)$, crossing the axis at $t=3$ s and reaching $-24$ m/s at $t=6$ s. It is momentarily at rest at $t=3$ s; the distance to that turning point is the area of the first triangle, $1/2(3)(24)=36$ m. By symmetry the return leg also has area $36$ m, so the total distance after $6$ s is $36+36=72$ m. The displacement after $6$ s is

$$\int_0^6 (24-8t)\,dt=\left[24t-4t^2\right]_0^6=144-144=0,$$

so the ball returns to its starting point at $t=6$ s.

**Q12.** The graph is a triangular hump above the axis on $[0,8]$ and an equal triangular hump below on $[8,16]$. The area above is $1/2(8)(8)=32$ and the area below is $1/2(8)(8)=32$, so the displacement is $32-32=0$ and the distance is $32+32=64$ m. Over $16$ s the average velocity is $\frac{0}{16}=0$ and the average speed is $\frac{64}{16}=4$ m/s. They differ because the runner reverses direction: the return trip cancels the outward displacement (giving zero average velocity) but still adds to the distance covered.

**Q13.** Here $v=3t^2-6t=3t(t-2)$, so on $0\le t\le 3$ the velocity is zero at $t=0$ and $t=2$ s. It is negative on $0<t<2$ and positive on $2<t\le 3$, changing sign at $t=2$. The displacement is

$$\int_0^3 (3t^2-6t)\,dt=[t^3-3t^2]_0^3=27-27=0\,\text{m}.$$

Splitting at $t=2$,

$$\int_0^2 (3t^2-6t)\,dt=-4,\int_2^3 (3t^2-6t)\,dt=4,$$

so the distance is $4+4=8$ m. The acceleration is $a=\frac{dv}{dt}=6t-6$.

**Q14.** The line through $(0,2)$ and $(5,12)$ has gradient $a=\frac{12-2}{5-0}=2\,\text{m/s}^2$, so $v=2t+2$. The region under it on $[0,5]$ is a trapezium with parallel sides 2 and 12 and width 5, so the displacement is $1/2(2+12)(5)=35$ m (equivalently $\int_0^5 (2t+2)\,dt=[t^2+2t]_0^5=35$). Since $v>0$ throughout, the average velocity is $\frac{35}{5}=7$ m/s.

**Q15.** On $[0,6]$ the graph rises to $(3,6)$ then falls back to $(6,0)$; on $[6,9]$ it lies on the $t$-axis; on $[9,12]$ it falls to $(12,-6)$. The particle is **at rest for** $6\le t\le 9$ **s** (and momentarily at $t=0$). The forward part is a triangle of area $1/2(6)(6)=18$ and the backward part a triangle of area $1/2(3)(6)=9$, so the displacement is $18-9=9$ m and the distance is $18+9=27$ m.

**Q16.** The velocity $v=4-t^2$ is zero at $t=2$ s on $0\le t\le 3$, positive before and negative after, so the particle changes direction at $t=2$ s. The displacement is

$$\int_0^3 (4-t^2)\,dt=\left[4t-\frac{t^3}{3}\right]_0^3=12-9=3\,\text{m}.$$

Splitting at $t=2$,

$$\int_0^2 (4-t^2)\,dt=\frac{16}{3},\int_2^3 (4-t^2)\,dt=-\frac{7}{3},$$

so the distance is $\frac{16}{3}+\frac{7}{3}=\frac{23}{3}$ m. The acceleration is $a=\frac{dv}{dt}=-2t$.

**Q17.** (a) Where the graph crosses the $t$-axis, $v$ changes sign, so the particle is momentarily at rest and reverses direction. (b) Where the graph is horizontal, the gradient is zero, so the acceleration is zero. (c) A steep positive gradient means a large positive acceleration (the velocity is increasing quickly). (d) The signed area between the graph and the axis is the displacement, and the total area (below counted as positive) is the distance. For $v=t^2-2t-3=(t-3)(t+1)$ the only zero in $[0,4]$ is $t=3$; the velocity is negative on $0\le t<3$ and positive on $3<t\le 4$. The displacement is

$$\int_0^4 (t^2-2t-3)\,dt=\left[\frac{t^3}{3}-t^2-3t\right]_0^4=\frac{64}{3}-28=-\frac{20}{3}\,\text{m}.$$

Splitting at $t=3$,

$$\int_0^3 (t^2-2t-3)\,dt=-9,\int_3^4 (t^2-2t-3)\,dt=\frac{7}{3},$$

so the distance is $9+\frac{7}{3}=\frac{34}{3}$ m.

**Q18.** Here $v=t^2-5t+4=(t-1)(t-4)$, so the velocity is zero at $t=1$ s and $t=4$ s, changing sign at each — the **direction changes at** $t=1$ **s and** $t=4$ **s**. The displacement is

$$\int_0^5 \left(t^2-5t+4\right)dt=\left[\frac{t^3}{3}-\frac{5t^2}{2}+4t\right]_0^5=\frac{125}{3}-\frac{125}{2}+20=-\frac{5}{6}\,\mathrm{m}.$$

Splitting at $t=1$ and $t=4$,

$$\int_0^1 \left(t^2-5t+4\right)dt=\frac{11}{6},\int_1^4 \left(t^2-5t+4\right)dt=-\frac{9}{2},\int_4^5 \left(t^2-5t+4\right)dt=\frac{11}{6},$$

so the distance is $\frac{11}{6}+\frac{9}{2}+\frac{11}{6}=\frac{49}{6}$ m. The acceleration is $a=\frac{dv}{dt}=2t-5$, which is zero at $t=\frac{5}{2}$ s; this is exactly where the velocity is a minimum, so the acceleration there is $0\,\mathrm{m/s^2}$.

## Theory

For a particle moving in a straight line, position $x$, velocity $v=\frac{dx}{dt}$ and acceleration $a=\frac{dv}{dt}$ are linked by calculus. Acceleration has several equivalent forms,

$$a=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(1/2\,v^2\right),$$

and choosing the right one turns a motion problem into a differential equation that **separates**. Earlier sections dealt with the rate depending on time. This section treats the two cases in which the rate depends on something else: **velocity as a function of position**, $v=f(x)$, and **acceleration as a function of velocity**, $a=f(v)$. In each case the equation cannot be antidifferentiated directly, because the right-hand side is not yet a known function of $t$; the reciprocal (separation-of-variables) method is needed.

**Velocity as a function of position: $v=f(x)$.**

Here the speed is prescribed at each position, so $\frac{dx}{dt}=f(x)$. The right-hand side is a function of $x$, not of $t$, so we cannot integrate with respect to $t$. Instead, turn the derivative upside down. Provided $f(x)\neq 0$,

$$\frac{dt}{dx}=\frac{1}{f(x)},\ t=\int\frac{1}{f(x)}\,dx+c.$$

This gives $t$ **as a function of** $x$, with one arbitrary constant fixed by an initial condition. Where the algebra allows, the relation is then rearranged to give $x$ **as a function of** $t$; otherwise the answer is left as $t$ in terms of $x$.

**Acceleration as a function of velocity: $a=f(v)$.**

When the acceleration depends on the velocity — as it does whenever a resistance grows with speed — two of the derivative forms are available, and the choice depends on **which variables the problem links**.

*Time form.* If the question involves **time**, use $a=\frac{dv}{dt}$:

$$\frac{dv}{dt}=f(v),\ \frac{dt}{dv}=\frac{1}{f(v)},\ t=\int\frac{1}{f(v)}\,dv+c.$$

This gives $t$ as a function of $v$, which is rearranged to give $v$ **as a function of** $t$.

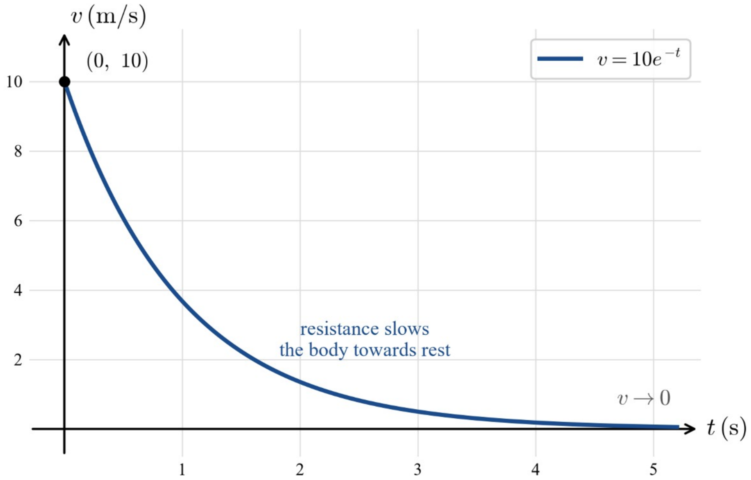
*Position form.* If the question links **velocity and position** and time does not appear, use $a=v\frac{dv}{dx}$:

$$v\frac{dv}{dx}=f(v),\ \frac{dx}{dv}=\frac{v}{f(v)},\ x=\int\frac{v}{f(v)}\,dv+c.$$

This relates $v$ and $x$ directly. The third form $\frac{d}{dx}\left(1/2\,v^2\right)$ is just another way of writing $v\frac{dv}{dx}$, and is convenient when $f$ depends on $v^2$. Both routes are separation of variables; always keep the $+c$ and apply the initial condition last.

**Resisted motion.**

A body moving through air or a liquid meets a **resistance** that opposes its motion and grows with speed. Two standard models are used: a resistance giving acceleration $a=-kv$ (roughly, low speeds) and $a=-kv^2$ (higher speeds), where $k>0$ is a constant and the minus sign shows the acceleration opposes the velocity. Solving these with the methods above reveals a telling difference. Under $a=-kv$ the **position form** gives $v\frac{dv}{dx}=-kv$, so $\frac{dv}{dx}=-k$ and $v=v_0-kx$ : the velocity falls **linearly** with position and the body comes to rest after a finite distance $\frac{v_0}{k}$. The **time form** gives $v=v_0e^{-kt}$, which decays towards zero but never quite reaches it.




The diagram shows the velocity–time curve for $a=-kv$ (here $v=10e^{-t}$): resistance slows the body ever closer to rest, the curve flattening onto the $t$-axis, $v\to 0$.

**Terminal velocity.**

When a body falls under gravity **against** resistance, the two effects compete. Taking down as positive with gravitational acceleration $g$ and linear resistance, $a=\frac{dv}{dt}=g-kv$. As the speed rises the resistance $kv$ grows until it balances gravity; then $a=0$ and the speed stops changing. This limiting speed is the **terminal velocity**

$$v_T=\frac{g}{k},\text{ where } a=g-kv_T=0.$$

Solving the equation gives $v=v_T\left(1-e^{-kt}\right)$, which rises from rest and flattens onto the horizontal asymptote $v=v_T$. The terminal velocity is the equilibrium of the motion — the value the velocity approaches but, in this model, never exactly attains.

