Specialist Mathematics Units 3 & 4

Chapter 6 — Techniques of integration

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Significant effort has been made to ensure this content is legal. Please send any areas of concern to admin@vwx.au with appropriate document/legal references so errors can be verified and changes be made.

Contents

Theory

Antidifferentiation reverses differentiation. Given a function f, an **antiderivative** of f is any function F whose derivative is f; that is, $F'(x)=f(x)$. For example, since $\frac{d}{dx}(x^3)=3x^2$, one antiderivative of $3x^2$ is x^3. But x^3+5 and x^3-2 also have derivative $3x^2$, because the derivative of a constant is zero. So an antiderivative is never unique: if F is one antiderivative of f, then $F(x)+c$ is an antiderivative for **every** constant c, and every antiderivative of f has this form.

The indefinite integral.

The family of all antiderivatives of f is written with the integral sign:

$$\int f(x)\,dx=F(x)+c\text{, where }F'(x)=f(x).$$

Reading this notation: $f(x)$ is the **integrand**, c is the **constant of integration**, and the right-hand side is a whole family of functions — the graphs $y=F(x)+c$ are vertical translates of one another. Because a definite value of c cannot be recovered from f alone, we always write $+c$ for an indefinite integral. An antiderivative can be checked at any time by differentiating it: if differentiating the answer returns the integrand, the antiderivative is correct.

The power rule.

For any rational $n\neq-1$,

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c.$$

Add one to the index, then divide by the new index. This works for negative and fractional indices too, so terms such as $\frac{1}{x^2}=x^{-2}$ and $\sqrt{x}=x^{1/2}$ are handled by first writing them as powers. The one index the rule cannot handle is $n=-1$, because then $n+1=0$ and we would be dividing by zero.

The exception $n=-1$.

The missing case is supplied by the derivative of the logarithm. Since $\frac{d}{dx}\log_e|x|=\frac{1}{x}$ for $x\neq0$,

$$\int\frac{1}{x}\,dx=\log_e|x|+c.$$

The absolute value allows negative x: the domain of $\frac{1}{x}$ includes $x<0$, and $\log_e|x|$ is defined there while $\log_e x$ is not.

Exponential and circular functions.

Reversing $\frac{d}{dx}e^{kx}=k\,e^{kx}$ and the derivatives of $\sin kx$, $\cos kx$ and $\tan kx$ gives, for any constant $k\neq0$,

$$\int e^{kx}\,dx=\frac{1}{k}e^{kx}+c\,,\int\sin kx\,dx=-\frac{1}{k}\cos kx+c\,,$$

$$\int\cos kx\,dx=\frac{1}{k}\sin kx+c\,,\int sec^2 kx\,dx=\frac{1}{k}\tan kx+c\,.$$

In each case the factor $\frac{1}{k}$ appears because differentiating the inside function kx produces a factor of k that must be divided out. Watch the sign: antidifferentiating $\sin$ introduces a minus sign, while antidifferentiating $\cos$ does not.

Standard antiderivatives.

Collecting these results, each row of the table gives a function f and an antiderivative F, with $\int f(x)\,dx = F(x) + c$. Here $k \neq 0$.

$f(x)$	An antiderivative $F(x)$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\log_e \lvert x \rvert$
e^{kx}	$\frac{1}{k} e^{kx}$
$\sin kx$	$-\frac{1}{k} \cos kx$
$\cos kx$	$\frac{1}{k} \sin kx$
$sec^2 kx$	$\frac{1}{k} \tan kx$

Linearity.

Antidifferentiation is **linear**: a sum is integrated term by term, and a constant multiple passes straight through the integral sign.

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx, \int k f(x)\,dx = k \int f(x)\,dx.$$

Only one constant of integration is needed for the whole expression, no matter how many terms it has.

Linear-argument forms.

Each standard form still applies when the variable x is replaced by a linear expression $ax+b$ (with $a \neq 0$). Reversing the chain rule simply divides by the coefficient a:

$f(x)$	An antiderivative $F(x)$
$(ax+b)^n (n \neq -1)$	$\frac{(ax+b)^{n+1}}{a(n+1)}$
$\frac{1}{ax+b}$	$\frac{1}{a} \log_e \lvert ax+b \rvert$
e^{ax+b}	$\frac{1}{a} e^{ax+b}$

so that

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\,(n \neq -1), \int \frac{1}{ax+b}\,dx = \frac{1}{a} \log_e |ax+b| + c,$$

$$\int e^{ax+b}\,dx = \frac{1}{a} e^{ax+b} + c.$$

The same “divide by a” pattern extends to the circular functions; for instance

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c,$$

and the standard forms $\sin kx$, $\cos kx$ are just the case $b=0$.

No product or quotient rule.

Unlike differentiation, antidifferentiation has **no** product or quotient rule. A product or quotient must first be turned into a sum of standard forms — by expanding brackets, or by splitting a single fraction into separate terms — before the rules above apply. Genuinely awkward integrands need the methods of later sections, such as substitution, partial fractions and integration by parts.

Finding a particular antiderivative.

A **boundary condition** (or initial condition) selects one member of the family. Given $f'(x)$ together with the value of f at one point, first antidifferentiate to obtain the general antiderivative $F(x)+c$, then substitute the known point and solve for c.

Two further standard antiderivatives — those of $\frac{1}{\sqrt{a^2-x^2}}$ and $\frac{1}{a^2+x^2}$, which reverse the derivatives of the inverse circular functions — are developed in a later section.

Worked examples

Example 1 — scaffolded

Find

$$\int\left(3x^2-\frac{4}{x^2}+\sqrt{x}\right)dx.$$

Working	Comment
Rewrite as powers of x: $-\frac{4}{x^2}=-4x^{-2}$ and $\sqrt{x}=x^{1/2}$.	Negative and fractional indices let the power rule apply.
$3x^2 \to \frac{3x^3}{3}=x^3$.	Raise the index to 3, then divide by 3.
$-4x^{-2} \to -4\cdot\frac{x^{-1}}{-1}=4x^{-1}=\frac{4}{x}$.	New index -1; dividing by -1 flips the sign.
$x^{1/2} \to \frac{x^{3/2}}{3/2}=\frac{2}{3}x^{3/2}$.	New index $3/2$; dividing by $3/2$ is multiplying by $2/3$.
Combine, with one constant: $x^3+\frac{4}{x}+\frac{2}{3}x^{3/2}+c$.	A single c covers the whole antiderivative.

Example 2 — scaffolded

Find

$$\int\left(4e^{2x}-\sin 3x+sec^2\frac{x}{2}\right)dx.$$

Working	Comment
$4e^{2x} \to 4\cdot\frac{1}{2}e^{2x}=2e^{2x}$.	Antiderivative of e^{kx} is $1/k\,e^{kx}$; here $k=2$.

Working	Comment
$-\sin 3x \to -\left(-\frac{1}{3}\cos 3x\right)=\frac{1}{3}\cos 3x$.	Antiderivative of $\sin kx$ is $-1/k\cos kx$; here $k=3$, and two minus signs give a plus.
$sec^2\frac{x}{2} \to \frac{1}{1/2}\tan\frac{x}{2}=2\tan\frac{x}{2}$.	Antiderivative of $sec^2 kx$ is $1/k\tan kx$; here $k=1/2$, so $1/1/2=2$.
Sum, with one constant: $2e^{2x}+\frac{1}{3}\cos 3x+2\tan\frac{x}{2}+c$.	Add the antiderivatives and a single c.

Example 3 — unscaffolded

Find

$$\int\left((3x-2)^3+\frac{1}{2x+1}+e^{4-x}\right)dx.$$

Every term contains a linear expression $ax+b$ inside a standard form, so after applying the basic rule we divide by the coefficient a. Term by term,

$$\int(3x-2)^3\,dx=\frac{(3x-2)^4}{3\cdot 4}+c=\frac{(3x-2)^4}{12}+c,$$

$$\int\frac{1}{2x+1}\,dx=\frac{1}{2}\log_e|2x+1|+c,$$

$$\int e^{4-x}\,dx=\frac{1}{-1}e^{4-x}+c=-e^{4-x}+c,$$

the last because here $a=-1$. Combining the three (one constant suffices),

$\therefore$

$$\int\left((3x-2)^3+\frac{1}{2x+1}+e^{4-x}\right)dx=\frac{(3x-2)^4}{12}+\frac{1}{2}\log_e|2x+1|-e^{4-x}+c.$$

Example 4 — unscaffolded

Find $f(x)$ given that $f'(x)=4e^{2x}-\sin x$ and $f(0)=5$.

Antidifferentiate to obtain the general form, then use the boundary condition to fix c:

$$f(x)=\int\left(4e^{2x}-\sin x\right)dx=2e^{2x}+\cos x+c.$$

Substituting $x=0$ and using $e^0=1$, $\cos 0=1$,

$$f(0)=2e^0+\cos 0+c=2(1)+1+c=3+c.$$

Since $f(0)=5$, we need $3+c=5$, so $c=2$.

$\therefore f(x)=2e^{2x}+\cos x+2$.

Practice questions

Scaffolded

Q1. Find

$$\int\left(12x^5-8x^3+6x-7\right)dx.$$

(a) Antidifferentiate each power term using the power rule. (b) Antidifferentiate the constant term -7. (c) Write the full antiderivative, with a single constant c.

Q2. Find

$$\int\left(\sqrt{x}+\frac{3}{x^2}\right)dx.$$

(a) Rewrite $\sqrt{x}$ and $\frac{3}{x^2}$ as powers of x. (b) Apply the power rule to each term. (c) State the antiderivative.

Q3. Find

$$\int\left(e^{5x}+\cos 4x\right)dx.$$

(a) Use $\int e^{kx}\,dx=1/k\,e^{kx}+c$ with $k=5$. (b) Use $\int\cos kx\,dx=1/k\sin kx+c$ with $k=4$. (c) Combine.

Q4. Find

$$\int\left(\frac{5}{x}+\sin 2x\right)dx.$$

(a) Antidifferentiate $\frac{5}{x}$, recalling that $\int 1/x\,dx=\log_e|x|+c$. (b) Antidifferentiate $\sin 2x$. (c) Combine.

Q5. Find

$$\int(2x+3)^5\,dx.$$

(a) Use the linear-argument form with $a=2$, $b=3$, $n=5$. (b) Simplify the constant $a(n+1)$. (c) State the antiderivative.

Q6. A curve has gradient $\frac{dy}{dx}=3x^2-4x$ and passes through the point $(1,2)$. (a) Antidifferentiate to find y in terms of x and a constant c. (b) Substitute $(1,2)$ to find c. (c) State the equation of the curve.

Unscaffolded

Q7. Find

$$\int\left(10x^4-\frac{6}{x^2}+1\right)dx.$$

Q8. Find

$$\int\left(\frac{1}{2}x^4-\sqrt[3]{x}\right)dx.$$

Q9. Find

$$\int\left(e^{-3x}+sec^2 2x\right)dx.$$

Q10. Find

$$\int\left(6\sin\frac{x}{2}-4\cos 3x\right)dx.$$

Q11. Find

$$\int \frac{7}{2x-5}\,dx.$$

Q12. Find

$$\int \left((4x-1)^3+e^{2x+1}\right)dx.$$

Q13. Find

$$\int \frac{1}{\sqrt{3x+4}}\,dx.$$

Q14. Find

$$\int x^2(3x-2)\,dx.$$

Q15. Find

$$\int \frac{x^3-4x+6}{x^2}\,dx.$$

Q16. Find $f(x)$ given that $f'(x)=6e^{3x}-2\sin x$ and $f(0)=5$.

Q17. Find

$$\int (x+2)(x-3)\,dx.$$

Q18. A curve has gradient $\frac{dy}{dx}=\frac{3}{3x+1}$ and passes through the point $(0,4)$. Find its equation.

Answers

Q1. $2x^6-2x^4+3x^2-7x+c$. **Q2.** $\frac{2}{3}x^{3/2}-\frac{3}{x}+c$. **Q3.** $\frac{1}{5}e^{5x}+\frac{1}{4}\sin 4x+c$. **Q4.** $5\log_e|x|-\frac{1}{2}\cos 2x+c$. **Q5.** $\frac{(2x+3)^6}{12}+c$. **Q6.** $y=x^3-2x^2+3$. **Q7.** $2x^5+\frac{6}{x}+x+c$. **Q8.** $\frac{x^5}{10}-\frac{3}{4}x^{4/3}+c$. **Q9.** $-\frac{1}{3}e^{-3x}+\frac{1}{2}\tan 2x+c$. **Q10.** $-12\cos\frac{x}{2}-\frac{4}{3}\sin 3x+c$. **Q11.** $\frac{7}{2}\log_e|2x-5|+c$. **Q12.** $\frac{(4x-1)^4}{16}+\frac{1}{2}e^{2x+1}+c$. **Q13.** $\frac{2}{3}\sqrt{3x+4}+c$. **Q14.** $\frac{3}{4}x^4-\frac{2}{3}x^3+c$. **Q15.** $\frac{x^2}{2}-4\log_e|x|-\frac{6}{x}+c$. **Q16.** $f(x)=2e^{3x}+2\cos x+1$. **Q17.** $\frac{x^3}{3}-\frac{x^2}{2}-6x+c$. **Q18.** $y=\log_e|3x+1|+4$.

Fully-worked solutions

Q1. Antidifferentiate term by term with the power rule, dividing each constant term straight through:

$$\int(12x^5-8x^3+6x-7)\,dx=\frac{12x^6}{6}-\frac{8x^4}{4}+\frac{6x^2}{2}-7x+c=2x^6-2x^4+3x^2-7x+c.$$

Q2. Write both terms as powers: $\sqrt{x}=x^{1/2}$ and $\frac{3}{x^2}=3x^{-2}$. Then

$$\int(x^{1/2}+3x^{-2})\,dx=\frac{x^{3/2}}{3/2}+3\cdot\frac{x^{-1}}{-1}+c=\frac{2}{3}x^{3/2}-3x^{-1}+c=\frac{2}{3}x^{3/2}-\frac{3}{x}+c.$$

Q3. Using the standard exponential and cosine forms with $k=5$ and $k=4$,

$$\int(e^{5x}+\cos 4x)\,dx=\frac{1}{5}e^{5x}+\frac{1}{4}\sin 4x+c.$$

Q4. The first term uses $\int 1/x\,dx=\log_e|x|+c$, the second the standard sine form with $k=2$:

$$\int\left(\frac{5}{x}+\sin 2x\right)dx=5\log_e|x|-\frac{1}{2}\cos 2x+c.$$

Q5. With $a=2$, $b=3$, $n=5$ the linear-argument power rule gives $a(n+1)=2\times 6=12$, so

$$\int(2x+3)^5\,dx=\frac{(2x+3)^6}{2(5+1)}+c=\frac{(2x+3)^6}{12}+c.$$

Q6. Antidifferentiating the gradient,

$$y=\int(3x^2-4x)\,dx=x^3-2x^2+c.$$

Substituting the point $(1,2)$: $2=1^3-2(1)^2+c=1-2+c=-1+c$, so $c=3$. Hence $y=x^3-2x^2+3$.

Q7. Write $\frac{6}{x^2}=6x^{-2}$. Then

$$\int(10x^4-6x^{-2}+1)\,dx=\frac{10x^5}{5}-6\cdot\frac{x^{-1}}{-1}+x+c=2x^5+6x^{-1}+x+c=2x^5+\frac{6}{x}+x+c.$$

Q8. Write $\sqrt[3]{x}=x^{1/3}$. Then

$$\int\left(\frac{1}{2}x^4 - x^{1/3}\right)dx = \frac{1}{2}\cdot\frac{x^5}{5} - \frac{x^{4/3}}{4/3} + c = \frac{x^5}{10} - \frac{3}{4}x^{4/3} + c.$$

Q9. Using the exponential form with $k = -3$ and the sec^2 form with $k = 2$,

$$\int\left(e^{-3x} + sec^2 2x\right)dx = \frac{1}{-3}e^{-3x} + \frac{1}{2}\tan 2x + c = -\frac{1}{3}e^{-3x} + \frac{1}{2}\tan 2x + c.$$

Q10. Here $\sin\frac{x}{2}$ has $k = 1/2$, so its antiderivative is $-\frac{1}{1/2}\cos\frac{x}{2} = -2\cos\frac{x}{2}$; and $\cos 3x$ has $k = 3$. Thus

$$\int\left(6\sin\frac{x}{2} - 4\cos 3x\right)dx = 6\left(-2\cos\frac{x}{2}\right) - 4\cdot\frac{1}{3}\sin 3x + c = -12\cos\frac{x}{2} - \frac{4}{3}\sin 3x + c.$$

Q11. Using $\int\frac{1}{ax+b}dx = \frac{1}{a}\log_e|ax+b| + c$ with $a = 2$,

$$\int\frac{7}{2x-5}dx = 7\cdot\frac{1}{2}\log_e|2x-5| + c = \frac{7}{2}\log_e|2x-5| + c.$$

Q12. For $(4x-1)^3$ use $a = 4$, $n = 3$, so $a(n+1) = 4\times 4 = 16$; for e^{2x+1} use $a = 2$:

$$\int\left((4x-1)^3 + e^{2x+1}\right)dx = \frac{(4x-1)^4}{16} + \frac{1}{2}e^{2x+1} + c.$$

Q13. Write the integrand as $(3x+4)^{-1/2}$. With $a = 3$, $n = -1/2$ so $n+1 = 1/2$ and $a(n+1) = 3\times 1/2 = 3/2$:

$$\int(3x+4)^{-1/2}dx = \frac{(3x+4)^{1/2}}{3/2} + c = \frac{2}{3}(3x+4)^{1/2} + c = \frac{2}{3}\sqrt{3x+4} + c.$$

Q14. There is no product rule, so expand first: $x^2(3x-2) = 3x^3 - 2x^2$. Then

$$\int(3x^3 - 2x^2)dx = \frac{3x^4}{4} - \frac{2x^3}{3} + c = \frac{3}{4}x^4 - \frac{2}{3}x^3 + c.$$

Q15. Split the single fraction into separate terms: $\frac{x^3 - 4x + 6}{x^2} = x - \frac{4}{x} + \frac{6}{x^2} = x - 4x^{-1} + 6x^{-2}$. Then

$$\int(x - 4x^{-1} + 6x^{-2})dx = \frac{x^2}{2} - 4\log_e|x| + 6\cdot\frac{x^{-1}}{-1} + c = \frac{x^2}{2} - 4\log_e|x| - \frac{6}{x} + c.$$

Q16. Antidifferentiating,

$$f(x) = \int\left(6e^{3x} - 2\sin x\right)dx = 2e^{3x} + 2\cos x + c.$$

Using $f(0) = 5$ with $e^0 = 1$ and $\cos 0 = 1$: $2(1) + 2(1) + c = 4 + c = 5$, so $c = 1$. Hence $f(x) = 2e^{3x} + 2\cos x + 1$.

Q17. Expand the product first: $(x+2)(x-3) = x^2 - x - 6$. Then

$$\int\left(x^2 - x - 6\right)dx = \frac{x^3}{3} - \frac{x^2}{2} - 6x + c.$$

Q18. Antidifferentiating the gradient with $a = 3$,

$$y = \int\frac{3}{3x+1}dx = 3\cdot\frac{1}{3}\log_e|3x+1| + c = \log_e|3x+1| + c.$$

Substituting $(0,4)$: since $3(0) + 1 = 1$ and $\log_e 1 = 0$, we get $4 = 0 + c$, so $c = 4$. Hence $y = \log_e|3x+1| + 4$.

Chapter 6 Techniques of integration — 6B Antiderivatives giving inverse circular functions

Theory

Antidifferentiation reverses differentiation, so every derivative rule read backwards supplies an antiderivative. The derivatives of the inverse circular functions produce three new standard forms. Recall that, for a constant $a>0$,

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}},\ \frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right)=-\frac{1}{\sqrt{a^2-x^2}},\ \frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)=\frac{a}{a^2+x^2}.$$

Reading each of these in reverse gives the antiderivative it came from.

