Specialist Mathematics Units 3 & 4

Chapter 7 — Applications of integration

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Theory

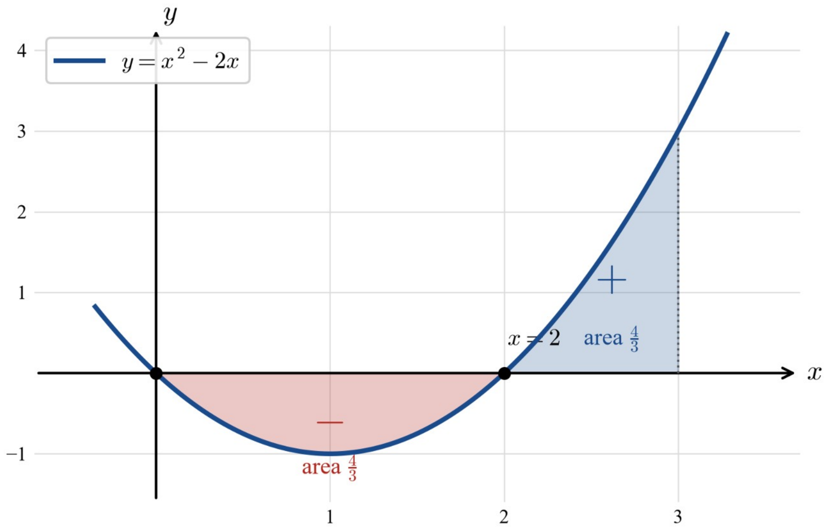
An earlier chapter built **antiderivatives**: given a function f, the indefinite integral $\int f(x)\,dx = F(x)+c$ names the whole family of functions whose derivative is f. This section attaches **limits** to the integral sign to produce instead a single *number*, the **definite integral**, and states the **fundamental theorem of calculus** — the result that ties antidifferentiation (an algebraic process) to area (a geometric one), and so explains why the same symbol $\int$ is used for both.

The definite integral as signed area.

For a function f that is continuous on $[a,b]$, the **definite integral**

$$\int_a^b f(x)\,dx$$

measures the **signed area** between the graph $y=f(x)$ and the x-axis from $x=a$ to $x=b$. "Signed" means that area **above** the axis (where $f(x)>0$) counts as **positive**, while area **below** the axis (where $f(x)<0$) counts as **negative**. The number $\int_a^b f(x)\,dx$ is therefore the area above the axis **minus** the area below it — a net total, not the physical amount of shading.



In the diagram the curve $y=x^2-2x$ dips below the axis on $0<x<2$ (a negative contribution) and rises above it on $2<x<3$ (a positive contribution). Because the two shaded pieces have equal size, the *signed* area over $[0,3]$ is zero, even though the *total* amount of region enclosed is not. To find a genuine (unsigned) area we must split the interval at each axis crossing, integrate over each piece, and add the **sizes** of the results — this is developed in Example 3.

The fundamental theorem of calculus, Part 1.

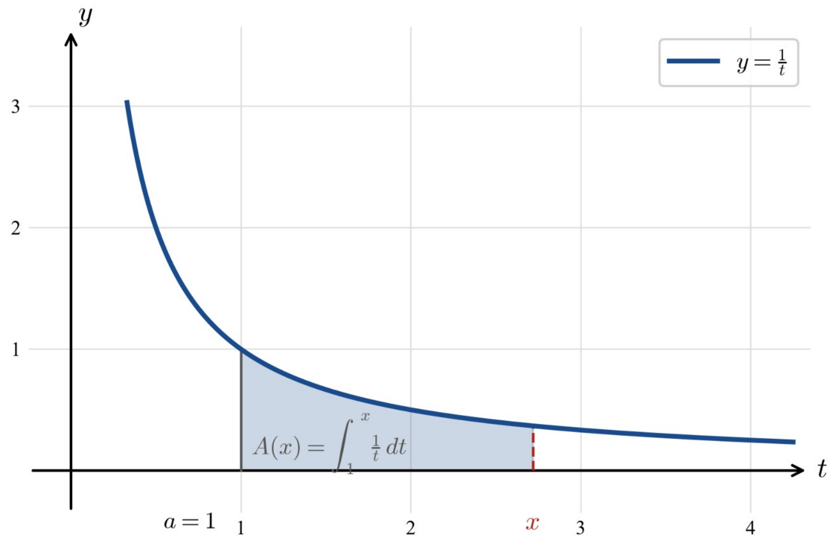
Fix the lower limit a and let the **upper** limit vary. This defines an **accumulation function**

$$A(x)=\int_a^x f(t)\,dt,$$

which records how much signed area has built up between a and the moving point x. A **dummy variable** t is used inside the integral so that it is not confused with the limit x. Part 1 of the fundamental theorem states that, for continuous f, this accumulation function is differentiable and

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

In words: **differentiating an integral with a variable upper limit gives back the integrand**, evaluated at that limit. So $A(x)$ is itself an antiderivative of f — the particular one with $A(a)=0$. The rate at which area accumulates at x is just the height $f(x)$ of the curve there, which is exactly what the picture of a thin strip of width dx and height $f(x)$ suggests.



For example, take $f(t)=\frac{1}{t}$ and $a=1$. The shaded region above shows the area accumulating from $t=1$ up to the moving limit x; Part 1 guarantees at once that $\frac{d}{dx}\int_1^x \frac{1}{t}\,dt=\frac{1}{x}$, without our having to evaluate the integral first (Example 4).

The fundamental theorem of calculus, Part 2.

Part 1 says the accumulation function is *an* antiderivative of f; Part 2 turns this into a practical rule for evaluating a definite integral using **any** antiderivative. If F is any antiderivative of f (so $F'(x)=f(x)$) on $[a,b]$, then

$$\int_a^b f(x)\,dx=[F(x)]_a^b=F(b)-F(a).$$

The square-bracket notation $[F(x)]_a^b$ is shorthand for "substitute the upper limit, then the lower limit, and subtract". The answer does not depend on which antiderivative is chosen: replacing F by $F+c$ gives $(F(b)+c)-(F(a)+c)=F(b)-F(a)$, so the constant cancels. For this reason **no $+c$ is written for a definite integral** — the result is a definite number.

Properties of definite integrals.

The following properties follow directly from Part 2 and are used constantly. For continuous functions and constants p,q:

$$\int_a^a f(x)\,dx=0,\quad \int_b^a f(x)\,dx=-\int_a^b f(x)\,dx,$$

$$\int_a^b f(x)\,dx+\int_b^c f(x)\,dx=\int_a^c f(x)\,dx,$$

$$\int_a^b (p f(x) + q g(x))\,dx = p\int_a^b f(x)\,dx + q\int_a^b g(x)\,dx.$$

The first says an interval of zero width encloses no area. The second says **reversing the limits reverses the sign**. The third is **additivity**: areas over adjacent intervals join, which is exactly what lets us split at an axis crossing. The fourth is **linearity**: a definite integral of a sum is the sum of the integrals, and a constant multiplier passes through the integral sign.

The average value of a function.

The **average value** (or mean value) of a continuous function f over $[a,b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This is the natural continuous analogue of averaging a list of numbers: the integral adds up the values of f across the interval, and dividing by the width $b-a$ shares the total out evenly. Geometrically, $\bar{f}$ is the **height of the rectangle** on the base $[a,b]$ whose area equals the (signed) area under the curve: $\bar{f}(b-a) = \int_a^b f(x)\,dx$. For a continuous function this average height is actually attained by f somewhere on the interval.

Worked examples

Example 1 — scaffolded

Evaluate $\int_1^3 (3x^2 - 2x)\,dx$.

Working	Comment
An antiderivative of $3x^2-2x$ is $F(x)=x^3-x^2$.	Antidifferentiate the integrand; check $F'(x)=3x^2-2x$. No $+c$ for a definite integral.
$\int_1^3 (3x^2-2x)\,dx = [x^3-x^2]_1^3$.	Write the antiderivative in square brackets with the limits attached.
$=(3^3-3^2)-(1^3-1^2)$.	Substitute the upper limit $x=3$, then the lower limit $x=1$, and subtract.
$=(27-9)-(1-1)=18-0=18$.	Evaluate. The definite integral is the exact number 18.

Example 2 — scaffolded

Find the average value of $f(x)=x^2$ over the interval $[0,3]$.

Working	Comment
$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$ with $a=0$, $b=3$.	Recall the average-value formula.
$\bar{f} = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\int_0^3 x^2\,dx$.	Substitute the endpoints; the width is $b-a=3$.
$\int_0^3 x^2\,dx = [\frac{x^3}{3}]_0^3 = \frac{27}{3} - 0 = 9$.	Evaluate the definite integral by Part 2.
$\bar{f} = \frac{1}{3}\times 9 = 3$.	Divide by the width. A rectangle of height 3 on $[0,3]$ has the same area, 9.

Example 3 — unscaffolded

Evaluate $\int_0^3 (x^2 - 2x)\,dx$, and find the **total area** enclosed between the curve $y = x^2 - 2x$ and the x-axis from $x = 0$ to $x = 3$.

The curve meets the axis where $x^2 - 2x = x(x-2) = 0$, that is at $x = 0$ and $x = 2$. Between them ($0 < x < 2$) the curve is below the axis, and for $2 < x < 3$ it is above. The definite integral adds **signed** area, so

$$\int_0^3 (x^2 - 2x)\,dx = [\frac{x^3}{3} - x^2]_0^3 = \left(\frac{27}{3} - 9\right) - 0 = 9 - 9 = 0.$$

The signed area is 0: the region below the axis exactly cancels the region above. For the **total** area we integrate over each piece and take the size of each. Below the axis,

$$\int_0^2 (x^2 - 2x)\,dx = [\frac{x^3}{3} - x^2]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3},$$

so that region has area $\frac{4}{3}$; and above the axis,

$$\int_2^3 (x^2 - 2x)\,dx = [\frac{x^3}{3} - x^2]_2^3 = (9 - 9) - \left(\frac{8}{3} - 4\right) = \frac{4}{3}.$$

Adding the two sizes gives total area $\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$.

$\therefore$ the definite integral is 0, while the total area between the curve and the axis is $\frac{8}{3}$.

Example 4 — unscaffolded

Let $A(x) = \int_1^x \frac{1}{t}\,dt$ for $x > 0$. Find $A'(x)$ using the fundamental theorem of calculus, and confirm the result by evaluating $A(x)$ directly.

By Part 1, differentiating an integral with a variable upper limit returns the integrand evaluated at that limit:

$$A'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

No antiderivative was needed for this. We can confirm it by evaluating the accumulation function with Part 2. Since an antiderivative of $\frac{1}{t}$ is $\log_e|t|$, and $t > 0$ throughout the interval,

$$A(x) = \int_1^x \frac{1}{t}\,dt = [\log_e t]_1^x = \log_e x - \log_e 1 = \log_e x.$$

Note that $A(1) = \log_e 1 = 0$, as it must be since $\int_1^1 \frac{1}{t}\,dt = 0$. Differentiating this closed form, $\frac{d}{dx}\log_e x = \frac{1}{x}$, which agrees with Part 1.

$\therefore A'(x) = \frac{1}{x}$, and $A(x) = \log_e x$.

Practice questions

Scaffolded

Q1. Evaluate $\int_0^2 (4x^3 - 6x)\,dx$. (a) Find an antiderivative $F(x)$ of the integrand. (b) Write the integral as $[F(x)]_0^2$. (c) Substitute the limits and evaluate.

Q2. Evaluate $\int_1^2 \left(2x + \frac{1}{x^2}\right) dx$. (a) Rewrite $\frac{1}{x^2}$ as a power of x and antidifferentiate each term. (b) Write the antiderivative in square brackets with the limits 1 and 2. (c) Substitute and simplify.

Q3. Evaluate $\int_0^{\pi/2} \cos x\,dx$. (a) State an antiderivative of $\cos x$. (b) Apply Part 2 with the limits 0 and $\frac{\pi}{2}$. (c) Use the exact values of $\sin$.

Q4. Find the average value of $f(x) = 6x^2$ over $[1, 2]$. (a) Write the average-value formula with $a = 1$, $b = 2$. (b) Evaluate $\int_1^2 6x^2\,dx$. (c) Divide by $b - a$.

Q5. The curve $y = x^2 - 4$ is drawn from $x = 0$ to $x = 3$. (a) Find where the curve crosses the x-axis on this interval. (b) Evaluate $\int_0^3 (x^2 - 4)\,dx$ (the signed area). (c) Find the total area between the curve and the x-axis from $x = 0$ to $x = 3$.

Q6. Let $G(x) = \int_2^x (t^3 + 1)\,dt$. (a) Use Part 1 of the fundamental theorem to write $G'(x)$. (b) Hence find $G'(2)$. (c) State the value of $G(2)$, giving a reason.

Unscaffolded

Q7. Evaluate $\int_1^4 (2x + 3)\,dx$.

Q8. Evaluate $\int_0^1 e^{2x}\,dx$, giving an exact answer.

Q9. Evaluate $\int_1^e \frac{1}{x}\,dx$.

Q10. Evaluate $\int_0^{\pi} \sin x\,dx$.

Q11. Evaluate $\int_{-1}^2 (3x^2 - 2x)\,dx$.

Q12. Evaluate $\int_1^3 \frac{2}{x}\,dx$, giving an exact answer.

Q13. For the curve $y = x^2 - 9$ from $x = 0$ to $x = 4$, evaluate $\int_0^4 (x^2 - 9)\,dx$ and find the total area between the curve and the x-axis.

Q14. Find the average value of $f(x) = e^x$ over $[0, 1]$, giving an exact answer.

Q15. Find $\frac{d}{dx}\int_0^x \sqrt{1+t^4}\,dt$.

Q16. Given that $\int_0^4 f(x)\,dx=10$ and $\int_0^1 f(x)\,dx=3$, find (a) $\int_1^4 f(x)\,dx$ and (b) $\int_4^0 f(x)\,dx$.

Q17. Given that $\int_1^2 f(x)\,dx=5$ and $\int_1^2 g(x)\,dx=-2$, evaluate $\int_1^2 (3f(x)-2g(x))\,dx$.

Q18. Find the value of $a>0$ for which $\int_0^a (2x+1)\,dx=6$.

Answers

Q1. 4. **Q2.** $\frac{7}{2}$. **Q3.** 1. **Q4.** 14. **Q5.** Crosses at $x=2$; signed area -3; total area $\frac{23}{3}$. **Q6.** (a) $G'(x)=x^3+1$; (b) $G'(2)=9$; (c) $G(2)=0$, since $\int_2^2 (t^3+1)\,dt=0$. **Q7.** 24. **Q8.** $\frac{1}{2}(e^2-1)$. **Q9.** 1. **Q10.** 2. **Q11.** 6. **Q12.** $2\log_e 3$. **Q13.** Signed area $-\frac{44}{3}$; total area $\frac{64}{3}$. **Q14.** $e-1$. **Q15.** $\sqrt{1+x^4}$. **Q16.** (a) 7; (b) -10. **Q17.** 19. **Q18.** $a=2$.

Fully-worked solutions

Q1. An antiderivative of $4x^3-6x$ is $F(x)=x^4-3x^2$. By Part 2,

$$\int_0^2 (4x^3-6x)\,dx=[x^4-3x^2]_0^2=(2^4-3\cdot 2^2)-0=(16-12)=4.$$

Q2. Write $\frac{1}{x^2}=x^{-2}$, whose antiderivative is $-x^{-1}=-\frac{1}{x}$; and $2x$ antidifferentiates to x^2. Then

$$\int_1^2 \left(2x+\frac{1}{x^2}\right)dx=[x^2-\frac{1}{x}]_1^2=\left(4-\frac{1}{2}\right)-(1-1)=\frac{7}{2}-0=\frac{7}{2}.$$

Q3. An antiderivative of $\cos x$ is $\sin x$, so

$$\int_0^{\pi/2} \cos x\,dx=[\sin x]_0^{\pi/2}=\sin\frac{\pi}{2}-\sin 0=1-0=1.$$

Q4. With $a=1$, $b=2$ the average value is

$$\acute{f}=\frac{1}{2-1}\int_1^2 6x^2\,dx=[2x^3]_1^2=2\cdot 2^3-2\cdot 1^3=16-2=14.$$

Since $b-a=1$, the average value equals the integral, 14.

Q5. (a) $x^2-4=0$ gives $x=\pm 2$; on $[0,3]$ the crossing is at $x=2$. (b) The signed area is

$$\int_0^3 (x^2-4)\,dx=[\frac{x^3}{3}-4x]_0^3=(9-12)-0=-3.$$

(c) The curve is below the axis on $[0,2]$ and above it on $[2,3]$. Over $[0,2]$,

$$\int_0^2 (x^2-4)\,dx=[\frac{x^3}{3}-4x]_0^2=\frac{8}{3}-8=-\frac{16}{3},$$

an area of $\frac{16}{3}$; over $[2,3]$,

$$\int_2^3 (x^2-4)\,dx=[\frac{x^3}{3}-4x]_2^3=(9-12)-\left(\frac{8}{3}-8\right)=\frac{7}{3}.$$

Total area $=\frac{16}{3}+\frac{7}{3}=\frac{23}{3}$.

Q6. (a) By Part 1, differentiating $\int_{2}^{x}(t^3+1)\,dt$ returns the integrand with t replaced by x: $G'(x)=x^3+1$. (b) Hence $G'(2)=2^3+1=9$. (c) $G(2)=\int_{2}^{2}(t^3+1)\,dt=0$, because the interval has zero width.

Q7. An antiderivative of $2x+3$ is x^2+3x, so

$$\int_{1}^{4}(2x+3)\,dx=[x^2+3x]_1^4=(16+12)-(1+3)=28-4=24.$$

Q8. Using $\int e^{2x}\,dx=1/2\,e^{2x}$,

$$\int_{0}^{1}e^{2x}\,dx=[\frac{1}{2}e^{2x}]_0^1=\frac{1}{2}e^2-\frac{1}{2}e^0=\frac{1}{2}(e^2-1).$$

Q9. An antiderivative of $\frac{1}{x}$ is $\log_e|x|$; on $[1,e]$ the argument is positive, so

$$\int_{1}^{e}\frac{1}{x}\,dx=[\log_e x]_1^e=\log_e e-\log_e 1=1-0=1.$$

Q10. An antiderivative of $\sin x$ is $-\cos x$, so

$$\int_{0}^{\pi}\sin x\,dx=[-\cos x]_0^\pi=(-\cos\pi)-(-\cos 0)=-(-1)-(-1)=1+1=2.$$

Q11. An antiderivative of $3x^2-2x$ is x^3-x^2, so

$$\int_{-1}^{2}(3x^2-2x)\,dx=[x^3-x^2]_{-1}^2=(8-4)-((-1)-1)=4-(-2)=6.$$

Q12. Taking the constant through the integral and using $\int\frac{1}{x}\,dx=\log_e|x|$,

$$\int_{1}^{3}\frac{2}{x}\,dx=[2\log_e x]_1^3=2\log_e 3-2\log_e 1=2\log_e 3.$$

Q13. The curve $y=x^2-9$ meets the axis at $x=3$ (within $[0,4]$). The signed area is

$$\int_{0}^{4}(x^2-9)\,dx=[\frac{x^3}{3}-9x]_0^4=\frac{64}{3}-36=-\frac{44}{3}.$$

Below the axis, on $[0,3]$,

$$\int_{0}^{3}(x^2-9)\,dx=[\frac{x^3}{3}-9x]_0^3=9-27=-18,$$

an area of 18; above the axis, on $[3,4]$,

$$\int_{3}^{4}(x^2-9)\,dx=[\frac{x^3}{3}-9x]_3^4=\left(\frac{64}{3}-36\right)-(9-27)=\frac{10}{3}.$$

Total area $=18+\frac{10}{3}=\frac{64}{3}$.