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## Worked examples

### *Example 1 — scaffolded*

A particle moves in a straight line with velocity $v=\frac{dx}{dt}=x+2$ (for $x\ge 0$), starting at $x=0$ when $t=0$. Find $t$ as a function of $x$, and hence $x$ as a function of $t$.

| Working | Comment |
|---|---|
| Velocity depends on position, so invert: $\frac{dt}{dx}=\frac{1}{x+2}$. | The right-hand side is a function of $x$ alone, ready to integrate with respect to $x$. |
| Integrate: $t=\int\frac{1}{x+2}\,dx=\log_e\lvert x+2\rvert+c$. | One indefinite integral, one constant $+c$. |
| Since $x\ge 0$, $\lvert x+2\rvert=x+2$. Apply $x=0$ at $t=0$: $0=\log_e 2+c$, so $c=-\log_e 2$. | The initial condition fixes the constant. |
| $t=\log_e(x+2)-\log_e 2=\log_e\frac{x+2}{2}$. | This is $t$ as a function of $x$. |

| Working | Comment |
|---|---|
| Invert: $e^t=\frac{x+2}{2}$, so $x=2e^t-2$. | Rearranged to give position as a function of time. Check: $\frac{dx}{dt}=2e^t=x+2$. |

### *Example 2 — scaffolded*

A body decelerates in a resisting medium so that $a=\frac{dv}{dt}=-2v$, with $v=10$ m/s when $t=0$. Find $v$ as a function of $t$, and the time taken to slow to 5 m/s.

| Working | Comment |
|---|---|
| Acceleration depends on velocity and time is involved, so use the time form and invert: $\frac{dt}{dv}=-\frac{1}{2v}$. | This is resisted motion $a=-kv$ with $k=2$. |
| Integrate: $t=\int -\frac{1}{2v}\,dv=-\frac{1}{2}\log_e\lvert v\rvert+c$. | Integrate with respect to $v$; keep the $+c$. |
| Since $v>0$, apply $v=10$ at $t=0$: $0=-\frac{1}{2}\log_e 10+c$, so $c=\frac{1}{2}\log_e 10$. | Use the initial condition. |
| $t=\frac{1}{2}\left(\log_e 10-\log_e v\right)=\frac{1}{2}\log_e\frac{10}{v}$, so $\frac{10}{v}=e^{2t}$ and $v=10e^{-2t}$. | Rearrange: velocity decays exponentially. |
| Set $v=5$: $5=10e^{-2t}$, so $e^{-2t}=\frac{1}{2}$ and $t=\frac{1}{2}\log_e 2$. | Substitute $v=5$ and solve for $t$. |

### *Example 3 — unscaffolded*

A particle decelerates through a resisting medium with acceleration $a=-\frac{1}{2}v^2$, and has velocity $v=20$ m/s at $x=0$. Find $v$ as a function of position $x$, and the distance travelled while the speed falls from 20 m/s to 5 m/s.

The acceleration depends on velocity and the question links velocity to position (time does not appear), so use the position form $a=v\frac{dv}{dx}$:

$$v\frac{dv}{dx}=-\frac{1}{2}v^2.$$

Dividing by $v$ (with $v\neq 0$) gives $\frac{dv}{dx}=-\frac{1}{2}v$, and inverting,

$$\frac{dx}{dv}=-\frac{2}{v},\ x=\int -\frac{2}{v}\,dv=-2\log_e\lvert v\rvert+c.$$

With $v=20>0$ at $x=0$: $0=-2\log_e 20+c$, so $c=2\log_e 20$ and $x=2\log_e\frac{20}{v}$. Inverting, $\frac{x}{2}=\log_e\frac{20}{v}$, so $v=20e^{-x/2}$. The speed reaches 5 m/s when $5=20e^{-x/2}$, that is $e^{-x/2}=\frac{1}{4}$, so $\frac{x}{2}=\log_e 4$ and $x=2\log_e 4=4\log_e 2$.

$\therefore v=20e^{-x/2}$, and the body travels $4\log_e 2\approx 2.77$ m while slowing from 20 m/s to 5 m/s.

### *Example 4 — unscaffolded*

A body is released from rest and falls so that its velocity satisfies $a=\frac{dv}{dt}=10-v$ (with $v$ in m/s and $t$ in seconds). Find $v$ as a function of $t$, and the terminal velocity.

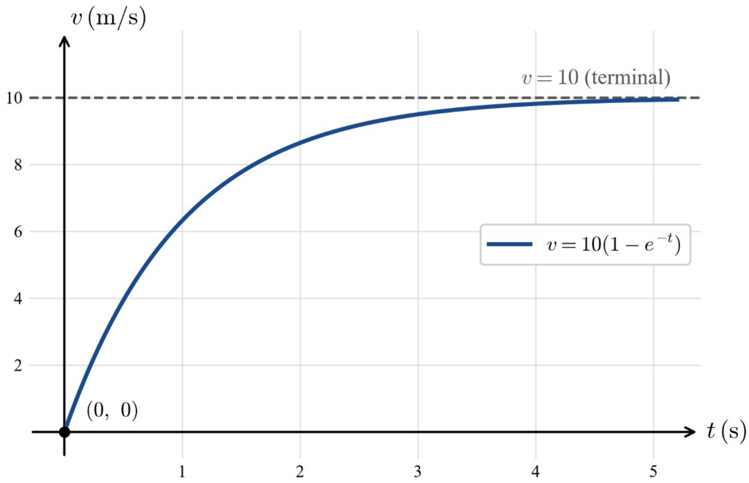
The acceleration depends on velocity and time is involved, so use the time form and invert:

$$\frac{dt}{dv}=\frac{1}{10-v},\ t=\int\frac{1}{10-v}\,dv=-\log_e|10-v|+c.$$

The body starts from rest and speeds up towards 10 m/s, so $v<10$ and $|10-v|=10-v$. The initial condition $v=0$ at $t=0$ gives $0=-\log_e 10+c$, so $c=\log_e 10$ and

$$t=\log_e 10-\log_e(10-v)=\log_e\frac{10}{10-v}.$$

Inverting, $e^t=\frac{10}{10-v}$, so $10-v=10e^{-t}$ and $v=10-10e^{-t}=10\left(1-e^{-t}\right)$. As $t\to\infty$, $e^{-t}\to 0$, so $v\to 10$ m/s: the terminal velocity, at which $a=10-v=0$ and the body falls at constant speed.




$\therefore v=10\left(1-e^{-t}\right)$, with terminal velocity 10 m/s.

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## Practice questions

### *Scaffolded*

**Q1.** A particle moves with $v=\frac{dx}{dt}=x+1$ and $x=0$ when $t=0$. (a) Write down $\frac{dt}{dx}$. (b) Integrate to find $t$ in terms of $x$, including $+c$. (c) Apply the initial condition to find $c$. (d) Rearrange to give $x$ as a function of $t$.

**Q2.** A particle moves with $v=\frac{dx}{dt}=4-x$ (with $x<4$) and $x=0$ when $t=0$. (a) Write $\frac{dt}{dx}=\frac{1}{4-x}$ and integrate to find $t$ in terms of $x$. (b) Apply the initial condition and rearrange to $x=4\left(1-e^{-t}\right)$. (c) State the value $x$ approaches as $t\to\infty$.

**Q3.** A body decelerates in a resisting medium with $a=\frac{dv}{dt}=-3v$ and $v=12$ m/s when $t=0$. (a) Write $\frac{dt}{dv}$ and integrate with respect to $v$. (b) Rearrange to show $v=12e^{-3t}$. (c) Find the time at which $v=4$ m/s.

**Q4.** A particle decelerates with $a=\frac{dv}{dt}=-v^2$ and $v=5$ m/s when $t=0$. (a) Write $\frac{dt}{dv}=-\frac{1}{v^2}$ and integrate. (b) Apply the initial condition to find the constant. (c) Rearrange to give $v$ as a function of $t$.

**Q5.** A particle has acceleration $a=-2v$, with $v=10$ m/s at $x=0$. Use the position form $a=v\frac{dv}{dx}$. (a) Show that $\frac{dv}{dx}=-2$. (b) Integrate to find $v$ as a function of $x$. (c) Find the distance travelled before the particle comes to rest.

**Q6.** A body released from rest falls so that $a=\frac{dv}{dt}=10-2v$ (with $v$ in m/s). (a) Write $\frac{dt}{dv}=\frac{1}{10-2v}$ and integrate. (b) Apply $v=0$ at $t=0$ and rearrange to $v=5\left(1-e^{-2t}\right)$. (c) State the terminal velocity.

***Unscaffolded***

**Q7.** A particle moves with $v=\frac{dx}{dt}=3x$ and $x=2$ when $t=0$. Find $x$ as a function of $t$.

**Q8.** A particle moves with $v=\frac{dx}{dt}=x^2$ and $x=1$ when $t=0$. Find $x$ as a function of $t$, and state the interval of $t$ for which it is valid.

**Q9.** A particle moves with $v=\frac{dx}{dt}=2\sqrt{x}$ and $x=1$ when $t=0$. Find $x$ as a function of $t$.

**Q10.** A particle moves with $v=\frac{dx}{dt}=x^2+1$ and $x=0$ when $t=0$. Find $x$ as a function of $t$.

**Q11.** A body decelerates with $a=\frac{dv}{dt}=-\frac{1}{4}v$ and $v=8$ m/s at $t=0$. Find $v$ as a function of $t$, and the time taken for the speed to halve.

**Q12.** A particle decelerates with $a=\frac{dv}{dt}=-\frac{1}{2}v^2$ and $v=4$ m/s at $t=0$. Find $v$ as a function of $t$.

**Q13.** A particle has acceleration $a=-v^2$, with $v=10$ m/s at $x=0$. Using $a=v\frac{dv}{dx}$, find $v$ as a function of $x$.

**Q14.** A particle has acceleration $a=-\frac{1}{3}v$, with $v=15$ m/s at $x=0$. Using $a=v\frac{dv}{dx}$, find $v$ as a function of $x$ and the distance travelled before it comes to rest.

**Q15.** A particle decelerates with $a=-\frac{1}{4}v^2$, with $v=8$ m/s at $x=0$. Using $a=v\frac{dv}{dx}$, find $v$ as a function of $x$ and the distance travelled while the speed falls from 8 m/s to 2 m/s.

**Q16.** A body released from rest falls with $a=\frac{dv}{dt}=10-5v$ (with $v$ in m/s). Find $v$ as a function of $t$ and the terminal velocity.

**Q17.** A ball is thrown vertically upward with speed 20 m/s through a resisting medium so that, taking up as positive, $a=\frac{dv}{dt}=-(10+v)$. Find $v$ as a function of $t$, and the time to reach the highest point.

