# Specialist Mathematics Units 3 & 4

## Chapter 5 — Differentiation and rational functions



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### Contents

### Theory

**The derivative as a gradient function.**

The gradient of a curve $y=f(x)$ changes from point to point. The **derivative** $f'(x)$ records the gradient of the tangent to the curve at each value of $x$. It is defined as the limiting gradient of a chord: taking a second point a horizontal distance $h$ away and letting $h$ shrink to zero,

$$f'(x)=\frac{\lim_{h\to 0} f(x+h)-f(x)}{h}.$$

Because this rule assigns exactly one gradient to each $x$, the derivative is itself a function of $x$ — the **gradient function** — also written $\frac{dy}{dx}$. Everything in this section rests on one idea: the *value* of $f'(x)$ is the *slope* of $y=f(x)$ at that $x$, so the graphs of $f$ and $f'$ are locked together.

**Standard rules (assumed from Mathematical Methods Units 3 & 4).**

Specialist Mathematics Units 3 & 4 assumes concurrent study or previous completion of Mathematical Methods Units 3 & 4, where the rules below are established. They are used throughout and are recapped only briefly here; $u$ and $v$ are functions of $x$.

| Rule | Statement |
|---|---|
| Power (with linearity) | $\frac{d}{dx}(x^n)=nx^{n-1}$, and derivatives add term by term |
| Product | $\frac{d}{dx}(uv)=u'v+uv'$ |
| Quotient | $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}$ |
| Chain | $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$, i.e. $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$ |

These let us differentiate polynomials, rational functions and composite functions — the functions whose related graphs are studied below.

**Reading the graph of $f$ from the graph of $f'$.**

Since $f'(x)$ *is* the slope of $f$, the sign of $f'$ tells us how $f$ is behaving:

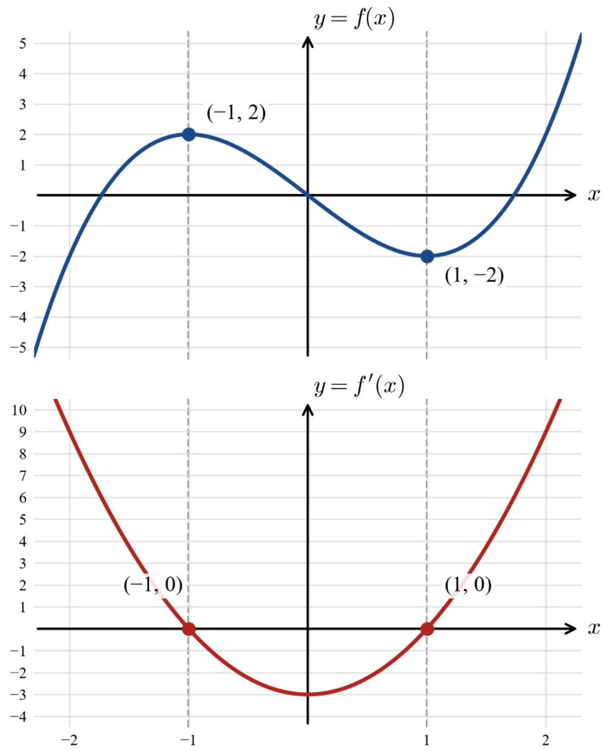
- where $f'(x)>0$, the tangents slope upward, so $f$ is **increasing**;
- where $f'(x)<0$, the tangents slope downward, so $f$ is **decreasing**;
- where $f'(x)=0$, the tangent is horizontal — a **stationary point**.

The *nature* of a stationary point is fixed by how $f'$ behaves on either side of it:

| Sign of $f'$ near the stationary point | Nature of the point on $y=f(x)$ |
|---|---|
| positive, then negative | local **maximum** |
| negative, then positive | local **minimum** |
| same sign on both sides (touches zero) | **stationary point of inflection** |

**The shape of the gradient function.**

For a polynomial $f$ of degree $n$, the gradient function $f'$ has degree $n-1$: a cubic has a quadratic (parabola) gradient function, a quadratic has a linear one, and so on. The **zeros of** $f'$ sit directly beneath (or above) the **stationary points of** $f$, and the **sign of** $f'$ matches the **slope** of $f$. The diagram shows $f(x)=x^3-3x$ above its gradient function $f'(x)=3x^2-3$: the stationary points of $f$ at $x=-1$ and $x=1$ line up exactly with the zeros of the parabola $f'$, and $f'$ is positive precisely where $f$ is rising.




**Reading an antiderivative graph.**

An **antiderivative** $F$ of $f$ is any function with $F'(x)=f(x)$. Reading this the other way round, $f$ now plays the role of the *gradient function of* $F$, so the same sign rules run in reverse:

- where $f(x)>0$, $F$ is increasing; where $f(x)<0$, $F$ is decreasing;
- where $f(x)=0$, $F$ has a stationary point, whose nature comes from the sign change of $f$ (positive-to-negative gives a local maximum of $F$, negative-to-positive a local minimum).

An antiderivative is only fixed up to a constant: replacing $F(x)$ by $F(x)+c$ shifts the graph vertically without changing any gradient, so a whole family of parallel curves shares the one gradient function $f$.

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## Worked examples

### *Example 1 — scaffolded*

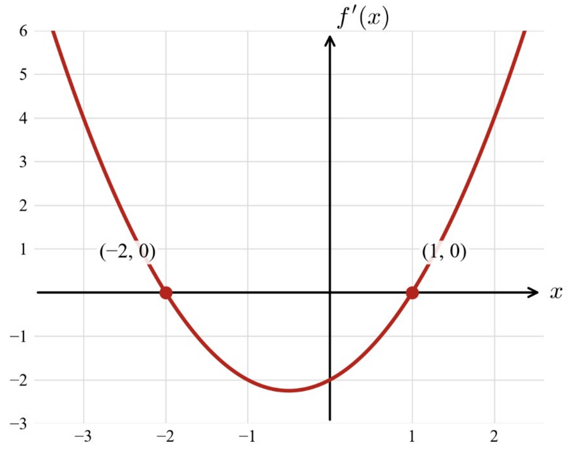
For $f(x)=x^3-6x^2+9x+1$, find $f'(x)$, locate the stationary points, and state where $f$ is increasing and where it is decreasing.

| Working | Comment |
|---|---|
| $f'(x)=3x^2-12x+9$. | Differentiate term by term with the power rule. |
| $f'(x)=3(x^2-4x+3)=3(x-1)(x-3)$. | Factorise to expose the zeros of the gradient function. |
| $f'(x)=0$ at $x=1$ and $x=3$. | Stationary points occur where $f'(x)=0$. |
| $f(1)=5$ and $f(3)=1$, giving $(1,5)$ and $(3,1)$. | Substitute back into $f$ for the $y$-coordinates. |

| Working | Comment |
| --- | --- |
| $f'(0)=9>0, f'(2)=-3<0, f'(4)=9>0$. | Test the sign of $f'$ on each interval. |
| So $f$ is increasing for $x<1$ and $x>3$, decreasing for $1<x<3$. | $f$ rises where $f'>0$ and falls where $f'<0$. |
| $(1,5)$ is a local maximum ($f'$ positive then negative); $(3,1)$ is a local minimum ($f'$ negative then positive). | The sign change of $f'$ fixes the nature of each point. |

***Example 2 — scaffolded***

The graph of the gradient function $y=f'(x)$ is shown. Use it to describe the behaviour of $f$.




| Working | Comment |
| --- | --- |
| The graph cuts the $x$-axis at $x=-2$ and $x=1$, so $f'(x)=0$ there. | Zeros of $f'$ mark the stationary points of $f$. |
| The parabola lies **below** the axis for $-2<x<1$, so $f'(x)<0$ there. | Read the sign of $f'$ straight off the graph. |
| It lies **above** the axis for $x<-2$ and for $x>1$, so $f'(x)>0$ there. | Both outer branches are positive. |
| Hence $f$ is decreasing for $-2<x<1$, and increasing for $x<-2$ and $x>1$. | $f$ rises where $f'>0$, falls where $f'<0$. |
| At $x=-2$, $f'$ changes from positive to negative: a local maximum. At $x=1$, $f'$ changes from negative to positive: a local minimum. | The sign change of $f'$ gives the nature. The $y$-values cannot be found from $f'$ alone. |

***Example 3 — unscaffolded***

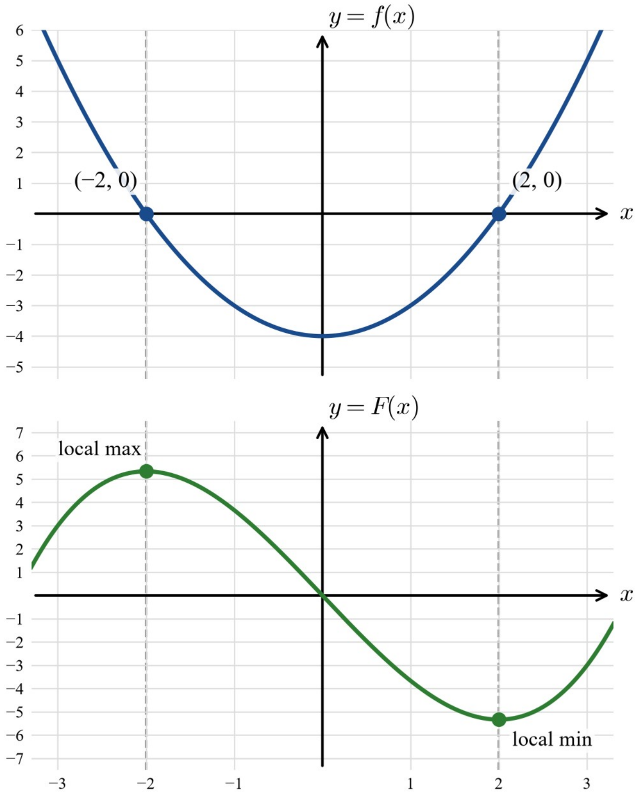
For $f(x)=x^3-3x^2-9x+5$, find the gradient function $f'(x)$ and describe the graph of $y=f'(x)$, explaining how its features correspond to those of $f$.

Differentiating, $f'(x)=3x^2-6x-9=3(x^2-2x-3)=3(x-3)(x+1)$. This gradient function is an **upward parabola**. Its zeros $x=-1$ and $x=3$ are the $x$-coordinates of the stationary points of $f$: between them $f'(x)<0$ so $f$ decreases, while outside them $f'(x)>0$ so $f$ increases. Therefore $f$ has a local maximum at $x=-1$ (where $f=10$) and a local minimum at $x=3$ (where $f=-22$). The vertex of the parabola is at $x=1$, midway between the zeros, where the gradient is least; there $f'(1)=-12$, the point at which $f$ is falling most steeply.

$\therefore f'(x)=3(x-3)(x+1)$ is an upward parabola with zeros at $x=-1$ and $x=3$ and least gradient $-12$ at $x=1$.

*Example 4 — unscaffolded*

The graph of $y=f(x)$, where $f(x)=x^2-4$, is shown in the upper panel. Sketch a graph of an antiderivative $F$, where $F'(x)=f(x)$, and describe its stationary points.




An antiderivative satisfies $F'(x)=f(x)$, so $f$ is the gradient function of $F$. Reading the sign of $f$: since $f(x)=x^2-4=0$ at $x=-2$ and $x=2$, the graph of $F$ has stationary points there. For $x<-2$, $f>0$ so $F$ is rising; for $-2<x<2$, $f<0$ so $F$ is falling; for $x>2$, $f>0$ so $F$ is rising again. Thus $f$ changes from positive to negative at $x=-2$, giving $F$ a local maximum, and from negative to positive at $x=2$, giving a local minimum. One antiderivative is $F(x)=\frac{x^3}{3}-4x$, with the local maximum $\left(-2,\frac{16}{3}\right)$ and local minimum $\left(2,-\frac{16}{3}\right)$ shown; any vertical shift $F(x)+c$ has the identical shape, since adding a constant does not change the gradient.

$\therefore F$ increases for $x<-2$ and $x>2$ and decreases for $-2<x<2$, with a local maximum at $x=-2$ and a local minimum at $x=2$ (for example $F(x)=\frac{x^3}{3}-4x$).