Q14. Over $[0,1]$ the width is 1, so

$$\acute{f}=\frac{1}{1-0}\int_0^1 e^x\,dx=[e^x]_0^1=e^1-e^0=e-1.$$

Q15. By Part 1 of the fundamental theorem, differentiating an integral with a variable upper limit returns the integrand evaluated at that limit — no antiderivative of $\sqrt{1+t^4}$ is required:

$$\frac{d}{dx}\int_0^x \sqrt{1+t^4}\,dt=\sqrt{1+x^4}.$$

Q16. (a) By additivity, $\int_0^1 f(x)\,dx+\int_1^4 f(x)\,dx=\int_0^4 f(x)\,dx$, so

$$\int_1^4 f(x)\,dx=\int_0^4 f(x)\,dx-\int_0^1 f(x)\,dx=10-3=7.$$

(b) Reversing the limits reverses the sign, so $\int_4^0 f(x)\,dx=-\int_0^4 f(x)\,dx=-10.$

Q17. By linearity,

$$\int_1^2 (3f(x)-2g(x))\,dx=3\int_1^2 f(x)\,dx-2\int_1^2 g(x)\,dx=3(5)-2(-2)=15+4=19.$$

Q18. Evaluating the definite integral in terms of a,

$$\int_0^a (2x+1)\,dx=[x^2+x]_0^a=a^2+a.$$

Setting $a^2+a=6$ gives $a^2+a-6=0$, i.e. $(a+3)(a-2)=0$, so $a=-3$ or $a=2$. Since $a>0$, $a=2$.

Theory

A definite integral measures the **signed area** trapped between a graph and the x-axis. This section turns that idea into a tool for finding the **geometric area** of a region — a positive quantity — no matter how the boundary curves are arranged: above or below the axis, between two curves, or bounded left and right rather than top and bottom. Throughout, the region is described by its **boundaries**, and the task is to choose limits and an integrand whose definite integral returns the area.

Area under a curve.

Suppose $y=f(x)$ is continuous and $f(x)\geq 0$ for $a\leq x\leq b$. The region enclosed by the curve, the x-axis, and the vertical lines $x=a$ and $x=b$ has area

$$A=\int_a^b f(x)\,dx.$$

This is exactly the signed area the definite integral computes, and here — because the curve lies on or above the axis — that signed area is already positive. As established in an earlier section, the fundamental theorem of calculus evaluates such an integral as $F(b)-F(a)$, where F is any antiderivative of f, so an exact area is obtained whenever an exact antiderivative is known.

Regions below the x-axis.

When the curve dips **below** the axis, $f(x)\leq 0$, and the definite integral is **negative** or zero: it still records the signed area, but a geometric area cannot be negative. The area of a region lying below the axis is the **magnitude** of the integral,

$$A=\left|\int_a^b f(x)\,dx\right|=-\int_a^b f(x)\,dx\,(f(x)\leq 0\,\text{on}\,[a,b]).$$

If a curve **crosses** the axis inside the interval, the parts above and below would partly cancel in a single integral, understating the true area. The remedy is to **split at every x-intercept**, integrate over each piece separately, and add the **magnitudes**:

$$A=\sum\left|\int_{x_k}^{x_{k+1}} f(x)\,dx\right|,$$

the sum running over the sub-intervals between consecutive intercepts. Find the intercepts by solving $f(x)=0$; on each piece the sign of f is constant, so each integral is either already positive or is made positive by a single change of sign.

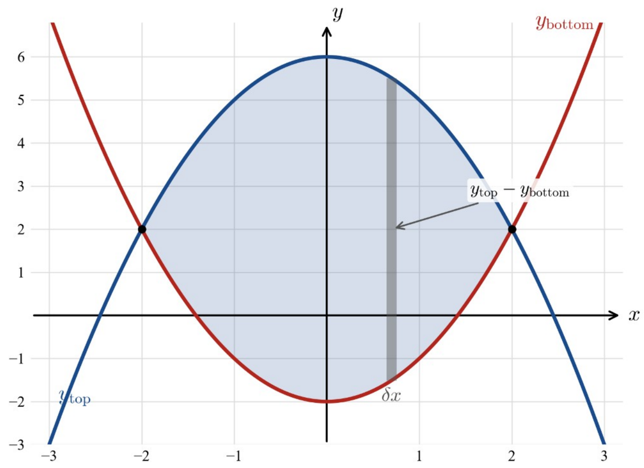
Area between two curves.

Let $y=y_{\text{top}}(x)$ and $y=y_{\text{bottom}}(x)$ be continuous with $y_{\text{top}}(x)\geq y_{\text{bottom}}(x)$ for $a\leq x\leq b$, meeting at $x=a$ and $x=b$. Picture the region as a stack of thin vertical strips: a strip at position x has width δx and height $y_{\text{top}}-y_{\text{bottom}}$, so its area is approximately $(y_{\text{top}}-y_{\text{bottom}})\delta x$. Summing the strips and refining gives

$$A=\int_a^b (y_{\text{top}}-y_{\text{bottom}})\,dx.$$

The height is measured **top minus bottom**, so the integrand is positive throughout and the result is a genuine area. Because only the **difference** of the two heights enters, this formula works even when part or all of the region lies

below the x-axis — the axis is irrelevant once the region is bounded by two curves. The limits a and b are the x-coordinates of the points where the curves meet, found by solving $y_{\text{top}}(x) = y_{\text{bottom}}(x)$.



Curves that cross.

If the two curves **cross** between the outer limits, neither is uniformly the upper curve, so a single "top minus bottom" integrand cannot be used. The intersections found by solving $y_1(x) = y_2(x)$ divide the interval into pieces; on each piece one curve is consistently above the other, but **which one may swap** from piece to piece. Integrate the correct difference (upper minus lower) over each piece and add the results. Equivalently, each piece contributes $\left|\int (y_1 - y_2)\, dx\right|$, so taking magnitudes automatically accounts for the swap.

Area with respect to y.

Some regions are described most naturally by curves written as x in terms of y — for example a "sideways" parabola $x = g(y)$, or a region bounded on the **left and right** rather than top and bottom. Then it is easier to slice with **horizontal** strips. A strip at height y has thickness δy and length $x_{\text{right}} - x_{\text{left}}$, giving

$$A = \int_c^d (x_{\text{right}} - x_{\text{left}})\, dy,$$

where c and d are the **y-coordinates** of the extreme points of the region, found by solving $x_{\text{right}}(y) = x_{\text{left}}(y)$. The length is measured **right minus left**, the mirror image of the top-minus-bottom rule. If the y-axis itself is a boundary, then $x_{\text{left}} = 0$ (or $x_{\text{right}} = 0$) on that side.

Choosing dx or dy.

Both slicings give the same area, so the choice is one of convenience.

Slice with vertical strips (dx) when…	Slice with horizontal strips (dy) when…
curves are given as $y = f(x)$	curves are given as $x = g(y)$
the region has a clear **top and bottom**	the region has a clear **left and right**
a single upper and single lower curve span the width	a single upper/lower would need the region split

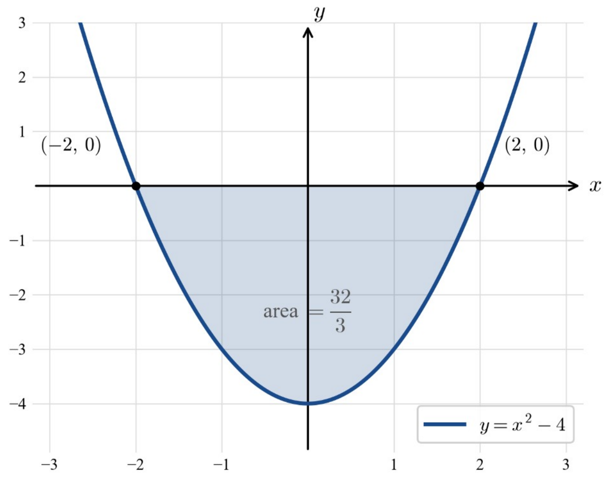
A sideways parabola such as $x = g(y)$ is the classic signal to integrate with respect to y: using dx there would force the region to be broken into awkward pieces, whereas a single dy-integral handles it at once.

Worked examples

Example 1 — scaffolded

Find the area of the region bounded by the curve $y = x^2 - 4$ and the x-axis.

Working	Comment
Solve $x^2-4=0$: $x=-2$ or $x=2$.	The curve meets the x-axis at these points, giving the limits.
For $-2<x<2$, $x^2-4<0$, so the region lies **below** the axis.	Test $x=0$: $0-4=-4<0$.
$\int_{-2}^{2}(x^2-4)\,dx=\left[\frac{x^3}{3}-4x\right]_{-2}^{2}$	Antidifferentiate; the integral will be negative.
$=\left(\frac{8}{3}-8\right)-\left(-\frac{8}{3}+8\right)=\frac{16}{3}-16=-\frac{32}{3}$.	Signed area is negative because the region is below the axis.
$A=\left\lvert-\frac{32}{3}\right\rvert=\frac{32}{3}$.	Take the magnitude to obtain the geometric area.



Example 2 — scaffolded

Find the area of the region enclosed between the line $y=x+2$ and the parabola $y=x^2$.

Working	Comment
Solve $x^2=x+2$: $x^2-x-2=0$, $(x-2)(x+1)=0$, so $x=-1$ or $x=2$.	Intersections give the limits $a=-1$, $b=2$.
At $x=0$: line gives 2, parabola gives 0, so the line is the **upper** curve.	Check which curve is on top between the limits.
$A=\int_{-1}^{2}\left((x+2)-x^2\right)dx$	Area $=\int\left(y_{\text{top}}-y_{\text{bottom}}\right)dx$.
$=\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2}=\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$	Evaluate the antiderivative at both limits.
$=\frac{10}{3}-\left(-\frac{7}{6}\right)=\frac{20}{6}+\frac{7}{6}=\frac{27}{6}=\frac{9}{2}$.	Top minus bottom keeps the integrand positive across $[-1,2]$, so this value is already the area.

Example 3 — unscaffolded

Find the total area of the region enclosed between the curves $y=x^3$ and $y=x$.

Solving $x^3=x$ gives $x^3-x=0$, so $x(x-1)(x+1)=0$ and the curves meet at $x=-1$, $x=0$ and $x=1$. Between these the curves **cross**, so the upper curve swaps at $x=0$. On $[-1,0]$, testing $x=-1/2$ gives $x^3=-1/8$ and $x=-1/2$, so $y=x^3$ is above; on $[0,1]$, testing $x=1/2$ gives $x^3=1/8$ and $x=1/2$, so $y=x$ is above. Integrating the upper-minus-lower difference over each piece,

$$A=\int_{-1}^{0}\left(x^3-x\right)dx+\int_{0}^{1}\left(x-x^3\right)dx=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1}.$$

The first bracket gives $0-(1/4-1/2)=1/4$ and the second gives $(1/2-1/4)-0=1/4$, so $A=1/4+1/4=1/2$.

$\therefore$ the enclosed area is $\frac{1}{2}$ (the two loops are equal by symmetry).

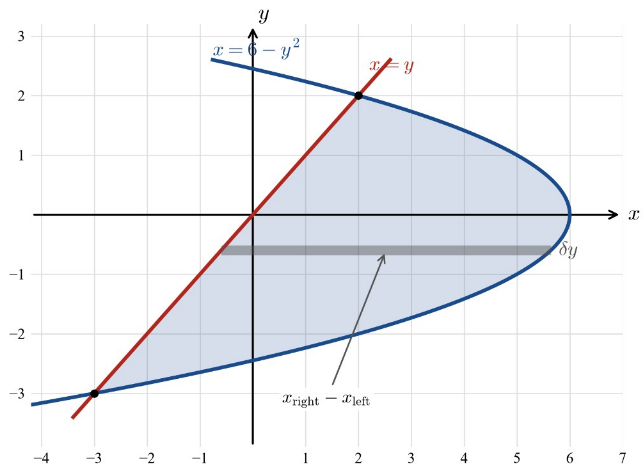
Example 4 — unscaffolded

Find the area of the region bounded by the parabola $x=6-y^2$ and the line $x=y$.

The parabola opens sideways, so it is easier to integrate with respect to y. Solving $6-y^2=y$ gives $y^2+y-6=0$, so $(y+3)(y-2)=0$ and the curves meet at $y=-3$ and $y=2$; these are the limits $c=-3$, $d=2$. At $y=0$ the parabola gives $x=6$ and the line gives $x=0$, so the parabola is the **right-hand** boundary and the line is the left-hand one. Using horizontal strips,

$$A=\int_{-3}^{2}\left((6-y^2)-y\right)dy=\left[6y-\frac{y^3}{3}-\frac{y^2}{2}\right]_{-3}^{2}.$$

At $y=2$ this is $12-8/3-2=22/3$, and at $y=-3$ it is $-18+9-9/2=-27/2$, so $A=22/3-(-27/2)=44/6+81/6=125/6$.



$\therefore$ the area is $\frac{125}{6}$.

Practice questions

Scaffolded

Q1. Find the area of the region bounded by $y=x^2$, the x-axis, and the lines $x=1$ and $x=3$. (a) Confirm that $y=x^2\geq 0$ on $[1,3]$, so no splitting is needed. (b) Write down the definite integral for the area. (c) Antidifferentiate and substitute the limits. (d) State the exact area.

Q2. Find the area of the region bounded by $y=x^2-1$, the x-axis, and the lines $x=0$ and $x=2$. (a) Solve $x^2-1=0$ to find where the curve crosses the axis in $[0,2]$. (b) Explain why the interval must be split at $x=1$. (c) Evaluate $\int_{0}^{1}(x^2-1)\,dx$ and $\int_{1}^{2}(x^2-1)\,dx$. (d) Add the magnitudes to obtain the total area.

Q3. Find the area of the region enclosed between $y=2x$ and $y=x^2$. (a) Solve $2x=x^2$ for the intersections. (b) Decide which curve is the upper one between the intersections. (c) Set up and evaluate $\int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$.

Q4. Find the total area enclosed between $y=4x$ and $y=x^3$. (a) Solve $4x=x^3$ to find the three intersections. (b) Determine which curve is on top on $[-2,0]$ and on $[0,2]$. (c) Write the area as a sum of two integrals of upper minus lower. (d) Evaluate to find the total area.

Q5. Find the area of the region bounded by the parabola $x=y^2$ and the line $x=9$. (a) Solve $y^2=9$ for the limits in y. (b) Identify the right-hand and left-hand boundaries. (c) Evaluate $\int_c^d (x_{\text{right}} - x_{\text{left}})\,dy$.

Q6. Find the area of the region bounded by the parabola $x=y^2$ and the line $x=2-y$. (a) Solve $y^2=2-y$ for the limits in y. (b) Identify the right-hand and left-hand boundaries at $y=0$. (c) Set up and evaluate the area integral with respect to y.

Unscaffolded

Q7. Find the area of the region bounded by $y=x^2+1$, the x-axis, and the lines $x=0$ and $x=3$.

Q8. Find the area of the region bounded by $y=x^2-16$ and the x-axis.

Q9. Find the total area of the region enclosed between the curve $y=x^3-x^2-2x$ and the x-axis.

Q10. Find the area of the region enclosed between the parabola $y=6x-x^2$ and the line $y=x$.

Q11. Find the area of the region enclosed between the parabolas $y=x^2$ and $y=8-x^2$.

Q12. Find the total area enclosed between $y=x^3$ and $y=9x$.

Q13. Find the area of the region bounded by the parabola $x=y^2$ and the line $x=y+2$.

Q14. Find the area of the region bounded by the parabola $x=y^2-4y$ and the y-axis.

Q15. Find the area of the region bounded by $y=\sqrt{x}$, the x-axis, and the line $x=4$.

Q16. Find the area of the region enclosed between $y=\cos x$ and $y=\sin x$ from $x=0$ to $x=\frac{\pi}{2}$.

Q17. Find the area of the region bounded by $y=e^x$, the line $y=x+1$, and the line $x=1$.

Q18. Find the area of the region bounded by the parabola $x=y^2+1$ and the line $x=2y+4$.

Answers

Q1. $\frac{26}{3}$. **Q2.** 2. **Q3.** $\frac{4}{3}$. **Q4.** 8. **Q5.** 36. **Q6.** $\frac{9}{2}$. **Q7.** 12. **Q8.** $\frac{256}{3}$. **Q9.** $\frac{37}{12}$. **Q10.** $\frac{125}{6}$. **Q11.** $\frac{64}{3}$. **Q12.** $\frac{81}{2}$. **Q13.** $\frac{9}{2}$. **Q14.** $\frac{32}{3}$. **Q15.** $\frac{16}{3}$. **Q16.** $2\sqrt{2}-2$. **Q17.** $e-\frac{5}{2}$. **Q18.** $\frac{32}{3}$.

Fully-worked solutions

Q1. On $[1,3]$ the curve $y=x^2$ is positive, so the area is a single integral:

$$A=\int_1^3 x^2\,dx=\left[\frac{x^3}{3}\right]_1^3=\frac{27}{3}-\frac{1}{3}=\frac{26}{3}.$$

Q2. Solving $x^2-1=0$ gives $x=1$ (in $[0,2]$), where the curve crosses the axis: below the axis on $[0,1]$ and above it on $[1,2]$, so the interval must be split there.

$$\int_0^1 (x^2-1)\,dx=\left[\frac{x^3}{3}-x\right]_0^1=\frac{1}{3}-1=-\frac{2}{3},\int_1^2 (x^2-1)\,dx=\left[\frac{x^3}{3}-x\right]_1^2=\frac{2}{3}-\left(-\frac{2}{3}\right)=\frac{4}{3}.$$

Adding the magnitudes, $A=|-2/3|+4/3=2/3+4/3=2$.

Q3. Solving $2x=x^2$ gives $x^2-2x=0$, so $x=0$ or $x=2$. At $x=1$ the line gives 2 and the parabola gives 1, so $y=2x$ is the upper curve. Hence

$$A=\int_0^2 \left(2x-x^2\right)dx=\left[x^2-\frac{x^3}{3}\right]_0^2=4-\frac{8}{3}=\frac{4}{3}.$$

Q4. Solving $4x=x^3$ gives $x^3-4x=0$, so $x(x-2)(x+2)=0$ and the intersections are $x=-2,0,2$. On $[-2,0]$, testing $x=-1$ gives $x^3=-1$ and $4x=-4$, so $y=x^3$ is on top; on $[0,2]$, testing $x=1$ gives $x^3=1$ and $4x=4$, so $y=4x$ is on top. Thus

$$A=\int_{-2}^0 \left(x^3-4x\right)dx+\int_0^2 \left(4x-x^3\right)dx=\left[\frac{x^4}{4}-2x^2\right]_{-2}^0+\left[2x^2-\frac{x^4}{4}\right]_0^2.$$

The first bracket gives $0-(4-8)=4$ and the second gives $8-4=4$, so $A=4+4=8$.

Q5. Solving $y^2=9$ gives $y=-3$ and $y=3$. At $y=0$ the line $x=9$ is to the right of the parabola $x=y^2=0$, so $x_{\text{right}}=9$ and $x_{\text{left}}=y^2$. Then

$$A=\int_{-3}^3 \left(9-y^2\right)dy=\left[9y-\frac{y^3}{3}\right]_{-3}^3=(27-9)-(-27+9)=18+18=36.$$

Q6. Solving $y^2=2-y$ gives $y^2+y-2=0$, so $(y+2)(y-1)=0$ and $y=-2$ or $y=1$. At $y=0$ the line gives $x=2$ and the parabola gives $x=0$, so $x_{\text{right}}=2-y$ and $x_{\text{left}}=y^2$. Hence

$$A=\int_{-2}^1 \left((2-y)-y^2\right)dy=\left[2y-\frac{y^2}{2}-\frac{y^3}{3}\right]_{-2}^1=\frac{7}{6}-\left(-\frac{10}{3}\right)=\frac{7}{6}+\frac{20}{6}=\frac{9}{2}.$$

Q7. On $[0,3]$ the curve $y=x^2+1$ is positive, so

$$A=\int_0^3 \left(x^2+1\right)dx=\left[\frac{x^3}{3}+x\right]_0^3=9+3=12.$$

Q8. Solving $x^2-16=0$ gives $x=\pm 4$, and between them $y=x^2-16<0$, so the whole region lies below the axis:

$$\int_{-4}^{4}\left(x^2-16\right)dx=\left[\frac{x^3}{3}-16x\right]_{-4}^{4}=\left(\frac{64}{3}-64\right)-\left(-\frac{64}{3}+64\right)=\frac{128}{3}-128=-\frac{256}{3},$$

so $A=|-256/3|=\dfrac{256}{3}$.

Q9. Solving $x^3-x^2-2x=0$ gives $x(x-2)(x+1)=0$, so the curve meets the axis at $x=-1,0,2$. Testing $x=-1/2$ gives $5/8>0$ and testing $x=1$ gives $-2<0$, so the curve is above the axis on $[-1,0]$ and below it on $[0,2]$:

$$\int_{-1}^{0}\left(x^3-x^2-2x\right)dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{-1}^{0}=0-\left(\frac{1}{4}+\frac{1}{3}-1\right)=\frac{5}{12},$$

$$\int_{0}^{2}\left(x^3-x^2-2x\right)dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{0}^{2}=4-\frac{8}{3}-4=-\frac{8}{3}.$$

Adding the magnitudes, $A=5/12+|-8/3|=5/12+32/12=37/12$ (the two loops are not equal here, so each piece must be evaluated separately).