**Q18.** A particle decelerates with $a=-\frac{1}{2}v^2$ and has $v=6$ m/s at $x=0$ when $t=0$. (a) Using $a=v\frac{dv}{dx}$, find $v$ as a function of $x$. (b) Find the distance travelled while the speed falls to 2 m/s. (c) Using $a=\frac{dv}{dt}$, find $v$ as a function of $t$.

### Answers

**Q1.** $t=\log_e(x+1)$; $x=e^t-1$. **Q2.** $x=4(1-e^{-t})$; $x\to 4$ as $t\to\infty$. **Q3.** $v=12e^{-3t}$; $t=\frac{1}{3}\log_e 3$ s. **Q4.** $v=\frac{5}{5t+1}$. **Q5.** $v=10-2x$; comes to rest after 5 m. **Q6.** $v=5(1-e^{-2t})$; terminal velocity 5 m/s. **Q7.** $x=2e^{3t}$. **Q8.** $x=\frac{1}{1-t}$, valid for $t<1$. **Q9.** $x=(t+1)^2$. **Q10.** $x=\tan t$. **Q11.** $v=8e^{-t/4}$; the speed halves after $t=4\log_e 2$ s. **Q12.** $v=\frac{4}{2t+1}$. **Q13.** $v=10e^{-x}$. **Q14.** $v=15-\frac{x}{3}$; comes to rest after 45 m. **Q15.** $v=8e^{-x/4}$; distance $8\log_e 2$ m. **Q16.** $v=2(1-e^{-5t})$; terminal velocity 2 m/s. **Q17.** $v=30e^{-t}-10$; highest point at $t=\log_e 3$ s. **Q18.** (a) $v=6e^{-x/2}$; (b) $x=2\log_e 3$ m; (c) $v=\frac{6}{3t+1}$.

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### Fully-worked solutions

**Q1.** Velocity depends on position, so invert and integrate with respect to $x$:

$$\frac{dt}{dx}=\frac{1}{x+1},\ t=\int\frac{1}{x+1}\,dx=\log_e|x+1|+c.$$

With $x=0$ at $t=0$: $0=\log_e 1+c=c$, so $c=0$ and $t=\log_e(x+1)$ (taking $x>-1$). Inverting, $e^t=x+1$, so $x=e^t-1$.

**Q2.** Invert and integrate:

$$\frac{dt}{dx}=\frac{1}{4-x},\ t=\int\frac{1}{4-x}\,dx=-\log_e|4-x|+c.$$

Since $x<4$, $|4-x|=4-x$. With $x=0$ at $t=0$: $0=-\log_e 4+c$, so $c=\log_e 4$ and $t=\log_e 4-\log_e(4-x)=\log_e\frac{4}{4-x}$. Inverting, $4-x=4e^{-t}$, so $x=4(1-e^{-t})$. As $t\to\infty$, $e^{-t}\to 0$, so $x\to 4$.

**Q3.** Use the time form and invert:

$$\frac{dt}{dv}=-\frac{1}{3v},\ t=\int-\frac{1}{3v}\,dv=-\frac{1}{3}\log_e|v|+c.$$

With $v=12$ at $t=0$: $c=\frac{1}{3}\log_e 12$, so $t=\frac{1}{3}\log_e\frac{12}{v}$ and $v=12e^{-3t}$. Setting $v=4$: $4=12e^{-3t}$, so $e^{-3t}=\frac{1}{3}$ and $t=\frac{1}{3}\log_e 3$ s.

**Q4.** Invert and integrate:

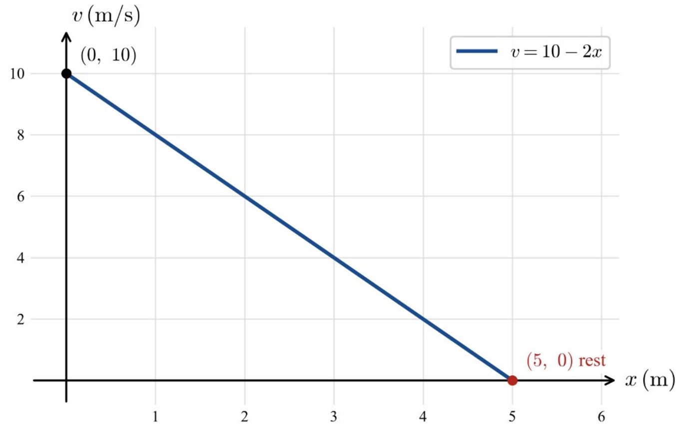
$$\frac{dt}{dv}=-\frac{1}{v^2}=-v^{-2},\ t=\int-v^{-2}\,dv=\frac{1}{v}+c.$$

With $v=5$ at $t=0$: $0=\frac{1}{5}+c$, so $c=-\frac{1}{5}$ and $t=\frac{1}{v}-\frac{1}{5}$. Then $\frac{1}{v}=t+\frac{1}{5}=\frac{5t+1}{5}$, so $v=\frac{5}{5t+1}$.

**Q5.** With $a=v\frac{dv}{dx}=-2v$, dividing by $v$ gives $\frac{dv}{dx}=-2$. Inverting, $\frac{dx}{dv}=-\frac{1}{2}$, so

$$x=\int-\frac{1}{2}\,dv=-\frac{1}{2}v+c.$$

With $v=10$ at $x=0$: $0=-5+c$, so $c=5$ and $x=5-\frac{1}{2}v$; rearranging, $v=10-2x$. The particle comes to rest when $v=0$: $0=10-2x$, so $x=5$ m.




**Q6.** Invert and integrate:

$$\frac{dt}{dv}=\frac{1}{10-2v},t=\int\frac{1}{10-2v}dv=-\frac{1}{2}\log_e|10-2v|+c.$$

The body starts from rest and speeds up towards 5 m/s, so $v<5$ and $10-2v>0$. With $v=0$ at $t=0$: $0=-\frac{1}{2}\log_e 10+c$, so $c=\frac{1}{2}\log_e 10$ and $t=\frac{1}{2}\log_e\frac{10}{10-2v}$. Inverting, $10-2v=10e^{-2t}$, so $v=5\left(1-e^{-2t}\right)$. As $t\to\infty$, $v\to 5$ m/s; equivalently the terminal velocity satisfies $a=10-2v=0$, giving $v=5$ m/s.

**Q7.** Invert and integrate:

$$\frac{dt}{dx}=\frac{1}{3x},t=\int\frac{1}{3x}dx=\frac{1}{3}\log_e|x|+c.$$

With $x=2$ at $t=0$: $c=-\frac{1}{3}\log_e 2$, so $t=\frac{1}{3}\log_e\frac{x}{2}$ and $\frac{x}{2}=e^{3t}$, giving $x=2e^{3t}$.

**Q8.** Invert and integrate:

$$\frac{dt}{dx}=\frac{1}{x^2}=x^{-2},t=\int x^{-2}dx=-\frac{1}{x}+c.$$

With $x=1$ at $t=0$: $0=-1+c$, so $c=1$ and $t=1-\frac{1}{x}$. Then $\frac{1}{x}=1-t$, so $x=\frac{1}{1-t}$, valid for $t<1$ (as $t\to 1^-$ the particle runs off to infinity).

**Q9.** Invert and integrate:

$$\frac{dt}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2}x^{-1/2},t=\int\frac{1}{2}x^{-1/2}dx=x^{1/2}+c=\sqrt{x}+c.$$

With $x=1$ at $t=0$: $0=1+c$, so $c=-1$ and $t=\sqrt{x}-1$. Then $\sqrt{x}=t+1$, so $x=(t+1)^2$ (for $t\geq 0$). Check: $\frac{dx}{dt}=2(t+1)=2\sqrt{x}$.

**Q10.** Invert and integrate:

$$\frac{dt}{dx}=\frac{1}{x^2+1},t=\int\frac{1}{x^2+1}dx=\tan^{-1}x+c.$$

With $x=0$ at $t=0$: $0=\tan^{-1}0+c=c$, so $c=0$ and $t=\tan^{-1}x$. Hence $x=\tan t$ (for $-\frac{\pi}{2}<t<\frac{\pi}{2}$).

**Q11.** Use the time form and invert:

$$\frac{dt}{dv}=-\frac{4}{v}, t=\int -\frac{4}{v}dv=-4\log_e|v|+c.$$

With $v=8$ at $t=0$: $c=4\log_e 8$, so $t=4\log_e\frac{8}{v}$ and $v=8e^{-t/4}$. The speed halves when $v=4$: $4=8e^{-t/4}$, so $e^{-t/4}=\frac{1}{2}$ and $t=4\log_e 2$ s.

**Q12.** Invert and integrate:

$$\frac{dt}{dv}=-\frac{2}{v^2}=-2v^{-2}, t=\int -2v^{-2}dv=\frac{2}{v}+c.$$

With $v=4$ at $t=0$: $0=\frac{2}{4}+c=\frac{1}{2}+c$, so $c=-\frac{1}{2}$ and $t=\frac{2}{v}-\frac{1}{2}$. Then $\frac{2}{v}=t+\frac{1}{2}=\frac{2t+1}{2}$, so $v=\frac{4}{2t+1}$.

**Q13.** With $a=v\frac{dv}{dx}=-v^2$, dividing by $v$ gives $\frac{dv}{dx}=-v$. Inverting,

$$\frac{dx}{dv}=-\frac{1}{v}, x=\int -\frac{1}{v}dv=-\log_e|v|+c.$$

With $v=10$ at $x=0$: $c=\log_e 10$, so $x=\log_e\frac{10}{v}$ and $v=10e^{-x}$.

**Q14.** With $a=v\frac{dv}{dx}=-\frac{1}{3}v$, dividing by $v$ gives $\frac{dv}{dx}=-\frac{1}{3}$. Inverting, $\frac{dx}{dv}=-3$, so

$$x=\int -3\,dv=-3v+c.$$

With $v=15$ at $x=0$: $0=-45+c$, so $c=45$ and $x=45-3v$; rearranging, $v=15-\frac{x}{3}$. It comes to rest when $v=0$: $x=45$ m.

**Q15.** With $a=v\frac{dv}{dx}=-\frac{1}{4}v^2$, dividing by $v$ gives $\frac{dv}{dx}=-\frac{1}{4}v$. Inverting,

$$\frac{dx}{dv}=-\frac{4}{v}, x=\int -\frac{4}{v}dv=-4\log_e|v|+c.$$

With $v=8$ at $x=0$: $c=4\log_e 8$, so $x=4\log_e\frac{8}{v}$ and $v=8e^{-x/4}$. The speed reaches 2 m/s when $2=8e^{-x/4}$, that is $e^{-x/4}=\frac{1}{4}$, so $\frac{x}{4}=\log_e 4$ and $x=4\log_e 4=8\log_e 2$ m.