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## Practice questions

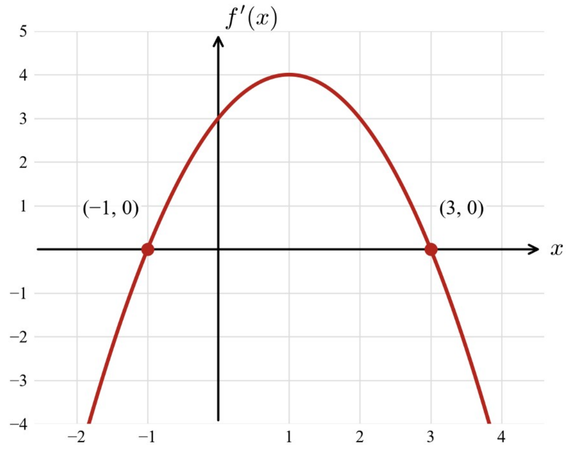
### *Scaffolded*

**Q1.** For $f(x)=x^3-12x+2$: (a) find $f'(x)$; (b) solve $f'(x)=0$ to find the stationary points; (c) find the $y$-coordinate and nature of each stationary point; (d) state where $f$ is increasing and where it is decreasing.

**Q2.** For the rational function $f(x)=\frac{x}{x-2}$ (with $x\neq 2$): (a) use the quotient rule to find $f'(x)$; (b) explain why $f'(x)<0$ for every $x$ in the domain; (c) hence describe the behaviour of $f$ on each side of $x=2$.

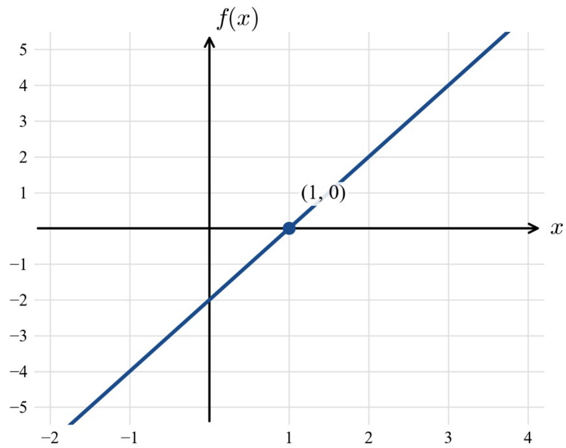
**Q3.** For $f(x)=(2x-1)^3$: (a) use the chain rule to find $f'(x)$; (b) explain why $f'(x)\geq 0$ for all $x$; (c) find the stationary point and state its nature; (d) describe where $f$ is increasing.

**Q4.** The graph of the gradient function $y=f'(x)$ is shown.




(a) State the values of $x$ where $f$ has stationary points.
(b) State where $f'(x)>0$ and where $f'(x)<0$.
(c) Hence state where $f$ is increasing and where it is decreasing.
(d) State the nature of each stationary point of $f$.

**Q5.** The graph of $y=f(x)$, where $f(x)=2x-2$, is shown; $F$ is an antiderivative with $F'(x)=f(x)$.




(a) State where $f(x)<0$ and where $f(x)>0$.
(b) Hence state where $F$ is decreasing and where it is increasing.
(c) State the $x$-coordinate and nature of the stationary point of $F$.
(d) Describe the overall shape of the graph of $F$.

**Q6.** For $f(x)=x^2e^x$: (a) use the product rule to find $f'(x)$; (b) factorise $f'(x)$; (c) find the $x$-coordinates of the stationary points; (d) state the nature of each and where $f$ is increasing.

### *Unscaffolded*

**Q7.** For $f(x)=2x^3-3x^2-12x+1$, find the coordinates and nature of all stationary points, and state where $f$ is increasing.

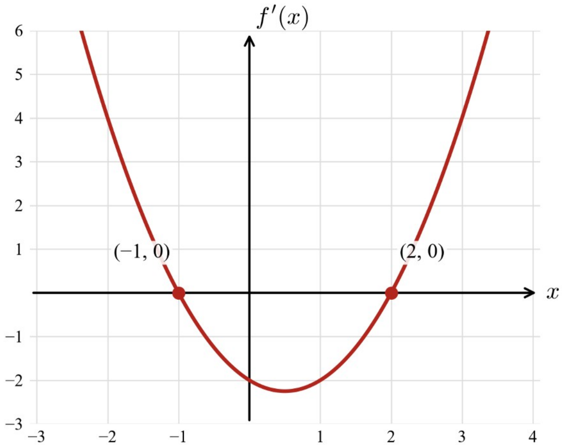
**Q8.** For $f(x)=x^4-8x^2+3$, find the coordinates and nature of all stationary points.

**Q9.** For $f(x)=(x^2-2x)^2$, find $f'(x)$ and the coordinates and nature of all stationary points.

**Q10.** For the rational function $f(x)=\dfrac{x^2}{x-1}$ (with $x\neq 1$), find $f'(x)$ and the coordinates and nature of all stationary points.

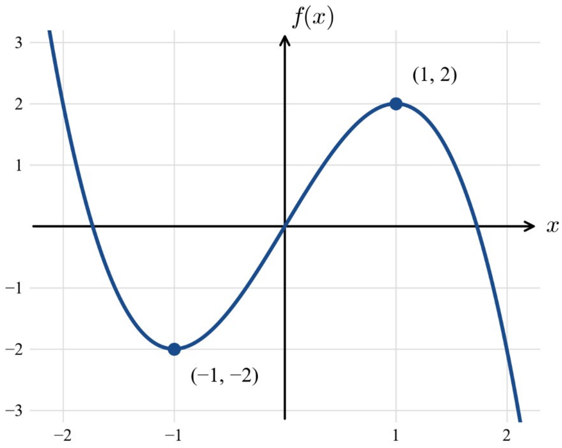
**Q11.** For $f(x)=(x+2)^2(x-1)$, find $f'(x)$ in factorised form and the coordinates and nature of all stationary points.

**Q12.** The graph of $y = f'(x)$ is shown.




Describe fully the behaviour of $f$: where it is increasing or decreasing, and the location and nature of its stationary points.

**Q13.** The graph of $y = f(x)$, where $f(x) = -x^3 + 3x$, is shown with its stationary points marked.




Find $f'(x)$ and describe the graph of the gradient function $y = f'(x)$, including its zeros and where it is positive or negative.

**Q14.** A function $f$ has gradient function $f'(x) = (x-1)(x-4)$. Without finding $f$, state where $f$ is increasing and where it is decreasing, and the nature of each stationary point.

**Q15.** The graph of $y = f(x)$ is the parabola $f(x) = x^2 - 1$. For an antiderivative $F$ with $F'(x) = f(x)$, state where $F$ is increasing and where it is decreasing, and the location and nature of its stationary points. Give one possible rule for $F$.

**Q16.** For $f(x) = x^3 - 3x^2 - 9x + 2$, find the coordinates and nature of all stationary points.

**Q17.** For the rational function $f(x) = \dfrac{x^2+3}{x}$ (with $x \neq 0$), find $f'(x)$ and the exact coordinates and nature of all stationary points.

**Q18.** A function $f$ has gradient function $f'(x) = 3x^2 - 6x$ and passes through the point $(0, 1)$. (a) Find the $x$-coordinates of the stationary points of $f$ and state the nature of each. (b) State where $f$ is increasing. (c) By antidifferentiating and using $f(0) = 1$, find $f(x)$, and hence the coordinates of the stationary points.

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## Answers

**Q1.** $f'(x)=3x^2-12$; $(-2,18)$ local maximum and $(2,-14)$ local minimum; increasing for $x<-2$ and $x>2$, decreasing for $-2<x<2$. **Q2.** $f'(x)=-\frac{2}{(x-2)^2}<0$; $f$ is strictly decreasing on $(-\infty,2)$ and on $(2,\infty)$; no stationary points. **Q3.** $f'(x)=6(2x-1)^2\geq 0$; stationary point of inflection at $(1/2,0)$; increasing for all $x$. **Q4.** Stationary points at $x=-1$ and $x=3$; $f'(x)>0$ on $-1<x<3$, $f'(x)<0$ for $x<-1$ and $x>3$; $f$ increasing on $-1<x<3$, decreasing outside; local minimum at $x=-1$, local maximum at $x=3$. **Q5.** $f<0$ for $x<1$, $f>0$ for $x>1$; $F$ decreasing for $x<1$, increasing for $x>1$; local minimum at $x=1$; $F$ is an upward-opening parabola (e.g. $F(x)=x^2-2x+c$). **Q6.** $f'(x)=x(x+2)e^x$; $(-2,4e^{-2})$ local maximum and $(0,0)$ local minimum; increasing for $x<-2$ and $x>0$, decreasing for $-2<x<0$. **Q7.** $(-1,8)$ local maximum, $(2,-19)$ local minimum; increasing for $x<-1$ and $x>2$. **Q8.** $(0,3)$ local maximum; $(-2,-13)$ and $(2,-13)$ local minima. **Q9.** $f'(x)=4x(x-1)(x-2)$; $(0,0)$ and $(2,0)$ local minima, $(1,1)$ local maximum. **Q10.** $f'(x)=\frac{x(x-2)}{(x-1)^2}$; $(0,0)$ local maximum, $(2,4)$ local minimum. **Q11.** $f'(x)=3x(x+2)$; $(-2,0)$ local maximum, $(0,-4)$ local minimum. **Q12.** $f$ increasing for $x<-1$ and $x>2$, decreasing for $-1<x<2$; local maximum at $x=-1$, local minimum at $x=2$. **Q13.** $f'(x)=3-3x^2=-3(x-1)(x+1)$; a downward parabola with zeros at $x=-1$ and $x=1$; $f'(x)>0$ for $-1<x<1$, $f'(x)<0$ for $x<-1$ and $x>1$. **Q14.** Increasing for $x<1$ and $x>4$, decreasing for $1<x<4$; local maximum at $x=1$, local minimum at $x=4$. **Q15.** $F$ increasing for $x<-1$ and $x>1$, decreasing for $-1<x<1$; local maximum at $x=-1$, local minimum at $x=1$; e.g. $F(x)=\frac{x^3}{3}-x$. **Q16.** $(-1,7)$ local maximum, $(3,-25)$ local minimum. **Q17.** $f'(x)=1-\frac{3}{x^2}=\frac{x^2-3}{x^2}$; $(-\sqrt{3},-2\sqrt{3})$ local maximum, $(\sqrt{3},2\sqrt{3})$ local minimum. **Q18.** (a) $x=0$ local maximum, $x=2$ local minimum; (b) increasing for $x<0$ and $x>2$; (c) $f(x)=x^3-3x^2+1$, stationary points $(0,1)$ and $(2,-3)$.

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## Fully-worked solutions

**Q1.** $f'(x)=3x^2-12=3(x^2-4)=3(x-2)(x+2)$, so $f'(x)=0$ at $x=-2$ and $x=2$. Then $f(-2)=-8+24+2=18$ and $f(2)=8-24+2=-14$. Testing the sign of $f'$: $f'(-3)=15>0$, $f'(0)=-12<0$, $f'(3)=15>0$, so $f$ is increasing for $x<-2$ and $x>2$ and decreasing for $-2<x<2$. Hence $(-2,18)$ is a local maximum ($f'$ positive then negative) and $(2,-14)$ a local minimum ($f'$ negative then positive).

**Q2.** With $u=x$ and $v=x-2$ (so $u'=1$, $v'=1$), the quotient rule gives

$$f'(x)=\frac{u'v-uv'}{v^2}=\frac{(1)(x-2)-(x)(1)}{(x-2)^2}=\frac{x-2-x}{(x-2)^2}=-\frac{2}{(x-2)^2}.$$

The numerator $-2$ is negative and the denominator $(x-2)^2$ is positive for all $x\neq 2$, so $f'(x)<0$ throughout the domain. Therefore $f$ is strictly decreasing on $(-\infty,2)$ and on $(2,\infty)$; since $f'$ is never zero, $f$ has no stationary points.

**Q3.** By the chain rule, $f'(x)=3(2x-1)^2\cdot 2=6(2x-1)^2$. A square is never negative, so $f'(x)\geq 0$ for all $x$, with equality only at $x=1/2$. Because $f'$ keeps the same (positive) sign on both sides of $x=1/2$, the point is a **stationary point of inflection**; as $f(1/2)=0$, it is $(1/2,0)$. The curve is increasing for all $x$ (with that single horizontal moment).