Q10. Solving $6x-x^2=x$ gives $5x-x^2=0$, so $x(5-x)=0$ and $x=0$ or $x=5$. At $x=1$ the parabola gives 5 and the line gives 1, so the parabola is on top. Hence

$$A=\int_{0}^{5}\left(\left(6x-x^2\right)-x\right)dx=\int_{0}^{5}\left(5x-x^2\right)dx=\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_{0}^{5}=\frac{125}{2}-\frac{125}{3}=\frac{125}{6}.$$

Q11. Solving $x^2=8-x^2$ gives $2x^2=8$, so $x=\pm 2$. Between them $y=8-x^2$ is the upper curve (at $x=0$ it gives 8 against 0), so

$$A=\int_{-2}^{2}\left(\left(8-x^2\right)-x^2\right)dx=\int_{-2}^{2}\left(8-2x^2\right)dx=\left[8x-\frac{2x^3}{3}\right]_{-2}^{2}=\left(16-\frac{16}{3}\right)-\left(-16+\frac{16}{3}\right)=32-\frac{32}{3}=\frac{64}{3}.$$

Q12. Solving $x^3=9x$ gives $x^3-9x=0$, so $x(x-3)(x+3)=0$ and the intersections are $x=-3,0,3$. By the symmetry of the two odd functions the loops on $[-3,0]$ and $[0,3]$ have equal area, and on $[0,3]$ the line $y=9x$ is on top, so

$$A=2\int_{0}^{3}\left(9x-x^3\right)dx=2\left[\frac{9x^2}{2}-\frac{x^4}{4}\right]_{0}^{3}=2\left(\frac{81}{2}-\frac{81}{4}\right)=2\cdot\frac{81}{4}=\frac{81}{2}.$$

Q13. The parabola opens sideways, so integrate with respect to y. Solving $y^2=y+2$ gives $y^2-y-2=0$, so $(y-2)(y+1)=0$ and $y=-1$ or $y=2$. At $y=0$ the line gives $x=2$ and the parabola gives $x=0$, so $x_{\text{right}}=y+2$ and $x_{\text{left}}=y^2$. Hence

$$A=\int_{-1}^{2}\left(\left(y+2\right)-y^2\right)dy=\left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2}=\frac{10}{3}-\left(-\frac{7}{6}\right)=\frac{9}{2}.$$

Q14. The y-axis is the line $x=0$. Solving $y^2-4y=0$ gives $y(y-4)=0$, so $y=0$ or $y=4$. On $0<y<4$ the parabola gives $x=y^2-4y<0$, so it lies to the left of the axis: $x_{\text{right}}=0$ and $x_{\text{left}}=y^2-4y$. Therefore

$$A=\int_{0}^{4}\left(0-\left(y^2-4y\right)\right)dy=\int_{0}^{4}\left(4y-y^2\right)dy=\left[2y^2-\frac{y^3}{3}\right]_{0}^{4}=32-\frac{64}{3}=\frac{32}{3}.$$

Q15. On $[0,4]$ the curve $y=\sqrt{x}=x^{1/2}$ is positive, so

$$A=\int_0^4 x^{1/2}\,dx=\left[\frac{2}{3}x^{3/2}\right]_0^4=\frac{2}{3}\cdot 4^{3/2}=\frac{2}{3}\cdot 8=\frac{16}{3}.$$

Q16. The curves cross where $\cos x=\sin x$, that is at $x=\frac{\pi}{4}$ in the interval. On $[0,\pi/4]$, $\cos x\ge\sin x$; on $[\pi/4,\pi/2]$, $\sin x\ge\cos x$. Splitting there,

$$A=\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)\,dx.$$

The first integral is $[\sin x+\cos x]_0^{\pi/4}=\sqrt{2}-1$, and the second is $[-\cos x-\sin x]_{\pi/4}^{\pi/2}=-1-(-\sqrt{2})=\sqrt{2}-1$. Hence $A=2(\sqrt{2}-1)=2\sqrt{2}-2$.

Q17. For $x\ge 0$, $e^x\ge x+1$ with equality only at $x=0$, so on $[0,1]$ the curve $y=e^x$ is the upper boundary, the line $y=x+1$ the lower, and $x=1$ closes the region on the right. Thus

$$A=\int_0^1\left(e^x-(x+1)\right)dx=\left[e^x-\frac{x^2}{2}-x\right]_0^1=\left(e-\frac{1}{2}-1\right)-1=e-\frac{5}{2}.$$

Q18. The parabola opens sideways, so integrate with respect to y. Solving $y^2+1=2y+4$ gives $y^2-2y-3=0$, so $(y-3)(y+1)=0$ and $y=-1$ or $y=3$. At $y=0$ the line gives $x=4$ and the parabola gives $x=1$, so $x_{\text{right}}=2y+4$ and $x_{\text{left}}=y^2+1$. Hence

$$A=\int_{-1}^3\left((2y+4)-(y^2+1)\right)dy=\int_{-1}^3(2y+3-y^2)\,dy=\left[y^2+3y-\frac{y^3}{3}\right]_{-1}^3=9-\left(-\frac{5}{3}\right)=\frac{32}{3}.$$

Chapter 7 Applications of integration — 7C Numerical and technology-assisted integration

Theory

The **fundamental theorem of calculus** evaluates a definite integral through an antiderivative: if $F'(x)=f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx=F(b)-F(a).$$

This is exact, but it depends on our being able to *write down* an antiderivative F in terms of the familiar functions — powers, exponentials, logarithms and circular functions. For many perfectly ordinary integrands that is impossible.

Integrands with no elementary antiderivative.

Some functions have no antiderivative that can be expressed as a finite combination of elementary functions. Three standard examples are

$$e^{-x^2},\frac{\sin x}{x},\sqrt{1+x^3}.$$

Each is smooth and perfectly well behaved, so the definite integral — the signed area under the graph — certainly exists. What fails is the *search for a formula*: there is no elementary F with $F'(x)=e^{-x^2}$, so the fundamental theorem cannot be applied directly. When this happens we must **estimate** the definite integral numerically, or hand it to **technology** (a CAS or numerical integrator). Recognising *in advance* that an integrand is of this kind — so that no time is wasted hunting for an antiderivative that does not exist — is an important skill.

A useful contrast: $\int x e^{-x^2}\,dx=-1/2\,e^{-x^2}+c$ **is** elementary, because the extra factor x is (up to a constant) the derivative of the inside function $-x^2$. Remove that factor and the integral of e^{-x^2} alone becomes non-elementary. A small change to the integrand can decide whether an exact answer is available at all.

The definite integral as signed area.

Every numerical method rests on the same idea: the definite integral $\int_a^b f(x)\,dx$ is the signed area between the graph $y=f(x)$ and the x-axis from $x=a$ to $x=b$. If we cannot find that area exactly, we approximate it by simple shapes — rectangles or trapezia — whose areas we *can* compute.

Divide $[a,b]$ into n equal strips, each of width

$$h=\frac{b-a}{n},$$

with **nodes** (dividing points) $x_0=a,x_1=a+h,x_2=a+2h,\ldots,x_n=b$. Write $f_i=f(x_i)$ for the **ordinate** (height) at the i-th node.

Rectangle sums.

The crudest estimate replaces each strip by a rectangle. The **left rectangle sum** uses the height at the left edge of each strip, and the **right rectangle sum** uses the height at the right edge:

$$L=h\left(f_0+f_1+\ldots+f_{n-1}\right),R=h\left(f_1+f_2+\ldots+f_n\right).$$

If f is **increasing** on $[a,b]$ then every left height is the smaller and every right height the larger, so

$$L \le \int_a^b f(x)\,dx \le R,$$

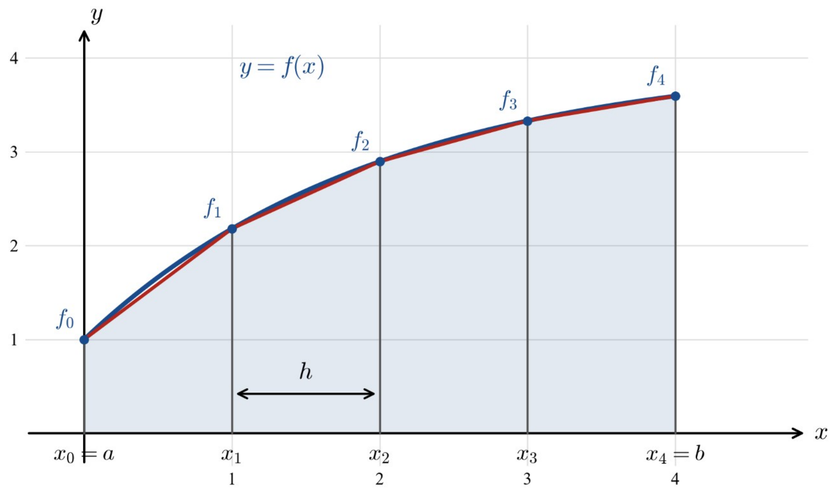
and the two sums *bracket* the true value (the inequalities reverse if f is decreasing). Rectangle sums are simple but coarse; their flat tops leave large gaps under a sloping curve.

The trapezium rule.

A far better estimate joins consecutive points on the curve by a straight **chord**, turning each strip into a **trapezium**. A trapezium of width h with parallel vertical sides f_{i-1} and f_i has area $h/2(f_{i-1}+f_i)$. Adding the n trapezia and collecting terms — every interior ordinate belongs to two adjacent trapezia, so is counted twice — gives the **trapezium rule**:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f_0+2f_1+2f_2+\ldots+2f_{n-1}+f_n\right],\quad h=\frac{b-a}{n}.$$

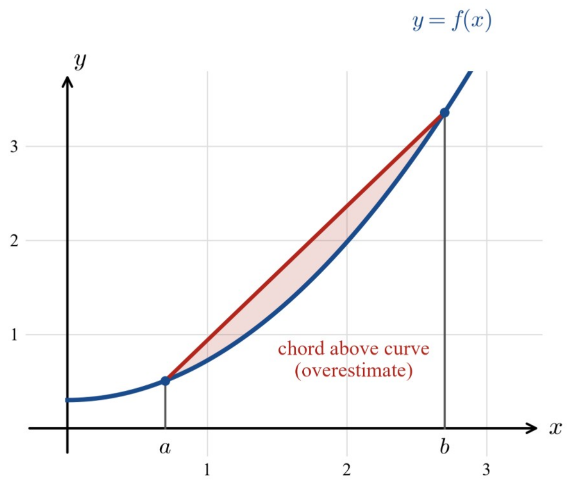
The two **end** ordinates f_0 and f_n have weight 1; every **interior** ordinate has weight 2. Equivalently the trapezium-rule estimate is exactly the average $1/2(L+R)$ of the left and right rectangle sums. The strips are shown below.



Accuracy: over- or under-estimate.

Because each trapezium replaces an arc of the curve by its chord, the sign of the error is governed by the **concavity**.

- If f is **concave up** (convex) on $[a,b]$, every chord lies *above* the curve, so each trapezium contains more than its strip and the trapezium rule **overestimates**.
- If f is **concave down** on $[a,b]$, every chord lies *below* the curve and the rule **underestimates**.



The error also shrinks as the strips get narrower: **doubling** the number of strips n (halving h) makes the chords hug the curve more closely, and the trapezium-rule error falls by a factor of about four. So a coarse estimate can always be refined by using more strips.

Using CAS and technology.

A CAS handles integration in two modes, and interpreting *which* you have is essential.

- **Symbolic (exact) integration** returns a formula. When an elementary antiderivative exists the CAS gives the exact value, e.g. $\int_1^2 1/x\,dx = \log_e 2$. When one does *not* exist, a CAS may still return an exact answer written with a **special function** (such as the error function or a sine-integral); for school purposes that output is treated as a name for the number, to be read off as a decimal.
- **Numerical integration** returns a decimal approximation directly, e.g. $\int_0^1 e^{-x^2}\,dx \approx 0.7468$. This is what technology reports whenever no useful closed form is available, and it is the value we quote.

The guiding principle for **where exact versus numerical is expected**: give an **exact** value whenever the integrand has an elementary antiderivative (use the fundamental theorem, by hand or by CAS); otherwise state a **numerical** estimate — from the trapezium rule or from technology — and say to how many figures it is reliable. A numerical answer should always be reported to a sensible accuracy and, where possible, checked against a second method.

Worked examples

Example 1 — scaffolded

Use the trapezium rule with $n=4$ strips to estimate $\int_1^3 \frac{1}{x}\,dx$, and compare with the exact value $\log_e 3$.

Working	Comment
$h=\frac{3-1}{4}=\frac{1}{2}$; nodes $x_0=1, x_1=3/2, x_2=2, x_3=5/2, x_4=3$.	Strip width $h=\frac{b-a}{n}$; five nodes for four strips.
$f_0=1, f_1=\frac{2}{3}, f_2=\frac{1}{2}, f_3=\frac{2}{5}, f_4=\frac{1}{3}$.	Ordinates $f_i=\frac{1}{x_i}$.
$\int_1^3 \frac{1}{x}\,dx \approx \frac{1/2}{2}\left[f_0+f_4+2(f_1+f_2+f_3)\right]$.	Ends weight 1, interior ordinates weight 2.
$=\frac{1}{4}\left[\frac{4}{3}+2\left(\frac{2}{3}+\frac{1}{2}+\frac{2}{5}\right)\right]=\frac{1}{4}\left[\frac{4}{3}+\frac{47}{15}\right]=\frac{1}{4}\cdot\frac{67}{15}=\frac{67}{60}$.	Exact rational arithmetic; $\frac{67}{60}\approx 1.1167$.
Exact value $\int_1^3 \frac{1}{x}\,dx=\log_e 3\approx 1.0986$.	Here an antiderivative *does* exist, so the exact value is available.
Estimate $1.1167 >$ exact 1.0986: an **over**estimate.	$\frac{1}{x}$ is concave up, so the chords lie above the curve.

Example 2 — scaffolded

Estimate $\int_0^1 e^{-x^2}\,dx$ using the trapezium rule with $n=4$, giving the answer to four decimal places.

Working	Comment
e^{-x^2} has no elementary antiderivative, so the	Recognise the integrand as non-elementary before

Working	Comment
fundamental theorem cannot be applied — a numerical estimate is required.	starting.
$h=\dfrac{1-0}{4}=0.25$; nodes $0,0.25,0.5,0.75,1$.	Four equal strips of width 0.25.
$f_0=1, f_1=e^{-0.0625}\approx 0.9394, f_2=e^{-0.25}\approx 0.7788, f_3=e^{-0}$.	Ordinates $f_i=e^{-x_i^2}$, to four decimal places.
$\int_0^1 e^{-x^2}\,dx\approx\dfrac{0.25}{2}[1+0.3679+2(0.9394+0.7788+0.5($.	Apply the rule with the end and interior weights.
$=0.125[1.3679+4.5760]=0.125\times 5.9439\approx 0.7430$.	The trapezium-rule estimate.
Technology gives $\int_0^1 e^{-x^2}\,dx\approx 0.7468$, so the estimate is a little low.	A CAS returns $\sqrt{\pi}/2\,erf(1)$, read as 0.7468.

Example 3 — unscaffolded

The integrand $\sqrt{1+x^3}$ has no elementary antiderivative. Using $n=4$ strips on $[0,2]$, find the left and right rectangle sums for $\int_0^2 \sqrt{1+x^3}\,dx$, use them to bound the integral, and hence state the trapezium-rule estimate.

With $h=\dfrac{2-0}{4}=0.5$ the nodes are $0,0.5,1,1.5,2$, and the ordinates $f_i=\sqrt{1+x_i^3}$ are

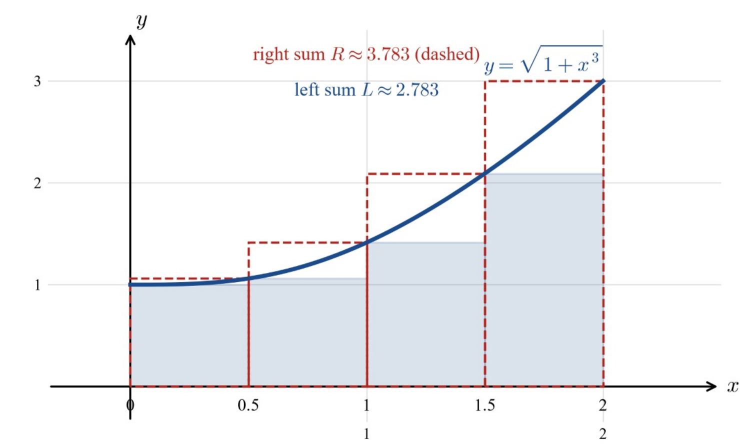
$$f_0=1, f_1=\sqrt{1.125}\approx 1.0607, f_2=\sqrt{2}\approx 1.4142, f_3=\sqrt{4.375}\approx 2.0917, f_4=3.$$

The function is increasing on $[0,2]$, so the left sum is a lower estimate and the right sum an upper estimate:

$$L=0.5(f_0+f_1+f_2+f_3)\approx 0.5\times 5.5666=2.7833,$$

$$R=0.5(f_1+f_2+f_3+f_4)\approx 0.5\times 7.5666=3.7833,$$

so that $2.7833\le\int_0^2 \sqrt{1+x^3}\,dx\le 3.7833$. The left rectangles sit under the curve and the right rectangles poke above it, as shown below.



The trapezium-rule estimate is their average,

$$\int_0^2 \sqrt{1+x^3}\,dx\approx\frac{L+R}{2}=\frac{2.7833+3.7833}{2}=3.2833.$$

$\therefore$ the trapezium rule gives ≈ 3.283; a CAS numerical integration returns ≈ 3.241, so the estimate is a slight overestimate (the curve is concave up).

Example 4 — unscaffolded

Decide whether each of $\int_0^2 x e^{-x^2} dx$ and $\int_0^2 e^{-x^2} dx$ can be evaluated exactly, and give its value.

For the first, notice that $\frac{d}{dx}\left(-1/2 e^{-x^2}\right) = -1/2 \cdot (-2x) e^{-x^2} = x e^{-x^2}$, so an elementary antiderivative *is* available and the fundamental theorem applies:

$$\int_0^2 x e^{-x^2} dx = \left[-1/2 e^{-x^2}\right]_0^2 = -1/2 e^{-4} + 1/2 = 1/2\left(1 - e^{-4}\right) \approx 0.4908.$$

For the second, the factor x is gone, and e^{-x^2} has no elementary antiderivative; no exact elementary value exists, so we integrate numerically. Technology gives

$$\int_0^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2} erf(2) \approx 0.8821.$$

$\therefore$ the first integral is exactly $1/2\left(1 - e^{-4}\right) \approx 0.4908$; the second must be evaluated numerically, giving ≈ 0.8821. The single factor x decides whether an exact answer is possible.

Practice questions

Scaffolded

Q1. Use the trapezium rule with $n = 4$ to estimate $\int_0^2 \frac{1}{1+x^2} dx$. (a) State the strip width h and the five nodes. (b) Evaluate the ordinates (four decimal places where needed). (c) Apply the trapezium-rule formula. (d) The exact value is $\tan^{-1} 2$. Compare your estimate with it.

Q2. Consider $\int_0^1 e^{x^2} dx$. (a) Explain why an elementary antiderivative is not available. (b) With $n = 4$, state h and the ordinates (four decimal places). (c) Apply the trapezium rule. (d) A CAS gives 1.4627; comment on the accuracy of your estimate.

Q3. Estimate $\int_0^1 \frac{1}{1+x} dx$ using $n = 4$ strips. (a) State the nodes and ordinates. (b) Find the left rectangle sum L. (c) Find the right rectangle sum R. (d) The exact value is $\log_e 2 \approx 0.6931$. Verify that it lies between R and L.

Q4. Use the trapezium rule with $n = 4$ to estimate $\int_0^{\pi} \sin x \, dx$. (a) State $h = \frac{\pi}{4}$ and the five nodes. (b) Give the exact ordinates (surd form). (c) Apply the trapezium-rule formula. (d) The exact value is 2. Is the estimate an over- or under-estimate, and why?