The three standard forms.

For $a>0$,

$$\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c,\ (-a<x<a),$$

$$\int\frac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+c,\ (-a<x<a),$$

$$\int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c.$$

The third result carries an extra factor of $\frac{1}{a}$ because $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)=\frac{a}{a^2+x^2}$, so we divide by a to undo it. The two square-root forms are valid only on the open interval $-a<x<a$, where the expression under the root is positive; the $\tan^{-1}$ form is valid for every real x.

The sine and cosine forms.

Because $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{x}{a}\right)\right)=0$, the two functions $\sin^{-1}\left(\frac{x}{a}\right)$ and $-\cos^{-1}\left(\frac{x}{a}\right)$ differ only by a constant. Both are therefore antiderivatives of $\frac{1}{\sqrt{a^2-x^2}}$, and either may be quoted — the difference is absorbed into c. It is the sign of the numerator that decides which name to use: a $+1$ numerator points to $\sin^{-1}$, a -1 numerator to $\cos^{-1}$.

Recognising the shape.

To match an integrand to a standard form, read off a from the constant. For $\frac{1}{\sqrt{a^2-x^2}}$ the number under the root is a^2, so $\sqrt{9-x^2}$ gives $a=3$ and $\sqrt{5-x^2}$ gives $a=\sqrt{5}$. For $\frac{1}{a^2+x^2}$ the constant added to x^2 is a^2, so $16+x^2$ gives $a=4$. Since $a>0$ we always take the positive square root.

Adjusting the numerator.

A constant factor in the numerator passes straight through the integral. If k is a constant,

$$\int\frac{k}{\sqrt{a^2-x^2}}\,dx=k\sin^{-1}\left(\frac{x}{a}\right)+c,\ \int\frac{k}{a^2+x^2}\,dx=\frac{k}{a}\tan^{-1}\left(\frac{x}{a}\right)+c.$$

Take the factor outside, apply the standard form, and keep a single constant c.

Completing the square.

When the denominator is a quadratic that is not yet a sum or difference of squares, complete the square first. Writing $x^2+bx+d=\left(x+\frac{b}{2}\right)^2+\left(d-\frac{b^2}{4}\right)$ turns the quadratic into a shifted square. If the leftover constant is positive, say a^2, the denominator becomes $(x+p)^2+a^2$ and the integrand matches the $\tan^{-1}$ form with x replaced by $x+p$:

$$\int \frac{1}{(x+p)^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x+p}{a}\right)+c.$$

This replacement is really the substitution $u=x+p$; because the shift does not change the differential, the standard form applies directly with $u=x+p$. The same idea works **under a root**: rearrange the quadratic into the form $a^2-(x+p)^2$ to reach a $\sin^{-1}$ (or $\cos^{-1}$) form,

$$\int \frac{1}{\sqrt{a^2-(x+p)^2}}\,dx=\sin^{-1}\left(\frac{x+p}{a}\right)+c.$$

Watch the sign of the x^2 term: when it is negative, factor -1 out of the x-terms before completing the square, so that the square appears with a minus sign and the constant with a plus — for example $8-2x-x^2=9-(x+1)^2$.

Definite integrals.

For a definite integral of any of these forms, find the antiderivative, then substitute the terminals and subtract, exactly as for other integrals. Standard terminals give exact multiples of π, since $\tan^{-1}(1)=\frac{\pi}{4}$, $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$, and so on.

The interval of integration must lie inside the domain of the antiderivative — for a $\sin^{-1}$ or $\cos^{-1}$ form the terminals must satisfy $-a<x<a$.

Worked examples

Example 1 — scaffolded

Find

$$\int \frac{1}{\sqrt{9-x^2}}\,dx.$$

Working	Comment
The integrand is $\frac{1}{\sqrt{9-x^2}}$; compare it with $\frac{1}{\sqrt{a^2-x^2}}$.	Match the shape of the standard sine form.
Here $a^2=9$, and since $a>0$, $a=3$.	Read a from the constant under the root.
An antiderivative is $\sin^{-1}\left(\frac{x}{3}\right)$.	Standard form $\sin^{-1}\left(\frac{x}{a}\right)$ with $a=3$.
Hence the integral is $\sin^{-1}\left(\frac{x}{3}\right)+c$.	Add the constant of integration.

Example 2 — scaffolded

Find

$$\int \frac{1}{x^2+4x+13}\,dx.$$

Working	Comment
The denominator does not yet match a^2+x^2, so complete the square.	The quadratic has no obvious a.
$x^2+4x+13=(x+2)^2+9$.	Half of 4 is 2, and $13-2^2=9$.
The integrand is $\dfrac{1}{(x+2)^2+9}$, matching $\dfrac{1}{(x+p)^2+a^2}$ with $p=2$, $a=3$.	A shifted $\tan^{-1}$ form.
An antiderivative is $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x+2}{3}\right)$.	Apply the standard tangent form, $x\to x+2$, $a=3$.
Hence the integral is $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x+2}{3}\right)+c$.	Add the constant of integration.

Example 3 — unscaffolded

Evaluate

$$\int_0^2 \frac{1}{4+x^2}\,dx.$$

The integrand matches $\dfrac{1}{a^2+x^2}$ with $a^2=4$, so $a=2$ and an antiderivative is $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)$. Evaluating between the terminals,

$$\int_0^2 \frac{1}{4+x^2}\,dx=\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2=\frac{1}{2}\tan^{-1}(1)-\frac{1}{2}\tan^{-1}(0)=\frac{1}{2}\cdot\frac{\pi}{4}-0=\frac{\pi}{8}.$$

$\therefore$ the definite integral equals $\dfrac{\pi}{8}$.

Example 4 — unscaffolded

Find

$$\int \frac{1}{\sqrt{8-2x-x^2}}\,dx.$$

The expression under the root is a quadratic with a negative x^2 term, so factor -1 from the x-terms and complete the square:

$$8-2x-x^2=-(x^2+2x-8)=-\left((x+1)^2-9\right)=9-(x+1)^2.$$

This is the difference-of-squares shape $a^2-(x+p)^2$ with $a=3$ and $p=1$, so the sine form applies with x replaced by $x+1$:

$$\int \frac{1}{\sqrt{8-2x-x^2}}\,dx=\int \frac{1}{\sqrt{9-(x+1)^2}}\,dx=\sin^{-1}\left(\frac{x+1}{3}\right)+c.$$

$\therefore$ the integral is $\sin^{-1}\left(\dfrac{x+1}{3}\right)+c$, valid for $-4<x<2$.

Practice questions

Scaffolded

Q1. Find

$$\int \frac{1}{\sqrt{25-x^2}}\,dx.$$

(a) Compare the integrand with $\dfrac{1}{\sqrt{a^2-x^2}}$ and state a. (b) Write down the matching standard antiderivative. (c) State the integral, including $+c$.

Q2. Find

$$\int \frac{1}{16+x^2}\,dx.$$

(a) Compare the integrand with $\dfrac{1}{a^2+x^2}$ and state a. (b) Write down the standard form $\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$ for this a. (c) State the integral, including $+c$.

Q3. Find

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx.$$

(a) Which inverse circular function has this expression as its derivative? (b) State the value of a. (c) Write the integral, including $+c$.

Q4. Find

$$\int \frac{1}{x^2+6x+10}\,dx.$$

(a) Complete the square in the denominator. (b) Identify a and the shift $x+p$. (c) Write the integral, including $+c$.

Q5. Find

$$\int \frac{4}{\sqrt{9-x^2}}\,dx.$$

(a) Take the constant factor 4 outside. (b) Write an antiderivative of $\dfrac{1}{\sqrt{9-x^2}}$. (c) State the integral, including $+c$.

Q6. Evaluate

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

(a) Write an antiderivative of the integrand. (b) Substitute the terminals $x=1$ and $x=0$. (c) State the exact value.

Unscaffolded

Q7. Find

$$\int \frac{1}{\sqrt{49-x^2}}\,dx.$$

Q8. Find

$$\int \frac{1}{36+x^2}\,dx.$$

Q9. Find

$$\int \frac{1}{\sqrt{5-x^2}}\,dx.$$

Q10. Find

$$\int \frac{-1}{\sqrt{7-x^2}}\,dx.$$

Q11. Find

$$\int \frac{5}{4+x^2}\,dx.$$

Q12. Find

$$\int \frac{3}{\sqrt{4-x^2}}\,dx.$$

Q13. Find

$$\int \frac{1}{x^2-2x+5}\,dx.$$

Q14. Find

$$\int \frac{1}{x^2+8x+20}\,dx.$$

Q15. Find

$$\int \frac{1}{\sqrt{3-2x-x^2}}\,dx.$$

Q16. Evaluate

$$\int_0^{\sqrt{3}} \frac{1}{3+x^2}\,dx.$$

Q17. Evaluate

$$\int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx.$$

Q18. Evaluate

$$\int_2^4 \frac{1}{x^2-4x+8}\,dx.$$

Answers

Q1. $\sin^{-1}\left(\frac{x}{5}\right)+c$. **Q2.** $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$. **Q3.** $\cos^{-1}x+c$. **Q4.** $\tan^{-1}(x+3)+c$. **Q5.** $4\sin^{-1}\left(\frac{x}{3}\right)+c$. **Q6.** $\frac{\pi}{4}$. **Q7.** $\sin^{-1}\left(\frac{x}{7}\right)+c$. **Q8.** $\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)+c$. **Q9.** $\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$. **Q10.** $\cos^{-1}\left(\frac{x}{\sqrt{7}}\right)+c$. **Q11.** $\frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$. **Q12.** $3\sin^{-1}\left(\frac{x}{2}\right)+c$. **Q13.** $\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c$. **Q14.** $\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+c$. **Q15.** $\sin^{-1}\left(\frac{x+1}{2}\right)+c$. **Q16.** $\frac{\pi\sqrt{3}}{12}$. **Q17.** $\frac{\pi}{6}$. **Q18.** $\frac{\pi}{8}$.

Fully-worked solutions

Q1. The integrand $\frac{1}{\sqrt{25-x^2}}$ matches $\frac{1}{\sqrt{a^2-x^2}}$ with $a^2=25$, so $a=5$. Hence

$$\int\frac{1}{\sqrt{25-x^2}}\,dx=\sin^{-1}\left(\frac{x}{5}\right)+c.$$

Q2. The integrand $\frac{1}{16+x^2}$ matches $\frac{1}{a^2+x^2}$ with $a^2=16$, so $a=4$. Using $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$,

$$\int\frac{1}{16+x^2}\,dx=\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c.$$

Q3. The expression $\frac{-1}{\sqrt{1-x^2}}$ is the derivative of $\cos^{-1}x$, matching $\frac{-1}{\sqrt{a^2-x^2}}$ with $a=1$. Hence

$$\int\frac{-1}{\sqrt{1-x^2}}\,dx=\cos^{-1}x+c.$$

Q4. Completing the square, $x^2+6x+10=(x+3)^2+1$, which is $(x+p)^2+a^2$ with $p=3$ and $a=1$. Therefore

$$\int\frac{1}{x^2+6x+10}\,dx=\int\frac{1}{(x+3)^2+1}\,dx=\tan^{-1}(x+3)+c.$$

Q5. Take the constant factor of 4 outside and apply the sine form with $a=3$:

$$\int\frac{4}{\sqrt{9-x^2}}\,dx=4\int\frac{1}{\sqrt{9-x^2}}\,dx=4\sin^{-1}\left(\frac{x}{3}\right)+c.$$

Q6. An antiderivative of $\frac{1}{1+x^2}$ is $\tan^{-1}x$ (the case $a=1$). Hence

$$\int_0^1\frac{1}{1+x^2}\,dx=[\tan^{-1}x]_0^1=\tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}-0=\frac{\pi}{4}.$$

Q7. Here $a^2=49$, so $a=7$ and

$$\int\frac{1}{\sqrt{49-x^2}}\,dx=\sin^{-1}\left(\frac{x}{7}\right)+c.$$

Q8. Here $a^2=36$, so $a=6$ and

$$\int \frac{1}{36+x^2}\,dx=\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)+c.$$

Q9. Here $a^2=5$, so $a=\sqrt{5}$ and

$$\int \frac{1}{\sqrt{5-x^2}}\,dx=\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+c.$$

Q10. The -1 numerator gives the cosine form, with $a^2=7$, so $a=\sqrt{7}$ and

$$\int \frac{-1}{\sqrt{7-x^2}}\,dx=\cos^{-1}\left(\frac{x}{\sqrt{7}}\right)+c.$$

Q11. Take the constant factor of 5 outside; the remaining integrand has $a=2$:

$$\int \frac{5}{4+x^2}\,dx=5\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c=\frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right)+c.$$

Q12. Take the constant factor of 3 outside; the sine form has $a=2$:

$$\int \frac{3}{\sqrt{4-x^2}}\,dx=3\sin^{-1}\left(\frac{x}{2}\right)+c.$$

Q13. Completing the square, $x^2-2x+5=(x-1)^2+4$, so $p=-1$ and $a=2$. Hence

$$\int \frac{1}{x^2-2x+5}\,dx=\int \frac{1}{(x-1)^2+4}\,dx=\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c.$$

Q14. Completing the square, $x^2+8x+20=(x+4)^2+4$, so $p=4$ and $a=2$. Hence

$$\int \frac{1}{x^2+8x+20}\,dx=\int \frac{1}{(x+4)^2+4}\,dx=\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+c.$$

Q15. The x^2 term is negative, so factor -1 from the x-terms and complete the square:

$$3-2x-x^2=-(x^2+2x-3)=-\left((x+1)^2-4\right)=4-(x+1)^2.$$

This is $a^2-(x+p)^2$ with $a=2$ and $p=1$, so

$$\int \frac{1}{\sqrt{3-2x-x^2}}\,dx=\int \frac{1}{\sqrt{4-(x+1)^2}}\,dx=\sin^{-1}\left(\frac{x+1}{2}\right)+c.$$

Q16. Here $a^2=3$, so $a=\sqrt{3}$ and an antiderivative is $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$. Then

$$\int_0^{\sqrt{3}} \frac{1}{3+x^2}\,dx=\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^{\sqrt{3}}=\frac{1}{\sqrt{3}}\tan^{-1}(1)-0=\frac{1}{\sqrt{3}}\cdot\frac{\pi}{4}=\frac{\pi}{4\sqrt{3}}=\frac{\pi\sqrt{3}}{12}.$$

Q17. Here $a^2=16$, so $a=4$ and an antiderivative is $\sin^{-1}\left(\frac{x}{4}\right)$. Then

$$\int_0^{2} \frac{1}{\sqrt{16-x^2}}\,dx=\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^2=\sin^{-1}\left(\frac{1}{2}\right)-0=\frac{\pi}{6}.$$

Q18. Completing the square, $x^2-4x+8=(x-2)^2+4$, so $a=2$ and an antiderivative is $\frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right)$. Then

$$\int_2^4 \frac{1}{x^2-4x+8}\,dx=\left[\frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right)\right]_2^4=\frac{1}{2}\tan^{-1}(1)-\frac{1}{2}\tan^{-1}(0)=\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}.$$

Theory

Antidifferentiation reverses differentiation, but many integrands are built by the **chain rule** and cannot be integrated by simply reversing a standard form. The technique of **substitution** (sometimes called the *reverse chain rule*) undoes the chain rule: it replaces the inner function of a composite by a single new variable u, turning a difficult integral in x into a simple one in u.

The substitution rule.

Recall the chain rule: if F is an antiderivative of f (so that $F' = f$), then

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x).$$

Antidifferentiating both sides with respect to x gives

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c.$$

Now put $u = g(x)$. Its derivative is $\frac{du}{dx} = g'(x)$, which we record in the **differential form** $du = g'(x)\,dx$. Replacing $g(x)$ by u and $g'(x)\,dx$ by du collapses the left-hand side to an integral in u alone:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + c.$$

This is the **substitution formula**. The whole method rests on choosing $u = g(x)$ so that the factor $g'(x)\,dx$ — or a constant multiple of it — is already present in the integrand.

Recognising when a substitution works.

A substitution simplifies an integral when the integrand is a **composite function** $f(g(x))$ multiplied by (a constant multiple of) the derivative $g'(x)$ of its inner function. The tell-tale sign is that *one factor is essentially the derivative of the "inside" of another factor*. For example:

Integrand	inner function $u = g(x)$	$\frac{du}{dx}$
$2x\sqrt{x^2+1}$	$u = x^2+1$	$2x$
xe^{x^2}	$u = x^2$	$2x$
$\frac{\log_e x}{x}$	$u = \log_e x$	$\frac{1}{x}$
$\sin^3 x\cos x$	$u = \sin x$	$\cos x$

In each row one factor of the integrand is the derivative of u (exactly, or up to a constant), so the substitution clears every x.

The method, step by step.

1. **Choose** $u = g(x)$ — usually the inner function of a composite, or an expression whose derivative appears elsewhere in the integrand.
2. **Differentiate** to get $\frac{du}{dx} = g'(x)$, and write $du = g'(x)\,dx$.
3. **Rewrite** the entire integral in terms of u. Every x and the dx must be replaced; if any x remains, the substitution is either the wrong choice or incomplete.

4. **Integrate** with respect to u.
5. **Back-substitute** $u=g(x)$ to return the answer to x, and add the constant $+c$.

Adjusting for a missing constant factor.

Often the derivative $g'(x)$ is present only up to a constant. Consider the integral of xe^{x^2}: the natural choice $u=x^2$ gives $du=2x\,dx$, whereas the integrand supplies only $x\,dx$. Since $x\,dx=1/2\,du$, we substitute that:

$$\int xe^{x^2}dx=\int e^u\cdot 1/2\,du=1/2\int e^u\,du=1/2\,e^u+c=1/2\,e^{x^2}+c.$$

Only a **constant** may be moved across the integral sign in this way; a substitution can never be rescued by moving a variable factor such as x outside the integral.

Back-substituting.

Because the original problem is stated in x, the final answer must also be in x: always replace u by $g(x)$ at the end. A quick and reliable check is to **differentiate the answer** — by the chain rule it should return the original integrand.

Three patterns that always yield to substitution.

Writing the method in general form gives three standard results worth recognising on sight (each is just $u=g(x)$):

Integral	substitution	result
$\int (g(x))^n g'(x)\,dx,\ n\neq -1$	$u=g(x)$	$\dfrac{(g(x))^{n+1}}{n+1}+c$
$\int \dfrac{g'(x)}{g(x)}dx$	$u=g(x)$	$\log_e\lvert g(x)\rvert+c$
$\int g'(x)e^{g(x)}dx$	$u=g(x)$	$e^{g(x)}+c$

Definite integrals by substitution, and substitutions in which the inner function's derivative is *not* already present, are treated in later sections; here every integrand is chosen so that a multiple of $g'(x)$ appears.

Worked examples

Example 1 — scaffolded

Find $\int 2x\sqrt{x^2+1}\,dx$.

Working	Comment
The integrand is a composite $\sqrt{x^2+1}$ times $2x$; let $u=x^2+1$.	Choose u to be the inner function of the composite.
$\dfrac{du}{dx}=2x$, so $du=2x\,dx$.	Differentiate; here $2x\,dx$ is exactly the remaining factor.
$\int 2x\sqrt{x^2+1}\,dx=\int\sqrt{u}\,du$.	Replace $\sqrt{x^2+1}$ by $\sqrt{u}$ and $2x\,dx$ by du; no x remains.
$\int u^{1/2}du=\dfrac{2}{3}u^{3/2}+c$.	Integrate $u^{1/2}$ by the power rule (add one to the index).
$=\dfrac{2}{3}(x^2+1)^{3/2}+c$.	Back-substitute $u=x^2+1$.

Example 2 — scaffolded

Find $\int\dfrac{x}{\sqrt{x^2+9}}dx$.