**Q4.** The graph of $f'$ cuts the $x$-axis at $x=-1$ and $x=3$, so $f$ has stationary points there. The parabola opens downward, lying **above** the axis for $-1<x<3$ (so $f'(x)>0$) and **below** it for $x<-1$ and $x>3$ (so $f'(x)<0$). Hence $f$ is increasing on $-1<x<3$ and decreasing for $x<-1$ and $x>3$. At $x=-1$, $f'$ changes from negative to positive —

a local minimum; at $x=3$, $f'$ changes from positive to negative — a local maximum. (The $y$-values cannot be determined from $f'$ alone.)

**Q5.** $f(x)=2x-2=0$ at $x=1$, with $f(x)<0$ for $x<1$ and $f(x)>0$ for $x>1$. Since $F'(x)=f(x)$, the graph of $F$ is decreasing where $f<0$ (that is, $x<1$) and increasing where $f>0$ ($x>1$). At $x=1$, $f$ changes from negative to positive, so $F$ has a local minimum there. Overall $F$ is an upward-opening parabola-shaped curve with its minimum at $x=1$; antidifferentiating, $F(x)=x^2-2x+c$ for any constant $c$.

**Q6.** With $u=x^2$ and $v=e^x$ (so $u'=2x$, $v'=e^x$), the product rule gives

$$f'(x)=u'v+uv'=2xe^x+x^2e^x=(x^2+2x)e^x=x(x+2)e^x.$$

Because $e^x>0$ always, the sign of $f'$ is the sign of $x(x+2)$, which is zero at $x=-2$ and $x=0$. Then $x(x+2)>0$ for $x<-2$ and $x>0$, and $x(x+2)<0$ for $-2<x<0$, so $f$ is increasing for $x<-2$ and $x>0$ and decreasing for $-2<x<0$. Thus $x=-2$ is a local maximum, $f(-2)=4e^{-2}$, and $x=0$ is a local minimum, $f(0)=0$: the points $(-2,4e^{-2})$ and $(0,0)$.

**Q7.** $f'(x)=6x^2-6x-12=6(x^2-x-2)=6(x-2)(x+1)$, zero at $x=-1$ and $x=2$. Then $f(-1)=-2-3+12+1=8$ and $f(2)=16-12-24+1=-19$. The upward parabola $f'$ is negative between the zeros and positive outside, so $f$ is increasing for $x<-1$ and $x>2$ and decreasing for $-1<x<2$. Hence $(-1,8)$ is a local maximum and $(2,-19)$ a local minimum.

**Q8.** $f'(x)=4x^3-16x=4x(x^2-4)=4x(x-2)(x+2)$, zero at $x=-2$, $x=0$ and $x=2$. The $y$-values are $f(-2)=16-32+3=-13$, $f(0)=3$ and $f(2)=16-32+3=-13$. Sign of $f'$: $f'(-3)=-60<0$, $f'(-1)=12>0$, $f'(1)=-12<0$, $f'(3)=60>0$. So $f$ falls, rises, falls, rises across the four intervals, giving local minima at $(-2,-13)$ and $(2,-13)$ and a local maximum at $(0,3)$ — the classic "W" shape of a positive quartic.

**Q9.** By the chain rule, $f'(x)=2(x^2-2x)(2x-2)=4(x^2-2x)(x-1)=4x(x-1)(x-2)$, zero at $x=0$, $x=1$ and $x=2$. Since $f(x)=(x^2-2x)^2\geq 0$ and equals 0 at $x=0$ and $x=2$, those are local minima $(0,0)$ and $(2,0)$. At $x=1$, $f(1)=(1-2)^2=1$. Checking signs, $f'(1/2)>0$ and $f'(3/2)<0$, so $(1,1)$ is a local maximum.

**Q10.** With $u=x^2$ and $v=x-1$, the quotient rule gives

$$f'(x)=\frac{2x(x-1)-x^2(1)}{(x-1)^2}=\frac{2x^2-2x-x^2}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}=\frac{x(x-2)}{(x-1)^2}.$$

The denominator is positive (for $x\neq 1$), so the sign of $f'$ follows $x(x-2)$: zero at $x=0$ and $x=2$, positive for $x<0$ and $x>2$, negative for $0<x<2$. Hence $f$ increases, decreases, then increases, giving a local maximum $(0,0)$ and a local minimum $(2,4)$ (using $f(2)=4/1=4$).

**Q11.** With $u=(x+2)^2$ and $v=x-1$ (so $u'=2(x+2)$, $v'=1$),

$$f'(x)=2(x+2)(x-1)+(x+2)^2=(x+2)[2(x-1)+(x+2)]=(x+2)(3x)=3x(x+2),$$

zero at $x=-2$ and $x=0$. Then $f(-2)=0$ and $f(0)=(2)^2(-1)=-4$. The sign of $f'$ is positive for $x<-2$, negative for $-2<x<0$ and positive for $x>0$, so $(-2,0)$ is a local maximum and $(0,-4)$ a local minimum.

**Q12.** The gradient function is an upward parabola with zeros at $x=-1$ and $x=2$ (where $f'=(x+1)(x-2)$). It is positive for $x<-1$ and $x>2$ and negative for $-1<x<2$. Hence $f$ is increasing for $x<-1$ and $x>2$ and decreasing for $-1<x<2$. At $x=-1$, $f'$ changes from positive to negative, a local maximum; at $x=2$, from negative to positive, a local minimum. (The $y$-values are not fixed by $f'$ alone.)

**Q13.** $f'(x)=-3x^2+3=3-3x^2=-3(x^2-1)=-3(x-1)(x+1)$. This is a **downward** parabola with zeros at $x=-1$ and $x=1$. It is positive between the zeros, $-1<x<1$, and negative outside, $x<-1$ and $x>1$. This agrees with the given curve: $f$ falls to the local minimum $(-1,-2)$, rises to the local maximum $(1,2)$, then falls again — exactly where $f'$ is negative, positive, negative in turn.

**Q14.** $f'(x)=(x-1)(x-4)$ is an upward parabola with zeros at $x=1$ and $x=4$, positive for $x<1$ and $x>4$ and negative for $1<x<4$. So $f$ is increasing for $x<1$ and $x>4$ and decreasing for $1<x<4$. At $x=1$, $f'$ changes from positive to negative — a local maximum; at $x=4$, from negative to positive — a local minimum.

**Q15.** Here $F'(x)=f(x)=x^2-1$, which is negative for $-1<x<1$ and positive for $x<-1$ and $x>1$. So $F$ is decreasing on $-1<x<1$ and increasing for $x<-1$ and $x>1$. At $x=-1$, $f$ changes from positive to negative, giving $F$ a local maximum; at $x=1$, $f$ changes from negative to positive, giving a local minimum. Antidifferentiating, $F(x)=\frac{x^3}{3}-x+c$; taking $c=0$ gives the local maximum $\left(-1,\frac{2}{3}\right)$ and local minimum $\left(1,-\frac{2}{3}\right)$.

**Q16.** $f'(x)=3x^2-6x-9=3(x^2-2x-3)=3(x-3)(x+1)$, zero at $x=-1$ and $x=3$. Then $f(-1)=-1-3+9+2=7$ and $f(3)=27-27-27+2=-25$. The upward parabola $f'$ is negative between the zeros and positive outside, so $(-1,7)$ is a local maximum and $(3,-25)$ a local minimum.

**Q17.** Writing $f(x)=\frac{x^2+3}{x}=x+\frac{3}{x}$, differentiate to get

$$f'(x)=1-\frac{3}{x^2}=\frac{x^2-3}{x^2}.$$

The denominator is positive, so the sign of $f'$ follows $x^2-3$: zero at $x=\pm\sqrt{3}$, positive for $|x|>\sqrt{3}$ and negative for $0<|x|<\sqrt{3}$. Hence $f$ has a local maximum at $x=-\sqrt{3}$ and a local minimum at $x=\sqrt{3}$. The exact values are $f(-\sqrt{3})=-\sqrt{3}+\frac{3}{-\sqrt{3}}=-\sqrt{3}-\sqrt{3}=-2\sqrt{3}$ and $f(\sqrt{3})=\sqrt{3}+\frac{3}{\sqrt{3}}=2\sqrt{3}$, giving $(-\sqrt{3},-2\sqrt{3})$ and $(\sqrt{3},2\sqrt{3})$.

**Q18.** (a) $f'(x)=3x^2-6x=3x(x-2)$, zero at $x=0$ and $x=2$. Since $f'(-1)=9>0$, $f'(1)=-3<0$ and $f'(3)=9>0$, the sign pattern is positive, negative, positive, so $x=0$ is a local maximum and $x=2$ a local minimum. (b) $f$ is increasing where $f'(x)>0$: for $x<0$ and $x>2$. (c) Antidifferentiating, $f(x)=x^3-3x^2+c$; the condition $f(0)=1$ gives $c=1$, so $f(x)=x^3-3x^2+1$. Then $f(0)=1$ and $f(2)=8-12+1=-3$, so the stationary points are $(0,1)$ (local maximum) and $(2,-3)$ (local minimum).

### Theory

Most curves so far have been written with $y$ as a function of $x$, so that $\frac{dy}{dx}$ is found by differentiating the rule directly. Some relations, however, are far simpler to write the other way round — with $x$ **as a function of** $y$,

$$x=f(y).$$

For example, $x=y^3+2y$ gives $x$ neatly in terms of $y$, but solving for $y$ in terms of $x$ is awkward. This section shows how to find the gradient $\frac{dy}{dx}$ of such a curve **without** first rearranging it into the form $y=g(x)$.

**The reciprocal result.**

Suppose the curve $x=f(y)$ also determines $y$ as a function of $x$ near a point of interest (so that a small step along the curve moves it past the vertical-line test there). Then $y$ is a function of $x$, and $x$ is in turn a function of $y$. Applying the **chain rule** to the identity $x=x$, differentiated with respect to $x$,

$$\frac{dx}{dx}=\frac{dx}{dy}\cdot\frac{dy}{dx}.$$

The left-hand side is 1, so

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=1,\text{giving}\ \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}},\ \frac{dx}{dy}\neq 0.$$

In words: **the derivative $\frac{dy}{dx}$ is the reciprocal of the derivative $\frac{dx}{dy}$.** This is exactly what the notation invites — the two derivatives behave, for this purpose, like a fraction and its reciprocal — but the result is a genuine consequence of the chain rule, not merely a trick of symbols.

**Using the result.**

To find $\frac{dy}{dx}$ for a curve given as $x=f(y)$:

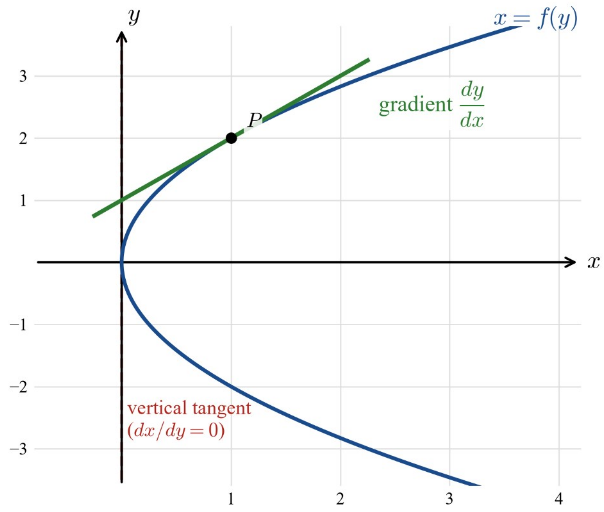
1. differentiate $x$ with respect to $y$ to obtain $\frac{dx}{dy}=f'(y)$;
2. take the reciprocal, $\frac{dy}{dx}=\frac{1}{f'(y)}$, which is usually a function of $y$;
3. to find the gradient at a particular point, substitute the $y$-coordinate of that point into $\frac{dy}{dx}$.

Because $\frac{dy}{dx}$ comes out in terms of $y$, it is natural here to locate a point by its $y$-coordinate and read off the matching $x$-coordinate from $x=f(y)$.