Q5. For each integral, state whether an elementary antiderivative exists and, if so, evaluate it exactly. (a) $\int_0^1 x e^{x^2} dx$. (b) $\int_0^1 e^{x^2} dx$. (c) $\int_1^2 \frac{1}{x} dx$.

Q6. A car's speed v (in m/s) is recorded each second:

t (s)	0	1	2	3	4
v (m/s)	0	3	5	6	6

(a) Explain why the distance travelled is $\int_0^4 v\,dt$.

(b) With $h=1$, apply the trapezium rule to the recorded values.

(c) State the estimated distance, with units.

Unscaffolded

Q7. Use the trapezium rule with $n=4$ to estimate $\int_0^2 e^{-x^2}\,dx$, giving four decimal places. A CAS gives 0.8821; state the discrepancy.

Q8. State which of the following have an elementary antiderivative, giving the antiderivative where one exists: (i) $\int e^{-x^2}\,dx$; (ii) $\int x e^{-x^2}\,dx$; (iii) $\int \frac{\sin x}{x}\,dx$; (iv) $\int x\cos(x^2)\,dx$.

Q9. Estimate $\int_1^2 \frac{1}{x}\,dx$ by the trapezium rule with $n=4$. Compare with the exact value $\log_e 2$ and state whether the estimate is an over- or under-estimate.

Q10. Use the trapezium rule with $n=4$ to estimate $\int_0^2 x^2\,dx$, whose exact value is $\frac{8}{3}$. Explain why the trapezium rule does not give the exact value here.

Q11. Estimate $\int_0^1 \frac{\sin x}{x}\,dx$ using $n=4$ strips, taking the integrand to equal 1 at $x=0$ (its limiting value). A CAS gives 0.9461.

Q12. For $\int_1^3 \log_e x\,dx$ with $n=4$, find the left and right rectangle sums, and confirm they bound the exact value $3\log_e 3-2$.

Q13. Use the trapezium rule with $n=4$ to estimate $\int_0^1 \cos(x^2)\,dx$. A CAS gives 0.9045.

Q14. Evaluate $\int_0^1 2x\cos(x^2)\,dx$ exactly, then explain why $\int_0^1 \cos(x^2)\,dx$ must instead be found using technology (a CAS gives 0.9045).

Q15. Estimate $\int_0^1 e^{-x^2}\,dx$ using the trapezium rule with $n=2$. Given that the $n=4$ estimate is 0.7430 and technology gives 0.7468, comment on how the accuracy changes with the number of strips.

Q16. Water flows into a tank at a rate r (in L/min) recorded every 2 minutes:

t (min)	0	2	4	6	8
r (L/min)	0	4	7	9	10

Estimate the volume $\int_0^8 r\,dt$ that flows in during the 8 minutes, using the trapezium rule.

Q17. Use the trapezium rule with $n=4$ to estimate $\int_0^1 e^x \, dx$, whose exact value is $e-1$. State whether the estimate is an over- or under-estimate and explain why.

Q18. The length of the curve $y=\sin x$ from $x=0$ to $x=\pi$ is $\int_0^{\pi} \sqrt{1+\cos^2 x} \, dx$. Explain why this cannot be evaluated with an elementary antiderivative, then estimate it using the trapezium rule with $n=4$.

Answers

Q1. $\frac{287}{260}\approx 1.1038$; exact $\tan^{-1}2\approx 1.1071$ (slight underestimate). **Q2.** (a) e^{x^2} has no elementary antiderivative; (c) ≈ 1.4907; (d) about $1.9\,\%$ above the CAS value 1.4627. **Q3.** $L=\frac{319}{420}\approx 0.7595$, $R=\frac{533}{840}\approx 0.6345$; and $R<\log_e 2<L$. **Q4.** $\frac{\pi(1+\sqrt{2})}{4}\approx 1.8961$; an underestimate, since $\sin x$ is concave down on $[0,\pi]$. **Q5.** (a) $\frac{1}{2}(e-1)\approx 0.8591$; (b) no elementary antiderivative — numerical only, ≈ 1.4627; (c) $\log_e 2\approx 0.6931$. **Q6.** 17 m. **Q7.** ≈ 0.8806; about 0.0015 below the CAS value 0.8821. **Q8.** (i) none; (ii) $-\frac{1}{2}e^{-x^2}+c$; (iii) none; (iv) $\frac{1}{2}\sin(x^2)+c$. **Q9.** $\frac{1171}{1680}\approx 0.6970$; exact $\log_e 2\approx 0.6931$ — an overestimate. **Q10.** $\frac{11}{4}=2.75$; exact $\frac{8}{3}\approx 2.6667$. The graph is curved, so the chords do not follow it exactly. **Q11.** ≈ 0.9445 (CAS 0.9461). **Q12.** $L\approx 1.0075$, $R\approx 1.5568$; and $L<3\log_e 3-2\approx 1.2958<R$. **Q13.** ≈ 0.8958 (CAS 0.9045). **Q14.** $\int_0^1 2x\cos(x^2)\,dx=\sin 1\approx 0.8415$; without the factor x, $\cos(x^2)$ has no elementary antiderivative. **Q15.** $n=2$ gives ≈ 0.7314; more strips give a closer estimate ($0.7314\to 0.7430\to 0.7468$). **Q16.** 50 L. **Q17.** ≈ 1.7272; exact $e-1\approx 1.7183$ — an overestimate, since e^x is concave up. **Q18.** $\frac{\pi(1+\sqrt{2}+\sqrt{6})}{4}\approx 3.8199$ (CAS 3.8202); $\sqrt{1+\cos^2 x}$ has no elementary antiderivative.

Fully-worked solutions

Q1. Here $h=\frac{2-0}{4}=0.5$, with nodes $0, 0.5, 1, 1.5, 2$. The ordinates $f_i=\frac{1}{1+x_i^2}$ are

$$f_0=1, f_1=0.8, f_2=0.5, f_3=1/3.25\approx 0.3077, f_4=0.2.$$

By the trapezium rule,

$$\int_0^2 \frac{1}{1+x^2}\,dx\approx\frac{0.5}{2}\left[1+0.2+2(0.8+0.5+0.3077)\right]=0.25\times 4.4154\approx 1.1038,$$

in exact form $\frac{287}{260}$. The exact value is $\tan^{-1}2\approx 1.1071$, so the estimate is slightly low.

Q2. (a) e^{x^2} has no antiderivative expressible with elementary functions (the factor needed to reverse the chain rule, a stray $2x$, is not present), so the fundamental theorem cannot be used. (b) With $h=0.25$ and nodes $0, 0.25, 0.5, 0.75, 1$,

$$f_0=1, f_1=e^{0.0625}\approx 1.0645, f_2=e^{0.25}\approx 1.2840, f_3=e^{0.5625}\approx 1.7551, f_4=e\approx 2.7183.$$

(c) Then

$$\int_0^1 e^{x^2}\,dx\approx\frac{0.25}{2}\left[1+2.7183+2(1.0645+1.2840+1.7551)\right]=0.125\times 11.9255\approx 1.4907.$$

(d) This is about $1.9\,\%$ above the CAS value 1.4627; the overestimate is expected because e^{x^2} is concave up.

Q3. With $h=0.25$ and nodes $0, 0.25, 0.5, 0.75, 1$, the ordinates $f_i=\frac{1}{1+x_i}$ are $1, 0.8, 0.6667, 0.5714, 0.5$. The left and right sums are

$$L=0.25(1+0.8+0.6667+0.5714)=0.25\times 3.0381\approx 0.7595=319/420,$$

$$R=0.25(0.8+0.6667+0.5714+0.5)=0.25\times 2.5381\approx 0.6345=533/840.$$

As $\frac{1}{1+x}$ is decreasing, the left rectangles are too tall and the right too short, so $R<\log_e 2<L$, i.e. $0.6345<0.6931<0.7595$, as required.

Q4. (a) $h=\frac{\pi}{4}$, with nodes $0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi$. (b) The exact ordinates are

$$f_0=0, f_1=\sqrt{2}/2, f_2=1, f_3=\sqrt{2}/2, f_4=0.$$

(c) By the trapezium rule,

$$\int_0^\pi \sin x\,dx\approx\frac{\pi/4}{2}\left[0+0+2\left(\sqrt{2}/2+1+\sqrt{2}/2\right)\right]=\frac{\pi}{8}\cdot 2\left(1+\sqrt{2}\right)=\frac{\pi\left(1+\sqrt{2}\right)}{4}\approx 1.8961.$$

(d) Since the exact value is 2, the estimate 1.8961 is an underestimate; $\sin x$ is concave down on $[0,\pi]$, so each chord lies below the curve.

Q5. (a) Because $\frac{d}{dx}\left(1/2\,e^{x^2}\right)=x\,e^{x^2}$, an antiderivative exists and

$$\int_0^1 x\,e^{x^2}\,dx=\left[1/2\,e^{x^2}\right]_0^1=1/2(e-1)\approx 0.8591.$$

(b) Removing the factor x leaves e^{x^2}, which has no elementary antiderivative; the integral can only be found numerically, ≈ 1.4627. (c) $\int_1^2 1/x\,dx=[\log_e x]_1^2=\log_e 2\approx 0.6931$.

Q6. (a) Speed is the *magnitude* of the velocity — the rate at which the distance travelled increases. Every recorded value of v is non-negative, so the car never reverses and each short interval contributes (speed) × (time) to the distance. Adding those contributions gives the area under the speed–time graph, $\int_0^4 v\,dt$. (b) With $h=1$ and recorded values $0,3,5,6,6$,

$$\int_0^4 v\,dt\approx\frac{1}{2}[0+6+2(3+5+6)]=\frac{1}{2}(6+28)=17.$$

(c) The estimated distance is 17 m.

Q7. With $h=0.5$ and nodes $0,0.5,1,1.5,2$, the ordinates $f_i=e^{-x_i^2}$ are

$$f_0=1, f_1=e^{-0.25}\approx 0.7788, f_2=e^{-1}\approx 0.3679, f_3=e^{-2.25}\approx 0.1054, f_4=e^{-4}\approx 0.0183.$$

Then

$$\int_0^2 e^{-x^2}\,dx\approx\frac{0.5}{2}\left[1+0.0183+2\left(0.7788+0.3679+0.1054\right)\right]=0.25\times 3.5225\approx 0.8806.$$

This is about 0.0015 below the CAS value 0.8821.

Q8. (i) e^{-x^2} has **no** elementary antiderivative. (ii) $\int x\,e^{-x^2}\,dx=-1/2\,e^{-x^2}+c$, since the factor x supplies (up to a constant) the derivative of the inside $-x^2$. (iii) $\frac{\sin x}{x}$ has **no** elementary antiderivative. (iv)

$\int x\cos(x^2)\,dx = 1/2\sin(x^2)+c$, because $\dfrac{d}{dx}\sin(x^2)=2x\cos(x^2)$. So (ii) and (iv) can be integrated exactly; (i) and (iii) cannot.

Q9. With $h=0.25$ and nodes $1, 1.25, 1.5, 1.75, 2$, the ordinates $f_i=\dfrac{1}{x_i}$ are $1, 0.8, 0.6667, 0.5714, 0.5$. Then

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.25}{2}\left[1+0.5+2(0.8+0.6667+0.5714)\right]=0.125\times 5.5762\approx 0.6970,$$

in exact form $\dfrac{1171}{1680}$. The exact value is $\log_e 2\approx 0.6931$, so the estimate is an overestimate — expected, as $\dfrac{1}{x}$ is concave up.

Q10. With $h=0.5$ and nodes $0, 0.5, 1, 1.5, 2$, the ordinates $f_i=x_i^2$ are $0, 0.25, 1, 2.25, 4$. Then

$$\int_0^2 x^2\,dx \approx \frac{0.5}{2}\left[0+4+2(0.25+1+2.25)\right]=0.25\times 11=\frac{11}{4}=2.75.$$

The exact value is $\dfrac{8}{3}\approx 2.6667$. The trapezium rule replaces the parabola by straight chords; since the graph is genuinely curved (concave up), the chords lie above it and the estimate is too large. The rule is exact only for straight-line graphs.

Q11. Taking $f(0)=1$, with $h=0.25$ and nodes $0, 0.25, 0.5, 0.75, 1$ the ordinates $f_i=\dfrac{\sin x_i}{x_i}$ are

$$f_0=1, f_1\approx 0.9896, f_2\approx 0.9589, f_3\approx 0.9089, f_4=\sin 1\approx 0.8415.$$

Then

$$\int_0^1 \frac{\sin x}{x}\,dx \approx \frac{0.25}{2}\left[1+0.8415+2(0.9896+0.9589+0.9089)\right]=0.125\times 7.5563\approx 0.9445,$$

close to the CAS value 0.9461.

Q12. With $h=0.5$ and nodes $1, 1.5, 2, 2.5, 3$, the ordinates $f_i=\log_e x_i$ are $0, 0.4055, 0.6931, 0.9163, 1.0986$. The rectangle sums are

$$L=0.5(0+0.4055+0.6931+0.9163)\approx 0.5\times 2.0149=1.0075,$$

$$R=0.5(0.4055+0.6931+0.9163+1.0986)\approx 0.5\times 3.1135=1.5568.$$

Since $\log_e x$ is increasing, $L<\int_1^3 \log_e x\,dx<R$. The exact value is $[x\log_e x-x]_1^3=3\log_e 3-2\approx 1.2958$, which indeed satisfies $1.0075<1.2958<1.5568$.

Q13. With $h=0.25$ and nodes $0, 0.25, 0.5, 0.75, 1$, the ordinates $f_i=\cos(x_i^2)$ are

$$f_0=1, f_1=\cos 0.0625\approx 0.99805, f_2=\cos 0.25\approx 0.96891, f_3=\cos 0.5625\approx 0.84592, f_4=\cos 1\approx 0.54030.$$

Then

$$\int_0^1 \cos(x^2)\,dx \approx \frac{0.25}{2}\left[1+0.54030+2(0.99805+0.96891+0.84592)\right]=0.125\times 7.16606\approx 0.8958,$$

close to the CAS value 0.9045. (A fifth decimal place is carried in the ordinates here so that the fourth decimal place of the estimate is secure.)

Q14. Since $\frac{d}{dx}\sin(x^2)=2x\cos(x^2)$,

$$\int_0^1 2x\cos(x^2)\,dx=[\sin(x^2)]_0^1=\sin 1-\sin 0=\sin 1\approx 0.8415.$$

The factor $2x$ is exactly what the chain rule needs, so this integral is elementary. Remove it and $\cos(x^2)$ has no elementary antiderivative, so $\int_0^1 \cos(x^2)\,dx$ can only be found numerically (a CAS gives 0.9045).

Q15. With $n=2$, $h=0.5$ and nodes $0, 0.5, 1$, the ordinates are $f_0=1, f_1=e^{-0.25}\approx 0.7788, f_2=e^{-1}\approx 0.3679$. Then

$$\int_0^1 e^{-x^2}\,dx\approx\frac{0.5}{2}\left[1+0.3679+2(0.7788)\right]=0.25\times 2.9255\approx 0.7314.$$

Comparing the three values $0.7314\,(n=2)$, $0.7430\,(n=4)$ and 0.7468 (technology): as the number of strips increases the estimate rises steadily towards the true value, so more strips give greater accuracy. Doubling n roughly quarters the error.

Q16. With $h=2$ and recorded rates $0, 4, 7, 9, 10$,

$$\int_0^8 r\,dt\approx\frac{2}{2}\left[0+10+2(4+7+9)\right]=1\times(10+40)=50.$$

So about 50 L flows into the tank during the 8 minutes.

Q17. With $h=0.25$ and nodes $0, 0.25, 0.5, 0.75, 1$, the ordinates $f_i=e^{x_i}$ are

$$f_0=1, f_1=e^{0.25}\approx 1.2840, f_2=e^{0.5}\approx 1.6487, f_3=e^{0.75}\approx 2.1170, f_4=e\approx 2.7183.$$

Then

$$\int_0^1 e^x\,dx\approx\frac{0.25}{2}\left[1+2.7183+2(1.2840+1.6487+2.1170)\right]=0.125\times 13.8177\approx 1.7272.$$

The exact value is $e-1\approx 1.7183$, so the estimate is an overestimate; e^x is concave up, so its chords lie above the curve.

Q18. The arc-length integrand $\sqrt{1+\cos^2 x}$ is not the derivative of any elementary function (it is an *elliptic* integral), so no exact elementary value exists and a numerical method is required. With $h=\frac{\pi}{4}$ and nodes $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, the ordinates $f_i=\sqrt{1+\cos^2 x_i}$ are

$$f_0=\sqrt{2}, f_1=\sqrt{3/2}\approx 1.2247, f_2=1, f_3=\sqrt{3/2}\approx 1.2247, f_4=\sqrt{2}.$$

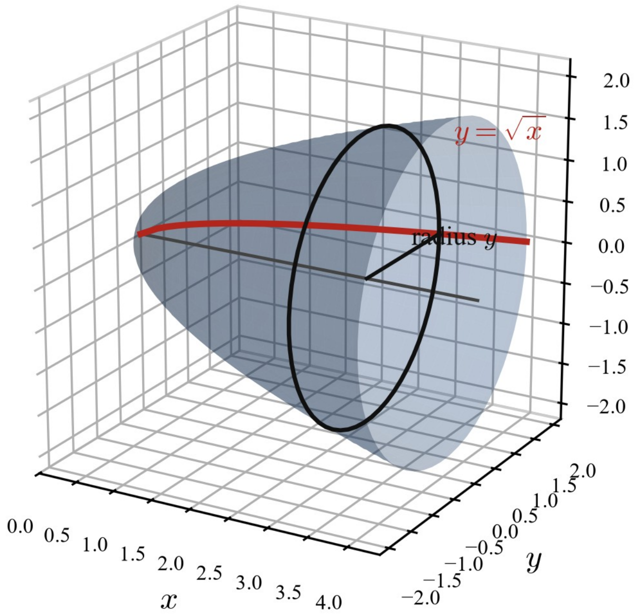
Then

$$\int_0^\pi \sqrt{1+\cos^2 x}\,dx\approx\frac{\pi/4}{2}\left[\sqrt{2}+\sqrt{2}+2(1.2247+1+1.2247)\right]=\frac{\pi}{8}\times 9.7272\approx 3.8199,$$

in exact form $\frac{\pi\left(1+\sqrt{2}+\sqrt{6}\right)}{4}$. A CAS numerical integration gives 3.8202, so the trapezium-rule estimate is accurate to three decimal places.

Theory

When a region of the plane is rotated one full turn about a fixed straight line, it sweeps out a three-dimensional shape called a **solid of revolution**. A bowl, a cone, a rugby ball and a sphere can all be produced this way. This section finds the **volume** of such a solid when a region is rotated about the x-axis or the y-axis, by adding up infinitely many thin circular slices with a definite integral.

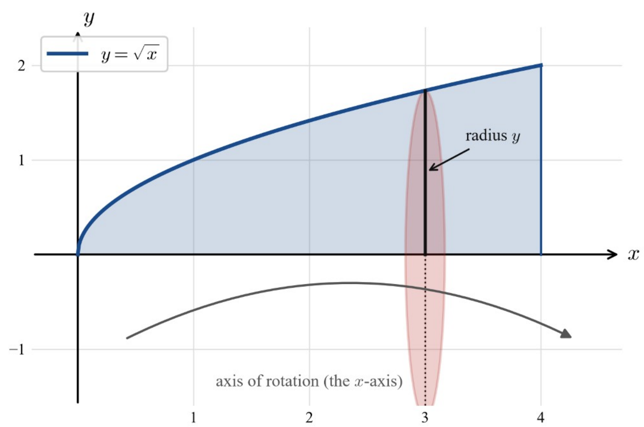


The disc method — rotation about the x-axis.

Consider the region bounded by the curve $y=f(x)$, the x-axis, and the vertical lines $x=a$ and $x=b$. Imagine cutting the region into thin vertical strips. A strip at position x, of width δx and height y, sweeps out a thin **disc** (a flat cylinder) when the region is rotated about the x-axis: the radius of the disc is the height y of the strip and its thickness is δx. The circular face has area πy^2, so the volume of the disc is approximately $\pi y^2 \delta x$. Adding the volumes of all such discs from $x=a$ to $x=b$ and letting the thickness shrink to zero replaces the sum by an integral:

$$V=\pi\int_a^b y^2\,dx\,,$$

where $y=f(x)$ must be written as a function of x before integrating. The radius of a typical disc is simply the distance from the axis to the curve, namely y.