**Q16.** Use the time form and invert:

$$\frac{dt}{dv}=\frac{1}{10-5v}, t=\int \frac{1}{10-5v}dv=-\frac{1}{5}\log_e|10-5v|+c.$$

The body starts from rest and speeds up towards 2 m/s, so $v<2$ and $10-5v>0$. With $v=0$ at $t=0$: $c=\frac{1}{5}\log_e 10$, so $t=\frac{1}{5}\log_e\frac{10}{10-5v}$. Inverting, $10-5v=10e^{-5t}$, so $v=2\left(1-e^{-5t}\right)$. As $t\to\infty$, $v\to 2$; equivalently $a=10-5v=0$ gives the terminal velocity $v=2$ m/s.

**Q17.** Use the time form and invert:

$$\frac{dt}{dv} = -\frac{1}{10+v}, t = \int -\frac{1}{10+v}\,dv = -\log_e|10+v| + c.$$

On the way up $v > 0$, so $10 + v > 0$. With $v = 20$ at $t = 0$: $c = \log_e 30$, so $t = \log_e \frac{30}{10+v}$. Inverting, $10 + v = 30e^{-t}$, so $v = 30e^{-t} - 10$. The highest point is reached when $v = 0$: $30e^{-t} = 10$, so $e^{-t} = \frac{1}{3}$ and $t = \log_e 3$ s.

**Q18.** (a) With $a = v\frac{dv}{dx} = -\frac{1}{2}v^2$, dividing by $v$ gives $\frac{dv}{dx} = -\frac{1}{2}v$. Inverting,

$$\frac{dx}{dv} = -\frac{2}{v}, x = \int -\frac{2}{v}\,dv = -2\log_e|v| + c.$$

With $v = 6$ at $x = 0$: $c = 2\log_e 6$, so $x = 2\log_e \frac{6}{v}$ and $v = 6e^{-x/2}$. (b) The speed reaches 2 m/s when $2 = 6e^{-x/2}$, so $e^{-x/2} = \frac{1}{3}$, $\frac{x}{2} = \log_e 3$ and $x = 2\log_e 3$ m. (c) Using the time form, $\frac{dt}{dv} = -\frac{2}{v^2} = -2v^{-2}$, so

$$t = \int -2v^{-2}\,dv = \frac{2}{v} + c.$$

With $v = 6$ at $t = 0$: $0 = \frac{1}{3} + c$, so $c = -\frac{1}{3}$ and $t = \frac{2}{v} - \frac{1}{3}$, giving $\frac{2}{v} = \frac{3t+1}{3}$ and $v = \frac{6}{3t+1}$.

### Theory

For a particle moving in a straight line, the **acceleration** is the rate of change of velocity with respect to time,

$$a=\frac{dv}{dt}.$$

This is the natural form when the velocity or the acceleration is known as a function of **time** $t$: antidifferentiating with respect to $t$ recovers $v$, and then $x$. Frequently, however, the acceleration is known instead as a function of **position** $x$ — for example a particle drawn towards a fixed point with a pull that depends on how far away it is. For such problems two further expressions for $a$ are needed. This section establishes the identity

$$a=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(1/2\,v^2\right)$$

and shows how to choose the most convenient form for the information given.

**The chain-rule form $v\frac{dv}{dx}$.**

The velocity $v$ is a function of position $x$, and $x$ is itself a function of time $t$, so the chain rule applies to $\frac{dv}{dt}$:

$$a=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=\frac{dv}{dx}\cdot v=v\frac{dv}{dx},$$

where the last step uses $\frac{dx}{dt}=v$. This form removes $t$ from the problem and relates acceleration directly to how the velocity changes with position.

**The form $\frac{d}{dx}\left(1/2\,v^2\right)$.**

Differentiating $1/2\,v^2$ with respect to $x$ — again by the chain rule, since $v$ depends on $x$ —

$$\frac{d}{dx}\left(1/2\,v^2\right)=1/2\cdot 2v\cdot\frac{dv}{dx}=v\frac{dv}{dx}=a.$$

So $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(1/2\,v^2\right)$ are two ways of writing the same thing, and all three expressions for $a$ are equal.

**Choosing the right form.**

The three forms let you match the method to the way the acceleration is presented.

| Acceleration given as | Convenient form | What it gives |
|---|---|---|
| a function of $t$ | $a=\frac{dv}{dt}$ | $v$ as a function of $t$ (antidifferentiate in $t$) |
| a function of $x$ | $a=v\frac{dv}{dx}$ or $a=\frac{d}{dx}\left(1/2\,v^2\right)$ | $v^2$ as a function of $x$ |
| a function of $v$ | either $a=\frac{dv}{dt}$ or $a=v\frac{dv}{dx}$ | a differential equation |

When the acceleration is a function of $v$, either form produces a differential equation that is solved by separation of variables, as treated in an earlier section; that case is not pursued again here.

**Using $\frac{d}{dx}\left(1/2v^2\right)$ when $a=f(x)$.**

If the acceleration is a known function of position, $a=f(x)$, then

$$\frac{d}{dx}\left(1/2v^2\right)=f(x),$$

so $1/2v^2$ is an antiderivative of $f(x)$:

$$1/2v^2=\int f(x)\,dx+c.$$

Doubling gives $v^2$ as a function of $x$, and the constant $c$ is found from a known speed at a known position. This is the quickest route to $v^2$ in terms of $x$, because it avoids handling the product $v\frac{dv}{dx}$; the middle form gives the same result through the first step $v\,dv=f(x)\,dx$.

**Constant acceleration as a special case.**

When $a$ is constant the position form reproduces the velocity–position formula derived earlier in this chapter. With $a$ constant, $1/2v^2=ax+c$; if $v=u$ at $x=0$ then $c=1/2u^2$, and doubling gives

$$v^2=u^2+2ax.$$

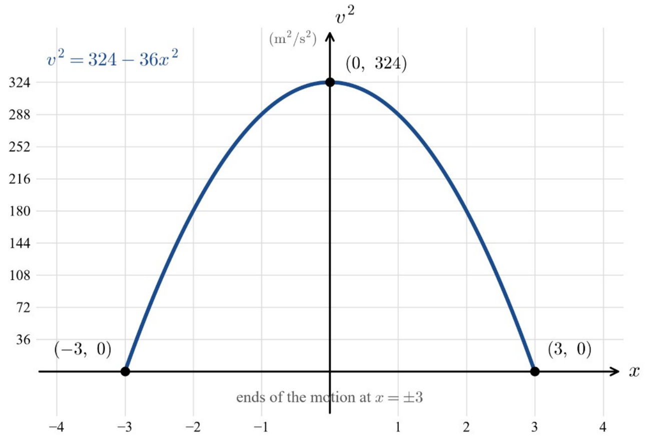
So the constant-acceleration relation is just the special case of $v^2$ as a function of $x$ in which the acceleration does not vary.

**Motion where acceleration depends on position.**

Writing $v^2$ as a function of $x$ tells you the speed at every position and locates where the particle is momentarily at rest, namely where $v=0$. Two standard patterns occur often.

- A particle drawn towards the origin with acceleration proportional to its displacement, $a=-n^2x$, produces $v^2=n^2\left(A^2-x^2\right)$, which is zero at $x=\pm A$. The particle oscillates between $x=-A$ and $x=A$; the number $A$ is the **amplitude** and the greatest speed, $nA$, occurs at the centre $x=0$.
- A particle drawn in by an inverse-square law $a=-\frac{k}{x^2}$ produces a $v^2$ that contains a $\frac{1}{x}$ term. The speed changes rapidly near the centre and levels off far away.

The graph of $v^2$ against $x$ makes the motion easy to read. For the oscillation $a=-36x$ with speed 18 m/s at the centre $x=0$, the relation $v^2=324-36x^2$ is a parabola: the value of $v^2$ is greatest at the centre and falls to zero at $x=\pm 3$, the two ends of the motion. Only the arc between those two zeros is drawn, because $v^2$ cannot be negative and the particle never reaches a position beyond a rest point.



## Worked examples

### *Example 1 — scaffolded*

A particle moves in a straight line with acceleration $a=4x+2$ (in $\mathrm{m/s^2}$, with $x$ in metres). When $x=0$ its velocity is 1 m/s. Find $v^2$ as a function of $x$, and hence the speed when $x=3$.

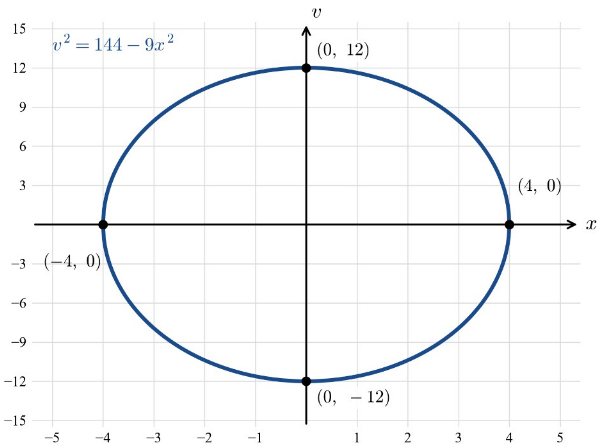
| Working | Comment |
|---|---|
| The acceleration is a function of $x$, so use $a=\frac{d}{dx}\left(1/2\,v^2\right)$. | Match the form to the information given. |
| $\frac{d}{dx}\left(1/2\,v^2\right)=4x+2$. | Set $\frac{d}{dx}\left(1/2\,v^2\right)$ equal to $a$. |
| $1/2\,v^2=\int(4x+2)\,dx=2x^2+2x+c$. | Antidifferentiate with respect to $x$; keep the $+c$. |
| $v^2=4x^2+4x+2c$. | Double both sides to isolate $v^2$. |
| $v=1$ at $x=0$: $1=0+0+2c$, so $2c=1$. | Use the known speed at a known position to find the constant. |
| $v^2=4x^2+4x+1=(2x+1)^2$. | The right-hand side is a perfect square here. |
| At $x=3$: $v^2=(2\times3+1)^2=7^2=49$, so $v=7$ m/s. | Substitute $x=3$ and take the positive square root for the speed. |

### *Example 2 — scaffolded*

A particle moves so that its acceleration is $a=-9x$ (in $\mathrm{m/s^2}$). As it passes through the centre $x=0$ its speed is 12 m/s. Find $v^2$ as a function of $x$, and the two positions at which the particle is momentarily at rest.

| Working | Comment |
|---|---|
| The acceleration is a function of $x$, so use $\frac{d}{dx}\left(1/2\,v^2\right)=-9x$. | Position-dependent acceleration: use the position form. |
| $1/2\,v^2=\int -9x\,dx=-9/2\,x^2+c$. | Antidifferentiate with respect to $x$. |
| $v^2=-9x^2+2c$. | Double to isolate $v^2$. |
| $v=12$ at $x=0$: $144=0+2c$, so $2c=144$. | Apply the given speed at the centre. |
| $v^2=144-9x^2=9\left(16-x^2\right)$. | This has the form $n^2\left(A^2-x^2\right)$ with $n=3$, $A=4$. |
| $v=0$ when $9x^2=144$, i.e. $x^2=16$, so $x=\pm4$. | Rest positions are where $v=0$. |

The particle oscillates between $x=-4$ and $x=4$, with greatest speed 12 m/s at the centre. The closed curve of $v$ against $x$ is shown below.