**When the result is valid.**

The formula requires $\frac{dx}{dy}\neq 0$. Geometrically, $\frac{dx}{dy}=0$ at a point where $x$ is **stationary in the $y$-direction**: as $y$ increases through that value, $x$ momentarily stops changing, so the tangent to the curve is **vertical**. Most often $x$ turns back there, as at the vertex of a sideways parabola; sometimes it merely pauses and then continues in the same direction, and the tangent is vertical in that case too. A vertical tangent has no ordinary gradient — $\frac{dy}{dx}$ is undefined there — which matches the fact that $\frac{1}{dx/dy}$ cannot be formed when $\frac{dx}{dy}=0$. So the points where $\frac{dx}{dy}=0$ are precisely the points where the curve has a vertical tangent, and these must be excluded.

The diagram shows a curve $x=f(y)$ with a tangent drawn at a point $P$. The gradient of that tangent is $\frac{dy}{dx}$, the reciprocal of $\frac{dx}{dy}$. At the left-most point of this curve the tangent is vertical, and there $\frac{dx}{dy}=0$.




**Differentiating "backwards".**

The result is especially useful when a function is easier to differentiate in the form $x=f(y)$ than as $y=g(x)$. For instance, the cube-root curve $y=\sqrt[3]{x}$ is awkward as written, but the same curve is $x=y^3$, a simple polynomial. Then $\frac{dx}{dy}=3y^2$, so

$$\frac{dy}{dx}=\frac{1}{3y^2},$$

and since $y=x^{1/3}$ this is $\frac{1}{3\left(x^{1/3}\right)^2}=\frac{1}{3x^{2/3}}$ — the derivative of $\sqrt[3]{x}$, obtained without differentiating a fractional power directly.

The same idea underlies the derivatives of the **inverse functions**: to differentiate $y=\sin^{-1}x$, for instance, it is far easier to write $x=\sin y$, differentiate to get $\frac{dx}{dy}=\cos y$, and then take the reciprocal. The inverse circular functions are developed fully in a later section; here the point is simply that $\frac{dy}{dx}=\frac{1}{dx/dy}$ is the tool that makes such derivatives accessible.

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## Worked examples

### *Example 1 — scaffolded*

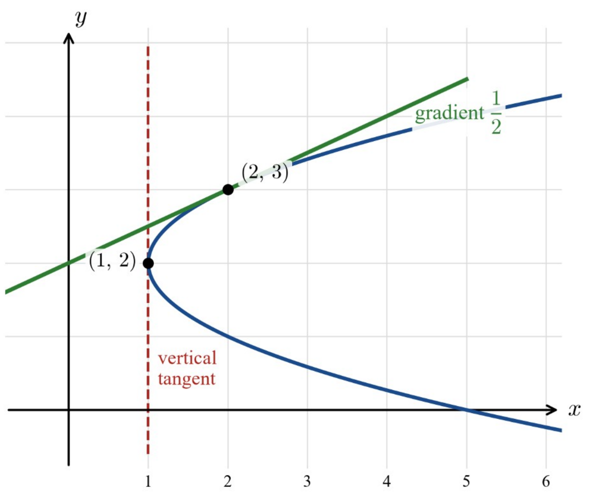
For the curve $x=y^3+2y$, find $\frac{dy}{dx}$ in terms of $y$, and hence find the gradient at the point where $y=1$.

| Working | Comment |
|---|---|
| $\dfrac{dx}{dy} = 3y^2 + 2.$ | Differentiate $x = f(y)$ with respect to $y$. |
| $\dfrac{dy}{dx} = \dfrac{1}{3y^2+2}.$ | Take the reciprocal, $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$. |
| Since $3y^2 + 2 \geq 2 > 0$ for all $y$, the gradient is defined everywhere. | $\dfrac{dx}{dy}$ is never $0$, so there is no vertical tangent. |
| At $y = 1$: $x = 1^3 + 2(1) = 3$, so the point is $(3, 1)$. | Read off the matching $x$-coordinate from $x = f(y)$. |
| $\dfrac{dy}{dx}\Big\vert_{y=1} = \dfrac{1}{3(1)^2+2} = \dfrac{1}{5}.$ | Substitute $y = 1$ into $\dfrac{dy}{dx}$. |

### *Example 2 — scaffolded*

The curve $x = y^2 - 4y + 5$ is a sideways parabola. Find $\dfrac{dy}{dx}$, state where it has a vertical tangent, and find the gradient at the point where $y = 3$.

| Working | Comment |
|---|---|
| $\dfrac{dx}{dy} = 2y - 4.$ | Differentiate with respect to $y$. |
| $\dfrac{dy}{dx} = \dfrac{1}{2y-4}$, provided $2y - 4 \neq 0$. | Reciprocal, valid where $\dfrac{dx}{dy} \neq 0$. |
| $\dfrac{dx}{dy} = 0$ when $y = 2$; then $x = 2^2 - 4(2) + 5 = 1$. | Solve $\dfrac{dx}{dy} = 0$ for the vertical tangent. |
| So the tangent is vertical at $(1, 2)$, the left-most point of the parabola. | At this point $\dfrac{dy}{dx}$ is undefined. |
| At $y = 3$: $x = 9 - 12 + 5 = 2$, and $\dfrac{dy}{dx} = \dfrac{1}{2(3)-4} = \dfrac{1}{2}$. | Gradient $\dfrac{1}{2}$ at the point $(2, 3)$. |




### *Example 3 — unscaffolded*

For the curve $x = y^3 - 3y$, find the coordinates of the points at which the tangent is vertical, and find the gradient at the point where $y = 2$.

Differentiating, $\dfrac{dx}{dy} = 3y^2 - 3 = 3(y^2 - 1)$. The tangent is vertical where $\dfrac{dx}{dy} = 0$, that is $y^2 - 1 = 0$, so $y = 1$ or $y = -1$. The corresponding $x$-values are $x = 1 - 3 = -2$ and $x = -1 + 3 = 2$, giving the points $(-2, 1)$ and $(2, -1)$.

Elsewhere $\frac{dy}{dx}=\frac{1}{3y^2-3}$. At $y=2$, $x=8-6=2$, and

$$\frac{dy}{dx}\Big|_{y=2}=\frac{1}{3(2)^2-3}=\frac{1}{9}.$$

$\therefore$ the tangent is vertical at $(-2,1)$ and $(2,-1)$, and the gradient at $(2,2)$ is $\frac{1}{9}$.

### *Example 4 — unscaffolded*

The curve $x=y^3$ can also be written $y=\sqrt[3]{x}$. Find the gradient at the point $(8,2)$ using $\frac{dy}{dx}=\frac{1}{dx/dy}$, and confirm it by differentiating $y=x^{1/3}$ directly.

Writing $x=y^3$ gives $\frac{dx}{dy}=3y^2$, so $\frac{dy}{dx}=\frac{1}{3y^2}$. At the point $(8,2)$ the $y$-coordinate is $2$ (and indeed $2^3=8$), so

$$\frac{dy}{dx}\Big|_{y=2}=\frac{1}{3(2)^2}=\frac{1}{12}.$$

Differentiating the explicit form $y=x^{1/3}$ instead, $\frac{dy}{dx}=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$, and at $x=8$,

$$\frac{dy}{dx}\Big|_{x=8}=\frac{1}{3\cdot 8^{2/3}}=\frac{1}{3\cdot 4}=\frac{1}{12},$$

which agrees. The reverse form $x=y^3$ needed only the power rule on a polynomial. The same reciprocal method also gives derivatives such as that of $y=\sin^{-1}x$, where the reverse form $x=\sin y$ is much simpler to differentiate.

$\therefore$ the gradient at $(8,2)$ is $\frac{1}{12}$.

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## Practice questions

### *Scaffolded*

**Q1.** For the curve $x=y^3+4y$, find $\frac{dy}{dx}$ and its value at $y=1$. (a) Find $\frac{dx}{dy}$. (b) Write down $\frac{dy}{dx}=\frac{1}{dx/dy}$. (c) Explain why $\frac{dy}{dx}$ is defined for every value of $y$. (d) Find the point where $y=1$ and the gradient there.

**Q2.** The curve $x=y^2+6y+10$ is a sideways parabola. (a) Find $\frac{dx}{dy}$. (b) Write down $\frac{dy}{dx}$, stating the value of $y$ to exclude. (c) Find the point at which the tangent is vertical. (d) Find the gradient at the point where $y=-1$.

**Q3.** For the curve $x=y^3-12y$: (a) Find $\frac{dx}{dy}$. (b) Solve $\frac{dx}{dy}=0$. (c) Hence state the coordinates of the two points with a vertical tangent. (d) Find the gradient at the point where $y=0$.

**Q4.** The curve $x=y^3$ is the same as $y=\sqrt[3]{x}$. (a) Find $\frac{dx}{dy}$. (b) Write down $\frac{dy}{dx}$ in terms of $y$. (c) Find the gradient at the point $(27,3)$. (d) Confirm your answer by differentiating $y=x^{1/3}$ directly.

**Q5.** The curve $x=\frac{1}{y}$ (with $y\neq 0$) is the same as $y=\frac{1}{x}$. (a) Find $\frac{dx}{dy}$. (b) Write down $\frac{dy}{dx}$ in terms of $y$. (c) Find the gradient at the point where $y=2$. (d) Confirm your answer by differentiating $y=\frac{1}{x}$ directly.

**Q6.** For the curve $x=y^3+y$: (a) Find $\frac{dx}{dy}$. (b) Write down $\frac{dy}{dx}$ in terms of $y$. (c) Find the point where $y=1$ and the gradient there. (d) Explain why the gradient at the point where $y=-1$ is the same value.

***Unscaffolded***

**Q7.** For the curve $x=y^3+5y-2$, find $\frac{dy}{dx}$ in terms of $y$.

**Q8.** For the curve $x=y^2-8y+3$, find $\frac{dy}{dx}$ and the coordinates of the point at which the tangent is vertical.

**Q9.** For the curve $x=2y^3-6y$, find the coordinates of the points at which the tangent is vertical.

**Q10.** Find the gradient of the curve $x=y^2+y$ at the point $(6,2)$.

**Q11.** For the curve $x=y^4+2y^2$, find $\frac{dy}{dx}$ in terms of $y$ and its value at the point where $y=1$.

**Q12.** The curve $x=\sqrt{y}$ (with $y\geq 0$) is the same as $y=x^2$ (with $x\geq 0$). Find $\frac{dy}{dx}$ in terms of $y$, express it in terms of $x$, and verify your answer by differentiating $y=x^2$ directly.

**Q13.** For the curve $x=(y+1)^3$, find $\frac{dy}{dx}$ and the coordinates of the point at which the tangent is vertical.

**Q14.** The curve $x=y^2-6y$ meets the $y$-axis where $x=0$. Find the gradient of the curve at each of the two points where $x=0$.

**Q15.** Find the equation of the tangent to the curve $x=y^3+2y$ at the point $(3,1)$, giving your answer in the form $y=mx+c$.

**Q16.** For the curve $x=3y-y^3$, find the coordinates of the points at which the tangent is vertical.

**Q17.** The curve $y=\sqrt[3]{x-1}$ can be written with $x$ as a function of $y$. Write $x$ in terms of $y$, and hence find the gradient at the point where $y=2$.

**Q18.** The curve $x=\sin y$, for $-\frac{\pi}{2}<y<\frac{\pi}{2}$, is the same as $y=\sin^{-1}x$. (a) Find $\frac{dx}{dy}$. (b) Hence write $\frac{dy}{dx}$ in terms of $y$. (c) Find the exact gradient at the point where $y=\frac{\pi}{6}$, and state the $x$-coordinate of that point.