Rotation about the y-axis.

If instead the region — now bounded by the curve, the y-axis and the horizontal lines $y=c$ and $y=d$ — is rotated about the y-axis, the thin discs are horizontal. A typical disc has radius x (the distance from the y-axis across to the curve) and thickness δy, so

$$V=\pi\int_c^d x^2\,dy.$$

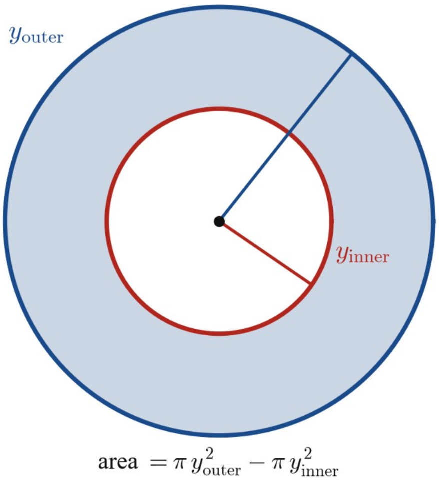
Here the curve must be rearranged so that x, or better x^2, is a function of y, and the terminals c and d are **y-values**. Choosing the axis of rotation therefore decides which variable to integrate with respect to: about the x-axis integrate y^2 with respect to x; about the y-axis integrate x^2 with respect to y.

Volumes between two curves — the washer method.

Suppose the region lies between an **outer** curve of height y_{outer} and an **inner** curve of height y_{inner}, with $y_{\text{outer}}\ge y_{\text{inner}}\ge 0$ throughout $[a,b]$. Rotating about the x-axis, each slice is now a **washer** (an annulus): a large disc of radius y_{outer} with a smaller disc of radius y_{inner} punched out of its centre. The area of the washer face is $\pi y_{\text{outer}}^2-\pi y_{\text{inner}}^2$, so

$$V=\pi\int_a^b\left(y_{\text{outer}}^2-y_{\text{inner}}^2\right)dx.$$

It is the **squares** of the radii that are subtracted, one from the other — the volume is not found from $(y_{\text{outer}}-y_{\text{inner}})^2$, which is a different quantity. The same reasoning about the y-axis gives $V=\pi\int_c^d\left(x_{\text{outer}}^2-x_{\text{inner}}^2\right)dy$.



Setting up the radius and the limits.

Three steps turn any such problem into a routine integral.

1. **Sketch** the region and mark the axis of rotation. This shows whether the slices are vertical discs (axis is the x-axis) or horizontal discs (axis is the y-axis), and whether one curve or two bound the region.
2. **Identify the radius.** For a single curve the radius is the distance from the axis to the curve (y about the x-axis, or x about the y-axis). For a region between two curves, find the outer and inner radii separately.
3. **Find the limits.** These are the values of the integrating variable at the ends of the region. Where two curves bound the region, the limits are the coordinates of the points of intersection, found by solving the two equations simultaneously.

Exact answers. Each volume is a single number, and throughout this section the boundaries are chosen so that it is **exact** — a multiple of π such as 8π or $\frac{2\pi}{15}$, or occasionally a multiple of π^2 such as $\frac{\pi^2}{2}$. Leave answers in exact form, and quote a volume in cubic units.

Worked examples

Example 1 — scaffolded

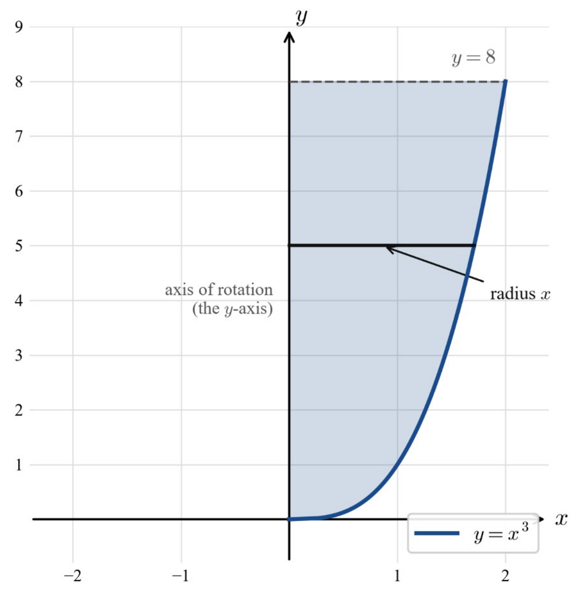
Find the volume of the solid generated when the region bounded by $y=\sqrt{x}$, the x-axis and the line $x=4$ is rotated about the x-axis. Give an exact answer.

Working	Comment
The region runs from $x=0$, where $\sqrt{x}=0$, to $x=4$; a typical disc has radius $y=\sqrt{x}$.	Rotation about the x-axis, so use the disc method with radius y.
$V=\pi\int_0^4 y^2\,dx$	Volume of a solid of revolution about the x-axis.
$y^2=(\sqrt{x})^2=x$	Square the radius and write it as a function of x.
$V=\pi\int_0^4 x\,dx=\pi\left[\frac{x^2}{2}\right]_0^4$	Antidifferentiate x.
$=\pi\left(\frac{16}{2}-0\right)=8\pi$	Evaluate: the volume is 8π cubic units.

Example 2 — scaffolded

The region in the first quadrant bounded by $y=x^3$, the y-axis and the line $y=8$ is rotated about the y-axis. Find the exact volume.

Working	Comment
$y=x^3 \Rightarrow x=y^{1/3}$, so $x^2=y^{2/3}$; the region runs from $y=0$ to $y=8$.	Rotation about the y-axis, so express x^2 as a function of y.
$V=\pi\int_0^8 x^2\,dy=\pi\int_0^8 y^{2/3}\,dy$	Volume of a solid of revolution about the y-axis; radius x.
$=\pi\left[\frac{3}{5}y^{5/3}\right]_0^8$	Antidifferentiate: $\int y^{2/3}\,dy=\frac{3}{5}y^{5/3}$.
$=\pi\cdot\frac{3}{5}\left(8^{5/3}-0\right)=\pi\cdot\frac{3}{5}\cdot 32$	$8^{5/3}=\left(8^{1/3}\right)^5=2^5=32$.
$=\frac{96\pi}{5}$	The volume is $\frac{96\pi}{5}$ cubic units.

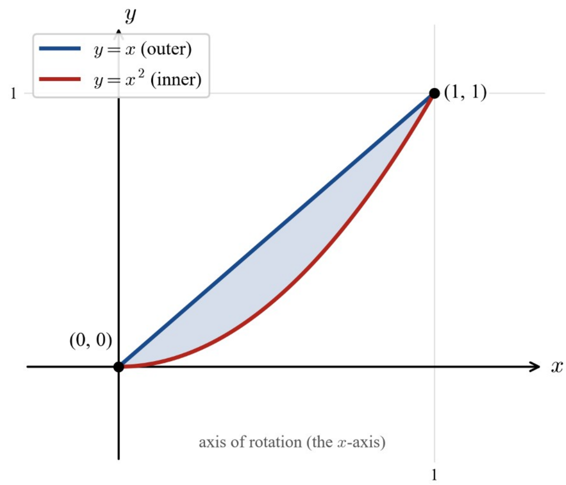


Example 3 — unscaffolded

The region enclosed between the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the exact volume of the resulting solid.

The curves meet where $x = x^2$, that is $x(1 - x) = 0$, giving $x = 0$ and $x = 1$. On the interval $[0, 1]$ the line lies above the parabola, $x \geq x^2$, so the line $y = x$ gives the **outer** radius and the parabola $y = x^2$ the **inner** radius. Rotating about the x-axis produces washers of outer radius x and inner radius x^2, so

$$V = \pi \int_0^1 \left(x^2 - (x^2)^2\right) dx = \pi \int_0^1 \left(x^2 - x^4\right) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$



$\therefore$ the solid of revolution has volume $\dfrac{2\pi}{15}$ cubic units.

Example 4 — unscaffolded

The region bounded by $y = \sin x$ and the x-axis between $x = 0$ and $x = \pi$ is rotated about the x-axis. Find the exact volume.

A typical disc has radius $y = \sin x$, so the volume is

$$V = \pi \int_0^\pi \sin^2 x \, dx.$$

The square is handled with the double-angle identity $\sin^2 x = 1/2(1-\cos 2x)$:

$$V = \pi \int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi}{2}[(\pi-0)-(0-0)] = \frac{\pi^2}{2},$$

using $\sin 2\pi = \sin 0 = 0$ to clear both terminals of the sine term.

$\therefore$ the volume is $\frac{\pi^2}{2}$ cubic units.

Practice questions

Scaffolded

Q1. The region bounded by $y = x^2$, the x-axis and the line $x = 2$ is rotated about the x-axis. (a) State the radius of a typical disc and the limits of integration. (b) Write the volume as $V = \pi \int_a^b y^2\,dx$ with the limits filled in. (c) Replace y^2 and antidifferentiate. (d) Evaluate to an exact multiple of π.

Q2. The region bounded by the line $y = 2x$, the x-axis and the line $x = 3$ is rotated about the x-axis, forming a cone. (a) State the radius y and the limits. (b) Write $V = \pi \int_0^3 y^2\,dx$ and replace y^2. (c) Antidifferentiate and evaluate. (d) Check the answer with the cone formula $V = 1/3\,\pi r^2 h$, using $r = 6$ and $h = 3$.

Q3. The region in the first quadrant bounded by $y = x^2$, the y-axis and the line $y = 4$ is rotated about the y-axis. (a) Rearrange $y = x^2$ to give x^2 in terms of y. (b) Write $V = \pi \int_c^d x^2\,dy$ with the limits filled in. (c) Antidifferentiate. (d) Evaluate exactly.

Q4. The region bounded by the line $y = 2x$, the y-axis and the line $y = 4$ is rotated about the y-axis, forming a cone. (a) Make x the subject and square it. (b) Write $V = \pi \int_0^4 x^2\,dy$. (c) Antidifferentiate and evaluate. (d) Confirm the answer with the cone formula, using $r = 2$ and $h = 4$.

Q5. The region enclosed between $y = \sqrt{x}$ and $y = x$ is rotated about the x-axis. (a) Find the x-coordinates where the two curves meet. (b) State which curve gives the outer radius and which the inner on $[0, 1]$. (c) Write $V = \pi \int_0^1 \left(y_{\text{outer}}^2 - y_{\text{inner}}^2\right)dx$ and simplify the integrand. (d) Integrate and evaluate.

Q6. The region bounded by $y = e^x$, the x-axis, the y-axis and the line $x = 1$ is rotated about the x-axis. (a) Square the radius to find y^2. (b) Write $V = \pi \int_0^1 e^{2x}\,dx$. (c) Antidifferentiate e^{2x}. (d) Evaluate to an exact answer in terms of e and π.

Unscaffolded

Q7. The region bounded by $y = x^2 + 1$, the x-axis, the y-axis and the line $x = 1$ is rotated about the x-axis. Find the exact volume.

Q8. The region bounded by $y = 3x$, the x-axis and the line $x = 2$ is rotated about the x-axis. Find the exact volume.

Q9. The region bounded by $y = \sqrt{4 - x^2}$ and the x-axis is rotated about the x-axis, generating a sphere. Find its exact volume.

Q10. The region in the first quadrant bounded by $y = x^2$, the y-axis and the line $y = 2$ is rotated about the y-axis. Find the exact volume.

Q11. The triangular region bounded by the line $y = 4 - x$ and the two coordinate axes is rotated about the y-axis. Find the exact volume.

Q12. The region enclosed between $y = 2x$ and $y = x^2$ is rotated about the x-axis. Find the exact volume.

Q13. The region enclosed between $y = x^2$ and $y = 8 - x^2$ is rotated about the x-axis. Find the exact volume.

Q14. The region bounded by $y = x^3$, the x-axis and the line $x = 2$ is rotated about the x-axis. Find the exact volume.

Q15. The region bounded by $y = \cos x$, the coordinate axes and the line $x = \frac{\pi}{2}$ is rotated about the x-axis. Find the exact volume.

Q16. The region bounded by $y = \frac{1}{\sqrt{x}}$, the x-axis and the lines $x = 1$ and $x = 4$ is rotated about the x-axis. Find the exact volume.

Q17. The region enclosed between $y = 2\sqrt{x}$ and $y = x$ is rotated about the x-axis. Find the exact volume.

Q18. The region enclosed between $y = x$ and $y = x^2$ is rotated about the y-axis. Find the exact volume.

Answers

Q1. $\frac{32\pi}{5}$. **Q2.** 36π. **Q3.** 8π. **Q4.** $\frac{16\pi}{3}$. **Q5.** $\frac{\pi}{6}$. **Q6.** $\frac{\pi(e^2-1)}{2}$. **Q7.** $\frac{28\pi}{15}$. **Q8.** 24π. **Q9.** $\frac{32\pi}{3}$. **Q10.** 2π. **Q11.** $\frac{64\pi}{3}$. **Q12.** $\frac{64\pi}{15}$. **Q13.** $\frac{512\pi}{3}$. **Q14.** $\frac{128\pi}{7}$. **Q15.** $\frac{\pi^2}{4}$. **Q16.** $\pi\log_e 4$. **Q17.** $\frac{32\pi}{3}$. **Q18.** $\frac{\pi}{6}$.

Fully-worked solutions

Q1. Rotation about the x-axis, so the radius is $y=x^2$ and the limits are $x=0$ to $x=2$. Then $y^2=(x^2)^2=x^4$, and

$$V=\pi\int_0^2 y^2\,dx=\pi\int_0^2 x^4\,dx=\pi\left[\frac{x^5}{5}\right]_0^2=\pi\cdot\frac{32}{5}=\frac{32\pi}{5}.$$

Q2. The radius is $y=2x$, with limits $x=0$ to $x=3$, so $y^2=4x^2$ and

$$V=\pi\int_0^3 (2x)^2\,dx=\pi\int_0^3 4x^2\,dx=4\pi\left[\frac{x^3}{3}\right]_0^3=4\pi\cdot\frac{27}{3}=36\pi.$$

As a check, the solid is a cone of base radius $r=2\times3=6$ and height $h=3$, and $1/3\,\pi r^2 h=1/3\,\pi(6)^2(3)=36\pi$, which agrees.

Q3. Rotation about the y-axis, so rearrange $y=x^2$ to $x^2=y$; the limits are $y=0$ to $y=4$. Then

$$V=\pi\int_0^4 x^2\,dy=\pi\int_0^4 y\,dy=\pi\left[\frac{y^2}{2}\right]_0^4=\pi\cdot\frac{16}{2}=8\pi.$$

Q4. Making x the subject of $y=2x$ gives $x=\frac{y}{2}$, so $x^2=\frac{y^2}{4}$, with limits $y=0$ to $y=4$. Then

$$V=\pi\int_0^4 x^2\,dy=\pi\int_0^4 \frac{y^2}{4}\,dy=\frac{\pi}{4}\left[\frac{y^3}{3}\right]_0^4=\frac{\pi}{4}\cdot\frac{64}{3}=\frac{16\pi}{3}.$$

As a check, the cone has base radius $r=2$ (where $y=4$, $x=2$) and height $h=4$, and $1/3\,\pi r^2 h=1/3\,\pi(2)^2(4)=\frac{16\pi}{3}$, which agrees.

Q5. The curves $y=\sqrt{x}$ and $y=x$ meet where $\sqrt{x}=x$; squaring, $x=x^2$, so $x(1-x)=0$ and $x=0$ or $x=1$. On $[0,1]$, $\sqrt{x}\ge x$, so $y=\sqrt{x}$ is the outer radius and $y=x$ the inner radius. Then

$$V=\pi\int_0^1 \left((\sqrt{x})^2-x^2\right)dx=\pi\int_0^1 \left(x-x^2\right)dx=\pi\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{6}.$$

Q6. The radius is $y=e^x$, so $y^2=(e^x)^2=e^{2x}$, with limits $x=0$ to $x=1$. Since $\int e^{2x}\,dx=1/2\,e^{2x}+c$,

$$V=\pi\int_0^1 e^{2x}\,dx=\pi\left[\frac{e^{2x}}{2}\right]_0^1=\frac{\pi}{2}\left(e^2-e^0\right)=\frac{\pi(e^2-1)}{2}.$$

Q7. Radius $y=x^2+1$, limits $x=0$ to $x=1$. Expanding the square, $y^2=(x^2+1)^2=x^4+2x^2+1$, so

$$V=\pi\int_0^1 \left(x^4+2x^2+1\right)dx=\pi\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_0^1=\pi\left(\frac{1}{5}+\frac{2}{3}+1\right)=\pi\cdot\frac{3+10+15}{15}=\frac{28\pi}{15}.$$

Q8. Radius $y=3x$, so $y^2=9x^2$, limits $x=0$ to $x=2$. Then

$$V=\pi\int_0^2 9x^2\,dx=9\pi\left[\frac{x^3}{3}\right]_0^2=9\pi\cdot\frac{8}{3}=24\pi.$$

(The solid is a cone with $r=6$, $h=2$, and $1/3\pi(6)^2(2)=24\pi$.)

Q9. The semicircle $y=\sqrt{4-x^2}$ meets the x-axis at $x=-2$ and $x=2$, and $y^2=4-x^2$. Rotating about the x-axis gives a sphere of radius 2:

$$V=\pi\int_{-2}^2 (4-x^2)\,dx=\pi\left[4x-\frac{x^3}{3}\right]_{-2}^2=\pi\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right]=\pi\left(16-\frac{16}{3}\right)=\frac{32\pi}{3}.$$

This matches the sphere formula $4/3\pi r^3=4/3\pi(2)^3=\dfrac{32\pi}{3}$.

Q10. Rotation about the y-axis, so $x^2=y$, with limits $y=0$ to $y=2$. Then

$$V=\pi\int_0^2 x^2\,dy=\pi\int_0^2 y\,dy=\pi\left[\frac{y^2}{2}\right]_0^2=\pi\cdot\frac{4}{2}=2\pi.$$

Q11. The line $y=4-x$ crosses the axes at $(4,0)$ and $(0,4)$, so the triangle has vertices $(0,0)$, $(4,0)$ and $(0,4)$. Rotating about the y-axis, the radius of a typical horizontal disc is $x=4-y$, with limits $y=0$ to $y=4$. Then

$$V=\pi\int_0^4 (4-y)^2\,dy=\pi\left[-\frac{(4-y)^3}{3}\right]_0^4=\pi\left(0-\left(-\frac{64}{3}\right)\right)=\frac{64\pi}{3}.$$

(This is a cone with $r=4$, $h=4$, and $1/3\pi(4)^2(4)=\dfrac{64\pi}{3}$.)

Q12. The curves $y=2x$ and $y=x^2$ meet where $2x=x^2$, so $x(x-2)=0$ and $x=0$ or $x=2$. On $[0,2]$, $2x\geq x^2$, so $y=2x$ is the outer radius and $y=x^2$ the inner. Then

$$V=\pi\int_0^2 \left((2x)^2-(x^2)^2\right)dx=\pi\int_0^2 \left(4x^2-x^4\right)dx=\pi\left[\frac{4x^3}{3}-\frac{x^5}{5}\right]_0^2=\pi\left(\frac{32}{3}-\frac{32}{5}\right)=\frac{64\pi}{15}.$$

Q13. The curves $y=x^2$ and $y=8-x^2$ meet where $x^2=8-x^2$, so $2x^2=8$, $x^2=4$ and $x=\pm 2$. On $[-2,2]$, $8-x^2\geq x^2$, so $y=8-x^2$ is the outer radius and $y=x^2$ the inner. Then

$$V=\pi\int_{-2}^2 \left((8-x^2)^2-(x^2)^2\right)dx=\pi\int_{-2}^2 \left(64-16x^2\right)dx,$$

since $(8-x^2)^2-x^4=64-16x^2+x^4-x^4=64-16x^2$. The integrand is even, so

$$V=2\pi\int_0^2 \left(64-16x^2\right)dx=2\pi\left[64x-\frac{16x^3}{3}\right]_0^2=2\pi\left(128-\frac{128}{3}\right)=2\pi\cdot\frac{256}{3}=\frac{512\pi}{3}.$$

Q14. Radius $y=x^3$, so $y^2=x^6$, limits $x=0$ to $x=2$. Then

$$V=\pi\int_0^2 x^6\,dx=\pi\left[\frac{x^7}{7}\right]_0^2=\pi\cdot\frac{128}{7}=\frac{128\pi}{7}.$$

Q15. Radius $y=\cos x$, limits $x=0$ to $x=\dfrac{\pi}{2}$. Using $\cos^2 x=1/2(1+\cos 2x)$,

$$V=\pi\int_0^{\pi/2}\cos^2 x\,dx=\pi\int_0^{\pi/2}\frac{1+\cos 2x}{2}\,dx=\frac{\pi}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}=\frac{\pi}{2}\left(\frac{\pi}{2}+0\right)=\frac{\pi^2}{4},$$

since $\sin\pi=\sin 0=0$.