Working	Comment
Let $u=x^2+9$, the expression under the root.	The inner function of the composite $\frac{1}{\sqrt{x^2+9}}$.
$\frac{du}{dx}=2x$, so $du=2x\,dx$ and $x\,dx=\frac{1}{2}du$.	The integrand has only $x\,dx$, so rescale by the constant $\frac{1}{2}$.
$\int \frac{x}{\sqrt{x^2+9}}dx=\int \frac{1}{\sqrt{u}}\cdot\frac{1}{2}du=\frac{1}{2}\int u^{-1/2}du$.	Rewrite fully in u; move the constant $\frac{1}{2}$ outside.
$\frac{1}{2}\int u^{-1/2}du=\frac{1}{2}\cdot 2u^{1/2}+c=u^{1/2}+c$.	Integrate $u^{-1/2}$ by the power rule.
$=\sqrt{x^2+9}+c$.	Back-substitute $u=x^2+9$.

Example 3 — unscaffolded

Find $\int \frac{\log_e x}{x}dx$.

The factor $\frac{1}{x}$ is the derivative of $\log_e x$, which suggests the substitution $u=\log_e x$. Then $\frac{du}{dx}=\frac{1}{x}$, so $du=\frac{1}{x}dx$, and the integral becomes

$$\int \frac{\log_e x}{x}dx=\int u\,du=\frac{u^2}{2}+c=\frac{(\log_e x)^2}{2}+c.$$

$\therefore \int \frac{\log_e x}{x}dx=\frac{(\log_e x)^2}{2}+c$.

Example 4 — unscaffolded

Find $\int \sin^3 x\cos x\,dx$.

Here $\sin^3 x=(\sin x)^3$ is a composite with inner function $\sin x$, and the remaining factor $\cos x$ is exactly its derivative. Let $u=\sin x$, so $\frac{du}{dx}=\cos x$ and $du=\cos x\,dx$. Then

$$\int \sin^3 x\cos x\,dx=\int u^3du=\frac{u^4}{4}+c=\frac{\sin^4 x}{4}+c.$$

$\therefore \int \sin^3 x\cos x\,dx=\frac{\sin^4 x}{4}+c$.

Practice questions

Scaffolded

Q1. Find $\int 2x(x^2+3)^4dx$. (a) Let $u=x^2+3$ and find $\frac{du}{dx}$, hence du. (b) Rewrite the integral in terms of u. (c) Integrate with respect to u. (d) Back-substitute to give the answer in terms of x.

Q2. Find $\int 3x^2e^{x^3}dx$. (a) Let $u=x^3$ and find du. (b) Show that the integral becomes $\int e^u du$. (c) Integrate and back-substitute.

Q3. Find $\int \frac{2x}{x^2+1}\,dx$. (a) Let $u=x^2+1$ and find du. (b) Write the integral as $\int \frac{1}{u}\,du$. (c) Integrate using $\int \frac{1}{u}\,du=\log_e|u|+c$, then back-substitute, explaining why the absolute value is unnecessary here.

Q4. Find $\int \cos^4 x \sin x\,dx$. (a) Let $u=\cos x$; show that $\frac{du}{dx}=-\sin x$, so $\sin x\,dx=-du$. (b) Rewrite the integral in terms of u. (c) Integrate and back-substitute.

Q5. Find $\int x\sqrt{x^2+4}\,dx$. (a) Let $u=x^2+4$; find du and express $x\,dx$ in terms of du. (b) Rewrite the integral in terms of u. (c) Integrate and back-substitute.

Q6. Find $\int \frac{(\log_e x)^2}{x}\,dx$. (a) Let $u=\log_e x$ and find du. (b) Rewrite the integral in terms of u. (c) Integrate and back-substitute.

Unscaffolded

Q7. Find $\int 4x^3(x^4+2)^5\,dx$.

Q8. Find $\int x^2 e^{x^3+1}\,dx$.

Q9. Find $\int \frac{x}{x^2+5}\,dx$.

Q10. Find $\int \sin^5 x \cos x\,dx$.

Q11. Find $\int (3x^2+2)(x^3+2x)^4\,dx$.

Q12. Find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Q13. Find $\int x^2\sqrt{x^3+1}\,dx$.

Q14. Find $\int \frac{\sin x}{\cos^2 x}\,dx$.

Q15. Find $\int \frac{\sin x}{\cos x}\,dx$.

Q16. Find $\int \frac{e^x}{e^x+1}\,dx$.

Q17. Find $\int \frac{6x}{(x^2+1)^2}\,dx$.

Q18. Find $\int \frac{x}{\sqrt{1-x^2}}\,dx$.

Answers

Q1. $\frac{(x^2+3)^5}{5}+c$. **Q2.** $e^{x^3}+c$. **Q3.** $\log_e(x^2+1)+c$. **Q4.** $-\frac{\cos^5 x}{5}+c$. **Q5.** $\frac{1}{3}(x^2+4)^{3/2}+c$. **Q6.** $\frac{(\log_e x)^3}{3}+c$. **Q7.** $\frac{(x^4+2)^6}{6}+c$. **Q8.** $\frac{1}{3}e^{x^3+1}+c$. **Q9.** $\frac{1}{2}\log_e(x^2+5)+c$. **Q10.** $\frac{\sin^6 x}{6}+c$. **Q11.** $\frac{(x^3+2x)^5}{5}+c$. **Q12.** $2e^{\sqrt{x}}+c$. **Q13.** $\frac{2}{9}(x^3+1)^{3/2}+c$. **Q14.** $\frac{1}{\cos x}+c$ (equivalently $sec\, x+c$). **Q15.** $-\log_e|\cos x|+c$. **Q16.** $\log_e(e^x+1)+c$. **Q17.** $-\frac{3}{x^2+1}+c$. **Q18.** $-\sqrt{1-x^2}+c$.

Fully-worked solutions

Q1. Let $u=x^2+3$, so $\frac{du}{dx}=2x$ and $du=2x\,dx$ — exactly the leftover factor. Then

$$\int 2x(x^2+3)^4\,dx=\int u^4\,du=\frac{u^5}{5}+c=\frac{(x^2+3)^5}{5}+c.$$

Q2. Let $u=x^3$, so $\frac{du}{dx}=3x^2$ and $du=3x^2\,dx$. The factor $3x^2\,dx$ is present, so

$$\int 3x^2e^{x^3}\,dx=\int e^u\,du=e^u+c=e^{x^3}+c.$$

Q3. Let $u=x^2+1$, so $du=2x\,dx$, again exactly the numerator. Then

$$\int\frac{2x}{x^2+1}\,dx=\int\frac{1}{u}\,du=\log_e|u|+c=\log_e(x^2+1)+c.$$

The absolute value is dropped because $u=x^2+1>0$ for every real x.

Q4. Let $u=\cos x$, so $\frac{du}{dx}=-\sin x$ and $\sin x\,dx=-du$. Then

$$\int\cos^4 x\sin x\,dx=\int u^4(-du)=-\frac{u^5}{5}+c=-\frac{\cos^5 x}{5}+c.$$

Q5. Let $u=x^2+4$, so $du=2x\,dx$ and $x\,dx=\frac{1}{2}du$. Then

$$\int x\sqrt{x^2+4}\,dx=\int\sqrt{u}\cdot\frac{1}{2}du=\frac{1}{2}\cdot\frac{2}{3}u^{3/2}+c=\frac{1}{3}(x^2+4)^{3/2}+c.$$

Q6. Let $u=\log_e x$, so $\frac{du}{dx}=\frac{1}{x}$ and $du=\frac{1}{x}dx$. Then

$$\int\frac{(\log_e x)^2}{x}\,dx=\int u^2\,du=\frac{u^3}{3}+c=\frac{(\log_e x)^3}{3}+c.$$

Q7. Let $u=x^4+2$, so $du=4x^3\,dx$ — exactly the leftover factor. Then

$$\int 4x^3(x^4+2)^5\,dx=\int u^5\,du=\frac{u^6}{6}+c=\frac{(x^4+2)^6}{6}+c.$$

Q8. Let $u = x^3 + 1$, so $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}du$. Then

$$\int x^2 e^{x^3+1}\,dx = \int e^u \cdot \frac{1}{3}du = \frac{1}{3}e^u + c = \frac{1}{3}e^{x^3+1} + c.$$

Q9. Let $u = x^2 + 5$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$. Then

$$\int \frac{x}{x^2+5}\,dx = \int \frac{1}{u} \cdot \frac{1}{2}du = \frac{1}{2}\log_e|u| + c = \frac{1}{2}\log_e(x^2+5) + c,$$

the absolute value being unnecessary since $x^2 + 5 > 0$.

Q10. Let $u = \sin x$, so $du = \cos x\,dx$. Then

$$\int \sin^5 x \cos x\,dx = \int u^5\,du = \frac{u^6}{6} + c = \frac{\sin^6 x}{6} + c.$$

Q11. Notice $\frac{d}{dx}(x^3 + 2x) = 3x^2 + 2$, exactly the first factor. Let $u = x^3 + 2x$, so $du = (3x^2 + 2)\,dx$. Then

$$\int (3x^2+2)(x^3+2x)^4\,dx = \int u^4\,du = \frac{u^5}{5} + c = \frac{(x^3+2x)^5}{5} + c.$$

Q12. Let $u = \sqrt{x} = x^{1/2}$, so $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and $\frac{1}{\sqrt{x}}dx = 2\,du$. Then

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int e^u \cdot 2\,du = 2e^u + c = 2e^{\sqrt{x}} + c.$$

Q13. Let $u = x^3 + 1$, so $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}du$. Then

$$\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u} \cdot \frac{1}{3}du = \frac{1}{3} \cdot \frac{2}{3}u^{3/2} + c = \frac{2}{9}(x^3+1)^{3/2} + c.$$

Q14. Let $u = \cos x$, so $\frac{du}{dx} = -\sin x$ and $\sin x\,dx = -du$. Writing $\frac{\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} \cdot \sin x$,

$$\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{1}{u^2}(-du) = -\int u^{-2}\,du = u^{-1} + c = \frac{1}{\cos x} + c.$$

Equivalently this is $sec\,x + c$.

Q15. Let $u = \cos x$, so $\frac{du}{dx} = -\sin x$ and $\sin x\,dx = -du$. Then

$$\int \frac{\sin x}{\cos x}\,dx = \int \frac{1}{u}(-du) = -\log_e|u| + c = -\log_e|\cos x| + c.$$

(Since $\frac{\sin x}{\cos x} = \tan x$, this is the standard result $\int \tan x\,dx = -\log_e|\cos x| + c$.)

Q16. Let $u = e^x + 1$, so $\frac{du}{dx} = e^x$ and $du = e^x\,dx$ — exactly the numerator. Then

$$\int \frac{e^x}{e^x+1}\,dx = \int \frac{1}{u}\,du = \log_e|u| + c = \log_e(e^x+1) + c,$$

with no absolute value needed since $e^x + 1 > 0$.

Q17. Let $u = x^2 + 1$, so $du = 2x\,dx$ and $6x\,dx = 3\,du$. Then

$$\int \frac{6x}{(x^2+1)^2}\,dx = \int \frac{3}{u^2}\,du = 3\int u^{-2}\,du = -\frac{3}{u} + c = -\frac{3}{x^2+1} + c.$$

Q18. Let $u = 1 - x^2$, so $\frac{du}{dx} = -2x$ and $x\,dx = -\frac{1}{2}\,du$. Then

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\right)du = -\frac{1}{2} \cdot 2u^{1/2} + c = -\sqrt{1-x^2} + c.$$

(Note the extra factor x makes this a substitution problem, distinct from the standard form $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$.)

Theory

Substitution was introduced in an earlier section as a way of reversing the chain rule: to antidifferentiate a product of the form $f(g(x))g'(x)$ we let $u=g(x)$, so that $du=g'(x)dx$, and the integral collapses to $\int f(u)du$. This section applies the same idea to **definite** integrals $\int_a^b f(x)dx$, where the extra care is entirely about the **terminals** — the lower and upper limits of integration.

The change-of-variable rule for definite integrals.

If $u=g(x)$ is differentiable and f is continuous, then

$$\int_a^b f(g(x))g'(x)dx=\int_{g(a)}^{g(b)} f(u)du.$$

The terminals $x=a$ and $x=b$ are x**-values**. Once the integrand and dx are rewritten in terms of u, the terminals must be rewritten too: the lower terminal becomes $u=g(a)$ and the upper terminal becomes $u=g(b)$. This is the single most important point of the section — **the limits are transformed along with the variable**.

Two equivalent methods.

There are two standard ways to organise the work.

Method 1 — change the terminals. Substitute for the variable **and** for the limits, evaluate the new integral entirely in u, and stop. Because the limits are already u-values, there is **no back-substitution**:

$$\int_a^b f(g(x))g'(x)dx=\int_{g(a)}^{g(b)} f(u)du=[F(u)]_{g(a)}^{g(b)},$$

where F is any antiderivative of f.

Method 2 — keep the original limits. First find the **indefinite** integral in u, then **back-substitute** $u=g(x)$ to return to x, and finally apply the **original** x-limits:

$$\int f(g(x))g'(x)dx=F(u)+c=F(g(x))+c, \int_a^b f(g(x))g'(x)dx=[F(g(x))]_a^b.$$

Both methods give the same value. Method 1 is usually quicker and avoids re-expressing the antiderivative in x; Method 2 is convenient when the antiderivative is wanted in its own right. When writing the indefinite step in Method 2, keep the constant of integration $+c$.

A common error.

Do **not** substitute for the variable while leaving the original x-limits on a u-integral. Writing $\int_0^2 2x(x^2+1)^3dx=\int_0^2 u^3du$ is wrong: the numbers 0 and 2 are x-values, but the integral on the right is in u. Either change the limits to $u=1$ and $u=5$ (Method 1), or return to x before substituting the limits (Method 2).

Reversed terminals.

After substituting, the new lower limit may be **larger** than the new upper limit — this happens whenever g is decreasing on the interval. That is not a problem: evaluate as usual, or use

$$\int_p^q f(u)\,du = -\int_q^p f(u)\,du$$

to swap the terminals and change the sign.

Even and odd functions.

Substitution also explains two useful symmetry shortcuts. Split a symmetric integral at 0:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx,$$

and in the first piece substitute $u=-x$, so $du=-dx$ and the terminals $x=-a, 0$ become $u=a, 0$:

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = \int_{0}^{a} f(-u)\,du.$$

Hence

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(-u)+f(u)]\,du.$$

For an **even** function, $f(-x)=f(x)$, so the bracket is $2f(u)$ and

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \; (f \text{ even}).$$

For an **odd** function, $f(-x)=-f(x)$, so the bracket is 0 and

$$\int_{-a}^{a} f(x)\,dx = 0 \; (f \text{ odd}).$$

Recognising symmetry can turn a long calculation into a single line — for instance $\int_{-a}^{a} x^3\,dx = 0$ because x^3 is odd.

Exact values.

A definite integral is a number. Throughout this section the terminals are chosen so that the answer is **exact** — a rational number, a surd such as $2-\sqrt{3}$, a logarithm such as $1/2 \log_e 2$, or a multiple of π such as $\frac{\pi}{6}$. Leave answers in exact form rather than as decimals, and remember that $\log_e 1 = 0$ often clears the lower terminal.

Worked examples

Example 1 — scaffolded

Evaluate $\int_0^2 2x(x^2+1)^3\,dx$ by changing the terminals to the u-variable.

Working	Comment
Let $u=x^2+1$, so $\frac{du}{dx}=2x$ and $du=2x\,dx$.	Choose u as the inside function; its derivative $2x$ is already a factor.
When $x=0$: $u=0^2+1=1$. When $x=2$: $u=2^2+1=5$.	Transform **both** terminals from x-values to u-values.

Working	Comment
$\int_0^2 2x(x^2+1)^3\,dx=\int_1^5 u^3\,du$	Replace $2x\,dx$ by du and the limits $0,2$ by $1,5$.
$=\left[\dfrac{u^4}{4}\right]_1^5=\dfrac{5^4}{4}-\dfrac{1^4}{4}=\dfrac{625-1}{4}=156$	Antidifferentiate in u and evaluate — no back-substitution is needed.

Example 2 — scaffolded

Evaluate $\int_0^1 \dfrac{x}{x^2+1}\,dx$ by finding the indefinite integral first, then applying the original limits.

Working	Comment
Let $u=x^2+1$, so $du=2x\,dx$, giving $x\,dx=\frac{1}{2}\,du$.	The same substitution; this time build the indefinite integral.
$\int \dfrac{x}{x^2+1}\,dx=\int \dfrac{1}{u}\cdot\dfrac{1}{2}\,du=\dfrac{1}{2}\log_e u+c$	Antidifferentiate in u; keep the constant $+c$.
$=\dfrac{1}{2}\log_e(x^2+1)+c$	**Back-substitute** $u=x^2+1$ to return to x.
$\int_0^1 \dfrac{x}{x^2+1}\,dx=\dfrac{1}{2}[\log_e(x^2+1)]_0^1$	Apply the **original** x-limits 0 and 1.
$=\dfrac{1}{2}(\log_e 2-\log_e 1)=\dfrac{1}{2}\log_e 2$	Since $\log_e 1=0$. This is the exact value.

Example 3 — unscaffolded

Evaluate $\int_0^1 \dfrac{x}{\sqrt{4-x^2}}\,dx$.

Let $u=4-x^2$, so $du=-2x\,dx$ and $x\,dx=-1/2\,du$. The terminals transform as $x=0\Rightarrow u=4$ and $x=1\Rightarrow u=3$, so the lower terminal becomes the *larger* value — the decreasing substitution reverses the limits. Changing to u,

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx=\int_4^3 \frac{-1/2}{\sqrt{u}}\,du=\frac{1}{2}\int_3^4 u^{-1/2}\,du=\frac{1}{2}[2\sqrt{u}]_3^4=[\sqrt{u}]_3^4=\sqrt{4}-\sqrt{3},$$

where $\int_4^3 f(u)\,du=-\int_3^4 f(u)\,du$ was used to put the terminals in the usual order.

$\therefore \int_0^1 \dfrac{x}{\sqrt{4-x^2}}\,dx=2-\sqrt{3}.$

Example 4 — unscaffolded

Evaluate $\int_{-2}^{2} x e^{x^2}\,dx$.

The integrand xe^{x^2} is the product of the odd function x and the even function e^{x^2}, so it is **odd**; hence the integral over the symmetric interval $[-2,2]$ is 0. The substitution makes this explicit: let $u=x^2$, so $du=2x\,dx$. Both terminals map to the *same* value, since $x=-2\Rightarrow u=4$ and $x=2\Rightarrow u=4$, giving

$$\int_{-2}^{2} xe^{x^2}\,dx=\frac{1}{2}\int_4^4 e^u\,du=0,$$

because an integral whose two terminals are equal is zero.

$$\therefore \int_{-2}^{2} x e^{x^2}\, dx = 0.$$

Practice questions

Scaffolded

Q1. Evaluate $\int_0^1 3x^2(x^3+1)^2\, dx$ by changing the terminals. (a) Let $u = x^3+1$ and find du. (b) Change both terminals to u-values. (c) Write the integral in u and antidifferentiate. (d) Evaluate to a single exact value.

Q2. Evaluate $\int_0^1 x\sqrt{x^2+1}\, dx$. (a) Let $u = x^2+1$; find du and hence $x\,dx$. (b) Change the terminals to u-values. (c) Write the integral as $1/2 \int u^{1/2}\, du$ and antidifferentiate. (d) State the exact value.

Q3. Evaluate $\int_0^2 \frac{x}{x^2+1}\, dx$ using the indefinite integral and the original limits. (a) Let $u = x^2+1$ and find the indefinite integral in u. (b) Back-substitute to write the antiderivative in x. (c) Apply the limits 0 and 2. (d) State the exact value.