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### Answers

**Q1.** $\frac{dy}{dx}=\frac{1}{3y^2+4}$; at $(5,1)$ the gradient is $\frac{1}{7}$. **Q2.** $\frac{dy}{dx}=\frac{1}{2y+6}$ ($y\neq -3$); vertical tangent at $(1,-3)$; gradient $\frac{1}{4}$ at $(5,-1)$. **Q3.** $\frac{dx}{dy}=3y^2-12=0$ at $y=\pm 2$; vertical tangents at $(-16,2)$ and $(16,-2)$; gradient $-\frac{1}{12}$ at $(0,0)$. **Q4.** $\frac{dy}{dx}=\frac{1}{3y^2}$; at $(27,3)$ the gradient is $\frac{1}{27}$ (agrees with $y=x^{1/3}$). **Q5.** $\frac{dy}{dx}=-y^2$; at $\left(\frac{1}{2},2\right)$ the gradient is $-4$ (agrees with $y=\frac{1}{x}$). **Q6.** $\frac{dy}{dx}=\frac{1}{3y^2+1}$; at $(2,1)$ the gradient is $\frac{1}{4}$; the same value occurs at $(-2,-1)$ because $\frac{dy}{dx}$ depends only on $y^2$. **Q7.** $\frac{dy}{dx}=\frac{1}{3y^2+5}$. **Q8.** $\frac{dy}{dx}=\frac{1}{2y-8}$; vertical tangent at $(-13,4)$. **Q9.** $(-4,1)$ and $(4,-1)$. **Q10.** $\frac{1}{5}$. **Q11.** $\frac{dy}{dx}=\frac{1}{4y^3+4y}$; at $(3,1)$ the gradient is $\frac{1}{8}$. **Q12.** $\frac{dy}{dx}=2\sqrt{y}=2x$; agrees with differentiating $y=x^2$. **Q13.** $\frac{dy}{dx}=\frac{1}{3(y+1)^2}$ ($y\neq -1$); vertical tangent at $(0,-1)$. **Q14.** At $(0,0)$ the gradient is $-\frac{1}{6}$; at $(0,6)$ the gradient is $\frac{1}{6}$. **Q15.** $y=\frac{1}{5}x+\frac{2}{5}$. **Q16.** $(2,1)$ and $(-2,-1)$. **Q17.** $x=y^3+1$; the gradient at the point $(9,2)$ is $\frac{1}{12}$. **Q18.** (a) $\frac{dx}{dy}=\cos y$; (b) $\frac{dy}{dx}=\frac{1}{\cos y}$ (valid for $-\frac{\pi}{2}<y<\frac{\pi}{2}$); (c) gradient $\frac{2\sqrt{3}}{3}$ at the point $\left(\frac{1}{2},\frac{\pi}{6}\right)$.

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### Fully-worked solutions

**Q1.** Differentiate: $\frac{dx}{dy}=3y^2+4$, so $\frac{dy}{dx}=\frac{1}{3y^2+4}$. Since $3y^2\geq 0$, the denominator satisfies $3y^2+4\geq 4>0$ for every $y$, so $\frac{dx}{dy}$ is never 0 and $\frac{dy}{dx}$ is defined for all $y$. At $y=1$, $x=1+4=5$, giving the point $(5,1)$, and $\frac{dy}{dx}=\frac{1}{3+4}=\frac{1}{7}$.

**Q2.** $\frac{dx}{dy}=2y+6$, so $\frac{dy}{dx}=\frac{1}{2y+6}$, which requires $2y+6\neq 0$, that is $y\neq -3$. The tangent is vertical where $2y+6=0$, i.e. $y=-3$; then $x=(-3)^2+6(-3)+10=9-18+10=1$, so the point is $(1,-3)$. At $y=-1$, $x=1-6+10=5$ and $\frac{dy}{dx}=\frac{1}{-2+6}=\frac{1}{4}$, a gradient of $\frac{1}{4}$ at $(5,-1)$.

**Q3.** $\frac{dx}{dy}=3y^2-12$. Setting $\frac{dx}{dy}=0$ gives $3y^2=12$, so $y^2=4$ and $y=\pm 2$. At $y=2$, $x=8-24=-16$; at $y=-2$, $x=-8+24=16$. So the tangent is vertical at $(-16,2)$ and $(16,-2)$. At $y=0$, $x=0$ and $\frac{dy}{dx}=\frac{1}{3(0)^2-12}=-\frac{1}{12}$, a gradient of $-\frac{1}{12}$ at $(0,0)$.

**Q4.** From $x=y^3$, $\frac{dx}{dy}=3y^2$, so $\frac{dy}{dx}=\frac{1}{3y^2}$. At $(27,3)$ the $y$-coordinate is 3 (and $3^3=27$), giving $\frac{dy}{dx}=\frac{1}{3(3)^2}=\frac{1}{27}$. Directly, from $y=x^{1/3}$, $\frac{dy}{dx}=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$, and at $x=27$ this is $\frac{1}{3\cdot 27^{2/3}}=\frac{1}{3\cdot 9}=\frac{1}{27}$, the same value.

**Q5.** From $x=y^{-1}$, $\frac{dx}{dy}=-y^{-2}=-\frac{1}{y^2}$, so $\frac{dy}{dx}=\frac{1}{-1/y^2}=-y^2$. At $y=2$, $x=\frac{1}{2}$ and $\frac{dy}{dx}=-(2)^2=-4$, a gradient of $-4$ at $\left(\frac{1}{2},2\right)$. Directly, from $y=x^{-1}$, $\frac{dy}{dx}=-x^{-2}=-\frac{1}{x^2}$; at $x=\frac{1}{2}$ this is $-\frac{1}{(1/2)^2}=-4$, confirming the result.

**Q6.** $\frac{dx}{dy}=3y^2+1$, so $\frac{dy}{dx}=\frac{1}{3y^2+1}$ (defined for all $y$, since $3y^2+1\geq 1$). At $y=1$, $x=1+1=2$ and $\frac{dy}{dx}=\frac{1}{3+1}=\frac{1}{4}$, so the point is $(2,1)$ with gradient $\frac{1}{4}$. At $y=-1$, $\frac{dy}{dx}=\frac{1}{3(-1)^2+1}=\frac{1}{4}$ as well: because $\frac{dy}{dx}$ depends on $y$ only through $y^2$, replacing $y$ by $-y$ leaves it unchanged.

**Q7.** $\frac{dx}{dy}=3y^2+5$, so $\frac{dy}{dx}=\frac{1}{3y^2+5}$. (The denominator is at least $5$, so the gradient is defined for all $y$.)

**Q8.** $\frac{dx}{dy}=2y-8$, so $\frac{dy}{dx}=\frac{1}{2y-8}$, provided $y\neq 4$. The tangent is vertical where $2y-8=0$, i.e. $y=4$; then $x=16-32+3=-13$, giving the point $(-13,4)$.

**Q9.** $\frac{dx}{dy}=6y^2-6$. Setting this to $0$ gives $y^2=1$, so $y=\pm 1$. At $y=1$, $x=2-6=-4$; at $y=-1$, $x=-2+6=4$. The tangent is vertical at $(-4,1)$ and $(4,-1)$.

**Q10.** $\frac{dx}{dy}=2y+1$, so $\frac{dy}{dx}=\frac{1}{2y+1}$. At the point $(6,2)$ the $y$-coordinate is 2 (and $2^2+2=6$), so $\frac{dy}{dx}=\frac{1}{2(2)+1}=\frac{1}{5}$.

**Q11.** $\frac{dx}{dy}=4y^3+4y$, so $\frac{dy}{dx}=\frac{1}{4y^3+4y}$. At $y=1$, $x=1+2=3$ and $\frac{dy}{dx}=\frac{1}{4+4}=\frac{1}{8}$, a gradient of $\frac{1}{8}$ at $(3,1)$.

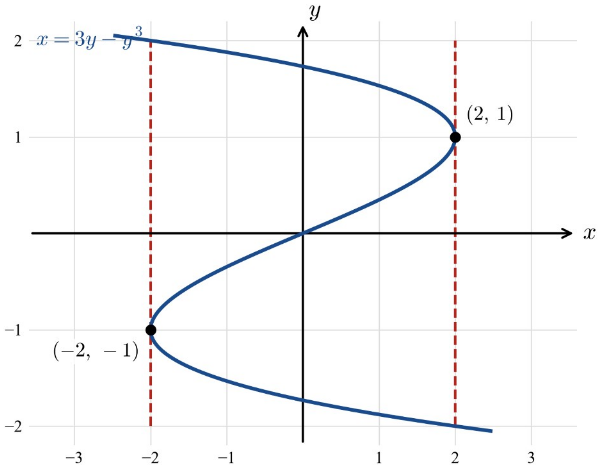
**Q12.** From $x=y^{1/2}$, $\frac{dx}{dy}=\frac{1}{2}y^{-1/2}=\frac{1}{2\sqrt{y}}$, so $\frac{dy}{dx}=2\sqrt{y}$. Since $x=\sqrt{y}$, this is $\frac{dy}{dx}=2x$. Differentiating $y=x^2$ directly gives $\frac{dy}{dx}=2x$, the same result.

**Q13.** $\frac{dx}{dy}=3(y+1)^2$, so $\frac{dy}{dx}=\frac{1}{3(y+1)^2}$, provided $y\neq -1$. The tangent is vertical where $3(y+1)^2=0$, i.e. $y=-1$; then $x=0$, giving the point $(0,-1)$.

**Q14.** On $x=0$: $y^2-6y=0$, so $y(y-6)=0$ and $y=0$ or $y=6$; the points are $(0,0)$ and $(0,6)$. Now $\frac{dx}{dy}=2y-6$, so $\frac{dy}{dx}=\frac{1}{2y-6}$. At $y=0$, $\frac{dy}{dx}=\frac{1}{-6}=-\frac{1}{6}$; at $y=6$, $\frac{dy}{dx}=\frac{1}{6}$.

**Q15.** $\frac{dx}{dy}=3y^2+2$, so $\frac{dy}{dx}=\frac{1}{3y^2+2}$. At the point $(3,1)$ the $y$-coordinate is 1, so the tangent's gradient is $m=\frac{1}{3+2}=\frac{1}{5}$. Using $y-y_1=m(x-x_1)$ with $(x_1,y_1)=(3,1)$: $y-1=\frac{1}{5}(x-3)$, so $y=\frac{1}{5}x-\frac{3}{5}+1=\frac{1}{5}x+\frac{2}{5}$.

**Q16.** $\frac{dx}{dy}=3-3y^2$. Setting this to $0$ gives $y^2=1$, so $y=\pm 1$. At $y=1$, $x=3-1=2$; at $y=-1$, $x=-3+1=-2$. The tangent is vertical at $(2,1)$ and $(-2,-1)$, where the curve turns back on itself — the right-most and left-most points of its middle branch. (The curve as a whole has no left- or right-most point: $x\to+\infty$ as $y\to-\infty$, and $x\to-\infty$ as $y\to+\infty$.)




**Q17.** From $y = \sqrt[3]{x-1}$, cubing gives $y^3 = x-1$, so $x = y^3 + 1$. Then $\dfrac{dx}{dy} = 3y^2$ and $\dfrac{dy}{dx} = \dfrac{1}{3y^2}$. At $y = 2$, $x = 8+1 = 9$, so the point is $(9, 2)$, and $\dfrac{dy}{dx} = \dfrac{1}{3(2)^2} = \dfrac{1}{12}$.

**Q18.** (a) From $x = \sin y$, $\dfrac{dx}{dy} = \cos y$. (b) Hence $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$; this is valid while $\cos y \neq 0$, that is for $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$. (c) At $y = \dfrac{\pi}{6}$, $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$, so

$$\frac{dy}{dx} = \frac{1}{\cos(\pi/6)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

The $x$-coordinate is $x = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$, so the point is $\left(\dfrac{1}{2}, \dfrac{\pi}{6}\right)$. (This gradient can also be written $\dfrac{1}{\sqrt{1-x^2}}$, which is the derivative of $\sin^{-1} x$ met in the next section.)

### Theory

The inverse circular functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are assumed from Units 1 & 2: each is the inverse of $\sin$, $\cos$ or $\tan$ restricted to an interval on which that function is one-to-one. This section finds their derivatives. The **principal branches**, on which each inverse is defined, are recapped below:

| $y=f(x)$ | domain | range (principal branch) |
|---|---|---|
| $y=\sin^{-1}x$ | $-1\le x\le 1$ | $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$ |
| $y=\cos^{-1}x$ | $-1\le x\le 1$ | $0\le y\le\pi$ |
| $y=\tan^{-1}x$ | $x\in\mathbb{R}$ | $-\frac{\pi}{2}< y<\frac{\pi}{2}$ |

**The key tool.**

Every derivation below rests on one idea: rewrite $y=\sin^{-1}x$ (say) as $x=\sin y$, differentiate, and use the result from an earlier section that, provided $\frac{dx}{dy}\neq 0$,

$$\frac{dy}{dx}=\frac{1}{dx/dy}.$$

**Derivative of $\sin^{-1}x$.**

Let $y=\sin^{-1}x$. Then $x=\sin y$ with $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$. Differentiating $x=\sin y$ with respect to $y$ gives $\frac{dx}{dy}=\cos y$, so

$$\frac{dy}{dx}=\frac{1}{\cos y}.$$

Now $\cos^2 y=1-\sin^2 y=1-x^2$, and on the principal branch $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$ the cosine is non-negative, so $\cos y=\sqrt{1-x^2}$. Therefore

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}},\ -1<x<1.$$

The endpoints $x=\pm 1$ are excluded: there $\sqrt{1-x^2}=0$, and the graph of $\sin^{-1}x$ has a vertical tangent, so the gradient is undefined.