Q16. Radius $y=\frac{1}{\sqrt{x}}$, so $y^2=\frac{1}{x}$, limits $x=1$ to $x=4$. Then

$$V=\pi\int_1^4\frac{1}{x}\,dx=\pi[\log_e x]_1^4=\pi(\log_e 4-\log_e 1)=\pi\log_e 4,$$

using $\log_e 1=0$.

Q17. The curves $y=2\sqrt{x}$ and $y=x$ meet where $2\sqrt{x}=x$; squaring, $4x=x^2$, so $x(x-4)=0$ and $x=0$ or $x=4$. On $[0,4]$, $2\sqrt{x}\geq x$, so $y=2\sqrt{x}$ is the outer radius and $y=x$ the inner. Then

$$V=\pi\int_0^4\left((2\sqrt{x})^2-x^2\right)dx=\pi\int_0^4\left(4x-x^2\right)dx=\pi\left[2x^2-\frac{x^3}{3}\right]_0^4=\pi\left(32-\frac{64}{3}\right)=\frac{32\pi}{3}.$$

Q18. Rotation is about the y-axis, so work with x as a function of y. The curves $y=x$ and $y=x^2$ meet at $(0,0)$ and $(1,1)$. Solving for x: the line gives $x=y$ and the parabola gives $x=\sqrt{y}$. For y in $[0,1]$, $\sqrt{y}\geq y$, so the parabola provides the outer radius $x=\sqrt{y}$ and the line the inner radius $x=y$. Then

$$V=\pi\int_0^1\left((\sqrt{y})^2-y^2\right)dy=\pi\int_0^1\left(y-y^2\right)dy=\pi\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{6}.$$

Theory

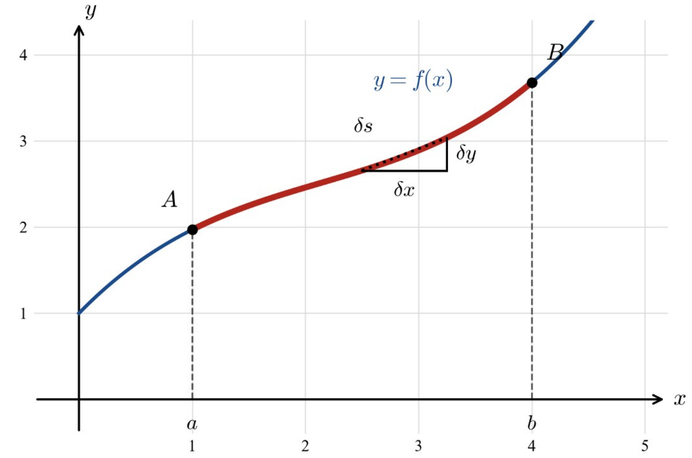
Earlier applications of integration measured the **area** of a region or the **volume** of a solid. Integration can also measure the **length of a curved arc** itself — for example the length of a cable hanging between two poles, or the distance travelled along a curved path. This length is called the **arc length**.

Approximating the arc by chords.

A curved arc has no length formula from elementary geometry, so we approximate it. Divide the arc into many short pieces. Each piece is very nearly a straight **chord**, and a chord with horizontal run δx and vertical rise δy has length

$$\delta s \approx \sqrt{(\delta x)^2 + (\delta y)^2}$$

by Pythagoras' theorem. Adding the lengths of all the chords and letting each δx shrink to zero turns the sum into an integral. This diagram shows a curve with the arc between A and B highlighted, together with one representative element whose sides are δx, δy and δs.



Arc length of $y = f(x)$.

Factor δx out of the root:

$$\delta s \approx \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\, \delta x.$$

As $\delta x \to 0$ the ratio $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$, and the sum of the elements becomes the integral

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

the length of the curve $y = f(x)$ from $x = a$ to $x = b$. If the curve is more naturally described as $x = g(y)$, the same argument with the roles of x and y swapped gives

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Arc length of a parametrically determined curve.

When a curve is given parametrically by $x = x(t)$ and $y = y(t)$, a short piece corresponds to a small change δt in the parameter, with $\delta x \approx \frac{dx}{dt}\delta t$ and $\delta y \approx \frac{dy}{dt}\delta t$. Then

$$\delta s \approx \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\delta t,$$

so the arc length between the parameter values $t = t_1$ and $t = t_2$ is

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

The Cartesian formula is the special case $x = t$. The Study Design places particular emphasis on this parametric form, because many curves (circles, ellipses, cycloids, spirals) are described far more easily by a parameter than by an equation in x and y. As a check, the circle $x = r\cos t$, $y = r\sin t$ has $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2\sin^2 t + r^2\cos^2 t = r^2$, so the integrand is the constant r and $L = r(t_2 - t_1)$ — exactly the familiar “arc length = radius × angle”.

Setting up and evaluating.

Every arc-length problem follows the same four steps:

1. differentiate to find $\frac{dy}{dx}$, or both $\frac{dx}{dt}$ and $\frac{dy}{dt}$;
2. form the integrand $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ or $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ (always the **positive** root);
3. attach the correct limits;
4. evaluate — exactly when the integral can be done by the methods of this course, otherwise numerically.

Curves engineered for an exact answer.

The square root usually blocks an exact integration, so examination curves are chosen so that $1 + \left(\frac{dy}{dx}\right)^2$ collapses to a **perfect square**. The key observation is that if $\frac{dy}{dx} = \frac{1}{2}\left(g - \frac{1}{g}\right)$ for some expression g, then the cross term cancels and

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(g^2 - 2 + \frac{1}{g^2}\right) = \frac{1}{4}\left(g^2 + 2 + \frac{1}{g^2}\right) = \frac{1}{4}\left(g + \frac{1}{g}\right)^2,$$

so the root simplifies exactly to $\frac{1}{2}\left(g + \frac{1}{g}\right)$. The table lists the standard building blocks used throughout this section.

curve y	$\frac{dy}{dx}$	$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
$\frac{x^3}{6} + \frac{1}{2x}$	$\frac{x^2}{2} - \frac{1}{2x^2}$	$\frac{x^2}{2} + \frac{1}{2x^2}$
$\frac{x^2}{4} - \frac{1}{2}\log_e x$	$\frac{x}{2} - \frac{1}{2x}$	$\frac{x}{2} + \frac{1}{2x}$
$\frac{2}{3}x^{3/2}$	$x^{1/2}$	$\sqrt{1 + x}$
$\frac{1}{3}(x^2 + 2)^{3/2}$	$x\sqrt{x^2 + 2}$	$x^2 + 1$

When the integral cannot be done exactly.

For most curves the integrand $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ cannot be antidifferentiated by the methods of this course, and this happens in two different ways.

For some curves no antiderivative exists in terms of familiar functions at all: the cubic $y=x^3$, a sine wave $y=\sin x$ and an ellipse all lead to integrals of this kind (the last two are called *elliptic* integrals). For other curves an antiderivative does exist, but finding it needs a technique beyond this course; the parabola $y=x^2$, with integrand $\sqrt{1+4x^2}$, is of this second kind.

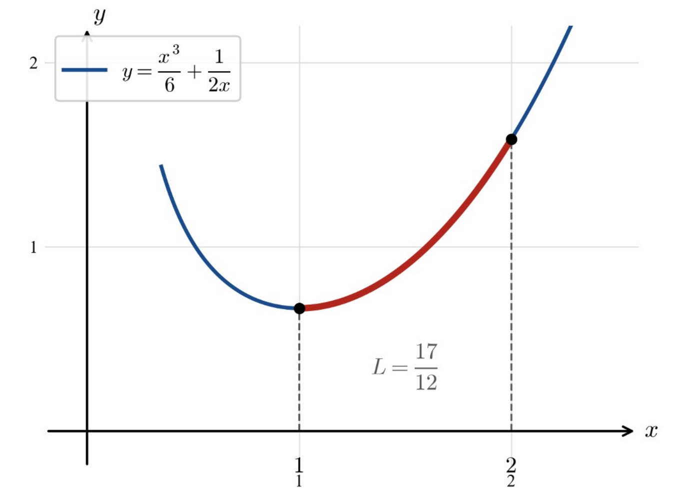
In both cases the integral is evaluated **numerically** using technology and the answer is quoted as a decimal approximation. Give an exact value whenever the integral can be evaluated by the methods of this course, and a decimal approximation otherwise.

Worked examples

Example 1 — scaffolded

Find the length of the curve $y=\frac{x^3}{6}+\frac{1}{2x}$ from $x=1$ to $x=2$.

Working	Comment
$\frac{dy}{dx}=\frac{x^2}{2}-\frac{1}{2x^2}$.	Differentiate; $\frac{1}{2x}=\frac{1}{2}x^{-1}$ gives $-\frac{1}{2}x^{-2}$.
$\left(\frac{dy}{dx}\right)^2=\frac{x^4}{4}-\frac{1}{2}+\frac{1}{4x^4}$, so	The $+1$ cancels the $-\frac{1}{2}$ cross term, leaving a perfect square.
$1+\left(\frac{dy}{dx}\right)^2=\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}=\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2$.	
$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{x^2}{2}+\frac{1}{2x^2}$, positive on $[1,2]$.	Take the positive root.
$L=\int_1^2\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)dx=\left[\frac{x^3}{6}-\frac{1}{2x}\right]_1^2$	Set up the arc-length integral and antidifferentiate.
$=\left(\frac{8}{6}-\frac{1}{4}\right)-\left(\frac{1}{6}-\frac{1}{2}\right)=\frac{13}{12}+\frac{1}{3}=\frac{17}{12}$.	Evaluate at the limits.

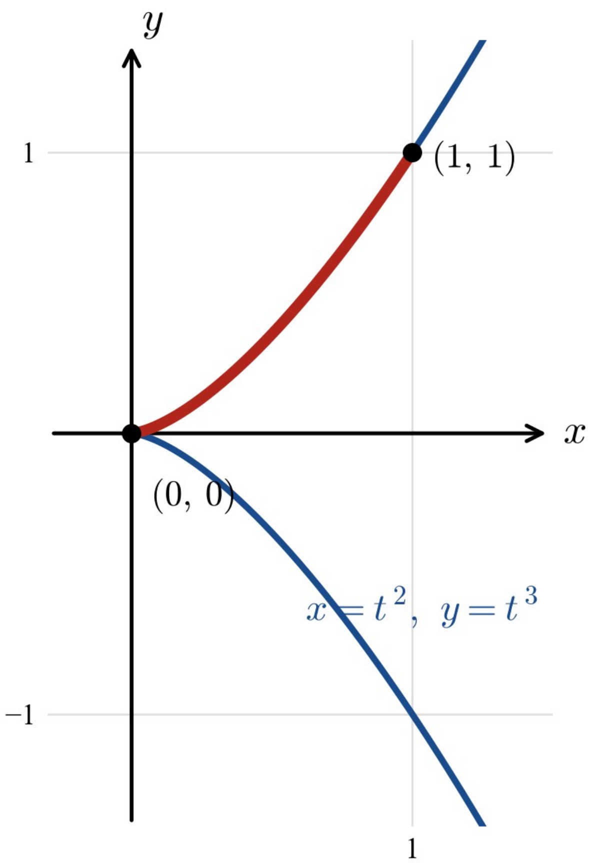


The length of the highlighted arc is $\frac{17}{12}$.

Example 2 — scaffolded

Find the length of the curve given parametrically by $x = t^2$, $y = t^3$ from $t = 0$ to $t = 1$.

Working	Comment
$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$.	Differentiate each parametric equation with respect to t.
$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 9t^4 = t^2(4 + 9t^2)$.	Add the squares and factor out t^2.
$\sqrt{t^2(4+9t^2)} = t\sqrt{4+9t^2}$ for $t \geq 0$.	The positive root, since $t \geq 0$ on $[0, 1]$.
$$L = \int_0^1 t\sqrt{4+9t^2}\,dt$$	Set up the parametric arc-length integral.
Let $u = 4 + 9t^2$, so $du = 18t\,dt$; $t = 0 \to u = 4$ and $t = 1 \to u = 13$. $$L = \frac{1}{18}\int_4^{13} \sqrt{u}\,du = \frac{1}{18} \cdot \frac{2}{3}\left[u^{3/2}\right]_4^{13}$$	Substitute to reach a standard power.
$= \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) = \frac{1}{27}\left(13\sqrt{13} - 8\right) = \frac{13\sqrt{13} - 8}{27}$.	$4^{3/2} = 8$ and $13^{3/2} = 13\sqrt{13}$.



The arc length is $\frac{13\sqrt{13} - 8}{27}$.

Example 3 — unscaffolded

Find the exact length of the curve $y = \frac{x^2}{4} - \frac{1}{2}\log_e x$ from $x = 1$ to $x = e$.

Differentiating, $\dfrac{dy}{dx}=\dfrac{x}{2}-\dfrac{1}{2x}$. Then

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4}\left(x-\frac{1}{x}\right)^2=1+\frac{1}{4}\left(x^2-2+\frac{1}{x^2}\right)=\frac{1}{4}\left(x^2+2+\frac{1}{x^2}\right)=\frac{1}{4}\left(x+\frac{1}{x}\right)^2,$$

and since $x+\dfrac{1}{x}>0$ on $[1,e]$ the positive root is $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)$. Therefore

$$L=\int_1^e \frac{1}{2}\left(x+\frac{1}{x}\right)dx=\frac{1}{2}\left[\frac{x^2}{2}+\log_e x\right]_1^e=\frac{1}{2}\left[\left(\frac{e^2}{2}+1\right)-\left(\frac{1}{2}+0\right)\right]=\frac{1}{2}\left(\frac{e^2}{2}+\frac{1}{2}\right).$$

$\therefore L=\dfrac{e^2+1}{4}$.

Example 4 — unscaffolded

A quarter of an ellipse is traced by $x=3\cos t$, $y=2\sin t$ for $0\le t\le\dfrac{\pi}{2}$. Find the length of this arc, correct to four decimal places.

Here $\dfrac{dx}{dt}=-3\sin t$ and $\dfrac{dy}{dt}=2\cos t$, so

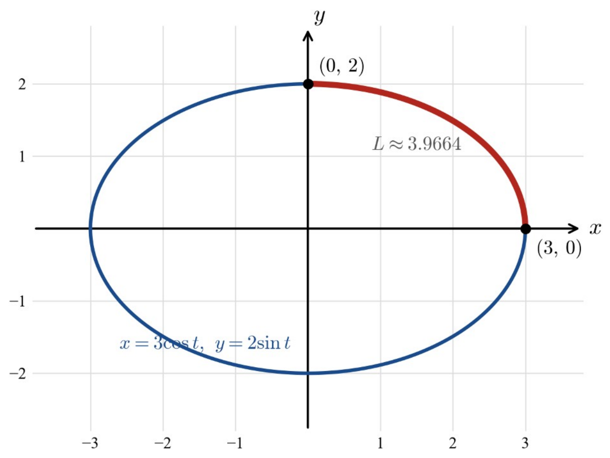
$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=9\sin^2 t+4\cos^2 t=4+5\sin^2 t,$$

using $\cos^2 t=1-\sin^2 t$. The arc length is therefore

$$L=\int_0^{\pi/2}\sqrt{9\sin^2 t+4\cos^2 t}\,dt=\int_0^{\pi/2}\sqrt{4+5\sin^2 t}\,dt.$$

This integrand has no antiderivative among the functions of this course (it is an *elliptic* integral), so the integral is evaluated numerically with technology:

$$L=\int_0^{\pi/2}\sqrt{4+5\sin^2 t}\,dt\approx 3.9664.$$



$\therefore L\approx 3.9664$, the length of the quarter-ellipse shown.

Practice questions

Scaffolded

Q1. Find the length of $y=\frac{2}{3}x^{3/2}$ from $x=0$ to $x=3$. (a) Find $\frac{dy}{dx}$. (b) Show that $1+\left(\frac{dy}{dx}\right)^2=1+x$. (c) Set up the arc-length integral and evaluate $\int_0^3 \sqrt{1+x}\,dx$. (d) State the length.

Q2. Find the length of $y=\frac{x^4}{8}+\frac{1}{4x^2}$ from $x=1$ to $x=2$. (a) Find $\frac{dy}{dx}$. (b) Show that $1+\left(\frac{dy}{dx}\right)^2=\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)^2$. (c) Integrate $\frac{x^3}{2}+\frac{1}{2x^3}$ from 1 to 2. (d) State the length.

Q3. Find the length of the arc given by $x=4\cos t,\ y=4\sin t$ from $t=0$ to $t=\frac{\pi}{3}$. (a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$. (b) Show that $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=16$. (c) Evaluate $\int_0^{\pi/3} 4\,dt$. (d) State the length, and confirm it equals radius × angle.

Q4. The curve $x=\cos^3 t,\ y=\sin^3 t$ for $0\le t\le\frac{\pi}{2}$ is one quarter of an astroid. Find its length. (a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$. (b) Show that $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=9\sin^2 t\cos^2 t$. (c) Hence show the integrand equals $3\sin t\cos t=\frac{3}{2}\sin 2t$ on this interval, and integrate it. (d) State the length.

Q5. Find the exact length of $y=\frac{x^2}{2}-\frac{1}{4}\log_e x$ from $x=1$ to $x=2$. (a) Find $\frac{dy}{dx}$. (b) Show that $1+\left(\frac{dy}{dx}\right)^2=\left(x+\frac{1}{4x}\right)^2$. (c) Integrate $x+\frac{1}{4x}$ from 1 to 2. (d) State the exact length.

Q6. Find the length of $y=\frac{1}{3}(x^2+2)^{3/2}$ from $x=0$ to $x=3$. (a) Show that $\frac{dy}{dx}=x\sqrt{x^2+2}$. (b) Show that $1+\left(\frac{dy}{dx}\right)^2=(x^2+1)^2$. (c) Evaluate $\int_0^3 (x^2+1)\,dx$. (d) State the length.

Unscaffolded

Q7. Find the exact length of $y=\frac{x^3}{3}+\frac{1}{4x}$ from $x=1$ to $x=2$.

Q8. Find the exact length of $y=\frac{2}{3}x^{3/2}$ from $x=0$ to $x=8$.

Q9. Find the exact length of $y=\frac{x^2}{4}-\frac{1}{2}\log_e x$ from $x=1$ to $x=3$.

Q10. The curve $x=t-\sin t,\ y=1-\cos t$ traces one arch of a cycloid as t runs from 0 to 2π. Show that the length of the arch is 8. (Hint: $1-\cos t=2\sin^2\frac{t}{2}$.)

Q11. The involute of a circle is given by $x=\cos t+t\sin t,\ y=\sin t-t\cos t$. Find the exact length of the arc from $t=0$ to $t=\pi$.

Q12. A logarithmic spiral is given by $x=e^t\cos t,\ y=e^t\sin t$. Find the exact length of the arc from $t=0$ to $t=3$.

Q13. Find the length of $y = x^3$ from $x = 0$ to $x = 1$, correct to four decimal places.

Q14. Find the length of one arch of $y = \sin x$ from $x = 0$ to $x = \pi$, correct to four decimal places.

Q15. Find the perimeter of the ellipse $x = 3\cos t$, $y = 2\sin t$ (for $0 \le t \le 2\pi$), correct to four decimal places.

Q16. Find the exact length of the curve $x = t^2$, $y = t^3$ from $t = 0$ to $t = 2$.

Q17. A garden path follows the curve $y = \frac{1}{10}x^2$, where x and y are in metres, for $0 \le x \le 20$. Find the length of the path, correct to the nearest $0.1\,\text{m}$.

Q18. An Archimedean spiral is given by $x = t\cos t$, $y = t\sin t$. (a) Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 + t^2$. (b) Hence find the length of the spiral from $t = 0$ to $t = 2\pi$, correct to four decimal places.