Q4. Evaluate $\int_0^{\pi/2} \sin^2 x \cos x\, dx$. (a) Let $u = \sin x$ and find du. (b) Change the terminals, using $\sin 0$ and $\sin \frac{\pi}{2}$. (c) Evaluate the resulting integral in u.

Q5. Evaluate $\int_0^{\log_e 2} \frac{e^x}{e^x+1}\, dx$. (a) Let $u = e^x+1$ and find du. (b) Change the terminals, using e^0 and $e^{\log_e 2}$. (c) Evaluate the integral in u, leaving the answer as a single logarithm.

Q6. Evaluate $\int_{-1}^{1} (4x^3+3x^2)\, dx$ using symmetry. (a) State which term is odd and which is even. (b) Write down the value of the odd term's integral over $[-1, 1]$. (c) Use $\int_{-1}^{1} f(x)\, dx = 2\int_0^1 f(x)\, dx$ for the even term and evaluate it. (d) State the total.

Unscaffolded

Q7. Evaluate $\int_0^1 2x(x^2+1)^4\, dx$.

Q8. Evaluate $\int_0^1 \frac{x^2}{(x^3+1)^2}\, dx$.

Q9. Evaluate $\int_0^2 \frac{x}{\sqrt{x^2+1}}\, dx$.

Q10. Evaluate $\int_0^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\, dx$.

Q11. Evaluate $\int_0^1 \frac{x}{x^2+3}\, dx$.

Q12. Evaluate $\int_0^{\pi/2} \sin x \cos^3 x \, d\,x$.

Q13. Evaluate $\int_0^{\pi/2} e^{\sin x} \cos x \, d\,x$.

Q14. Evaluate $\int_0^{2} \frac{1}{x^2+4} \, d\,x$ using the substitution $u=\frac{x}{2}$.

Q15. Evaluate $\int_0^{1} \frac{1}{\sqrt{4-x^2}} \, d\,x$ using the substitution $u=\frac{x}{2}$.

Q16. Evaluate $\int_{-3}^{3} \frac{x^3}{x^2+1} \, d\,x$.

Q17. Evaluate $\int_{-2}^{2} \left(x^5-4x^3+x^2\right) d\,x$.

Q18. Evaluate $\int_1^{e} \frac{\log_e x}{x} \, d\,x$.

Answers

Q1. $\frac{7}{3}$. **Q2.** $\frac{2\sqrt{2}-1}{3}$. **Q3.** $\frac{1}{2}\log_e 5$. **Q4.** $\frac{1}{3}$. **Q5.** $\log_e \frac{3}{2}$. **Q6.** 2. **Q7.** $\frac{31}{5}$. **Q8.** $\frac{1}{6}$. **Q9.** $\sqrt{5}-1$. **Q10.** $2-\sqrt{2}$. **Q11.** $\frac{1}{2}\log_e \frac{4}{3}$. **Q12.** $\frac{1}{4}$. **Q13.** $e-1$. **Q14.** $\frac{\pi}{8}$. **Q15.** $\frac{\pi}{6}$. **Q16.** 0. **Q17.** $\frac{16}{3}$. **Q18.** $\frac{1}{2}$.

Fully-worked solutions

Q1. Let $u=x^3+1$, so $du=3x^2\,dx$. The terminals become $x=0 \Rightarrow u=1$ and $x=1 \Rightarrow u=2$. Then

$$\int_0^1 3x^2(x^3+1)^2\,dx=\int_1^2 u^2\,du=\left[\frac{u^3}{3}\right]_1^2=\frac{8}{3}-\frac{1}{3}=\frac{7}{3}.$$

Q2. Let $u=x^2+1$, so $du=2x\,dx$ and $x\,dx=1/2\,du$. The terminals become $x=0 \Rightarrow u=1$ and $x=1 \Rightarrow u=2$. Then

$$\int_0^1 x\sqrt{x^2+1}\,dx=\frac{1}{2}\int_1^2 u^{1/2}\,du=\frac{1}{2}\cdot\frac{2}{3}[u^{3/2}]_1^2=\frac{1}{3}\left(2^{3/2}-1\right)=\frac{2\sqrt{2}-1}{3},$$

using $2^{3/2}=2\sqrt{2}$.

Q3. Let $u=x^2+1$, so $du=2x\,dx$ and $x\,dx=1/2\,du$. The indefinite integral is

$$\int\frac{x}{x^2+1}\,dx=\frac{1}{2}\int\frac{1}{u}\,du=\frac{1}{2}\log_e u+c=\frac{1}{2}\log_e\left(x^2+1\right)+c.$$

Applying the original limits 0 and 2,

$$\int_0^2\frac{x}{x^2+1}\,dx=\frac{1}{2}[\log_e(x^2+1)]_0^2=\frac{1}{2}\left(\log_e 5-\log_e 1\right)=\frac{1}{2}\log_e 5.$$

Q4. Let $u=\sin x$, so $du=\cos x\,dx$. The terminals become $x=0 \Rightarrow u=\sin 0=0$ and $x=\pi/2 \Rightarrow u=\sin\pi/2=1$. Then

$$\int_0^{\pi/2}\sin^2 x\cos x\,dx=\int_0^1 u^2\,du=\left[\frac{u^3}{3}\right]_0^1=\frac{1}{3}.$$

Q5. Let $u=e^x+1$, so $du=e^x\,dx$. The terminals become $x=0 \Rightarrow u=e^0+1=2$ and $x=\log_e 2 \Rightarrow u=e^{\log_e 2}+1=2+1=3$. Then

$$\int_0^{\log_e 2}\frac{e^x}{e^x+1}\,dx=\int_2^3\frac{1}{u}\,du=[\log_e u]_2^3=\log_e 3-\log_e 2=\log_e\frac{3}{2}.$$

Q6. The term $4x^3$ is odd and the term $3x^2$ is even. Over the symmetric interval $[-1,1]$ the odd term integrates to 0, while the even term satisfies $\int_{-1}^1 3x^2\,dx=2\int_0^1 3x^2\,dx$. Hence

$$\int_{-1}^1(4x^3+3x^2)\,dx=0+2\int_0^1 3x^2\,dx=2[x^3]_0^1=2(1)=2.$$

Q7. Let $u=x^2+1$, so $du=2x\,dx$. The terminals become $x=0 \Rightarrow u=1$ and $x=1 \Rightarrow u=2$. Then

$$\int_0^1 2x(x^2+1)^4\,dx=\int_1^2 u^4\,du=\left[\frac{u^5}{5}\right]_1^2=\frac{32}{5}-\frac{1}{5}=\frac{31}{5}.$$

Q8. Let $u=x^3+1$, so $du=3x^2\,dx$ and $x^2\,dx=1/3\,du$. The terminals become $x=0\Rightarrow u=1$ and $x=1\Rightarrow u=2$. Then

$$\int_0^1 \frac{x^2}{(x^3+1)^2}\,dx=\frac{1}{3}\int_1^2 u^{-2}\,du=\frac{1}{3}\left[-\frac{1}{u}\right]_1^2=\frac{1}{3}\left(-\frac{1}{2}+1\right)=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}.$$

Q9. Let $u=x^2+1$, so $du=2x\,dx$ and $x\,dx=1/2\,du$. The terminals become $x=0\Rightarrow u=1$ and $x=2\Rightarrow u=5$. Then

$$\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx=\frac{1}{2}\int_1^5 u^{-1/2}\,du=\frac{1}{2}[2\sqrt{u}]_1^5=\sqrt{5}-\sqrt{1}=\sqrt{5}-1.$$

Q10. Let $u=4-x^2$, so $du=-2x\,dx$ and $x\,dx=-1/2\,du$. The terminals become $x=0\Rightarrow u=4$ and $x=\sqrt{2}\Rightarrow u=4-2=2$. Then

$$\int_0^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\,dx=-\frac{1}{2}\int_4^2 u^{-1/2}\,du=\frac{1}{2}\int_2^4 u^{-1/2}\,du=\frac{1}{2}[2\sqrt{u}]_2^4=\sqrt{4}-\sqrt{2}=2-\sqrt{2}.$$

Q11. Let $u=x^2+3$, so $du=2x\,dx$ and $x\,dx=1/2\,du$. The terminals become $x=0\Rightarrow u=3$ and $x=1\Rightarrow u=4$. Then

$$\int_0^1 \frac{x}{x^2+3}\,dx=\frac{1}{2}\int_3^4 \frac{1}{u}\,du=\frac{1}{2}[\log_e u]_3^4=\frac{1}{2}(\log_e 4-\log_e 3)=\frac{1}{2}\log_e\frac{4}{3}.$$

Q12. Let $u=\cos x$, so $du=-\sin x\,dx$. The terminals become $x=0\Rightarrow u=\cos 0=1$ and $x=\pi/2\Rightarrow u=\cos\pi/2=0$. Then

$$\int_0^{\pi/2} \sin x\cos^3 x\,dx=-\int_1^0 u^3\,du=\int_0^1 u^3\,du=\left[\frac{u^4}{4}\right]_0^1=\frac{1}{4}.$$

Q13. Let $u=\sin x$, so $du=\cos x\,dx$. The terminals become $x=0\Rightarrow u=0$ and $x=\pi/2\Rightarrow u=1$. Then

$$\int_0^{\pi/2} e^{\sin x}\cos x\,dx=\int_0^1 e^u\,du=[e^u]_0^1=e-1.$$

Q14. Let $u=\frac{x}{2}$, so $dx=2\,du$ and $x^2+4=4u^2+4=4(u^2+1)$. The terminals become $x=0\Rightarrow u=0$ and $x=2\Rightarrow u=1$. Then, using the standard form $\int\frac{1}{u^2+1}\,du=\tan^{-1}u+c$ from an earlier section,

$$\int_0^2 \frac{1}{x^2+4}\,dx=\int_0^1 \frac{2}{4(u^2+1)}\,du=\frac{1}{2}\int_0^1 \frac{1}{u^2+1}\,du=\frac{1}{2}[\tan^{-1}u]_0^1=\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}.$$

Q15. Let $u=\frac{x}{2}$, so $dx=2\,du$ and $4-x^2=4-4u^2=4(1-u^2)$, giving $\sqrt{4-x^2}=2\sqrt{1-u^2}$. The terminals become $x=0\Rightarrow u=0$ and $x=1\Rightarrow u=1/2$. Then, using $\int\frac{1}{\sqrt{1-u^2}}\,du=\sin^{-1}u+c$ from an earlier section,

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx=\int_0^{1/2} \frac{2}{2\sqrt{1-u^2}}\,du=\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du=[\sin^{-1}u]_0^{1/2}=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}.$$

Q16. The integrand $\frac{x^3}{x^2+1}$ has an odd numerator and an even denominator, so it is an **odd** function: replacing x by $-x$ gives $\frac{-x^3}{x^2+1}$. The interval $[-3,3]$ is symmetric about 0, so

$$\int_{-3}^{3} \frac{x^3}{x^2+1} dx = 0.$$

Q17. The terms x^5 and $-4x^3$ are odd, so each integrates to 0 over the symmetric interval $[-2, 2]$; only the even term x^2 survives, and $\int_{-2}^{2} x^2 dx = 2\int_{0}^{2} x^2 dx$. Hence

$$\int_{-2}^{2} (x^5 - 4x^3 + x^2) dx = 2\int_{0}^{2} x^2 dx = 2\left[\frac{x^3}{3}\right]_0^2 = 2 \cdot \frac{8}{3} = \frac{16}{3}.$$

Q18. Let $u = \log_e x$, so $du = \frac{1}{x} dx$. The terminals become $x = 1 \Rightarrow u = \log_e 1 = 0$ and $x = e \Rightarrow u = \log_e e = 1$. Then

$$\int_{1}^{e} \frac{\log_e x}{x} dx = \int_{0}^{1} u\, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

Theory

Powers and products of $\sin$ and $\cos$ rarely have an antiderivative that can be written straight down: there is no rule in a table for $\cos^2 x$ read off directly. The technique of this section is to first **rewrite the integrand using a trigonometric identity** so that every term becomes a **standard form** whose antiderivative is known — a constant multiple of $\cos k x$, $\sin k x$ or $sec^2 k x$. The whole skill lies in **choosing the identity that removes the obstacle** — a square, a product, or an odd power — and the paragraphs below sort the common integrands by the identity each one needs.

The standard antiderivatives used throughout (each with constant $k \neq 0$) are

$$\int \cos k x \, d x = \frac{1}{k} \sin k x + c \,, \int \sin k x \, d x = -\frac{1}{k} \cos k x + c \,, \int sec^2 k x \, d x = \frac{1}{k} \tan k x + c \,.$$

Even powers: the double-angle identities.

The double-angle formula for cosine can be written in three equivalent ways,

$$\cos 2 x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \,.$$

Rearranging the last two forms isolates each squared term:

$$\cos^2 x = \frac{1 + \cos 2 x}{2} \,, \sin^2 x = \frac{1 - \cos 2 x}{2} \,.$$

Each right-hand side is a **sum of a constant and a single cosine**, both of which integrate directly. Hence

$$\int \cos^2 x \, d x = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2 x \right) d x = \frac{x}{2} + \frac{1}{4} \sin 2 x + c \,,$$

$$\int \sin^2 x \, d x = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2 x \right) d x = \frac{x}{2} - \frac{1}{4} \sin 2 x + c \,.$$

The same identities work for a **general argument** $k x$: replacing x by $k x$ gives $\cos^2 k x = \dfrac{1 + \cos 2 k x}{2}$, so, for example,

$$\int \cos^2 k x \, d x = \frac{x}{2} + \frac{1}{4 k} \sin 2 k x + c \,.$$

The identity $\sin 2 x = 2 \sin x \cos x$.

Read in reverse, the sine double-angle identity turns a product of a sine and a cosine of the *same* angle into a single sine of the double angle:

$$\sin x \cos x = 1/2 \sin 2 x \,, \text{so} \int \sin x \cos x \, d x = \frac{1}{2} \int \sin 2 x \, d x = -\frac{1}{4} \cos 2 x + c \,.$$

More generally $\sin k x \cos k x = 1/2 \sin 2 k x$. (Differentiating $-1/4 \cos 2 x$ returns $1/2 \sin 2 x = \sin x \cos x$, confirming the result.)

Odd powers: split off one factor and use $\sin^2 x + \cos^2 x = 1$.

For an **odd** power of $\sin$ or $\cos$, peel off one factor to pair with the differential, then convert the remaining **even** power to the other function using $\sin^2 x = 1 - \cos^2 x$ (or $\cos^2 x = 1 - \sin^2 x$). This sets up a substitution. For $\sin^3 x$, write $\sin^3 x = \sin x \left(1 - \cos^2 x\right)$ and substitute $u = \cos x$, for which $\frac{du}{dx} = -\sin x$:

$$\int \sin^3 x \, dx = \int \sin x \left(1 - \cos^2 x\right) dx = -\int \left(1 - u^2\right) du = -u + \frac{u^3}{3} + c = -\cos x + \frac{\cos^3 x}{3} + c.$$

The key is to keep one factor of the odd function beside the differential; the Pythagorean identity converts what remains. Choosing u to be the function whose power is even is exactly what makes the substitution work.

The identity $\tan^2 x = sec^2 x - 1$**.**

There is no direct antiderivative for $\tan^2 x$, but rearranging the Pythagorean identity $sec^2 x = 1 + \tan^2 x$ gives $\tan^2 x = sec^2 x - 1$, and $sec^2 x$ integrates to $\tan x$:

$$\int \tan^2 x \, dx = \int \left(sec^2 x - 1\right) dx = \tan x - x + c.$$

With a general argument, $\tan^2 kx = sec^2 kx - 1$, so

$$\int \tan^2 kx \, dx = \frac{1}{k} \tan kx - x + c.$$

Choosing the identity.

The table summarises which identity clears each obstacle.

Integrand	Obstacle	Identity to use	Rewritten as
$\cos^2 x$, $\sin^2 x$ (even power)	a square	double angle $\cos 2x = 2\cos^2 x - 1 = 1 - 2$	constant $+ \cos 2x$
$\sin x \cos x$ (same angle)	a product	$\sin 2x = 2 \sin x \cos x$	$1/2 \sin 2x$
$\sin^3 x$, $\cos^3 x$ (odd power)	an odd power	$\sin^2 x + \cos^2 x = 1$, then substitute	one factor × even power
$\tan^2 x$	no direct form	$\tan^2 x = sec^2 x - 1$	$sec^2 x - 1$

Every answer can be checked by **differentiating it**: the derivative of a correct antiderivative simplifies back to the original integrand. This is the reliable test used in the worked solutions below.

Worked examples

Example 1 — scaffolded

Find

$$\int \sin^2 x \, dx.$$

Working	Comment
The integrand is an even power of $\sin$, so use the double-angle identity $\cos 2x = 1 - 2\sin^2 x$, rearranged to $\sin^2 x = \frac{1 - \cos 2x}{2}$.	Choose the identity that turns the square into a first-power cosine.
Split into two terms: $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$.	Now every term has a standard antiderivative.

Working	Comment
The antiderivative of $\frac{1}{2}$ is $\frac{1}{2}x$; the antiderivative of $\frac{1}{2}\cos 2x$ is $\frac{1}{2}\cdot\frac{1}{2}\sin 2x=\frac{1}{4}\sin 2x$.	Integrate term by term; the antiderivative of $\cos 2x$ is $1/2\sin 2x$.
Combine, keeping the minus sign on the cosine term: $\frac{x}{2}-\frac{1}{4}\sin 2x+c$.	Subtract the second antiderivative and add the constant $+c$.

$$\int \sin^2 x\,dx=\frac{x}{2}-\frac{1}{4}\sin 2x+c.$$

Check: $\frac{d}{dx}\left(\frac{x}{2}-\frac{1}{4}\sin 2x\right)=\frac{1}{2}-\frac{1}{2}\cos 2x=\frac{1-\cos 2x}{2}=\sin^2 x$, as required.

Example 2 — scaffolded

Find

$$\int \sin^3 x\,dx.$$

Working	Comment
The power is **odd**, so split off one factor: $\sin^3 x=\sin x\left(\sin^2 x\right)$.	Peel one $\sin x$ from the odd power.
Convert the remaining square with $\sin^2 x=1-\cos^2 x$: $\sin^3 x=\sin x\left(1-\cos^2 x\right)$.	Everything except one $\sin x$ is now written in $\cos x$.
This is set up for the substitution $u=\cos x$, for which $\frac{du}{dx}=-\sin x$.	Take u to be the function carrying the even power, so its derivative supplies the leftover $\sin x$.
In terms of u the integrand is $-\left(1-u^2\right)$, a polynomial whose antiderivative is $-u+\frac{u^3}{3}$.	Integrate term by term in u.
Resubstitute $u=\cos x$ and add the constant: $-\cos x+\frac{\cos^3 x}{3}+c$.	Always return to the original variable.

$$\int \sin^3 x\,dx=\int \sin x\left(1-\cos^2 x\right)dx=-\int\left(1-u^2\right)du=-\cos x+\frac{\cos^3 x}{3}+c\,,u=\cos x.$$

Check: $\frac{d}{dx}\left(-\cos x+\frac{\cos^3 x}{3}\right)=\sin x-\cos^2 x\sin x=\sin x\left(1-\cos^2 x\right)=\sin^3 x$.

Example 3 — unscaffolded

Find

$$\int \cos^4 x\,dx.$$

A fourth power is still an even power, so the double-angle identity applies — but it must be used **twice**. Writing $\cos^4 x=\left(\cos^2 x\right)^2$ and substituting $\cos^2 x=\frac{1+\cos 2x}{2}$,

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right).$$

The leftover square $\cos^2 2x$ is itself an even power; applying the identity again with argument $2x$ gives $\cos^2 2x = \frac{1+\cos 4x}{2}$, so

$$\cos^4 x = \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right) = \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x.$$

Every term is now a standard form, so

$$\int \cos^4 x\,dx = \frac{3x}{8}+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c.$$

Differentiating returns $3/8+1/2\cos 2x+1/8\cos 4x=\cos^4 x$, confirming the result.

