Specialist Mathematics Units 3 & 4

Chapter 8 — Differential equations

This work is licensed under the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of this license, visit http://creativecommons.org/licenses/by-nc-nd/4.0/ or send a letter to Creative Commons, PO Box 1866, Mountain View, CA 94042, USA.

Attribution: Classbasics Pty Ltd, vwx.au

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Significant effort has been made to ensure this content is legal. Please send any areas of concern to admin@vwx.au with appropriate document/legal references so errors can be verified and changes be made.

Contents

Theory

A **differential equation** (DE) is an equation that connects an unknown function with one or more of its **derivatives**. Instead of asking "what number x satisfies this equation?", a differential equation asks "what *function* $y=y(x)$ satisfies this relationship between y and its rates of change?" For example,

$$\frac{dy}{dx}=2x,\ \frac{dy}{dx}=3y,\ \frac{d^2y}{dx^2}+9y=0$$

are all differential equations. The first says the gradient of the graph at each point equals $2x$; the second says the rate of change of y is three times y itself; the third links a function to its second derivative. Differential equations arise whenever a quantity is described by *how it changes* — population growth, cooling, motion, radioactive decay — rather than by an explicit formula.

Order of a differential equation.

The **order** of a differential equation is the order of the **highest derivative** that appears in it. Thus $\frac{dy}{dx}=2x$ and $\frac{dy}{dx}=3y$ are **first-order** (the highest derivative is the first derivative), while $\frac{d^2y}{dx^2}+9y=0$ is **second-order**. The order counts *which* derivative is highest, not how many times a derivative is multiplied by itself: the equation $\left(\frac{dy}{dx}\right)^3+y=0$ is still first-order, because no derivative beyond the first occurs. This chapter is concerned mainly with first-order equations.

Solutions of a differential equation.

A **solution** of a differential equation is a function that satisfies it: when the function and its derivatives are substituted, the two sides become identically equal. Because a solution is a function (not a single number), the way to *check* a proposed solution is always the same — differentiate it as many times as the equation requires, substitute into the equation, and confirm that the left-hand side reduces to the right-hand side. This process is called **verification of a solution**, and it does not require us to know how the solution was found.

For instance, to test whether $y=e^{3x}$ solves $\frac{dy}{dx}=3y$, differentiate to get $\frac{dy}{dx}=3e^{3x}$, and compare with $3y=3e^{3x}$. The two agree for every x, so $y=e^{3x}$ is a solution. Verification works equally for solutions containing an unknown constant, and for second-order equations, where the second derivative must be formed as well.

General solutions and the family of solution curves.

The simplest differential equations have the form $\frac{dy}{dx}=f(x)$, where the right-hand side depends on x alone. Such an equation is solved by **antidifferentiation**:

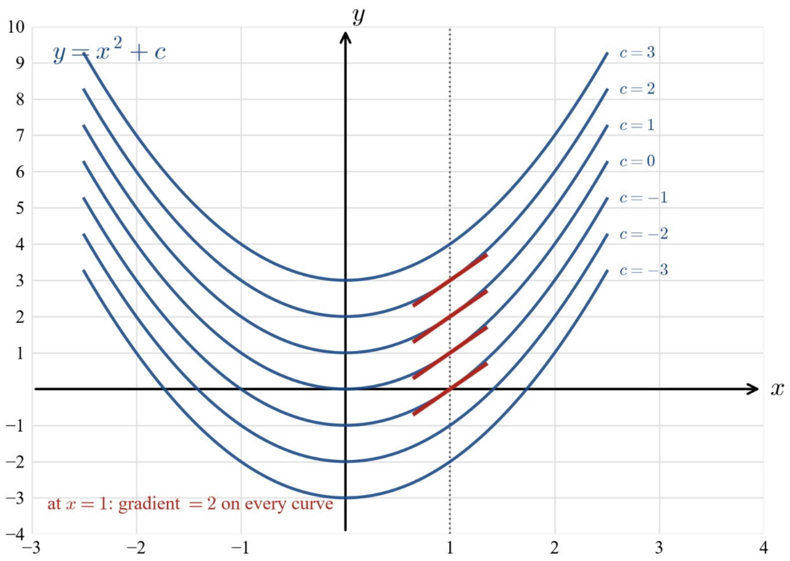
$$\frac{dy}{dx}=f(x)\Rightarrow y=\int f(x)\,dx.$$

Because antidifferentiation introduces an **arbitrary constant**, the answer is not a single function but a whole family. For example, solving $\frac{dy}{dx}=2x$ gives

$$y=\int 2x\,dx=x^2+c,$$

where c may be any real number. This is the **general solution**: it contains one arbitrary constant and so describes infinitely many functions at once. A first-order equation has a general solution with **one** arbitrary constant; a second-order equation needs **two**.

Graphed together, the members of the general solution form a **family of solution curves**. Here the curves $y=x^2+c$ are all vertical translations of the parabola $y=x^2$: changing c slides the curve up or down without changing its shape. At any fixed value of x every curve in the family has the **same gradient** $2x$ — which is exactly what the differential equation demands.

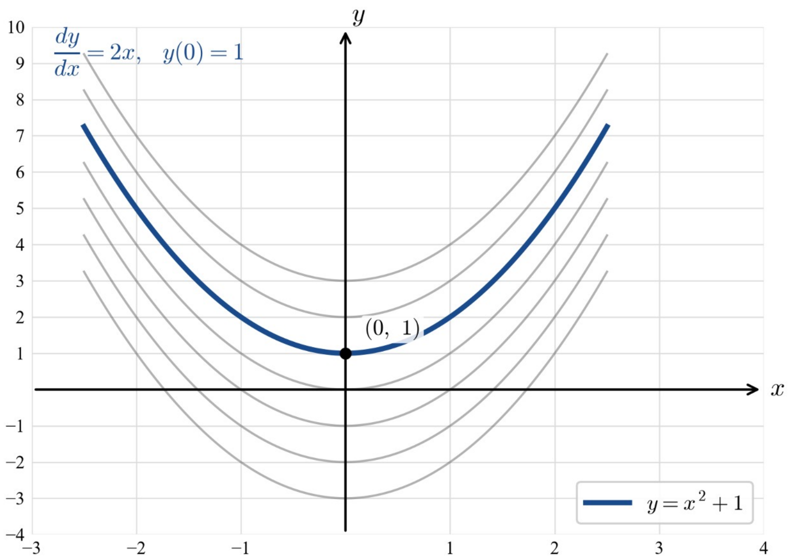


Particular solutions and initial conditions.

To single out **one** curve from the family we need extra information, usually the value of y at a particular value of x. Such a statement — for example "$y=1$ when $x=0$" — is called an **initial condition** (or a **boundary condition**). Substituting it into the general solution fixes the arbitrary constant and produces a **particular solution**. Continuing the example above, if $y=1$ when $x=0$ then

$$1=0^2+c\Rightarrow c=1,$$

so the particular solution is $y=x^2+1$ — the single curve of the family that passes through $(0,1)$, drawn in bold below.



Forming a differential equation from a rate of change.

Many situations are described by stating *how fast* a quantity changes. Translating such a description into symbols produces a differential equation. The key is to read “the rate of change of Q with respect to time” as the derivative $\frac{dQ}{dt}$, and “is proportional to …” as “$= k \times \ldots$” for a **constant of proportionality** k.

Two standard translations recur throughout the chapter:

- **Proportional growth or decay.** “The rate at which a population P increases is proportional to the current population” becomes $\frac{dP}{dt} = kP$, with $k > 0$. If instead a quantity *decreases* at a rate proportional to its size (as in radioactive decay), the sign is negative: $\frac{dm}{dt} = -km$.
- **Newton’s law of cooling.** “The rate of cooling of an object is proportional to the amount by which its temperature T exceeds the surrounding temperature T_s” becomes
$$\frac{dT}{dt} = -k(T - T_s), k > 0.$$
The quantity $T - T_s$ is the **temperature excess**; the minus sign records that a body hotter than its surroundings ($T > T_s$) cools, so T decreases.

Each of these is a first-order differential equation. In this section we form such equations and verify proposed solutions; systematic techniques for solving them are developed in the sections that follow.

Worked examples

Example 1 — scaffolded

Verify that $y = 3e^{4x}$ is a solution of $\frac{dy}{dx} = 4y$, and state the order of the equation.

Working	Comment
The highest derivative present is $\frac{dy}{dx}$, so the equation is first-order.	Order = order of the highest derivative.
Differentiate the proposed solution: $\frac{dy}{dx} = 3 \times 4e^{4x} = 12e^{4x}$.	Differentiate $y = 3e^{4x}$ using the chain rule.
Right-hand side: $4y = 4 \times 3e^{4x} = 12e^{4x}$.	Compute the other side of the equation from the same y.
Both sides equal $12e^{4x}$, so $y = 3e^{4x}$ satisfies the equation and is a solution.	Left side equals right side for all x: verification complete.

Example 2 — scaffolded

Show that $y = \cos 3x$ is a solution of $\frac{d^2y}{dx^2} + 9y = 0$, and state the order of the equation.

Working	Comment
The highest derivative is $\frac{d^2y}{dx^2}$, so the equation is second-order.	Order counts the highest derivative present.
First derivative: $\frac{dy}{dx} = -3\sin 3x$.	Differentiate once, using the chain rule.

Working	Comment
Second derivative: $\frac{d^2y}{dx^2} = -9\cos 3x$.	Differentiate again.
Substitute: $\frac{d^2y}{dx^2} + 9y = -9\cos 3x + 9\cos 3x = 0$.	The left-hand side reduces to 0, matching the right-hand side.
So $y = \cos 3x$ satisfies the equation and is a solution.	Verification complete.

Example 3 — unscaffolded

Find the general solution of $\frac{dy}{dx} = 3x^2 - 2$, and hence the particular solution for which $y = 4$ when $x = 1$.

The right-hand side depends on x alone, so the equation is solved by antidifferentiation:

$$y = \int (3x^2 - 2)\,dx = x^3 - 2x + c.$$

This is the general solution — it carries one arbitrary constant c, as expected for a first-order equation. Applying the initial condition $y = 4$ when $x = 1$,

$$4 = 1^3 - 2(1) + c = -1 + c \Rightarrow c = 5.$$

Substituting $c = 5$ selects the one curve through $(1, 4)$.

∴ the general solution is $y = x^3 - 2x + c$, and the particular solution through $(1, 4)$ is $y = x^3 - 2x + 5$.

Example 4 — unscaffolded

A cup of coffee cools in a room kept at 20°C. The rate at which the coffee cools is proportional to the amount by which its temperature T (measured in °C) exceeds the room temperature. Write a differential equation for T at time t minutes, state its order, and verify that $T = 20 + 75e^{-kt}$ is a solution for any positive constant k.

The rate of cooling is $\frac{dT}{dt}$, and the temperature excess above the room is $T - 20$. “Proportional to the excess”, with cooling meaning that T decreases, gives

$$\frac{dT}{dt} = -k(T - 20), k > 0,$$

a first-order differential equation. To verify the proposed solution, differentiate $T = 20 + 75e^{-kt}$ with respect to t:

$$\frac{dT}{dt} = 75 \times (-k)e^{-kt} = -75ke^{-kt}.$$

Substituting the same T into the right-hand side,

$$-k(T - 20) = -k\left((20 + 75e^{-kt}) - 20\right) = -k \times 75e^{-kt} = -75ke^{-kt}.$$

Both sides agree, so $T = 20 + 75e^{-kt}$ is a solution. At $t = 0$ it gives $T = 95$, a plausible starting temperature of 95°C, and as $t \to \infty$ the temperature falls towards 20°C.

∴ $\frac{dT}{dt} = -k(T - 20)$ is first-order, and $T = 20 + 75e^{-kt}$ satisfies it.

Practice questions

Scaffolded

Q1. Verify that $y=5e^{3x}$ is a solution of $\frac{dy}{dx}=3y$, and state the order of the equation. (a) Find $\frac{dy}{dx}$ by differentiating y. (b) Compute $3y$. (c) State whether y is a solution, and give the order.

Q2. Consider the differential equation $\frac{d^2y}{dx^2}-4y=0$ and the function $y=e^{2x}$. (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. (b) Substitute into $\frac{d^2y}{dx^2}-4y$ and simplify. (c) State whether y is a solution, and give the order of the equation.

Q3. Find the general solution of $\frac{dy}{dx}=4x+1$. (a) Antidifferentiate the right-hand side. (b) Remember to include the arbitrary constant. (c) Write the general solution.

Q4. Find the particular solution of $\frac{dy}{dx}=3x^2$ for which $y=2$ when $x=1$. (a) Write the general solution (include $+c$). (b) Substitute $x=1$, $y=2$ to find c. (c) State the particular solution.

Q5. A colony of bacteria grows so that its rate of increase is proportional to the number P present. (a) Write the derivative that represents the rate of increase of P with respect to time t. (b) Write "proportional to the number present" as an expression, using $k>0$. (c) State the differential equation and its order.

Q6. The functions $y=x^2+c$ form a family, one for each constant c. (a) Differentiate $y=x^2+c$. (b) Explain why every member of the family satisfies $\frac{dy}{dx}=2x$. (c) Find the member of the family that passes through $(2,7)$.

Unscaffolded

Q7. State the order of each differential equation. (a) $\frac{dy}{dx}=x^2-y$ (b) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=e^x$ (c) $\left(\frac{dy}{dx}\right)^3+y=0$

Q8. Verify that $y=2e^{-x}+3$ is a solution of $\frac{dy}{dx}=3-y$.

Q9. Verify that $y=\sin 2x$ is a solution of $\frac{d^2y}{dx^2}+4y=0$.

Q10. Verify that $y=x^3$ is a solution of $x\frac{dy}{dx}=3y$.

Q11. Find the general solution of $\frac{dy}{dx}=6x^2-4x+1$.

Q12. Find the particular solution of $\frac{dy}{dx}=e^{2x}$ for which $y=0$ when $x=0$, giving an exact answer.

Q13. Find the particular solution of $\frac{dy}{dx}=\frac{4}{x}$ (for $x>0$) for which $y=1$ when $x=1$.

Q14. The mass m of a radioactive sample decays at a rate proportional to the mass present. Write a differential equation for m at time t, using $k>0$, and state its order.

Q15. A metal bar at temperature θ cools in surroundings kept at 18°C, the rate of cooling being proportional to the temperature excess above the surroundings. Write a differential equation for θ at time t, using $k>0$.

Q16. Verify that $y = A e^{x} + B e^{-x}$ is a solution of $\frac{d^2 y}{d x^2} - y = 0$ for all constants A and B.

Q17. Verify that $y = A e^{-2x}$ is a solution of $\frac{d y}{d x} + 2 y = 0$ for any constant A, and find the value of A for which $y = 6$ when $x = 0$.

Q18. A tank drains so that the rate of change of the volume V of water is proportional to the square root of the volume. Write a differential equation for V at time t, using $k > 0$, and state its order.

Answers

Q1. $\frac{dy}{dx}=15e^{3x}=3y$, so y is a solution; first-order. **Q2.** $\frac{d^2y}{dx^2}-4y=4e^{2x}-4e^{2x}=0$, so y is a solution; second-order. **Q3.** $y=2x^2+x+c$. **Q4.** $y=x^3+1$. **Q5.** $\frac{dP}{dt}=kP$, $k>0$; first-order. **Q6.** $\frac{dy}{dx}=2x$ for every c; through $(2,7)$, $c=3$, so $y=x^2+3$. **Q7.** (a) first-order; (b) second-order; (c) first-order. **Q8.** $\frac{dy}{dx}=-2e^{-x}$ and $3-y=-2e^{-x}$, so y is a solution. **Q9.** $\frac{d^2y}{dx^2}+4y=-4\sin 2x+4\sin 2x=0$, so y is a solution. **Q10.** $x\frac{dy}{dx}=x(3x^2)=3x^3=3y$, so y is a solution. **Q11.** $y=2x^3-2x^2+x+c$. **Q12.** $y=\frac{1}{2}(e^{2x}-1)$. **Q13.** $y=4\log_e x+1$. **Q14.** $\frac{dm}{dt}=-km$, $k>0$; first-order. **Q15.** $\frac{d\theta}{dt}=-k(\theta-18)$, $k>0$. **Q16.** $\frac{d^2y}{dx^2}=Ae^x+Be^{-x}=y$, so $\frac{d^2y}{dx^2}-y=0$; y is a solution. **Q17.** $\frac{dy}{dx}+2y=-2Ae^{-2x}+2Ae^{-2x}=0$; $A=6$, so $y=6e^{-2x}$. **Q18.** $\frac{dV}{dt}=-k\sqrt{V}$, $k>0$; first-order.

Fully-worked solutions

Q1. Differentiating, $\frac{dy}{dx}=5\times 3e^{3x}=15e^{3x}$. The right-hand side is $3y=3\times 5e^{3x}=15e^{3x}$. Both sides equal $15e^{3x}$, so $y=5e^{3x}$ satisfies the equation and is a solution. The highest derivative is the first, so the equation is first-order.

Q2. From $y=e^{2x}$, $\frac{dy}{dx}=2e^{2x}$ and $\frac{d^2y}{dx^2}=4e^{2x}$. Then

$$\frac{d^2y}{dx^2}-4y=4e^{2x}-4e^{2x}=0,$$

which matches the right-hand side, so $y=e^{2x}$ is a solution. The highest derivative is the second, so the equation is second-order.

Q3. The right-hand side depends on x alone, so antidifferentiate:

$$y=\int(4x+1)\,dx=2x^2+x+c.$$

The arbitrary constant c makes this the general solution.

Q4. Antidifferentiating gives the general solution

$$y=\int 3x^2\,dx=x^3+c.$$

Substituting $x=1$, $y=2$: $2=1^3+c$, so $c=1$. The particular solution is $y=x^3+1$.

Q5. The rate of increase of P is $\frac{dP}{dt}$. “Proportional to the number present” means it equals kP for a positive constant k. Hence $\frac{dP}{dt}=kP$, a first-order differential equation.

Q6. Differentiating $y=x^2+c$ gives $\frac{dy}{dx}=2x$, since the constant c has derivative 0. The result does not involve c, so every member of the family satisfies $\frac{dy}{dx}=2x$. For the member through $(2,7)$, substitute: $7=2^2+c=4+c$, so $c=3$ and $y=x^2+3$.

Q7. The order is the order of the highest derivative present. (a) The highest is $\frac{dy}{dx}$, so the equation is first-order. (b) The highest is $\frac{d^2y}{dx^2}$, so it is second-order. (c) Only the first derivative appears (here cubed, but still no derivative beyond the first), so it is first-order.

Q8. From $y=2e^{-x}+3$, $\frac{dy}{dx}=-2e^{-x}$. The right-hand side is $3-y=3-(2e^{-x}+3)=-2e^{-x}$. Both sides equal $-2e^{-x}$, so $y=2e^{-x}+3$ is a solution.

Q9. From $y=\sin 2x$, $\frac{dy}{dx}=2\cos 2x$ and $\frac{d^2y}{dx^2}=-4\sin 2x$. Then

$$\frac{d^2y}{dx^2}+4y=-4\sin 2x+4\sin 2x=0,$$

so $y=\sin 2x$ is a solution.

Q10. From $y=x^3$, $\frac{dy}{dx}=3x^2$. Then $x\frac{dy}{dx}=x\times 3x^2=3x^3$, while $3y=3x^3$. The two sides are equal, so $y=x^3$ is a solution.

Q11. Antidifferentiate term by term:

$$y=\int(6x^2-4x+1)\,dx=2x^3-2x^2+x+c.$$

Q12. The general solution is

$$y=\int e^{2x}\,dx=\frac{1}{2}e^{2x}+c.$$

Substituting $x=0$, $y=0$: $0=\frac{1}{2}e^0+c=\frac{1}{2}+c$, so $c=-\frac{1}{2}$. Hence

$$y=\frac{1}{2}e^{2x}-\frac{1}{2}=\frac{1}{2}(e^{2x}-1).$$

Q13. For $x>0$ the general solution is

$$y=\int\frac{4}{x}\,dx=4\log_e x+c.$$

Substituting $x=1$, $y=1$: since $\log_e 1=0$, we get $1=0+c$, so $c=1$ and $y=4\log_e x+1$.

Q14. The rate of change of the mass is $\frac{dm}{dt}$. Decay at a rate proportional to the mass present, with the mass decreasing, gives $\frac{dm}{dt}=-km$ for a positive constant k. This is a first-order differential equation.

Q15. The rate of change of temperature is $\frac{d\theta}{dt}$, and the temperature excess above the surroundings is $\theta-18$. Cooling at a rate proportional to this excess gives $\frac{d\theta}{dt}=-k(\theta-18)$, with $k>0$ (the bar cools while $\theta>18$).

Q16. From $y=Ae^x+Be^{-x}$, $\frac{dy}{dx}=Ae^x-Be^{-x}$ and $\frac{d^2y}{dx^2}=Ae^x+Be^{-x}$. The second derivative equals y, so

$$\frac{d^2 y}{dx^2} - y = \left(Ae^x + Be^{-x}\right) - \left(Ae^x + Be^{-x}\right) = 0$$

for all A and B. Hence $y = Ae^x + Be^{-x}$ is a solution. (Being second-order, the equation has a general solution with two arbitrary constants.)

Q17. From $y = Ae^{-2x}$, $\frac{dy}{dx} = -2Ae^{-2x}$. Then

$$\frac{dy}{dx} + 2y = -2Ae^{-2x} + 2Ae^{-2x} = 0$$

for any A, so $y = Ae^{-2x}$ is a solution. Applying $y = 6$ when $x = 0$: $6 = Ae^0 = A$, so $A = 6$ and $y = 6e^{-2x}$.

Q18. The rate of change of volume is $\frac{dV}{dt}$, and it is proportional to $\sqrt{V}$. Since the tank drains (the volume decreases), the sign is negative, giving $\frac{dV}{dt} = -k\sqrt{V}$ with $k > 0$. This is a first-order differential equation.

Theory

A **differential equation** (DE) is an equation that involves a derivative of an unknown function. The simplest type — and the one studied here — has the form

$$\frac{dy}{dx}=f(x),$$

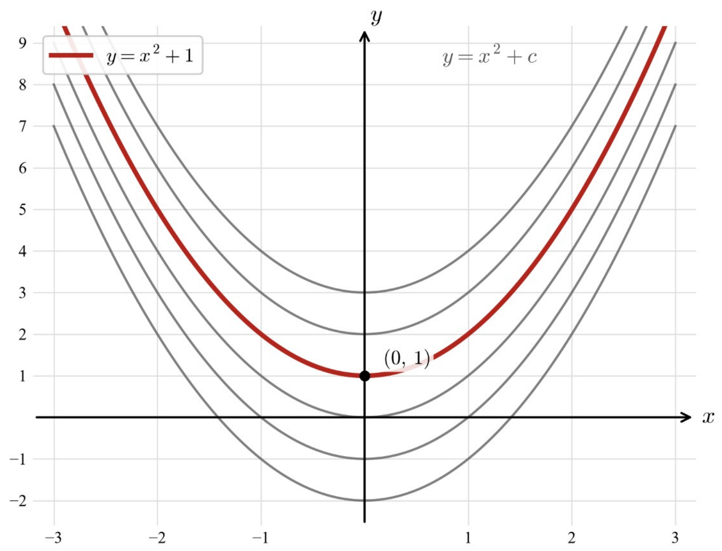
where the right-hand side depends on x alone. To **solve** it means to find every function $y=y(x)$ whose derivative is $f(x)$; this is exactly the task of antidifferentiation from an earlier chapter. So a DE of this form is really an integration problem in disguise, and the harder the function $f(x)$, the more of our integration toolkit it calls on.

The general solution.

Because $\frac{dy}{dx}=f(x)$ says that y is an antiderivative of f, we recover y by integrating both sides with respect to x:

$$y=\int f(x)\,dx+c.$$

The arbitrary constant c is essential. Any two antiderivatives of f differ only by a constant, so a first-order DE has not one solution but a whole **family** of them — one curve for each value of c. This family is the **general solution**. Geometrically the solution curves are vertical translates of one another: at any given x they all share the same gradient $f(x)$, so they never meet and sit stacked parallel in the y-direction.



The figure shows the general solution of $\frac{dy}{dx}=2x$, namely $y=x^2+c$: a family of congruent parabolas, each obtained by sliding the curve $y=x^2$ up or down.

The particular solution.

To select one member of the family we need one extra piece of information — an **initial condition** (or **boundary condition**), usually the value of y at a stated x, written $y(x_0)=y_0$. Substituting this into the general solution fixes c, and the result is the **particular solution**: the single curve of the family that passes through (x_0,y_0) (drawn highlighted above, through $(0,1)$). In practice:

1. antidifferentiate to obtain the general solution, remembering the $+c$;
2. substitute the initial condition;

3. solve for c;
4. write out the particular solution.

Second-order equations $\frac{d^2y}{dx^2}=f(x)$.

An equation that prescribes the **second** derivative,

$$\frac{d^2y}{dx^2}=f(x),$$

is solved by antidifferentiating **twice**. The first integration produces $\frac{dy}{dx}$ and introduces one constant; the second produces y and introduces a second constant:

$$\frac{dy}{dx}=\int f(x)\,dx+c_1,\ y=\int\left(\frac{dy}{dx}\right)dx+c_2.$$

Because there are now **two** arbitrary constants, **two** conditions are needed to pin them down — for example the values of $\frac{dy}{dx}$ and of y at a single point, or the value of y at two different points. When one condition involves $\frac{dy}{dx}$ it is usually cleanest to find c_1 immediately after the first integration, then c_2 after the second.

When $f(x)$ needs an integration technique.

The right-hand side $f(x)$ may be any function we can integrate, so every method of the earlier chapters can be needed: **substitution** for a function such as xe^{x^2} or $\frac{x}{\sqrt{x^2+a^2}}$; **partial fractions** for a rational function such as $\frac{1}{(x-a)(x-b)}$; and **trigonometric identities** for a function such as $\sin^2 x$ or $\cos^2 x$, rewritten with a double-angle formula before integrating. The differential equation is only ever as hard as the integral on its right-hand side.

Interpreting the solution.

Since $\frac{dy}{dx}$ is the gradient of $y=y(x)$, a DE of this form is a statement about the gradient at every point. Reading it back gives the shape of the solution curve: stationary points occur where $f(x)=0$, the curve rises where $f(x)>0$ and falls where $f(x)<0$, and the initial condition fixes the one curve of the correct height. In an applied setting the same equation may model a rate of change — for instance $\frac{dV}{dt}=f(t)$ for a volume V changing with time t — and the particular solution then predicts the quantity itself.

Worked examples

Example 1 — scaffolded

Solve the differential equation $\frac{dy}{dx}=3x^2-4x+5$ given that $y=2$ when $x=1$.

Working	Comment
The equation has the form $\frac{dy}{dx}=f(x)$, so antidifferentiate: $y=\int(3x^2-4x+5)\,dx$.	Integrate both sides with respect to x.
$y=x^3-2x^2+5x+c$.	General solution — a family with one constant c.

Working	Comment
Substitute $y=2$, $x=1$: $2=(1)^3-2(1)^2+5(1)+c$.	Apply the initial condition to find c.
$2=4+c$, so $c=-2$.	Solve for the constant.
$y=x^3-2x^2+5x-2$.	Particular solution: the curve through $(1,2)$.

Example 2 — scaffolded

Solve $\dfrac{d^2y}{dx^2}=6x-4$ given that $\dfrac{dy}{dx}=3$ and $y=5$ when $x=1$.

Working	Comment
Integrate once: $\dfrac{dy}{dx}=\int(6x-4)\,dx=3x^2-4x+c_1$.	First antidifferentiation introduces c_1.
Use $\dfrac{dy}{dx}=3$ at $x=1$: $3=3-4+c_1$, so $c_1=4$.	The gradient condition fixes c_1.
Hence $\dfrac{dy}{dx}=3x^2-4x+4$.	The first derivative is now fully known.
Integrate again: $y=\int(3x^2-4x+4)\,dx=x^3-2x^2+4x+c_2$.	Second antidifferentiation introduces c_2.
Use $y=5$ at $x=1$: $5=1-2+4+c_2$, so $c_2=2$.	The value condition fixes c_2.
$y=x^3-2x^2+4x+2$.	Particular solution satisfying both conditions.

Example 3 — unscaffolded

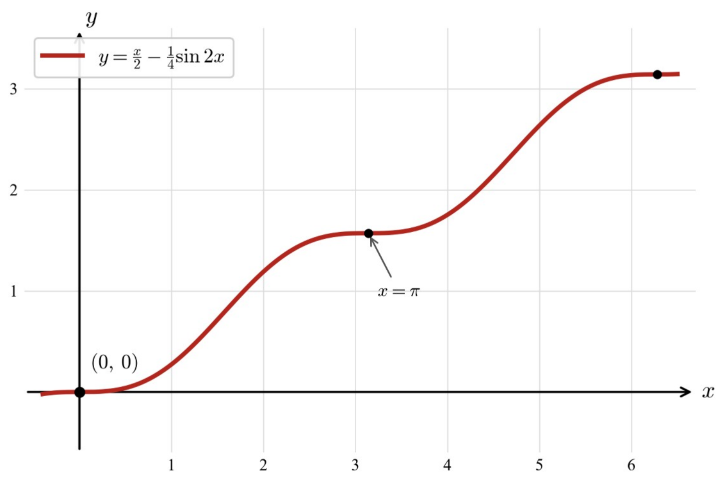
Solve $\dfrac{dy}{dx}=\sin^2 x$ given that $y=0$ when $x=0$, and describe the shape of the solution curve.

The integrand $\sin^2 x$ is not a standard antiderivative, so first rewrite it with the double-angle identity $\sin^2 x=\dfrac{1-\cos 2x}{2}$. Then

$$y=\int\sin^2 x\,dx=\int\frac{1-\cos 2x}{2}\,dx=\frac{x}{2}-\frac{1}{4}\sin 2x+c.$$

Applying the condition $y=0$ at $x=0$, and using $\sin 0=0$, gives $0=0-0+c$, so $c=0$. The particular solution is therefore

$$y=\frac{x}{2}-\frac{1}{4}\sin 2x.$$



Because $\sin^2 x \geq 0$, the gradient is never negative, so the curve rises steadily; it has a horizontal tangent wherever $\sin x = 0$ (at $x = 0, \pi, 2\pi, \ldots$) and otherwise climbs, which the graph confirms.

$\therefore y = \frac{x}{2} - \frac{1}{4}\sin 2x.$

Example 4 — unscaffolded

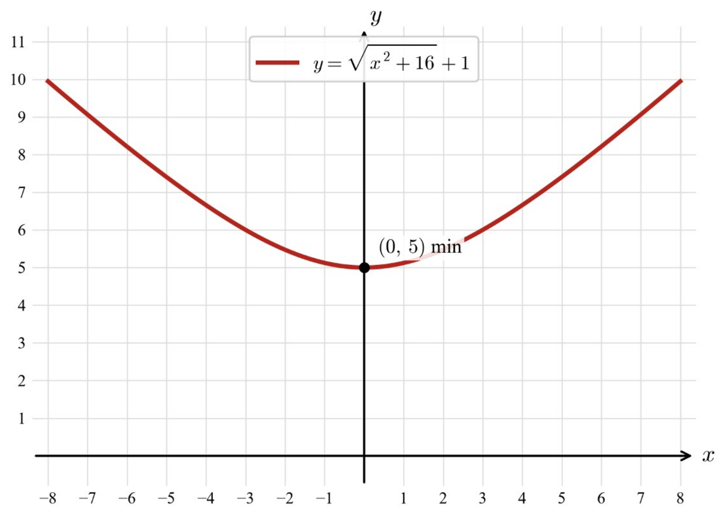
The gradient of a curve at each point is $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+16}}$, and the curve passes through $(0,5)$. Find its equation and interpret the point $(0,5)$.

The right-hand side calls for the substitution $u = x^2 + 16$, so that $du = 2x\,dx$, that is $x\,dx = 1/2\,du$. Then

$$y = \int \frac{x}{\sqrt{x^2+16}}\,dx = \int \frac{1}{2}u^{-1/2}\,du = u^{1/2} + c = \sqrt{x^2+16} + c.$$

The curve passes through $(0,5)$, so $5 = \sqrt{0+16} + c = 4 + c$, giving $c = 1$. Hence

$$y = \sqrt{x^2+16} + 1.$$



At $x = 0$ the gradient is $\frac{0}{\sqrt{16}} = 0$, while $\frac{dy}{dx} < 0$ for $x < 0$ and $\frac{dy}{dx} > 0$ for $x > 0$. So $(0,5)$ is the **minimum point** of the curve: the smallest value the function attains.

$\therefore y = \sqrt{x^2+16} + 1$, with minimum point $(0,5)$.

Practice questions

Scaffolded

Q1. Solve $\frac{dy}{dx} = 4x - 3$ given that $y = 5$ when $x = 2$. (a) Antidifferentiate to write the general solution, including $+c$. (b) Substitute the initial condition and solve for c. (c) State the particular solution.

Q2. A curve with gradient $\frac{dy}{dx} = 6x^2 + 2x - 4$ passes through $(1,0)$. (a) Find the general solution. (b) Use the point $(1,0)$ to find c. (c) State the equation of the curve.

Q3. Solve $\frac{dy}{dx}=e^{2x}+1$ given that $y=3$ when $x=0$. (a) Antidifferentiate each term to obtain the general solution. (b) Apply the initial condition. (c) State the particular solution.

Q4. Solve $\frac{d^2y}{dx^2}=12x+2$ given that $\frac{dy}{dx}=3$ and $y=1$ when $x=0$. (a) Integrate once and use the gradient condition to find c_1. (b) Integrate again and use the value condition to find c_2. (c) State the particular solution.

Q5. Solve $\frac{dy}{dx}=\cos^2 x$ given that $y=0$ when $x=0$. (a) Rewrite $\cos^2 x$ using the identity $\cos^2 x=\frac{1+\cos 2x}{2}$. (b) Antidifferentiate to obtain the general solution. (c) Apply the initial condition and state the particular solution.

Q6. Solve $\frac{dy}{dx}=x e^{x^2}$ given that $y=2$ when $x=0$. (a) Use the substitution $u=x^2$ to rewrite the integral. (b) Antidifferentiate and return to x. (c) Apply the initial condition and state the particular solution.

Unscaffolded

Q7. Solve $\frac{dy}{dx}=5x^4-6x+1$ given that $y=4$ when $x=1$.

Q8. Solve $\frac{dy}{dx}=\frac{4}{x}+2x$ (for $x>0$) given that $y=3$ when $x=1$.

Q9. Solve $\frac{dy}{dx}=\sin 3x+2\cos x$ given that $y=1$ when $x=0$.

Q10. Find the general solution of $\frac{dy}{dx}=6\sqrt{x}-\frac{1}{x^2}$ (for $x>0$).

Q11. Solve $\frac{dy}{dx}=2\sin^2 x$ given that $y=1$ when $x=0$.

Q12. Solve $\frac{dy}{dx}=x(x^2-1)^4$ given that $y=2$ when $x=1$.

Q13. Find the general solution of $\frac{dy}{dx}=\frac{1}{(x-1)(x+2)}$ using partial fractions.

Q14. Solve $\frac{d^2y}{dx^2}=e^x$ given that $\frac{dy}{dx}=2$ and $y=3$ when $x=0$.

Q15. Solve $\frac{d^2y}{dx^2}=\cos x$ given that $\frac{dy}{dx}=0$ and $y=2$ when $x=0$.

Q16. Solve $\frac{dy}{dx}=\frac{x}{x^2+4}$ given that $y=0$ when $x=0$.

Q17. The gradient of a curve at any point is $\frac{dy}{dx}=3x^2-12x+9$. (a) Find the x-coordinates of the stationary points of the curve. (b) Given that the curve passes through $(0,4)$, find its equation.