Answers

Q1. $L=\frac{14}{3}$. **Q2.** $L=\frac{33}{16}$. **Q3.** $L=\frac{4\pi}{3}$. **Q4.** $L=\frac{3}{2}$. **Q5.** $L=\frac{3}{2}+\frac{1}{4}\log_e 2$. **Q6.** $L=12$. **Q7.** $L=\frac{59}{24}$. **Q8.** $L=\frac{52}{3}$. **Q9.** $L=2+\frac{1}{2}\log_e 3$. **Q10.** $L=8$. **Q11.** $L=\frac{\pi^2}{2}$. **Q12.** $L=\sqrt{2}(e^3-1)$. **Q13.** $L\approx 1.5479$. **Q14.** $L\approx 3.8202$. **Q15.** $L\approx 15.8654$. **Q16.** $L=\frac{80\sqrt{10}-8}{27}$. **Q17.** $L\approx 46.5\,\text{m}$. **Q18.** (a) shown below; (b) $L\approx 21.2563$.

Fully-worked solutions

Q1. $\frac{dy}{dx}=x^{1/2}$, so $1+\left(\frac{dy}{dx}\right)^2=1+x$. Then

$$L=\int_0^3 \sqrt{1+x}\,dx=\left[\frac{2}{3}(1+x)^{3/2}\right]_0^3=\frac{2}{3}\left(4^{3/2}-1^{3/2}\right)=\frac{2}{3}(8-1)=\frac{14}{3}.$$

Q2. $\frac{dy}{dx}=\frac{x^3}{2}-\frac{1}{2x^3}$, so

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^6}{4}-\frac{1}{2}+\frac{1}{4x^6}=\frac{x^6}{4}+\frac{1}{2}+\frac{1}{4x^6}=\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)^2.$$

The positive root is $\frac{x^3}{2}+\frac{1}{2x^3}$, so

$$L=\int_1^2\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx=\left[\frac{x^4}{8}-\frac{1}{4x^2}\right]_1^2=\left(2-\frac{1}{16}\right)-\left(\frac{1}{8}-\frac{1}{4}\right)=\frac{31}{16}+\frac{1}{8}=\frac{33}{16}.$$

Q3. $\frac{dx}{dt}=-4\sin t$ and $\frac{dy}{dt}=4\cos t$, so $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=16\sin^2 t+16\cos^2 t=16$. Then

$$L=\int_0^{\pi/3}\sqrt{16}\,dt=\int_0^{\pi/3}4\,dt=4\cdot\frac{\pi}{3}=\frac{4\pi}{3}.$$

This equals radius × angle $=4\times\frac{\pi}{3}$, as expected for a circular arc.

Q4. $\frac{dx}{dt}=-3\cos^2 t\sin t$ and $\frac{dy}{dt}=3\sin^2 t\cos t$, so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t=9\sin^2 t\cos^2 t\left(\cos^2 t+\sin^2 t\right)=9\sin^2 t\cos^2 t.$$

On $0\le t\le\frac{\pi}{2}$ both $\sin t$ and $\cos t$ are non-negative, so the positive root is $3\sin t\cos t=\frac{3}{2}\sin 2t$. Then

$$L=\int_0^{\pi/2}\frac{3}{2}\sin 2t\,dt=\left[-\frac{3}{4}\cos 2t\right]_0^{\pi/2}=-\frac{3}{4}(\cos\pi-\cos 0)=-\frac{3}{4}(-1-1)=\frac{3}{2}.$$

Q5. $\frac{dy}{dx}=x-\frac{1}{4x}$, so

$$1+\left(\frac{dy}{dx}\right)^2=1+x^2-\frac{1}{2}+\frac{1}{16x^2}=x^2+\frac{1}{2}+\frac{1}{16x^2}=\left(x+\frac{1}{4x}\right)^2.$$

The positive root is $x+\dfrac{1}{4x}$, so

$$L=\int_1^2\left(x+\frac{1}{4x}\right)dx=\left[\frac{x^2}{2}+\frac{1}{4}\log_e x\right]_1^2=\left(2+\frac{1}{4}\log_e 2\right)-\left(\frac{1}{2}+0\right)=\frac{3}{2}+\frac{1}{4}\log_e 2.$$

Q6. With $y=\dfrac{1}{3}(x^2+2)^{3/2}$, the chain rule gives $\dfrac{dy}{dx}=\dfrac{1}{3}\cdot\dfrac{3}{2}(x^2+2)^{1/2}\cdot 2x=x\sqrt{x^2+2}$. Then $\left(\dfrac{dy}{dx}\right)^2=x^2(x^2+2)=x^4+2x^2$, so

$$1+\left(\frac{dy}{dx}\right)^2=x^4+2x^2+1=(x^2+1)^2,$$

whose positive root is x^2+1. Therefore

$$L=\int_0^3(x^2+1)\,dx=\left[\frac{x^3}{3}+x\right]_0^3=(9+3)-0=12.$$

Q7. $\dfrac{dy}{dx}=x^2-\dfrac{1}{4x^2}$, so

$$1+\left(\frac{dy}{dx}\right)^2=1+x^4-\frac{1}{2}+\frac{1}{16x^4}=x^4+\frac{1}{2}+\frac{1}{16x^4}=\left(x^2+\frac{1}{4x^2}\right)^2.$$

The positive root is $x^2+\dfrac{1}{4x^2}$, so

$$L=\int_1^2\left(x^2+\frac{1}{4x^2}\right)dx=\left[\frac{x^3}{3}-\frac{1}{4x}\right]_1^2=\left(\frac{8}{3}-\frac{1}{8}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{7}{3}+\frac{1}{8}=\frac{59}{24}.$$

Q8. $\dfrac{dy}{dx}=x^{1/2}$, so $1+\left(\dfrac{dy}{dx}\right)^2=1+x$. Then

$$L=\int_0^8\sqrt{1+x}\,dx=\left[\frac{2}{3}(1+x)^{3/2}\right]_0^8=\frac{2}{3}\left(9^{3/2}-1\right)=\frac{2}{3}(27-1)=\frac{52}{3}.$$

Q9. As in Example 3, $\dfrac{dy}{dx}=\dfrac{x}{2}-\dfrac{1}{2x}$ and $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)$. Then

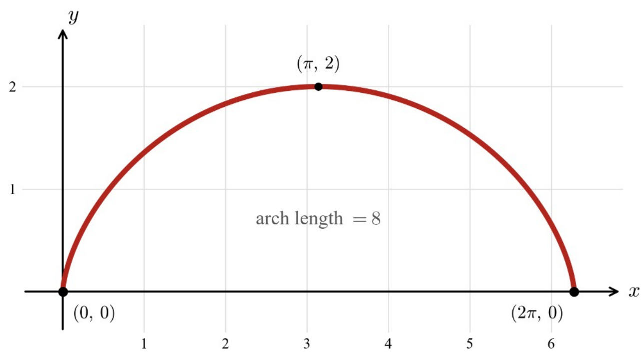
$$L=\int_1^3\frac{1}{2}\left(x+\frac{1}{x}\right)dx=\frac{1}{2}\left[\frac{x^2}{2}+\log_e x\right]_1^3=\frac{1}{2}\left[\left(\frac{9}{2}+\log_e 3\right)-\frac{1}{2}\right]=\frac{1}{2}\left(4+\log_e 3\right)=2+\frac{1}{2}\log_e 3.$$

Q10. $\dfrac{dx}{dt}=1-\cos t$ and $\dfrac{dy}{dt}=\sin t$, so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(1-\cos t)^2+\sin^2 t=1-2\cos t+\cos^2 t+\sin^2 t=2-2\cos t.$$

Using $1-\cos t=2\sin^2\frac{t}{2}$ gives $2-2\cos t=4\sin^2\frac{t}{2}$, and on $0\le t\le 2\pi$ we have $\sin\frac{t}{2}\ge 0$, so the positive root is $2\sin\frac{t}{2}$. Therefore

$$L=\int_0^{2\pi} 2\sin\frac{t}{2}\,dt=\left[-4\cos\frac{t}{2}\right]_0^{2\pi}=-4(\cos\pi-\cos 0)=-4(-1-1)=8.$$



Q11. $\frac{dx}{dt}=-\sin t+\sin t+t\cos t=t\cos t$ and $\frac{dy}{dt}=\cos t-\cos t+t\sin t=t\sin t$, so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=t^2\cos^2 t+t^2\sin^2 t=t^2.$$

On $0\le t\le\pi$, $t\ge 0$, so the positive root is t. Therefore

$$L=\int_0^{\pi} t\,dt=\left[\frac{t^2}{2}\right]_0^{\pi}=\frac{\pi^2}{2}.$$

Q12. $\frac{dx}{dt}=e^t\cos t-e^t\sin t=e^t(\cos t-\sin t)$ and $\frac{dy}{dt}=e^t\sin t+e^t\cos t=e^t(\sin t+\cos t)$, so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=e^{2t}\left[(\cos t-\sin t)^2+(\sin t+\cos t)^2\right]=e^{2t}\cdot 2=2e^{2t},$$

since the bracket expands to $2(\cos^2 t+\sin^2 t)=2$. The positive root is $\sqrt{2}e^t$, so

$$L=\int_0^{3}\sqrt{2}e^t\,dt=\sqrt{2}\left[e^t\right]_0^3=\sqrt{2}(e^3-1).$$

Q13. $\frac{dy}{dx}=3x^2$, so $1+\left(\frac{dy}{dx}\right)^2=1+9x^4$. The integral

$$L=\int_0^{1}\sqrt{1+9x^4}\,dx$$

has no elementary antiderivative, so evaluating it numerically with technology gives $L\approx 1.5479$.

Q14. $\frac{dy}{dx}=\cos x$, so $1+\left(\frac{dy}{dx}\right)^2=1+\cos^2 x$. The integral

$$L=\int_0^{\pi}\sqrt{1+\cos^2 x}\,dx$$

is a (non-elementary) elliptic integral, evaluated numerically as $L\approx 3.8202$.

Q15. $\frac{dx}{dt}=-3\sin t$ and $\frac{dy}{dt}=2\cos t$, so the perimeter is

$$L=\int_0^{2\pi}\sqrt{9\sin^2 t+4\cos^2 t}\,dt=\int_0^{2\pi}\sqrt{4+5\sin^2 t}\,dt.$$

This elliptic integral has no elementary antiderivative; numerically $L\approx 15.8654$. (By symmetry it is four times the quarter-arc of Example 4, but the check must use the unrounded quarter-arc: $4\times 3.96635\ldots=15.86543\ldots$, which is 15.8654 to four decimal places. Multiplying the rounded 3.9664 instead would give 15.8656, wrong in the final digit.)

Q16. As in Example 2, $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=t^2(4+9t^2)$ and the positive root is $t\sqrt{4+9t^2}$ for $t\geq 0$. With $u=4+9t^2$, $du=18t\,dt$; $t=0\rightarrow u=4$ and $t=2\rightarrow u=40$, so

$$L=\int_0^2 t\sqrt{4+9t^2}\,dt=\frac{1}{18}\int_4^{40}\sqrt{u}\,du=\frac{1}{27}\left[u^{3/2}\right]_4^{40}=\frac{1}{27}\left(40^{3/2}-8\right).$$

Since $40^{3/2}=40\sqrt{40}=80\sqrt{10}$, this is $L=\frac{80\sqrt{10}-8}{27}$.

Q17. $\frac{dy}{dx}=\frac{x}{5}$, so $1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{25}$. The length of the path is

$$L=\int_0^{20}\sqrt{1+\frac{x^2}{25}}\,dx,$$

an integral that cannot be antidifferentiated by the methods of this course, so it is evaluated numerically as $L\approx 46.4678$, that is $46.5\,\text{m}$ to the nearest $0.1\,\text{m}$.

Q18. (a) $\frac{dx}{dt}=\cos t-t\sin t$ and $\frac{dy}{dt}=\sin t+t\cos t$, so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(\cos t-t\sin t)^2+(\sin t+t\cos t)^2.$$

Expanding, the cross terms $-2t\sin t\cos t$ and $+2t\sin t\cos t$ cancel, leaving

$$\cos^2 t+t^2\sin^2 t+\sin^2 t+t^2\cos^2 t=(\cos^2 t+\sin^2 t)+t^2(\sin^2 t+\cos^2 t)=1+t^2.$$

(b) The positive root is $\sqrt{1+t^2}$, so

$$L=\int_0^{2\pi}\sqrt{1+t^2}\,dt\approx 21.2563,$$

this being another integrand that cannot be antidifferentiated by the methods of this course, so it is evaluated numerically with technology.

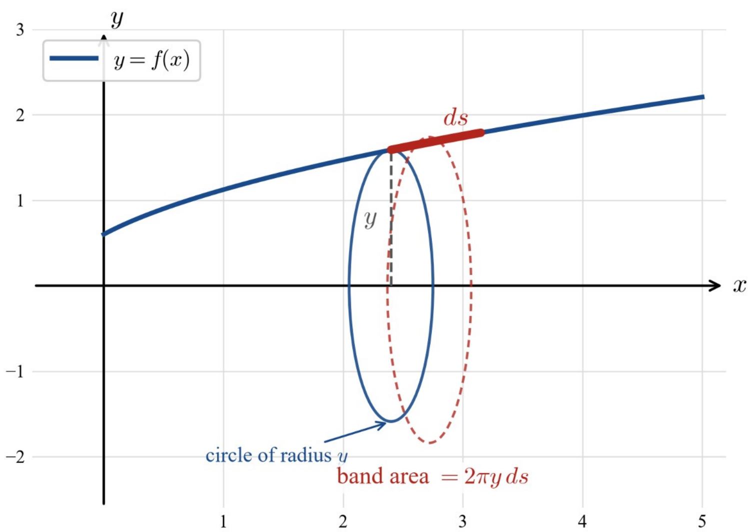
Theory

Rotating a curve one full turn about a straight line sweeps out a **surface of revolution**: rotating $y=f(x)$ about the x-axis produces a shape such as a cone, a sphere or a trumpet-shaped bell. This section finds the **area of that curved surface**. (Rotating the *region* under the curve produces a *solid*, whose volume is treated separately; here it is only the surface swept by the curve itself that is measured.)

The surface element.

Take a short piece of the curve, of length ds, at height y above the x-axis. When the curve is rotated a full turn, this little arc traces out a thin **band** — a narrow frustum — of radius y and slant width ds. Cutting the band and rolling it out flat gives a strip whose length is the circumference of the circle the arc travelled around, $2\pi y$, and whose width is ds, so its area is

$$dS = 2\pi y\, ds.$$

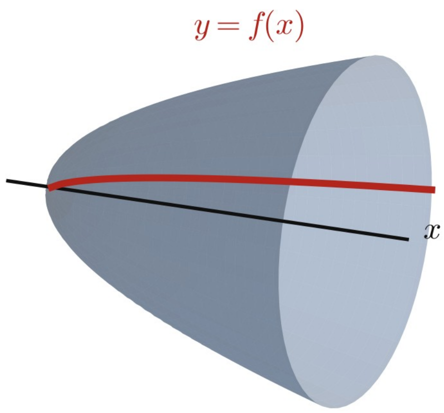


Rotation about the x-axis.

In an earlier section the arc-length element was found to be $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. Substituting this into $dS=2\pi y\,ds$ and adding the bands from $x=a$ to $x=b$ gives the surface area

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,$$

where $y=f(x)\geq 0$ on $[a,b]$ so that y is the non-negative radius of each band.



Parametric form.

If the curve is given parametrically by $x = x(t)$, $y = y(t)$ for $t_1 \le t \le t_2$, the arc-length element is $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, and the same $2\pi y\, ds$ idea gives

$$S = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Rotation about the y-axis.

If instead the curve is rotated about the y-axis, the radius of each band is the horizontal distance x, so $dS = 2\pi x\, ds$ and

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The rule is the same in every case: the radius is the distance from the point on the curve to the **axis of rotation**, and ds is the arc-length element.

Choosing the curve so the integral is exact.

The extra factor of y (or x) makes these integrands harder than the arc-length ones, so the curve must be chosen with care if an exact answer is wanted. Two classic choices always work.

- A **straight line** gives a **cone** or **frustum**, for which $\sqrt{1 + (dy/dx)^2}$ is constant. The result can be checked against the geometry formulas $\pi r \ell$ (cone) and $\pi(r_1 + r_2)\ell$ (frustum), where ℓ is the slant height.
- The semicircle $y = \sqrt{r^2 - x^2}$ gives a **sphere**. Here the awkward square root cancels and the integrand collapses to the constant $2\pi r$, so a band's area depends only on how wide it is along the axis, and the whole sphere has area

$$S = \int_{-r}^{r} 2\pi r\, dx = 4\pi r^2.$$

For many other curves the integrand $2\pi y \sqrt{1 + (dy/dx)^2}$ has no elementary antiderivative, and the definite integral is evaluated **numerically using technology**. Every displayed surface integral still carries its full integrand; only its evaluation may be numerical.

Method.

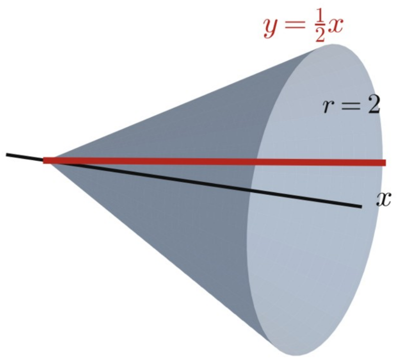
1. Identify the axis of rotation and the radius: y for rotation about the x-axis, x for rotation about the y-axis.
2. Differentiate to form $\sqrt{1+(dy/dx)^2}$ (or the parametric $\sqrt{(dx/dt)^2+(dy/dt)^2}$).
3. Substitute into $S=\int_a^b 2\pi y\,ds$ (about the x-axis) or $S=\int_a^b 2\pi x\,ds$ (about the y-axis), insert the limits, and simplify the integrand — often a square root cancels.
4. Evaluate exactly where possible, otherwise numerically; check against a cone or sphere formula when the shape allows.

Worked examples

Example 1 — scaffolded

The line $y=\frac{1}{2}x$, $0\le x\le 4$, is rotated about the x-axis. Find the area of the surface generated, and check the result against the cone formula.

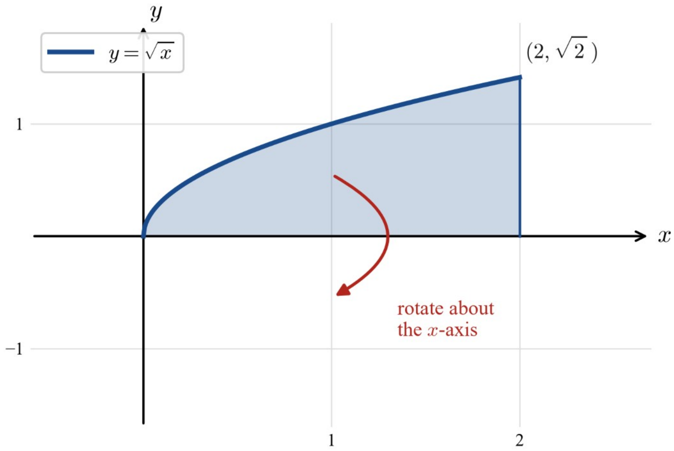
Working	Comment
$\frac{dy}{dx}=\frac{1}{2}$, so $1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4}=\frac{5}{4}$ and $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\sqrt{5}}{2}$.	A straight line gives a constant root.
$S=\int_0^4 2\pi\cdot\frac{x}{2}\cdot\frac{\sqrt{5}}{2}\,dx=\frac{\pi\sqrt{5}}{2}\int_0^4 x\,dx$	Radius is $y=\frac{x}{2}$; take the constants out.
$=\frac{\pi\sqrt{5}}{2}\left[\frac{x^2}{2}\right]_0^4=\frac{\pi\sqrt{5}}{2}\cdot 8=4\sqrt{5}\pi$	Evaluate the definite integral.
Base radius $r=y(4)=2$; slant $\ell=\sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$.	The line from $(0,0)$ to $(4,2)$ is the slant of the cone.
$\pi r\ell=\pi(2)(2\sqrt{5})=4\sqrt{5}\pi$, matching the integral.	The geometry formula confirms $S=4\sqrt{5}\pi$.



Example 2 — scaffolded

Find the area of the surface formed when $y=\sqrt{x}$, $0\le x\le 2$, is rotated about the x-axis.