***Example 3 — unscaffolded***

A particle moves along the positive $x$-axis with acceleration $a = -\frac{8}{x^2}$ (in m/s$^2$, for $x > 0$). When $x = 1$ its velocity is 3 m/s in the positive direction. Find $v^2$ as a function of $x$, and the position at which the particle first comes to rest.

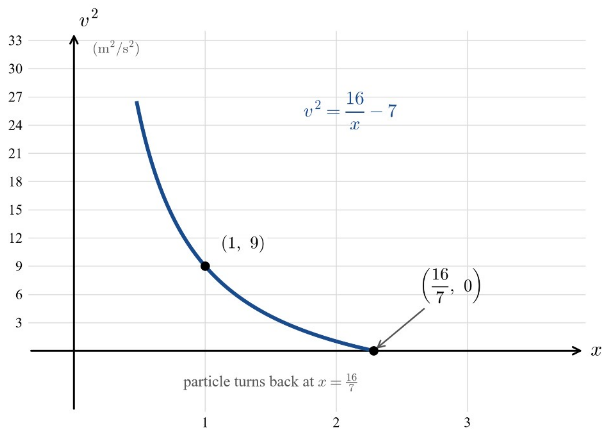
The acceleration is a function of position, so use the position form $\frac{d}{dx}\left(1/2\, v^2\right) = -\frac{8}{x^2}$. Antidifferentiating with respect to $x$,

$$1/2\, v^2 = \int -8x^{-2}\,dx = 8x^{-1} + c = \frac{8}{x} + c.$$

Doubling gives $v^2 = \frac{16}{x} + 2c$. The condition $v = 3$ at $x = 1$ gives $9 = 16 + 2c$, so $2c = -7$ and

$$v^2 = \frac{16}{x} - 7.$$

The particle is momentarily at rest where $v = 0$, that is where $\frac{16}{x} = 7$, giving $x = \frac{16}{7}$. Because the acceleration is negative throughout, the particle slows as it moves out, so this is the greatest distance reached before it turns back. The graph of $v^2$ against $x$ falls steeply and meets $v^2 = 0$ at $x = \frac{16}{7}$.




$\therefore v^2 = \frac{16}{x} - 7$, and the particle first comes to rest at $x = \frac{16}{7}$ m.

### *Example 4 — unscaffolded*

A particle moves in a straight line with acceleration $a=8-4x$ (in $\mathrm{m/s^2}$). It starts **from rest** at $x=5$. Find $v^2$ as a function of $x$, the other position at which the particle is at rest, and its maximum speed.

Since the acceleration is given in terms of $x$, use $\frac{d}{dx}\left(1/2\,v^2\right)=8-4x$. Antidifferentiating,

$$1/2\,v^2=\int(8-4x)\,dx=8x-2x^2+c,$$

so $v^2=16x-4x^2+2c$. The particle starts from rest, so $v=0$ at $x=5$: $0=80-100+2c$, giving $2c=20$. Hence

$$v^2=-4x^2+16x+20=-4\left(x^2-4x-5\right)=4(5-x)(x+1).$$

This is zero at $x=5$ (the start) and at $x=-1$, so the particle oscillates over $-1\le x\le 5$. The greatest value of $v^2$ occurs at the centre of this interval, $x=2$, where $v^2=4(3)(3)=36$.

$\therefore v^2=-4x^2+16x+20$; the other rest position is $x=-1$, and the maximum speed is $\sqrt{36}=6$ m/s at $x=2$.

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## Practice questions

### *Scaffolded*

**Q1.** Establish the identity for acceleration. (a) Starting from $a=\frac{dv}{dt}$ and using the chain rule with $x$ as the intermediate variable, show that $a=v\frac{dv}{dx}$. (b) By differentiating $1/2\,v^2$ with respect to $x$, show that $\frac{d}{dx}\left(1/2\,v^2\right)=v\frac{dv}{dx}$. (c) Hence write down the three equivalent forms for $a$. (d) State which form is most convenient when $a$ is given as a function of $x$.

**Q2.** A particle has acceleration $a=6x+2$ (in $\mathrm{m/s^2}$), and $v=2$ m/s when $x=0$. (a) State which form of $a$ to use, and why. (b) Antidifferentiate to find $1/2\,v^2$ in terms of $x$. (c) Use the given values to find the constant, and write $v^2$ in terms of $x$. (d) Find $v$ when $x=2$.

**Q3.** A particle has acceleration $a=-16x$ (in $\mathrm{m/s^2}$), and its speed at the centre $x=0$ is $20$ m/s. (a) Write $\frac{d}{dx}\left(1/2\,v^2\right)=-16x$ and antidifferentiate. (b) Find the constant from the given speed, and write $v^2$ in terms of $x$. (c) Find the amplitude of the motion (where $v=0$). (d) Find the speed when $x=3$.

**Q4.** A particle moves along the positive $x$-axis with acceleration $a=-\frac{6}{x^2}$ (in $\mathrm{m/s^2}$, $x>0$), and $v=4$ m/s when $x=1$. (a) Write $\frac{d}{dx}\left(1/2\,v^2\right)=-\frac{6}{x^2}$ and antidifferentiate. (b) Find the constant, and write $v^2$ in terms of $x$. (c) Describe what happens to $v^2$ as $x$ becomes very large. (d) State the limiting speed the particle approaches.

**Q5.** A particle has acceleration $a=6t+4$ (in $\mathrm{m/s^2}$, with $t$ in seconds), and $v=3$ m/s when $t=0$. (a) State which form of $a$ is appropriate when $a$ is a function of $t$. (b) Antidifferentiate with respect to $t$ to find $v$ in terms of $t$. (c) Use the given values to find the constant. (d) Find $v$ when $t=2$.

**Q6.** A particle has acceleration $a=4-x$ (in $\mathrm{m/s^2}$) and starts from rest at $x=9$. (a) Write $\frac{d}{dx}\left(1/2\,v^2\right)=4-x$ and antidifferentiate. (b) Find the constant from the rest condition, and write $v^2$ in terms of $x$. (c) Find the two positions at which the particle is at rest, and its maximum speed. (d) Find the speed when $x=1$.

### *Unscaffolded*

**Q7.** State the three equivalent forms for the acceleration $a$ of a particle moving in a straight line, and explain which form you would choose if the acceleration were given (i) as a function of $t$, and (ii) as a function of $x$.

**Q8.** A particle has acceleration $a=3x^2$ (in $\mathrm{m/s^2}$), and $v=2$ m/s when $x=0$. Find $v^2$ as a function of $x$, and the speed when $x=2$.

**Q9.** A particle has acceleration $a=-25x$ (in $\mathrm{m/s^2}$), and speed 30 m/s at the centre $x=0$. Find $v^2$ as a function of $x$, the amplitude of the motion, and the speed when $x=4$.

**Q10.** A particle moves along the positive $x$-axis with acceleration $a=-\dfrac{50}{x^2}$ (in $\mathrm{m/s^2}$, $x>0$), and $v=6$ m/s when $x=2$. Find $v^2$ as a function of $x$, and the position at which the particle comes to rest.

**Q11.** A particle has acceleration $a=6t-12$ (in $\mathrm{m/s^2}$, $t$ in seconds), and $v=9$ m/s when $t=0$. Find $v$ as a function of $t$, and the times at which the particle is momentarily at rest.

**Q12.** The acceleration of a particle is given as a function of its velocity, $a=f(v)$. Explain, naming the appropriate form(s) for $a$, how the velocity could be found either as a function of $t$ or as a function of $x$, and note where the resulting differential equations were solved.

**Q13.** A particle has acceleration $a=18-9x$ (in $\mathrm{m/s^2}$) and starts from rest at $x=6$. Find $v^2$ as a function of $x$, the interval over which the particle oscillates, and its maximum speed.

**Q14.** A particle moves along the positive $x$-axis with acceleration $a=-\dfrac{8}{x^2}$ (in $\mathrm{m/s^2}$, $x>0$). It has speed 2 m/s when $x=4$ and is moving towards the origin. Find $v^2$ as a function of $x$, and its speed when $x=1$.

**Q15.** A particle has acceleration $a=9x+3$ (in $\mathrm{m/s^2}$), and $v=1$ m/s when $x=0$. Find $v^2$ as a function of $x$, show that $v=3x+1$ for $x\geq 0$, and find $v$ when $x=3$.

**Q16.** A particle has acceleration $a=-4x$ (in $\mathrm{m/s^2}$). When $x=3$ its speed is 8 m/s. Find $v^2$ as a function of $x$, the amplitude of the motion, and the maximum speed.

**Q17.** A particle has acceleration $a=2x-6$ (in $\mathrm{m/s^2}$). When $x=0$ its velocity is 4 m/s in the positive direction. Find $v^2$ as a function of $x$, and the position at which the particle first comes to rest.

**Q18.** A particle moves in a straight line with acceleration $a=-n^2x$. As it passes through $x=0$ its speed is 15 m/s, and the amplitude of the motion is 5 m. (a) Find $v^2$ in terms of $x$ and $n$. (b) Use the amplitude to find $n$. (c) State the maximum speed and where it occurs. (d) Find the speed when $x=3$.