Q18. Solve $\frac{d^2y}{dx^2}=6x+2$ given the two boundary conditions $y=3$ when $x=0$ and $y=8$ when $x=1$.

Answers

Q1. $y=2x^2-3x+3$. **Q2.** $y=2x^3+x^2-4x+1$. **Q3.** $y=\frac{1}{2}e^{2x}+x+\frac{5}{2}$. **Q4.** $y=2x^3+x^2+3x+1$. **Q5.** $y=\frac{x}{2}+\frac{1}{4}\sin 2x$. **Q6.** $y=\frac{1}{2}e^{x^2}+\frac{3}{2}$. **Q7.** $y=x^5-3x^2+x+5$. **Q8.** $y=4\log_e x+x^2+2$. **Q9.** $y=-\frac{1}{3}\cos 3x+2\sin x+\frac{4}{3}$. **Q10.** $y=4x^{\frac{3}{2}}+\frac{1}{x}+c$. **Q11.** $y=x-\frac{1}{2}\sin 2x+1$. **Q12.** $y=\frac{(x^2-1)^5}{10}+2$. **Q13.** $y=\frac{1}{3}\log_e\left|\frac{x-1}{x+2}\right|+c$. **Q14.** $y=e^x+x+2$. **Q15.** $y=-\cos x+3$. **Q16.** $y=\frac{1}{2}\log_e(x^2+4)-\log_e 2$ (equivalently $\frac{1}{2}\log_e\frac{x^2+4}{4}$). **Q17.** (a) $x=1$ and $x=3$; (b) $y=x^3-6x^2+9x+4$. **Q18.** $y=x^3+x^2+3x+3$.

Fully-worked solutions

Q1. Antidifferentiating,

$$y=\int(4x-3)\,dx=2x^2-3x+c.$$

Substituting $y=5$, $x=2$ gives $5=2(4)-3(2)+c=2+c$, so $c=3$. Hence $y=2x^2-3x+3$.

Q2. The general solution is

$$y=\int(6x^2+2x-4)\,dx=2x^3+x^2-4x+c.$$

The curve passes through $(1,0)$, so $0=2+1-4+c=-1+c$, giving $c=1$. Hence $y=2x^3+x^2-4x+1$.

Q3. Integrating term by term, and using $\int e^{2x}\,dx=1/2\,e^{2x}$,

$$y=\int(e^{2x}+1)\,dx=\frac{1}{2}e^{2x}+x+c.$$

At $x=0$, $y=3$: $3=1/2\,e^0+0+c=1/2+c$, so $c=5/2$. Hence $y=\frac{1}{2}e^{2x}+x+\frac{5}{2}$.

Q4. Integrating once,

$$\frac{dy}{dx}=\int(12x+2)\,dx=6x^2+2x+c_1.$$

At $x=0$, $\frac{dy}{dx}=3$, so $3=0+0+c_1$, giving $c_1=3$ and $\frac{dy}{dx}=6x^2+2x+3$. Integrating again,

$$y=\int(6x^2+2x+3)\,dx=2x^3+x^2+3x+c_2.$$

At $x=0$, $y=1$, so $1=c_2$. Hence $y=2x^3+x^2+3x+1$.

Q5. Using $\cos^2 x=\frac{1+\cos 2x}{2}$,

$$y=\int\cos^2 x\,dx=\int\frac{1+\cos 2x}{2}\,dx=\frac{x}{2}+\frac{1}{4}\sin 2x+c.$$

At $x=0$, $y=0$ gives $0=0+0+c$, so $c=0$. Hence $y=\frac{x}{2}+\frac{1}{4}\sin 2x$.

Q6. With $u=x^2$, $du=2x\,dx$, so $x\,dx=1/2\,du$ and

$$y = \int x e^{x^2}\,dx = \int \frac{1}{2} e^u\,du = \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c.$$

At $x=0$, $y=2$: $2 = 1/2\, e^0 + c = 1/2 + c$, so $c = 3/2$. Hence $y = \frac{1}{2} e^{x^2} + \frac{3}{2}$.

Q7. Antidifferentiating,

$$y = \int (5x^4 - 6x + 1)\,dx = x^5 - 3x^2 + x + c.$$

At $x=1$, $y=4$: $4 = 1 - 3 + 1 + c = -1 + c$, so $c = 5$. Hence $y = x^5 - 3x^2 + x + 5$.

Q8. Using $\int \frac{4}{x}\,dx = 4\log_e x$ for $x > 0$,

$$y = \int \left(\frac{4}{x} + 2x\right) dx = 4\log_e x + x^2 + c.$$

At $x=1$, $y=3$: $3 = 4\log_e 1 + 1 + c = 0 + 1 + c$, so $c = 2$. Hence $y = 4\log_e x + x^2 + 2$.

Q9. Antidifferentiating each term,

$$y = \int (\sin 3x + 2\cos x)\,dx = -\frac{1}{3}\cos 3x + 2\sin x + c.$$

At $x=0$, $y=1$: $1 = -1/3\cos 0 + 2\sin 0 + c = -1/3 + c$, so $c = 4/3$. Hence $y = -\frac{1}{3}\cos 3x + 2\sin x + \frac{4}{3}$.

Q10. Writing $6\sqrt{x} = 6x^{1/2}$ and $\frac{1}{x^2} = x^{-2}$,

$$y = \int (6x^{1/2} - x^{-2})\,dx = 6\cdot\frac{x^{3/2}}{3/2} + x^{-1} + c = 4x^{\frac{3}{2}} + \frac{1}{x} + c.$$

There is no initial condition, so this general solution (with its arbitrary constant c) is the answer.

Q11. Using $2\sin^2 x = 1 - \cos 2x$,

$$y = \int 2\sin^2 x\,dx = \int (1 - \cos 2x)\,dx = x - \frac{1}{2}\sin 2x + c.$$

At $x=0$, $y=1$: $1 = 0 - 0 + c$, so $c = 1$. Hence $y = x - \frac{1}{2}\sin 2x + 1$.

Q12. With $u = x^2 - 1$, $du = 2x\,dx$, so $x\,dx = 1/2\,du$ and

$$y = \int x(x^2-1)^4\,dx = \int \frac{1}{2} u^4\,du = \frac{1}{2}\cdot\frac{u^5}{5} + c = \frac{(x^2-1)^5}{10} + c.$$

At $x=1$, $y=2$: since $x^2 - 1 = 0$ there, $2 = 0 + c$, so $c = 2$. Hence $y = \frac{(x^2-1)^5}{10} + 2$.

Q13. Write $\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$. Then $1 = A(x+2) + B(x-1)$. Putting $x=1$ gives $3A = 1$, so $A = 1/3$; putting $x = -2$ gives $-3B = 1$, so $B = -1/3$. Therefore

$$y = \int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2}\right) dx = \frac{1}{3}\log_e|x-1| - \frac{1}{3}\log_e|x+2| + c.$$

Combining the logarithms, $y=\frac{1}{3}\log_e\left|\frac{x-1}{x+2}\right|+c$.

Q14. Integrating once,

$$\frac{dy}{dx}=\int e^x\,dx=e^x+c_1.$$

At $x=0$, $\frac{dy}{dx}=2$: $2=e^0+c_1=1+c_1$, so $c_1=1$ and $\frac{dy}{dx}=e^x+1$. Integrating again,

$$y=\int(e^x+1)\,dx=e^x+x+c_2.$$

At $x=0$, $y=3$: $3=e^0+0+c_2=1+c_2$, so $c_2=2$. Hence $y=e^x+x+2$.

Q15. Integrating once,

$$\frac{dy}{dx}=\int\cos x\,dx=\sin x+c_1.$$

At $x=0$, $\frac{dy}{dx}=0$: $0=\sin 0+c_1=c_1$, so $c_1=0$ and $\frac{dy}{dx}=\sin x$. Integrating again,

$$y=\int\sin x\,dx=-\cos x+c_2.$$

At $x=0$, $y=2$: $2=-\cos 0+c_2=-1+c_2$, so $c_2=3$. Hence $y=-\cos x+3$.

Q16. With $u=x^2+4$, $du=2x\,dx$, so $x\,dx=1/2\,du$ and

$$y=\int\frac{x}{x^2+4}\,dx=\int\frac{1}{2u}\,du=\frac{1}{2}\log_e(x^2+4)+c$$

(the argument $x^2+4>0$, so no modulus is needed). At $x=0$, $y=0$: $0=1/2\log_e 4+c$, so $c=-1/2\log_e 4=-\log_e 2$. Hence $y=\frac{1}{2}\log_e(x^2+4)-\log_e 2$.

Q17. (a) The stationary points occur where the gradient is zero: $3x^2-12x+9=3(x^2-4x+3)=3(x-1)(x-3)=0$, so $x=1$ and $x=3$. (b) Antidifferentiating,

$$y=\int(3x^2-12x+9)\,dx=x^3-6x^2+9x+c.$$

The curve passes through $(0,4)$, so $4=0-0+0+c$, giving $c=4$. Hence $y=x^3-6x^2+9x+4$.

Q18. Integrating once,

$$\frac{dy}{dx}=\int(6x+2)\,dx=3x^2+2x+c_1,$$

and integrating again,

$$y=\int(3x^2+2x+c_1)\,dx=x^3+x^2+c_1x+c_2.$$

The condition $y=3$ at $x=0$ gives $c_2=3$. The condition $y=8$ at $x=1$ gives $1+1+c_1+3=8$, that is $5+c_1=8$, so $c_1=3$. Hence $y=x^3+x^2+3x+3$.

Theory

A **differential equation** is an equation linking a function to its derivatives; a **solution** is a function that satisfies it. The previous section solved equations of the form $\frac{dy}{dx}=f(x)$, where the rate of change depends only on the independent variable x and the solution is found by antidifferentiating directly. This section treats the companion case

$$\frac{dy}{dx}=f(y),$$

in which the rate of change depends only on the **dependent** variable y. Such equations model any quantity whose rate of growth or decay is governed by its own current size — a population, a radioactive sample, or a cooling body — and they cannot be integrated by simply antidifferentiating the right-hand side with respect to x, because the right-hand side is not (yet) a known function of x.

The reciprocal method.

The key idea is that the reciprocal of $\frac{dy}{dx}$ is $\frac{dx}{dy}$. Provided $f(y)\neq 0$, turning the equation upside down gives

$$\frac{dx}{dy}=\frac{1}{f(y)},$$

and now the right-hand side is a function of y alone. Integrating **with respect to** y,

$$x=\int\frac{1}{f(y)}\,dy+c.$$

This produces x as a function of y. It is the same result reached by **separating the variables** — writing $\frac{1}{f(y)}\,dy=dx$ and integrating each side — and the two descriptions are interchangeable. Where the algebra allows, the final relation is rearranged to give y **explicitly** as a function of x; otherwise the answer is left as x in terms of y.

The constant of integration and the general solution.

A single indefinite integral introduces one arbitrary constant c, so the working above describes a whole **family** of solution curves — the **general solution**. Each choice of c (equivalently, each choice of the constant A or B that appears after rearranging) selects one member of the family. A **boundary** or **initial condition**, such as a known value $y=y_0$ when $x=x_0$, fixes that constant and singles out the one **particular solution** passing through the given point. Always write $+c$ on the general solution, then substitute the condition to evaluate it.

Equilibrium solutions.

Wherever $f(y)=0$ the right-hand side vanishes, so $\frac{dy}{dx}=0$ and y is constant. These constant **equilibrium solutions** are worth noting separately, because the reciprocal method divides by $f(y)$ and so cannot produce them directly. In the cooling model below, the equilibrium is the ambient temperature the body settles towards.

Classic case 1: $\frac{dy}{dx}=ky$ (exponential change).

When $f(y)=ky$ for a constant k,

$$\frac{dx}{dy}=\frac{1}{ky},x=\int\frac{1}{ky}dy=\frac{1}{k}\log_e|y|+c.$$

Multiplying by k and exponentiating, $|y|=e^{k(x-c)}=e^{-kc}e^{kx}$, so

$$y=Ae^{kx},$$

where the constant $A=\pm e^{-kc}$. This is **exponential growth** when $k>0$ and **exponential decay** when $k<0$. Putting $x=0$ shows $A=y(0)$, the initial value. The defining feature is that the rate of change is proportional to the amount present.

Classic case 2: $\frac{dy}{dx}=k(y-A)$ (approach to an ambient value).

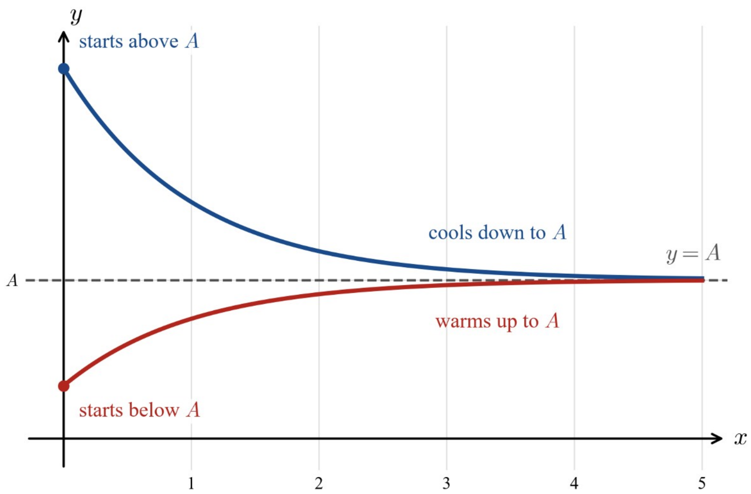
Newton's law of cooling states that a body loses heat at a rate proportional to the difference between its temperature and that of its surroundings. With A the ambient value and k a negative constant,

$$\frac{dx}{dy}=\frac{1}{k(y-A)},x=\int\frac{1}{k(y-A)}dy=\frac{1}{k}\log_e|y-A|+c.$$

Rearranging as before, $y-A=Be^{kx}$, so

$$y=A+Be^{kx}.$$

Because $k<0$, the term $Be^{kx}\to 0$ as $x\to\infty$, and $y\to A$: the solution curve **approaches the horizontal asymptote** $y=A$, which is exactly the equilibrium $f(A)=0$. The constant $B=y(0)-A$ measures the initial gap from ambient. A body starting above A cools down towards it; one starting below A warms up towards it.



The diagram shows both cases of $\frac{dy}{dx}=k(y-A)$ with $k<0$: whether y begins above or below the ambient value A, the difference $y-A$ decays exponentially and the curve flattens onto the asymptote $y=A$.

Worked examples

Example 1 — scaffolded

Find the general solution of $\frac{dy}{dx}=4y$, and hence the particular solution with $y(0)=3$.

Working	Comment
Invert the derivative: $\frac{dx}{dy}=\frac{1}{4y}$.	The right-hand side is now a function of y alone, ready to integrate with respect to y.

Working	Comment
Integrate: $x=\int \frac{1}{4y}\,dy=\frac{1}{4}\log_e\lvert y\rvert+c$.	One indefinite integral, one constant $+c$: the general solution.
Rearrange: $4(x-c)=\log_e\lvert y\rvert$, so $\lvert y\rvert=e^{4x-4c}$ and $y=Ae^{4x}$.	Exponentiate; the constant $A=\pm e^{-4c}$ absorbs the sign and the constant.
Apply $y(0)=3$: $3=Ae^0=A$, so $A=3$.	The initial condition selects the one curve through $(0,3)$.
$\therefore y=3e^{4x}$.	Check: $\frac{dy}{dx}=12e^{4x}=4\left(3e^{4x}\right)=4y$, as required.

Example 2 — scaffolded

A quantity satisfies $\frac{dy}{dx}=-2(y-15)$ with $y(0)=40$. Solve for y and state its long-term value.

Working	Comment
Invert: $\frac{dx}{dy}=\frac{1}{-2(y-15)}=-\frac{1}{2(y-15)}$.	This is the case $\frac{dy}{dx}=k(y-A)$ with $k=-2$, $A=15$.
Integrate: $x=\int -\frac{1}{2(y-15)}\,dy=-\frac{1}{2}\log_e\lvert y-15\rvert+c$.	Integrate with respect to y; keep the $+c$.
Rearrange: $-2(x-c)=\log_e\lvert y-15\rvert$, so $y-15=Be^{-2x}$ and $y=15+Be^{-2x}$.	Exponentiate; $B=\pm e^{2c}$ is the new constant.
Apply $y(0)=40$: $40=15+B$, so $B=25$.	Use the initial condition.
$\therefore y=15+25e^{-2x}$. As $x\to\infty$, $e^{-2x}\to 0$, so $y\to 15$.	The solution decays onto the equilibrium $y=15$ (where $f(15)=0$).

Example 3 — unscaffolded

Solve $\frac{dy}{dx}=y^2$ given that $y=1$ when $x=0$, giving y explicitly.

Because the right-hand side depends only on y, invert the derivative and integrate with respect to y:

$$\frac{dx}{dy}=\frac{1}{y^2}=y^{-2},x=\int y^{-2}\,dy=-\frac{1}{y}+c.$$

The initial condition $y(0)=1$ gives $0=-\frac{1}{1}+c$, so $c=1$ and $x=1-\frac{1}{y}$. This is x as a function of y; solving for y, $\frac{1}{y}=1-x$, hence

$$y=\frac{1}{1-x}.$$

As a check, $\frac{dy}{dx}=\frac{1}{(1-x)^2}=y^2$. The solution is valid for $x<1$; as $x\to 1^-$ the curve increases without bound.

$\therefore y=\frac{1}{1-x}$, for $x<1$.

Example 4 — unscaffolded

A liquid cools in a room kept at 20°C. Its temperature T °C after t minutes obeys Newton's law of cooling $\frac{dT}{dt}=-\frac{1}{5}(T-20)$ with $T(0)=90$. Find T as a function of t, and the time taken to reach 55°C.

The rate depends only on T, so invert the derivative and integrate with respect to T:

$$\frac{dt}{dT}=\frac{1}{-\frac{1}{5}(T-20)}=-\frac{5}{T-20}, t=\int-\frac{5}{T-20}dT=-5\log_e|T-20|+c.$$

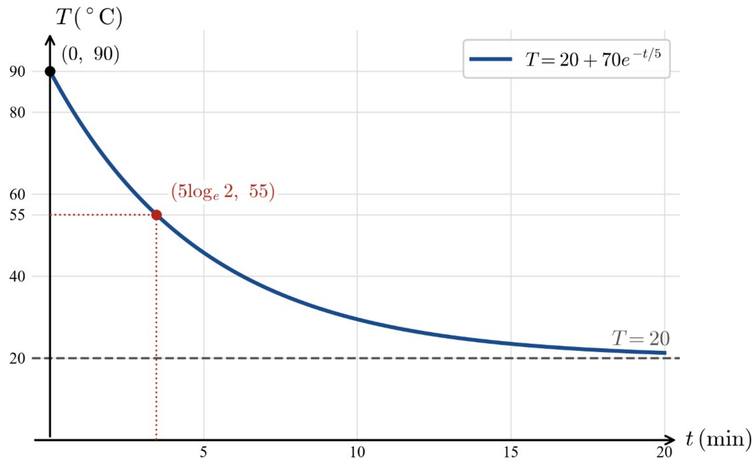
The liquid stays hotter than the room, so $T>20$ and $|T-20|=T-20$. Rearranging, $\log_e(T-20)=\frac{c-t}{5}$, hence $T-20=Be^{-t/5}$ and

$$T=20+Be^{-t/5}.$$

The initial temperature gives $90=20+B$, so $B=70$ and $T=20+70e^{-t/5}$. Setting $T=55$,

$$55=20+70e^{-t/5} \quad e^{-t/5}=\frac{1}{2} \quad -\frac{t}{5}=\log_e\frac{1}{2} \quad t=5\log_e 2.$$

The diagram shows the temperature falling from 90°C and levelling onto the room temperature 20°C, the horizontal asymptote.



$\therefore T=20+70e^{-t/5}$, reaching 55°C after $t=5\log_e 2\approx3.47$ minutes.

Practice questions

Scaffolded

Q1. Find the particular solution of $\frac{dy}{dx}=5y$ with $y(0)=2$. (a) Write down $\frac{dx}{dy}$. (b) Integrate with respect to y to find x in terms of y, including $+c$. (c) Rearrange to the general solution $y=Ae^{5x}$. (d) Use the initial condition to find A and state y.

Q2. Solve $\frac{dy}{dx}=-3y$ with $y(0)=8$. (a) Write $\frac{dx}{dy}$ and integrate with respect to y. (b) Rearrange to $y=Ae^{-3x}$. (c) Apply $y(0)=8$ to find A and state y.

Q3. Solve $\frac{dy}{dx}=2(y-6)$ with $y(0)=10$. (a) Write $\frac{dx}{dy}=\frac{1}{2(y-6)}$ and integrate with respect to y. (b) Rearrange to $y=6+Be^{2x}$. (c) Use $y(0)=10$ to find B and state y.

Q4. A body cools according to $\frac{dy}{dx}=-(y-30)$ with $y(0)=100$. (a) Integrate to find x in terms of y. (b) Rearrange to $y=30+Be^{-x}$. (c) Find B and state the value y approaches as $x\to\infty$.

Q5. Solve $\frac{dy}{dx}=y^2$ with $y(0)=2$, giving y explicitly. (a) Write $\frac{dx}{dy}=\frac{1}{y^2}$ and integrate with respect to y. (b) Use $y(0)=2$ to find c. (c) Rearrange to give y as a function of x.

Q6. A cup of tea cools in a room at 25°C. Its temperature T °C after t minutes satisfies $\frac{dT}{dt}=-\frac{1}{10}(T-25)$ with $T(0)=85$. (a) Integrate with respect to T to find t in terms of T. (b) Rearrange to $T=25+Be^{-t/10}$ and find B. (c) Find the time at which $T=55$°C.

Unscaffolded

Q7. Solve $\frac{dy}{dx}=7y$ with $y(0)=4$.

Q8. Solve $\frac{dy}{dx}=-\frac{1}{2}y$ with $y(0)=6$.

Q9. Find the general solution of $\frac{dy}{dx}=3(y-2)$.

Q10. Solve $\frac{dy}{dx}=4-y$ with $y(0)=1$, and state the value y approaches as $x\to\infty$.

Q11. Solve $\frac{dy}{dx}=y^2$ with $y(1)=1$, giving y explicitly.

Q12. Solve $\frac{dy}{dx}=\frac{1}{y}$ with $y(0)=2$, taking the positive branch.

Q13. Solve $\frac{dy}{dx}=\sqrt{y}$ with $y(0)=1$, giving y explicitly.

Q14. Solve $\frac{dy}{dx}=2(y+3)$ with $y(0)=-1$.

Q15. A metal casting at 200°C cools in air at 25°C so that its temperature T °C after t hours satisfies $\frac{dT}{dt}=-\frac{1}{4}(T-25)$. Find T as a function of t, and its temperature after 4 hours (exact value).

Q16. A colony of P bacteria grows so that $\frac{dP}{dt}=\frac{1}{10}P$, where t is in hours and $P(0)=500$. Find P as a function of t, and the time at which the colony first reaches 1000.

Q17. Solve $\frac{dy}{dx}=y^2+1$ with $y(0)=0$, giving y explicitly.

Q18. A sample of N cells satisfies $\frac{dN}{dt}=kN$. It is observed that $N=100$ when $t=0$ and $N=400$ when $t=2$ (with t in hours). (a) Find k. (b) Find N as a function of t. (c) Find N when $t=5$.

Answers

Q1. $y=2e^{5x}$. **Q2.** $y=8e^{-3x}$. **Q3.** $y=6+4e^{2x}$. **Q4.** $y=30+70e^{-x}$; $y\to 30$ as $x\to\infty$. **Q5.** $y=\frac{2}{1-2x}$. **Q6.** $T=25+60e^{-t/10}$; $t=10\log_e 2$ minutes. **Q7.** $y=4e^{7x}$. **Q8.** $y=6e^{-x/2}$. **Q9.** $y=2+Ae^{3x}$. **Q10.** $y=4-3e^{-x}$; $y\to 4$ as $x\to\infty$. **Q11.** $y=\frac{1}{2-x}$. **Q12.** $y=\sqrt{2x+4}$. **Q13.** $y=\frac{(x+2)^2}{4}$. **Q14.** $y=-3+2e^{2x}$. **Q15.** $T=25+175e^{-t/4}$; after 4 hours $T=25+\frac{175}{e}$°C. **Q16.** $P=500e^{t/10}$; $t=10\log_e 2$ hours. **Q17.** $y=\tan x$. **Q18.** (a) $k=\log_e 2$; (b) $N=100\times 2^t$; (c) $N=3200$.

Fully-worked solutions

Q1. Invert and integrate with respect to y:

$$\frac{dx}{dy}=\frac{1}{5y},x=\int\frac{1}{5y}dy=\frac{1}{5}\log_e|y|+c.$$

Then $5(x-c)=\log_e|y|$, so $y=Ae^{5x}$ with $A=\pm e^{-5c}$. Since $y(0)=2$ gives $A=2$, the solution is $y=2e^{5x}$.

Q2. Invert and integrate:

$$\frac{dx}{dy}=\frac{1}{-3y}=-\frac{1}{3y},x=\int-\frac{1}{3y}dy=-\frac{1}{3}\log_e|y|+c.$$

Rearranging, $-3(x-c)=\log_e|y|$, so $y=Ae^{-3x}$. From $y(0)=8$, $A=8$, giving $y=8e^{-3x}$ (exponential decay).

Q3. With $f(y)=2(y-6)$,

$$\frac{dx}{dy}=\frac{1}{2(y-6)},x=\int\frac{1}{2(y-6)}dy=\frac{1}{2}\log_e|y-6|+c.$$

Then $2(x-c)=\log_e|y-6|$, so $y-6=Be^{2x}$ and $y=6+Be^{2x}$. Applying $y(0)=10$ gives $10=6+B$, so $B=4$ and $y=6+4e^{2x}$.

Q4. With $f(y)=-(y-30)$,

$$\frac{dx}{dy}=-\frac{1}{y-30},x=\int-\frac{1}{y-30}dy=-\log_e|y-30|+c.$$

Rearranging, $y-30=Be^{-x}$, so $y=30+Be^{-x}$. From $y(0)=100$, $B=70$, hence $y=30+70e^{-x}$. As $x\to\infty$, $e^{-x}\to 0$, so $y\to 30$.

Q5. Invert and integrate:

$$\frac{dx}{dy}=\frac{1}{y^2}=y^{-2},x=\int y^{-2}dy=-\frac{1}{y}+c.$$

From $y(0)=2$: $0=-\frac{1}{2}+c$, so $c=\frac{1}{2}$ and $x=\frac{1}{2}-\frac{1}{y}$. Then $\frac{1}{y}=\frac{1}{2}-x=\frac{1-2x}{2}$, so $y=\frac{2}{1-2x}$.

Q6. With $f(T)=-\frac{1}{10}(T-25)$,

$$\frac{dt}{dT}=-\frac{10}{T-25},t=\int-\frac{10}{T-25}dT=-10\log_e|T-25|+c.$$

Since $T>25$, rearranging gives $T-25=Be^{-t/10}$, so $T=25+Be^{-t/10}$. From $T(0)=85$, $B=60$, hence $T=25+60e^{-t/10}$. Setting $T=55$:

$$30=60e^{-t/10} \quad e^{-t/10}=\frac{1}{2} \quad t=10\log_e 2\approx 6.93\text{ minutes}.$$

Q7. $\frac{dx}{dy}=\frac{1}{7y}$, so $x=\int\frac{1}{7y}dy=\frac{1}{7}\log_e|y|+c$, giving $y=Ae^{7x}$. From $y(0)=4$, $A=4$, so $y=4e^{7x}$.

Q8. $\frac{dx}{dy}=-\frac{2}{y}$, so $x=\int-\frac{2}{y}dy=-2\log_e|y|+c$, giving $y=Ae^{-x/2}$. From $y(0)=6$, $A=6$, so $y=6e^{-x/2}$.

Q9. With $f(y)=3(y-2)$,

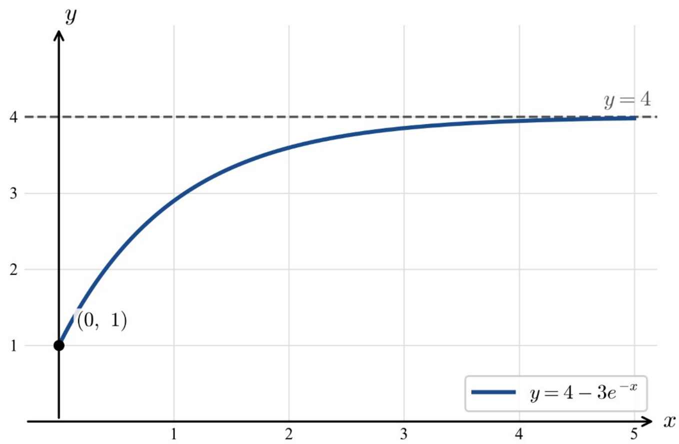
$$\frac{dx}{dy}=\frac{1}{3(y-2)}, x=\int\frac{1}{3(y-2)}dy=\frac{1}{3}\log_e|y-2|+c.$$

Then $3(x-c)=\log_e|y-2|$, so $y-2=Ae^{3x}$ and the general solution is $y=2+Ae^{3x}$ (with A arbitrary; no initial condition is given).

Q10. Write $\frac{dy}{dx}=4-y=-(y-4)$. Then

$$\frac{dx}{dy}=-\frac{1}{y-4}, x=\int-\frac{1}{y-4}dy=-\log_e|y-4|+c,$$

so $y-4=Be^{-x}$ and $y=4+Be^{-x}$. From $y(0)=1$, $1=4+B$, so $B=-3$ and $y=4-3e^{-x}$. As $x\to\infty$, $y\to 4$ (rising onto the asymptote from below).



Q11. As in Example 3, $x=-\frac{1}{y}+c$. The condition $y(1)=1$ gives $1=-1+c$, so $c=2$ and $x=2-\frac{1}{y}$. Then $\frac{1}{y}=2-x$, so $y=\frac{1}{2-x}$.

Q12. With $f(y)=\frac{1}{y}$, the reciprocal is $\frac{dx}{dy}=y$, so

$$x=\int y\,dy=\frac{y^2}{2}+c.$$

From $y(0)=2$: $0=\frac{4}{2}+c$, so $c=-2$ and $x=\frac{y^2}{2}-2$. Then $y^2=2x+4$, and taking the positive branch, $y=\sqrt{2x+4}$.

Q13. With $f(y)=\sqrt{y}=y^{1/2}$,

$$\frac{dx}{dy}=y^{-1/2}, x=\int y^{-1/2}\,dy=2y^{1/2}+c=2\sqrt{y}+c.$$

From $y(0)=1$: $0=2+c$, so $c=-2$ and $x=2\sqrt{y}-2$. Then $\sqrt{y}=\frac{x+2}{2}$, so $y=\frac{(x+2)^2}{4}$ (valid for $x\ge -2$). Check: $\frac{dy}{dx}=\frac{x+2}{2}=\sqrt{y}$.

Q14. With $f(y)=2(y+3)$,

$$\frac{dx}{dy}=\frac{1}{2(y+3)}, x=\int \frac{1}{2(y+3)}\,dy=\frac{1}{2}\log_e|y+3|+c.$$

Then $y+3=Ae^{2x}$, so $y=-3+Ae^{2x}$. From $y(0)=-1$: $-1=-3+A$, so $A=2$ and $y=-3+2e^{2x}$.

Q15. With $f(T)=-\frac{1}{4}(T-25)$,

$$\frac{dt}{dT}=-\frac{4}{T-25}, t=\int -\frac{4}{T-25}\,dT=-4\log_e|T-25|+c.$$

Since $T>25$, $T-25=Be^{-t/4}$, so $T=25+Be^{-t/4}$. From $T(0)=200$, $B=175$, hence $T=25+175e^{-t/4}$. After 4 hours, $t=4$ gives

$$T=25+175e^{-1}=25+\frac{175}{e}\approx 89.4\,°\mathrm{C}.$$

Q16. With $f(P)=\frac{1}{10}P$,

$$\frac{dt}{dP}=\frac{10}{P}, t=\int \frac{10}{P}\,dP=10\log_e|P|+c,$$

so $P=Ae^{t/10}$. From $P(0)=500$, $A=500$ and $P=500e^{t/10}$. Setting $P=1000$:

$$1000=500e^{t/10} \quad e^{t/10}=2 \quad t=10\log_e 2\approx 6.93\,\text{hours}.$$

Q17. With $f(y)=y^2+1$,

$$\frac{dx}{dy}=\frac{1}{y^2+1}, x=\int \frac{1}{y^2+1}\,dy=\tan^{-1}y+c.$$

From $y(0)=0$: $0=\tan^{-1}0+c=c$, so $c=0$ and $x=\tan^{-1}y$. Hence $y=\tan x$ (for $-\frac{\pi}{2}<x<\frac{\pi}{2}$). Check: $\frac{dy}{dx}=sec^2x=1+\tan^2x=1+y^2$.

Q18. (a) The equation $\frac{dN}{dt}=kN$ has solution $N=Ae^{kt}$; with $N(0)=100$, $A=100$, so $N=100e^{kt}$. Then $N(2)=400$ gives

$$400=100e^{2k} \quad e^{2k}=4 \quad 2k=\log_e 4 \quad k=\frac{1}{2}\log_e 4=\log_e 2.$$

(b) Hence $N=100e^{(\log_e 2)t}=100\times 2^t$. (c) When $t=5$, $N=100\times 2^5=100\times 32=3200$.

Theory

Many real situations are described not by a formula for a quantity but by a statement about **how fast it changes** — its rate of change. A cooling drink, a growing population, salt washing out of a tank, a falling body slowed by the air: in each case the *rate* is given in words, and that description translates directly into a **first-order differential equation**. This section is about **modelling**: turning a described rate of change into a DE, solving it with the methods of earlier sections, applying the given condition, and **interpreting** the answer — the long-term value it approaches, or the time it takes to reach a stated level.

The modelling cycle.

1. **Define the variable** (with units) and let its derivative $\frac{dQ}{dt}$ be the rate of change with respect to time t.
2. **Translate** the description into a DE. Two phrases occur again and again: "the rate of change is *proportional to* the amount" becomes $\frac{dQ}{dt}=kQ$; a tank or store obeys the **balance law** $\frac{dQ}{dt}=(\text{rate in})-(\text{rate out})$.
3. **Solve** the DE. Almost all of these models are separable, so separate the variables and integrate both sides — each displayed integral keeps its integrand — add $+c$, and rearrange. Then use the initial or boundary condition to fix c (a **particular solution**).
4. **Interpret** the solution: read off the long-term behaviour as $t\to\infty$, and answer questions such as the time to reach a given level.

Exponential growth and decay.

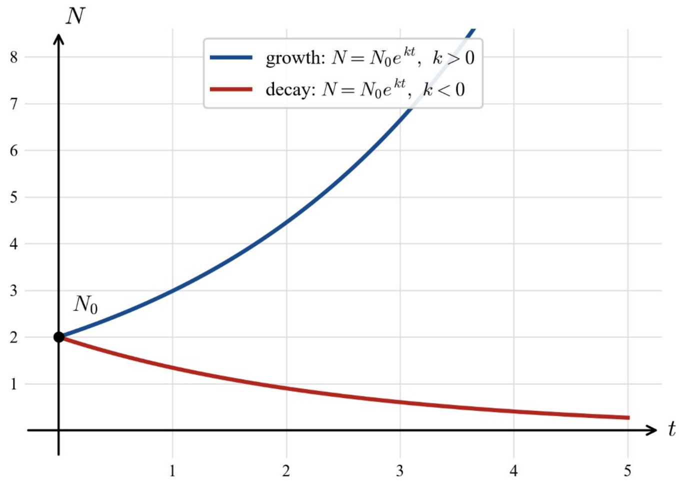
When the rate of change of a quantity is proportional to the amount present,

$$\frac{dN}{dt}=kN.$$

Separating the variables and integrating (an earlier section),

$$\int\frac{1}{N}\,dN=\int k\,dt \qquad \log_e N=kt+c\,(N>0),$$

so $N=e^{kt+c}=Ae^{kt}$, where $A=N(0)$ is the initial amount. If $k>0$ the quantity **grows** without bound; if $k<0$ it **decays** towards 0. The time for the amount to double (growth) or halve (decay) is constant, and is found by solving $e^{k\tau}=2$ or $e^{k\tau}=1/2$.