**Derivative of $\cos^{-1}x$.**

Let $y=\cos^{-1}x$, so $x=\cos y$ with $0\le y\le\pi$. Then $\frac{dx}{dy}=-\sin y$, giving

$$\frac{dy}{dx}=\frac{1}{-\sin y}=-\frac{1}{\sin y}.$$

Here $\sin^2 y=1-\cos^2 y=1-x^2$, and on $0\le y\le\pi$ the sine is non-negative, so $\sin y=\sqrt{1-x^2}$. Therefore

$$\frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{1-x^2}},\ -1<x<1.$$

This is the exact negative of the $\sin^{-1}$ derivative — a fact that reappears in Q17.

**Derivative of $\tan^{-1} x$.**

Let $y = \tan^{-1} x$, so $x = \tan y$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Then

$$\frac{dx}{dy} = sec^2 y = 1 + \tan^2 y = 1 + x^2,$$

using the identity $sec^2 y = 1 + \tan^2 y$. Hence

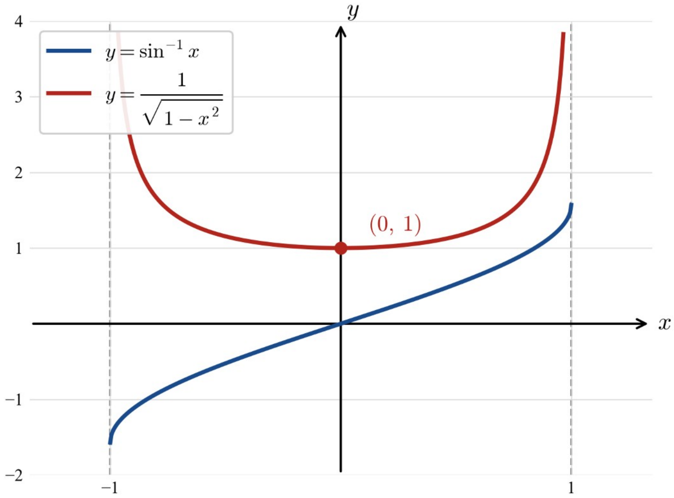
$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, x \in \mathbb{R}.$$

Unlike the other two, this derivative is defined for **all** real $x$: the denominator $1+x^2$ is never zero, matching the fact that the graph of $\tan^{-1} x$ has no vertical tangents.

**Standard results.**

| $f(x)$ | $f'(x)$ | valid for |
|---|---|---|
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ | $-1 < x < 1$ |
| $\cos^{-1} x$ | $-\frac{1}{\sqrt{1-x^2}}$ | $-1 < x < 1$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ | $x \in \mathbb{R}$ |

The figure below shows $y = \sin^{-1} x$ together with its gradient function $y = \frac{1}{\sqrt{1-x^2}}$. Where $\sin^{-1} x$ is steepest (near $x = \pm 1$) the gradient function grows without bound; at $x = 0$ the curve $y = \sin^{-1} x$ has gradient $1$, and the gradient function passes through $(0, 1)$.




**The general forms with a constant $a$.**

Let $a > 0$. Replacing $x$ by $\frac{x}{a}$ and applying the chain rule with $u = \frac{x}{a}$ (so $\frac{du}{dx} = \frac{1}{a}$) gives, for the sine,

$$\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{1-(x/a)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{(a^2-x^2)/a^2}} \cdot \frac{1}{a} = \frac{a}{\sqrt{a^2-x^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}},$$

where $\sqrt{(a^2-x^2)/a^2}=\dfrac{\sqrt{a^2-x^2}}{a}$ because $a>0$. The same substitution applied to $\cos^{-1}$ and $\tan^{-1}$ yields the companion results:

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}},\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right)=-\frac{1}{\sqrt{a^2-x^2}},(-a<x<a),$$

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)=\frac{a}{a^2+x^2},(x\in\mathbb{R}).$$

For the $\tan^{-1}$ form the working is $\dfrac{1}{1+(x/a)^2}\cdot\dfrac{1}{a}=\dfrac{a^2}{a^2+x^2}\cdot\dfrac{1}{a}=\dfrac{a}{a^2+x^2}$.

**The chain rule for a general inside function.**

More generally, if $u=g(x)$ is differentiable then

$$\frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\frac{d}{dx}\cos^{-1}u=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}.$$

The forms with $\dfrac{x}{a}$ are just the special case $u=\dfrac{x}{a}$. **Domain of validity:** for $\sin^{-1}u$ and $\cos^{-1}u$ the result holds only where $-1<u<1$ (so that $1-u^2>0$); the $\tan^{-1}u$ result holds wherever $g$ is differentiable. In practice, find the values of $x$ for which the derivative is defined by solving the inequality $-1<g(x)<1$ for the two square-root forms. Take care to distinguish this set from the domain of $y$ itself: $y=\sin^{-1}u$ is defined whenever $-1\le u\le 1$, but its derivative is undefined at every $x$ for which $u=\pm 1$.

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## Worked examples

### *Example 1 — scaffolded*

Differentiate $y=\sin^{-1}(4x)$, and state the values of $x$ for which the derivative is defined.

| Working | Comment |
|---|---|
| $y=\sin^{-1}u$ where $u=4x$. | Recognise a composite; name the inside function $u$. |
| $\dfrac{du}{dx}=4$. | Differentiate the inside function. |
| $\dfrac{dy}{du}=\dfrac{1}{\sqrt{1-u^2}}$. | Standard derivative of $\sin^{-1}$. |
| $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\dfrac{1}{\sqrt{1-u^2}}\cdot 4=\dfrac{4}{\sqrt{1-16x^2}}$. | Chain rule, then substitute $u=4x$ so $u^2=16x^2$. |
| Defined when $1-16x^2>0$, i.e. $-\dfrac{1}{4}<x<\dfrac{1}{4}$. | The expression under the square root must be strictly positive. |

### *Example 2 — scaffolded*

Differentiate $y=\tan^{-1}\left(\dfrac{x}{3}\right)$, and state the values of $x$ for which the derivative is defined.

| Working | Comment |
|---|---|
| $y=\tan^{-1}u$ where $u=\dfrac{x}{3}$, so $\dfrac{du}{dx}=\dfrac{1}{3}$. | Identify the inside function and its derivative. |

| Working | Comment |
| --- | --- |
| $\frac{dy}{dx} = \frac{1}{1+u^2} \cdot \frac{1}{3} = \frac{1}{1+x^2/9} \cdot \frac{1}{3}$. | Chain rule with the derivative of $\tan^{-1}$. |
| $= \frac{9}{9+x^2} \cdot \frac{1}{3} = \frac{3}{9+x^2}$. | Multiply numerator and denominator of the first fraction by 9, then simplify. |
| This matches $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$ with $a=3$. | Cross-check against the general form. |
| Defined for all real $x$, since $9+x^2>0$ always. | The $\tan^{-1}$ forms have no domain restriction. |

### *Example 3 — unscaffolded*

Differentiate $y=\cos^{-1}(x^2)$ and state the values of $x$ for which the derivative is defined.

Write $u=x^2$, so $\frac{du}{dx}=2x$. Using $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = -\frac{2x}{\sqrt{1-x^4}}.$$

The result is valid where $1-x^4>0$, that is $x^4<1$, which gives $-1<x<1$.

$\therefore \frac{dy}{dx} = -\frac{2x}{\sqrt{1-x^4}}$, valid for $-1<x<1$.

### *Example 4 — unscaffolded*

Differentiate $y=x\tan^{-1}x$.

This is a product, so apply the product rule with $u=x$ and $v=\tan^{-1}x$. Then $\frac{du}{dx}=1$ and $\frac{dv}{dx}=\frac{1}{1+x^2}$, so

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = (1)\tan^{-1}x + x \cdot \frac{1}{1+x^2} = \tan^{-1}x + \frac{x}{1+x^2}.$$

Both terms are defined for every real $x$.

$\therefore \frac{dy}{dx} = \tan^{-1}x + \frac{x}{1+x^2}$, for all real $x$.

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## Practice questions

### *Scaffolded*

**Q1.** Derive the derivative of $\tan^{-1}x$ starting from $x=\tan y$. (a) Write $y=\tan^{-1}x$ as $x=\tan y$ and state the range of $y$. (b) Find $\frac{dx}{dy}$. (c) Use the identity $sec^2 y = 1+\tan^2 y$ to write $\frac{dx}{dy}$ in terms of $x$. (d) Hence state $\frac{dy}{dx}$ and the values of $x$ for which it is defined.

**Q2.** Differentiate $y=\sin^{-1}(5x)$. (a) Identify the inside function $u$ and find $\frac{du}{dx}$. (b) Write down $\frac{dy}{du}$. (c) Apply the chain rule to find $\frac{dy}{dx}$. (d) State the values of $x$ for which $\frac{dy}{dx}$ is defined.

**Q3.** Differentiate $y=\cos^{-1}\left(\frac{x}{4}\right)$. (a) Compare with $\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right)$ and state $a$. (b) Write down $\frac{dy}{dx}$. (c) State the values of $x$ for which $\frac{dy}{dx}$ is defined.

**Q4.** Differentiate $y=\tan^{-1}\left(\frac{x}{2}\right)$. (a) State the value of $a$. (b) Write $\frac{dy}{dx}$ using the form $\frac{a}{a^2+x^2}$. (c) State the values of $x$ for which $\frac{dy}{dx}$ is defined.

**Q5.** Differentiate $y=\sin^{-1}(x^3)$. (a) Let $u=x^3$ and find $\frac{du}{dx}$. (b) Apply the chain rule to find $\frac{dy}{dx}$. (c) State the values of $x$ for which $\frac{dy}{dx}$ is defined.

**Q6.** Consider the curve $y=\tan^{-1}(3x)$. (a) Find $\frac{dy}{dx}$. (b) Find the gradient of the curve at $x=0$. (c) Hence find the equation of the tangent to the curve at the origin.

***Unscaffolded***

**Q7.** Differentiate $y=\cos^{-1}(5x)$ and state the values of $x$ for which the derivative is defined.

**Q8.** Differentiate $y=\sin^{-1}\left(\frac{x}{7}\right)$ and state the values of $x$ for which the derivative is defined.

**Q9.** Differentiate $y=\tan^{-1}\left(\frac{x}{6}\right)$ and state the values of $x$ for which the derivative is defined.

**Q10.** Differentiate $y=\sin^{-1}(2x)$ and state the values of $x$ for which the derivative is defined.

**Q11.** Differentiate $y=\tan^{-1}(x^2)$.

**Q12.** Differentiate $y=\cos^{-1}(3x)$ and state the values of $x$ for which the derivative is defined.

**Q13.** Differentiate $y=x\sin^{-1}x$.

**Q14.** Differentiate $y=\tan^{-1}(\sqrt{x})$ and state the values of $x$ for which the derivative is defined.

**Q15.** The curve $y=\sin^{-1}\left(\frac{x}{4}\right)$ is defined for $-4\le x\le 4$. Find the exact gradient of the curve at $x=2$.

**Q16.** Differentiate $y=\left(\tan^{-1}x\right)^2$.

**Q17.** Show that $\frac{d}{dx}\left(\sin^{-1}x+\cos^{-1}x\right)=0$ for $-1<x<1$, and explain what this tells you about the value of $\sin^{-1}x+\cos^{-1}x$.