$\therefore \int \cos^4 x\,dx = \frac{3x}{8}+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c$.

Example 4 — unscaffolded

Evaluate

$$\int_0^{\pi/4} \tan^2 x\,dx.$$

There is no direct antiderivative of $\tan^2 x$, so rewrite it with the Pythagorean identity $sec^2 x = 1+\tan^2 x$, giving $\tan^2 x = sec^2 x - 1$. Since $sec^2 x$ integrates to $\tan x$,

$$\int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4} \left(sec^2 x - 1\right)dx = [\tan x - x]_0^{\pi/4}.$$

Evaluating at the terminals, with $\tan\frac{\pi}{4}=1$ and $\tan 0 = 0$,

$$[\tan x - x]_0^{\pi/4} = \left(1-\frac{\pi}{4}\right)-(0-0) = 1-\frac{\pi}{4}.$$

$\therefore$ the value of the integral is $1-\frac{\pi}{4}$.

Practice questions

Scaffolded

Q1. Find $\int \cos^2 x\,dx$. (a) Use the double-angle identity to write $\cos^2 x$ in the form $\frac{1+\cos 2x}{2}$. (b) Split the result into two separate terms. (c) Antidifferentiate each term. (d) State the antiderivative, including $+c$.

Q2. Find $\int \sin^2 2x\,dx$. (a) Write $\sin^2 2x$ using the double-angle identity (replace x by $2x$). (b) Split into two terms. (c) Antidifferentiate, remembering that the antiderivative of $\cos 4x$ is $1/4 \sin 4x$. (d) State the antiderivative with $+c$.

Q3. Find $\int \sin 3x \cos 3x\,dx$. (a) The two angles are equal; which double-angle identity turns $\sin\theta\cos\theta$ into a single sine? (b) Write $\sin 3x\cos 3x$ as a multiple of $\sin 6x$. (c) Antidifferentiate. (d) State the antiderivative with $+c$.

Q4. Find $\int \cos^3 x\,dx$. (a) The power is odd; split off one factor as $\cos x\left(\cos^2 x\right)$. (b) Replace $\cos^2 x$ using $\cos^2 x = 1 - \sin^2 x$. (c) Substitute $u = \sin x$ and antidifferentiate the polynomial in u. (d) Resubstitute and state the antiderivative with $+c$.

Q5. Evaluate $\int_0^{\pi/4} \cos^2 x\,dx$. (a) Use the double-angle identity to write $\cos^2 x = \dfrac{1+\cos 2x}{2}$. (b) Antidifferentiate to obtain $\dfrac{x}{2} + \dfrac{1}{4}\sin 2x$. (c) Substitute the terminals $x = \dfrac{\pi}{4}$ and $x = 0$, using $\sin\dfrac{\pi}{2} = 1$. (d) State the exact value.

Q6. Find $\int \tan^2 x\,dx$. (a) Rewrite $\tan^2 x$ using $sec^2 x = 1 + \tan^2 x$. (b) Recall the antiderivative of $sec^2 x$. (c) Antidifferentiate $sec^2 x - 1$. (d) State the antiderivative with $+c$.

Unscaffolded

Q7. Find $\int \sin^2 3x\,dx$.

Q8. Find $\int \cos^2 \dfrac{x}{2}\,dx$.

Q9. Find $\int \sin^3 2x\,dx$.

Q10. Find $\int \sin^2 x \cos^2 x\,dx$.

Q11. Find $\int \cos^5 x\,dx$.

Q12. Find $\int \sin^4 x\,dx$.

Q13. Find $\int \tan^2 3x\,dx$.

Q14. Evaluate

$$\int_0^{\pi/2} \sin^2 x\,dx.$$

Q15. Evaluate

$$\int_0^{\pi/6} \cos^2 3x\,dx.$$

Q16. Evaluate

$$\int_0^{\pi/2} \sin^3 x\,dx.$$

Q17. Find $\int \left(\cos^2 x - \sin^2 x\right)dx$. (Choose the identity before integrating.)

Q18. Find $\int \sin^5 x\,dx$.

Answers

Q1. $\frac{x}{2}+\frac{1}{4}\sin 2x+c$. **Q2.** $\frac{x}{2}-\frac{1}{8}\sin 4x+c$. **Q3.** $-\frac{1}{12}\cos 6x+c$. **Q4.** $\sin x-\frac{1}{3}\sin^3 x+c$. **Q5.** $\frac{\pi}{8}+\frac{1}{4}$. **Q6.** $\tan x-x+c$. **Q7.** $\frac{x}{2}-\frac{1}{12}\sin 6x+c$. **Q8.** $\frac{x}{2}+\frac{1}{2}\sin x+c$. **Q9.** $-\frac{1}{2}\cos 2x+\frac{1}{6}\cos^3 2x+c$. **Q10.** $\frac{x}{8}-\frac{1}{32}\sin 4x+c$. **Q11.** $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c$. **Q12.** $\frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c$. **Q13.** $\frac{1}{3}\tan 3x-x+c$. **Q14.** $\frac{\pi}{4}$. **Q15.** $\frac{\pi}{12}$. **Q16.** $\frac{2}{3}$. **Q17.** $\frac{1}{2}\sin 2x+c$. **Q18.** $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+c$.

Fully-worked solutions

Q1. Even power of cosine, so $\cos^2 x=\frac{1+\cos 2x}{2}$. Then

$$\int \cos^2 x\,dx=\int\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx=\frac{x}{2}+\frac{1}{4}\sin 2x+c.$$

Differentiating gives $1/2+1/2\cos 2x=\cos^2 x$.

Q2. With argument $2x$, $\sin^2 2x=\frac{1-\cos 4x}{2}$. Then

$$\int \sin^2 2x\,dx=\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)dx=\frac{x}{2}-\frac{1}{8}\sin 4x+c,$$

since the antiderivative of $\cos 4x$ is $1/4\sin 4x$.

Q3. Equal angles, so $\sin 3x\cos 3x=1/2\sin 6x$. Then

$$\int \sin 3x\cos 3x\,dx=\frac{1}{2}\int \sin 6x\,dx=\frac{1}{2}\left(-\frac{1}{6}\cos 6x\right)+c=-\frac{1}{12}\cos 6x+c.$$

Q4. Odd power: split $\cos^3 x=\cos x(1-\sin^2 x)$. With $u=\sin x$ (so the leftover $\cos x$ matches $\frac{du}{dx}=\cos x$),

$$\int \cos^3 x\,dx=\int(1-u^2)\,du=u-\frac{u^3}{3}+c=\sin x-\frac{1}{3}\sin^3 x+c.$$

Q5. Even power: $\cos^2 x=\frac{1+\cos 2x}{2}$, with antiderivative $\frac{x}{2}+\frac{1}{4}\sin 2x$. Then

$$\int_0^{\pi/4}\cos^2 x\,dx=\left[\frac{x}{2}+\frac{1}{4}\sin 2x\right]_0^{\pi/4}=\left(\frac{\pi}{8}+\frac{1}{4}\sin\frac{\pi}{2}\right)-0=\frac{\pi}{8}+\frac{1}{4},$$

since $\sin\frac{\pi}{2}=1$.

Q6. $\tan^2 x=sec^2 x-1$, so

$$\int \tan^2 x\,dx=\int(sec^2 x-1)\,dx=\tan x-x+c.$$

Q7. $\sin^2 3x=\frac{1-\cos 6x}{2}$, so

$$\int \sin^2 3x\,dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right) dx = \frac{x}{2} - \frac{1}{12}\sin 6x + c.$$

Q8. Using the double angle with argument $x/2$ (since $\cos x = 2\cos^2 x/2 - 1$), $\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$. Then

$$\int \cos^2 \frac{x}{2}\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos x\right) dx = \frac{x}{2} + \frac{1}{2}\sin x + c.$$

Q9. Odd power: split $\sin^3 2x = \sin 2x(1 - \cos^2 2x)$. With $u = \cos 2x$, $\frac{du}{dx} = -2\sin 2x$, so the factor $\sin 2x$ contributes a factor $-1/2$ under the substitution:

$$\int \sin^3 2x\,dx = -\frac{1}{2}\int (1 - u^2)\,du = -\frac{1}{2}\left(u - \frac{u^3}{3}\right) + c = -\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x + c.$$

Q10. Two identities in turn. First $\sin^2 x\cos^2 x = (\sin x\cos x)^2 = (1/2\sin 2x)^2 = 1/4\sin^2 2x$; then $\sin^2 2x = \frac{1-\cos 4x}{2}$, so $\sin^2 x\cos^2 x = \frac{1-\cos 4x}{8}$. Hence

$$\int \sin^2 x\cos^2 x\,dx = \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) dx = \frac{x}{8} - \frac{1}{32}\sin 4x + c.$$

Q11. Odd power: split $\cos^5 x = \cos x(1 - \sin^2 x)^2$. With $u = \sin x$ (so the leftover $\cos x$ matches $\frac{du}{dx} = \cos x$),

$$\int \cos^5 x\,dx = \int (1 - u^2)^2\,du = \int (1 - 2u^2 + u^4)\,du = u - \frac{2u^3}{3} + \frac{u^5}{5} + c,$$

so, resubstituting $u = \sin x$,

$$\int \cos^5 x\,dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c.$$

Q12. Even fourth power, so apply the double-angle identity twice, as in Example 3. From $\sin^2 x = \frac{1-\cos 2x}{2}$,

$$\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x,$$

using $\cos^2 2x = \frac{1+\cos 4x}{2}$. Hence

$$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$$

Q13. $\tan^2 3x = sec^2 3x - 1$, and the antiderivative of $sec^2 3x$ is $1/3\tan 3x$, so

$$\int \tan^2 3x\,dx = \int (sec^2 3x - 1)\,dx = \frac{1}{3}\tan 3x - x + c.$$

Q14. $\sin^2 x = \frac{1-\cos 2x}{2}$, giving the antiderivative $\frac{x}{2} - \frac{1}{4}\sin 2x$. Then

$$\int_0^{\pi/2} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \left(\frac{\pi}{4} - \frac{1}{4}\sin \pi\right) - 0 = \frac{\pi}{4},$$

since $\sin \pi = 0$.

Q15. $\cos^2 3x = \dfrac{1+\cos 6x}{2}$, giving the antiderivative $\dfrac{x}{2} + \dfrac{1}{12}\sin 6x$. Then

$$\int_0^{\pi/6} \cos^2 3x \, dx = \left[\frac{x}{2} + \frac{1}{12}\sin 6x\right]_0^{\pi/6} = \left(\frac{\pi}{12} + \frac{1}{12}\sin \pi\right) - 0 = \frac{\pi}{12}.$$

Q16. Odd power: $\sin^3 x = \sin x (1 - \cos^2 x)$ gives the antiderivative $-\cos x + 1/3 \cos^3 x$. Then

$$\int_0^{\pi/2} \sin^3 x \, dx = \left[-\cos x + \frac{1}{3}\cos^3 x\right]_0^{\pi/2} = (0+0) - \left(-1 + \frac{1}{3}\right) = \frac{2}{3}.$$

Q17. Recognise the integrand as a double-angle form: $\cos^2 x - \sin^2 x = \cos 2x$. Then

$$\int (\cos^2 x - \sin^2 x) \, dx = \int \cos 2x \, dx = \frac{1}{2}\sin 2x + c.$$

Choosing the identity first avoids integrating each square separately.

Q18. Odd power: split $\sin^5 x = \sin x (\sin^2 x)^2 = \sin x (1 - \cos^2 x)^2$. With $u = \cos x$,

$$\int \sin^5 x \, dx = -\int (1-u^2)^2 \, du = -\int (1 - 2u^2 + u^4) \, du = -\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + c,$$

so

$$\int \sin^5 x \, dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c.$$

Theory

Substitution reverses the chain rule. In the earlier work on basic substitution the integrand already contained an inner function together with (a multiple of) its derivative, so replacing that inner function by u made the integral collapse to a standard form. This section treats the harder cases, where the derivative is **not** sitting there ready to use. The strategy changes: instead of hunting for a derivative, we look at the integrand, decide which part is **awkward**, set u equal to that part, and then rewrite *everything* — the awkward expression, any leftover x's, and dx itself — in terms of u. If the result is a standard form or a polynomial in u, the substitution has done its job.

Choosing u as an inner expression or a whole denominator.

A good first guess for u is the expression that blocks integration: a whole denominator, the quantity under a root, or the argument of an exponential. Once u is chosen, three things must be expressed in u:

1. the awkward expression itself (that is why u was chosen);
2. any remaining x's — solve $u=\dots$ for x and substitute;
3. the differential dx — differentiate u and rearrange.

For example, with $u=1+\sqrt{x}$ we get $\sqrt{x}=u-1$, so $x=(u-1)^2$ and $dx=2(u-1)\,du$; every trace of x can then be removed. Choosing u as the surd in this way **rationalises** the integrand — it clears the root. The same idea handles $u=1+e^x$ (then $e^x=u-1$) and $u=\tan x$ (then dx is supplied through $du=sec^2 x\,dx$).

Changing the limits for a definite integral.

For a definite integral there are two equally valid finishes. Either back-substitute to return to x and use the original limits, or — usually neater — **change the limits** to values of u (or θ) as the substitution is made, and never return to x. If $u=g(x)$ then the limit $x=a$ becomes $u=g(a)$.

Trigonometric substitution.

When the awkward part is a **sum or difference of squares under a root**, a substitution that turns it into a single squared term is powerful, because the Pythagorean identities then simplify the root exactly.

awkward part	substitution	becomes	because
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$a\cos\theta$	$a^2-a^2\sin^2\theta=a^2\cos^2\theta$
a^2+x^2	$x=a\tan\theta$	$a^2 sec^2\theta$	$a^2+a^2\tan^2\theta=a^2 sec^2\theta$

For $x=a\sin\theta$ we take $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, where $\cos\theta\ge0$, so $\sqrt{a^2-x^2}=a\cos\theta$ (the positive root), and $dx=a\cos\theta\,d\theta$. A $\sqrt{a^2-x^2}$ integrand then usually produces a $\cos^2\theta$ integral, which is handled by the double-angle identities

$$\cos^2\theta=\frac{1+\cos2\theta}{2},\ \sin^2\theta=\frac{1-\cos2\theta}{2}.$$

For $x=a\tan\theta$ we have $dx=a\,sec^2\theta\,d\theta$. To return to x after a trigonometric substitution, read the ratios off a **reference right triangle** built from the substitution (for $x=a\sin\theta$ the opposite side is x and the hypotenuse is a).

Landing on a standard form.

Many harder substitutions are engineered to reduce the integral to one of the inverse circular standard forms

$$\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\frac{x}{a}+c,\ \int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c,$$

or to a simple power or logarithm. The guiding question throughout is the same: *what is the awkward part, and which substitution removes it?*

Worked examples

Example 1 — scaffolded

Find

$$\int \frac{1}{1+\sqrt{x}}\,dx.$$

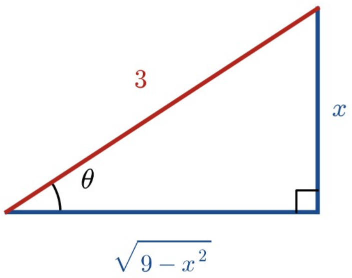
Working	Comment
The surd $\sqrt{x}$ is the awkward part, so let $u=1+\sqrt{x}$.	Set u equal to the expression that blocks integration.
Then $\sqrt{x}=u-1$, so $x=(u-1)^2$ and $dx=2(u-1)\,du$.	Solve for x, then differentiate to get dx in terms of du.
$\int \frac{1}{1+\sqrt{x}}\,dx=\int \frac{1}{u}\cdot 2(u-1)\,du=2\int \frac{u-1}{u}\,du$	Replace $1+\sqrt{x}$ by u and dx by $2(u-1)\,du$.
$=2\int\left(1-\frac{1}{u}\right)du=2(u-\log_e u)+c$	Split the fraction; $u=1+\sqrt{x}>0$, so no modulus is needed.
$=2(1+\sqrt{x})-2\log_e(1+\sqrt{x})+c=2\sqrt{x}-2\log_e(1+\sqrt{x})$	Back-substitute $u=1+\sqrt{x}$; the constant 2 is absorbed into c.

Example 2 — scaffolded

Find

$$\int \sqrt{9-x^2}\,dx.$$

Working	Comment
The awkward part is $\sqrt{9-x^2}$, so let $x=3\sin\theta$, with $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.	For $\sqrt{a^2-x^2}$ try $x=a\sin\theta$; here $a=3$.
Then $dx=3\cos\theta\,d\theta$ and $\sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=3\cos\theta$.	$9-9\sin^2\theta=9\cos^2\theta$; $\cos\theta\ge 0$ on this range.
$\int\sqrt{9-x^2}\,dx=\int 3\cos\theta\cdot 3\cos\theta\,d\theta=9\int\cos^2\theta\,d\theta$	Substitute both $\sqrt{9-x^2}$ and dx.
$=9\int\frac{1+\cos 2\theta}{2}\,d\theta=\frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+c$	Use $\cos^2\theta=\frac{1+\cos 2\theta}{2}$.
$\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}=\frac{2x\sqrt{9-x^2}}{9}$.	Read $\sin\theta=\frac{x}{3}$ and $\cos\theta=\frac{\sqrt{9-x^2}}{3}$ from the triangle below.
$=\frac{9}{2}\sin^{-1}\frac{x}{3}+\frac{9}{4}\cdot\frac{2x\sqrt{9-x^2}}{9}+c=\frac{9}{2}\sin^{-1}\frac{x}{3}+\frac{1}{2}x\sqrt{9-x}$	$\theta=\sin^{-1}\frac{x}{3}$; simplify the second term.



The reference triangle for $x=3\sin\theta$: the opposite side is x, the hypotenuse is 3, and by Pythagoras the adjacent side is $\sqrt{9-x^2}$, giving $\sin\theta$ and $\cos\theta$ directly.

Example 3 — unscaffolded

Find

$$\int \frac{1}{(x^2+4)^{3/2}}\,dx.$$

The factor $(x^2+4)^{3/2}$ points to a tangent substitution, which turns the sum x^2+4 into a single squared term. Let $x=2\tan\theta$ with $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=2\sec^2\theta\,d\theta$ and, using $1+\tan^2\theta=\sec^2\theta$,

$$x^2+4=4\tan^2\theta+4=4\sec^2\theta,\ (x^2+4)^{3/2}=(4\sec^2\theta)^{3/2}=8\sec^3\theta.$$

Substituting,

$$\int \frac{1}{(x^2+4)^{3/2}}\,dx=\int \frac{2\sec^2\theta}{8\sec^3\theta}\,d\theta=\frac{1}{4}\int\cos\theta\,d\theta=\frac{1}{4}\sin\theta+c.$$

The substitution $x=2\tan\theta$ gives a right triangle with opposite side x, adjacent side 2 and hypotenuse $\sqrt{x^2+4}$, so $\sin\theta=\frac{x}{\sqrt{x^2+4}}$. Therefore

$$\therefore \int \frac{1}{(x^2+4)^{3/2}}\,dx=\frac{x}{4\sqrt{x^2+4}}+c.$$

Example 4 — unscaffolded

Find

$$\int \sec^4 x\,dx.$$

The power of $\sec x$ is even, so peel off one $\sec^2 x$ to pair with dx and rewrite the rest with $\sec^2 x=1+\tan^2 x$. Let $u=\tan x$, so $du=\sec^2 x\,dx$. Then

$$\int \sec^4 x\,dx=\int \sec^2 x\cdot\sec^2 x\,dx=\int(1+\tan^2 x)\sec^2 x\,dx=\int(1+u^2)\,du.$$

Integrating in u and then returning to x via $u=\tan x$,

$$\int(1+u^2)\,du=u+\frac{1}{3}u^3+c,$$

so

$$\therefore \int \sec^4 x\,dx=\tan x+\frac{1}{3}\tan^3 x+c.$$

Practice questions

Scaffolded

Q1. Find

$$\int \frac{1}{2+\sqrt{x}}\,dx.$$

(a) Let $u=2+\sqrt{x}$; write x and dx in terms of u. (b) Rewrite the integrand in terms of u. (c) Integrate with respect to u. (d) Substitute back to give the answer in x.