Newton's law of cooling.

An object at temperature T in surroundings held at the constant temperature T_s loses (or gains) heat at a rate proportional to the temperature difference:

$$\frac{dT}{dt}=-k(T-T_s), k>0.$$

Separating,

$$\int\frac{1}{T-T_s}dT=\int -k\,dt \quad \log_e|T-T_s|=-kt+c,$$

so $T-T_s=Ae^{-kt}$ and

$$T=T_s+(T_0-T_s)e^{-kt},$$

where $T_0=T(0)$. As $t\to\infty$, $e^{-kt}\to 0$, so $T\to T_s$: the object approaches the surrounding temperature. If $T_0>T_s$ it cools; if $T_0<T_s$ the same equation describes it **warming** up.

Mixing (dilution) problems.

A tank holds a dissolved substance of amount $Q(t)$ in a volume V of liquid. Liquid flows in and out at the **same** rate, so V stays constant. The balance law is

$$\frac{dQ}{dt}=(\text{rate in})-(\text{rate out}),$$

where rate in $=(\text{inflow concentration})\times(\text{inflow rate})$ and rate out $=\frac{Q}{V}\times(\text{outflow rate})$, since the well-stirred mixture leaving has the current concentration $\frac{Q}{V}$. This gives a separable DE of the form $\frac{dQ}{dt}=a-bQ$. The **long-term (steady-state)** amount is found by setting $\frac{dQ}{dt}=0$, which gives $Q=\frac{a}{b}$.

Motion with resistance.

For a body moving in a straight line with velocity $v(t)$, Newton's second law gives mass $\times\frac{dv}{dt}=(\text{driving force})-(\text{resistance})$. When the resistance is proportional to the speed, a body falling under gravity satisfies

$$\frac{dv}{dt}=g-kv.$$

At $v=0$ the resistance vanishes, so the constant term is the initial acceleration g; the falling-body models here take $g=10\,\text{m/s}^2$, a convenient rounding of 9.8. The **terminal velocity** is the constant speed at which the acceleration is zero: setting $\frac{dv}{dt}=0$ gives $v=\frac{g}{k}$, and the velocity rises towards it. With no driving force, $\frac{dv}{dt}=-kv$ gives exponential slowing towards rest.

Method.

1. Define the variable and write the DE from the described rate of change.
2. Separate the variables and integrate; add $+c$.
3. Apply the initial/boundary condition to find c (and any second condition to find k).
4. Interpret: state the limit as $t\to\infty$, and solve for the time to reach a given level.

Worked examples

Example 1 — scaffolded

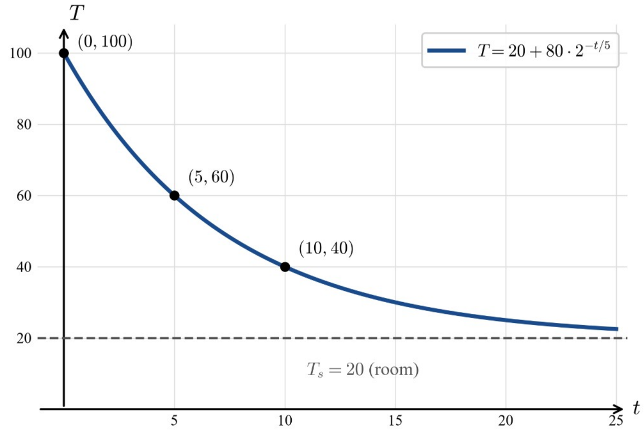
A bacterial culture grows so that its rate of increase is proportional to the number present. Initially there are 500 bacteria, and the number doubles every 4 hours. Find the number P after t hours, the population after 12 hours, and the time taken to reach 8000.

Working	Comment
Let P be the number after t hours. Rate proportional to the number present: $\frac{dP}{dt}=kP$.	Translate "rate proportional to amount" into a DE.
Separate and integrate: $\int \frac{1}{P}\,dP=\int k\,dt$, so $\log_e P=kt+c\ (P>0)$.	Separation of variables (earlier section); keep the integrand.
Then $P=e^{kt+c}=Ae^{kt}$; at $t=0$, $P=500$, so $A=500$ and $P=500e^{kt}$.	$A=e^c$; apply the initial condition.
Doubles every 4 h: $P(4)=1000$, so $1000=500e^{4k}$, giving $e^{4k}=2$ and $k=\frac{\log_e 2}{4}$.	The second condition fixes k.
$P=500e^{(\log_e 2)t/4}=500\cdot 2^{t/4}$. After 12 h, $P=500\cdot 2^3=4000$.	$e^{(\log_e 2)t/4}=2^{t/4}$; substitute $t=12$.
For $P=8000$: $2^{t/4}=16=2^4$, so $t=16$ hours.	Interpret: time to reach a given level.

Example 2 — scaffolded

A metal bar at $100\,°\mathrm{C}$ is placed in a room kept at $20\,°\mathrm{C}$. After 5 minutes it has cooled to $60\,°\mathrm{C}$. Taking Newton's law of cooling $\frac{dT}{dt}=-k(T-20)$, find T after t minutes, the temperature after 10 minutes, and the long-term temperature.

Working	Comment
Let T be the temperature (°C) after t minutes: $\frac{dT}{dt}=-k(T-20)$, $k>0$.	Rate proportional to the temperature above the room.
Separate: $\int \frac{1}{T-20}\,dT=\int -k\,dt$, so $\log_e(T-20)=-kt+c\ (T>20)$.	Separation; the integrand is retained.
Then $T-20=Ae^{-kt}$, i.e. $T=20+Ae^{-kt}$. At $t=0$, $T=100$, so $A=80$.	Apply the initial temperature $T_0=100$.
At $t=5$, $T=60$: $60=20+80e^{-5k}$, so $e^{-5k}=\frac{1}{2}$ and $k=\frac{\log_e 2}{5}$.	The second condition fixes k.
$T=20+80e^{-(\log_e 2)t/5}=20+80\cdot 2^{-t/5}$. After 10 min, $T=20+80\cdot 2^{-2}=40$.	Temperature after 10 minutes is $40\,°\mathrm{C}$.
As $t\to\infty$, $2^{-t/5}\to 0$, so $T\to 20$: the bar approaches room temperature.	Long-term interpretation.



Example 3 — unscaffolded

A tank holds 200 litres of pure water. Brine containing 0.25 kg of salt per litre flows in at 8 L/min, and the well-stirred mixture flows out at 8 L/min. Find the mass Q of salt after t minutes, the long-term amount of salt, and the time taken to reach 25 kg.

Let Q be the mass of salt (kg) after t minutes. The inflow and outflow rates are equal, so the volume stays at 200 L. The salt enters at rate $\text{in} = 0.25 \times 8 = 2$ kg/min and leaves at rate $\text{out} = \frac{Q}{200} \times 8 = \frac{Q}{25}$ kg/min, so the balance law gives

$$\frac{dQ}{dt} = 2 - \frac{Q}{25} = \frac{50 - Q}{25}.$$

Separating the variables and integrating,

$$\int \frac{1}{50 - Q}\, dQ = \int \frac{1}{25}\, dt \quad -\log_e(50 - Q) = \frac{t}{25} + c.$$

At $t = 0$, $Q = 0$, so $-\log_e 50 = c$. Hence $\log_e(50 - Q) = \log_e 50 - \frac{t}{25}$, giving $50 - Q = 50e^{-t/25}$ and

$$Q = 50\left(1 - e^{-t/25}\right).$$

As $t \to \infty$, $e^{-t/25} \to 0$, so $Q \to 50$ kg — the steady state, which is also obtained directly by setting $\frac{dQ}{dt} = 0$. To reach 25 kg, $25 = 50\left(1 - e^{-t/25}\right)$ gives $e^{-t/25} = \frac{1}{2}$, so $t = 25\log_e 2 \approx 17.3$ min.

$\therefore Q = 50\left(1 - e^{-t/25}\right)$ kg, approaching 50 kg, and it reaches 25 kg after $25\log_e 2 \approx 17.3$ minutes.

Example 4 — unscaffolded

A body falls from rest under gravity, with air resistance proportional to its speed, so that its velocity v (m/s) after t seconds satisfies $\frac{dv}{dt} = 10 - 2v$. Find the terminal velocity, an expression for v, and the time taken to reach half the terminal velocity.

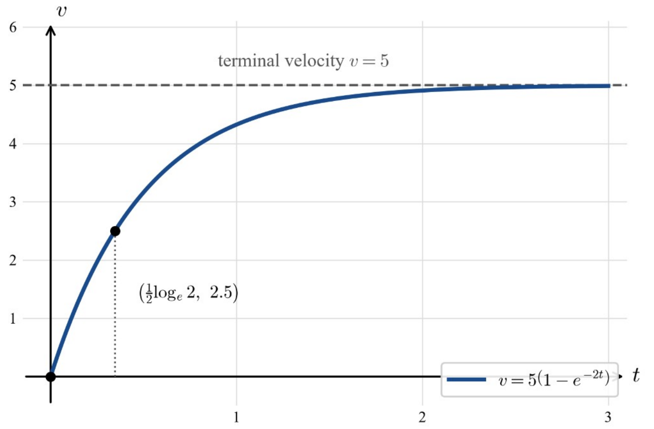
The terminal velocity is the constant speed at which the acceleration is zero. Setting $\frac{dv}{dt} = 0$ gives $10 - 2v = 0$, so $v = 5$ m/s. To find $v(t)$, separate the variables and integrate:

$$\int \frac{1}{10 - 2v}\, dv = \int 1\, dt \quad -\frac{1}{2}\log_e(10 - 2v) = t + c.$$

At $t = 0$, $v = 0$, so $-\frac{1}{2}\log_e 10 = c$. Then $\log_e(10 - 2v) = \log_e 10 - 2t$, giving $10 - 2v = 10e^{-2t}$ and

$$v = 5\left(1 - e^{-2t}\right).$$

As $t \to \infty$, $e^{-2t} \to 0$, so $v \to 5$ m/s, confirming the terminal velocity. Half of this is 2.5 m/s: $2.5 = 5\left(1 - e^{-2t}\right)$ gives $e^{-2t} = \frac{1}{2}$, so $t = \frac{\log_e 2}{2} \approx 0.35$ s.



$\therefore v = 5\left(1 - e^{-2t}\right)$ m/s, with terminal velocity 5 m/s, reached at half its value after $\frac{\log_e 2}{2} \approx 0.35$ seconds.

Practice questions

Scaffolded

Q1. A radioactive isotope decays at a rate proportional to the mass present: $\frac{dm}{dt} = -km$. Initially there are 80 g, and the half-life is 6 days (so 40 g remain after 6 days). (a) Separate and solve the DE for m in terms of t and k. (b) Use the half-life to find k. (c) Find the mass remaining after 18 days. (d) Find when 5 g remain.

Q2. A cup of coffee at $85\,°\mathrm{C}$ is placed in a room at $25\,°\mathrm{C}$; after 4 minutes it has cooled to $55\,°\mathrm{C}$. Model it with $\frac{dT}{dt} = -k(T - 25)$. (a) Solve the DE for T in terms of t, A and k. (b) Apply the conditions to find A and k. (c) Find the temperature after 12 minutes. (d) State the long-term temperature.

Q3. A tank holds 50 L of water in which 10 kg of dye is dissolved. Pure water enters at 2 L/min and the well-stirred mixture leaves at 2 L/min. (a) Explain why $\frac{dQ}{dt} = -\frac{Q}{25}$. (b) Separate and solve for Q. (c) Apply $Q(0) = 10$. (d) Find when 2.5 kg of dye remains.

Q4. A body falling from rest has velocity v (m/s) after t seconds satisfying $\frac{dv}{dt} = 10 - 4v$. (a) Find the terminal velocity by setting $\frac{dv}{dt} = 0$. (b) Separate and solve the DE for v. (c) Apply $v(0) = 0$. (d) Find the time taken to reach 2 m/s.

Q5. A population of fish grows so that $\frac{dP}{dt} = kP$. In the year $t = 0$ there were 3000 fish, and after 3 years there were 6000. (a) Solve the DE for P. (b) Find k. (c) Predict the population after 9 years. (d) Find when the population first reaches 48000.

Q6. A cold drink at $5\,°\mathrm{C}$ is placed in a room at $25\,°\mathrm{C}$; after 10 minutes it has warmed to $15\,°\mathrm{C}$. Newton's law still applies: $\frac{dT}{dt} = -k(T-25)$. (a) Solve the DE for T. (b) Apply the conditions (note A is negative here). (c) Find the temperature after 20 minutes. (d) State the long-term temperature.

Unscaffolded

Q7. The number of cells N satisfies $\frac{dN}{dt} = kN$. There are 200 cells initially and 800 after 4 hours. Find N in terms of t, and the number after 10 hours.

Q8. The mass of a decaying substance halves every 8 years; initially there are 12 g. Write down the DE model, solve for m, and find the mass after 24 years.

Q9. An object at $200\,°\mathrm{C}$ cools in air kept at $20\,°\mathrm{C}$; after 15 minutes it is at $110\,°\mathrm{C}$. Find the temperature after 30 minutes and the long-term temperature.

Q10. A body cools from $80\,°\mathrm{C}$ to $50\,°\mathrm{C}$ in 12 minutes in surroundings at $20\,°\mathrm{C}$, with $\frac{dT}{dt} = -k(T-20)$. Find k, and how long after the start it takes to reach $35\,°\mathrm{C}$.

Q11. A vat holds 400 L of brine containing 25 kg of salt. Pure water enters at 8 L/min and the well-stirred mixture leaves at 8 L/min. Find Q in terms of t, and the time until 5 kg of salt remains.

Q12. A tank holds 300 L of pure water. Brine of concentration 0.2 kg/L enters at 6 L/min, and the well-stirred mixture leaves at 6 L/min. Find Q in terms of t, and the long-term amount of salt in the tank.

Q13. A falling object has velocity v (m/s) satisfying $\frac{dv}{dt} = 10 - 5v$, with $v(0) = 0$. Find the terminal velocity, an expression for v, and the time to reach 1 m/s.

Q14. A boat's engine is switched off, and its velocity satisfies $\frac{dv}{dt} = -\frac{1}{2}v$ with $v(0) = 8$ m/s. Find v in terms of t, and how long it takes to slow to 2 m/s.

Q15. A colony grows at a rate proportional to its size, from 1500 to 4500 in 5 hours. Find the doubling time and the size of the colony after 10 hours.

Q16. A body is discovered at $27\,°\mathrm{C}$ in a room kept at $17\,°\mathrm{C}$; one hour later it has cooled to $22\,°\mathrm{C}$. Cooling follows $\frac{dT}{dt} = -k(T-17)$, with t measured in hours from discovery. Assuming the body was at $37\,°\mathrm{C}$ at the moment of death, find how long before discovery death occurred.

Q17. A tank holds 100 L of pure water. Salt solution of concentration 0.25 kg/L enters at 5 L/min, and the well-stirred mixture leaves at 5 L/min. Find Q in terms of t, the long-term concentration of salt, and the time for the concentration to reach 0.2 kg/L.

Q18. A skydiver's speed v (m/s) satisfies $\frac{dv}{dt} = 10 - \frac{v}{5}$, with $v(0) = 0$. Find the terminal velocity, an expression for v, and the time taken to reach 45 m/s (that is, 90% of the terminal velocity).

Answers

Q1. $m = 80\,e^{-kt} = 80 \cdot 2^{-t/6}$ with $k = \dfrac{\log_e 2}{6}$; after 18 days, $m = 10$ g; 5 g remain after 24 days. **Q2.** $T = 25 + 60 \cdot 2^{-t/4}$ with $A = 60$, $k = \dfrac{\log_e 2}{4}$; after 12 min, $T = 32.5\,°\mathrm{C}$; long-term $25\,°\mathrm{C}$. **Q3.** $Q = 10\,e^{-t/25}$; 2.5 kg remains after $50 \log_e 2 = 25 \log_e 4 \approx 34.7$ min. **Q4.** terminal velocity 2.5 m/s; $v = 2.5\left(1 - e^{-4t}\right)$; reaches 2 m/s after $\dfrac{\log_e 5}{4} \approx 0.40$ s. **Q5.** $P = 3000 \cdot 2^{t/3}$ with $k = \dfrac{\log_e 2}{3}$; after 9 years, 24000; reaches 48000 after 12 years. **Q6.** $T = 25 - 20 \cdot 2^{-t/10}$ with $A = -20$, $k = \dfrac{\log_e 2}{10}$; after 20 min, $T = 20\,°\mathrm{C}$; long-term $25\,°\mathrm{C}$. **Q7.** $N = 200 \cdot 2^{t/2}$ with $k = \dfrac{\log_e 2}{2}$; after 10 hours, 6400. **Q8.** $\dfrac{dm}{dt} = -km$, $k = \dfrac{\log_e 2}{8}$; $m = 12 \cdot 2^{-t/8}$; after 24 years, 1.5 g. **Q9.** $T = 20 + 180 \cdot 2^{-t/15}$; after 30 min, $T = 65\,°\mathrm{C}$; long-term $20\,°\mathrm{C}$. **Q10.** $k = \dfrac{\log_e 2}{12}$; reaches $35\,°\mathrm{C}$ after 24 minutes. **Q11.** $Q = 25\,e^{-t/50}$; 5 kg remains after $50 \log_e 5 \approx 80.5$ min. **Q12.** $Q = 60\left(1 - e^{-t/50}\right)$; long-term 60 kg. **Q13.** terminal velocity 2 m/s; $v = 2\left(1 - e^{-5t}\right)$; reaches 1 m/s after $\dfrac{\log_e 2}{5} \approx 0.14$ s. **Q14.** $v = 8\,e^{-t/2}$; slows to 2 m/s after $4 \log_e 2 = 2 \log_e 4 \approx 2.77$ s. **Q15.** doubling time $\dfrac{5 \log_e 2}{\log_e 3} \approx 3.15$ hours; after 10 hours, 13500. **Q16.** $k = \log_e 2$; the body was at $37\,°\mathrm{C}$ one hour before discovery, so death occurred about 1 hour before it was found. **Q17.** $Q = 25\left(1 - e^{-t/20}\right)$; long-term concentration 0.25 kg/L; the concentration reaches 0.2 kg/L after $20 \log_e 5 \approx 32.2$ min. **Q18.** terminal velocity 50 m/s; $v = 50\left(1 - e^{-t/5}\right)$; reaches 45 m/s after $5 \log_e 10 \approx 11.5$ s.

Fully-worked solutions

Q1. (a) With $\dfrac{dm}{dt} = -km$, separate and integrate:

$$\int \frac{1}{m}\,dm = \int -k\,dt \quad \log_e m = -kt + c,$$

so $m = A\,e^{-kt}$. At $t = 0$, $m = 80$, so $A = 80$ and $m = 80\,e^{-kt}$. (b) Half-life 6 days: $m(6) = 40$, so $40 = 80\,e^{-6k}$, giving $e^{-6k} = \dfrac{1}{2}$ and $k = \dfrac{\log_e 2}{6}$. Then $m = 80 \cdot 2^{-t/6}$. (c) After 18 days, $m = 80 \cdot 2^{-3} = \dfrac{80}{8} = 10$ g. (d) For $m = 5$: $2^{-t/6} = \dfrac{5}{80} = \dfrac{1}{16} = 2^{-4}$, so $\dfrac{t}{6} = 4$ and $t = 24$ days.

Q2. (a) Separating $\dfrac{dT}{dt} = -k(T - 25)$,

$$\int \frac{1}{T - 25}\,dT = \int -k\,dt \quad \log_e(T - 25) = -kt + c,$$

so $T = 25 + A\,e^{-kt}$. (b) At $t = 0$, $T = 85$, so $A = 60$. At $t = 4$, $T = 55$: $55 = 25 + 60\,e^{-4k}$, so $e^{-4k} = \dfrac{1}{2}$ and $k = \dfrac{\log_e 2}{4}$; thus $T = 25 + 60 \cdot 2^{-t/4}$. (c) After 12 min, $T = 25 + 60 \cdot 2^{-3} = 25 + 7.5 = 32.5\,°\mathrm{C}$. (d) As $t \to \infty$, $2^{-t/4} \to 0$, so $T \to 25\,°\mathrm{C}$ (room temperature).

Q3. (a) Pure water enters, so $\text{rate in} = 0$. The volume stays at 50 L, and the mixture leaves at 2 L/min carrying concentration $\frac{Q}{50}$, so $\text{rate out} = \frac{Q}{50} \times 2 = \frac{Q}{25}$. Hence $\frac{dQ}{dt} = 0 - \frac{Q}{25} = -\frac{Q}{25}$. (b) Separating,

$$\int \frac{1}{Q}\, dQ = \int -\frac{1}{25}\, dt \quad \log_e Q = -\frac{t}{25} + c,$$

so $Q = A e^{-t/25}$. (c) $Q(0) = 10$ gives $A = 10$, so $Q = 10 e^{-t/25}$. (d) For $Q = 2.5$: $e^{-t/25} = \frac{2.5}{10} = \frac{1}{4}$, so $-\frac{t}{25} = \log_e \frac{1}{4} = -\log_e 4$, giving $t = 25 \log_e 4 = 50 \log_e 2 \approx 34.7$ min.

Q4. (a) Setting $\frac{dv}{dt} = 0$: $10 - 4v = 0$, so $v = 2.5$ m/s (terminal velocity). (b) Separating,

$$\int \frac{1}{10 - 4v}\, dv = \int 1\, dt \quad -\frac{1}{4} \log_e(10 - 4v) = t + c.$$

(c) At $t = 0$, $v = 0$, so $-\frac{1}{4} \log_e 10 = c$. Then $\log_e(10 - 4v) = \log_e 10 - 4t$, giving $10 - 4v = 10 e^{-4t}$ and $v = 2.5(1 - e^{-4t})$. (d) For $v = 2$: $2 = 2.5(1 - e^{-4t})$, so $e^{-4t} = \frac{1}{5}$ and $t = \frac{\log_e 5}{4} \approx 0.40$ s.

Q5. (a) $\frac{dP}{dt} = kP$ separates to

$$\int \frac{1}{P}\, dP = \int k\, dt \quad \log_e P = kt + c,$$

so $P = A e^{kt}$; with $P(0) = 3000$, $P = 3000 e^{kt}$. (b) $P(3) = 6000$: $6000 = 3000 e^{3k}$, so $e^{3k} = 2$ and $k = \frac{\log_e 2}{3}$; hence $P = 3000 \cdot 2^{t/3}$. (c) After 9 years, $P = 3000 \cdot 2^3 = 24000$. (d) For $P = 48000$: $2^{t/3} = 16 = 2^4$, so $\frac{t}{3} = 4$ and $t = 12$ years.

Q6. (a) As in Q2, $T = 25 + A e^{-kt}$. (b) At $t = 0$, $T = 5$, so $A = 5 - 25 = -20$. At $t = 10$, $T = 15$: $15 = 25 - 20 e^{-10k}$, so $-10 = -20 e^{-10k}$, giving $e^{-10k} = \frac{1}{2}$ and $k = \frac{\log_e 2}{10}$; thus $T = 25 - 20 \cdot 2^{-t/10}$. (c) After 20 min, $T = 25 - 20 \cdot 2^{-2} = 25 - 5 = 20\,°\text{C}$. (d) As $t \to \infty$, $2^{-t/10} \to 0$, so $T \to 25\,°\text{C}$. The drink warms up towards room temperature.

Q7. Separating $\frac{dN}{dt} = kN$ gives

$$\int \frac{1}{N}\, dN = \int k\, dt \quad \log_e N = kt + c,$$

so $N = A e^{kt}$ with $A = N(0) = 200$. From $N(4) = 800$: $800 = 200 e^{4k}$, so $e^{4k} = 4$ and $k = \frac{\log_e 4}{4} = \frac{\log_e 2}{2}$; hence $N = 200 \cdot 2^{t/2}$. After 10 hours, $N = 200 \cdot 2^5 = 6400$.

Q8. "Rate of decay proportional to the amount" gives $\frac{dm}{dt} = -km$, so as in Q1, $m = A e^{-kt}$ with $A = 12$. Halving every 8 years means $m(8) = 6$: $6 = 12 e^{-8k}$, so $e^{-8k} = \frac{1}{2}$ and $k = \frac{\log_e 2}{8}$; thus $m = 12 \cdot 2^{-t/8}$. After 24 years, $m = 12 \cdot 2^{-3} = \frac{12}{8} = 1.5$ g.

Q9. With $\frac{dT}{dt} = -k(T-20)$, separation gives $T = 20 + Ae^{-kt}$; at $t=0$, $T=200$, so $A=180$. From $T(15)=110$: $110 = 20 + 180e^{-15k}$, so $90 = 180e^{-15k}$, giving $e^{-15k} = \frac{1}{2}$ and $k = \frac{\log_e 2}{15}$; hence $T = 20 + 180 \cdot 2^{-t/15}$. After 30 min, $T = 20 + 180 \cdot 2^{-2} = 20 + 45 = 65\,°\mathrm{C}$. As $t \to \infty$, $T \to 20\,°\mathrm{C}$.

Q10. As before, $T = 20 + Ae^{-kt}$ with $A = 80 - 20 = 60$. From $T(12) = 50$: $50 = 20 + 60e^{-12k}$, so $30 = 60e^{-12k}$, giving $e^{-12k} = \frac{1}{2}$ and $k = \frac{\log_e 2}{12}$; hence $T = 20 + 60 \cdot 2^{-t/12}$. For $T = 35$: $35 = 20 + 60 \cdot 2^{-t/12}$, so $2^{-t/12} = \frac{15}{60} = \frac{1}{4} = 2^{-2}$, giving $\frac{t}{12} = 2$ and $t = 24$ minutes.

Q11. Pure water enters, so rate in $= 0$; the volume stays at 400 L and the mixture leaves at 8 L/min, so rate out $= \frac{Q}{400} \times 8 = \frac{Q}{50}$. Thus $\frac{dQ}{dt} = -\frac{Q}{50}$, and separating,

$$\int \frac{1}{Q}\,dQ = \int -\frac{1}{50}\,dt \quad \log_e Q = -\frac{t}{50} + c,$$

so $Q = 25e^{-t/50}$ (using $Q(0) = 25$). For $Q = 5$: $e^{-t/50} = \frac{1}{5}$, so $t = 50\log_e 5 \approx 80.5$ min.

Q12. The salt enters at rate in $= 0.2 \times 6 = 1.2$ kg/min, and leaves at rate out $= \frac{Q}{300} \times 6 = \frac{Q}{50}$, so

$$\frac{dQ}{dt} = 1.2 - \frac{Q}{50} = \frac{60 - Q}{50}.$$

Separating,

$$\int \frac{1}{60 - Q}\,dQ = \int \frac{1}{50}\,dt \quad -\log_e(60 - Q) = \frac{t}{50} + c,$$

and $Q(0) = 0$ gives $c = -\log_e 60$. Then $60 - Q = 60e^{-t/50}$, so $Q = 60\left(1 - e^{-t/50}\right)$. As $t \to \infty$, $Q \to 60$ kg (equivalently, setting $\frac{dQ}{dt} = 0$ gives $Q = 60$).

Q13. Terminal velocity: $10 - 5v = 0$ gives $v = 2$ m/s. Separating,

$$\int \frac{1}{10 - 5v}\,dv = \int 1\,dt \quad -\frac{1}{5}\log_e(10 - 5v) = t + c.$$

With $v(0) = 0$, $c = -\frac{1}{5}\log_e 10$, so $\log_e(10 - 5v) = \log_e 10 - 5t$, giving $10 - 5v = 10e^{-5t}$ and $v = 2\left(1 - e^{-5t}\right)$. For $v = 1$: $e^{-5t} = \frac{1}{2}$, so $t = \frac{\log_e 2}{5} \approx 0.14$ s.

Q14. Separating $\frac{dv}{dt} = -\frac{1}{2}v$,

$$\int \frac{1}{v}\,dv = \int -\frac{1}{2}\,dt \quad \log_e v = -\frac{t}{2} + c,$$

so $v = Ae^{-t/2}$ with $A = v(0) = 8$; hence $v = 8e^{-t/2}$. For $v = 2$: $e^{-t/2} = \frac{1}{4}$, so $\frac{t}{2} = \log_e 4$ and $t = 2\log_e 4 = 4\log_e 2 \approx 2.77$ s.

Q15. $\frac{dP}{dt} = kP$ gives $P = Ae^{kt}$ with $A = 1500$. From $P(5) = 4500$: $4500 = 1500e^{5k}$, so $e^{5k} = 3$ and $k = \frac{\log_e 3}{5}$. The doubling time τ satisfies $e^{k\tau} = 2$, so $\tau = \frac{\log_e 2}{k} = \frac{5\log_e 2}{\log_e 3} \approx 3.15$ hours. After 10 hours, $P = 1500e^{10k} = 1500(e^{5k})^2 = 1500 \cdot 3^2 = 13500$.

Q16. With $\frac{dT}{dt} = -k(T - 17)$, separation gives $T = 17 + Ae^{-kt}$. Taking $t = 0$ at discovery, $T(0) = 27$, so $A = 10$. From $T(1) = 22$: $22 = 17 + 10e^{-k}$, so $5 = 10e^{-k}$, giving $e^{-k} = \frac{1}{2}$ and $k = \log_e 2$; thus $T = 17 + 10 \cdot 2^{-t}$. Setting $T = 37$ (temperature at death): $37 = 17 + 10 \cdot 2^{-t}$, so $2^{-t} = 2 = 2^1$, giving $-t = 1$ and $t = -1$. The negative time means the body was at $37\,°\text{C}$ one hour *before* discovery, so death occurred about 1 hour before the body was found.

Q17. The salt enters at rate $\text{in} = 0.25 \times 5 = 1.25$ kg/min and leaves at $\text{rate out} = \frac{Q}{100} \times 5 = \frac{Q}{20}$, so

$$\frac{dQ}{dt} = 1.25 - \frac{Q}{20} = \frac{25 - Q}{20}.$$

Separating,

$$\int \frac{1}{25 - Q}\,dQ = \int \frac{1}{20}\,dt \quad -\log_e(25 - Q) = \frac{t}{20} + c,$$

and $Q(0) = 0$ gives $c = -\log_e 25$. Then $25 - Q = 25e^{-t/20}$, so $Q = 25(1 - e^{-t/20})$. As $t \to \infty$, $Q \to 25$ kg, so the long-term concentration is $\frac{25}{100} = 0.25$ kg/L (the same as the inflow). The concentration is $\frac{Q}{100}$; it reaches 0.2 kg/L when $Q = 20$: $20 = 25(1 - e^{-t/20})$ gives $e^{-t/20} = \frac{1}{5}$, so $t = 20\log_e 5 \approx 32.2$ min.

Q18. Terminal velocity: $10 - \frac{v}{5} = 0$ gives $v = 50$ m/s. Since $10 - \frac{v}{5} = \frac{50 - v}{5}$, separating,

$$\int \frac{5}{50 - v}\,dv = \int 1\,dt \quad -5\log_e(50 - v) = t + c.$$

With $v(0) = 0$, $c = -5\log_e 50$, so $\log_e(50 - v) = \log_e 50 - \frac{t}{5}$, giving $50 - v = 50e^{-t/5}$ and $v = 50(1 - e^{-t/5})$. For $v = 45$ (that is, $90\,\%$ of 50): $45 = 50(1 - e^{-t/5})$, so $e^{-t/5} = 0.1 = \frac{1}{10}$, giving $t = 5\log_e 10 \approx 11.5$ s.

Theory

A population that reproduces freely, with a growth rate proportional to its current size, obeys $\frac{dP}{dt} = kP$ and grows **exponentially** without bound. In reality no population can grow without limit: food, space and other resources place a ceiling on how large it can become. The **logistic model** modifies exponential growth so that the growth rate falls away as the population approaches this ceiling, producing growth that is rapid at first and then levels off.

The logistic differential equation.

Let $P = P(t)$ be the size of a population at time t. The **logistic differential equation** is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where $k > 0$ and $M > 0$ are constants. The constant k is the **relative growth rate** for small populations, and M is the **carrying capacity** — the largest population the environment can sustain. When P is small the bracket $1 - \frac{P}{M}$ is close to 1, so $\frac{dP}{dt} \approx kP$ and growth is nearly exponential; as P rises towards M the bracket shrinks towards 0, so the growth rate is throttled back.

Expanding the bracket and writing $r = \frac{k}{M}$ gives the equivalent **factored form**

$$\frac{dP}{dt} = \frac{k}{M}P(M - P) = rP(M - P),$$

which some problems state directly. The two forms describe the same model: reading off $k = rM$ and the carrying capacity M recovers the first form from the second.

Equilibrium (constant) solutions.

A constant population neither grows nor shrinks, so $\frac{dP}{dt} = 0$. Setting the right-hand side to zero,

$$kP\left(1 - \frac{P}{M}\right) = 0 \Rightarrow P = 0 \text{ or } P = M.$$

These two constant solutions are the **equilibria**. The equilibrium $P = 0$ is *unstable*: any small population moves away from it and grows. The equilibrium $P = M$ is *stable*: whether the population starts below or above M, it is drawn towards the carrying capacity as time passes.

Solving by separation of variables and partial fractions.

Because the right-hand side depends only on P, the equation separates. Dividing by $P\left(1 - \frac{P}{M}\right)$ and integrating,

$$\int \frac{1}{P\left(1 - \frac{P}{M}\right)}\, dP = \int k\, dt.$$

The integrand on the left is a rational function of P that must be split into **partial fractions** before it can be integrated. Writing the factor over a common denominator and decomposing,

$$\frac{1}{P\left(1-\frac{P}{M}\right)}=\frac{M}{P(M-P)}=\frac{1}{P}+\frac{1}{M-P},$$

since $\frac{1}{P}+\frac{1}{M-P}=\frac{(M-P)+P}{P(M-P)}=\frac{M}{P(M-P)}$; both numerators are 1. Integrating each simple piece (taking $0<P<M$, so that P and $M-P$ are positive and no modulus signs are needed),

$$\int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP=\int k\,dt \Rightarrow \log_e P-\log_e(M-P)=kt+c.$$

Combining the logarithms and exponentiating,

$$\log_e\frac{P}{M-P}=kt+c \Rightarrow \frac{P}{M-P}=e^c e^{kt}.$$

Taking reciprocals gives $\frac{M-P}{P}=\frac{M}{P}-1=Ae^{-kt}$, where $A=e^{-c}$ is a positive constant. Hence $\frac{M}{P}=1+Ae^{-kt}$, and solving for P gives the **logistic solution**

$$P(t)=\frac{M}{1+Ae^{-kt}}.$$

Finding the constants.

The constant A is fixed by the initial population. Putting $t=0$ in the solution, $P(0)=\frac{M}{1+A}$, so if $P(0)=P_0$ then

$$A=\frac{M-P_0}{P_0}.$$

A second condition (a known population P at some later time t) is then substituted into $P(t)=\frac{M}{1+Ae^{-kt}}$ and the resulting equation solved for k, usually giving an exact answer of the form $k=\frac{1}{t}\log_e(\cdots)$.

The sigmoid curve and the point of fastest growth.