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### Answers

**Q1.** $a=v\frac{dv}{dx}=\frac{d}{dx}\left(1/2v^2\right)$; the $\frac{d}{dx}\left(1/2v^2\right)$ form is most convenient when $a=f(x)$. **Q2.** $v^2=6x^2+4x+4$; $v=6$ m/s when $x=2$. **Q3.** $v^2=400-16x^2$; amplitude 5 m; $v=16$ m/s when $x=3$. **Q4.** $v^2=\frac{12}{x}+4$; as $x$ grows large $v^2\to 4$, so the limiting speed is 2 m/s. **Q5.** $v=3t^2+4t+3$; $v=23$ m/s when $t=2$. **Q6.** $v^2=(9-x)(x+1)$; at rest at $x=-1$ and $x=9$; maximum speed 5 m/s at $x=4$; $v=4$ m/s when $x=1$. **Q7.** $a=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(1/2v^2\right)$; (i) use $\frac{dv}{dt}$; (ii) use $v\frac{dv}{dx}$ or $\frac{d}{dx}\left(1/2v^2\right)$. **Q8.** $v^2=2x^3+4$; $v=2\sqrt{5}$ m/s when $x=2$. **Q9.** $v^2=900-25x^2$; amplitude 6 m; $v=10\sqrt{5}$ m/s when $x=4$. **Q10.** $v^2=\frac{100}{x}-14$; at rest at $x=\frac{50}{7}$ m. **Q11.** $v=3t^2-12t+9$; at rest at $t=1$ s and $t=3$ s. **Q12.** Use $a=\frac{dv}{dt}$ for $v(t)$ or $a=v\frac{dv}{dx}$ for $v(x)$; each is a separable differential equation of the kind solved in an earlier section. **Q13.** $v^2=9(6-x)(x+2)$; oscillates over $-2\le x\le 6$; maximum speed 12 m/s at $x=2$. **Q14.** $v^2=\frac{16}{x}$; $v=4$ m/s when $x=1$. **Q15.** $v^2=(3x+1)^2$, so $v=3x+1$ for $x\ge 0$; $v=10$ m/s when $x=3$. **Q16.** $v^2=100-4x^2$; amplitude 5 m; maximum speed 10 m/s. **Q17.** $v^2=2(x-2)(x-4)$; first at rest at $x=2$ m. **Q18.** (a) $v^2=225-n^2x^2$; (b) $n=3$; (c) maximum speed 15 m/s at $x=0$; (d) $v=12$ m/s.

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### Fully-worked solutions

**Q1.** (a) Since $v$ is a function of $x$ and $x$ is a function of $t$, the chain rule gives

$$a=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dx},$$

using $\frac{dx}{dt}=v$. (b) Differentiating $1/2v^2$ with respect to $x$ (chain rule),

$$\frac{d}{dx}\left(1/2v^2\right)=1/2\cdot 2v\cdot\frac{dv}{dx}=v\frac{dv}{dx}.$$

(c) Combining these, $a=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(1/2v^2\right)$. (d) When $a=f(x)$, the form $\frac{d}{dx}\left(1/2v^2\right)$ is most convenient, because antidifferentiating with respect to $x$ gives $v^2$ directly.

**Q2.** The acceleration is a function of $x$, so use $\frac{d}{dx}\left(1/2v^2\right)=6x+2$. Then

$$1/2v^2=\int(6x+2)\,dx=3x^2+2x+c,$$

so $v^2=6x^2+4x+2c$. With $v=2$ at $x=0$: $4=2c$, so $2c=4$ and $v^2=6x^2+4x+4$. At $x=2$, $v^2=24+8+4=36$, so $v=6$ m/s.

**Q3.** Using $\frac{d}{dx}\left(1/2v^2\right)=-16x$,

$$1/2v^2=\int-16x\,dx=-8x^2+c,$$

so $v^2=-16x^2+2c$. With $v=20$ at $x=0$: $400=2c$, giving $v^2=400-16x^2=16\left(25-x^2\right)$. The particle is at rest where $v=0$, i.e. $x^2=25$, so $x=\pm 5$ and the amplitude is 5 m. At $x=3$, $v^2=16(25-9)=16\times 16=256$, so $v=16$ m/s.

**Q4.** Using $\frac{d}{dx}\left(1/2\,v^2\right)=-\frac{6}{x^2}$,

$$1/2\,v^2=\int -6x^{-2}\,dx=6x^{-1}+c=\frac{6}{x}+c\,,$$

so $v^2=\frac{12}{x}+2c$. With $v=4$ at $x=1$: $16=12+2c$, so $2c=4$ and $v^2=\frac{12}{x}+4$. As $x$ becomes very large, $\frac{12}{x}\to 0$, so $v^2\to 4$; the limiting speed is $\sqrt{4}=2$ m/s.

**Q5.** The acceleration is a function of $t$, so use $a=\frac{dv}{dt}=6t+4$ and antidifferentiate with respect to $t$:

$$v=\int(6t+4)\,dt=3t^2+4t+c\,.$$

With $v=3$ at $t=0$: $c=3$, so $v=3t^2+4t+3$. At $t=2$, $v=12+8+3=23$ m/s.

**Q6.** Using $\frac{d}{dx}\left(1/2\,v^2\right)=4-x$,

$$1/2\,v^2=\int(4-x)\,dx=4x-1/2\,x^2+c\,,$$

so $v^2=8x-x^2+2c$. The particle starts from rest, so $v=0$ at $x=9$: $0=72-81+2c$, giving $2c=9$. Then

$$v^2=-x^2+8x+9=-(x-9)(x+1)=(9-x)(x+1).$$

So $v=0$ at $x=9$ and $x=-1$; the particle oscillates over $-1\le x\le 9$. The greatest $v^2$ is at the centre $x=4$, where $v^2=(5)(5)=25$, so the maximum speed is 5 m/s. At $x=1$, $v^2=(8)(2)=16$, so $v=4$ m/s.

**Q7.** The three equivalent forms are

$$a=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(1/2\,v^2\right).$$

(i) If $a$ is a function of $t$, use $a=\frac{dv}{dt}$ and antidifferentiate with respect to $t$. (ii) If $a$ is a function of $x$, use $a=v\frac{dv}{dx}$ or $a=\frac{d}{dx}\left(1/2\,v^2\right)$, which give $v^2$ as a function of $x$.

**Q8.** Using $\frac{d}{dx}\left(1/2\,v^2\right)=3x^2$,

$$1/2\,v^2=\int 3x^2\,dx=x^3+c\,,$$

so $v^2=2x^3+2c$. With $v=2$ at $x=0$: $4=2c$, so $v^2=2x^3+4$. At $x=2$, $v^2=16+4=20$, so $v=\sqrt{20}=2\sqrt{5}$ m/s.

**Q9.** Using $\frac{d}{dx}\left(1/2\,v^2\right)=-25x$,

$$1/2\,v^2=\int -25x\,dx=-25/2\,x^2+c\,,$$

so $v^2=-25x^2+2c$. With $v=30$ at $x=0$: $900=2c$, giving $v^2=900-25x^2=25\left(36-x^2\right)$. Rest occurs where $x^2=36$, so $x=\pm 6$ and the amplitude is 6 m. At $x=4$, $v^2=25(36-16)=25\times 20=500$, so $v=\sqrt{500}=10\sqrt{5}$ m/s.

**Q10.** Using $\frac{d}{dx}\left(1/2\,v^2\right)=-\frac{50}{x^2}$,

$$1/2\,v^2 = \int -50x^{-2}\,dx = 50x^{-1} + c = \frac{50}{x} + c,$$

so $v^2 = \frac{100}{x} + 2c$. With $v = 6$ at $x = 2$: $36 = 50 + 2c$, so $2c = -14$ and $v^2 = \frac{100}{x} - 14$. The particle is at rest where $\frac{100}{x} = 14$, i.e. $x = \frac{100}{14} = \frac{50}{7}$ m.

**Q11.** The acceleration is a function of $t$, so use $\frac{dv}{dt} = 6t - 12$:

$$v = \int (6t - 12)\,dt = 3t^2 - 12t + c.$$

With $v = 9$ at $t = 0$: $c = 9$, so $v = 3t^2 - 12t + 9 = 3(t-1)(t-3)$. Hence $v = 0$ at $t = 1$ s and $t = 3$ s.

**Q12.** When $a = f(v)$, either derivative form may be used. Writing $\frac{dv}{dt} = f(v)$ gives a differential equation relating $v$ and $t$, whose solution is $v$ as a function of $t$; writing $v\frac{dv}{dx} = f(v)$ gives one relating $v$ and $x$, whose solution is $v$ as a function of $x$. Both are separable differential equations, of the kind solved by separation of variables in an earlier section.

**Q13.** Using $\frac{d}{dx}(1/2\,v^2) = 18 - 9x$,

$$1/2\,v^2 = \int (18 - 9x)\,dx = 18x - 9/2\,x^2 + c,$$

so $v^2 = 36x - 9x^2 + 2c$. The particle starts from rest, so $v = 0$ at $x = 6$: $0 = 216 - 324 + 2c$, giving $2c = 108$. Then

$$v^2 = -9x^2 + 36x + 108 = -9(x^2 - 4x - 12) = 9(6 - x)(x + 2).$$

So $v = 0$ at $x = 6$ and $x = -2$; the particle oscillates over $-2 \le x \le 6$. The greatest $v^2$ is at the centre $x = 2$, where $v^2 = 9(4)(4) = 144$, so the maximum speed is 12 m/s.

**Q14.** Using $\frac{d}{dx}(1/2\,v^2) = -\frac{8}{x^2}$,

$$1/2\,v^2 = \int -8x^{-2}\,dx = 8x^{-1} + c = \frac{8}{x} + c,$$

so $v^2 = \frac{16}{x} + 2c$. With $v = 2$ at $x = 4$: $4 = 4 + 2c$, so $2c = 0$ and $v^2 = \frac{16}{x}$. At $x = 1$, $v^2 = 16$, so $v = 4$ m/s (the speed increases as the particle moves in towards the origin).

**Q15.** Using $\frac{d}{dx}(1/2\,v^2) = 9x + 3$,

$$1/2\,v^2 = \int (9x + 3)\,dx = 9/2\,x^2 + 3x + c,$$

so $v^2 = 9x^2 + 6x + 2c$. With $v = 1$ at $x = 0$: $1 = 2c$, so $v^2 = 9x^2 + 6x + 1 = (3x + 1)^2$. For $x \ge 0$, $3x + 1 > 0$, so $v = 3x + 1$. At $x = 3$, $v = 3 \times 3 + 1 = 10$ m/s.

**Q16.** Using $\frac{d}{dx}(1/2\,v^2) = -4x$,

$$1/2\,v^2 = \int -4x\,dx = -2x^2 + c,$$

so $v^2 = -4x^2 + 2c$. With $v=8$ at $x=3$: $64 = -36 + 2c$, so $2c = 100$ and $v^2 = 100 - 4x^2 = 4(25 - x^2)$. Rest occurs where $x^2 = 25$, so $x = \pm 5$ and the amplitude is 5 m. The maximum $v^2$ is at the centre $x=0$, where $v^2 = 100$, so the maximum speed is 10 m/s.

**Q17.** Using $\frac{d}{dx}(1/2v^2) = 2x - 6$,

$$1/2v^2 = \int (2x-6)\,dx = x^2 - 6x + c,$$

so $v^2 = 2x^2 - 12x + 2c$. With $v=4$ at $x=0$: $16 = 2c$, giving

$$v^2 = 2x^2 - 12x + 16 = 2(x^2 - 6x + 8) = 2(x-2)(x-4).$$

Starting at $x=0$ with positive velocity, the particle moves in the positive direction while $v^2 > 0$. The first place where $v=0$ is $x=2$, so the particle first comes to rest at $x=2$ m. (It cannot enter $2 < x < 4$, where $v^2$ would be negative.)