Q2. Find

$$\int \frac{e^{2x}}{1+e^{x}}\,dx.$$

(a) Let $u=1+e^{x}$; show that $e^{2x}=(u-1)^2$ and $dx=\frac{du}{u-1}$. (b) Show the integrand reduces to $\frac{u-1}{u}$. (c) Integrate with respect to u. (d) Back-substitute to give the answer in x.

Q3. Find

$$\int \sqrt{4-x^2}\,dx.$$

(a) Let $x=2\sin\theta$; write dx and $\sqrt{4-x^2}$ in terms of θ. (b) Show the integrand becomes $4\cos^2\theta$. (c) Use $\cos^2\theta=\frac{1+\cos 2\theta}{2}$ to integrate. (d) Use $\sin\theta=\frac{x}{2}$ and $\cos\theta=\frac{\sqrt{4-x^2}}{2}$ to write the answer in x.

Q4. Find

$$\int \frac{1}{(9+x^2)^{3/2}}\,dx.$$

(a) Let $x=3\tan\theta$; write dx and $9+x^2$ in terms of θ. (b) Show the integrand (with dx) reduces to $\frac{1}{9}\cos\theta\,d\theta$. (c) Integrate to obtain $\frac{1}{9}\sin\theta+c$. (d) Use $\sin\theta=\frac{x}{\sqrt{9+x^2}}$ to finish.

Q5. Find

$$\int \frac{sec^2 x}{(1+\tan x)^2}\,dx.$$

(a) Let $u=1+\tan x$; find du. (b) Show the integrand becomes $\frac{1}{u^2}$. (c) Integrate with respect to u. (d) Back-substitute.

Q6. Find

$$\int x\sqrt{x-1}\,dx.$$

(a) Let $u=x-1$; write x and dx in terms of u. (b) Show the integrand becomes $(u+1)\sqrt{u}=u^{3/2}+u^{1/2}$. (c) Integrate with respect to u. (d) Back-substitute.

Unscaffolded

Q7. Find

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx.$$

Q8. Find

$$\int \frac{1}{1+e^x}\,dx.$$

(Hint: write $\frac{1}{1+e^x}=\frac{e^{-x}}{e^{-x}+1}$.)

Q9. Evaluate

$$\int_0^2 \sqrt{4-x^2}\,dx$$

using the substitution $x=2\sin\theta$, and interpret the result as an area.

Q10. Find

$$\int \frac{1}{(1+x^2)^2}\,dx.$$

Q11. Find

$$\int \frac{sec^2 x}{\sqrt{\tan x}}\,dx.$$

Q12. Find

$$\int \frac{x}{\sqrt{2x+1}}\,dx.$$

Q13. Find

$$\int \frac{e^x}{(1+e^x)^2}\,dx.$$

Q14. Find

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx.$$

Q15. Find

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx.$$

Q16. Evaluate

$$\int_0^{\log_e\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx.$$

(Hint: let $u=e^x$ to reach a standard form.)

Q17. Find

$$\int \frac{1}{e^x+e^{-x}}\,dx.$$

(Hint: multiply the integrand by $\frac{e^x}{e^x}$.)

Q18. Evaluate, as an exact value,

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx.$$

Answers

Q1. $2\sqrt{x}-4\log_e(2+\sqrt{x})+c$. **Q2.** $e^x-\log_e(1+e^x)+c$. **Q3.** $2\sin^{-1}\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+c$. **Q4.** $\frac{x}{9\sqrt{9+x^2}}+c$. **Q5.** $-\frac{1}{1+\tan x}+c$. **Q6.** $\frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+c$. **Q7.** $x-2\sqrt{x}+2\log_e(1+\sqrt{x})+c$. **Q8.** $-\log_e(1+e^{-x})+c$ (equivalently $x-\log_e(1+e^x)+c$). **Q9.** π. **Q10.** $\frac{1}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}+c$. **Q11.** $2\sqrt{\tan x}+c$. **Q12.** $\frac{1}{6}(2x+1)^{3/2}-\frac{1}{2}(2x+1)^{1/2}+c$. **Q13.** $-\frac{1}{1+e^x}+c$. **Q14.** $2\log_e(1+\sqrt{x})+c$. **Q15.** $2\sin^{-1}\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}+c$. **Q16.** $\frac{\pi}{12}$. **Q17.** $\tan^{-1}(e^x)+c$. **Q18.** $\frac{\pi}{3}-\frac{\sqrt{3}}{2}$.

Fully-worked solutions

Q1. Let $u=2+\sqrt{x}$, so $\sqrt{x}=u-2$, $x=(u-2)^2$ and $dx=2(u-2)\,du$. Then

$$\int\frac{1}{2+\sqrt{x}}\,dx=\int\frac{2(u-2)}{u}\,du=2\int\left(1-\frac{2}{u}\right)du=2(u-2\log_e u)+c.$$

Since $u=2+\sqrt{x}>0$, substitute back:

$$=2(2+\sqrt{x})-4\log_e(2+\sqrt{x})+c=2\sqrt{x}-4\log_e(2+\sqrt{x})+c,$$

absorbing the constant 4 into c.

Q2. Let $u=1+e^x$, so $e^x=u-1$, $e^{2x}=(u-1)^2$ and, from $du=e^x\,dx$, $dx=\frac{du}{u-1}$. Then

$$\int\frac{e^{2x}}{1+e^x}\,dx=\int\frac{(u-1)^2}{u}\cdot\frac{du}{u-1}=\int\frac{u-1}{u}\,du=\int\left(1-\frac{1}{u}\right)du=u-\log_e u+c.$$

Back-substituting $u=1+e^x>0$,

$$=(1+e^x)-\log_e(1+e^x)+c=e^x-\log_e(1+e^x)+c.$$

Q3. Let $x=2\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dx=2\cos\theta\,d\theta$ and $\sqrt{4-x^2}=2\cos\theta$, so

$$\int\sqrt{4-x^2}\,dx=\int 2\cos\theta\cdot 2\cos\theta\,d\theta=4\int\cos^2\theta\,d\theta=4\int\frac{1+\cos 2\theta}{2}\,d\theta=2\left(\theta+\frac{1}{2}\sin 2\theta\right)+c.$$

With $\theta=\sin^{-1}\frac{x}{2}$ and $\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}=\frac{x\sqrt{4-x^2}}{2}$,

$$=2\sin^{-1}\frac{x}{2}+2\cdot\frac{1}{2}\cdot\frac{x\sqrt{4-x^2}}{2}+c=2\sin^{-1}\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+c.$$

Q4. Let $x=3\tan\theta$, so $dx=3\sec^2\theta\,d\theta$ and $9+x^2=9\sec^2\theta$, giving $(9+x^2)^{3/2}=27\sec^3\theta$. Then

$$\int\frac{1}{(9+x^2)^{3/2}}\,dx=\int\frac{3\sec^2\theta}{27\sec^3\theta}\,d\theta=\frac{1}{9}\int\cos\theta\,d\theta=\frac{1}{9}\sin\theta+c.$$

From the triangle for $x=3\tan\theta$, $\sin\theta=\frac{x}{\sqrt{9+x^2}}$, so

$$=\frac{1}{9}\cdot\frac{x}{\sqrt{9+x^2}}+c=\frac{x}{9\sqrt{9+x^2}}+c.$$

Q5. Let $u=1+\tan x$, so $du=sec^2 x\,dx$. Then

$$\int\frac{sec^2 x}{(1+\tan x)^2}dx=\int\frac{1}{u^2}du=-\frac{1}{u}+c=-\frac{1}{1+\tan x}+c.$$

Q6. Let $u=x-1$, so $x=u+1$ and $dx=du$. Then

$$\int x\sqrt{x-1}\,dx=\int(u+1)\sqrt{u}\,du=\int\left(u^{3/2}+u^{1/2}\right)du=\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+c.$$

Back-substituting $u=x-1$,

$$=\frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+c.$$

Q7. Let $u=1+\sqrt{x}$, so $\sqrt{x}=u-1$, $x=(u-1)^2$ and $dx=2(u-1)\,du$. Then

$$\int\frac{\sqrt{x}}{1+\sqrt{x}}dx=\int\frac{u-1}{u}\cdot2(u-1)\,du=2\int\frac{(u-1)^2}{u}du=2\int\left(u-2+\frac{1}{u}\right)du=u^2-4u+2\log_e u+c.$$

Back-substituting $u=1+\sqrt{x}$ and expanding $(1+\sqrt{x})^2-4(1+\sqrt{x})$,

$$=(1+\sqrt{x})^2-4(1+\sqrt{x})+2\log_e(1+\sqrt{x})+c=x-2\sqrt{x}+2\log_e(1+\sqrt{x})+c,$$

where the numerical constants are absorbed into c.

Q8. Write $\frac{1}{1+e^x}=\frac{e^{-x}}{e^{-x}+1}$ and let $u=1+e^{-x}$, so $du=-e^{-x}dx$. Then

$$\int\frac{1}{1+e^x}dx=\int\frac{e^{-x}}{1+e^{-x}}dx=\int\frac{-1}{u}du=-\log_e u+c=-\log_e(1+e^{-x})+c.$$

Since $1+e^{-x}=\frac{e^x+1}{e^x}$, this equals $x-\log_e(1+e^x)+c$.

Q9. Let $x=2\sin\theta$, so $dx=2\cos\theta\,d\theta$ and $\sqrt{4-x^2}=2\cos\theta$. The limits change with the substitution: $x=0$ gives $\theta=0$, and $x=2$ gives $\sin\theta=1$, so $\theta=\frac{\pi}{2}$. Then

$$\int_0^2\sqrt{4-x^2}\,dx=\int_0^{\pi/2}4\cos^2\theta\,d\theta=4\int_0^{\pi/2}\frac{1+\cos2\theta}{2}d\theta=2\left[\theta+\frac{1}{2}\sin2\theta\right]_0^{\pi/2}=2\left(\frac{\pi}{2}+0\right)=\pi.$$

This is the area under $y=\sqrt{4-x^2}$ from $x=0$ to $x=2$, a quarter of the disc of radius 2: $\frac{1}{4}\pi(2)^2=\pi$, as found.

Q10. Let $x=\tan\theta$, so $dx=sec^2\theta\,d\theta$ and $1+x^2=sec^2\theta$, giving $(1+x^2)^2=sec^4\theta$. Then

$$\int\frac{1}{(1+x^2)^2}dx=\int\frac{sec^2\theta}{sec^4\theta}d\theta=\int\cos^2\theta\,d\theta=\int\frac{1+\cos2\theta}{2}d\theta=\frac{1}{2}\left(\theta+\frac{1}{2}\sin2\theta\right)+c.$$

With $\theta = \tan^{-1} x$ and $\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$,

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{4} \cdot \frac{2x}{1+x^2} + c = \frac{1}{2}\tan^{-1} x + \frac{x}{2(1+x^2)} + c.$$

Q11. Let $u = \tan x$, so $du = sec^2 x\, dx$. Then

$$\int \frac{sec^2 x}{\sqrt{\tan x}}\, dx = \int \frac{1}{\sqrt{u}}\, du = \int u^{-1/2}\, du = 2u^{1/2} + c = 2\sqrt{\tan x} + c.$$

Q12. Let $u = 2x+1$, so $x = \frac{u-1}{2}$ and $dx = \frac{1}{2}du$. Then

$$\int \frac{x}{\sqrt{2x+1}}\, dx = \int \frac{(u-1)/2}{\sqrt{u}} \cdot \frac{1}{2}du = \frac{1}{4}\int \left(u^{1/2} - u^{-1/2}\right) du = \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + c.$$

Back-substituting $u = 2x+1$,

$$= \frac{1}{6}(2x+1)^{3/2} - \frac{1}{2}(2x+1)^{1/2} + c.$$

Q13. Let $u = 1 + e^x$, so $du = e^x dx$. Then

$$\int \frac{e^x}{(1+e^x)^2}\, dx = \int \frac{1}{u^2}\, du = -\frac{1}{u} + c = -\frac{1}{1+e^x} + c.$$

Q14. Let $u = 1 + \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}dx$, giving $\frac{dx}{\sqrt{x}} = 2du$. Then

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\, dx = \int \frac{1}{1+\sqrt{x}} \cdot \frac{dx}{\sqrt{x}} = \int \frac{2}{u}\, du = 2\log_e u + c = 2\log_e(1+\sqrt{x}) + c.$$

Q15. Let $x = 2\sin\theta$, so $dx = 2\cos\theta\, d\theta$, $\sqrt{4-x^2} = 2\cos\theta$ and $x^2 = 4\sin^2\theta$. Then

$$\int \frac{x^2}{\sqrt{4-x^2}}\, dx = \int \frac{4\sin^2\theta}{2\cos\theta} \cdot 2\cos\theta\, d\theta = 4\int \sin^2\theta\, d\theta = 4\int \frac{1-\cos 2\theta}{2}\, d\theta = 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + c.$$

With $\theta = \sin^{-1}\frac{x}{2}$ and $\sin 2\theta = \frac{x\sqrt{4-x^2}}{2}$,

$$= 2\sin^{-1}\frac{x}{2} - \frac{1}{2}x\sqrt{4-x^2} + c.$$

Q16. Let $u = e^x$, so $du = e^x dx$ and $e^{2x} = u^2$. The limits change: $x = 0$ gives $u = 1$, and $x = \log_e\sqrt{3}$ gives $u = \sqrt{3}$. Then

$$\int_0^{\log_e\sqrt{3}} \frac{e^x}{1+e^{2x}}\, dx = \int_1^{\sqrt{3}} \frac{1}{1+u^2}\, du = \left[\tan^{-1} u\right]_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

Q17. Multiplying numerator and denominator by e^x gives $\frac{1}{e^x + e^{-x}} = \frac{e^x}{e^{2x}+1}$. Let $u = e^x$, so $du = e^x dx$ and $e^{2x} = u^2$. Then

$$\int \frac{1}{e^x + e^{-x}}\, dx = \int \frac{e^x}{1+e^{2x}}\, dx = \int \frac{1}{1+u^2}\, du = \tan^{-1} u + c = \tan^{-1}(e^x) + c.$$

Q18. The antiderivative found in Q15 is $F(x)=2\sin^{-1}\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}$. Evaluating between 0 and 1,

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx=\left[2\sin^{-1}\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}\right]_0^1=2\sin^{-1}\frac{1}{2}-\frac{1}{2}\sqrt{3}-0=2\cdot\frac{\pi}{6}-\frac{\sqrt{3}}{2}=\frac{\pi}{3}-\frac{\sqrt{3}}{2}.$$

Theory

A **rational function** is a quotient $\frac{P(x)}{Q(x)}$ of two polynomials. Most rational functions cannot be antidifferentiated as they stand, but when the denominator factorises the quotient can first be rewritten as a **sum of simpler fractions**, called **partial fractions**. Each simple piece then integrates to a logarithm, a reciprocal power, or an inverse tangent. The method reverses the familiar process of adding algebraic fractions over a common denominator: instead of combining $\frac{2}{x-1}-\frac{1}{x+3}$ into one fraction, we start from the single fraction and recover the pieces.

Proper and improper rational functions.

A rational function is **proper** when the degree of the numerator is **strictly less** than the degree of the denominator, and **improper** otherwise. Partial-fraction decomposition applies only to proper fractions, so an **improper** rational function must be **divided first** by polynomial (long) division, writing

$$\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)},$$

where $S(x)$ is the quotient polynomial and the remainder fraction $\frac{R(x)}{Q(x)}$ is now proper. The polynomial $S(x)$ is integrated term by term and the proper remainder is split into partial fractions. **Always check the degrees before decomposing.**

Standard integrals.

Each partial fraction falls into one of the forms in this table, which are used throughout the section.

integrand	antiderivative
$\frac{1}{x-a}$	$\log_e\lvert x-a\rvert+c$
$\frac{1}{(x-a)^2}$	$-\frac{1}{x-a}+c$
$\frac{f'(x)}{f(x)}$	$\log_e\lvert f(x)\rvert+c$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+c$

The third row is the key to the quadratic case: if the numerator is a multiple of the derivative of the denominator, the integral is a logarithm of the denominator.

Case 1 — distinct linear factors.

If $Q(x)$ is a product of distinct linear factors, each factor contributes one partial fraction with a **constant** numerator. For two factors,

$$\frac{P(x)}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}.$$

Integrating term by term gives a **sum of logarithms**:

$$\int\frac{P(x)}{(x-a)(x-b)}dx=A\log_e\lvert x-a\rvert+B\log_e\lvert x-b\rvert+c.$$

Finding the constants. Multiply both sides by the denominator $(x-a)(x-b)$ to clear fractions, giving the identity $P(x)=A(x-b)+B(x-a)$. There are two efficient ways to find A and B:

- **Cover-up (substitution).** Substitute the value of x that makes one factor zero. Putting $x=a$ kills the B term and leaves $P(a)=A(a-b)$, so $A=\frac{P(a)}{a-b}$; putting $x=b$ gives B. Equivalently, "cover up" the factor $(x-a)$ in the original fraction and evaluate what remains at $x=a$.
- **Equating coefficients.** Expand the right-hand side and match the coefficients of each power of x, then solve the resulting simultaneous equations.

Case 2 — a repeated linear factor.

A factor $(x-a)$ that occurs **twice** contributes **two** partial fractions, one for each power up to the repetition:

$$\frac{P(x)}{(x-a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}.$$

Here the cover-up rule finds B directly (multiply by $(x-a)^2$ and put $x=a$), while A is found by equating coefficients. Integrating uses the first two rows of the table:

$$\int\left(\frac{A}{x-a}+\frac{B}{(x-a)^2}\right)dx=A\log_e|x-a|-\frac{B}{x-a}+c.$$

More generally a factor $(x-a)$ and a distinct factor may appear together, as in $\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ — one term for the single factor and two for the repeated one.

Case 3 — a linear numerator over an irreducible quadratic.

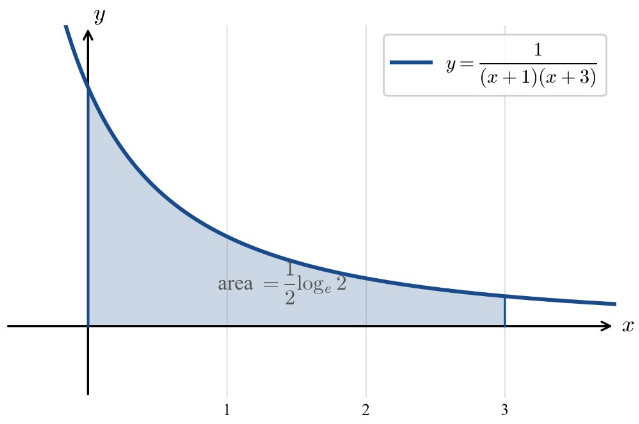
A quadratic x^2+bx+c is **irreducible** (has no real linear factors) when its discriminant $b^2-4c<0$. A fraction $\frac{px+q}{x^2+bx+c}$ with such a denominator does **not** split into linear pieces; instead it is integrated in **two parts**:

1. Write the numerator as a multiple of the derivative of the denominator, plus a constant. Since $\frac{d}{dx}(x^2+bx+c)=2x+b$, the first part integrates to a **logarithm** by the $\frac{f'}{f}$ rule.
2. **Complete the square** on the denominator, $x^2+bx+c=(x+b/2)^2+k^2$, so the remaining constant-over-quadratic part integrates to an **inverse tangent** by the last row of the table.

The result is a combination $1/2\log_e(\text{quadratic})+1/k\tan^{-1}(\ldots)$. Because the quadratic is always positive here, its logarithm needs no modulus signs.