For $0<P_0<M$ the solution is an increasing **S-shaped (sigmoid)** curve. As $t\to\infty$, $e^{-kt}\to 0$, so

$$P(t)\to\frac{M}{1+0}=M,$$

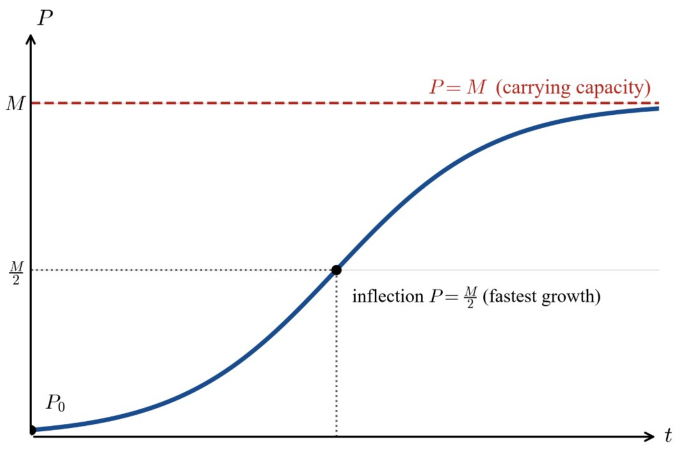
and the line $P=M$ is a horizontal asymptote — the population approaches the carrying capacity but never exceeds it. The curve is concave up while the population is small and concave down as it nears M, so it has a single **point of inflection**. Differentiating the logistic equation with respect to t (using the chain rule on P^2),

$$\frac{d^2P}{dt^2}=k\left(1-\frac{2P}{M}\right)\frac{dP}{dt},$$

which is zero, while $\frac{dP}{dt}\neq 0$, exactly when $P=\frac{M}{2}$. So the population is **growing fastest when it reaches half the carrying capacity**. The maximum growth rate there is

$$\left.\frac{dP}{dt}\right|_{P=M/2}=k\cdot\frac{M}{2}\left(1-\frac{1}{2}\right)=\frac{kM}{4}.$$

Below $P = \dfrac{M}{2}$ the growth rate is still increasing; above it the growth rate decreases as the population coasts towards M. The diagram shows the general shape: the initial value P_0, the carrying-capacity asymptote $P = M$, and the inflection at $P = \dfrac{M}{2}$.



Worked examples

Example 1 — scaffolded

A population of fish P (in thousands) in a lake is modelled by $\dfrac{dP}{dt} = 0.4P\left(1 - \dfrac{P}{50}\right)$, where t is measured in years. State the carrying capacity, the equilibrium populations, the population at which the stock grows fastest, and that maximum growth rate.

Working	Comment
Compare with $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$: here $k = 0.4$ and $M = 50$.	Read the relative growth rate k and the carrying capacity M from the equation.
Carrying capacity is $M = 50$ thousand fish.	The population levels off at M in the long term.
Equilibria where $\dfrac{dP}{dt} = 0$: $0.4P\left(1 - \dfrac{P}{50}\right) = 0$, so $P = 0$ or $P = 50$.	A constant population needs zero growth rate.
Fastest growth at $P = \dfrac{M}{2} = 25$ thousand fish.	The rate $kP\left(1 - \dfrac{P}{M}\right)$ is greatest halfway to the carrying capacity.
Maximum rate $= 0.4(25)\left(1 - \dfrac{25}{50}\right) = 5$ thousand fish per year.	Substitute $P = 25$; equivalently $\dfrac{kM}{4} = \dfrac{0.4 \times 50}{4} = 5$.

Example 2 — scaffolded

A colony of insects satisfies $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{800}\right)$ with $P(0) = 200$, where t is in days. Solve for $P(t)$, then find the population after 10 days (to the nearest insect) and the long-term population.

Working	Comment
Here $M = 800$ and $k = 0.2$. Separate the variables:	Put every P on the left, every t on the right.

Working	Comment
$\dfrac{1}{P\left(1-\dfrac{P}{800}\right)}dP=0.2\,dt$.	
Split into partial fractions: $\dfrac{1}{P\left(1-\dfrac{P}{800}\right)}=\dfrac{800}{P(800-P)}=\dfrac{1}{P}+\dfrac{1}{800-P}$.	Each factor contributes one term; both numerators are 1.
Integrate: $\int\left(\dfrac{1}{P}+\dfrac{1}{800-P}\right)dP=\int 0.2\,dt$, giving $\log_e P-\log_e(800-P)=0.2t+c$.	With $0<P<800$ both P and $800-P$ are positive, so no modulus is needed.
So $\log_e\dfrac{P}{800-P}=0.2t+c$ and $\dfrac{800-P}{P}=Ae^{-0.2t}$ with $A=e^{-c}$.	Exponentiate, then take reciprocals to isolate P.
Then $\dfrac{800}{P}=1+Ae^{-0.2t}$, so $P=\dfrac{800}{1+Ae^{-0.2t}}$; at $t=0$, $P=200$ gives $A=\dfrac{800-200}{200}=3$.	Use $A=\dfrac{M-P_0}{P_0}$.
$\therefore P(t)=\dfrac{800}{1+3e^{-0.2t}}$; then $P(10)=\dfrac{800}{1+3e^{-2}}\approx 569$ insects, and $P\to 800$ as $t\to\infty$.	The long-term population is the carrying capacity $M=800$.

Example 3 — unscaffolded

A herd of animals in a reserve satisfies $\dfrac{dP}{dt}=0.3P\left(1-\dfrac{P}{600}\right)$ with $P(0)=100$, where t is in years. Solve for $P(t)$ using separation of variables and partial fractions, and hence find the population after 5 years (to the nearest animal) and the limiting population.

Here $M=600$ and $k=0.3$. Separating the variables and decomposing the left-hand integrand into partial fractions,

$$\int\frac{1}{P\left(1-\frac{P}{600}\right)}dP=\int 0.3\,dt,\ \frac{1}{P\left(1-\frac{P}{600}\right)}=\frac{600}{P(600-P)}=\frac{1}{P}+\frac{1}{600-P}.$$

Integrating each side, with $0<P<600$ so that P and $600-P$ are positive,

$$\int\left(\frac{1}{P}+\frac{1}{600-P}\right)dP=\int 0.3\,dt\Rightarrow\log_e P-\log_e(600-P)=0.3t+c.$$

Combining and exponentiating gives $\dfrac{P}{600-P}=e^c e^{0.3t}$, and taking reciprocals, $\dfrac{600}{P}-1=Ae^{-0.3t}$ with $A=e^{-c}$. Hence

$$P(t)=\frac{600}{1+Ae^{-0.3t}}.$$

The initial condition $P(0)=100$ gives $A=\dfrac{600-100}{100}=5$. Then $P(5)=\dfrac{600}{1+5e^{-1.5}}\approx 284$ animals, and as $t\to\infty$, $e^{-0.3t}\to 0$, so $P\to 600$.

$\therefore P(t)=\dfrac{600}{1+5e^{-0.3t}}$, with $P(5)\approx 284$ and a limiting population of 600.

Example 4 — unscaffolded

A bacterial culture grows logistically with carrying capacity $M=1000$ (million cells), following $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{1000}\right)$. Initially $P(0)=100$ and after 2 hours $P(2)=250$. Find A and k, and the time at which the culture is growing fastest.

The general solution is $P(t)=\dfrac{1000}{1+Ae^{-kt}}$. The initial value $P(0)=100$ gives

$$A=\frac{1000-100}{100}=9.$$

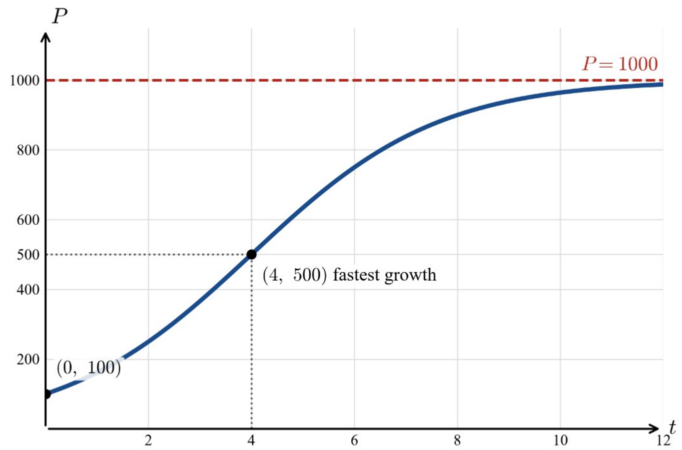
Using the second reading $P(2)=250$,

$$250=\frac{1000}{1+9e^{-2k}}\Rightarrow 1+9e^{-2k}=4\Rightarrow e^{-2k}=\frac{1}{3},$$

so $-2k=\log_e\dfrac{1}{3}=-\log_e 3$ and $k=\dfrac{1}{2}\log_e 3$. The culture grows fastest at the inflection $P=\dfrac{M}{2}=500$, where $Ae^{-kt}=1$; that is, $9e^{-kt}=1$, giving $e^{-kt}=\dfrac{1}{9}$ and

$$t=\frac{\log_e 9}{k}=\frac{2\log_e 3}{1/2\log_e 3}=4\,\text{hours}.$$

The curve below shows the model, rising from $P_0=100$ towards the asymptote $P=1000$ with its steepest point at $(4, 500)$.



$\therefore A=9$ and $k=\dfrac{1}{2}\log_e 3$, and the culture grows fastest after 4 hours, when $P=500$.

Practice questions

Scaffolded

Q1. A population follows $\dfrac{dP}{dt}=0.6P\left(1-\dfrac{P}{300}\right)$. (a) State the carrying capacity M. (b) Find the equilibrium populations. (c) State the population at which growth is fastest. (d) Find the maximum growth rate.

Q2. A population satisfies $\frac{dP}{dt}=0.25P\left(1-\frac{P}{500}\right)$ with $P(0)=100$. (a) Write down M and k. (b) Find $A=\frac{M-P_0}{P_0}$. (c) Write $P(t)$ in the form $\frac{M}{1+Ae^{-kt}}$. (d) State the limiting population.

Q3. Solve the general logistic equation $\frac{dP}{dt}=kP\left(1-\frac{P}{M}\right)$ by separation of variables. (a) Separate the variables to write an integral in P equal to an integral in t. (b) Show that $\frac{1}{P\left(1-\frac{P}{M}\right)}=\frac{1}{P}+\frac{1}{M-P}$. (c) Integrate both sides, taking $0<P<M$. (d) Rearrange to obtain $P=\frac{M}{1+Ae^{-kt}}$.

Q4. A population obeys $\frac{dP}{dt}=kP\left(1-\frac{P}{2000}\right)$ with $P(0)=250$ and $P(3)=1000$. (a) Find A. (b) Substitute $P(3)=1000$ into $P(t)=\frac{2000}{1+Ae^{-kt}}$. (c) Solve for k, giving an exact value. (d) State $P(t)$.

Q5. A logistic population is given by $P(t)=\frac{900}{1+8e^{-0.5t}}$. (a) Find the initial population. (b) State the limiting population. (c) State the population when growth is fastest. (d) Find the exact time at which growth is fastest.

Q6. The logistic equation is $\frac{dP}{dt}=kP\left(1-\frac{P}{M}\right)$. (a) Expand the right-hand side to $kP-\frac{k}{M}P^2$. (b) Differentiate with respect to t to show $\frac{d^2P}{dt^2}=k\left(1-\frac{2P}{M}\right)\frac{dP}{dt}$. (c) Explain why $\frac{d^2P}{dt^2}=0$, with $\frac{dP}{dt}\neq 0$, gives $P=\frac{M}{2}$. (d) State how the growth rate behaves for $P<\frac{M}{2}$ and for $P>\frac{M}{2}$.

Unscaffolded

Q7. For $\frac{dP}{dt}=0.8P\left(1-\frac{P}{750}\right)$, state the carrying capacity, the equilibrium populations, and the population at which growth is fastest.

Q8. Solve $\frac{dP}{dt}=0.4P\left(1-\frac{P}{1000}\right)$ with $P(0)=200$, giving $P(t)$ in the form $\frac{M}{1+Ae^{-kt}}$.

Q9. A logistic model gives $P(t)=\frac{450}{1+8e^{-0.2t}}$. Find the initial population and the limiting population.

Q10. A population is modelled by $P(t)=\frac{800}{1+7e^{-0.5t}}$. Find the exact time at which the population is growing fastest.

Q11. A population obeys $\frac{dP}{dt}=kP\left(1-\frac{P}{5000}\right)$ with $P(0)=500$ and $P(4)=2500$. Find A and the exact value of k.

Q12. Solve $\frac{dP}{dt}=0.1P\left(1-\frac{P}{400}\right)$ with $P(0)=50$ by separation of variables and partial fractions. Hence find $P(t)$ and the population when $t=20$ (to the nearest whole number).

Q13. A population satisfies $\frac{dP}{dt}=0.001P(400-P)$. Write this in the form $kP\left(1-\frac{P}{M}\right)$, stating k and M, and give the equilibrium populations and the population at which growth is fastest.

Q14. A population obeys $\frac{dP}{dt} = rP(M-P)$ with $r = 0.0005$ and carrying capacity $M = 2000$. Find $\frac{dP}{dt}$ when $P = 500$.

Q15. A population is modelled by $P(t) = \frac{600}{1+5e^{-0.4t}}$. Find the exact time at which $P = 400$, and give its value to two decimal places.

Q16. Solve $\frac{dP}{dt} = 0.002P(300-P)$ with $P(0) = 60$, giving $P(t)$ in the form $\frac{M}{1+Ae^{-kt}}$.

Q17. A logistic population has carrying capacity $M = 4000$ and relative growth rate $k = 0.5$. Find the population at which it grows fastest and the maximum value of $\frac{dP}{dt}$.

Q18. A bacterial population grows logistically with carrying capacity 1000, initial value $P(0) = 100$ and $P(1) = 250$. Find A and the exact value of k; find the population and the exact time when growth is fastest; and state the limiting population.

Answers

Q1. $M=300$; equilibria $P=0$ and $P=300$; fastest at $P=150$; maximum rate 45. **Q2.** $M=500$, $k=0.25$, $A=4$; $P(t)=\dfrac{500}{1+4e^{-0.25t}}$; limiting population 500. **Q3.** $P=\dfrac{M}{1+Ae^{-kt}}$ (derivation). **Q4.** $A=7$, $k=\dfrac{1}{3}\log_e 7$; $P(t)=\dfrac{2000}{1+7e^{-kt}}$. **Q5.** $P_0=100$; limiting population 900; fastest at $P=450$ when $t=2\log_e 8=6\log_e 2$. **Q6.** $\dfrac{d^2P}{dt^2}=k\left(1-\dfrac{2P}{M}\right)\dfrac{dP}{dt}$; inflection (fastest growth) at $P=\dfrac{M}{2}$. **Q7.** $M=750$; equilibria $P=0$ and $P=750$; fastest at $P=375$. **Q8.** $A=4$; $P(t)=\dfrac{1000}{1+4e^{-0.4t}}$. **Q9.** $P_0=50$; limiting population 450. **Q10.** $t=2\log_e 7\approx 3.89$. **Q11.** $A=9$; $k=\dfrac{1}{4}\log_e 9=\dfrac{1}{2}\log_e 3$. **Q12.** $A=7$; $P(t)=\dfrac{400}{1+7e^{-0.1t}}$; $P(20)\approx 205$. **Q13.** $k=0.4$, $M=400$; equilibria $P=0$ and $P=400$; fastest at $P=200$. **Q14.** $\dfrac{dP}{dt}=375$. **Q15.** $t=\dfrac{5}{2}\log_e 10\approx 5.76$. **Q16.** $k=0.6$, $M=300$, $A=4$; $P(t)=\dfrac{300}{1+4e^{-0.6t}}$. **Q17.** Fastest at $P=2000$; maximum rate $\dfrac{kM}{4}=500$. **Q18.** $A=9$, $k=\log_e 3$; fastest at $P=500$ when $t=2$; limiting population 1000.

Fully-worked solutions

Q1. Comparing $\dfrac{dP}{dt}=0.6P\left(1-\dfrac{P}{300}\right)$ with the standard form gives $k=0.6$ and $M=300$. (a) The carrying capacity is $M=300$. (b) Setting $0.6P\left(1-\dfrac{P}{300}\right)=0$ gives the equilibria $P=0$ and $P=300$. (c) Growth is fastest at $P=\dfrac{M}{2}=150$. (d) The maximum rate is $\dfrac{kM}{4}=\dfrac{0.6\times 300}{4}=45$ (equivalently $0.6(150)(1-150/300)=45$).

Q2. (a) $M=500$ and $k=0.25$. (b) With $P_0=100$, $A=\dfrac{M-P_0}{P_0}=\dfrac{500-100}{100}=4$. (c) Substituting into the logistic solution, $P(t)=\dfrac{500}{1+4e^{-0.25t}}$. (d) As $t\to\infty$, $e^{-0.25t}\to 0$, so $P\to 500$; the limiting population is the carrying capacity 500.

Q3. (a) Dividing by $P\left(1-\dfrac{P}{M}\right)$ and integrating,

$$\int \frac{1}{P\left(1-\frac{P}{M}\right)}\,dP=\int k\,dt.$$

(b) Over a common denominator, $\dfrac{1}{P}+\dfrac{1}{M-P}=\dfrac{(M-P)+P}{P(M-P)}=\dfrac{M}{P(M-P)}$, and since $P\left(1-\dfrac{P}{M}\right)=\dfrac{P(M-P)}{M}$, we have $\dfrac{1}{P\left(1-\frac{P}{M}\right)}=\dfrac{M}{P(M-P)}=\dfrac{1}{P}+\dfrac{1}{M-P}$. (c) Integrating, with $0<P<M$,

$$\int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP=\int k\,dt \Rightarrow \log_e P-\log_e(M-P)=kt+c.$$

(d) Then $\log_e \frac{P}{M-P} = kt + c$, so $\frac{M-P}{P} = e^{-c} e^{-kt} = A e^{-kt}$ with $A = e^{-c}$. Hence $\frac{M}{P} = 1 + A e^{-kt}$, giving $P = \frac{M}{1 + A e^{-kt}}$.

Q4. (a) With $M = 2000$ and $P_0 = 250$, $A = \frac{2000 - 250}{250} = 7$. (b) The solution is $P(t) = \frac{2000}{1 + 7e^{-kt}}$; the condition $P(3) = 1000$ gives $1000 = \frac{2000}{1 + 7e^{-3k}}$. (c) So $1 + 7e^{-3k} = 2$, hence $7e^{-3k} = 1$ and $e^{-3k} = \frac{1}{7}$; then $-3k = \log_e \frac{1}{7} = -\log_e 7$, so $k = \frac{1}{3} \log_e 7$. (d) Therefore $P(t) = \frac{2000}{1 + 7e^{-kt}}$ with $k = \frac{1}{3} \log_e 7$.

Q5. (a) $P(0) = \frac{900}{1+8} = \frac{900}{9} = 100$. (b) As $t \to \infty$, $e^{-0.5t} \to 0$, so $P \to 900$; the limiting population is 900. (c) Comparing with $\frac{M}{1 + A e^{-kt}}$ gives $M = 900$, so growth is fastest at $P = \frac{M}{2} = 450$. (d) There $A e^{-kt} = 1$, i.e. $8e^{-0.5t} = 1$, so $e^{-0.5t} = \frac{1}{8}$ and $-0.5t = -\log_e 8$; hence $t = \frac{\log_e 8}{0.5} = 2\log_e 8 = 6\log_e 2 \approx 4.16$.

Q6. (a) Expanding, $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) = kP - \frac{k}{M}P^2$. (b) Differentiating with respect to t and using the chain rule on P^2,

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt} - \frac{2k}{M}P\frac{dP}{dt} = k\left(1 - \frac{2P}{M}\right)\frac{dP}{dt}.$$

(c) While the population is changing, $\frac{dP}{dt} \neq 0$, so $\frac{d^2P}{dt^2} = 0$ requires $1 - \frac{2P}{M} = 0$, that is $P = \frac{M}{2}$. This is the point of inflection. (d) For $P < \frac{M}{2}$ the factor $1 - \frac{2P}{M} > 0$, so $\frac{d^2P}{dt^2} > 0$ and the growth rate is increasing; for $P > \frac{M}{2}$ it is negative, so the growth rate is decreasing.

Q7. Comparing with the standard form, $k = 0.8$ and $M = 750$. The carrying capacity is 750; the equilibria are $P = 0$ and $P = 750$; and growth is fastest at $P = \frac{M}{2} = 375$.

Q8. Here $M = 1000$, $k = 0.4$ and $P_0 = 200$, so $A = \frac{1000 - 200}{200} = 4$. Separating variables and using $\frac{1}{P\left(1 - \frac{P}{1000}\right)} = \frac{1}{P} + \frac{1}{1000 - P}$,

$$\int \left(\frac{1}{P} + \frac{1}{1000 - P}\right) dP = \int 0.4\, dt \Rightarrow \log_e \frac{P}{1000 - P} = 0.4t + c,$$

which rearranges to $P(t) = \frac{1000}{1 + 4e^{-0.4t}}$.

Q9. Writing $P(t) = \frac{450}{1 + 8e^{-0.2t}}$ in the form $\frac{M}{1 + A e^{-kt}}$ gives $M = 450$ and $A = 8$. The initial population is $P(0) = \frac{450}{1+8} = 50$, and as $t \to \infty$ the limiting population is $M = 450$.

Q10. With $M=800$ and $A=7$, growth is fastest at $P=\frac{M}{2}=400$, where $A\,e^{-kt}=1$. So $7\,e^{-0.5t}=1$, giving $e^{-0.5t}=\frac{1}{7}$ and $-0.5t=-\log_e 7$; hence $t=\frac{\log_e 7}{0.5}=2\log_e 7\approx 3.89$.

Q11. With $M=5000$ and $P_0=500$, $A=\frac{5000-500}{500}=9$, so $P(t)=\frac{5000}{1+9e^{-kt}}$. The condition $P(4)=2500$ gives $2500=\frac{5000}{1+9e^{-4k}}$, so $1+9e^{-4k}=2$, hence $9e^{-4k}=1$ and $e^{-4k}=\frac{1}{9}$. Then $-4k=-\log_e 9$, so $k=\frac{1}{4}\log_e 9=\frac{1}{4}(2\log_e 3)=\frac{1}{2}\log_e 3$.

Q12. Here $M=400$, $k=0.1$ and $P_0=50$, so $A=\frac{400-50}{50}=7$. Separating variables and decomposing,

$$\int \frac{1}{P\left(1-\frac{P}{400}\right)}\,dP=\int 0.1\,dt\,,\quad \frac{1}{P\left(1-\frac{P}{400}\right)}=\frac{1}{P}+\frac{1}{400-P}\,,$$

so integrating (with $0<P<400$) gives $\log_e P-\log_e(400-P)=0.1t+c$, which rearranges to $P(t)=\frac{400}{1+7e^{-0.1t}}$. At $t=20$, $P(20)=\frac{400}{1+7e^{-2}}\approx 205$.

Q13. Factor out to match the standard form: $0.001P(400-P)=0.001\times 400\times P\left(1-\frac{P}{400}\right)=0.4P\left(1-\frac{P}{400}\right)$, so $k=0.4$ and $M=400$. The equilibria are $P=0$ and $P=400$, and growth is fastest at $P=\frac{M}{2}=200$.

Q14. Substituting $r=0.0005$, $M=2000$ and $P=500$ into $\frac{dP}{dt}=rP(M-P)$,

$$\frac{dP}{dt}=0.0005\times 500\times(2000-500)=0.0005\times 500\times 1500=375\,.$$

Q15. Set $\frac{600}{1+5e^{-0.4t}}=400$, so $1+5e^{-0.4t}=\frac{600}{400}=\frac{3}{2}$, giving $5e^{-0.4t}=\frac{1}{2}$ and $e^{-0.4t}=\frac{1}{10}$. Then $-0.4t=\log_e\frac{1}{10}=-\log_e 10$, so $t=\frac{\log_e 10}{0.4}=\frac{5}{2}\log_e 10\approx 5.76$.

Q16. Match to the standard form: $0.002P(300-P)=0.002\times 300\times P\left(1-\frac{P}{300}\right)=0.6P\left(1-\frac{P}{300}\right)$, so $k=0.6$ and $M=300$. With $P_0=60$, $A=\frac{300-60}{60}=4$. Separating variables and using $\frac{1}{P\left(1-\frac{P}{300}\right)}=\frac{1}{P}+\frac{1}{300-P}$,

$$\int\left(\frac{1}{P}+\frac{1}{300-P}\right)dP=\int 0.6\,dt\Rightarrow \log_e\frac{P}{300-P}=0.6t+c\,,$$

which rearranges to $P(t)=\frac{300}{1+4e^{-0.6t}}$.

Q17. Growth is fastest at $P=\frac{M}{2}=\frac{4000}{2}=2000$. The maximum growth rate is $\frac{kM}{4}=\frac{0.5\times4000}{4}=500$ (equivalently $0.5(2000)(1-2000/4000)=500$).

Q18. The solution is $P(t)=\frac{1000}{1+Ae^{-kt}}$ with $M=1000$. From $P(0)=100$, $A=\frac{1000-100}{100}=9$. Using $P(1)=250$,

$$250=\frac{1000}{1+9e^{-k}}\Rightarrow1+9e^{-k}=4\Rightarrow e^{-k}=\frac{1}{3},$$

so $k=\log_e 3$. Growth is fastest at $P=\frac{M}{2}=500$, where $9e^{-kt}=1$, i.e. $e^{-kt}=\frac{1}{9}$; then $t=\frac{\log_e 9}{k}=\frac{\log_e 9}{\log_e 3}=\frac{2\log_e 3}{\log_e 3}=2$. As $t\to\infty$, $P\to1000$, the limiting population.

Theory

Sections earlier in this chapter solved the special first-order equations $\frac{dy}{dx}=f(x)$ (integrate straight away) and $\frac{dy}{dx}=f(y)$ (invert and integrate). This section develops the general technique that covers both, and much more: **separation of variables**, which applies to any first-order differential equation whose right-hand side can be written as *a function of* x *multiplied by a function of* y.

Separable differential equations.

A first-order differential equation is **separable** if it can be written in the form

$$\frac{dy}{dx}=g(x)h(y),$$

that is, the derivative equals a function of x alone times a function of y alone. To **recognise** a separable equation, try to factor the right-hand side into an x-part and a y-part. For instance

$$\frac{dy}{dx}=x^2y^3,\ \frac{dy}{dx}=e^x\sin y,\ \frac{dy}{dx}=\frac{x+1}{y-2}$$

are all separable (the last has $g(x)=x+1$ and $h(y)=\frac{1}{y-2}$). The two simpler types are special cases: $\frac{dy}{dx}=f(x)$ takes $h(y)=1$, and $\frac{dy}{dx}=f(y)$ takes $g(x)=1$. By contrast

$$\frac{dy}{dx}=x+y,\ \frac{dy}{dx}=x^2+y^2,\ \frac{dy}{dx}=\sin(x+y)$$

cannot be separated, because each right-hand side mixes x and y in a way that does not factor; a different method would be required.

The separation method.

Suppose $\frac{dy}{dx}=g(x)h(y)$ with $h(y)\neq 0$. Divide both sides by $h(y)$ and multiply by dx to collect all the y's on the left and all the x's on the right:

$$\frac{1}{h(y)}\,dy=g(x)\,dx.$$

Now integrate both sides — the left with respect to y, the right with respect to x:

$$\int\frac{1}{h(y)}\,dy=\int g(x)\,dx+c.$$

A single constant of integration c is written on the right; a constant on each side would simply combine into one. The result is a relationship between x and y — the **general solution** — containing one arbitrary constant.

Why integrating both sides works.

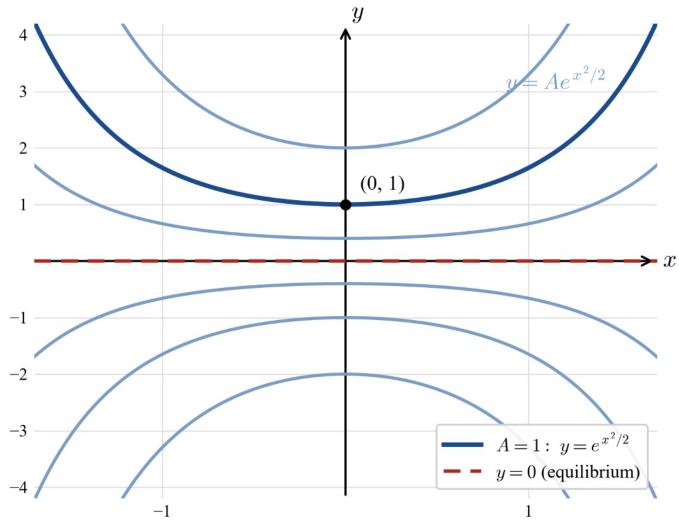
Treating $\frac{dy}{dx}$ as a ratio of dy and dx is a convenient shorthand; the step is justified by the chain rule. Starting from $\frac{1}{h(y)}\frac{dy}{dx}=g(x)$ and integrating both sides with respect to x,

$$\int \frac{1}{h(y)} \frac{dy}{dx} dx = \int g(x) dx.$$

On the left, $\frac{dy}{dx} dx$ is exactly the substitution that replaces the variable of integration by y, giving $\int \frac{1}{h(y)} dy$. So the informal "multiply across by dx" is a genuine change of variable, not a trick.

General and particular solutions.

After integrating, solve for y in terms of x where possible to obtain the **explicit** general solution $y = \ldots$; if y cannot be isolated cleanly, the **implicit** general solution (an equation relating x and y) is perfectly acceptable. Either way it carries one arbitrary constant and describes a whole **family** of solution curves. Given an initial or boundary **condition** — a known point (x_0, y_0) on the required curve — substitute it to evaluate the constant; this selects the single **particular solution** through that point. The figure below shows the family $y = A e^{x^2/2}$ — the general solution of $\frac{dy}{dx} = xy$, solved in Example 1 — one curve for each value of the constant A; a condition such as $y(0) = 1$ picks out the single member drawn in bold.



Equilibrium solutions and lost solutions.

Dividing by $h(y)$ is only valid where $h(y) \neq 0$. Any value $y = a$ for which $h(a) = 0$ makes the right-hand side zero, so $\frac{dy}{dx} = 0$ there and the constant function $y = a$ is itself a solution — an **equilibrium solution**. Because these constant solutions are removed when both sides are divided by $h(y)$, they can be **lost** during separation and must be checked separately. Often an equilibrium solution reappears as a limiting member of the general family (for example when the arbitrary constant is set to zero), and it can then be absorbed back into the general solution.

Integrals on the y-side: standard forms and partial fractions.

The x-side $\int g(x) dx$ is integrated with the usual rules. The y-side $\int \frac{1}{h(y)} dy$ may need a standard form, and when $\frac{1}{h(y)}$ is a rational function with distinct linear factors in its denominator, **partial fractions**. The forms that occur most often are

$\frac{1}{h(y)}$	$\int \frac{1}{h(y)} dy$
$\frac{1}{y}$	$\log_e \lvert y \rvert + c$

$\frac{1}{h(y)}$	$\int \frac{1}{h(y)}\,dy$
$\frac{1}{y^2}$	$-\frac{1}{y}+c$
$\frac{1}{a^2+y^2}$	$\frac{1}{a}\tan^{-1}\frac{y}{a}+c$
$\frac{1}{\sqrt{a^2-y^2}}$	$\sin^{-1}\frac{y}{a}+c$
$\frac{1}{(y-a)(y-b)}$	$\frac{1}{a-b}\log_e\left\lvert\frac{y-a}{y-b}\right\rvert+c$

The final row is obtained by writing $\frac{1}{(y-a)(y-b)}=\frac{1}{a-b}\left(\frac{1}{y-a}-\frac{1}{y-b}\right)$ and integrating each term. Choosing $h(y)$ from this list is what keeps a separable equation solvable by hand.

Method.

1. Confirm the equation is separable: write $\frac{dy}{dx}=g(x)h(y)$, identifying g and h.
2. Note any **equilibrium solutions** $y=a$ where $h(a)=0$.
3. Separate: $\frac{1}{h(y)}\,dy=g(x)\,dx$, then integrate both sides, $\int\frac{1}{h(y)}\,dy=\int g(x)\,dx+c$ (use partial fractions or a standard form on the y-side as needed).
4. This is the general solution; solve for y if an explicit form is wanted, and restore any lost equilibrium solution.
5. If a condition is given, substitute it to find c and state the particular solution.

Worked examples

Example 1 — scaffolded

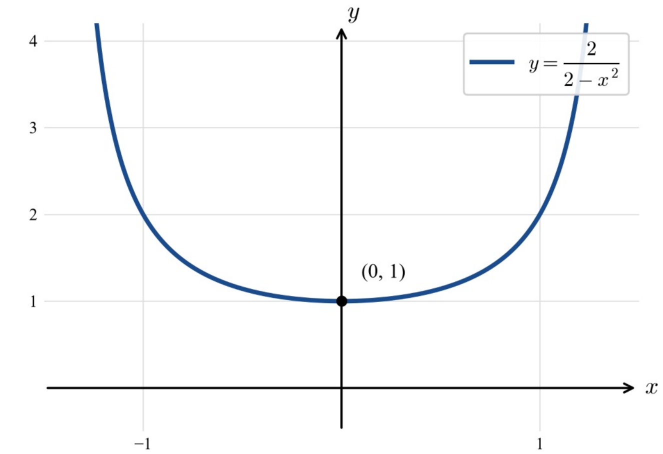
Find the general solution of $\frac{dy}{dx}=xy$, and state any equilibrium solution.

Working	Comment
The right side is $x\cdot y$: a function of x times a function of y, so the equation is separable with $g(x)=x$ and $h(y)=y$.	Check separability first.
$h(y)=y=0$ gives the equilibrium solution $y=0$.	Note it before dividing by $h(y)$.
Assuming $y\neq 0$, separate: $\frac{1}{y}\,dy=x\,dx$.	Collect y on the left, x on the right.
$\int\frac{1}{y}\,dy=\int x\,dx+c$	Integrate both sides; one constant c.
$\log_e\lvert y\rvert=\frac{x^2}{2}+c$	Standard antiderivatives.
$\lvert y\rvert=e^{x^2/2+c}=e^c e^{x^2/2}$, so $y=Ae^{x^2/2}$ with $A=\pm e^c$.	Exponentiate both sides.
Allowing $A=0$ recovers the lost solution $y=0$, so $y=Ae^{x^2/2}$ for any constant A.	The equilibrium is absorbed into the family.

Example 2 — scaffolded

Solve $\dfrac{dy}{dx}=xy^2$ given that $y=1$ when $x=0$.

Working	Comment
Separable with $g(x)=x$, $h(y)=y^2$; the equilibrium $y=0$ does not meet the condition $y(0)=1$.	Identify g, h and equilibria.
Assuming $y\neq 0$, separate: $\dfrac{1}{y^2}dy=x\,dx$.	Divide by y^2.
$\int y^{-2}dy=\int x\,dx+c$	Write $\dfrac{1}{y^2}=y^{-2}$.
$-\dfrac{1}{y}=\dfrac{x^2}{2}+c$	Integrate each side.
$y=1$ at $x=0$: $-\dfrac{1}{1}=0+c$, so $c=-1$.	Apply the condition to find c.
$-\dfrac{1}{y}=\dfrac{x^2}{2}-1=\dfrac{x^2-2}{2}$, so $y=\dfrac{-2}{x^2-2}=\dfrac{2}{2-x^2}$.	Solve for y (explicit form).
This is valid for $-\sqrt{2}<x<\sqrt{2}$, the interval containing $x=0$; $y\to\infty$ at both ends.	State where the particular solution holds.



Example 3 — unscaffolded

Solve $\dfrac{dy}{dx}=x\left(y^2-4\right)$ given that $y=0$ when $x=0$.