**Q18.** (a) Using $\frac{d}{dx}(1/2v^2) = -n^2 x$,

$$1/2v^2 = \int -n^2 x\,dx = -n^2/2x^2 + c,$$

so $v^2 = -n^2x^2 + 2c$. With $v=15$ at $x=0$: $225 = 2c$, so $v^2 = 225 - n^2x^2$. (b) The amplitude is where $v=0$: $x^2 = \frac{225}{n^2}$, so $x = \frac{15}{n}$. Setting $\frac{15}{n} = 5$ gives $n=3$, and $v^2 = 225 - 9x^2$. (c) The maximum speed is at the centre $x=0$, where $v^2 = 225$, so it is 15 m/s. (d) At $x=3$, $v^2 = 225 - 9 \times 9 = 225 - 81 = 144$, so $v = 12$ m/s.

## Topic 4 Test A — Technology-active

*One bound reference and one approved technology (CAS calculator or software) are permitted. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.*

**Question 1** (7 marks)

Consider the function $f(x)=\frac{x^2-9}{x^2-4}$.

**a.** Show that $f'(x)=\frac{10x}{(x^2-4)^2}$. 2 marks

**b.** Find the coordinates of the stationary point of the graph of $y=f(x)$ and determine its nature. 3 marks

**c.** State the equations of all asymptotes of the graph of $y=f(x)$. 2 marks

**Question 2** (8 marks)

**a.** Use the substitution $u=\sin x$ to find $\int \frac{\cos x}{1+\sin^2 x}\,dx$. 3 marks

**b.** Using partial fractions, find $\int \frac{4x-1}{(x+5)(x-1)}\,dx$. 3 marks

**c.** Evaluate $\int_0^{3\sqrt{2}} \frac{1}{\sqrt{36-x^2}}\,dx$. 2 marks

**Question 3** (7 marks)

Let $C$ be the curve with equation $y=\frac{6}{x}$, and let $R$ be the region bounded by $C$, the $x$-axis and the lines $x=1$ and $x=3$

.

**a.** Find the exact area of $R$. 2 marks

**b.** Find the exact volume of the solid generated when $R$ is rotated about the $x$-axis. 3 marks

**c.** Find the length of the arc of $C$ from $x=1$ to $x=3$, correct to four decimal places. 2 marks

**Question 4** (8 marks)

A vertical shaft is being bored into rock. The deeper the shaft, the more slowly it advances: its depth $D$ metres, $t$ hours after boring begins, satisfies

$$\frac{dD}{dt}=\frac{k}{1+D},$$

where $k$ is a positive constant. The shaft starts at the rock surface, so $D=0$ when $t=0$.

**a.** Given that the shaft is deepening at a rate of 1 m/h at the instant its depth is 2 m, find the value of $k$. 2 marks

**b.** Using Euler's method with a step size of 0.5 hours, estimate the depth of the shaft after 1 hour. 3 marks

**c.** The exact solution of the differential equation is $D=\sqrt{1+6t}-1$. Find the depth after 1 hour, correct to one decimal place, and hence find the error in the estimate from part **b.** 3 marks

**Question 5** (6 marks)

A particle moves in a straight line. Its acceleration at time $t$ seconds is $a = 6t - 24$, measured in $\mathrm{m/s^2}$. When $t = 0$ the particle is at the origin and is moving with velocity $21$ m/s.

**a.** Find the velocity $v$ of the particle at time $t$. 2 marks

**b.** Find the position $x$ of the particle at time $t$. 2 marks

**c.** Find the distance travelled by the particle during the first $3$ seconds. 2 marks

**Question 6** (4 marks)

A curve is defined implicitly by $x^2 - x\,y + 2\,y^2 = 16$.

**a.** Use implicit differentiation to show that $\frac{dy}{dx} = \frac{y - 2x}{4y - x}$. 2 marks

**b.** A point moves along the curve so that, as it passes through $(2, 3)$, its $x$-coordinate is increasing at $5$ units per second. Find the rate of change of its $y$-coordinate at that instant. 2 marks

*End of Topic 4 Test A*

### *Test A — answers and solutions*

**Question 1.**

**a.** By the quotient rule, with numerator $x^2-9$ and denominator $x^2-4$,

$$f'(x)=\frac{2x\left(x^2-4\right)-\left(x^2-9\right)(2x)}{\left(x^2-4\right)^2}=\frac{2x\left[\left(x^2-4\right)-\left(x^2-9\right)\right]}{\left(x^2-4\right)^2}=\frac{10x}{\left(x^2-4\right)^2}.$$

**b.** Since $\left(x^2-4\right)^2>0$ for all $x\neq\pm2$, $f'(x)=0$ only when $10x=0$, i.e. $x=0$. Then $f(0)=\frac{-9}{-4}=\frac{9}{4}$, so the stationary point is $\left(0,\frac{9}{4}\right)$. The sign of $f'(x)$ is the sign of $x$, so $f'(x)<0$ for $-2<x<0$ and $f'(x)>0$ for $0<x<2$. The gradient changes from negative to positive, so $\left(0,\frac{9}{4}\right)$ is a local minimum.

**c.** The denominator $x^2-4=(x-2)(x+2)$ is zero at $x=\pm2$, and the numerator is not zero there, so the vertical asymptotes are $x=-2$ and $x=2$. As $x\to\pm\infty$,

$$f(x)=\frac{x^2-9}{x^2-4}=\frac{1-\frac{9}{x^2}}{1-\frac{4}{x^2}}\to1,$$

so the horizontal asymptote is $y=1$.

**Question 2.**

**a.** With $u=\sin x$, $\frac{du}{dx}=\cos x$, so $\cos x\,dx=du$ and

$$\int\frac{\cos x}{1+\sin^2 x}\,dx=\int\frac{1}{1+u^2}\,du=\tan^{-1}(u)+c=\tan^{-1}(\sin x)+c.$$

**b.** Write $\frac{4x-1}{(x+5)(x-1)}=\frac{A}{x+5}+\frac{B}{x-1}$, so $4x-1=A(x-1)+B(x+5)$. Setting $x=1$ gives $3=6B$, so $B=\frac{1}{2}$; setting $x=-5$ gives $-21=-6A$, so $A=\frac{7}{2}$. Hence

$$\int\frac{4x-1}{(x+5)(x-1)}\,dx=\int\left(\frac{7/2}{x+5}+\frac{1/2}{x-1}\right)dx=\frac{7}{2}\log_e|x+5|+\frac{1}{2}\log_e|x-1|+c.$$

**c.** Recognising $\frac{d}{dx}\sin^{-1}\left(\frac{x}{6}\right)=\frac{1}{\sqrt{36-x^2}}$,

$$\int_0^{3\sqrt{2}}\frac{1}{\sqrt{36-x^2}}\,dx=\left[\sin^{-1}\left(\frac{x}{6}\right)\right]_0^{3\sqrt{2}}=\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)-0=\frac{\pi}{4}.$$

**Question 3.**

**a.** On $[1,3]$ the curve $y=\frac{6}{x}$ lies above the $x$-axis, so

$$\text{Area}=\int_1^3\frac{6}{x}\,dx=\left[6\log_e x\right]_1^3=6\log_e 3-0=6\log_e 3.$$

**b.** By the disc method, with radius $y=\frac{6}{x}$,

$$V=\pi\int_1^3\left(\frac{6}{x}\right)^2 dx=\pi\int_1^3\frac{36}{x^2}dx=36\pi\left[-\frac{1}{x}\right]_1^3=36\pi\left(-\frac{1}{3}+1\right)=24\pi.$$

**c.** With $y=\frac{6}{x}$, $\frac{dy}{dx}=-\frac{6}{x^2}$, so $1+\left(\frac{dy}{dx}\right)^2=1+\frac{36}{x^4}$. The arc length is

$$L=\int_1^3\sqrt{1+\frac{36}{x^4}}\,dx=4.6089\,(\text{to 4 d.p.}),$$

evaluated with technology.

**Question 4.**

**a.** The shaft deepens at 1 m/h when $D=2$, so $\frac{dD}{dt}=1$ at $D=2$:

$$1=\frac{k}{1+2}=\frac{k}{3}\Rightarrow k=3.$$

**b.** With $\frac{dD}{dt}=\frac{3}{1+D}$, step $h=0.5$ and starting value $D=0$ at $t=0$, Euler's method gives $D_{n+1}=D_n+h\frac{dD}{dt}$:

| $t$ | $D$ | $\frac{dD}{dt}=\frac{3}{1+D}$ | $D+h\frac{dD}{dt}$ |
|---|---|---|---|
| 0 | 0 | 3 | $0+0.5(3)=1.5$ |
| 0.5 | 1.5 | 1.2 | $1.5+0.5(1.2)=2.1$ |

Hence the estimated depth after 1 hour is $D\approx 2.1$ m.

**c.** From $D=\sqrt{1+6t}-1$, at $t=1$,

$$D(1)=\sqrt{7}-1=1.6457\ldots\approx 1.6\,\text{m}.$$

The Euler estimate was 2.1 m, so the error is $2.1-1.6=0.5$ m; Euler's method has overestimated the depth, because the solution curve is concave down and each step follows the tangent.

**Question 5.**

**a.** The velocity is the antiderivative of the acceleration:

$$v=\int(6t-24)\,dt=3t^2-24t+c.$$

Since $v=21$ when $t=0$, $c=21$, so $v=3t^2-24t+21$.

**b.** The position is the antiderivative of the velocity:

$$x=\int\left(3t^2-24t+21\right)dt=t^3-12t^2+21t+d.$$

Since $x=0$ when $t=0$, $d=0$, so $x=t^3-12t^2+21t$.

**c.** Factorising, $v=3t^2-24t+21=3(t-1)(t-7)$, so the particle is momentarily at rest at $t=1$ (the other zero, $t=7$, is outside the interval). The particle therefore reverses direction at $t=1$, and the distance travelled is not the same as the displacement. Evaluating the position,

$$x(0)=0,\ x(1)=1-12+21=10,\ x(3)=27-108+63=-18.$$

The particle moves $10$ m in the positive direction and then $|-18-10|=28$ m in the negative direction, so the distance travelled is $10+28=38$ m.

**Question 6.**

**a.** Differentiating $x^2-xy+2y^2=16$ term by term with respect to $x$, and treating $y$ as a function of $x$ (the middle term needs the product rule),

$$2x-\left(y+x\frac{dy}{dx}\right)+4y\frac{dy}{dx}=0\Rightarrow(4y-x)\frac{dy}{dx}=y-2x,$$

so $\dfrac{dy}{dx}=\dfrac{y-2x}{4y-x}$.