Definite integrals.

Once the antiderivative is found, a definite integral is evaluated by the fundamental theorem of calculus, $\int_a^b f(x)\,dx=F(b)-F(a)$. Logarithm answers combine neatly using $\log_e M-\log_e N=\log_e\frac{M}{N}$, and inverse-tangent answers use exact values such as $\tan^{-1}1=\frac{\pi}{4}$, so a partial-fraction integral over an interval produces an **exact** value — a single logarithm, a multiple of π, or a sum of the two. Graphically, such a value is the exact area between the curve and the x-axis over the interval. The figure shows the region under $y=\frac{1}{(x+1)(x+3)}$ from $x=0$ to $x=3$, whose area works out to the exact value $1/2\log_e 2$.



Worked examples

Example 1 — scaffolded

Find $\int \frac{x+7}{(x-1)(x+3)}\,dx$.

Working	Comment
The fraction is proper (numerator degree 1, denominator degree 2), with distinct linear factors, so write $\frac{x+7}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$.	Check the degrees, then set up one constant over each factor.
Multiply by $(x-1)(x+3)$: $x+7 = A(x+3) + B(x-1)$.	Clear the denominators to get an identity in x.
Put $x=1$: $8 = A(4)$, so $A=2$.	Cover-up: $x=1$ removes the B term.
Put $x=-3$: $4 = B(-4)$, so $B=-1$.	Cover-up: $x=-3$ removes the A term.
So $\frac{x+7}{(x-1)(x+3)} = \frac{2}{x-1} - \frac{1}{x+3}$.	Substitute the constants back.
$\int\left(\frac{2}{x-1} - \frac{1}{x+3}\right)dx = 2\log_e\lvert x-1\rvert - \log_e\lvert x+3\rvert + c$.	Integrate each term with $\int \frac{1}{x-a}\,dx = \log_e\lvert x-a\rvert$.

Example 2 — scaffolded

Find $\int \frac{4x+1}{(x+2)^2}\,dx$.

Working	Comment
Repeated linear factor, so write $\frac{4x+1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$.	One term for each power up to the repetition.
Multiply by $(x+2)^2$: $4x+1 = A(x+2) + B$.	Clear the denominator.
Put $x=-2$: $-7 = B$, so $B=-7$.	Cover-up finds B directly.
Equate coefficients of x: $4 = A$.	The only x-term on the right is Ax.
So $\frac{4x+1}{(x+2)^2} = \frac{4}{x+2} - \frac{7}{(x+2)^2}$.	Substitute $A=4$, $B=-7$.
$\int\left(\frac{4}{x+2} - \frac{7}{(x+2)^2}\right)dx = 4\log_e\lvert x+2\rvert + \frac{7}{x+2} + c$.	Using $\int (x+2)^{-2}\,dx = -\frac{1}{x+2}$, so -7 times it gives

Working	Comment
	$+\dfrac{7}{x+2}$.

Example 3 — unscaffolded

Find $\int \dfrac{x^2+4x+1}{x^2+x-2}\,dx$.

The numerator and denominator have the **same degree**, so the fraction is improper and must be divided first. Dividing x^2+4x+1 by x^2+x-2 gives a quotient 1 and remainder $3x+3$:

$$\frac{x^2+4x+1}{x^2+x-2}=1+\frac{3x+3}{x^2+x-2}=1+\frac{3x+3}{(x-1)(x+2)}.$$

Decompose the proper remainder as $\dfrac{3x+3}{(x-1)(x+2)}=\dfrac{A}{x-1}+\dfrac{B}{x+2}$, so $3x+3=A(x+2)+B(x-1)$. Putting $x=1$ gives $6=3A$, so $A=2$; putting $x=-2$ gives $-3=-3B$, so $B=1$. Hence

$$\int \frac{x^2+4x+1}{x^2+x-2}\,dx=\int\left(1+\frac{2}{x-1}+\frac{1}{x+2}\right)dx=x+2\log_e|x-1|+\log_e|x+2|+c.$$

$\therefore \int \dfrac{x^2+4x+1}{x^2+x-2}\,dx=x+2\log_e|x-1|+\log_e|x+2|+c$.

Example 4 — unscaffolded

Find $\int \dfrac{x+4}{x^2+2x+5}\,dx$.

The denominator has discriminant $2^2-4(5)=-16<0$, so it is irreducible and there are no linear partial fractions. Split the integrand using the derivative of the denominator, $\dfrac{d}{dx}(x^2+2x+5)=2x+2$. Write the numerator as $x+4=1/2(2x+2)+3$, so

$$\frac{x+4}{x^2+2x+5}=\frac{1}{2}\cdot\frac{2x+2}{x^2+2x+5}+\frac{3}{x^2+2x+5}.$$

The first part integrates to a logarithm by the $\dfrac{f'}{f}$ rule. For the second, complete the square: $x^2+2x+5=(x+1)^2+4$, giving an inverse tangent with $a=2$. Therefore

$$\int \frac{x+4}{x^2+2x+5}\,dx=\frac{1}{2}\log_e(x^2+2x+5)+3\cdot\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c.$$

$\therefore \int \dfrac{x+4}{x^2+2x+5}\,dx=\dfrac{1}{2}\log_e(x^2+2x+5)+\dfrac{3}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right)+c$.

Practice questions

Scaffolded

Q1. Find $\int \dfrac{7}{(x-3)(x+4)}\,dx$. (a) Write the integrand as $\dfrac{A}{x-3}+\dfrac{B}{x+4}$ and clear the denominators. (b) Use the cover-up rule to find A and B. (c) Rewrite the integrand as a sum of partial fractions. (d) Integrate term by term.

Q2. Find $\int \frac{2x+1}{(x-2)(x+3)}dx$. (a) Set up $\frac{2x+1}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$. (b) Find A and B. (c) Integrate to obtain a sum of logarithms.

Q3. Find $\int \frac{2x+3}{(x+1)^2}dx$. (a) Write the integrand as $\frac{A}{x+1}+\frac{B}{(x+1)^2}$. (b) Find A and B. (c) Integrate, using $\int (x+1)^{-2}dx=-\frac{1}{x+1}$.

Q4. Find $\int \frac{4x+5}{x^2+1}dx$. (a) The denominator is irreducible; its derivative is $2x$. Write $4x+5=2(2x)+5$. (b) Split the integrand into a $\frac{f'}{f}$ part and a constant-over-quadratic part. (c) Integrate the logarithm part. (d) Integrate the inverse-tangent part.

Q5. Find $\int \frac{x^2+3x+3}{x^2+x-2}dx$. (a) The fraction is improper; divide to show it equals $1+\frac{2x+5}{x^2+x-2}$. (b) Factorise the denominator and decompose $\frac{2x+5}{(x-1)(x+2)}$. (c) Find the constants. (d) Integrate term by term.

Q6. Evaluate $\int_3^4 \frac{1}{(x-1)(x-2)}dx$. (a) Decompose $\frac{1}{(x-1)(x-2)}$ into partial fractions. (b) Write down the antiderivative. (c) Substitute the limits $x=4$ and $x=3$. (d) Combine the logarithms into a single exact value.

Unscaffolded

Q7. Find $\int \frac{2x-3}{(x+1)(x-4)}dx$.

Q8. Find $\int \frac{5x+3}{x^2+x}dx$.

Q9. Find $\int \frac{x-4}{(x-2)^2}dx$.

Q10. Find $\int \frac{x+1}{x^2+4x+13}dx$.

Q11. Evaluate $\int_0^2 \frac{1}{(x+1)(x+3)}dx$, giving an exact value.

Q12. Find $\int \frac{x^3}{x^2-1}dx$.

Q13. Find $\int \frac{3x-1}{x(x-1)^2}dx$.

Q14. Find $\int \frac{1}{x^2+6x+10}dx$.

Q15. Find $\int \frac{5x-4}{x^2-x-2}dx$.

Q16. Evaluate $\int_{-1}^0 \frac{1}{x^2+2x+2}dx$, giving an exact value.

Q17. Find $\int \frac{2x^2+3x}{x+1}\,dx$.

Q18. Evaluate $\int_0^1 \frac{x+1}{x^2+1}\,dx$, giving an exact value.

Answers

Q1. $\log_e|x-3|-\log_e|x+4|+c$. **Q2.** $\log_e|x-2|+\log_e|x+3|+c$. **Q3.** $2\log_e|x+1|-\dfrac{1}{x+1}+c$. **Q4.** $2\log_e(x^2+1)+5\tan^{-1}x+c$. **Q5.** $x+\dfrac{7}{3}\log_e|x-1|-\dfrac{1}{3}\log_e|x+2|+c$. **Q6.** $\log_e\dfrac{4}{3}$. **Q7.** $\log_e|x+1|+\log_e|x-4|+c$. **Q8.** $3\log_e|x|+2\log_e|x+1|+c$. **Q9.** $\log_e|x-2|+\dfrac{2}{x-2}+c$. **Q10.** $\dfrac{1}{2}\log_e(x^2+4x+13)-\dfrac{1}{3}\tan^{-1}\left(\dfrac{x+2}{3}\right)+c$. **Q11.** $\dfrac{1}{2}\log_e\dfrac{9}{5}$. **Q12.** $\dfrac{x^2}{2}+\dfrac{1}{2}\log_e|x-1|+\dfrac{1}{2}\log_e|x+1|+c$. **Q13.** $-\log_e|x|+\log_e|x-1|-\dfrac{2}{x-1}+c$. **Q14.** $\tan^{-1}(x+3)+c$. **Q15.** $2\log_e|x-2|+3\log_e|x+1|+c$. **Q16.** $\dfrac{\pi}{4}$. **Q17.** $x^2+x-\log_e|x+1|+c$. **Q18.** $\dfrac{1}{2}\log_e 2+\dfrac{\pi}{4}$.

Fully-worked solutions

Q1. The fraction is proper with distinct linear factors. Write $\dfrac{7}{(x-3)(x+4)}=\dfrac{A}{x-3}+\dfrac{B}{x+4}$, so $7=A(x+4)+B(x-3)$. Putting $x=3$ gives $7=7A$, so $A=1$; putting $x=-4$ gives $7=-7B$, so $B=-1$. Hence

$$\int\frac{7}{(x-3)(x+4)}\,dx=\int\left(\frac{1}{x-3}-\frac{1}{x+4}\right)dx=\log_e|x-3|-\log_e|x+4|+c.$$

Q2. Write $\dfrac{2x+1}{(x-2)(x+3)}=\dfrac{A}{x-2}+\dfrac{B}{x+3}$, so $2x+1=A(x+3)+B(x-2)$. Putting $x=2$ gives $5=5A$, so $A=1$; putting $x=-3$ gives $-5=-5B$, so $B=1$. Hence

$$\int\frac{2x+1}{(x-2)(x+3)}\,dx=\int\left(\frac{1}{x-2}+\frac{1}{x+3}\right)dx=\log_e|x-2|+\log_e|x+3|+c.$$

Q3. Repeated linear factor: $\dfrac{2x+3}{(x+1)^2}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}$, so $2x+3=A(x+1)+B$. Putting $x=-1$ gives $1=B$; equating coefficients of x gives $A=2$. Then

$$\int\left(\frac{2}{x+1}+\frac{1}{(x+1)^2}\right)dx=2\log_e|x+1|-\frac{1}{x+1}+c,$$

using $\int(x+1)^{-2}\,dx=-\dfrac{1}{x+1}$.

Q4. The denominator x^2+1 is irreducible with derivative $2x$. Splitting the numerator, $4x+5=2(2x)+5$, so

$$\frac{4x+5}{x^2+1}=\frac{2(2x)}{x^2+1}+\frac{5}{x^2+1}.$$

The first term integrates by the $\dfrac{f'}{f}$ rule to $2\log_e(x^2+1)$ (no modulus needed as $x^2+1>0$); the second is a standard inverse tangent with $a=1$. Hence

$$\int\frac{4x+5}{x^2+1}\,dx=2\log_e(x^2+1)+5\tan^{-1}x+c.$$

Q5. Same degree top and bottom, so divide first: $\frac{x^2+3x+3}{x^2+x-2}=1+\frac{2x+5}{x^2+x-2}$, since $(x^2+3x+3)-(x^2+x-2)=2x+5$. Factorising, $x^2+x-2=(x-1)(x+2)$, and $\frac{2x+5}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$ gives $2x+5=A(x+2)+B(x-1)$. Putting $x=1$ gives $7=3A$, so $A=\frac{7}{3}$; putting $x=-2$ gives $1=-3B$, so $B=-\frac{1}{3}$. Therefore

$$\int\frac{x^2+3x+3}{x^2+x-2}\,dx=\int\left(1+\frac{7/3}{x-1}-\frac{1/3}{x+2}\right)dx=x+\frac{7}{3}\log_e|x-1|-\frac{1}{3}\log_e|x+2|+c.$$

Q6. Decompose $\frac{1}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$, so $1=A(x-2)+B(x-1)$. Putting $x=1$ gives $A=-1$; putting $x=2$ gives $B=1$. The antiderivative is $-\log_e|x-1|+\log_e|x-2|=\log_e\frac{|x-2|}{|x-1|}$. On $[3,4]$ both $x-1$ and $x-2$ are positive, so

$$\int_3^4\frac{1}{(x-1)(x-2)}\,dx=\left[\log_e\frac{x-2}{x-1}\right]_3^4=\log_e\frac{2}{3}-\log_e\frac{1}{2}=\log_e\left(\frac{2}{3}\cdot 2\right)=\log_e\frac{4}{3}.$$

Q7. Write $\frac{2x-3}{(x+1)(x-4)}=\frac{A}{x+1}+\frac{B}{x-4}$, so $2x-3=A(x-4)+B(x+1)$. Putting $x=-1$ gives $-5=-5A$, so $A=1$; putting $x=4$ gives $5=5B$, so $B=1$. Hence

$$\int\frac{2x-3}{(x+1)(x-4)}\,dx=\log_e|x+1|+\log_e|x-4|+c.$$

Q8. Factorise the denominator: $x^2+x=x(x+1)$. Then $\frac{5x+3}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$, so $5x+3=A(x+1)+Bx$. Putting $x=0$ gives $A=3$; putting $x=-1$ gives $-2=-B$, so $B=2$. Hence

$$\int\frac{5x+3}{x^2+x}\,dx=3\log_e|x|+2\log_e|x+1|+c.$$

Q9. Repeated factor: $\frac{x-4}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$, so $x-4=A(x-2)+B$. Putting $x=2$ gives $B=-2$; equating coefficients of x gives $A=1$. Hence

$$\int\frac{x-4}{(x-2)^2}\,dx=\log_e|x-2|+\frac{2}{x-2}+c,$$

since $\int-2(x-2)^{-2}\,dx=\frac{2}{x-2}$.

Q10. The denominator has discriminant $4^2-4(13)=-36<0$, so it is irreducible with derivative $2x+4$. Write the numerator as $x+1=1/2(2x+4)-1$, so

$$\frac{x+1}{x^2+4x+13}=\frac{1}{2}\cdot\frac{2x+4}{x^2+4x+13}-\frac{1}{x^2+4x+13}.$$

Completing the square, $x^2+4x+13=(x+2)^2+9$, so the second part is an inverse tangent with $a=3$. Therefore

$$\int\frac{x+1}{x^2+4x+13}\,dx=\frac{1}{2}\log_e(x^2+4x+13)-\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c.$$

Q11. Decompose $\frac{1}{(x+1)(x+3)}=\frac{A}{x+1}+\frac{B}{x+3}$, giving $1=A(x+3)+B(x+1)$. Putting $x=-1$ gives $A=\frac{1}{2}$; putting $x=-3$ gives $B=-\frac{1}{2}$. The antiderivative is $\frac{1}{2}\log_e\frac{x+1}{x+3}$, so

$$\int_0^2 \frac{1}{(x+1)(x+3)}\,dx=\frac{1}{2}\left[\log_e\frac{x+1}{x+3}\right]_0^2=\frac{1}{2}\left(\log_e\frac{3}{5}-\log_e\frac{1}{3}\right)=\frac{1}{2}\log_e\frac{9}{5}.$$

Q12. The fraction is improper (degree 3 over degree 2), so divide: $x^3=x(x^2-1)+x$, giving $\frac{x^3}{x^2-1}=x+\frac{x}{x^2-1}$. Decompose $\frac{x}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$, so $x=A(x+1)+B(x-1)$. Putting $x=1$ gives $A=\frac{1}{2}$; putting $x=-1$ gives $B=\frac{1}{2}$. Hence

$$\int\frac{x^3}{x^2-1}\,dx=\int\left(x+\frac{1/2}{x-1}+\frac{1/2}{x+1}\right)dx=\frac{x^2}{2}+\frac{1}{2}\log_e|x-1|+\frac{1}{2}\log_e|x+1|+c.$$

Q13. A single factor and a repeated factor: $\frac{3x-1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$, so $3x-1=A(x-1)^2+Bx(x-1)+Cx$. Putting $x=0$ gives $A=-1$; putting $x=1$ gives $C=2$; comparing coefficients of x^2 gives $A+B=0$, so $B=1$. Therefore

$$\int\frac{3x-1}{x(x-1)^2}\,dx=\int\left(-\frac{1}{x}+\frac{1}{x-1}+\frac{2}{(x-1)^2}\right)dx=-\log_e|x|+\log_e|x-1|-\frac{2}{x-1}+c.$$

Q14. The denominator has discriminant $6^2-4(10)=-4<0$, so complete the square: $x^2+6x+10=(x+3)^2+1$. This is a standard inverse tangent with $a=1$, so

$$\int\frac{1}{x^2+6x+10}\,dx=\tan^{-1}(x+3)+c.$$

Q15. Factorise the denominator: $x^2-x-2=(x-2)(x+1)$. Then $\frac{5x-4}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$, so $5x-4=A(x+1)+B(x-2)$. Putting $x=2$ gives $6=3A$, so $A=2$; putting $x=-1$ gives $-9=-3B$, so $B=3$. Hence

$$\int\frac{5x-4}{x^2-x-2}\,dx=2\log_e|x-2|+3\log_e|x+1|+c.$$

Q16. The denominator has discriminant $2^2-4(2)=-4<0$; completing the square, $x^2+2x+2=(x+1)^2+1$, so the antiderivative is $\tan^{-1}(x+1)$. Therefore

$$\int_{-1}^{0}\frac{1}{x^2+2x+2}\,dx=[\tan^{-1}(x+1)]_{-1}^{0}=\tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}-0=\frac{\pi}{4}.$$

Q17. The fraction is improper (degree 2 over degree 1), so divide: $2x^2+3x=(x+1)(2x+1)-1$, giving $\frac{2x^2+3x}{x+1}=2x+1-\frac{1}{x+1}$. Hence

$$\int\frac{2x^2+3x}{x+1}\,dx=\int\left(2x+1-\frac{1}{x+1}\right)dx=x^2+x-\log_e|x+1|+c.$$

Q18. The denominator x^2+1 is irreducible with derivative $2x$. Split the numerator:

$$\frac{x+1}{x^2+1}=\frac{x}{x^2+1}+\frac{1}{x^2+1},$$

where the first term is $1/2\cdot\frac{2x}{x^2+1}$. Integrating, $\frac{1}{2}\log_e(x^2+1)+\tan^{-1}x$, so

$$\int_0^1\frac{x+1}{x^2+1}dx=\left[\frac{1}{2}\log_e(x^2+1)+\tan^{-1}x\right]_0^1=\left(\frac{1}{2}\log_e 2+\frac{\pi}{4}\right)-(0+0)=\frac{1}{2}\log_e 2+\frac{\pi}{4}.$$

Theory

Substitution reverses the chain rule; **integration by parts** reverses the **product rule**. It is the standard technique for integrating a *product* of two functions of different types — for example $x\,e^x$, $x\sin x$ or $e^x \sin x$ — where no substitution simplifies the integrand.

Deriving the rule.