The equation is separable with $g(x)=x$ and $h(y)=y^2-4$, whose zeros give the equilibrium solutions $y=2$ and $y=-2$; neither meets $y(0)=0$. Separating and using partial fractions on the y-side, with $\dfrac{1}{y^2-4}=\dfrac{1}{(y-2)(y+2)}=\dfrac{1}{4}\left(\dfrac{1}{y-2}-\dfrac{1}{y+2}\right)$,

$$\int\frac{1}{y^2-4}dy=\int x\,dx+c\Rightarrow\frac{1}{4}\log_e\left|\frac{y-2}{y+2}\right|=\frac{x^2}{2}+c.$$

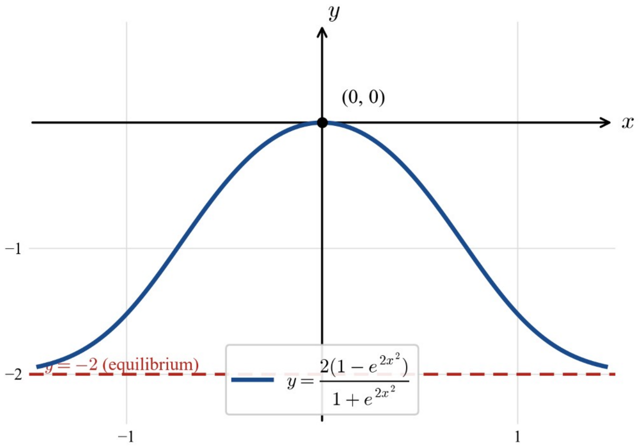
Applying $y=0$ at $x=0$ gives $\dfrac{1}{4}\log_e\left|\dfrac{-2}{2}\right|=\dfrac{1}{4}\log_e 1=0$, so $c=0$. Then $\log_e\left|\dfrac{y-2}{y+2}\right|=2x^2$, and since near the origin $\dfrac{y-2}{y+2}<0$,

$$\frac{y-2}{y+2}=-e^{2x^2}\Rightarrow y-2=-e^{2x^2}(y+2)\Rightarrow y\left(1+e^{2x^2}\right)=2\left(1-e^{2x^2}\right).$$

Hence the particular solution is

$$y=\frac{2\left(1-e^{2x^2}\right)}{1+e^{2x^2}}.$$

As $x\to 0$ this gives $y=0$, as required, and as $|x|$ grows it approaches the equilibrium value $y=-2$ from above.



$$\therefore y=\frac{2\left(1-e^{2x^2}\right)}{1+e^{2x^2}}.$$

Example 4 — unscaffolded

Solve $\dfrac{dy}{dx}=\dfrac{1+y^2}{1+x^2}$ given that $y=1$ when $x=0$.

Here $g(x)=\dfrac{1}{1+x^2}$ and $h(y)=1+y^2$. Since $1+y^2>0$ for every y, $h(y)$ is never zero and there are **no equilibrium solutions**. Separating the variables,

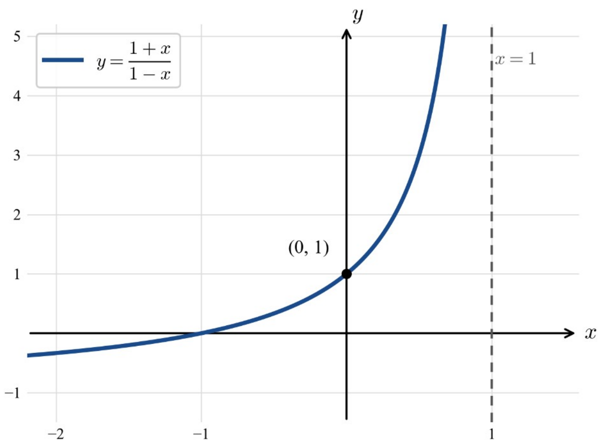
$$\int\frac{1}{1+y^2}\,dy=\int\frac{1}{1+x^2}\,dx+c,$$

and both sides are standard forms:

$$\tan^{-1}(y)=\tan^{-1}(x)+c.$$

The condition $y=1$ at $x=0$ gives $\tan^{-1}(1)=\tan^{-1}(0)+c$, so $c=\dfrac{\pi}{4}$. Therefore $\tan^{-1}(y)=\tan^{-1}(x)+\dfrac{\pi}{4}$, and taking the tangent of both sides with the identity $\tan\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{\tan\theta+1}{1-\tan\theta}$,

$$y=\tan\left(\tan^{-1}(x)+\frac{\pi}{4}\right)=\frac{x+1}{1-x}.$$



$\therefore y = \dfrac{1+x}{1-x}$ (valid for $x < 1$, the interval containing $x = 0$).

Practice questions

Scaffolded

Q1. Find the general solution of $\dfrac{dy}{dx} = 3x^2 y$. (a) Explain why the equation is separable, naming $g(x)$ and $h(y)$. (b) Separate the variables. (c) Integrate both sides. (d) State the general solution and the equilibrium solution.

Q2. Find the general solution of $\dfrac{dy}{dx} = \dfrac{2y}{x}$. (a) Separate the variables. (b) Integrate both sides. (c) State the general solution in the form $y = \ldots$.

Q3. Solve $\dfrac{dy}{dx} = \dfrac{x}{y}$ given that $y = 3$ when $x = 0$. (a) Separate the variables. (b) Integrate both sides to obtain an implicit relation between x and y. (c) Apply the condition and state y explicitly.

Q4. Solve $\dfrac{dy}{dx} = e^{x-y}$ given that $y = \log_e 2$ when $x = 0$. (a) Use an index law to write e^{x-y} as a function of x times a function of y. (b) Separate the variables and integrate both sides. (c) Apply the condition and state y explicitly.

Q5. Find the general solution of $\dfrac{dy}{dx} = y^2 - 4$. (a) State the two equilibrium solutions. (b) Separate the variables and write the y-side using partial fractions. (c) Integrate both sides. (d) State the general solution in implicit form.

Q6. Solve $\dfrac{dy}{dx} = 1 + y^2$ given that $y = 0$ when $x = 0$. (a) Explain why there are no equilibrium solutions. (b) Separate the variables and integrate both sides. (c) Apply the condition and state y explicitly.

Unscaffolded

Q7. Find the general solution of $\dfrac{dy}{dx} = -2xy$, and state the equilibrium solution.

Q8. Solve $\dfrac{dy}{dx} = 4x^3 y$ given that $y = 2$ when $x = 0$.

Q9. Find the general solution of $\dfrac{dy}{dx} = (2x+1)y$.

Q10. Solve $\dfrac{dy}{dx} = \dfrac{x}{y+1}$ given that $y = 1$ when $x = 0$.

Q11. Solve $\frac{dy}{dx}=\frac{3x^2}{2y}$ given that $y=2$ when $x=1$.

Q12. Solve $\frac{dy}{dx}=\frac{e^x}{y}$ given that $y=2$ when $x=0$.

Q13. Solve $\frac{dy}{dx}=y\cos x$ given that $y=1$ when $x=0$.

Q14. Find the general solution of $\frac{dy}{dx}=y^2-y$, and state the equilibrium solutions.

Q15. Solve $\frac{dy}{dx}=x\left(9-y^2\right)$ given that $y=0$ when $x=0$.

Q16. Solve $\frac{dy}{dx}=\frac{1+y^2}{x}$ given that $y=0$ when $x=1$.

Q17. Solve $\frac{dy}{dx}=xe^y$ given that $y=0$ when $x=0$.

Q18. (a) Explain why $\frac{dy}{dx}=x+y$ cannot be solved by separation of variables. (b) For $\frac{dy}{dx}=x(y-4)$, state the equilibrium solution and find the general solution.

Answers

Q1. $y = A e^{x^3}$; equilibrium $y = 0$. **Q2.** $y = A x^2$. **Q3.** $y = \sqrt{x^2 + 9}$. **Q4.** $y = \log_e(e^x + 1)$. **Q5.** $\frac{1}{4} \log_e \left| \frac{y-2}{y+2} \right| = x + c$; equilibria $y = 2$, $y = -2$. **Q6.** $y = \tan x$, valid for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **Q7.** $y = A e^{-x^2}$; equilibrium $y = 0$. **Q8.** $y = 2 e^{x^4}$. **Q9.** $y = A e^{x^2 + x}$. **Q10.** $y = \sqrt{x^2 + 4} - 1$. **Q11.** $y = \sqrt{x^3 + 3}$. **Q12.** $y = \sqrt{2 e^x + 2}$. **Q13.** $y = e^{\sin x}$. **Q14.** $\log_e \left| \frac{y-1}{y} \right| = x + c$, i.e. $y = \frac{1}{1 - A e^x}$; equilibria $y = 0$, $y = 1$. **Q15.** $y = \frac{3(e^{3x^2} - 1)}{e^{3x^2} + 1}$. **Q16.** $y = \tan(\log_e x)$, valid for $e^{-\pi/2} < x < e^{\pi/2}$. **Q17.** $y = \log_e \left(\frac{2}{2 - x^2} \right)$. **Q18.** (a) $x + y$ does not factor into a function of x times a function of y. (b) equilibrium $y = 4$; $y = 4 + A e^{x^2/2}$.

Fully-worked solutions

Q1. (a) The right side is $3x^2 \cdot y$, so the equation is separable with $g(x) = 3x^2$ and $h(y) = y$. (b) With $y \neq 0$, $\frac{1}{y} dy = 3x^2 dx$. (c) Integrating,

$$\int \frac{1}{y} dy = \int 3x^2 dx + c \Rightarrow \log_e |y| = x^3 + c.$$

(d) Exponentiating, $y = A e^{x^3}$ with $A = \pm e^c$; allowing $A = 0$ restores the equilibrium solution $y = 0$ (from $h(y) = y = 0$).

Q2. (a) With $y \neq 0$, $\frac{1}{y} dy = \frac{2}{x} dx$. (b) Integrating,

$$\int \frac{1}{y} dy = \int \frac{2}{x} dx + c \Rightarrow \log_e |y| = 2 \log_e |x| + c = \log_e x^2 + c.$$

(c) Exponentiating, $|y| = e^c x^2$, so $y = A x^2$ (with $A = \pm e^c$, and $A = 0$ giving the equilibrium $y = 0$).

Q3. (a) Writing $\frac{dy}{dx} = x \cdot \frac{1}{y}$ and separating, $y\, dy = x\, dx$. (b) Integrating,

$$\int y\, dy = \int x\, dx + c \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c,$$

so $y^2 = x^2 + C$ where $C = 2c$. (c) With $y = 3$ at $x = 0$, $9 = 0 + C$, so $C = 9$ and $y^2 = x^2 + 9$. Taking the branch through $(0, 3)$, $y = \sqrt{x^2 + 9}$.

Q4. (a) By the index law $e^{x-y} = e^x e^{-y}$, so $g(x) = e^x$ and $h(y) = e^{-y}$. (b) Separating, $e^y dy = e^x dx$, and integrating,

$$\int e^y dy = \int e^x dx + c \Rightarrow e^y = e^x + c.$$

(c) With $y = \log_e 2$ at $x = 0$: $e^{\log_e 2} = e^0 + c$, i.e. $2 = 1 + c$, so $c = 1$. Then $e^y = e^x + 1$, giving $y = \log_e(e^x + 1)$.

Q5. (a) $h(y) = y^2 - 4 = 0$ at $y = 2$ and $y = -2$, the equilibrium solutions. (b) With partial fractions $\frac{1}{y^2 - 4} = \frac{1}{4} \left(\frac{1}{y-2} - \frac{1}{y+2} \right)$, so $\frac{1}{y^2 - 4} dy = dx$. (c) Integrating,

$$\int \frac{1}{y^2-4}\,dy=\int 1\,dx+c \Rightarrow \frac{1}{4}\log_e\left|\frac{y-2}{y+2}\right|=x+c.$$

(d) This is the general solution in implicit form (the equilibria $y=\pm 2$ are the limiting members).

Q6. (a) $h(y)=1+y^2\geq 1>0$ for all y, so $h(y)$ is never zero and there are no constant (equilibrium) solutions. (b) Separating, $\frac{1}{1+y^2}\,dy=dx$, and

$$\int \frac{1}{1+y^2}\,dy=\int 1\,dx+c \Rightarrow \tan^{-1}(y)=x+c.$$

(c) With $y=0$ at $x=0$, $\tan^{-1}(0)=0+c$, so $c=0$ and $\tan^{-1}(y)=x$; hence $y=\tan x$. Since $\tan^{-1}(y)$ only takes values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, this solution is valid for $-\frac{\pi}{2}<x<\frac{\pi}{2}$, the interval containing $x=0$.

Q7. Separable with $g(x)=-2x$, $h(y)=y$; equilibrium $y=0$. With $y\neq 0$,

$$\int \frac{1}{y}\,dy=\int -2x\,dx+c \Rightarrow \log_e|y|=-x^2+c,$$

so $y=Ae^{-x^2}$ (with $A=0$ giving the equilibrium $y=0$).

Q8. Separating $\frac{dy}{dx}=4x^3y$ gives

$$\int \frac{1}{y}\,dy=\int 4x^3\,dx+c \Rightarrow \log_e|y|=x^4+c,$$

so $y=Ae^{x^4}$. With $y=2$ at $x=0$, $2=Ae^0=A$, hence $y=2e^{x^4}$.

Q9. Separating $\frac{dy}{dx}=(2x+1)y$ gives

$$\int \frac{1}{y}\,dy=\int (2x+1)\,dx+c \Rightarrow \log_e|y|=x^2+x+c,$$

so the general solution is $y=Ae^{x^2+x}$ (with $A=0$ the equilibrium $y=0$).

Q10. Writing $\frac{dy}{dx}=x\cdot\frac{1}{y+1}$ and separating, $(y+1)\,dy=x\,dx$, so

$$\int (y+1)\,dy=\int x\,dx+c \Rightarrow \frac{(y+1)^2}{2}=\frac{x^2}{2}+c,$$

giving $(y+1)^2=x^2+C$. With $y=1$ at $x=0$, $(2)^2=0+C$, so $C=4$ and $(y+1)^2=x^2+4$. The branch through $(0,1)$ has $y+1>0$, so $y=\sqrt{x^2+4}-1$.

Q11. Separating $\frac{dy}{dx}=\frac{3x^2}{2y}$ gives $2y\,dy=3x^2\,dx$, so

$$\int 2y\,dy=\int 3x^2\,dx+c \Rightarrow y^2=x^3+c.$$

With $y=2$ at $x=1$, $4=1+c$, so $c=3$ and $y^2=x^3+3$. The branch through $(1,2)$ is $y=\sqrt{x^3+3}$.

Q12. Writing $\frac{dy}{dx}=e^x\cdot\frac{1}{y}$ and separating, $y\,dy=e^x\,dx$, so

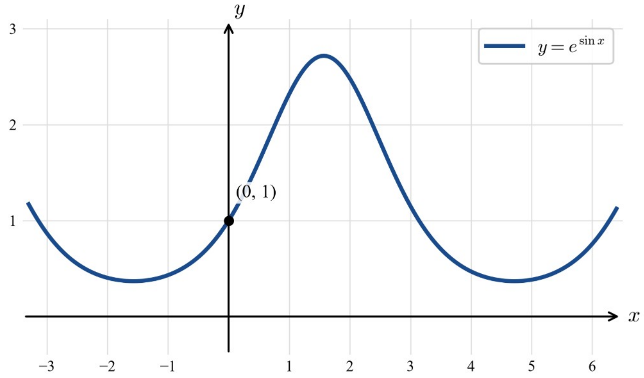
$$\int y\,dy = \int e^x\,dx + c \Rightarrow \frac{y^2}{2} = e^x + c.$$

With $y=2$ at $x=0$, $\frac{4}{2} = e^0 + c$, i.e. $2 = 1 + c$, so $c = 1$. Then $y^2 = 2e^x + 2$, and the branch through $(0,2)$ is $y = \sqrt{2e^x + 2}$.

Q13. Separable with $g(x) = \cos x$, $h(y) = y$; with $y \neq 0$,

$$\int \frac{1}{y}\,dy = \int \cos x\,dx + c \Rightarrow \log_e|y| = \sin x + c,$$

so $y = Ae^{\sin x}$. With $y=1$ at $x=0$, $1 = Ae^{\sin 0} = A$, hence $y = e^{\sin x}$.



Q14. $h(y) = y^2 - y = y(y-1) = 0$ at $y=0$ and $y=1$, the equilibrium solutions. With partial fractions $\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}$, so

$$\int \frac{1}{y^2 - y}\,dy = \int 1\,dx + c \Rightarrow \log_e\left|\frac{y-1}{y}\right| = x + c.$$

Exponentiating, $\frac{y-1}{y} = Ae^x$, and solving for y gives $y = \frac{1}{1 - Ae^x}$ (with the equilibria $y=0$, $y=1$ as limiting cases).

Q15. $h(y) = 9 - y^2$, equilibria $y = \pm 3$; the condition $y(0) = 0$ needs neither. With $\frac{1}{9 - y^2} = \frac{1}{6}\left(\frac{1}{3-y} + \frac{1}{3+y}\right)$,

$$\int \frac{1}{9 - y^2}\,dy = \int x\,dx + c \Rightarrow \frac{1}{6}\log_e\left|\frac{3+y}{3-y}\right| = \frac{x^2}{2} + c.$$

With $y=0$ at $x=0$, $\frac{1}{6}\log_e 1 = 0$, so $c = 0$. Then $\log_e\left|\frac{3+y}{3-y}\right| = 3x^2$, and since $-3 < y < 3$ near the origin, $\frac{3+y}{3-y} = e^{3x^2}$. Solving,

$$3 + y = e^{3x^2}(3 - y) \Rightarrow y\left(1 + e^{3x^2}\right) = 3\left(e^{3x^2} - 1\right) \Rightarrow y = \frac{3\left(e^{3x^2} - 1\right)}{e^{3x^2} + 1}.$$

Q16. Writing $\frac{dy}{dx} = \frac{1}{x}\left(1 + y^2\right)$ and separating, $\frac{1}{1+y^2}\,dy = \frac{1}{x}\,dx$, so

$$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{x}\,dx + c \Rightarrow \tan^{-1}(y) = \log_e|x| + c.$$

With $y=0$ at $x=1$, $\tan^{-1}(0)=\log_e 1+c$, so $c=0$. The condition is given at $x=1>0$, so $\log_e|x|=\log_e x$ and $\tan^{-1}(y)=\log_e x$, hence $y=\tan(\log_e x)$. Since $\tan^{-1}(y)$ only takes values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, this requires $-\frac{\pi}{2}<\log_e x<\frac{\pi}{2}$, so the particular solution is valid for $e^{-\pi/2}<x<e^{\pi/2}$ — the interval containing $x=1$, with $y\to\pm\infty$ at the two ends.

Q17. Writing $\frac{dy}{dx}=xe^{y}$ and separating, $e^{-y}\,dy=x\,dx$, so

$$\int e^{-y}\,dy=\int x\,dx+c\Rightarrow -e^{-y}=\frac{x^2}{2}+c.$$

With $y=0$ at $x=0$, $-e^{0}=0+c$, so $c=-1$. Then $-e^{-y}=\frac{x^2}{2}-1$, i.e. $e^{-y}=1-\frac{x^2}{2}=\frac{2-x^2}{2}$. Taking logarithms, $-y=\log_e\left(\frac{2-x^2}{2}\right)$, so $y=\log_e\left(\frac{2}{2-x^2}\right)$ (valid for $|x|<\sqrt{2}$).

Q18. (a) Separation requires the right side to factor as a function of x times a function of y. The expression $x+y$ is a **sum**, not such a product, and no algebra rewrites it as $g(x)h(y)$; the variables cannot be separated, so a different method is needed. (b) The equation $\frac{dy}{dx}=x(y-4)$ has $h(y)=y-4=0$ at $y=4$, the equilibrium solution. For $y\neq 4$,

$$\int\frac{1}{y-4}\,dy=\int x\,dx+c\Rightarrow \log_e|y-4|=\frac{x^2}{2}+c,$$

so $y-4=Ae^{x^2/2}$, giving $y=4+Ae^{x^2/2}$ (with $A=0$ the equilibrium $y=4$).

Theory

Many differential equations that arise in applications come from a **related-rates** situation: a quantity changes with time, and its rate is known in terms of *another* quantity through some geometry or physical law. The **chain rule** links the two rates into a single differential equation, which is then solved by **separation of variables**.

Combining rates with the chain rule.

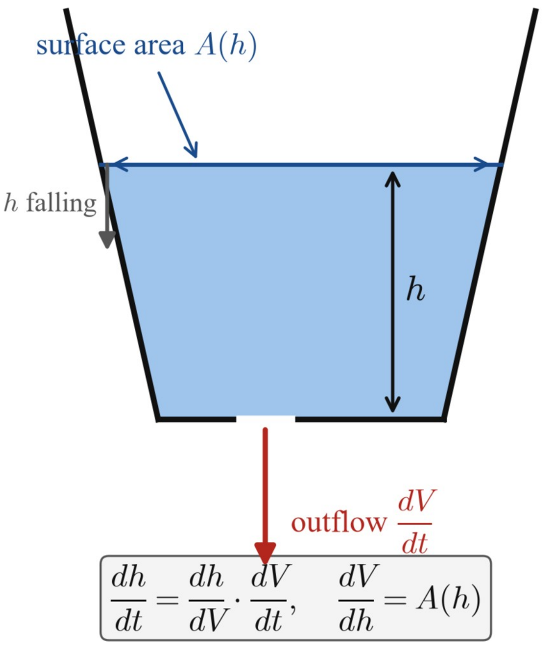
Suppose water drains from a tank and we want the depth h as a function of time t. Two facts are usually available. First, the shape of the tank relates the volume V of liquid to the depth h, and differentiating gives $\frac{dV}{dh}$; because a thin layer of liquid of thickness dh has volume (surface area)$\times dh$, this derivative is exactly the horizontal **cross-sectional area** at the surface,

$$\frac{dV}{dh}=A(h).$$

Second, the outflow gives the rate $\frac{dV}{dt}$ at which volume leaves the tank. The chain rule combines them into one equation for the depth:

$$\frac{dh}{dt}=\frac{dh}{dV}\cdot\frac{dV}{dt}=\frac{1}{dV/dh}\cdot\frac{dV}{dt}.$$

Since $\frac{dV}{dh}=A(h)$ and $\frac{dV}{dt}$ is known, the right-hand side is a function of h alone — a first-order differential equation $\frac{dh}{dt}=g(h)$.



Torricelli's law.

When a tank empties through a small hole in its base, the speed of the escaping liquid depends on the depth of liquid above the hole, and **Torricelli's law** states that the volume leaves at a rate proportional to $\sqrt{h}$,

$$\frac{dV}{dt} = -k\sqrt{h}\,(k>0),$$

the minus sign recording that the volume is decreasing. Combining this with $\frac{dV}{dh} = A(h)$ gives $\frac{dh}{dt} = -\frac{k\sqrt{h}}{A(h)}$; for a tank of constant cross-section A this is the separable equation $\frac{dh}{dt} = -\frac{k}{A}\sqrt{h}$.

Growing and shrinking spheres.

For a sphere, $V = \frac{4}{3}\pi r^3$ and the surface area is $S = 4\pi r^2$, so $\frac{dV}{dr} = 4\pi r^2 = S$. If the volume changes at a rate **proportional to the surface area**, $\frac{dV}{dt} = \pm kS = \pm k(4\pi r^2)$, then the chain rule gives

$$\frac{dr}{dt} = \frac{dV/dt}{dV/dr} = \frac{\pm k(4\pi r^2)}{4\pi r^2} = \pm k,$$

so the radius changes at a **constant rate** — the model for an evaporating raindrop, a shrinking mothball or a growing hailstone. If instead the volume changes at a constant rate Q (a balloon inflated by a pump), then $\frac{dr}{dt} = \frac{Q}{4\pi r^2}$, which is separable in r and t.

Solving by separation of variables.

Once the model has the form $\frac{dh}{dt} = g(h)$, separate the variables and integrate,

$$\int \frac{1}{g(h)}\,dh = \int 1\,dt,$$

add the constant $+c$ to the general solution, and then use the **initial or boundary condition** to find c for the particular solution. Finally answer the question asked — for a draining tank the **time to empty** is the value of t that makes $h=0$.

Method.

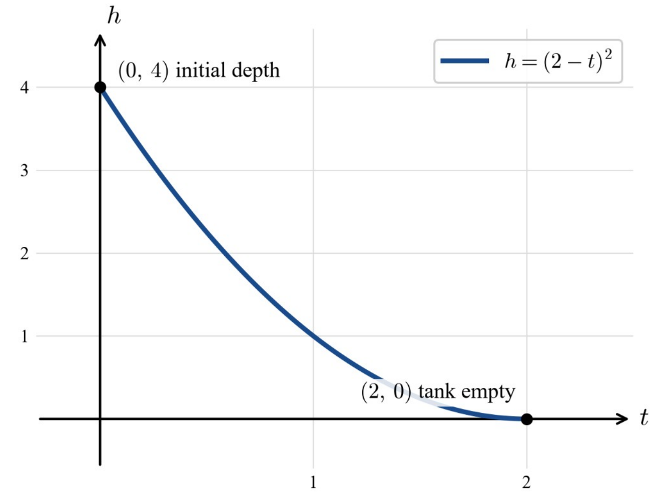
1. From the geometry, write V in terms of h (or r) and differentiate to get $\frac{dV}{dh} = A(h)$ (or $\frac{dV}{dr} = 4\pi r^2$).
2. Write down the given rate $\frac{dV}{dt}$ (Torricelli, constant, or proportional to surface area).
3. Combine with the chain rule $\frac{dh}{dt} = \frac{1}{dV/dh} \cdot \frac{dV}{dt}$ to obtain a differential equation in one variable.
4. Separate the variables and integrate, including $+c$.
5. Apply the condition to find c, then evaluate the quantity asked for.

Worked examples

Example 1 — scaffolded

A cylindrical tank has constant cross-sectional area 3 square metres. Water drains through a hole in its base under Torricelli's law, $\frac{dV}{dt} = -6\sqrt{h}$ cubic metres per minute, where h is the depth in metres. Initially the depth is 4 m. Find h as a function of t and the time for the tank to empty.

Working	Comment
For a uniform tank $V=3h$, so $\frac{dV}{dh}=3$.	The volume is area × depth; $\frac{dV}{dh}$ equals the cross-sectional area.
$\frac{dh}{dt}=\frac{1}{dV/dh}\cdot\frac{dV}{dt}=\frac{1}{3}(-6\sqrt{h})=-2\sqrt{h}$.	The chain rule combines the two rates into one differential equation.
$\int\frac{1}{\sqrt{h}}\,dh=\int-2\,dt\Rightarrow 2\sqrt{h}=-2t+c$.	Separate the variables and integrate; $+c$ on the general solution.
$h(0)=4$: $2\sqrt{4}=c$, so $c=4$. Then $2\sqrt{h}=4-2t$, giving $\sqrt{h}=2-t$ and $h=(2-t)^2$.	Apply the initial condition to fix c.
Empty when $h=0$: $(2-t)^2=0$, so $t=2$.	Set $h=0$ and solve for t.



The tank empties 2 minutes after draining begins.

Example 2 — scaffolded

A tank is shaped so that at depth h metres its horizontal surface has area $A(h)=2h$ square metres. Liquid is pumped out at the constant rate 4 cubic metres per minute, and the initial depth is 4 m. Find the depth after 3 minutes and the time for the tank to empty.

Working	Comment
$\frac{dV}{dh}=A(h)=2h$.	The rate of change of volume with depth is the surface area.
$\frac{dh}{dt}=\frac{1}{2h}(-4)=-\frac{2}{h}$.	Chain rule; the outflow rate is -4 as the volume falls.
$\int h\,dh=\int-2\,dt\Rightarrow\frac{h^2}{2}=-2t+c$.	Separate and integrate; keep $+c$.
$h(0)=4$: $\frac{16}{2}=c$, so $c=8$. Then $h^2=16-4t$ and $h=\sqrt{16-4t}$.	The initial condition fixes c; take the positive root since $h\geq 0$.
After 3 min: $h=\sqrt{16-12}=\sqrt{4}=2$ m. Empty when $h=0$: $16-4t=0$, so $t=4$.	Evaluate both quantities asked for.

After 3 minutes the depth is 2 m, and the tank is empty after 4 minutes.

Example 3 — unscaffolded

A spherical mothball evaporates so that its volume decreases at a rate proportional to its surface area; measurements give $\frac{dV}{dt} = -8\pi r^2$ cubic millimetres per minute, where r is the radius in millimetres. The initial radius is 10 mm. Find r as a function of t and the time for the mothball to disappear.

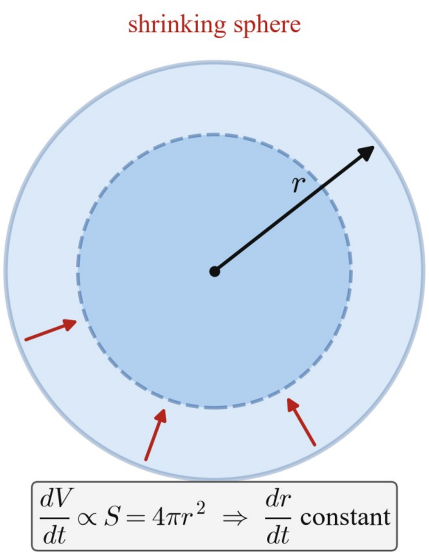
For a sphere $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, so $\frac{dV}{dr} = 4\pi r^2$. By the chain rule $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$. Setting this equal to the given rate, the factor $4\pi r^2$ cancels:

$$4\pi r^2 \frac{dr}{dt} = -8\pi r^2 \Rightarrow \frac{dr}{dt} = -2,$$

so the radius falls at the constant rate of 2 mm per minute. Separating and integrating,

$$\int 1\, dr = \int -2\, dt \Rightarrow r = -2t + c.$$

With $r(0) = 10$ the constant is $c = 10$, so $r = 10 - 2t$. The mothball disappears when $r = 0$, that is $10 - 2t = 0$, giving $t = 5$.



$\therefore r = 10 - 2t$, and the mothball has completely evaporated after 5 minutes.

Example 4 — unscaffolded

A spherical balloon is inflated with air at the constant rate of 8π cubic centimetres per second. When $t = 0$ the radius is 1 cm. Find the rate at which the radius is increasing when $r = 2$ cm, and the time for the radius to reach 4 cm.

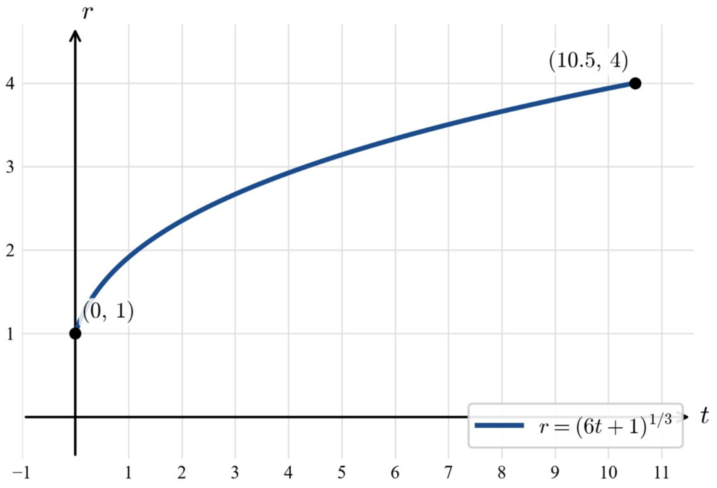
With $V = \frac{4}{3}\pi r^3$ we have $\frac{dV}{dr} = 4\pi r^2$, so the chain rule gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 8\pi \Rightarrow \frac{dr}{dt} = \frac{2}{r^2}.$$

When $r = 2$ this is $\frac{dr}{dt} = \frac{2}{2^2} = \frac{1}{2}$ centimetre per second. To find r as a function of t, separate the variables:

$$\int r^2\, dr = \int 2\, dt \Rightarrow \frac{r^3}{3} = 2t + c.$$

With $r(0)=1$ the constant is $c=\frac{1}{3}$, so $r^3=6t+1$ and $r=(6t+1)^{1/3}$. The radius reaches 4 cm when $r^3=64$, that is $6t+1=64$, giving $t=\frac{63}{6}=\frac{21}{2}$.



$\therefore \frac{dr}{dt}=\frac{1}{2}$ cm/s when $r=2$ cm, and the radius reaches 4 cm after 10.5 seconds.

Practice questions

Scaffolded

Q1. A cylindrical tank of constant cross-sectional area 2 square metres drains under Torricelli's law $\frac{dV}{dt}=-\sqrt{h}$ cubic metres per minute; initially $h=16$ m. (a) Write down $\frac{dV}{dh}$ and hence form the differential equation for $\frac{dh}{dt}$. (b) Separate the variables and integrate, including $+c$. (c) Use $h(0)=16$ to find c. (d) Find the time for the tank to empty.

Q2. A tank has horizontal surface area $A(h)=4h$ square metres at depth h, and is drained at the constant rate 16 cubic metres per minute; initially $h=8$ m. (a) State $\frac{dV}{dh}$ and form the differential equation for $\frac{dh}{dt}$. (b) Separate and integrate $(+c)$. (c) Apply the initial condition. (d) Find the time for the tank to empty.

Q3. A spherical mothball loses volume at a rate proportional to its surface area, $\frac{dV}{dt}=-k(4\pi r^2)$. Its radius falls from 6 mm to 4 mm in the first 2 days. (a) Using $V=\frac{4}{3}\pi r^3$, show that $\frac{dr}{dt}=-k$. (b) Solve for r, including $+c$, and apply $r(0)=6$. (c) Use the measurement to find k. (d) Find the time for the mothball to disappear completely.

Q4. A spherical balloon is inflated at the constant rate 12π cubic centimetres per second; at $t=0$ its radius is 1 cm. (a) Show that $\frac{dr}{dt}=\frac{3}{r^2}$. (b) Separate the variables and integrate $(+c)$. (c) Apply the initial condition to write r^3 in terms of t. (d) Find the time for the radius to reach 4 cm.

Q5. A cylindrical tank of cross-sectional area 5 square metres drains under Torricelli's law $\frac{dV}{dt}=-10\sqrt{h}$ cubic metres per minute; initially $h=9$ m. (a) Form the differential equation for $\frac{dh}{dt}$ and solve it, applying $h(0)=9$. (b) Write h as a function of t. (c) Find the time for the depth to fall to 1 m. (d) Find the time for the tank to empty.

Q6. A spherical hailstone grows so that its volume increases at a rate proportional to its surface area, $\frac{dV}{dt}=2\pi r^2$ cubic millimetres per minute; its initial radius is 2 mm. (a) Show that $\frac{dr}{dt}=\frac{1}{2}$. (b) Solve for r and apply the initial condition. (c) Find the time for the radius to reach 8 mm.

Unscaffolded

Q7. A cylindrical tank of cross-sectional area 2 square metres drains under Torricelli's law $\frac{dV}{dt}=-2\sqrt{h}$ cubic metres per minute. If the initial depth is 25 m, find the time for the tank to empty.

Q8. A tank has horizontal surface area $A(h)=6h$ square metres at depth h and is drained at the constant rate 12 cubic metres per minute; the initial depth is 6 m. Find the depth after 5 minutes and the time for the tank to empty.

Q9. A spherical balloon is inflated with gas at the constant rate 4π cubic centimetres per second. When its radius is 2 cm, find the further time for the radius to grow to 5 cm.

Q10. A spherical pill dissolves so that its volume decreases at a rate proportional to its surface area. Its radius falls from 8 mm to 5 mm in 6 minutes. Find the total time for the pill to dissolve completely.

Q11. A conical tank has its apex at the bottom and is shaped so that the water radius equals the water depth, $r=h$, giving $V=\frac{\pi}{3}h^3$. Water drains under Torricelli's law $\frac{dV}{dt}=-\pi\sqrt{h}$ cubic metres per minute. If the initial depth is 4 m, find the time for the tank to empty.