**Q18.** Differentiate $y=x^2\cos^{-1}x$.

### Answers

**Q1.** $\frac{dx}{dy}=sec^2 y=1+x^2$, so $\frac{dy}{dx}=\frac{1}{1+x^2}$, defined for all real $x$. **Q2.** $\frac{dy}{dx}=\frac{5}{\sqrt{1-25x^2}}$, defined for $-\frac{1}{5}<x<\frac{1}{5}$. **Q3.** $a=4$; $\frac{dy}{dx}=-\frac{1}{\sqrt{16-x^2}}$, defined for $-4<x<4$. **Q4.** $a=2$; $\frac{dy}{dx}=\frac{2}{4+x^2}$, defined for all real $x$. **Q5.** $\frac{dy}{dx}=\frac{3x^2}{\sqrt{1-x^6}}$, defined for $-1<x<1$. **Q6.** (a) $\frac{dy}{dx}=\frac{3}{1+9x^2}$; (b) gradient $=3$; (c) $y=3x$. **Q7.** $\frac{dy}{dx}=-\frac{5}{\sqrt{1-25x^2}}$, defined for $-\frac{1}{5}<x<\frac{1}{5}$. **Q8.** $\frac{dy}{dx}=\frac{1}{\sqrt{49-x^2}}$, defined for $-7<x<7$. **Q9.** $\frac{dy}{dx}=\frac{6}{36+x^2}$, defined for all real $x$. **Q10.** $\frac{dy}{dx}=\frac{2}{\sqrt{1-4x^2}}$, defined for $-\frac{1}{2}<x<\frac{1}{2}$. **Q11.** $\frac{dy}{dx}=\frac{2x}{1+x^4}$. **Q12.** $\frac{dy}{dx}=-\frac{3}{\sqrt{1-9x^2}}$, defined for $-\frac{1}{3}<x<\frac{1}{3}$. **Q13.** $\frac{dy}{dx}=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$. **Q14.** $\frac{dy}{dx}=\frac{1}{2\sqrt{x}(1+x)}$, defined for $x>0$. **Q15.** $\frac{\sqrt{3}}{6}$. **Q16.** $\frac{dy}{dx}=\frac{2\tan^{-1}x}{1+x^2}$. **Q17.** The derivative is $\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}=0$, so the sum is constant on $-1<x<1$; since it equals $\frac{\pi}{2}$ at $x=0$, $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1<x<1$. Checking the endpoints directly ($\frac{\pi}{2}+0$ at $x=1$ and $-\frac{\pi}{2}+\pi$ at $x=-1$) extends the result to all $-1\le x\le 1$. **Q18.** $\frac{dy}{dx}=2x\cos^{-1}x-\frac{x^2}{\sqrt{1-x^2}}$.

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### Fully-worked solutions

**Q1.** Let $y=\tan^{-1}x$, so $x=\tan y$ with $-\frac{\pi}{2}<y<\frac{\pi}{2}$ (the range of $\tan^{-1}$). Differentiating $x=\tan y$ with respect to $y$, $\frac{dx}{dy}=sec^2 y$. Using $sec^2 y=1+\tan^2 y$ and $\tan y=x$,

$$\frac{dx}{dy}=1+\tan^2 y=1+x^2.$$

By the earlier result $\frac{dy}{dx}=\frac{1}{dx/dy}$, so $\frac{dy}{dx}=\frac{1}{1+x^2}$. Since $1+x^2>0$ for every real $x$, the derivative is defined for all real $x$.

**Q2.** Let $u=5x$, so $y=\sin^{-1}u$ and $\frac{du}{dx}=5$. Then $\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}}$, and by the chain rule

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-u^2}}\cdot 5=\frac{5}{\sqrt{1-25x^2}}.$$

This requires $1-25x^2>0$, i.e. $x^2<\frac{1}{25}$, so $-\frac{1}{5}<x<\frac{1}{5}$.

**Q3.** Comparing $y=\cos^{-1}\left(\frac{x}{4}\right)$ with $\cos^{-1}\left(\frac{x}{a}\right)$ gives $a=4$. Using $\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right)=-\frac{1}{\sqrt{a^2-x^2}}$,

$$\frac{dy}{dx}=-\frac{1}{\sqrt{16-x^2}}.$$

This is defined where $16-x^2>0$, i.e. $-4<x<4$.

**Q4.** Here $a=2$. Using $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)=\frac{a}{a^2+x^2}$,

$$\frac{dy}{dx}=\frac{2}{2^2+x^2}=\frac{2}{4+x^2}.$$

Since $4+x^2>0$ for every real $x$, the derivative is defined for all real $x$.

**Q5.** Let $u=x^3$, so $\frac{du}{dx}=3x^2$. With $\frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$,

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(x^3)^2}}\cdot 3x^2=\frac{3x^2}{\sqrt{1-x^6}}.$$

The result is valid where $1-x^6>0$, i.e. $x^6<1$, which gives $-1<x<1$.

**Q6.** (a) With $u=3x$, $\frac{du}{dx}=3$, so $\frac{dy}{dx}=\frac{1}{1+(3x)^2}\cdot 3=\frac{3}{1+9x^2}$. (b) At $x=0$, $\frac{dy}{dx}=\frac{3}{1+0}=3$. (c) At $x=0$, $y=\tan^{-1}(0)=0$, so the tangent passes through the origin with gradient $3$. Its equation is $y=3x$.

**Q7.** Let $u=5x$, so $\frac{du}{dx}=5$ and $\frac{d}{dx}\cos^{-1}u=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$. Hence

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-25x^2}}\cdot 5=-\frac{5}{\sqrt{1-25x^2}},$$

defined where $1-25x^2>0$, i.e. $-\frac{1}{5}<x<\frac{1}{5}$.

**Q8.** With $a=7$, $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$, so

$$\frac{dy}{dx}=\frac{1}{\sqrt{49-x^2}},$$

defined where $49-x^2>0$, i.e. $-7<x<7$.

**Q9.** With $a=6$, $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)=\frac{a}{a^2+x^2}$, so

$$\frac{dy}{dx}=\frac{6}{36+x^2},$$

which is defined for all real $x$, since $36+x^2>0$.

**Q10.** Let $u=2x$, so $\frac{du}{dx}=2$. Then

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x)^2}}\cdot 2=\frac{2}{\sqrt{1-4x^2}},$$

defined where $1-4x^2>0$, i.e. $-\frac{1}{2}<x<\frac{1}{2}$.

**Q11.** Let $u=x^2$, so $\frac{du}{dx}=2x$. Using $\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$,

$$\frac{dy}{dx}=\frac{1}{1+(x^2)^2}\cdot 2x=\frac{2x}{1+x^4}.$$

This is defined for all real $x$, since $1+x^4>0$.

**Q12.** Let $u=3x$, so $\frac{du}{dx}=3$. Then

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-(3x)^2}}\cdot 3=-\frac{3}{\sqrt{1-9x^2}},$$

defined where $1-9x^2>0$, i.e. $-\frac{1}{3}<x<\frac{1}{3}$.

**Q13.** This is a product, so apply the product rule with $u=x$ and $v=\sin^{-1}x$. Then $\frac{du}{dx}=1$ and $\frac{dv}{dx}=\frac{1}{\sqrt{1-x^2}}$, so

$$\frac{dy}{dx}=(1)\sin^{-1}x+x\cdot\frac{1}{\sqrt{1-x^2}}=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}.$$

(The second term restricts the derivative to $-1<x<1$.)

**Q14.** Let $u=\sqrt{x}=x^{1/2}$, so $\frac{du}{dx}=\frac{1}{2\sqrt{x}}$. Using $\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$ and $u^2=x$,

$$\frac{dy}{dx}=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}(1+x)}.$$

This requires $\sqrt{x}$ and $\frac{1}{\sqrt{x}}$ to be defined, so $x>0$.

**Q15.** Differentiating $y=\sin^{-1}\left(\frac{x}{4}\right)$ with $a=4$ gives $\frac{dy}{dx}=\frac{1}{\sqrt{16-x^2}}$. At $x=2$,

$$\frac{dy}{dx}=\frac{1}{\sqrt{16-4}}=\frac{1}{\sqrt{12}}=\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}.$$

**Q16.** Let $u=\tan^{-1}x$, so $y=u^2$ and $\frac{du}{dx}=\frac{1}{1+x^2}$. By the chain rule $\frac{dy}{dx}=2u\frac{du}{dx}$, so

$$\frac{dy}{dx}=2\tan^{-1}x\cdot\frac{1}{1+x^2}=\frac{2\tan^{-1}x}{1+x^2}.$$

**Q17.** Differentiating term by term on $-1<x<1$,

$$\frac{d}{dx}\left(\sin^{-1}x+\cos^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}+\left(-\frac{1}{\sqrt{1-x^2}}\right)=0.$$

A function with zero derivative on an interval is constant there, so $\sin^{-1}x+\cos^{-1}x$ is constant on $-1<x<1$. Evaluating at $x=0$ gives $\sin^{-1}(0)+\cos^{-1}(0)=0+\frac{\pi}{2}=\frac{\pi}{2}$, so $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1<x<1$. The argument says nothing about $x=\pm 1$, where neither derivative exists, so evaluate there directly: $\sin^{-1}(1)+\cos^{-1}(1)=\frac{\pi}{2}+0=\frac{\pi}{2}$ and $\sin^{-1}(-1)+\cos^{-1}(-1)=-\frac{\pi}{2}+\pi=\frac{\pi}{2}$. Hence $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for all $x$ in $[-1,1]$.

**Q18.** This is a product, so apply the product rule with $u = x^2$ and $v = \cos^{-1} x$. Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$, so

$$\frac{dy}{dx} = (2x)\cos^{-1} x + x^2\left(-\frac{1}{\sqrt{1-x^2}}\right) = 2x\cos^{-1} x - \frac{x^2}{\sqrt{1-x^2}}.$$

(The second term restricts the derivative to $-1 < x < 1$.)

## Theory

Differentiating a function $y=f(x)$ produces a new function, the derivative $f'(x)$, which records the **gradient** of the original curve at each point. Because $f'$ is itself a function of $x$, it too can be differentiated. The result is the **second derivative**: the derivative of the derivative. It measures how fast the gradient is changing.

**Notation.**

Several equivalent symbols are used for the second derivative of $y=f(x)$:

$$f''(x), \frac{d^2y}{dx^2}, y''.$$

The form $\frac{d^2y}{dx^2}$ comes from applying the operator $\frac{d}{dx}$ twice,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right),$$

so the small 2's are **not powers**: the one on $d$ in the numerator and the one on $x$ in the denominator are simply a shorthand for "differentiate with respect to $x$ twice". In the same way $f''(x)$ means $\frac{d}{dx}\left(f'(x)\right)$, and the process can be repeated to give third and higher derivatives $f'''(x)$, $\frac{d^3y}{dx^3}$, and so on; this section works only with the second.

**Computing a second derivative.**

To find $f''(x)$, differentiate $f(x)$ to get $f'(x)$, then differentiate that result once more. Every rule already available — the rules for powers, the product, quotient and chain rules — may be needed at each stage. For example, if $f(x)=x^5$ then $f'(x)=5x^4$ and $f''(x)=20x^3$; if $f(x)=e^{kx}$ then $f'(x)=ke^{kx}$ and $f''(x)=k^2e^{kx}$; and if $f(x)=\sin kx$ then $f'(x)=k\cos kx$ and $f''(x)=-k^2\sin kx$.