Working	Comment
$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$, so $1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4x}=\frac{4x+1}{4x}$.	Differentiate $y=x^{1/2}$.
$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\sqrt{4x+1}}{2\sqrt{x}}$, so	The $\sqrt{x}$ cancels — the integrand is exact.
$2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=2\pi\sqrt{x}\cdot\frac{\sqrt{4x+1}}{2\sqrt{x}}=\pi\sqrt{4x+1}$.	
$S=\int_0^2 \pi\sqrt{4x+1}\,dx=\pi\left[\frac{1}{4}\cdot\frac{2}{3}(4x+1)^{3/2}\right]_0^2$	Integrate $(4x+1)^{1/2}$ (raise the power, divide by 4).
$=\frac{\pi}{6}\left[(9)^{3/2}-(1)^{3/2}\right]=\frac{\pi}{6}(27-1)=\frac{13\pi}{3}$	$9^{3/2}=27$; simplify.



Example 3 — unscaffolded

Show that rotating the semicircle $y=\sqrt{r^2-x^2}$, $-r\le x\le r$, about the x-axis generates a sphere of surface area $4\pi r^2$.

Differentiating, $\frac{dy}{dx}=\frac{-x}{\sqrt{r^2-x^2}}$, so

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{r^2-x^2}=\frac{r^2}{r^2-x^2},\ \sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{r}{\sqrt{r^2-x^2}}.$$

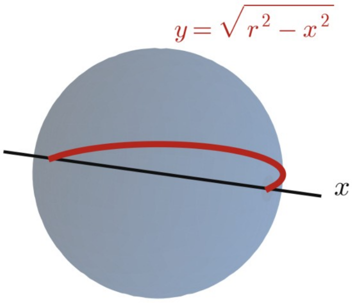
The radius factor $y=\sqrt{r^2-x^2}$ then cancels the square root:

$$2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=2\pi\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}=2\pi r,$$

a constant. Hence

$$S=\int_{-r}^{r}2\pi r\,dx=2\pi r[x]_{-r}^{r}=2\pi r(2r)=4\pi r^2.$$

Because the integrand is constant, the area of any zone cut between two planes $x=a$ and $x=b$ is $2\pi r(b-a)$ — it depends only on the width of the band along the axis, not on where it sits on the sphere.



$\therefore$ the surface generated is a sphere of area $S = 4\pi r^2$.

Example 4 — unscaffolded

The quarter circle $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \frac{\pi}{2}$, is rotated about the x-axis. Find the area of the surface generated.

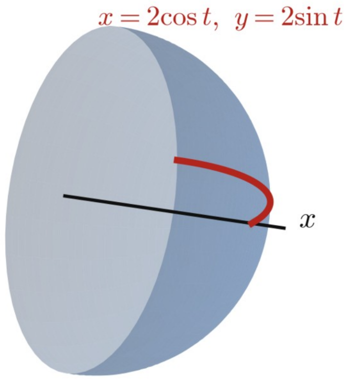
The curve is given parametrically, so use $S = \int_{t_1}^{t_2} 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$. Here $\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = 2\cos t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\sin^2 t + 4\cos^2 t = 4, \sqrt{(dx/dt)^2 + (dy/dt)^2} = 2.$$

Therefore

$$S = \int_0^{\pi/2} 2\pi (2\sin t)(2)\, dt = 8\pi \int_0^{\pi/2} \sin t\, dt = 8\pi [-\cos t]_0^{\pi/2} = 8\pi (0 - (-1)) = 8\pi.$$

The arc runs from $(2, 0)$ to $(0, 2)$, so it sweeps the curved surface of a hemisphere of radius 2; the geometry formula $2\pi r^2 = 2\pi (2)^2 = 8\pi$ agrees.



$\therefore S = 8\pi$.

Practice questions

Scaffolded

Q1. The line $y=3x$, $0 \le x \le 1$, is rotated about the x-axis. (a) Write down $\frac{dy}{dx}$ and hence $\sqrt{1+(dy/dx)^2}$. (b) Set up the surface-area integral. (c) Evaluate it to find S. (d) Check the answer with the cone formula $\pi r \ell$.

Q2. Find the surface area when $y=2\sqrt{x}$, $0 \le x \le 3$, is rotated about the x-axis. (a) Find $\frac{dy}{dx}$ and $\sqrt{1+(dy/dx)^2}$. (b) Show that $2\pi y\sqrt{1+(dy/dx)^2} = 4\pi\sqrt{x+1}$. (c) Integrate from $x=0$ to $x=3$.

Q3. Find the surface area when $y=x^3$, $0 \le x \le 1$, is rotated about the x-axis. (a) Write down the integrand $2\pi y\sqrt{1+(dy/dx)^2}$. (b) Use the substitution $u=1+9x^4$; find du and change the limits. (c) Evaluate the integral in u and simplify.

Q4. The curve $y=\frac{1}{2}x^2$, $0 \le x \le \sqrt{3}$, is rotated about the y**-axis**. (a) State the radius of a band and write $\sqrt{1+(dy/dx)^2}$. (b) Set up $S=\int_0^{\sqrt{3}} 2\pi x\sqrt{1+(dy/dx)^2}\,dx$. (c) Evaluate it (the substitution $u=1+x^2$ helps).

Q5. The arc $y=\sqrt{25-x^2}$, $1 \le x \le 4$, is rotated about the x-axis. (a) Find $\frac{dy}{dx}$ and $\sqrt{1+(dy/dx)^2}$. (b) Show that the integrand $2\pi y\sqrt{1+(dy/dx)^2}$ is constant. (c) Evaluate S, and confirm it equals $2\pi r \times (\text{band width})$.

Q6. The quarter circle $x=3\cos t$, $y=3\sin t$, $0 \le t \le \frac{\pi}{2}$, is rotated about the x-axis. (a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$. (b) Show that $\sqrt{(dx/dt)^2+(dy/dt)^2}=3$. (c) Evaluate the surface integral. (d) Confirm the answer using the hemisphere formula $2\pi r^2$.

Unscaffolded

Q7. Find the surface area when $y=\frac{4}{3}x$, $0 \le x \le 3$, is rotated about the x-axis, and confirm it with the cone formula.

Q8. The line $y=x+1$, $0 \le x \le 3$, is rotated about the x-axis, generating a frustum. Find the surface area, and check it against $\pi(r_1+r_2)\ell$.

Q9. Find the exact surface area when $y=\sqrt{x}$, $0 \le x \le 6$, is rotated about the x-axis.

Q10. The arc $y=\sqrt{9-x^2}$, $2 \le x \le 3$, is rotated about the x-axis, generating a spherical cap. Find its surface area.

Q11. The curve $x=t^2$, $y=2t$, $0 \le t \le 2$, is rotated about the x-axis. Find the exact surface area.

Q12. The curve $y=x^2$, $0 \le x \le 1$, is rotated about the x-axis. Write down the surface integral, and use technology to evaluate it to three decimal places.

Q13. The line $y=2x$, $0 \le x \le 2$, is rotated about the y**-axis**. Find the surface area, and check it with the cone formula.

Q14. Find the exact surface area when $y=3\sqrt{x}$, $0 \le x \le 4$, is rotated about the x-axis.

Q15. The line $y=\frac{1}{2}x+1$, $0 \le x \le 4$, is rotated about the x-axis. Find the surface area of the frustum generated.

Q16. The curve $x = r\cos t$, $y = r\sin t$, $0 \le t \le \pi$, is rotated about the x-axis. Show from the parametric surface integral that the area generated is $4\pi r^2$.

Q17. The curve $y = \sin x$, $0 \le x \le \pi$, is rotated about the x-axis. Write the surface integral, and use the substitution $u = \cos x$ to reduce it to $2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du$; hence evaluate S to three decimal places.

Q18. A metal drinking cup is modelled by rotating $y = \sqrt{x}$, $1 \le x \le 4$, about the x-axis. Find the exact area of its curved surface.

Answers

Q1. $S=3\sqrt{10}\pi$. **Q2.** $S=\frac{56\pi}{3}$. **Q3.** $S=\frac{\pi}{27}(10\sqrt{10}-1)$. **Q4.** $S=\frac{14\pi}{3}$. **Q5.** $S=30\pi$. **Q6.** $S=18\pi$. **Q7.** $S=20\pi$. **Q8.** $S=15\sqrt{2}\pi$. **Q9.** $S=\frac{62\pi}{3}$. **Q10.** $S=6\pi$. **Q11.** $S=\frac{8\pi}{3}(5\sqrt{5}-1)$. **Q12.** $S\approx 3.810$. **Q13.** $S=4\sqrt{5}\pi$. **Q14.** $S=49\pi$. **Q15.** $S=8\sqrt{5}\pi$. **Q16.** $S=4\pi r^2$. **Q17.** $S=2\pi\left(\sqrt{2}+\log_e\left(1+\sqrt{2}\right)\right)\approx 14.424$. **Q18.** $S=\frac{\pi}{6}(17\sqrt{17}-5\sqrt{5})$.

Fully-worked solutions

Q1. (a) $\frac{dy}{dx}=3$, so $\sqrt{1+(dy/dx)^2}=\sqrt{1+9}=\sqrt{10}$ (constant). (b), (c) With radius $y=3x$,

$$S=\int_0^1 2\pi(3x)\sqrt{10}\,dx=6\sqrt{10}\pi\int_0^1 x\,dx=6\sqrt{10}\pi\left[\frac{x^2}{2}\right]_0^1=3\sqrt{10}\pi.$$

(d) The base radius is $r=y(1)=3$ and the slant is $\ell=\sqrt{1^2+3^2}=\sqrt{10}$, so $\pi r\ell=\pi(3)\sqrt{10}=3\sqrt{10}\pi$, confirming the answer.

Q2. (a) $\frac{dy}{dx}=\frac{1}{\sqrt{x}}$, so $1+(dy/dx)^2=1+\frac{1}{x}=\frac{x+1}{x}$ and $\sqrt{1+(dy/dx)^2}=\frac{\sqrt{x+1}}{\sqrt{x}}$. (b) $2\pi y\sqrt{1+(dy/dx)^2}=2\pi(2\sqrt{x})\cdot\frac{\sqrt{x+1}}{\sqrt{x}}=4\pi\sqrt{x+1}$. (c) Then

$$S=\int_0^3 4\pi\sqrt{x+1}\,dx=4\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3=\frac{8\pi}{3}\left[(4)^{3/2}-(1)^{3/2}\right]=\frac{8\pi}{3}(8-1)=\frac{56\pi}{3}.$$

Q3. (a) $\frac{dy}{dx}=3x^2$, so the integrand is $2\pi y\sqrt{1+(dy/dx)^2}=2\pi x^3\sqrt{1+9x^4}$. (b) Let $u=1+9x^4$, so $du=36x^3\,dx$ and $x^3\,dx=\frac{du}{36}$; the limits become $x=0\Rightarrow u=1$ and $x=1\Rightarrow u=10$. (c) Then

$$S=\int_0^1 2\pi x^3\sqrt{1+9x^4}\,dx=\frac{2\pi}{36}\int_1^{10}\sqrt{u}\,du=\frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10}=\frac{\pi}{27}\left(10^{3/2}-1\right)=\frac{\pi}{27}(10\sqrt{10}-1).$$

Q4. (a) Rotation about the y-axis, so the radius is x; with $\frac{dy}{dx}=x$, $\sqrt{1+(dy/dx)^2}=\sqrt{1+x^2}$. (b), (c) Then

$$S=\int_0^{\sqrt{3}} 2\pi x\sqrt{1+x^2}\,dx=2\pi\left[\frac{1}{3}(1+x^2)^{3/2}\right]_0^{\sqrt{3}}=\frac{2\pi}{3}\left[(4)^{3/2}-(1)^{3/2}\right]=\frac{2\pi}{3}(8-1)=\frac{14\pi}{3},$$

using $u=1+x^2$, $du=2x\,dx$.

Q5. (a) $\frac{dy}{dx}=\frac{-x}{\sqrt{25-x^2}}$, so $1+(dy/dx)^2=\frac{25}{25-x^2}$ and $\sqrt{1+(dy/dx)^2}=\frac{5}{\sqrt{25-x^2}}$. (b) The radius cancels the root: $2\pi y\sqrt{1+(dy/dx)^2}=2\pi\sqrt{25-x^2}\cdot\frac{5}{\sqrt{25-x^2}}=10\pi$, a constant. (c) Hence

$$S=\int_1^4 10\pi\,dx=10\pi[x]_1^4=10\pi(4-1)=30\pi.$$

This equals $2\pi r \times (\text{band width}) = 2\pi(5)(3) = 30\pi$, since the sphere has radius $r=5$ and the band spans $4-1=3$ along the axis.

Q6. (a) $\dfrac{dx}{dt} = -3\sin t$, $\dfrac{dy}{dt} = 3\cos t$. (b) $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = 9\sin^2 t + 9\cos^2 t = 9$, so $\sqrt{(dx/dt)^2 + (dy/dt)^2} = 3$. (c) Then

$$S = \int_0^{\pi/2} 2\pi(3\sin t)(3)\,dt = 18\pi \int_0^{\pi/2} \sin t\,dt = 18\pi[-\cos t]_0^{\pi/2} = 18\pi(0-(-1)) = 18\pi.$$

(d) The arc sweeps a hemisphere of radius 3, and $2\pi r^2 = 2\pi(3)^2 = 18\pi$, confirming the answer.

Q7. With $\dfrac{dy}{dx} = \dfrac{4}{3}$, $\sqrt{1+(dy/dx)^2} = \sqrt{1+\dfrac{16}{9}} = \sqrt{\dfrac{25}{9}} = \dfrac{5}{3}$. Then

$$S = \int_0^3 2\pi\left(\frac{4}{3}x\right)\frac{5}{3}\,dx = \frac{40\pi}{9}\int_0^3 x\,dx = \frac{40\pi}{9}\left[\frac{x^2}{2}\right]_0^3 = \frac{40\pi}{9}\cdot\frac{9}{2} = 20\pi.$$

Check: base radius $r = y(3) = 4$, slant $\ell = \sqrt{3^2+4^2} = 5$, so $\pi r \ell = \pi(4)(5) = 20\pi$.

Q8. With $\dfrac{dy}{dx} = 1$, $\sqrt{1+(dy/dx)^2} = \sqrt{2}$. Then

$$S = \int_0^3 2\pi(x+1)\sqrt{2}\,dx = 2\sqrt{2}\pi\int_0^3 (x+1)\,dx = 2\sqrt{2}\pi\left[\frac{x^2}{2}+x\right]_0^3 = 2\sqrt{2}\pi\left(\frac{9}{2}+3\right) = 15\sqrt{2}\pi.$$

Check: the frustum has radii $r_1 = y(0) = 1$ and $r_2 = y(3) = 4$, and slant $\ell = \sqrt{3^2+3^2} = 3\sqrt{2}$, so $\pi(r_1+r_2)\ell = \pi(1+4)(3\sqrt{2}) = 15\sqrt{2}\pi$.

Q9. As in Example 2, with $y = \sqrt{x}$ the integrand is $\pi\sqrt{4x+1}$, so

$$S = \int_0^6 \pi\sqrt{4x+1}\,dx = \pi\left[\frac{1}{6}(4x+1)^{3/2}\right]_0^6 = \frac{\pi}{6}\left[(25)^{3/2} - (1)^{3/2}\right] = \frac{\pi}{6}(125-1) = \frac{62\pi}{3}.$$

Q10. With $y = \sqrt{9-x^2}$, exactly as in Example 3 (radius $r=3$),

$$2\pi y\sqrt{1+(dy/dx)^2} = 2\pi\sqrt{9-x^2}\cdot\frac{3}{\sqrt{9-x^2}} = 6\pi,$$

so

$$S = \int_2^3 6\pi\,dx = 6\pi[x]_2^3 = 6\pi(3-2) = 6\pi.$$

(Equivalently, $2\pi r \times \text{width} = 2\pi(3)(1) = 6\pi$.)

Q11. $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 2$, so $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = 4t^2+4 = 4(t^2+1)$ and $\sqrt{(dx/dt)^2+(dy/dt)^2} = 2\sqrt{t^2+1}$. Then

$$S = \int_0^2 2\pi(2t)\left(2\sqrt{t^2+1}\right)dt = 8\pi\int_0^2 t\sqrt{t^2+1}\,dt = 8\pi\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^2 = \frac{8\pi}{3}\left[(5)^{3/2}-1\right] = \frac{8\pi}{3}(5\sqrt{5}-1).$$

Q12. With $\dfrac{dy}{dx} = 2x$, the surface integral is

$$S=\int_0^1 2\pi x^2\sqrt{1+4x^2}\,dx.$$

This integrand has no simple elementary antiderivative at this level, so it is evaluated with technology: $S\approx 3.810$.

Q13. Rotation about the y-axis, so the radius is x; with $\frac{dy}{dx}=2$, $\sqrt{1+(dy/dx)^2}=\sqrt{5}$. Then

$$S=\int_0^2 2\pi x\sqrt{5}\,dx=2\sqrt{5}\pi\int_0^2 x\,dx=2\sqrt{5}\pi\left[\frac{x^2}{2}\right]_0^2=4\sqrt{5}\pi.$$

Check: rotating about the y-axis, the cone has base radius 2 (the largest x), height $y(2)=4$ and slant $\ell=\sqrt{2^2+4^2}=\sqrt{20}=2\sqrt{5}$, so $\pi r\ell=\pi(2)(2\sqrt{5})=4\sqrt{5}\pi$.

Q14. $\frac{dy}{dx}=\frac{3}{2\sqrt{x}}$, so $1+(dy/dx)^2=1+\frac{9}{4x}=\frac{4x+9}{4x}$ and $\sqrt{1+(dy/dx)^2}=\frac{\sqrt{4x+9}}{2\sqrt{x}}$. The radius cancels the $\sqrt{x}$:

$$2\pi y\sqrt{1+(dy/dx)^2}=2\pi(3\sqrt{x})\cdot\frac{\sqrt{4x+9}}{2\sqrt{x}}=3\pi\sqrt{4x+9}.$$

Then

$$S=\int_0^4 3\pi\sqrt{4x+9}\,dx=3\pi\left[\frac{1}{6}(4x+9)^{3/2}\right]_0^4=\frac{\pi}{2}\left[(25)^{3/2}-(9)^{3/2}\right]=\frac{\pi}{2}(125-27)=49\pi.$$

Q15. With $\frac{dy}{dx}=\frac{1}{2}$, $\sqrt{1+(dy/dx)^2}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$. Then

$$S=\int_0^4 2\pi\left(\frac{1}{2}x+1\right)\frac{\sqrt{5}}{2}\,dx=\pi\sqrt{5}\int_0^4\left(\frac{x}{2}+1\right)dx=\pi\sqrt{5}\left[\frac{x^2}{4}+x\right]_0^4=\pi\sqrt{5}(4+4)=8\sqrt{5}\pi.$$

(Check: frustum radii $r_1=1$, $r_2=3$, slant $\ell=\sqrt{4^2+2^2}=2\sqrt{5}$, giving $\pi(1+3)(2\sqrt{5})=8\sqrt{5}\pi$.)

Q16. $\frac{dx}{dt}=-r\sin t$, $\frac{dy}{dt}=r\cos t$, so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=r^2\sin^2 t+r^2\cos^2 t=r^2,\sqrt{(dx/dt)^2+(dy/dt)^2}=r.$$

Then

$$S=\int_0^\pi 2\pi(r\sin t)(r)\,dt=2\pi r^2\int_0^\pi \sin t\,dt=2\pi r^2[-\cos t]_0^\pi=2\pi r^2(1+1)=4\pi r^2,$$

the surface area of a sphere of radius r, as expected.

Q17. With $\frac{dy}{dx}=\cos x$, the surface integral is

$$S=\int_0^\pi 2\pi\sin x\sqrt{1+\cos^2 x}\,dx.$$

Let $u=\cos x$, so $du=-\sin x\,dx$; the limits become $x=0\Rightarrow u=1$ and $x=\pi\Rightarrow u=-1$. Then

$$S=2\pi\int_1^{-1}\sqrt{1+u^2}(-du)=2\pi\int_{-1}^1\sqrt{1+u^2}\,du.$$

This is the harder-substitution integral of $\sqrt{1+u^2}$; its exact value is $2\pi\left(\sqrt{2}+\log_e(1+\sqrt{2})\right)$, which by technology is $S \approx 14.424$.

Q18. With $y=\sqrt{x}$ the integrand is $\pi\sqrt{4x+1}$ (as in Example 2), so over $1 \le x \le 4$,

$$S=\int_1^4 \pi\sqrt{4x+1}\,dx=\pi\left[\frac{1}{6}(4x+1)^{3/2}\right]_1^4=\frac{\pi}{6}\left[(17)^{3/2}-(5)^{3/2}\right]=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right).$$