Q12. A hemispherical bowl of radius 3 m (flat face uppermost) is full of water. At depth h its horizontal surface has area $A(h)=\pi(6h-h^2)$ square metres, and water drains at the constant rate π cubic metres per minute. Form the differential equation for h and find the time for the bowl to empty.

Q13. A cylindrical tank of cross-sectional area 5 square metres drains under Torricelli's law $\frac{dV}{dt}=-k\sqrt{h}$. The depth falls from 9 m to 4 m during the first 5 minutes. Find the constant k and the total time for the tank to empty.

Q14. A spherical balloon is inflated at the constant rate 16π cubic centimetres per second; at $t=0$ its radius is 2 cm. (a) Find the rate at which the radius is increasing when $r=4$ cm. (b) Find the time for the radius to reach 8 cm.

Q15. A long trough is shaped so that $\frac{dV}{dh}=3h$ square metres at depth h. It drains under Torricelli's law $\frac{dV}{dt}=-3\sqrt{h}$ cubic metres per minute. If the initial depth is 9 m, find the time for the trough to empty.

Q16. A conical tank (apex at the bottom, $r=h$, so $V=\frac{\pi}{3}h^3$) is drained at the constant rate π cubic metres per minute. If the initial depth is 3 m, find the time for the tank to empty.

Q17. A tank is shaped so that its horizontal surface area at depth h is $A(h)=3h^2$ square metres. It is drained at the constant rate 6 cubic metres per minute, and the initial depth is 3 m. Find the time for the tank to empty.

Q18. A spherical hailstone grows so that its volume increases at a rate proportional to its surface area. Its radius grows from 1 mm to 4 mm in the first 3 minutes. Find the constant rate at which the radius increases, and the time for the radius to reach 10 mm.

Answers

Q1. $h=\left(4-\frac{t}{4}\right)^2$; the tank empties after 16 minutes. **Q2.** $h=\sqrt{64-8t}$; the tank empties after 8 minutes. **Q3.** $k=1$ mm/day; the mothball disappears after 6 days. **Q4.** $r^3=9t+1$; the radius reaches 4 cm after 7 seconds. **Q5.** $h=(3-t)^2$; depth 1 m after 2 minutes; empties after 3 minutes. **Q6.** $r=2+\frac{t}{2}$; the radius reaches 8 mm after 12 minutes. **Q7.** The tank empties after 10 minutes. **Q8.** Depth 4 m after 5 minutes; the tank empties after 9 minutes. **Q9.** 39 seconds. **Q10.** The pill dissolves after 16 minutes. **Q11.** The tank empties after $\frac{64}{5}=12.8$ minutes. **Q12.** $\frac{dh}{dt}=-\frac{1}{6h-h^2}$; the bowl empties after 18 minutes. **Q13.** $k=2$; the tank empties after 15 minutes. **Q14.** (a) $\frac{1}{4}$ cm/s; (b) 42 seconds. **Q15.** The trough empties after 18 minutes. **Q16.** The tank empties after 9 minutes. **Q17.** The tank empties after $\frac{9}{2}=4.5$ minutes. **Q18.** 1 mm/min; the radius reaches 10 mm after 9 minutes.

Fully-worked solutions

Q1. The tank is uniform, so $V=2h$ and $\frac{dV}{dh}=2$. The chain rule gives $\frac{dh}{dt}=\frac{1}{2}\left(-\sqrt{h}\right)=-\frac{1}{2}\sqrt{h}$. Separating,

$$\int\frac{1}{\sqrt{h}}\,dh=\int-\frac{1}{2}\,dt\Rightarrow 2\sqrt{h}=-\frac{1}{2}t+c.$$

Applying $h(0)=16$: $2\sqrt{16}=8=c$, so $2\sqrt{h}=8-\frac{1}{2}t$, that is $\sqrt{h}=4-\frac{t}{4}$ and $h=\left(4-\frac{t}{4}\right)^2$. The tank is empty when $h=0$: $4-\frac{t}{4}=0$, giving $t=16$ minutes.

Q2. Here $\frac{dV}{dh}=A(h)=4h$, so $\frac{dh}{dt}=\frac{1}{4h}(-16)=-\frac{4}{h}$. Separating,

$$\int h\,dh=\int-4\,dt\Rightarrow\frac{h^2}{2}=-4t+c.$$

With $h(0)=8$: $\frac{64}{2}=32=c$, so $h^2=64-8t$ and $h=\sqrt{64-8t}$. The tank empties when $h=0$: $64-8t=0$, giving $t=8$ minutes.

Q3. (a) Since $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dr}=4\pi r^2$, so $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$. Equating to $-k\left(4\pi r^2\right)$ and cancelling $4\pi r^2$ gives $\frac{dr}{dt}=-k$. (b) Separating, $\int 1\,dr=\int-k\,dt$, so $r=-kt+c$; with $r(0)=6$, $c=6$ and $r=6-kt$. (c) The radius falls from 6 to 4 in 2 days: $4=6-2k$, so $k=1$ mm/day, and $r=6-t$. (d) The mothball disappears when $r=0$: $6-t=0$, giving $t=6$ days.

Q4. (a) With $\frac{dV}{dr}=4\pi r^2$, the chain rule gives $4\pi r^2\frac{dr}{dt}=12\pi$, so $\frac{dr}{dt}=\frac{12\pi}{4\pi r^2}=\frac{3}{r^2}$. (b) Separating,

$$\int r^2\,dr=\int 3\,dt\Rightarrow\frac{r^3}{3}=3t+c.$$

(c) With $r(0)=1$: $\frac{1}{3}=c$, so $r^3=9t+1$. (d) The radius reaches 4 cm when $r^3=64$: $9t+1=64$, giving $t=7$ seconds.

Q5. The tank is uniform, so $\frac{dV}{dh}=5$ and $\frac{dh}{dt}=\frac{1}{5}(-10\sqrt{h})=-2\sqrt{h}$. Separating,

$$\int\frac{1}{\sqrt{h}}dh=\int-2\,dt\Rightarrow 2\sqrt{h}=-2t+c.$$

With $h(0)=9$: $2\sqrt{9}=6=c$, so $2\sqrt{h}=6-2t$, that is $\sqrt{h}=3-t$ and $h=(3-t)^2$. The depth is 1 m when $\sqrt{h}=1$: $3-t=1$, so $t=2$ minutes; the tank empties when $\sqrt{h}=0$: $3-t=0$, so $t=3$ minutes.

Q6. (a) With $\frac{dV}{dr}=4\pi r^2$, $4\pi r^2\frac{dr}{dt}=2\pi r^2$, so $\frac{dr}{dt}=\frac{2\pi r^2}{4\pi r^2}=\frac{1}{2}$. (b) Separating, $\int 1\,dr=\int\frac{1}{2}dt$, so $r=\frac{t}{2}+c$; with $r(0)=2$, $c=2$ and $r=2+\frac{t}{2}$. (c) The radius reaches 8 mm when $2+\frac{t}{2}=8$, so $\frac{t}{2}=6$ and $t=12$ minutes.

Q7. The tank is uniform, so $\frac{dV}{dh}=2$ and $\frac{dh}{dt}=\frac{1}{2}(-2\sqrt{h})=-\sqrt{h}$. Separating,

$$\int\frac{1}{\sqrt{h}}dh=\int-1\,dt\Rightarrow 2\sqrt{h}=-t+c.$$

With $h(0)=25$: $2\sqrt{25}=10=c$, so $2\sqrt{h}=10-t$ and $h=\left(\frac{10-t}{2}\right)^2$. The tank empties when $h=0$: $10-t=0$, giving $t=10$ minutes.

Q8. Here $\frac{dV}{dh}=A(h)=6h$, so $\frac{dh}{dt}=\frac{1}{6h}(-12)=-\frac{2}{h}$. Separating,

$$\int h\,dh=\int-2\,dt\Rightarrow\frac{h^2}{2}=-2t+c.$$

With $h(0)=6$: $\frac{36}{2}=18=c$, so $h^2=36-4t$. After 5 minutes $h^2=36-20=16$, so $h=4$ m; the tank empties when $h=0$: $36-4t=0$, giving $t=9$ minutes.

Q9. With $\frac{dV}{dr}=4\pi r^2$, $4\pi r^2\frac{dr}{dt}=4\pi$, so $\frac{dr}{dt}=\frac{1}{r^2}$. Separating,

$$\int r^2\,dr=\int 1\,dt\Rightarrow\frac{r^3}{3}=t+c.$$

Taking $t=0$ when $r=2$: $\frac{8}{3}=c$, so $r^3=3t+8$. The radius reaches 5 cm when $r^3=125$: $3t+8=125$, so $3t=117$ and $t=39$ seconds.

Q10. Volume decreasing at a rate proportional to surface area gives $\frac{dr}{dt}=-k$ (as in Q3), so $r=r_0-kt$. Taking $t=0$ at $r_0=8$, the radius falls to 5 in 6 minutes: $5=8-6k$, so $k=\frac{1}{2}$ mm/min and $r=8-\frac{t}{2}$. The pill dissolves when $r=0$: $8-\frac{t}{2}=0$, giving $t=16$ minutes.

Q11. For the cone $V=\frac{\pi}{3}h^3$, so $\frac{dV}{dh}=\pi h^2$. Then $\frac{dh}{dt}=\frac{1}{\pi h^2}(-\pi\sqrt{h})=-\frac{\sqrt{h}}{h^2}=-h^{-3/2}$ (the π cancels). Separating,

$$\int h^{3/2}\,dh=\int-1\,dt\Rightarrow\frac{2}{5}h^{5/2}=-t+c.$$

With $h(0)=4$: $\frac{2}{5}\left(4^{5/2}\right)=\frac{2}{5}(32)=\frac{64}{5}=c$, so $\frac{2}{5}h^{5/2}=\frac{64}{5}-t$. The tank empties when $h=0$, giving $t=\frac{64}{5}=12.8$ minutes.

Q12. Here $\frac{dV}{dh}=A(h)=\pi\left(6h-h^2\right)$, so

$$\frac{dh}{dt}=\frac{1}{\pi\left(6h-h^2\right)}(-\pi)=-\frac{1}{6h-h^2}.$$

Separating,

$$\int\left(6h-h^2\right)dh=\int-1\,dt\Rightarrow 3h^2-\frac{h^3}{3}=-t+c.$$

The bowl is full when $h=3$ at $t=0$: $3(9)-\frac{27}{3}=27-9=18=c$, so $3h^2-\frac{h^3}{3}=18-t$. The bowl empties when $h=0$: $0=18-t$, giving $t=18$ minutes.

Q13. The tank is uniform, so $\frac{dV}{dh}=5$ and $\frac{dh}{dt}=\frac{1}{5}\left(-k\sqrt{h}\right)=-\frac{k}{5}\sqrt{h}$. Separating,

$$\int\frac{1}{\sqrt{h}}dh=\int-\frac{k}{5}dt\Rightarrow 2\sqrt{h}=-\frac{k}{5}t+c.$$

With $h(0)=9$: $2\sqrt{9}=6=c$, so $2\sqrt{h}=6-\frac{k}{5}t$. The depth is 4 m at $t=5$: $2\sqrt{4}=4=6-\frac{k}{5}(5)=6-k$, so $k=2$. Then $2\sqrt{h}=6-\frac{2}{5}t$, and the tank empties when $\sqrt{h}=0$: $6=\frac{2}{5}t$, giving $t=15$ minutes.

Q14. With $\frac{dV}{dr}=4\pi r^2$, $4\pi r^2\frac{dr}{dt}=16\pi$, so $\frac{dr}{dt}=\frac{4}{r^2}$. (a) When $r=4$: $\frac{dr}{dt}=\frac{4}{16}=\frac{1}{4}$ cm/s. (b) Separating,

$$\int r^2\,dr=\int 4\,dt\Rightarrow\frac{r^3}{3}=4t+c.$$

With $r(0)=2$: $\frac{8}{3}=c$, so $r^3=12t+8$. The radius reaches 8 cm when $r^3=512$: $12t+8=512$, so $12t=504$ and $t=42$ seconds.

Q15. Here $\frac{dV}{dh}=3h$, so $\frac{dh}{dt}=\frac{1}{3h}\left(-3\sqrt{h}\right)=-\frac{\sqrt{h}}{h}=-\frac{1}{\sqrt{h}}$. Separating,

$$\int\sqrt{h}\,dh=\int-1\,dt\Rightarrow\frac{2}{3}h^{3/2}=-t+c.$$

With $h(0)=9$: $\frac{2}{3}\left(9^{3/2}\right)=\frac{2}{3}(27)=18=c$, so $\frac{2}{3}h^{3/2}=18-t$. The trough empties when $h=0$, giving $t=18$ minutes.

Q16. For the cone $V=\frac{\pi}{3}h^3$, so $\frac{dV}{dh}=\pi h^2$ and $\frac{dh}{dt}=\frac{1}{\pi h^2}(-\pi)=-\frac{1}{h^2}$. Separating,

$$\int h^2\,dh=\int-1\,dt\Rightarrow\frac{h^3}{3}=-t+c.$$

With $h(0)=3$: $\frac{27}{3}=9=c$, so $h^3=27-3t$. The tank empties when $h=0$: $27-3t=0$, giving $t=9$ minutes.

Q17. Here $\frac{dV}{dh} = A(h) = 3h^2$, so $\frac{dh}{dt} = \frac{1}{3h^2}(-6) = -\frac{2}{h^2}$. Separating,

$$\int h^2\, dh = \int -2\, dt \Rightarrow \frac{h^3}{3} = -2t + c.$$

With $h(0) = 3$: $\frac{27}{3} = 9 = c$, so $h^3 = 27 - 6t$. The tank empties when $h = 0$: $27 - 6t = 0$, giving $t = \frac{9}{2} = 4.5$ minutes.

Q18. Volume increasing at a rate proportional to surface area gives $\frac{dr}{dt} = k$ (as in Q6, with +), so $r = r_0 + kt$. Taking $t = 0$ at $r_0 = 1$, the radius grows to 4 in 3 minutes: $4 = 1 + 3k$, so $k = 1$ mm/min and $r = 1 + t$. The radius reaches 10 mm when $1 + t = 10$, giving $t = 9$ minutes.

Chapter 8 Differential equations — 8H Using a definite integral to solve a differential equation

Theory

Earlier sections solved a differential equation of the form $\frac{dy}{dx}=f(x)$ by **antidifferentiation**: find an antiderivative F of f, write the general solution $y=F(x)+c$, then use the initial condition to pin down c. That plan only works when f actually has an antiderivative we can write down. This section develops a form of the solution that works **even when no such formula exists** — the solution written as a **definite integral**.

The solution as a definite integral.

Suppose $\frac{dy}{dx}=f(x)$ with the initial condition $y(x_0)=y_0$. By the Fundamental Theorem of Calculus, the solution can be written directly as

$$y(x)=y_0+\int_{x_0}^{x} f(t)\,dt.$$

The upper limit of the integral is the variable x, so the variable of integration must be given a **different letter** — here t. This t is a **dummy variable**: it is used up by the integration and does not appear in the answer. (Writing $\int_{x_0}^{x} f(x)\,dx$ would clash, using x both as a limit and as the variable of integration.)

Why this is the solution.

Two checks confirm it. First, differentiate: by the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_{x_0}^{x} f(t)\,dt=f(x),$$

and y_0 is a constant, so $\frac{dy}{dx}=f(x)$ as required. Second, at $x=x_0$ the integral runs from x_0 to x_0, so

$$\int_{x_0}^{x_0} f(t)\,dt=0,\text{giving } y(x_0)=y_0.$$

So the single formula packages **both** the differential equation and the initial condition — there is no arbitrary constant c left to find.

The separated form.

The same idea applies after **separation of variables**. For a separable equation $\frac{dy}{dx}=g(x)h(y)$, divide by $h(y)$ and integrate each side between its initial value and its current value:

$$\int_{y_0}^{y} \frac{1}{h(s)}\,ds=\int_{x_0}^{x} g(t)\,dt.$$

Each side carries its **own** dummy variable — s for the y-integral, t for the x-integral — while the endpoints y and x sit outside. This is the analogue of the formula above: an equation linking y to x through two definite integrals.

When is this the preferred approach?

Whenever the integrand has a **non-elementary antiderivative** — no formula built from powers, roots, exponentials, logarithms and trigonometric functions. Standard examples are

$$e^{-x^2}, \frac{\sin x}{x}, \sqrt{1+x^3}, \cos(x^2), \frac{1}{\log_e x}.$$

For each of these, antidifferentiation simply stalls. The definite-integral form is then the **exact** solution, and its value at a particular x is found **numerically** by technology.

Evaluating y at a particular x.

To find y at $x=a$, substitute the value into the **upper limit** and evaluate the definite integral numerically:

$$y(a)=y_0+\int_{x_0}^{a} f(t)\,dt.$$

The result is quoted as a decimal approximation.

Contrast: when a closed form exists.

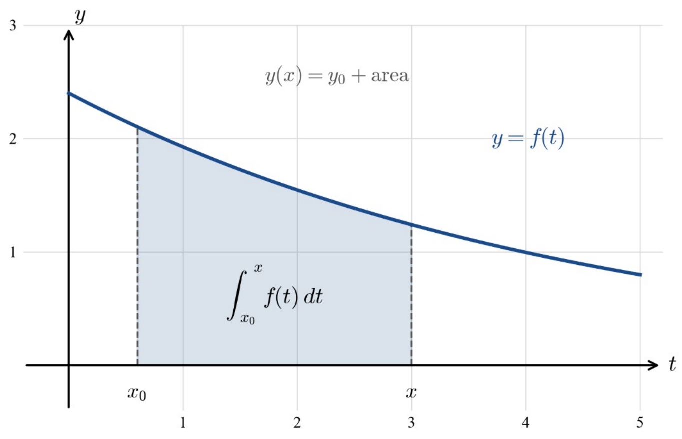
If f does have an elementary antiderivative F, then $\int_{x_0}^{x} f(t)\,dt=F(x)-F(x_0)$, and the definite-integral solution collapses to the familiar closed form

$$y(x)=F(x)+\left[y_0-F(x_0)\right].$$

The two descriptions agree — the definite integral is just the more general one. So give an **exact closed form** whenever the antiderivative is elementary, and a **numerically-evaluated definite integral** otherwise.

The accumulation picture.

Because $\int_{x_0}^{x} f(t)\,dt$ is the signed area under $y=f(t)$ from $t=x_0$ to $t=x$, the solution value $y(x)$ is simply the starting value y_0 **plus the area accumulated so far**. As x increases, more area is added and y builds up from y_0.



Worked examples

Example 1 — scaffolded

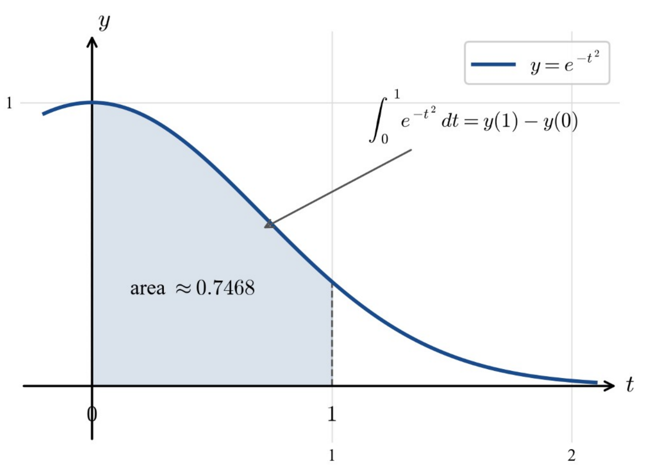
Solve $\frac{dy}{dx}=3x^2+1$ with $y(1)=2$. Write the solution as a definite integral, evaluate it to obtain a closed form, and hence find $y(3)$.

Working	Comment
$y(x)=2+\int_1^x (3t^2+1)\,dt$.	Use $y(x)=y_0+\int_{x_0}^x f(t)\,dt$ with $y_0=2$, $x_0=1$ and dummy variable t.
The integrand $3t^2+1$ has antiderivative t^3+t, so $$y(x)=2+\left[t^3+t\right]_1^x=2+(x^3+x)-(1+1).$$	Here the antiderivative is elementary, so the integral evaluates to a closed form.
$y(x)=x^3+x$.	Simplify: the constants $2-2$ cancel.
Check: $y(1)=1^3+1=2$.	The closed form satisfies the initial condition.
$y(3)=3^3+3=30$.	Substitute $x=3$; equivalently $2+\int_1^3 (3t^2+1)\,dt=2+28=30$.

Example 2 — scaffolded

Solve $\frac{dy}{dx}=e^{-x^2}$ with $y(0)=1$. Express the solution as a definite integral, and find $y(1)$ correct to four decimal places.

Working	Comment
$y(x)=1+\int_0^x e^{-t^2}\,dt$.	Here $y_0=1$, $x_0=0$; write the integrand with the dummy variable t.
The function e^{-t^2} has **no antiderivative** among the elementary functions, so the solution must be left in this definite-integral form.	This is the signal for the method: antidifferentiation cannot proceed.
$y(1)=1+\int_0^1 e^{-t^2}\,dt$.	Substitute $x=1$ into the upper limit.
Evaluating the definite integral numerically with technology, $\int_0^1 e^{-t^2}\,dt\approx 0.7468$.	Technology returns the accumulated area under $y=e^{-t^2}$ from 0 to 1.
$y(1)\approx 1+0.7468=1.7468$.	Add the starting value $y_0=1$.



Example 3 — unscaffolded

A curve passes through $(0,2)$ and has gradient $\frac{dy}{dx}=\sqrt{1+x^3}$ at every point. Write the equation of the curve as a definite integral, and hence find the y-coordinate when $x=2$, correct to four decimal places.

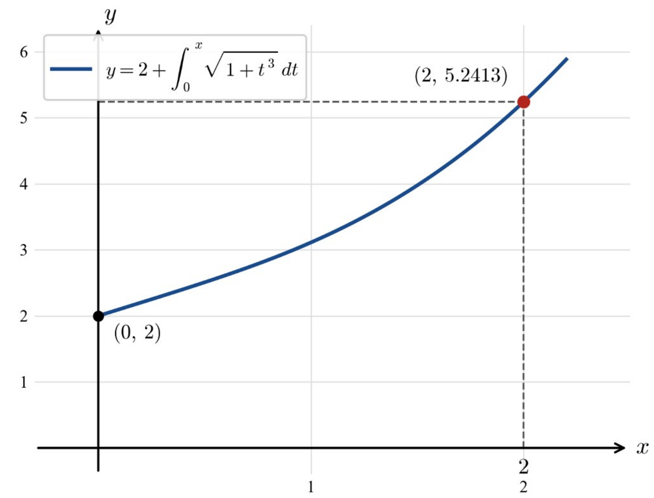
With $y_0 = 2$ and $x_0 = 0$, the solution is

$$y(x) = 2 + \int_0^x \sqrt{1+t^3}\, dt,$$

written with the dummy variable t. The integrand $\sqrt{1+t^3}$ has no elementary antiderivative, so this definite-integral form is the exact equation of the curve. At $x = 2$,

$$y(2) = 2 + \int_0^2 \sqrt{1+t^3}\, dt \approx 2 + 3.2413 = 5.2413,$$

the definite integral being evaluated numerically with technology.



$\therefore$ the curve is $y = 2 + \int_0^x \sqrt{1+t^3}\, dt$, and $y \approx 5.2413$ when $x = 2$.

Example 4 — unscaffolded

Solve $\dfrac{dy}{dx} = e^{-x^2} y$ with $y(0) = 2$, expressing the solution using a definite integral, and find $y(1)$ correct to four decimal places.

This equation is separable, of the form $\dfrac{dy}{dx} = g(x)h(y)$ with $g(x) = e^{-x^2}$ and $h(y) = y$. Since $y > 0$ here, divide by y and integrate each side between its initial and current value:

$$\int_2^y \frac{1}{s}\, ds = \int_0^x e^{-t^2}\, dt,$$

with dummy variables s on the left and t on the right. The left-hand integral is elementary,

$$\int_2^y \frac{1}{s}\, ds = \log_e y - \log_e 2 = \log_e \frac{y}{2},$$

while the right-hand integrand e^{-t^2} is not elementary and stays as a definite integral. Hence

$$\log_e \frac{y}{2} = \int_0^x e^{-t^2}\, dt, \text{so } y = 2\exp\left(\int_0^x e^{-t^2}\, dt\right).$$

At $x=1$, technology gives $\int_0^1 e^{-t^2}\,dt \approx 0.7468$, so $y(1)=2e^{0.7468}\approx 4.2206$.

$\therefore y=2\exp\left(\int_0^x e^{-t^2}\,dt\right)$ and $y(1)\approx 4.2206$.

Practice questions

Scaffolded

Q1. Solve $\frac{dy}{dx}=2x$ with $y(0)=3$. (a) Write the solution as a definite integral $y(x)=y_0+\int_{x_0}^{x} f(t)\,dt$, using the dummy variable t. (b) Evaluate the integral to obtain a closed form for $y(x)$. (c) Find $y(2)$. (d) Verify that your closed form satisfies $y(0)=3$.

Q2. Solve $\frac{dy}{dx}=4x^3$ with $y(1)=0$. (a) Write the solution as a definite integral with dummy variable t. (b) Evaluate it to find a closed form. (c) Find $y(2)$. (d) State whether a numerical method was needed, and why.

Q3. Solve $\frac{dy}{dx}=e^{-x^2}$ with $y(0)=0$. (a) Write the solution as a definite integral with dummy variable t. (b) Explain why the integral cannot be evaluated in closed form. (c) Using technology, evaluate $\int_0^2 e^{-t^2}\,dt$ to four decimal places. (d) Hence state $y(2)$.

Q4. Solve $\frac{dy}{dx}=e^{x^2}$ with $y(0)=1$. (a) Write $y(x)$ as a definite integral with dummy variable t. (b) Explain why the answer must be left as a definite integral. (c) Find $y(1)$ correct to four decimal places.

Q5. Solve the separable equation $\frac{dy}{dx}=2xy$ with $y(0)=1$, where $y>0$. (a) Separate the variables and write the definite-integral form $\int_{y_0}^{y}\frac{1}{s}\,ds=\int_{x_0}^{x}2t\,dt$. (b) Evaluate both integrals. (c) Solve for y to obtain a closed form. (d) Find $y(1)$ correct to four decimal places.

Q6. Solve the separable equation $\frac{dy}{dx}=y\cos(x^2)$ with $y(0)=1$, where $y>0$. (a) Separate the variables and write the definite-integral form, using dummy variables s and t. (b) Evaluate the left-hand integral. (c) Solve for y, leaving the right-hand side as a definite integral. (d) Find $y(1)$ correct to four decimal places.

Unscaffolded

Q7. Solve $\frac{dy}{dx}=6x^2+2$ with $y(1)=5$; give a closed form for y and hence $y(2)$.

Q8. Solve $\frac{dy}{dx}=\cos x$ with $y(0)=2$; give a closed form and hence $y\left(\frac{\pi}{2}\right)$.

Q9. Solve $\frac{dy}{dx}=\frac{1}{x}$ with $y(1)=0$ for $x>0$; give a closed form and hence $y(e)$.

Q10. Solve $\frac{dy}{dx}=e^{-x^2}$ with $y(0)=4$; express y as a definite integral and find $y(1)$ correct to four decimal places.

Q11. Solve $\frac{dy}{dx}=\frac{\sin x}{x}$ with $y(1)=0$; express y as a definite integral and find $y(3)$ correct to four decimal places.

Q12. Solve $\frac{dy}{dx}=\sqrt{1+x^3}$ with $y(0)=0$; express y as a definite integral and find $y(3)$ correct to four decimal places.

Q13. Solve $\frac{dy}{dx}=\sin(x^2)$ with $y(0)=1$; express y as a definite integral and find $y(2)$ correct to four decimal places.

Q14. Solve $\frac{dy}{dx}=2xe^{x^2}$ with $y(0)=2$. (a) Show that this integrand does have an elementary antiderivative, and find the closed form for y. (b) Hence find $y(1)$, giving an exact value. (c) Contrast with Q4: explain why e^{x^2} has no elementary antiderivative but $2xe^{x^2}$ does.

Q15. Solve the separable equation $\frac{dy}{dx}=\frac{e^{-x^2}}{y}$ with $y(0)=1$, where $y>0$. Show that $y=\sqrt{1+2\int_0^x e^{-t^2}\,dt}$ and find $y(1)$ correct to four decimal places.

Q16. Solve the separable equation $\frac{dy}{dx}=ye^{-x^2}$ with $y(0)=3$, where $y>0$; express y using a definite integral and find $y(1)$ correct to four decimal places.

Q17. Water flows into a tank at a rate of $\frac{dV}{dt}=e^{-t^2}$ cubic metres per minute, where t is the time in minutes. The tank initially holds $V=5$ cubic metres. Express V as a definite integral and find the volume after 2 minutes, correct to four decimal places.

Q18. A function y satisfies $\frac{dy}{dx}=\frac{1}{\log_e x}$ for $x>1$, with $y(2)=0$. (a) Explain why the solution cannot be written in closed form. (b) Write y as a definite integral with dummy variable t. (c) Find $y(4)$ correct to four decimal places.

Answers

Q1. $y=x^2+3$; $y(2)=7$. **Q2.** $y=x^4-1$; $y(2)=15$; no numerical method needed — the antiderivative is elementary. **Q3.** $y(x)=\int_0^x e^{-t^2}\,dt$; $\int_0^2 e^{-t^2}\,dt\approx 0.8821$, so $y(2)\approx 0.8821$. **Q4.** $y(x)=1+\int_0^x e^{t^2}\,dt$; $y(1)\approx 2.4627$. **Q5.** $y=e^{x^2}$; $y(1)=e\approx 2.7183$. **Q6.** $y=\exp\left(\int_0^x \cos(t^2)\,dt\right)$; $y(1)\approx 2.4708$. **Q7.** $y=2x^3+2x+1$; $y(2)=21$. **Q8.** $y=2+\sin x$; $y\left(\frac{\pi}{2}\right)=3$. **Q9.** $y=\log_e x$; $y(e)=1$. **Q10.** $y(x)=4+\int_0^x e^{-t^2}\,dt$; $y(1)\approx 4.7468$. **Q11.** $y(x)=\int_1^x \frac{\sin t}{t}\,dt$; $y(3)\approx 0.9026$. **Q12.** $y(x)=\int_0^x \sqrt{1+t^3}\,dt$; $y(3)\approx 7.3414$. **Q13.** $y(x)=1+\int_0^x \sin(t^2)\,dt$; $y(2)\approx 1.8048$. **Q14.** $y=e^{x^2}+1$; $y(1)=e+1\approx 3.7183$. **Q15.** $y=\sqrt{1+2\int_0^x e^{-t^2}\,dt}$; $y(1)\approx 1.5791$. **Q16.** $y=3\exp\left(\int_0^x e^{-t^2}\,dt\right)$; $y(1)\approx 6.3309$. **Q17.** $V(t)=5+\int_0^t e^{-s^2}\,ds$; $V(2)\approx 5.8821$ cubic metres. **Q18.** $y(x)=\int_2^x \frac{1}{\log_e t}\,dt$; $y(4)\approx 1.9224$.

Fully-worked solutions

Q1. With $y_0=3$, $x_0=0$ and $f(t)=2t$,

$$y(x)=3+\int_0^x 2t\,dt=3+\left[t^2\right]_0^x=3+x^2.$$

The antiderivative is elementary, so the closed form is $y=x^2+3$. Then $y(2)=2^2+3=7$, and $y(0)=0+3=3$, matching the initial condition.

Q2. With $y_0=0$, $x_0=1$,

$$y(x)=0+\int_1^x 4t^3\,dt=\left[t^4\right]_1^x=x^4-1.$$

So $y=x^4-1$ and $y(2)=2^4-1=15$. No numerical method was needed: $4t^3$ has the elementary antiderivative t^4, so the definite integral evaluates exactly.

Q3. With $y_0=0$, $x_0=0$, the solution is

$$y(x)=0+\int_0^x e^{-t^2}\,dt=\int_0^x e^{-t^2}\,dt.$$

The integrand e^{-t^2} has no elementary antiderivative, so the integral cannot be written in closed form and must be evaluated numerically. Technology gives $\int_0^2 e^{-t^2}\,dt\approx 0.8821$, so $y(2)\approx 0.8821$.

Q4. With $y_0=1$, $x_0=0$,

$$y(x)=1+\int_0^x e^{t^2}\,dt.$$

The integrand e^{t^2} has no elementary antiderivative, so the answer must be left as a definite integral. Evaluating numerically, $\int_0^1 e^{t^2}\,dt \approx 1.4627$, so $y(1)\approx 1+1.4627=2.4627$.

Q5. The equation is separable. Dividing by $y>0$ and integrating each side between its initial and current value,

$$\int_1^y \frac{1}{s}\,ds=\int_0^x 2t\,dt.$$

Both integrals are elementary:

$$[\log_e s]_1^y=[t^2]_0^x \Rightarrow \log_e y=x^2.$$

Hence $y=e^{x^2}$, a closed form. Then $y(1)=e^1=e\approx 2.7183$.

Q6. Separating and integrating each side between its initial and current value,

$$\int_1^y \frac{1}{s}\,ds=\int_0^x \cos(t^2)\,dt.$$

The left-hand integral is elementary, $[\log_e s]_1^y=\log_e y$, while $\cos(t^2)$ has no elementary antiderivative, so the right-hand side stays as a definite integral:

$$\log_e y=\int_0^x \cos(t^2)\,dt \Rightarrow y=\exp\left(\int_0^x \cos(t^2)\,dt\right).$$

At $x=1$, technology gives $\int_0^1 \cos(t^2)\,dt\approx 0.9045$, so $y(1)=e^{0.9045}\approx 2.4708$.

Q7. With $y_0=5$, $x_0=1$,

$$y(x)=5+\int_1^x (6t^2+2)\,dt=5+[2t^3+2t]_1^x=5+(2x^3+2x)-4=2x^3+2x+1.$$

So $y=2x^3+2x+1$ and $y(2)=16+4+1=21$.

Q8. With $y_0=2$, $x_0=0$,

$$y(x)=2+\int_0^x \cos t\,dt=2+[\sin t]_0^x=2+\sin x.$$

So $y=2+\sin x$ and $y\left(\frac{\pi}{2}\right)=2+\sin\frac{\pi}{2}=2+1=3$.

Q9. With $y_0=0$, $x_0=1$ and $x>0$,

$$y(x)=0+\int_1^x \frac{1}{t}\,dt=[\log_e t]_1^x=\log_e x-\log_e 1=\log_e x.$$

So $y=\log_e x$ and $y(e)=\log_e e=1$.

Q10. With $y_0=4$, $x_0=0$,

$$y(x)=4+\int_0^x e^{-t^2}\,dt.$$

Since e^{-t^2} has no elementary antiderivative, evaluate numerically: $\int_0^1 e^{-t^2}\,dt \approx 0.7468$, so $y(1)\approx 4+0.7468=4.7468$
.

Q11. With $y_0=0$, $x_0=1$,

$$y(x)=0+\int_1^x \frac{\sin t}{t}\,dt=\int_1^x \frac{\sin t}{t}\,dt.$$

The integrand $\frac{\sin t}{t}$ has no elementary antiderivative, so evaluate numerically: $\int_1^3 \frac{\sin t}{t}\,dt\approx 0.9026$, giving $y(3)\approx 0.9026$.

Q12. With $y_0=0$, $x_0=0$,

$$y(x)=\int_0^x \sqrt{1+t^3}\,dt.$$

The integrand $\sqrt{1+t^3}$ has no elementary antiderivative, so evaluate numerically: $\int_0^3 \sqrt{1+t^3}\,dt\approx 7.3414$, giving $y(3)\approx 7.3414$.