If u and v are differentiable functions of x, the product rule gives

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides with respect to x and using the fact that integrating $\frac{d}{dx}(uv)$ returns uv,

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx.$$

Rearranging to make the first integral the subject gives the **integration by parts formula**

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

In the shorthand $du = \frac{du}{dx}\,dx$ and $dv = \frac{dv}{dx}\,dx$ this reads

$$\int u\,dv = uv - \int v\,du.$$

The idea is to trade the integral $\int u\,dv$ that we cannot do for the integral $\int v\,du$ that we can. This only helps if the new integral is **simpler** than the original, and that depends entirely on how we split the integrand into u and dv.

Choosing u and dv.

We must label one factor u (which we will **differentiate**, to get du) and the other dv (which we will **integrate**, to get v). A good choice makes u get simpler when differentiated and makes dv something we can integrate. A reliable guideline is **LIATE**: choose u to be whichever factor comes **first** in the list below, and let the rest of the integrand be dv.

Priority for u	Type	Examples
L	Logarithmic	$\log_e x$
I	Inverse circular	$\tan^{-1} x$, $\sin^{-1} x$
A	Algebraic (powers)	x, x^2, $x+1$
T	Trigonometric	$\sin x$, $\cos x$
E	Exponential	e^x, e^{2x}

So in $\int x\,e^x\,dx$ the power x (**A**) is chosen as u ahead of the exponential e^x (**E**); differentiating x gives the constant 1, which collapses the new integral. Logs and inverse circular functions sit at the top of the list because they are awkward to integrate but easy to differentiate, so they are almost always taken as u.

The four steps. To evaluate $\int u\,dv$:

1. choose u and dv using LIATE;
2. differentiate to find du and integrate dv to find v (no constant is needed for v at this stage);

3. substitute into $\int u\,dv = uv - \int v\,du$;
4. evaluate the remaining integral $\int v\,du$ and add the constant $+c$.

The special case $dv = dx$.

Some functions are not obviously products at all — $\log_e x$ and $\tan^{-1} x$ are the important examples, since we have no direct antiderivative for either. We create a product by writing the integrand as $(\log_e x)\cdot 1$ and taking

$$u = \log_e x,\, dv = 1\,dx \Rightarrow v = x.$$

Because $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ and $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ are simple, the new integral $\int v\,du$ becomes an easy rational function. The same trick handles $\int \tan^{-1} x\,dx$.

Applying parts twice.

If u is a power such as x^2, one application lowers the power to x^1 but still leaves a product. Applying integration by parts a **second time** lowers it to a constant and finishes the job; $\int x^2 e^x\,dx$ needs two applications.

The cyclic case.

For $\int e^x \sin x\,dx$ neither factor simplifies on differentiation, so parts never terminates. Applying it **twice** — choosing u to be the same *type* both times — makes the **original integral reappear** on the right. Calling the unknown integral I, we obtain an equation of the form $I = (\text{terms}) - I$, which we then **solve algebraically** for I (remembering the $+c$).

Definite integrals.

For a definite integral the boundary term uv is evaluated at the limits:

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx.$$

Substitute the limits into uv first, then evaluate the remaining definite integral. No constant of integration is needed for a definite integral.

Every indefinite result below can be checked by **differentiating the answer**: the derivative must return the original integrand. This is the surest way to catch a slip in a sign or a coefficient.

Worked examples

Example 1 — scaffolded

Find $\int x e^x\,dx$.

Working	Comment
The integrand is a product of x (**A**) and e^x (**E**).	Identify the two factors and their LIATE types.
Take $u = x$ and $dv = e^x\,dx$.	**A** comes before **E**, so $u = x$.
$\frac{du}{dx} = 1$, so $du = dx$; and $v = \int e^x\,dx = e^x$.	Differentiate u; integrate dv (no constant needed for v).
$\int x e^x\,dx = x e^x - \int e^x \cdot 1\,dx$.	Substitute into $\int u\,dv = uv - \int v\,du$.
$= x e^x - e^x + c$.	The remaining integral $\int e^x\,dx = e^x$; add $+c$.
$= e^x(x-1) + c$.	Factorise. Check: $\frac{d}{dx}\left(e^x(x-1)\right) = e^x(x-1) + e^x = x e^x$.

Example 2 — scaffolded

Find $\int x \sin x \, dx$.

Working	Comment
Product of x (**A**) and $\sin x$ (**T**).	Identify the factors.
Take $u = x$ and $dv = \sin x \, dx$.	**A** before **T**, so $u = x$.
$du = dx$; and $v = \int \sin x \, dx = -\cos x$.	Watch the sign: $\int \sin x \, dx = -\cos x$.
$\int x \sin x \, dx = x(-\cos x) - \int (-\cos x) \, dx$.	Substitute into the formula.
$= -x \cos x + \int \cos x \, dx$.	Two negatives make the remaining integral + $\int \cos x \, dx$.
$= -x \cos x + \sin x + c$.	Check: $\frac{d}{dx}(-x \cos x + \sin x) = -\cos x + x \sin x + \cos x = x \sin x$.

Example 3 — unscaffolded

Find $\int \log_e x \, dx$.

There is no standard antiderivative of $\log_e x$, so we manufacture a product by writing the integrand as $(\log_e x) \cdot 1$ and applying integration by parts with $dv = dx$. Take $u = \log_e x$ (type **L**) and $dv = 1 \, dx$, so that

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx, v = \int 1 \, dx = x.$$

Substituting into $\int u \, dv = uv - \int v \, du$,

$$\int \log_e x \, dx = x \log_e x - \int x \cdot \frac{1}{x} dx = x \log_e x - \int 1 \, dx = x \log_e x - x + c.$$

Differentiating the answer checks it: $\frac{d}{dx}(x \log_e x - x) = \log_e x + x \cdot \frac{1}{x} - 1 = \log_e x$.

$\therefore \int \log_e x \, dx = x \log_e x - x + c$.

Example 4 — unscaffolded

Find $\int e^x \sin x \, dx$.

Neither factor simplifies when differentiated, so this is the **cyclic** case. Write $I = \int e^x \sin x \, dx$ and apply parts with $u = e^x$, $dv = \sin x \, dx$, giving $du = e^x \, dx$ and $v = -\cos x$:

$$I = -e^x \cos x - \int (-\cos x) e^x \, dx = -e^x \cos x + \int e^x \cos x \, dx.$$

Apply parts a **second time** to $\int e^x \cos x \, dx$, keeping u as the exponential: $u = e^x$, $dv = \cos x \, dx$, so $du = e^x \, dx$ and $v = \sin x$:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx = e^x \sin x - I.$$

Substituting this back, the original integral I reappears:

$$I = -e^x \cos x + e^x \sin x - I.$$

Solving for I by adding I to both sides gives $2I = e^x \sin x - e^x \cos x$, so

$$\int e^x \sin x \, dx = 1/2 \, e^x (\sin x - \cos x) + c.$$

$\therefore \int e^x \sin x\, dx = 1/2\, e^x (\sin x - \cos x) + c.$

Practice questions

Scaffolded

Q1. Find $\int x e^{2x}\, dx$. (a) State u and dv using LIATE. (b) Find du and v (recall $\int e^{2x}\, dx = 1/2\, e^{2x}$). (c) Apply $\int u\, dv = uv - \int v\, du$. (d) Evaluate the remaining integral and add $+c$.

Q2. Find $\int x \cos x\, dx$. (a) State u and dv. (b) Find du and v. (c) Substitute into the formula and evaluate the remaining integral.

Q3. Find $\int \tan^{-1} x\, dx$. (a) Take $u = \tan^{-1} x$, $dv = 1\, dx$; write down du and v. (b) Apply the formula to reach $x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx$. (c) Evaluate $\int \frac{x}{1+x^2}\, dx$ (the top is half the derivative of the bottom) and finish.

Q4. Find $\int x^2 e^x\, dx$ by applying integration by parts twice. (a) With $u = x^2$, $dv = e^x\, dx$, reduce the integral to $x^2 e^x - 2\int x e^x\, dx$. (b) Use the result of Example 1 for $\int x e^x\, dx$. (c) Combine and factorise.

Q5. Evaluate $\int_0^1 x e^x\, dx$. (a) Use the antiderivative $e^x (x-1)$ from Example 1. (b) Substitute the limits into $[e^x (x-1)]_0^1$ and simplify.

Q6. Find $\int e^x \cos x\, dx$ (the cyclic case). (a) Let $I = \int e^x \cos x\, dx$; apply parts with $u = e^x$, $dv = \cos x\, dx$. (b) Apply parts again to the new integral, keeping $u = e^x$. (c) Form an equation in I and solve for I, adding $+c$.

Unscaffolded

Q7. Find $\int x e^{-x}\, dx$.

Q8. Find $\int x \sin 2x\, dx$.

Q9. Find $\int 2x \log_e x\, dx$.

Q10. Find $\int x^2 \sin x\, dx$.

Q11. Find $\int x^3 \log_e x\, dx$.

Q12. Find $\int (x+1) e^x\, dx$.

Q13. Find $\int x \cos 3x\, dx$.

Q14. Evaluate $\int_1^e \log_e x\, dx$.

Q15. Evaluate $\int_0^\pi x \sin x\, dx$.

Q16. Find $\int x^2 e^{-x}\, dx$.

Q17. Find $\int e^{2x} \sin x\, dx$.

Q18. Evaluate $\int_0^1 \tan^{-1} x\, dx$, giving an exact answer.

Answers

Q1. $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$ **Q2.** $x\sin x+\cos x+c$ **Q3.** $x\tan^{-1}x-\frac{1}{2}\log_e(1+x^2)+c$ **Q4.** $e^x(x^2-2x+2)+c$ **Q5.** 1 **Q6.** $\frac{1}{2}e^x(\sin x+\cos x)+c$ **Q7.** $-e^{-x}(x+1)+c$ **Q8.** $-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+c$ **Q9.** $x^2\log_e x-\frac{1}{2}x^2+c$ **Q10.** $-x^2\cos x+2x\sin x+2\cos x+c$ **Q11.** $\frac{1}{4}x^4\log_e x-\frac{1}{16}x^4+c$ **Q12.** xe^x+c **Q13.** $\frac{1}{3}x\sin 3x+\frac{1}{9}\cos 3x+c$ **Q14.** 1 **Q15.** π **Q16.** $-e^{-x}(x^2+2x+2)+c$ **Q17.** $\frac{1}{5}e^{2x}(2\sin x-\cos x)+c$ **Q18.** $\frac{\pi}{4}-\frac{1}{2}\log_e 2$

Fully-worked solutions

Q1. Take $u=x$ (**A**) and $dv=e^{2x}dx$ (**E**), so $du=dx$ and $v=\int e^{2x}dx=1/2e^{2x}$. Then

$$\int xe^{2x}dx=x\cdot 1/2e^{2x}-\int 1/2e^{2x}dx=1/2xe^{2x}-1/2\cdot 1/2e^{2x}+c=1/2xe^{2x}-1/4e^{2x}+c.$$

Check: $\frac{d}{dx}\left(1/2xe^{2x}-1/4e^{2x}\right)=1/2e^{2x}+xe^{2x}-1/2e^{2x}=xe^{2x}$.

Q2. Take $u=x$ and $dv=\cos x\,dx$, so $du=dx$ and $v=\sin x$. Then

$$\int x\cos x\,dx=x\sin x-\int\sin x\,dx=x\sin x-(-\cos x)+c=x\sin x+\cos x+c.$$

Check: $\frac{d}{dx}(x\sin x+\cos x)=\sin x+x\cos x-\sin x=x\cos x$.

Q3. Take $u=\tan^{-1}x$ (**I**) and $dv=1\,dx$, so $du=\frac{1}{1+x^2}dx$ and $v=x$. Then

$$\int\tan^{-1}x\,dx=x\tan^{-1}x-\int\frac{x}{1+x^2}dx.$$

In the remaining integral the numerator x is half the derivative of the denominator $1+x^2$, so $\int\frac{x}{1+x^2}dx=1/2\log_e(1+x^2)$ (and $1+x^2>0$, so no modulus is needed). Hence

$$\int\tan^{-1}x\,dx=x\tan^{-1}x-1/2\log_e(1+x^2)+c.$$

Q4. Take $u=x^2$ and $dv=e^x dx$, so $du=2x\,dx$ and $v=e^x$:

$$\int x^2e^x dx=x^2e^x-\int 2xe^x dx=x^2e^x-2\int xe^x dx.$$

From Example 1, $\int xe^x dx=e^x(x-1)$, so

$$\int x^2e^x dx=x^2e^x-2e^x(x-1)+c=e^x\left(x^2-2x+2\right)+c.$$

Check: $\frac{d}{dx}\left(e^x(x^2-2x+2)\right)=e^x(x^2-2x+2)+e^x(2x-2)=x^2e^x$.

Q5. Using the antiderivative $e^x(x-1)$ from Example 1,

$$\int_0^1 xe^x dx=[e^x(x-1)]_0^1=e^1(1-1)-e^0(0-1)=0-(-1)=1.$$

Q6. Let $I=\int e^x\cos x\,dx$. Apply parts with $u=e^x$, $dv=\cos x\,dx$ ($du=e^x dx$, $v=\sin x$):

$$I = e^x \sin x - \int e^x \sin x \, dx.$$

Apply parts to $\int e^x \sin x \, dx$ with $u = e^x$, $dv = \sin x \, dx$ ($v = -\cos x$):

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + I.$$

Substituting back, $I = e^x \sin x - (-e^x \cos x + I) = e^x \sin x + e^x \cos x - I$. Hence $2I = e^x(\sin x + \cos x)$ and

$$\int e^x \cos x \, dx = 1/2 \, e^x(\sin x + \cos x) + c.$$

Q7. Take $u = x$ and $dv = e^{-x} dx$, so $du = dx$ and $v = \int e^{-x} dx = -e^{-x}$:

$$\int x e^{-x} dx = -x e^{-x} - \int(-e^{-x}) dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + c = -e^{-x}(x+1) + c.$$

Check: $\dfrac{d}{dx}\left(-e^{-x}(x+1)\right) = e^{-x}(x+1) - e^{-x} = x e^{-x}$.

Q8. Take $u = x$ and $dv = \sin 2x \, dx$, so $du = dx$ and $v = -1/2 \cos 2x$:

$$\int x \sin 2x \, dx = -1/2 \, x \cos 2x - \int(-1/2 \cos 2x) dx = -1/2 \, x \cos 2x + 1/2 \int \cos 2x \, dx = -1/2 \, x \cos 2x + 1/4 \sin 2x + c.$$

Check: $\dfrac{d}{dx}(-1/2 \, x \cos 2x + 1/4 \sin 2x) = -1/2 \cos 2x + x \sin 2x + 1/2 \cos 2x = x \sin 2x$.

Q9. Take $u = \log_e x$ (**L**) and $dv = 2x \, dx$, so $du = \dfrac{1}{x} dx$ and $v = x^2$:

$$\int 2x \log_e x \, dx = x^2 \log_e x - \int x^2 \cdot \frac{1}{x} dx = x^2 \log_e x - \int x \, dx = x^2 \log_e x - 1/2 \, x^2 + c.$$

Check: $\dfrac{d}{dx}(x^2 \log_e x - 1/2 \, x^2) = 2x \log_e x + x - x = 2x \log_e x$.

Q10. Take $u = x^2$, $dv = \sin x \, dx$, so $du = 2x \, dx$ and $v = -\cos x$:

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$

From Q2, $\int x \cos x \, dx = x \sin x + \cos x$, so

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x) + c = -x^2 \cos x + 2x \sin x + 2 \cos x + c.$$

Q11. Take $u = \log_e x$ and $dv = x^3 dx$, so $du = \dfrac{1}{x} dx$ and $v = 1/4 \, x^4$:

$$\int x^3 \log_e x \, dx = 1/4 \, x^4 \log_e x - \int 1/4 \, x^4 \cdot \frac{1}{x} dx = 1/4 \, x^4 \log_e x - 1/4 \int x^3 dx = 1/4 \, x^4 \log_e x - 1/16 \, x^4 + c.$$

Check: $\dfrac{d}{dx}(1/4 \, x^4 \log_e x - 1/16 \, x^4) = x^3 \log_e x + 1/4 \, x^3 - 1/4 \, x^3 = x^3 \log_e x$.

Q12. Take $u = x + 1$ and $dv = e^x dx$, so $du = dx$ and $v = e^x$:

$$\int (x+1) e^x dx = (x+1) e^x - \int e^x dx = (x+1) e^x - e^x + c = x e^x + c.$$

Check: $\dfrac{d}{dx}(x e^x) = e^x + x e^x = (x+1) e^x$.

Q13. Take $u = x$ and $dv = \cos 3x \, dx$, so $du = dx$ and $v = 1/3 \sin 3x$:

$$\int x \cos 3x \, dx = 1/3 \, x \sin 3x - \int 1/3 \sin 3x \, dx = 1/3 \, x \sin 3x - 1/3(-1/3 \cos 3x) + c = 1/3 \, x \sin 3x + 1/9 \cos 3x + c.$$

Check: $\frac{d}{dx}\left(1/3\, x\sin 3x + 1/9\cos 3x\right) = 1/3\sin 3x + x\cos 3x - 1/3\sin 3x = x\cos 3x$.

Q14. Using $\int \log_e x\, dx = x\log_e x - x$ from Example 3,

$$\int_1^e \log_e x\, dx = [x\log_e x - x]_1^e = (e\cdot 1 - e) - (1\cdot 0 - 1) = 0 - (-1) = 1.$$

Q15. Using $\int x\sin x\, dx = -x\cos x + \sin x$ from Example 2,

$$\int_0^\pi x\sin x\, dx = [-x\cos x + \sin x]_0^\pi = (-\pi\cos\pi + \sin\pi) - (0 + \sin 0) = (-\pi(-1) + 0) - 0 = \pi.$$

Q16. Take $u = x^2$, $dv = e^{-x}dx$, so $du = 2x\, dx$ and $v = -e^{-x}$:

$$\int x^2 e^{-x} dx = -x^2 e^{-x} + \int 2x e^{-x} dx = -x^2 e^{-x} + 2\int x e^{-x} dx.$$

From Q7, $\int x e^{-x} dx = -e^{-x}(x+1)$, so

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - 2e^{-x}(x+1) + c = -e^{-x}\left(x^2 + 2x + 2\right) + c.$$

Check: $\frac{d}{dx}\left(-e^{-x}(x^2 + 2x + 2)\right) = e^{-x}(x^2 + 2x + 2) - e^{-x}(2x + 2) = x^2 e^{-x}$.

Q17. Let $I = \int e^{2x}\sin x\, dx$. Apply parts with $u = e^{2x}$, $dv = \sin x\, dx$ ($du = 2e^{2x}dx$, $v = -\cos x$):

$$I = -e^{2x}\cos x + \int 2e^{2x}\cos x\, dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\, dx.$$

Apply parts to $\int e^{2x}\cos x\, dx$ with $u = e^{2x}$, $dv = \cos x\, dx$ ($v = \sin x$):

$$\int e^{2x}\cos x\, dx = e^{2x}\sin x - \int 2e^{2x}\sin x\, dx = e^{2x}\sin x - 2I.$$

Substituting back, $I = -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2I\right) = 2e^{2x}\sin x - e^{2x}\cos x - 4I$. Hence $5I = e^{2x}(2\sin x - \cos x)$ and

$$\int e^{2x}\sin x\, dx = 1/5\, e^{2x}(2\sin x - \cos x) + c.$$

Q18. Using $\int \tan^{-1} x\, dx = x\tan^{-1} x - 1/2\log_e(1 + x^2)$ from Q3,

$$\int_0^1 \tan^{-1} x\, dx = [x\tan^{-1} x - 1/2\log_e(1 + x^2)]_0^1 = (1\cdot\pi/4 - 1/2\log_e 2) - (0 - 1/2\log_e 1) = \frac{\pi}{4} - 1/2\log_e 2.$$