**b.** At $(2,3)$, $\dfrac{dy}{dx}=\dfrac{3-2(2)}{4(3)-2}=\dfrac{-1}{10}=-\dfrac{1}{10}$. By the chain rule, $\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}$. With $\dfrac{dx}{dt}=5$,

$$\frac{dy}{dt}=-\frac{1}{10}\times 5=-\frac{1}{2}.$$

The $y$-coordinate is decreasing at $\dfrac{1}{2}$ unit per second.

## Topic 4 Test B — Technology-free

*Technology-free. No calculator. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given. Where more than one mark is available, appropriate working must be shown.*

**Question 1** (7 marks)

**a.** Differentiate $y = \sin^{-1}(1-x)$ with respect to $x$. 2 marks

**b.** For $f(x) = x^3 - 9x^2 + 24x$, find the coordinates of the point of inflection and state the intervals on which the graph is concave up and concave down. 3 marks

**c.** The ellipse $2x^2 + y^2 = 33$ passes through the point $(2, 5)$. Find the gradient of the tangent to the ellipse at that point. 2 marks

**Question 2** (8 marks)

**a.** Find $\int \frac{1}{\sqrt{25-4x^2}}\,dx$. 2 marks

**b.** Find $\int \sin^2\left(x + \frac{\pi}{4}\right)dx$. 3 marks

**c.** Use integration by parts to evaluate $\int_0^{\pi/2} x\cos 2x\,dx$. 3 marks

**Question 3** (7 marks)

**a.** Find the exact area of the region bounded by the curve $y = \sin 2x$, the $x$-axis, and the lines $x = 0$ and $x = \frac{\pi}{4}$. 2 marks

**b.** The region bounded by the curve $y = \sqrt{\sin x}$ and the $x$-axis, from $x = 0$ to $x = \frac{\pi}{2}$, is rotated about the $x$-axis. Find the exact volume of the solid of revolution. 3 marks

**c.** Let $H(x) = \int_0^x \frac{1}{2+\sin t}\,dt$. Write down $H'(x)$, find the gradient of the graph of $y = H(x)$ at $x = \frac{\pi}{2}$, and explain why $H$ is increasing for all real $x$. 2 marks

**Question 4** (8 marks)

**a.** Solve the differential equation $\frac{dy}{dx} = \frac{2x}{y^2}$ given that $y = 1$ when $x = 0$, expressing $y$ as a function of $x$. 3 marks

**b.** Solve the differential equation $\frac{dy}{dx} = \frac{y}{1+x^2}$ given that $y = 2$ when $x = 0$ and $y > 0$, expressing $y$ as a function of $x$. 3 marks

**c.** A population $N$ of native fish satisfies the logistic equation $\frac{dN}{dt} = 0.35N\left(1 - \frac{N}{1200}\right)$. State the carrying capacity, and find the value of $N$ at which the population is growing most rapidly, justifying your answer by differentiation. 2 marks

**Question 5** (6 marks)

A particle moves in a straight line with acceleration $a = -3x^2$ (in m/s$^2$), where $x$ is its displacement in metres from a fixed origin $O$. When $x = 0$ the particle is moving with velocity 4 m/s in the positive direction.

**a.** Using the result $a = v\dfrac{dv}{dx}$, show that $v^2 = 16 - 2x^3$. 3 marks

**b.** Find the exact speed of the particle when $x = 1$. 1 mark

**c.** Find the maximum displacement of the particle from $O$. 2 marks

**Question 6** (4 marks)

Consider the graph of $y = \dfrac{x+1}{x-2}$.

**a.** State the equations of the vertical and horizontal asymptotes. 2 marks

**b.** Find the coordinates of the axis intercepts, and state whether the function is increasing or decreasing on each branch, justifying your answer. 2 marks

*End of Topic 4 Test B*

### *Test B — answers and solutions*

**Question 1.**

**a.** With $y=\sin^{-1}(1-x)$, apply $\frac{d}{dx}\sin^{-1}(u)=\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$ with $u=1-x$, so $\frac{du}{dx}=-1$:

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-(1-x)^2}}=\frac{-1}{\sqrt{1-(1-2x+x^2)}}=-\frac{1}{\sqrt{2x-x^2}}.$$

**b.** $f'(x)=3x^2-18x+24$ and $f''(x)=6x-18$. Setting $f''(x)=0$ gives $x=3$, and $f(3)=27-81+72=18$, so the point of inflection is $(3,18)$. Since $f''(x)<0$ for $x<3$ and $f''(x)>0$ for $x>3$, the graph is concave down on $x<3$ and concave up on $x>3$ (confirming a genuine change of concavity).

**c.** Differentiating $2x^2+y^2=33$ implicitly, $4x+2y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{2x}{y}$. At $(2,5)$,

$$\frac{dy}{dx}=-\frac{2(2)}{5}=-\frac{4}{5}.$$

**Question 2.**

**a.** Taking the factor 4 out of the surd, $\sqrt{25-4x^2}=2\sqrt{\frac{25}{4}-x^2}$, so with $a=\frac{5}{2}$,

$$\int\frac{1}{\sqrt{25-4x^2}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{5}{2}\right)^2-x^2}}\,dx=\frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right)+c.$$

**b.** Using the double-angle identity $\sin^2\theta=\frac{1-\cos 2\theta}{2}$ with $\theta=x+\frac{\pi}{4}$, and $\cos\left(2x+\frac{\pi}{2}\right)=-\sin 2x$,

$$\int\sin^2\left(x+\frac{\pi}{4}\right)dx=\int\frac{1-\cos\left(2x+\frac{\pi}{2}\right)}{2}\,dx=\int\frac{1+\sin 2x}{2}\,dx=\frac{x}{2}-\frac{\cos 2x}{4}+c.$$

**c.** Let $u=x$ and $\frac{dv}{dx}=\cos 2x$, so $\frac{du}{dx}=1$ and $v=\frac{\sin 2x}{2}$. Then

$$\int_0^{\pi/2}x\cos 2x\,dx=\left[\frac{x\sin 2x}{2}\right]_0^{\pi/2}-\int_0^{\pi/2}\frac{\sin 2x}{2}\,dx=0+\left[\frac{\cos 2x}{4}\right]_0^{\pi/2}=\frac{-1}{4}-\frac{1}{4}=-\frac{1}{2}.$$

The boundary term vanishes because $\sin\pi=0$ at $x=\frac{\pi}{2}$ and $x=0$ at the lower limit.

**Question 3.**

**a.** On $\left[0,\frac{\pi}{4}\right]$ the curve $y=\sin 2x$ lies above the $x$-axis, so

$$\text{Area}=\int_0^{\pi/4}\sin 2x\,dx=\left[-\frac{\cos 2x}{2}\right]_0^{\pi/4}=-\frac{\cos\frac{\pi}{2}}{2}+\frac{\cos 0}{2}=0+\frac{1}{2}=\frac{1}{2}.$$

**b.** By the disc method, with radius $y=\sqrt{\sin x}$ so that $y^2=\sin x$,

$$V=\pi\int_0^{\pi/2}\left(\sqrt{\sin x}\right)^2 dx=\pi\int_0^{\pi/2}\sin x\, dx=\pi\left[-\cos x\right]_0^{\pi/2}=\pi(0+1)=\pi.$$

**c.** By the fundamental theorem of calculus, $H'(x)=\dfrac{1}{2+\sin x}$. At $x=\dfrac{\pi}{2}$, $\sin x=1$, so the gradient is $H'\left(\dfrac{\pi}{2}\right)=\dfrac{1}{3}$. Since $-1\le\sin x\le 1$, the denominator satisfies $2+\sin x\ge 1>0$, so $H'(x)>0$ for all real $x$ and $H$ is increasing everywhere.

**Question 4.**

**a.** Separating the variables, $y^2\,dy=2x\,dx$, so

$$\int y^2\,dy=\int 2x\,dx\Rightarrow\frac{y^3}{3}=x^2+c\Rightarrow y^3=3x^2+C.$$

Using $y=1$ when $x=0$ gives $1=C$, so $y^3=3x^2+1$ and $y=\left(3x^2+1\right)^{1/3}$.

**b.** Separating the variables, $\dfrac{1}{y}\,dy=\dfrac{1}{1+x^2}\,dx$, so

$$\int\frac{1}{y}\,dy=\int\frac{1}{1+x^2}\,dx\Rightarrow\log_e y=\tan^{-1}x+c\,(y>0).$$

Using $y=2$ when $x=0$ gives $\log_e 2=0+c$, so $c=\log_e 2$ and $\log_e y=\tan^{-1}x+\log_e 2$. Hence $y=2e^{\tan^{-1}x}$.

**c.** The logistic equation $\dfrac{dN}{dt}=0.35N\left(1-\dfrac{N}{1200}\right)$ has carrying capacity $N=1200$ (the non-zero value making $\dfrac{dN}{dt}=0$). Writing the growth rate as a function of $N$, $g(N)=0.35N-\dfrac{0.35N^2}{1200}$, so

$$g'(N)=0.35-\frac{0.7N}{1200}=0\Rightarrow N=\frac{0.35\times 1200}{0.7}=600.$$

Since $g''(N)=-\dfrac{0.7}{1200}<0$, this is a maximum, so the population grows most rapidly at $N=600$, half the carrying capacity.

**Question 5.**

**a.** Using $a=v\dfrac{dv}{dx}=-3x^2$, separate the variables:

$$\int v\,dv=\int -3x^2\,dx\Rightarrow\frac{v^2}{2}=-x^3+c\Rightarrow v^2=-2x^3+C.$$

When $x=0$, $v=4$, so $16=C$. Hence $v^2=16-2x^3$, as required.

**b.** When $x=1$, $v^2=16-2(1)^3=14$, so the speed is $|v|=\sqrt{14}$ m/s.

**c.** The maximum displacement occurs when the particle is momentarily at rest, i.e. $v=0$: $16-2x^3=0$, so $x^3=8$ and $x=2$. The maximum displacement from $O$ is 2 m.

**Question 6.**

**a.** The denominator is zero at $x=2$, giving the vertical asymptote $x=2$. As $x\to\pm\infty$, $y=\dfrac{x+1}{x-2}\to 1$, so the horizontal asymptote is $y=1$.

**b.** The $x$-intercept occurs where $x+1=0$, i.e. $(-1,0)$; the $y$-intercept is $y=\frac{0+1}{0-2}=-\frac{1}{2}$, i.e. $\left(0,-\frac{1}{2}\right)$.

Differentiating by the quotient rule,

$$\frac{dy}{dx}=\frac{(x-2)(1)-(x+1)(1)}{(x-2)^2}=\frac{-3}{(x-2)^2}.$$

Since $(x-2)^2>0$ for all $x\neq 2$, $\frac{dy}{dx}<0$ everywhere it is defined, so the function is decreasing on each branch.