Q13. With $y_0=1$, $x_0=0$,

$$y(x)=1+\int_0^x \sin(t^2)\,dt.$$

Since $\sin(t^2)$ has no elementary antiderivative, evaluate numerically: $\int_0^2 \sin(t^2)\,dt\approx 0.8048$, so $y(2)\approx 1+0.8048=1.8048$.

Q14. (a) The integrand is a chain-rule derivative: $\frac{d}{dt}e^{t^2}=2te^{t^2}$, so $2te^{t^2}$ has the elementary antiderivative e^{t^2}. Hence

$$y(x)=2+\int_0^x 2te^{t^2}\,dt=2+\left[e^{t^2}\right]_0^x=2+e^{x^2}-1=e^{x^2}+1.$$

(b) Then $y(1)=e^1+1=e+1$, an exact value. (c) The extra factor $2x$ in $2xe^{x^2}$ is exactly the derivative of the exponent x^2, so the expression matches the chain rule and integrates back to e^{x^2}. Without that factor, e^{x^2} on its own has no elementary antiderivative (as in Q4), and its integral must be left in definite-integral form.

Q15. The equation is separable. Multiplying by $y>0$ and integrating each side between its initial and current value,

$$\int_1^y s\,ds=\int_0^x e^{-t^2}\,dt \Rightarrow \left[\frac{s^2}{2}\right]_1^y=\int_0^x e^{-t^2}\,dt,$$

so $\frac{y^2-1}{2}=\int_0^x e^{-t^2}\,dt$. Solving for the positive root,

$$y=\sqrt{1+2\int_0^x e^{-t^2}\,dt}.$$

At $x=1$, $\int_0^1 e^{-t^2}\,dt\approx 0.7468$, so $y(1)=\sqrt{1+2(0.7468)}=\sqrt{2.4936}\approx 1.5791$.

Q16. Separating and integrating each side between its initial and current value,

$$\int_3^y \frac{1}{s}\,ds=\int_0^x e^{-t^2}\,dt \Rightarrow \log_e y-\log_e 3=\int_0^x e^{-t^2}\,dt,$$

so $\log_e \dfrac{y}{3}=\int_0^x e^{-t^2}\,dt$ and hence

$$y=3\exp\left(\int_0^x e^{-t^2}\,dt\right).$$

At $x=1$, $\int_0^1 e^{-t^2}\,dt\approx 0.7468$, so $y(1)=3e^{0.7468}\approx 6.3309$.

Q17. With initial volume $V=5$ at $t=0$ and rate $\dfrac{dV}{dt}=e^{-t^2}$, the volume is the accumulation

$$V(t)=5+\int_0^t e^{-s^2}\,ds,$$

using the dummy variable s. The integrand e^{-s^2} has no elementary antiderivative, so evaluate numerically: $\int_0^2 e^{-s^2}\,ds\approx 0.8821$, giving $V(2)\approx 5+0.8821=5.8821$ cubic metres.

Q18. (a) The integrand $\dfrac{1}{\log_e x}$ has no antiderivative among the elementary functions, so the solution cannot be written in closed form. (b) With $y_0=0$, $x_0=2$,

$$y(x)=0+\int_2^x \frac{1}{\log_e t}\,dt=\int_2^x \frac{1}{\log_e t}\,dt,$$

valid for $x>1$ (the integrand is defined and continuous there, away from $t=1$). (c) Evaluating numerically, $\int_2^4 \dfrac{1}{\log_e t}\,dt\approx 1.9224$, so $y(4)\approx 1.9224$.

Theory

Most differential equations that arise in applications cannot be solved exactly: separation of variables and integrating factors only reach a small family of equations, and even a harmless-looking equation such as $\frac{dy}{dx}=x+y^2$ has no solution expressible in terms of the standard functions. When an exact solution is out of reach we fall back on a **numerical method** — a rule that generates a table of approximate values of the solution, one point at a time. The simplest such rule is **Euler's method**.

The problem.

Suppose we are given a first-order differential equation together with an initial condition,

$$\frac{dy}{dx}=f(x,y),\; y(x_0)=y_0.$$

The initial condition pins the solution curve to the single known point (x_0,y_0). The differential equation itself does not give us y directly, but it *does* tell us the **gradient** of the solution curve at every point of the plane: at (x,y) the curve must have slope $f(x,y)$. Euler's method uses this gradient information to step forward from the known point in small horizontal increments.

The idea — follow the tangent one step.

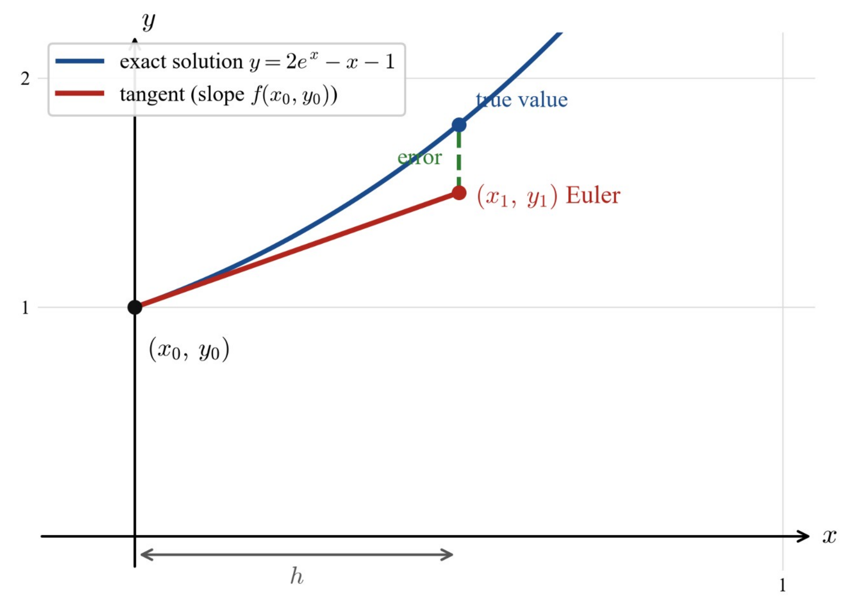
At the starting point (x_0,y_0) the solution curve has gradient $f(x_0,y_0)$, so its tangent line there is

$$y=y_0+f(x_0,y_0)(x-x_0).$$

If we move a short horizontal distance h — the **step size** — to $x_1=x_0+h$ and stay on this tangent, we arrive at the estimate

$$y_1=y_0+hf(x_0,y_0).$$

For a small step the tangent hugs the curve, so (x_1,y_1) lies close to the true solution. We now *pretend* (x_1,y_1) is exact, read off the new gradient $f(x_1,y_1)$ there, and take another step of size h. Repeating this gives a chain of points joined by short straight segments — the **Euler polyline** — that tracks the solution curve.



The iteration.

Writing x_n and y_n for the coordinates after n steps, each new point is produced from the previous one by

$$y_{n+1}=y_n+hf(x_n,y_n),x_{n+1}=x_n+h.$$

Because x advances by the fixed amount h each time, after n steps $x_n=x_0+nh$. To reach a target value $x=b$ from x_0 in equal steps of size h we therefore need $n=\frac{b-x_0}{h}$ steps.

Laying out the work in a table.

The iteration is best organised as a table, one row per step. Each row records the current n, x_n and y_n, evaluates the gradient $f(x_n,y_n)$, and forms the next value $y_{n+1}=y_n+hf(x_n,y_n)$. As a first illustration, take

$$\frac{dy}{dx}=x+y,y(0)=1,$$

and apply Euler's method with the (deliberately large) step size $h=0.5$ to estimate $y(1)$. Two steps are needed, since x must move from 0 to 1:

n	x_n	y_n	$f(x_n,y_n)=x_n+y_n$	$y_{n+1}=y_n+hf(x_n,y_n$
0	0	1	$0+1=1$	$1+0.5(1)=1.5$
1	0.5	1.5	$0.5+1.5=2$	$1.5+0.5(2)=2.5$

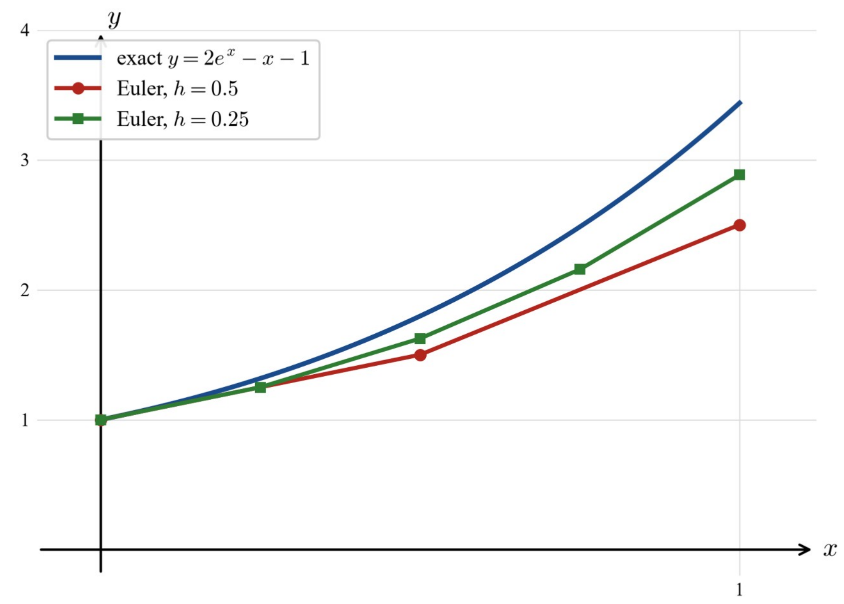
So Euler's method gives $y(1)\approx 2.5$. This equation happens to have the exact solution $y=2e^x-x-1$ (which can be checked by substitution), giving the true value $y(1)=2e-2\approx 3.437$. The crude estimate 2.5 is well short of this — a warning that a large step size can be very inaccurate.

Why the estimate is only approximate.

At every step the polyline leaves the curve and follows a straight tangent, so a small **error** is introduced. Worse, each step starts from the *previous* estimate rather than from the true curve, so errors accumulate as the steps march forward. Whether Euler's method **under**- or **over**-estimates depends on how the solution curve bends: where the curve is concave up (bending upwards) each tangent lies below it and Euler **under**-estimates, while where the curve is concave down each tangent lies above it and Euler **over**-estimates.

The effect of step size.

The one control we have over the accuracy is the step size h. A smaller h makes each tangent segment shorter, so the polyline is forced to correct its direction more often and clings more tightly to the true curve. Euler's method is a **first-order** method: roughly speaking, **halving the step size halves the total error** (while doubling the number of steps, and hence the arithmetic). The figure below shows the same equation $\frac{dy}{dx}=x+y$, $y(0)=1$, solved with $h=0.5$ and again with $h=0.25$: the finer polyline sits noticeably closer to the exact curve $y=2e^x-x-1$.



Comparing with the exact solution.

When an exact solution *is* available, evaluating it at the target point gives the true value, and the **error** of the Euler estimate is

$$\text{error} = |y_{\text{exact}} - y_{\text{Euler}}|.$$

Watching this error shrink as h is reduced is the standard way to see the method converging. In genuine applications, of course, we turn to Euler's method precisely *because* no exact solution is at hand; there we improve confidence by halving h and checking that the estimate settles down.

In summary. To apply Euler's method: identify $f(x, y)$, the starting point (x_0, y_0) and the step size h; then fill a table row by row using $y_{n+1} = y_n + hf(x_n, y_n)$ and $x_{n+1} = x_n + h$ until x_n reaches the required value. The last y_n is the estimate.

Worked examples

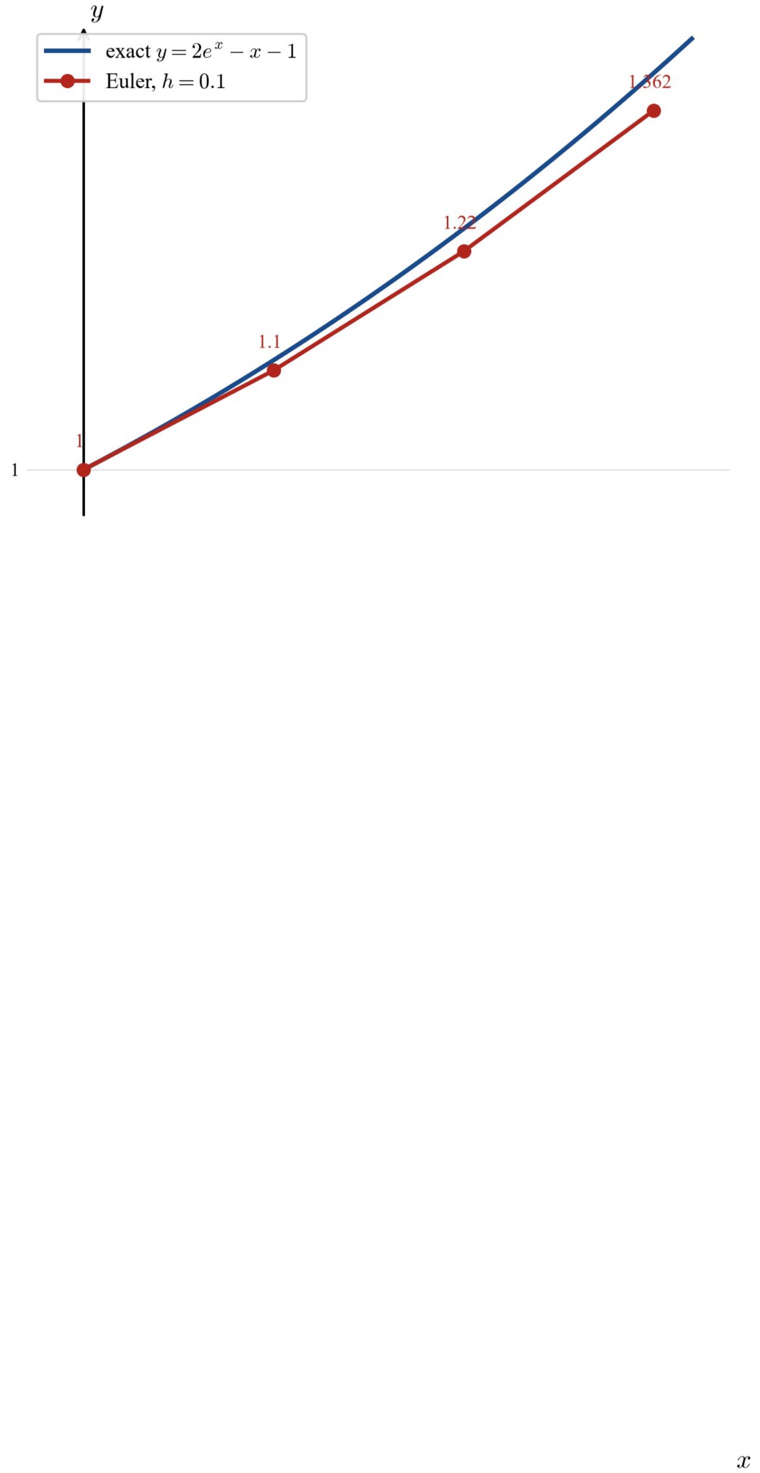
Example 1 — scaffolded

For $\dfrac{dy}{dx} = x + y$ with $y(0) = 1$, use Euler's method with step size $h = 0.1$ to estimate $y(0.3)$, giving each value exactly.

Working	Comment
Here $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$. To reach $x = 0.3$ needs $\dfrac{0.3 - 0}{0.1} = 3$ steps.	Read off f, the starting point and the step size, and count the steps.
Step $n = 0$: $f(x_0, y_0) = 0 + 1 = 1$, so $y_1 = 1 + 0.1(1) = 1.1$ at $x_1 = 0.1$.	Apply $y_1 = y_0 + hf(x_0, y_0)$; advance x by h.
Step $n = 1$: $f(x_1, y_1) = 0.1 + 1.1 = 1.2$, so $y_2 = 1.1 + 0.1(1.2) = 1.22$ at $x_2 = 0.2$.	Use the *estimated* point $(0.1, 1.1)$ to get the next gradient.
Step $n = 2$: $f(x_2, y_2) = 0.2 + 1.22 = 1.42$, so $y_3 = 1.22 + 0.1(1.42) = 1.362$ at $x_3 = 0.3$.	One more step lands on the target $x = 0.3$.
Hence $y(0.3) \approx 1.362$.	The final y-value is the estimate.

The three steps are collected in the table below.

n	x_n	y_n	$f(x_n, y_n) = x_n + y_n$	$y_{n+1} = y_n + 0.1f(x_n, y$
0	0	1	1	1.1
1	0.1	1.1	1.2	1.22
2	0.2	1.22	1.42	1.362



This equation has exact solution $y = 2e^x - x - 1$, so the true value is $y(0.3) = 2e^{0.3} - 1.3 \approx 1.3997$. The Euler estimate 1.362 falls just below it, as expected for a curve that is concave up.

Example 2 — scaffolded

For $\frac{dy}{dx} = x - y$ with $y(0) = 2$, use Euler's method with step size $h = 0.2$ to estimate $y(0.6)$.

Working	Comment
Here $f(x,y)=x-y$, $x_0=0$, $y_0=2$ and $h=0.2$; reaching $x=0.6$ needs 3 steps.	Note that the gradient is now *negative* near the start, so y will fall.
Step $n=0$: $f=0-2=-2$, so $y_1=2+0.2(-2)=1.6$ at $x_1=0.2$.	A negative gradient gives a downward step.
Step $n=1$: $f=0.2-1.6=-1.4$, so $y_2=1.6+0.2(-1.4)=1.32$ at $x_2=0.4$.	Re-evaluate f at the new estimated point.
Step $n=2$: $f=0.4-1.32=-0.92$, so $y_3=1.32+0.2(-0.92)=1.136$ at $x_3=0.6$.	Third step reaches $x=0.6$.
Hence $y(0.6)\approx 1.136$.	State the estimate.

n	x_n	y_n	$f(x_n,y_n)=x_n-y_n$	$y_{n+1}=y_n+0.2f(x_n,y$
0	0	2	-2	1.6
1	0.2	1.6	-1.4	1.32
2	0.4	1.32	-0.92	1.136

The exact solution is $y=x-1+3e^{-x}$, giving $y(0.6)=-0.4+3e^{-0.6}\approx 1.2464$, so the estimate 1.136 again lies a little below the true value.

Example 3 — unscaffolded

The curve $y=y(x)$ satisfies $\dfrac{dy}{dx}=xy$ with $y(0)=1$. Use Euler's method with step size $h=0.2$ to estimate $y(0.6)$.

Here $f(x,y)=xy$, and three steps of size 0.2 take x from 0 to 0.6. Notice that at the start $f(0,1)=0$, so the very first step is flat; the curve only begins to climb once $x>0$.

n	x_n	y_n	$f(x_n,y_n)=x_n y_n$	$y_{n+1}=y_n+0.2f(x_n,y$
0	0	1	0	1
1	0.2	1	0.2	1.04
2	0.4	1.04	0.416	1.1232

This equation is separable, with exact solution $y=e^{x^2/2}$; the true value is $y(0.6)=e^{0.18}\approx 1.1972$, so Euler's method has underestimated it — the flat first step held the estimate back while the real curve had already begun to rise.

$\therefore y(0.6)\approx 1.1232$.

Example 4 — unscaffolded

For $\dfrac{dy}{dx}=y$ with $y(0)=1$, estimate $y(1)$ using Euler's method, first with $h=0.5$ and then with $h=0.25$, and compare each with the exact value $y=e^x$.

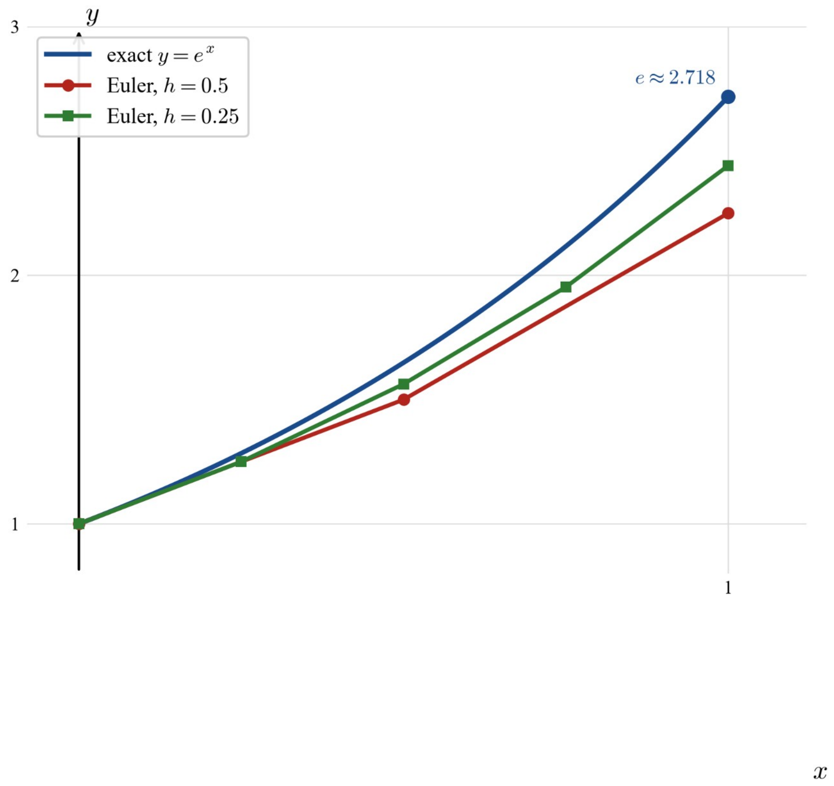
With $f(x,y)=y$, the step size $h=0.5$ requires two steps:

n	x_n	y_n	$f(x_n,y_n)=y_n$	$y_{n+1}=y_n+0.5f(x_n,y$
0	0	1	1	1.5
1	0.5	1.5	1.5	2.25

giving $y(1)\approx 2.25$. Halving the step to $h=0.25$ requires four steps:

n	x_n	y_n	$f(x_n, y_n) = y_n$	$y_{n+1} = y_n + 0.25 f(x_n,$
0	0	1	1	1.25
1	0.25	1.25	1.25	1.5625
2	0.5	1.5625	1.5625	1.953125
3	0.75	1.953125	1.953125	2.44140625

giving $y(1) \approx 2.4414$. The exact value is $y(1) = e \approx 2.7183$, so the errors are $|2.7183 - 2.25| \approx 0.468$ and $|2.7183 - 2.4414| \approx 0.277$. Halving the step size has cut the error almost in half — the hallmark of a first-order method — at the cost of doubling the number of steps.



$\therefore$ with $h = 0.5$, $y(1) \approx 2.25$; with $h = 0.25$, $y(1) \approx 2.4414$; the finer step is the more accurate.

Practice questions

Scaffolded

Q1. For $\dfrac{dy}{dx} = x + y$ with $y(0) = 1$, use Euler's method with $h = 0.2$ to estimate $y(0.4)$. (a) Write down $f(x, y)$, the starting point and the number of steps needed. (b) Carry out step $n = 0$ to find y_1 at $x_1 = 0.2$. (c) Carry out step $n = 1$ to find y_2 at $x_2 = 0.4$. (d) State the estimate for $y(0.4)$.

Q2. For $\dfrac{dy}{dx} = y - x$ with $y(0) = 2$, use Euler's method with $h = 0.1$ to estimate $y(0.3)$. (a) Identify $f(x, y)$ and the number of steps. (b) Complete a table with columns n, x_n, y_n, $f(x_n, y_n)$ and y_{n+1} for $n = 0, 1, 2$. (c) State the estimate for $y(0.3)$.

Q3. A quantity grows according to $\dfrac{dy}{dx} = 6 - 2y$ with $y(0) = 0$. (a) Use Euler's method with $h = 0.1$ to estimate $y(0.3)$, showing the table. (b) The exact solution is $y = 3\left(1 - e^{-2x}\right)$. Evaluate $y(0.3)$ exactly to four decimal places. (c) State whether Euler's method has over- or under-estimated, and explain the answer in terms of how the curve bends.

Q4. For $\frac{dy}{dx}=xy$ with $y(0)=1$, use Euler's method with the large step size $h=0.5$ to estimate $y(1)$. (a) Complete the table for $n=0,1$. (b) State the estimate. (c) The exact solution is $y=e^{x^2/2}$; find $y(1)$ to four decimal places and comment on the size of the error.

Q5. For $\frac{dy}{dx}=x^2-y$ with $y(0)=1$, use Euler's method with $h=0.2$ to estimate $y(0.4)$. (a) Complete the table for $n=0,1$. (b) State the estimate for $y(0.4)$.

Q6. A quantity decays according to $\frac{dy}{dx}=-y$ with $y(0)=4$. (a) Use Euler's method with $h=0.25$ to estimate $y(0.5)$, showing the table. (b) The exact solution is $y=4e^{-x}$. Find $y(0.5)$ to four decimal places and state the error in the Euler estimate.

Unscaffolded

Q7. For $\frac{dy}{dx}=x-y$ with $y(0)=1$, use Euler's method with $h=0.25$ to estimate $y(0.5)$.

Q8. For $\frac{dy}{dx}=2y$ with $y(0)=3$, use Euler's method with $h=0.2$ to estimate $y(0.6)$.

Q9. For $\frac{dy}{dx}=1+y$ with $y(0)=0$, use Euler's method with $h=0.25$ to estimate $y(0.75)$.

Q10. For $\frac{dy}{dx}=xy$ with $y(0)=2$, use Euler's method with $h=0.5$ to estimate $y(1.5)$.

Q11. For $\frac{dy}{dx}=y$ with $y(0)=1$, estimate $y(0.6)$ using Euler's method with (i) $h=0.3$ and (ii) $h=0.2$. Given the exact value $y(0.6)=e^{0.6}\approx 1.8221$, find the error of each estimate and comment.

Q12. A population of P thousand animals is modelled by the logistic equation $\frac{dP}{dt}=0.4P\left(1-\frac{P}{5}\right)$ with $P(0)=1$, where t is in years. Use Euler's method with $h=0.5$ to estimate $P(1)$, giving your answer in thousands to three decimal places.

Q13. For $\frac{dy}{dx}=3-y$ with $y(0)=1$, use Euler's method with $h=0.2$ to estimate $y(0.6)$. Given the exact solution $y=3-2e^{-x}$, state whether the estimate is an over- or under-estimate.

Q14. For $\frac{dy}{dx}=x+y$ with $y(0)=0$, use Euler's method with $h=0.1$ to estimate $y(0.3)$.

Q15. A cup of coffee cools according to $\frac{d\theta}{dt}=-0.2(\theta-20)$, where θ is the temperature in °C and t is in minutes, with $\theta(0)=90$. Use Euler's method with a step size of 1 minute to estimate the temperature after 3 minutes.

Q16. For $\frac{dy}{dx}=y^2$ with $y(0)=1$, use Euler's method with $h=0.2$ to estimate $y(0.4)$.

Q17. For $\frac{dy}{dx}=x+y$ with $y(0)=1$, estimate $y(0.2)$ using Euler's method with (i) a single step $h=0.2$ and (ii) two steps of $h=0.1$. Given the exact value $y(0.2)=2e^{0.2}-1.2\approx 1.2428$, compare the two errors.

Q18. For $\frac{dy}{dx}=\frac{y}{x+1}$ with $y(0)=2$, use Euler's method with $h=0.5$ to estimate $y(1)$. Verify that this matches the exact solution $y=2(x+1)$, and explain geometrically why Euler's method is exact here.

Answers

Q1. $y(0.4)\approx 1.48$. **Q2.** $y(0.3)\approx 2.631$. **Q3.** (a) $y(0.3)\approx 1.464$; (b) 1.3536; (c) an over-estimate — the curve is concave down, so each tangent lies above it. **Q4.** (a)/(b) $y(1)\approx 1.25$; (c) exact 1.6487, error ≈ 0.399 (large, because h is large). **Q5.** $y(0.4)\approx 0.648$. **Q6.** (a) $y(0.5)\approx 2.25$; (b) exact 2.4261, error ≈ 0.1761. **Q7.** $y(0.5)\approx 0.625$. **Q8.** $y(0.6)\approx 8.232$. **Q9.** $y(0.75)\approx 0.953125$. **Q10.** $y(1.5)\approx 3.75$. **Q11.** (i) 1.69, error ≈ 0.1321; (ii) 1.728, error ≈ 0.0941; the smaller step is more accurate. **Q12.** $P(1)\approx 1.338$ thousand. **Q13.** $y(0.6)\approx 1.976$; an over-estimate. **Q14.** $y(0.3)\approx 0.031$. **Q15.** $\theta(3)\approx 55.84$ °C. **Q16.** $y(0.4)\approx 1.488$. **Q17.** (i) 1.2, error ≈ 0.0428; (ii) 1.22, error ≈ 0.0228; halving h roughly halves the error. **Q18.** $y(1)=4$, which equals the exact value $2(1+1)=4$.

Fully-worked solutions

Q1. $f(x,y)=x+y$, $(x_0,y_0)=(0,1)$, $h=0.2$; reaching $x=0.4$ needs 2 steps.

n	x_n	y_n	$f(x_n,y_n)=x_n+y_n$	$y_{n+1}=y_n+0.2f$
0	0	1	1	$1+0.2(1)=1.2$
1	0.2	1.2	1.4	$1.2+0.2(1.4)=1.48$

Hence $y(0.4)\approx 1.48$. (The exact solution $y=2e^x-x-1$ gives $y(0.4)\approx 1.5836$, so this under-estimates.)

Q2. $f(x,y)=y-x$, $(x_0,y_0)=(0,2)$, $h=0.1$; 3 steps to $x=0.3$.

n	x_n	y_n	$f(x_n,y_n)=y_n-x_n$	$y_{n+1}=y_n+0.1f$
0	0	2	2	2.2
1	0.1	2.2	2.1	2.41
2	0.2	2.41	2.21	2.631

Hence $y(0.3)\approx 2.631$.

Q3. (a) $f(x,y)=6-2y$, $(x_0,y_0)=(0,0)$, $h=0.1$; 3 steps.

n	x_n	y_n	$f(x_n,y_n)=6-2y_n$	$y_{n+1}=y_n+0.1f$
0	0	0	6	0.6
1	0.1	0.6	4.8	1.08
2	0.2	1.08	3.84	1.464

So $y(0.3)\approx 1.464$. (b) Exactly, $y(0.3)=3(1-e^{-0.6})\approx 1.3536$. (c) Since $1.464>1.3536$ the estimate is an **over-estimate**: this solution is concave down (it rises towards the value 3 but ever more slowly), so each tangent segment sits above the curve and Euler's method overshoots.

Q4. (a) $f(x,y)=xy$, $(x_0,y_0)=(0,1)$, $h=0.5$; 2 steps to $x=1$.

n	x_n	y_n	$f(x_n,y_n)=x_n y_n$	$y_{n+1}=y_n+0.5f$
0	0	1	0	1
1	0.5	1	0.5	1.25

(b) $y(1)\approx 1.25$. (c) The exact solution is $y=e^{x^2/2}$, so $y(1)=e^{0.5}\approx 1.6487$; the error is about 0.399, which is large because the step size 0.5 is coarse — the flat first step ($f=0$) badly misjudges a curve that is already curving upward.

Q5. $f(x,y)=x^2-y$, $(x_0,y_0)=(0,1)$, $h=0.2$; 2 steps.

n	x_n	y_n	$f(x_n, y_n) = x_n^2 - y_n$	$y_{n+1} = y_n + 0.2f$
0	0	1	-1	0.8
1	0.2	0.8	$0.04 - 0.8 = -0.76$	0.648

Hence $y(0.4) \approx 0.648$.

Q6. (a) $f(x, y) = -y$, $(x_0, y_0) = (0, 4)$, $h = 0.25$; 2 steps.

n	x_n	y_n	$f(x_n, y_n) = -y_n$	$y_{n+1} = y_n + 0.25f$
0	0	4	-4	3
1	0.25	3	-3	2.25

So $y(0.5) \approx 2.25$. (b) Exactly $y(0.5) = 4e^{-0.5} \approx 2.4261$, so the error is $|2.4261 - 2.25| \approx 0.1761$.

Q7. $f(x, y) = x - y$, $(x_0, y_0) = (0, 1)$, $h = 0.25$; 2 steps.

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + 0.25f$
0	0	1	-1	0.75
1	0.25	0.75	-0.5	0.625

Hence $y(0.5) \approx 0.625$.

Q8. $f(x, y) = 2y$, $(x_0, y_0) = (0, 3)$, $h = 0.2$; 3 steps.

n	x_n	y_n	$f(x_n, y_n) = 2y_n$	$y_{n+1} = y_n + 0.2f$
0	0	3	6	4.2
1	0.2	4.2	8.4	5.88
2	0.4	5.88	11.76	8.232

Hence $y(0.6) \approx 8.232$. (The exact value $3e^{1.2} \approx 9.9604$ shows how severely a coarse step under-estimates fast growth.)

Q9. $f(x, y) = 1 + y$, $(x_0, y_0) = (0, 0)$, $h = 0.25$; 3 steps.

n	x_n	y_n	$f(x_n, y_n) = 1 + y_n$	$y_{n+1} = y_n + 0.25f$
0	0	0	1	0.25
1	0.25	0.25	1.25	0.5625
2	0.5	0.5625	1.5625	0.953125

Hence $y(0.75) \approx 0.953125$.

Q10. $f(x, y) = xy$, $(x_0, y_0) = (0, 2)$, $h = 0.5$; 3 steps to $x = 1.5$.

n	x_n	y_n	$f(x_n, y_n) = x_n y_n$	$y_{n+1} = y_n + 0.5f$
0	0	2	0	2
1	0.5	2	1	2.5
2	1	2.5	2.5	3.75

Hence $y(1.5) \approx 3.75$.

Q11. With $f(x, y) = y$ and $y(0) = 1$:

(i) $h=0.3$ (two steps): $y_1=1+0.3(1)=1.3$, then $y_2=1.3+0.3(1.3)=1.69$. So $y(0.6)\approx1.69$, with error $|1.8221-1.69|\approx0.1321$.

(ii) $h=0.2$ (three steps): $y_1=1.2$, $y_2=1.2+0.2(1.2)=1.44$, $y_3=1.44+0.2(1.44)=1.728$. So $y(0.6)\approx1.728$, with error $|1.8221-1.728|\approx0.0941$.

The smaller step size gives the smaller error; reducing h from 0.3 to 0.2 has cut the error by roughly a third, consistent with a first-order method.

Q12. $f(t,P)=0.4P\left(1-\frac{P}{5}\right)$, $(t_0,P_0)=(0,1)$, $h=0.5$; 2 steps to $t=1$.

n	t_n	P_n	$f(t_n,P_n)=0.4P_n(1-$	$P_{n+1}=P_n+0.5f$
0	0	1	$0.4(1)(0.8)=0.32$	1.16
1	0.5	1.16	$0.4(1.16)(0.768)=($	1.338176

Hence $P(1)\approx1.338$ thousand animals. (The exact logistic value is about 1.358 thousand, so Euler under-estimates slightly.)

Q13. $f(x,y)=3-y$, $(x_0,y_0)=(0,1)$, $h=0.2$; 3 steps.

n	x_n	y_n	$f(x_n,y_n)=3-y_n$	$y_{n+1}=y_n+0.2f$
0	0	1	2	1.4
1	0.2	1.4	1.6	1.72
2	0.4	1.72	1.28	1.976

So $y(0.6)\approx1.976$. The exact value is $y(0.6)=3-2e^{-0.6}\approx1.9024$; since $1.976>1.9024$ this is an **over-estimate** (the solution rises towards 3 but is concave down, so the tangents lie above the curve).

Q14. $f(x,y)=x+y$, $(x_0,y_0)=(0,0)$, $h=0.1$; 3 steps.

n	x_n	y_n	$f(x_n,y_n)=x_n+y_n$	$y_{n+1}=y_n+0.1f$
0	0	0	0	0
1	0.1	0	0.1	0.01
2	0.2	0.01	0.21	0.031

Hence $y(0.3)\approx0.031$.

Q15. $f(t,\theta)=-0.2(\theta-20)$, $(t_0,\theta_0)=(0,90)$, $h=1$; 3 steps to $t=3$.

n	t_n	θ_n	$f(t_n,\theta_n)=-0.2(\theta_n-$	$\theta_{n+1}=\theta_n+1\cdot f$
0	0	90	-14	76
1	1	76	-11.2	64.8
2	2	64.8	-8.96	55.84

So the estimated temperature after 3 minutes is $\theta(3)\approx55.84$ °C. (The exact solution $\theta=20+70e^{-0.2t}$ gives about 58.42 °C, so the coarse one-minute step under-estimates the temperature.)

Q16. $f(x,y)=y^2$, $(x_0,y_0)=(0,1)$, $h=0.2$; 2 steps.

n	x_n	y_n	$f(x_n,y_n)=y_n^2$	$y_{n+1}=y_n+0.2f$
0	0	1	1	1.2

n	x_n	y_n	$f(x_n, y_n) = y_n^2$	$y_{n+1} = y_n + 0.2f$
1	0.2	1.2	1.44	1.488

Hence $y(0.4) \approx 1.488$. (The exact solution $y = \frac{1}{1-x}$ gives $y(0.4) = \frac{1}{0.6} \approx 1.6667$; the estimate under-shoots this rapidly steepening curve.)

Q17. With $f(x, y) = x + y$ and $y(0) = 1$:

(i) One step of $h = 0.2$: $y_1 = 1 + 0.2(0+1) = 1.2$, so $y(0.2) \approx 1.2$, error $|1.2428 - 1.2| \approx 0.0428$.

(ii) Two steps of $h = 0.1$: $y_1 = 1 + 0.1(1) = 1.1$, then $f(0.1, 1.1) = 1.2$ and $y_2 = 1.1 + 0.1(1.2) = 1.22$, so $y(0.2) \approx 1.22$, error $|1.2428 - 1.22| \approx 0.0228$.

Halving the step size has roughly halved the error (from 0.0428 to 0.0228), illustrating the first-order accuracy of Euler's method.

Q18. $f(x, y) = \frac{y}{x+1}$, $(x_0, y_0) = (0, 2)$, $h = 0.5$; 2 steps to $x = 1$.

n	x_n	y_n	$f(x_n, y_n) = \frac{y_n}{x_n + 1}$	$y_{n+1} = y_n + 0.5f$
0	0	2	$\frac{2}{1} = 2$	3
1	0.5	3	$\frac{3}{1.5} = 2$	4

So $y(1) \approx 4$, which is exactly $2(1+1) = 4$. The exact solution $y = 2(x+1)$ is a straight line of constant gradient 2; every Euler point lands precisely on this line, where the slope $f(x, y) = \frac{y}{x+1}$ evaluates to exactly 2. Because the tangent segments coincide with the line itself, no error is ever introduced, and Euler's method returns the exact value.

Theory

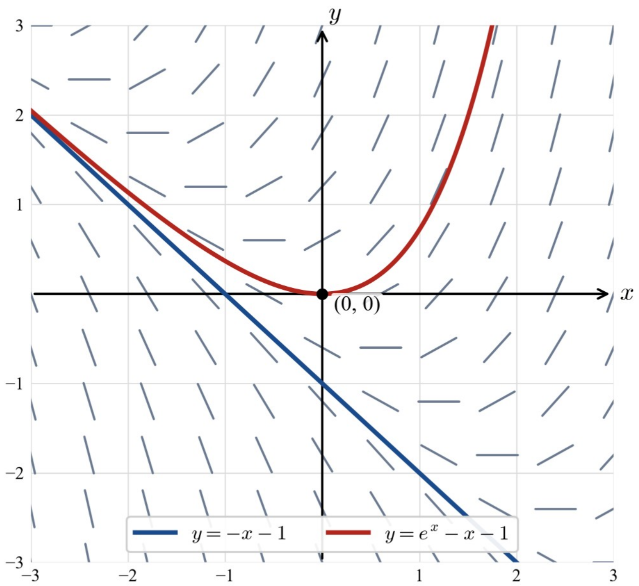
A **first-order differential equation** written in the form

$$\frac{dy}{dx}=f(x,y)$$

does something remarkable: at *every* point (x,y) of the plane it tells you the **gradient** $f(x,y)$ that a solution curve through that point must have. It does not tell you the solution outright, but it fixes the *direction* of travel everywhere. A **slope field** (also called a **direction field**) makes this visible: at a grid of points we draw a **short line segment** whose gradient is the value $f(x,y)$ at that point. Threading a curve so that it stays tangent to these segments traces out a solution.

Constructing a slope field.

To build a slope field by hand, choose a grid of points, and at each point (x,y) compute the number $m=f(x,y)$. This number is the gradient of the segment drawn there — steep and positive for large positive m, steep and negative for large negative m, horizontal when $m=0$, and (in the limit) vertical where f is undefined. Every segment is drawn the **same short length**; only its *direction* carries meaning, so the picture reads as a field of directions rather than of vectors. For example, for $\frac{dy}{dx}=x+y$ the gradient at $(1,1)$ is $1+1=2$, at $(0,0)$ it is 0, and at $(-1,-1)$ it is -2.



The field above is $\frac{dy}{dx}=x+y$. Two solution curves are threaded through it: the straight line $y=-x-1$ and the curve $y=e^x-x-1$ passing through the origin. Notice how each curve is tangent to the little segments at every point it crosses.

Where solutions increase or decrease.

The sign of $f(x,y)$ reads straight off the field.

- Where $f(x,y)>0$ the segments slope **upward**, so a solution curve passing through is **increasing** there.
- Where $f(x,y)<0$ the segments slope **downward**, so a solution is **decreasing** there.
- Where $f(x,y)=0$ the segments are **horizontal**, so a solution has a stationary point (or is momentarily flat) there.

Equilibrium (constant) solutions.

For a DE of the form $\frac{dy}{dx}=f(y)$ — the right-hand side depending on y only — the gradient is the same all along any horizontal line $y=k$. If $f(k)=0$ then the horizontal line $y=k$ is itself a solution: a constant, or **equilibrium**, solution. On the slope field an equilibrium shows up as a full row of horizontal segments, and no solution curve ever crosses it. Solutions on one side that move *towards* the line describe a **stable** equilibrium; solutions that move *away* describe an **unstable** equilibrium.

Isoclines.

An **isocline** is a curve on which the gradient takes a single constant value: the set of points where $f(x,y)=k$. Every segment sitting on an isocline is drawn with the *same* gradient k, so isoclines are the quickest way to sketch a field by hand — pick a value of k, draw the curve $f(x,y)=k$, and rule a set of parallel segments of gradient k along it. The zero-slope isocline $f(x,y)=0$ is especially useful, since it collects every point where solutions are momentarily flat.

Sketching a particular solution curve.

Given a starting point (x_0,y_0) — an initial condition — a particular solution is sketched by **following the field**: from (x_0,y_0), move off in the direction of the local segment, continually adjusting the direction to match the segments as you go, so that the curve stays tangent to the field throughout. Two guiding facts keep the sketch honest: a solution curve **never crosses an equilibrium line**, and (because only one direction is prescribed at each point) distinct solution curves **never cross one another**.

Matching a differential equation to its slope field.

To decide which DE produced a given field, look for structural clues.

- If the segments are **identical along every vertical line** (the field looks the same as you slide up or down a column), the gradient does not depend on y, so f is a function of x alone: $\frac{dy}{dx}=f(x)$.
- If the segments are **identical along every horizontal line**, the gradient does not depend on x, so f is a function of y alone: $\frac{dy}{dx}=f(y)$, and any horizontal equilibrium lines confirm the zeros of f.
- Read off **where the segments are horizontal** (the zero-slope isocline) and **where they are steep**, and check the **sign** of the gradient in each region. These features, together with any symmetry, usually single out one equation.

The rest of this section applies these ideas: constructing a field from a grid, reading increase, decrease and equilibria off a field, sketching a particular solution through a given point, and matching an equation to its field.

Worked examples

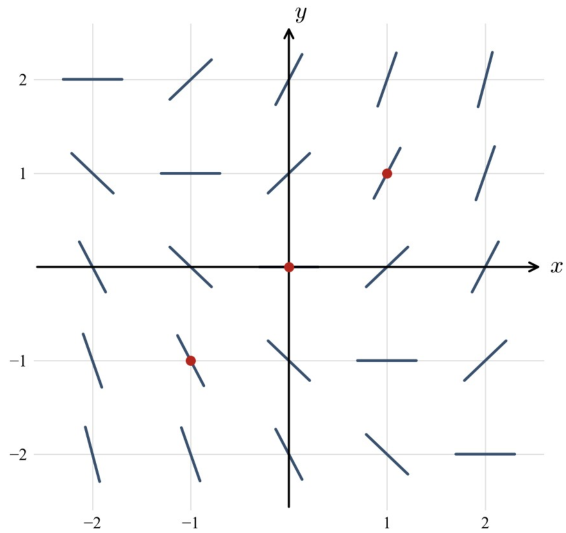
Example 1 — scaffolded

Construct the slope field of $\frac{dy}{dx}=x+y$ at the grid points with $x,y\in\{-1,0,1\}$, and describe one feature of the field.

Here $f(x,y)=x+y$, so the gradient at a grid point is just the sum of its coordinates. Evaluating f at each of the nine points gives the table of gradients (rows are y, columns are x):

$y \setminus x$	-1	0	1
1	0	1	2
0	-1	0	1
-1	-2	-1	0

Working	Comment
At $(1,1)$: $f=1+1=2$.	Substitute the coordinates into $f(x,y)=x+y$.
At $(0,0)$: $f=0+0=0$, so the segment is horizontal.	A zero gradient draws a flat segment.
At $(-1,-1)$: $f=-1+(-1)=-2$.	A steep, downward segment.
The gradient is 0 at $(-1,1)$, $(0,0)$ and $(1,-1)$ — all on the line $y=-x$.	These points share $x+y=0$: the zero-slope isocline.
Draw a short segment of the tabulated gradient at each point, every segment the same length.	Only direction matters, so lengths are equal.



The field is symmetric about the line $y=-x$: along that line every segment is horizontal, and moving away from it the segments steepen — upward above the line and downward below it.

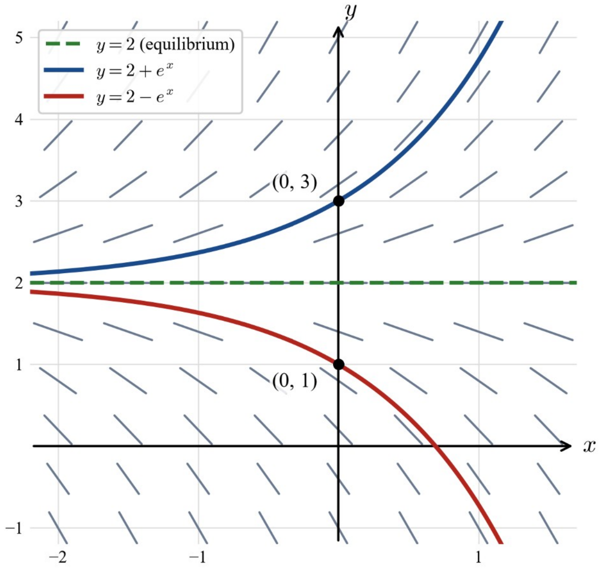
Example 2 — scaffolded

For $\dfrac{dy}{dx}=y-2$, find the equilibrium solution, state where solutions increase and where they decrease, and describe the solution through $(0,3)$.

The right-hand side $f(y)=y-2$ depends on y only, so the field is the same along every horizontal line.

Working	Comment
Equilibrium where $f(y)=0$: $y-2=0$, so $y=2$.	Constant solutions occur where the gradient is zero.
$y=2$ is a solution: if $y=2$ then $\dfrac{dy}{dx}=2-2=0$, matching the flat line.	Check: the horizontal line $y=2$ has gradient 0 everywhere.
For $y>2$: $f(y)=y-2>0$, so solutions are **increasing**.	Above the line the segments slope upward.
For $y<2$: $f(y)=y-2<0$, so solutions are **decreasing**.	Below the line the segments slope downward.
The solution through $(0,3)$ starts just above $y=2$ and, since it is increasing there, rises and moves **away** from $y=2$.	Solutions move away, so $y=2$ is an unstable equilibrium.

The family of solutions is $y=2+Ce^x$; the one through $(0,3)$ has $C=1$, giving $y=2+e^x$, while the one through $(0,1)$ has $C=-1$, giving $y=2-e^x$. Both are shown running away from the equilibrium line.



Example 3 — unscaffolded

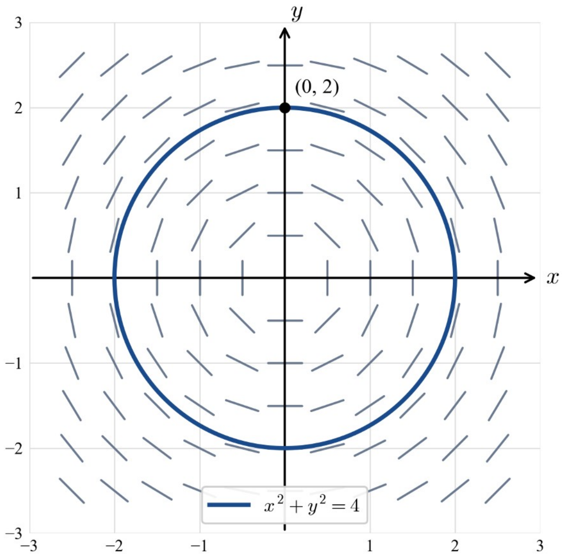
The slope field of $\frac{dy}{dx} = -\frac{x}{y}$ is shown below. Sketch the particular solution curve that passes through $(0, 2)$, and verify it.

At each point the gradient is $-\frac{x}{y}$. On the positive y-axis (where $x = 0$) the gradient is 0, so the segments there are horizontal; as x increases the segments tilt, and near $y = 0$ the gradient is large in magnitude, so the segments stand almost vertical. The gradient $-\frac{x}{y}$ is exactly the gradient perpendicular to the line joining the point to the origin, which suggests the solution curves are **circles centred at O**.

Following the field from $(0, 2)$ — moving off horizontally, then curving downward to the right and upward to the left — traces the circle of radius 2, $x^2 + y^2 = 4$. To verify, differentiate $x^2 + y^2 = 4$ implicitly:

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y},$$

which is the given equation. The point fits, since $0^2 + 2^2 = 4$.

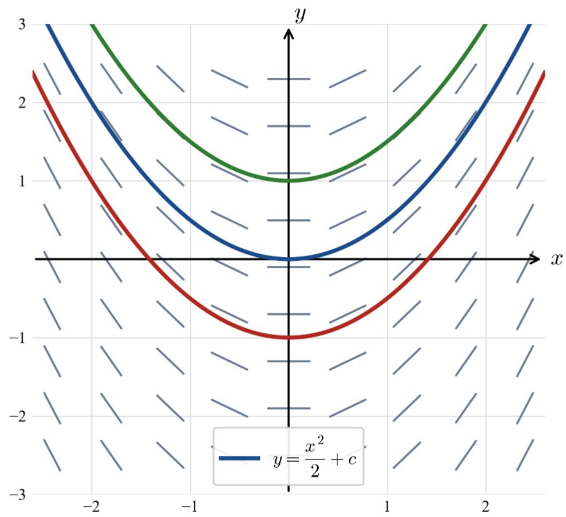


$\therefore$ the solution through $(0, 2)$ is the circle $x^2 + y^2 = 4$.

Example 4 — unscaffolded

A slope field has the following features: along every vertical line the segments are **identical**; on the y-axis every segment is **horizontal**; for $x>0$ the segments slope upward and steepen as x grows, and for $x<0$ they slope downward. Determine which of $\frac{dy}{dx}=y$, $\frac{dy}{dx}=x$ or $\frac{dy}{dx}=x+y$ produced it.

Because the segments are identical along every vertical line, the gradient does not change as y varies — it depends on x **only**. This rules out $\frac{dy}{dx}=y$ (which would be constant along *horizontal* lines) and $\frac{dy}{dx}=x+y$ (which changes with y). That leaves $\frac{dy}{dx}=x$, and its features agree: the gradient is 0 when $x=0$ (horizontal on the y-axis), positive and increasing for $x>0$, and negative for $x<0$. Its solutions are the parabolas $y=\frac{x^2}{2}+c$, one for each value of c, all tangent to the field.



$\therefore$ the field is that of $\frac{dy}{dx}=x$.

Practice questions

Scaffolded

Q1. Consider $\frac{dy}{dx}=2x$. (a) Find the gradient of the field at $(0,1)$ and at $(3,-2)$. (b) Explain why the segments are identical along every vertical line. (c) State where in the plane the segments are horizontal. (d) Verify that $y=x^2+c$ is a solution for every constant c.

Q2. Consider $\frac{dy}{dx}=y-1$. (a) Find the equilibrium (constant) solution. (b) State where solutions are increasing and where they are decreasing. (c) Is the equilibrium stable or unstable? (d) Verify that $y=1+Ce^x$ satisfies the equation.

Q3. Consider $\frac{dy}{dx}=4-y^2$. (a) Find the two equilibrium solutions. (b) Determine the sign of $\frac{dy}{dx}$ for $-2<y<2$, and hence whether solutions there increase or decrease. (c) Determine the sign of $\frac{dy}{dx}$ for $y>2$ and for $y<-2$. (d) Describe the behaviour of the solution through $(0,0)$ as x increases.

Q4. Consider $\frac{dy}{dx}=x+y$. (a) Find the equation of the zero-slope isocline (where the segments are horizontal). (b) Find the isocline on which every segment has gradient 1. (c) Explain why these isoclines are parallel straight lines. (d) Verify that $y=-x-1$ is a solution, and state its constant gradient.

Q5. Consider $\frac{dy}{dx}=\frac{x}{2}$. (a) State the gradient of the field on the y-axis. (b) Describe how the gradient changes as x increases through positive values. (c) Verify that $y=\frac{x^2}{4}+c$ is the family of solutions. (d) Find the particular solution through $(0,1)$.

Q6. Three differential equations are $\frac{dy}{dx}=x$, $\frac{dy}{dx}=y$ and $\frac{dy}{dx}=-y$. (a) Which one has a field that is identical along every horizontal line and whose solutions grow without bound away from $y=0$? (b) Which one has a field identical along every vertical line? (c) Which one has the equilibrium $y=0$ with every nearby solution decaying towards it?

Unscaffolded

Q7. For $\frac{dy}{dx}=x-y$, compute the gradient of the slope field at each of the points $(0,0)$, $(1,0)$, $(0,1)$ and $(2,1)$.

Q8. For $\frac{dy}{dx}=y+1$, find the equilibrium solution and state whether solutions above and below it move towards or away from it.

Q9. For $\frac{dy}{dx}=xy$, state the two families of points where the segments are horizontal, identify the constant solution, and verify that $y=e^{x^2/2}$ is a solution.

Q10. Verify that $y=e^{2x}$ is a solution of $\frac{dy}{dx}=2y$, and state the equilibrium solution of this equation.

Q11. A slope field is horizontal at every point of the line $y=x$, slopes upward below that line and downward above it. Which of $\frac{dy}{dx}=x+y$, $\frac{dy}{dx}=x-y$ or $\frac{dy}{dx}=xy$ produced it? Justify your choice.

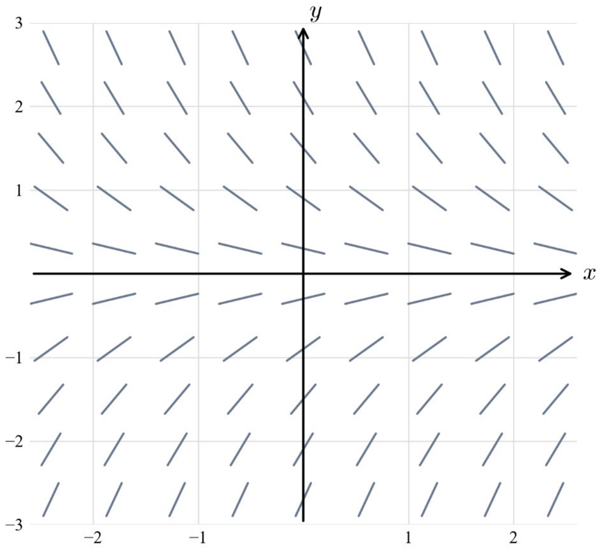
Q12. For $\frac{dy}{dx}=\cos x$, state a value of x at which the segments are horizontal, and verify that $y=\sin x+c$ is the family of solutions.

Q13. For $\frac{dy}{dx}=-\frac{y}{x}$ (with $x\neq 0$), verify that $y=\frac{2}{x}$ is a solution, and describe the shape of the solution curves.

Q14. For $\frac{dy}{dx}=1-y$, find the equilibrium solution, state whether it is stable or unstable, and find the particular solution through $(0,3)$ given that the family of solutions is $y=1+Ce^{-x}$.

Q15. For $\frac{dy}{dx}=2x-y$, describe the family of isoclines $2x-y=k$, and identify the one isocline that is itself a solution curve.

Q16. The slope field of $\frac{dy}{dx}=-y$ is shown below.



(a) State the equilibrium solution and say whether it is stable or unstable.

(b) Describe the behaviour of a solution as x increases.

(c) Given the family of solutions is $y = Ce^{-x}$, find and verify the particular solution through $(0, 2)$, and sketch it on the field above.

Q17. For $\frac{dy}{dx} = \frac{x}{y}$, verify that the solution through $(0, 2)$ is $y = \sqrt{4 + x^2}$, and state the implicit equation of the family of solution curves.

Q18. Four differential equations are $\frac{dy}{dx} = x$, $\frac{dy}{dx} = y$, $\frac{dy}{dx} = -y$ and $\frac{dy}{dx} = x + y$. (a) Find the gradient each of the four fields assigns at the point $(2, -2)$, and state which two of the fields agree there. (b) Which two have the horizontal equilibrium line $y = 0$? (c) Verify that $y = e^x - x - 1$ satisfies exactly one of the four equations, and state which one. (d) Hence explain why no solution curve of that equation can pass through $(2, -2)$ with gradient 2.

Answers

Q1. (a) 0 and 6; (b) $f=2x$ is independent of y; (c) on the y-axis, $x=0$; (d) $y=x^2+c$ gives $\frac{dy}{dx}=2x$. **Q2.** (a) $y=1$; (b) increasing for $y>1$, decreasing for $y<1$; (c) unstable; (d) satisfies the DE. **Q3.** (a) $y=2$ and $y=-2$; (b) $\frac{dy}{dx}>0$, increasing; (c) $\frac{dy}{dx}<0$ in both, decreasing; (d) rises from 0 towards 2. **Q4.** (a) $y=-x$; (b) $y=1-x$; (c) all have gradient -1; (d) solution with constant gradient -1. **Q5.** (a) 0; (b) positive and increasing; (c) satisfies the DE; (d) $y=\frac{x^2}{4}+1$. **Q6.** (a) $\frac{dy}{dx}=y$; (b) $\frac{dy}{dx}=x$; (c) $\frac{dy}{dx}=-y$. **Q7.** $0, 1, -1, 1$. **Q8.** $y=-1$; solutions move away (unstable). **Q9.** Horizontal where $x=0$ or $y=0$; constant solution $y=0$; $y=e^{x^2/2}$ satisfies the DE. **Q10.** $\frac{dy}{dx}=2e^{2x}=2y$; equilibrium $y=0$. **Q11.** $\frac{dy}{dx}=x-y$ (horizontal on $y=x$). **Q12.** $x=\frac{\pi}{2}$ (any odd multiple); $y=\sin x+c$ satisfies the DE. **Q13.** $y=\frac{2}{x}$ satisfies the DE; solutions are rectangular hyperbolas $xy=c$. **Q14.** $y=1$, stable; $y=1+2e^{-x}$. **Q15.** Parallel lines of gradient 2; the isocline $2x-y=2$, i.e. $y=2x-2$. **Q16.** (a) $y=0$, stable; (b) $y\to 0$; (c) $y=2e^{-x}$. **Q17.** $y=\sqrt{4+x^2}$ satisfies the DE; family $y^2-x^2=c$. **Q18.** (a) $2, -2, 2$ and 0; the fields of $\frac{dy}{dx}=x$ and $\frac{dy}{dx}=-y$ agree there; (b) $\frac{dy}{dx}=y$ and $\frac{dy}{dx}=-y$; (c) $\frac{dy}{dx}=x+y$; (d) that equation fixes the gradient at $(2,-2)$ as 0, and only one gradient is prescribed at each point.

Fully-worked solutions

Q1. With $f(x,y)=2x$: (a) at $(0,1)$, $f=2\times 0=0$; at $(3,-2)$, $f=2\times 3=6$. (b) The gradient $f=2x$ contains no y, so at a fixed x the gradient is the same for every y; hence the segments repeat identically up and down each vertical line. (c) The segments are horizontal where $2x=0$, i.e. on the y-axis $x=0$. (d) If $y=x^2+c$ then $\frac{dy}{dx}=2x$, which is exactly the right-hand side, for every constant c.

Q2. With $f(y)=y-1$: (a) The equilibrium is where $y-1=0$, so $y=1$; then $\frac{dy}{dx}=0$ and the constant $y=1$ is a solution. (b) For $y>1$, $y-1>0$, so solutions increase; for $y<1$, $y-1<0$, so solutions decrease. (c) Above the line solutions increase (moving away) and below it they decrease (also moving away), so $y=1$ is **unstable**. (d) If $y=1+Ce^x$ then $\frac{dy}{dx}=Ce^x$, while $y-1=Ce^x$; the two agree, so the family satisfies the equation.

Q3. With $f(y)=4-y^2$: (a) $4-y^2=0$ gives $y^2=4$, so the equilibria are $y=2$ and $y=-2$. (b) For $-2<y<2$, $y^2<4$, so $4-y^2>0$: the gradient is positive and solutions are increasing. (c) For $y>2$, $y^2>4$, so $4-y^2<0$; likewise for $y<-2$. In both outer regions solutions are decreasing. (d) The solution through $(0,0)$ lies in the strip $-2<y<2$, where the gradient is positive, so y increases from 0. As it rises it approaches $y=2$ from below, where the gradient shrinks to 0; the solution levels off towards the stable line $y=2$ without crossing it.

Q4. With $f(x,y)=x+y$: (a) Horizontal segments occur where $x+y=0$, i.e. the line $y=-x$. (b) Gradient 1 occurs where $x+y=1$, i.e. the line $y=1-x$. (c) Each isocline $x+y=k$ rearranges to $y=-x+k$, a line of gradient -1; different k give parallel lines. (d) If $y=-x-1$ then $\frac{dy}{dx}=-1$, while $x+y=x+(-x-1)=-1$; the two agree, so $y=-x-1$ is a solution, and its gradient is the constant -1.

Q5. With $f(x)=\frac{x}{2}$: (a) On the y-axis $x=0$, so the gradient is 0: the segments are horizontal. (b) As x increases through positive values $\frac{x}{2}$ increases, so the segments slope upward more and more steeply. (c) If $y=\frac{x^2}{4}+c$ then $\frac{dy}{dx}=\frac{2x}{4}=\frac{x}{2}$, matching the equation for every c. (d) Requiring $y=1$ when $x=0$ gives $1=0+c$, so $c=1$ and the particular solution is $y=\frac{x^2}{4}+1$.

Q6. (a) A field identical along every **horizontal** line depends on y only; of the three, that is $\frac{dy}{dx}=y$ or $\frac{dy}{dx}=-y$. Solutions that grow without bound away from $y=0$ need a positive gradient above the axis, so the equation is $\frac{dy}{dx}=y$ (its solutions $y=Ce^{x}$ grow). (b) A field identical along every **vertical** line depends on x only: $\frac{dy}{dx}=x$. (c) Decay towards the equilibrium $y=0$ needs a gradient pointing back to the axis (negative above, positive below), which is $\frac{dy}{dx}=-y$ (solutions $y=Ce^{-x}$ decay to 0).

Q7. With $f(x,y)=x-y$: at $(0,0)$, $f=0-0=0$; at $(1,0)$, $f=1-0=1$; at $(0,1)$, $f=0-1=-1$; at $(2,1)$, $f=2-1=1$.

Q8. With $f(y)=y+1$: the equilibrium is where $y+1=0$, i.e. $y=-1$. For $y>-1$, $y+1>0$, so solutions increase (move up, away from the line); for $y<-1$, $y+1<0$, so solutions decrease (move down, away). Both sides move away, so $y=-1$ is unstable.

Q9. With $f(x,y)=xy$: the gradient is 0 where $xy=0$, i.e. wherever $x=0$ (the y-axis) or $y=0$ (the x-axis). Along $y=0$ the gradient is 0 everywhere, so the constant $y=0$ is a solution. For $y=e^{x^2/2}$, $\frac{dy}{dx}=xe^{x^2/2}=x\cdot y=xy$, which is the given equation, so it is a solution.

Q10. For $y=e^{2x}$, $\frac{dy}{dx}=2e^{2x}$; since $2y=2e^{2x}$, the two are equal and $y=e^{2x}$ solves $\frac{dy}{dx}=2y$. The equilibrium is where $2y=0$, i.e. $y=0$.

Q11. The field is horizontal on the whole line $y=x$, so the zero-slope isocline is $x-y=0$. Of the three equations only $\frac{dy}{dx}=x-y$ vanishes there; $x+y=0$ is horizontal on $y=-x$, and $xy=0$ is horizontal on the two axes, so both of those are ruled out. Checking the signs of $x-y$ confirms the fit: below the line $y<x$, so $x-y>0$ and the segments slope **upward**; above the line $y>x$, so $x-y<0$ and they slope **downward**. This matches the description, so the equation is $\frac{dy}{dx}=x-y$.

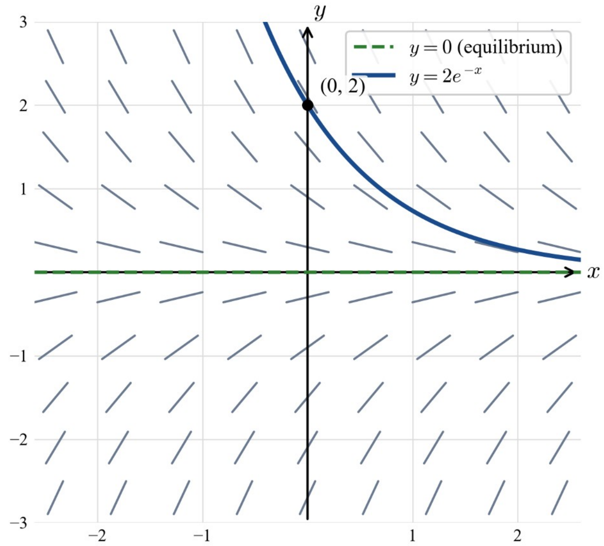
Q12. For $\frac{dy}{dx}=\cos x$ the segments are horizontal where $\cos x=0$, for instance at $x=\frac{\pi}{2}$ (and any odd multiple of $\frac{\pi}{2}$). For $y=\sin x+c$, $\frac{dy}{dx}=\cos x$, matching the equation for every constant c, so this is the family of solutions.

Q13. For $y=\frac{2}{x}$, $\frac{dy}{dx}=-\frac{2}{x^2}$. The right-hand side is $-\frac{y}{x}=-\frac{1}{x}\cdot\frac{2}{x}=-\frac{2}{x^2}$, which is equal, so $y=\frac{2}{x}$ is a solution. Since $xy=2$ here, and in general the solutions satisfy $xy=c$, the solution curves are **rectangular hyperbolas** with the axes as asymptotes.

Q14. With $f(y)=1-y$: the equilibrium is where $1-y=0$, i.e. $y=1$. For $y>1$, $1-y<0$ (solutions fall towards the line); for $y<1$, $1-y>0$ (solutions rise towards the line). Both sides move **towards** $y=1$, so it is **stable**. For the particular solution, put $y=3$ at $x=0$ into $y=1+Ce^{-x}$: $3=1+C$, so $C=2$ and $y=1+2e^{-x}$. (Check: $\frac{dy}{dx}=-2e^{-x}$ and $1-y=1-(1+2e^{-x})=-2e^{-x}$, which agree.)

Q15. Each isocline $2x-y=k$ rearranges to $y=2x-k$: a straight line of gradient 2, and different values of k give a family of parallel lines. On the isocline $2x-y=k$ every segment has gradient k, while the line $y=2x-k$ runs with its own gradient 2; the line can only coincide with the field direction when these agree, i.e. when $k=2$. That is the isocline $2x-y=2$, i.e. $y=2x-2$. Checking: $\frac{dy}{dx}=2$, while $2x-y=2x-(2x-2)=2$; the two agree, so $y=2x-2$ is itself a solution curve.

Q16. With $f(y)=-y$: (a) The equilibrium is where $-y=0$, i.e. $y=0$. Above the axis $-y<0$ (solutions fall towards it) and below it $-y>0$ (solutions rise towards it), so $y=0$ is **stable**. (b) As x increases, every solution is driven towards the axis, so $y\to 0$: the solutions decay. (c) Putting $y=2$ at $x=0$ into $y=Ce^{-x}$ gives $2=C$, so $y=2e^{-x}$. Check: $\frac{dy}{dx}=-2e^{-x}=-y$, as required. Sketched on the field, it starts at $(0,2)$ and decays onto the equilibrium line $y=0$ without ever reaching it:



Q17. For $y=\sqrt{4+x^2}$,

$$\frac{dy}{dx}=\frac{1}{2}(4+x^2)^{-1/2}\cdot 2x=\frac{x}{\sqrt{4+x^2}}=\frac{x}{y},$$

which is the given equation, and at $x=0$, $y=\sqrt{4}=2$, so it passes through $(0,2)$. Squaring gives $y^2=4+x^2$, so the family of solution curves is $y^2-x^2=c$ (here $c=4$): a family of hyperbolas.

Q18. (a) At $(2,-2)$: for $\frac{dy}{dx}=x$, $f=2$; for $\frac{dy}{dx}=y$, $f=-2$; for $\frac{dy}{dx}=-y$, $f=-(-2)=2$; and for $\frac{dy}{dx}=x+y$, $f=2+(-2)=0$. So the fields of $\frac{dy}{dx}=x$ and $\frac{dy}{dx}=-y$ draw the **same** segment, of gradient 2, at that point: a single point never settles which field you are looking at. (b) A horizontal equilibrium line $y=0$ needs $f=0$ all along $y=0$; both $\frac{dy}{dx}=y$ and $\frac{dy}{dx}=-y$ vanish there, so each has the constant solution $y=0$. (On $y=0$ the other two both give $f=x$, which is non-zero away from the origin, so neither has that equilibrium.) (c) For $y=e^x-x-1$, $\frac{dy}{dx}=e^x-1$. Testing the four right-hand sides in turn: $e^x-1=x$ holds only at $x=0$, not for all x; $e^x-1=y=e^x-x-1$ would require $x=0$; and $e^x-1=-y=-e^x+x+1$ would require $2e^x=x+2$, which fails for all but isolated values of x (at

$x = 1$, $2e \approx 5.44$ while $x + 2 = 3$). For the fourth, $x + y = x + (e^x - x - 1) = e^x - 1$, which is exactly $\frac{dy}{dx}$. So the curve satisfies $\frac{dy}{dx} = x + y$, and that equation alone — it is the curved solution threaded through the field in the theory. (d) A differential equation assigns **exactly one** gradient to each point of the plane, and a solution curve must be tangent to the field there. From part (a), $\frac{dy}{dx} = x + y$ gives $2 + (-2) = 0$ at $(2, -2)$, so every solution of it through that point has gradient 0; gradient 2 is impossible. (Gradient 2 at $(2, -2)$ belongs to the fields of $\frac{dy}{dx} = x$ and $\frac{dy}{dx} = -y$, not to this one.)