The **derivatives of the inverse circular functions**, established in an earlier section, are also differentiated a second time in the same way. Writing $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ for the inverse circular (arcsine, arccosine, arctangent) functions, recall

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}, \frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}, \frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}.$$

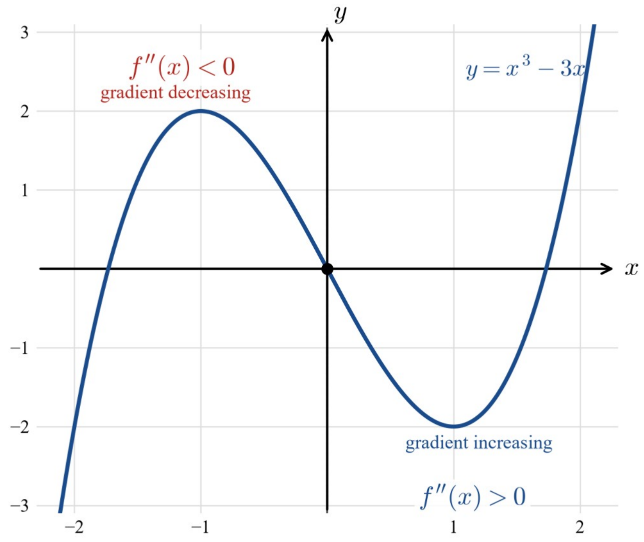
Differentiating each of these again (using the chain rule on the powers of $1-x^2$, or the quotient rule) gives, for instance,

$$\frac{d^2}{dx^2}\left(\tan^{-1}x\right)=\frac{d}{dx}\left((1+x^2)^{-1}\right)=-\frac{2x}{(1+x^2)^2}.$$

**Interpreting the second derivative.**

Since $f'(x)$ is the rate of change of $y$ with respect to $x$, the second derivative $f''(x)$ is the **rate of change of the gradient**. Reading its sign:

- where $f''(x)>0$, the gradient $f'(x)$ is **increasing** — the curve bends upward;
- where $f''(x)<0$, the gradient $f'(x)$ is **decreasing** — the curve bends downward.




The diagram shows $y = x^3 - 3x$, for which $f''(x) = 6x$. To the left of the origin $f''(x) < 0$: the gradient is falling and the curve bends downward. To the right $f''(x) > 0$: the gradient is rising and the curve bends upward. The precise language of **concavity**, and the points where the bending changes (**points of inflection**, where $f''$ changes sign), are developed fully in a later section; here we note only this link between the sign of $f''$ and the shape of the graph.

**Acceleration.**

One important use of the second derivative is in motion. If a particle has displacement $x(t)$ at time $t$, its **velocity** is the rate of change of displacement, $\dot{x} = \frac{dx}{dt}$, and its **acceleration** is the rate of change of velocity,

$$\ddot{x} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

So acceleration is the second derivative of displacement with respect to time (the dot notation $\dot{x}$, $\ddot{x}$ is standard for differentiation with respect to time). This kinematic application is taken up in detail in a later chapter; here it simply illustrates why a second derivative is worth computing.

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## Worked examples

### *Example 1 — scaffolded*

For $f(x) = 2x^4 - 3x^3 + x - 5$, find $f''(x)$, evaluate $f''(1)$, and state what the sign of $f''(1)$ tells you about the gradient at $x = 1$.

| Working | Comment |
|---|---|
| $f'(x) = 8x^3 - 9x^2 + 1.$ | Differentiate once, term by term. |
| $f''(x) = 24x^2 - 18x.$ | Differentiate $f'(x)$ again to get the second derivative. |
| $f''(1) = 24(1)^2 - 18(1) = 24 - 18 = 6.$ | Substitute $x = 1$. |
| $f''(1) = 6 > 0$, so at $x = 1$ the gradient $f'(x)$ is increasing. | The sign of $f''$ is the rate of change of the gradient. |

### *Example 2 — scaffolded*

For $y = x^2 e^x$, find $\frac{d^2y}{dx^2}$.

| Working | Comment |
|---|---|
| $\frac{dy}{dx}=2xe^x+x^2e^x=e^x(x^2+2x)$. | Product rule with $u=x^2$, $v=e^x$: $(uv)'=u'v+uv'$. |
| $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(e^x(x^2+2x)\right)$. | Differentiate $\frac{dy}{dx}$ once more. |
| $=e^x(x^2+2x)+e^x(2x+2)$. | Product rule again, on $e^x$ and $(x^2+2x)$. |
| $=e^x(x^2+4x+2)$. | Collect like terms inside the bracket. |

### *Example 3 — unscaffolded*

For $y=3\sin 2x$, find $\frac{d^2y}{dx^2}$ and show that it equals $-4y$.

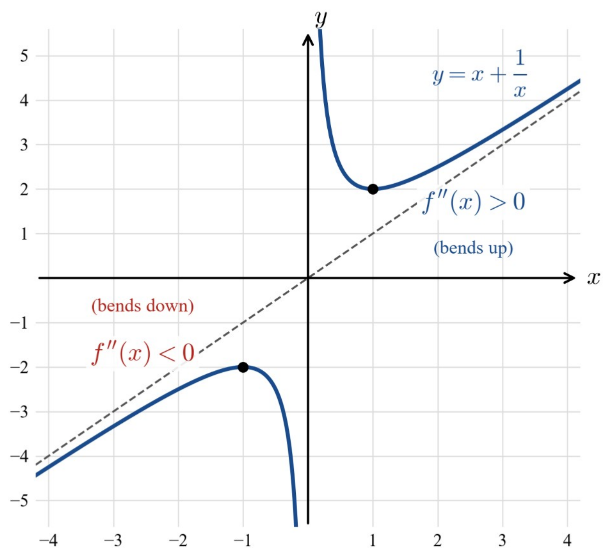
Differentiating with the chain rule, $\frac{dy}{dx}=3(\cos 2x)(2)=6\cos 2x$. Differentiating again, $\frac{d^2y}{dx^2}=6(-\sin 2x)(2)=-12\sin 2x$. Since $y=3\sin 2x$, we have $-4y=-4(3\sin 2x)=-12\sin 2x$, which is exactly $\frac{d^2y}{dx^2}$.

$$\therefore \frac{d^2y}{dx^2}=-12\sin 2x=-4y.$$

### *Example 4 — unscaffolded*

For the rational function $f(x)=x+\frac{1}{x}$, find $f''(x)$, determine where it is positive and where it is negative, and describe what this says about the graph.

Writing $f(x)=x+x^{-1}$, the first derivative is $f'(x)=1-x^{-2}$, and differentiating again gives $f''(x)=2x^{-3}=\frac{2}{x^3}$. The domain excludes $x=0$. For $x>0$ the cube $x^3>0$, so $f''(x)>0$: the gradient is increasing and the curve bends upward. For $x<0$ the cube $x^3<0$, so $f''(x)<0$: the gradient is decreasing and the curve bends downward.




$$\therefore f''(x)=\frac{2}{x^3}$$, which is positive for $x>0$ and negative for $x<0$.

## Practice questions

### *Scaffolded*

**Q1.** For $f(x)=x^5-2x^3+4x$: (a) find $f'(x)$; (b) find $f''(x)$; (c) evaluate $f''(1)$.

**Q2.** For $y=e^{3x}$: (a) find $\frac{dy}{dx}$; (b) find $\frac{d^2y}{dx^2}$; (c) evaluate $\frac{d^2y}{dx^2}$ at $x=0$.

**Q3.** For $y=\cos 4x$: (a) find $\frac{dy}{dx}$; (b) find $\frac{d^2y}{dx^2}$; (c) express $\frac{d^2y}{dx^2}$ as a multiple of $y$.

**Q4.** For $y=xe^{2x}$: (a) find $\frac{dy}{dx}$ (product rule); (b) find $\frac{d^2y}{dx^2}$; (c) evaluate $\frac{d^2y}{dx^2}$ at $x=0$.

**Q5.** For $y=\sin^{-1}x$: (a) write down $\frac{dy}{dx}$; (b) find $\frac{d^2y}{dx^2}$; (c) evaluate $\frac{d^2y}{dx^2}$ at $x=\frac{1}{2}$.

**Q6.** For $f(x)=x^3-6x^2+9x+1$: (a) find $f'(x)$; (b) find $f''(x)$; (c) find the values of $x$ for which $f''(x)>0$ and for which $f''(x)<0$, and state what this says about the gradient.

### *Unscaffolded*

**Q7.** For $f(x)=4x^3-7x^2+2x-9$, find $f''(x)$ and evaluate $f''(2)$.

**Q8.** For $y=x-\frac{4}{x^2}$, find $\frac{d^2y}{dx^2}$.

**Q9.** For $y=e^{-2x}$, find $\frac{d^2y}{dx^2}$ and show that it equals $4y$.

**Q10.** For $y=2\sin 3x+\cos 3x$, find $\frac{d^2y}{dx^2}$ and show that it equals $-9y$.

**Q11.** For $y=x^2e^{-x}$, find $\frac{d^2y}{dx^2}$ and evaluate it at $x=0$.

**Q12.** For $y=\frac{x}{x+1}$, find $\frac{d^2y}{dx^2}$.

**Q13.** For $y=\tan^{-1}x$, find $\frac{d^2y}{dx^2}$ and evaluate it at $x=1$.

**Q14.** For $y=e^x\sin x$, find $\frac{d^2y}{dx^2}$.

**Q15.** For $y=(2x-1)^4$, find $\frac{d^2y}{dx^2}$ and evaluate it at $x=1$.

**Q16.** For $y=\cos^{-1}x$, find $\frac{d^2y}{dx^2}$ and evaluate it at $x=\frac{1}{2}$.

**Q17.** A particle moves in a straight line with displacement $x(t)=t^3-6t^2+5t$ metres at time $t$ seconds. Its acceleration is $\ddot{x}=\frac{d^2x}{dt^2}$. (a) Find $\ddot{x}$ in terms of $t$. (b) Find the acceleration when $t=1$. (c) Find the time at which the acceleration is zero.

**Q18.** For $f(x) = x^4 - 6x^2$, find $f''(x)$, determine the values of $x$ for which $f''(x) > 0$ and for which $f''(x) < 0$, and describe what this says about the gradient.

## Answers

**Q1.** (a) $f'(x)=5x^4-6x^2+4$; (b) $f''(x)=20x^3-12x$; (c) $f''(1)=8$. **Q2.** (a) $\frac{dy}{dx}=3e^{3x}$; (b) $\frac{d^2y}{dx^2}=9e^{3x}$; (c) 9. **Q3.** (a) $-4\sin 4x$; (b) $-16\cos 4x$; (c) $\frac{d^2y}{dx^2}=-16y$. **Q4.** (a) $\frac{dy}{dx}=e^{2x}(1+2x)$; (b) $\frac{d^2y}{dx^2}=4e^{2x}(x+1)$; (c) 4. **Q5.** (a) $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$; (b) $\frac{d^2y}{dx^2}=\frac{x}{(1-x^2)^{3/2}}$; (c) $\frac{4\sqrt{3}}{9}$. **Q6.** (a) $f'(x)=3x^2-12x+9$; (b) $f''(x)=6x-12$; (c) $f''(x)>0$ for $x>2$ and $f''(x)<0$ for $x<2$; the gradient is increasing for $x>2$ and decreasing for $x<2$. **Q7.** $f''(x)=24x-14$; $f''(2)=34$. **Q8.** $\frac{d^2y}{dx^2}=-\frac{24}{x^4}$. **Q9.** $\frac{d^2y}{dx^2}=4e^{-2x}=4y$. **Q10.** $\frac{d^2y}{dx^2}=-18\sin 3x-9\cos 3x=-9y$. **Q11.** $\frac{d^2y}{dx^2}=e^{-x}(x^2-4x+2)$; at $x=0$ it is 2. **Q12.** $\frac{d^2y}{dx^2}=-\frac{2}{(x+1)^3}$. **Q13.** $\frac{d^2y}{dx^2}=-\frac{2x}{(1+x^2)^2}$; at $x=1$ it is $-\frac{1}{2}$. **Q14.** $\frac{d^2y}{dx^2}=2e^x\cos x$. **Q15.** $\frac{d^2y}{dx^2}=48(2x-1)^2$; at $x=1$ it is 48. **Q16.** $\frac{d^2y}{dx^2}=-\frac{x}{(1-x^2)^{3/2}}$; at $x=\frac{1}{2}$ it is $-\frac{4\sqrt{3}}{9}$. **Q17.** (a) $\ddot{x}=6t-12$; (b) $-6\text{ m/s}^2$; (c) $t=2$ s. **Q18.** $f''(x)=12x^2-12$; $f''(x)>0$ for $x<-1$ or $x>1$, and $f''(x)<0$ for $-1<x<1$; the gradient is increasing where $f''>0$ and decreasing where $f''<0$.

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## Fully-worked solutions

**Q1.** Differentiate term by term: $f'(x)=5x^4-6x^2+4$. Differentiate again: $f''(x)=20x^3-12x$. At $x=1$, $f''(1)=20(1)^3-12(1)=20-12=8$.

**Q2.** With $y=e^{3x}$, the chain rule gives $\frac{dy}{dx}=3e^{3x}$, and differentiating again $\frac{d^2y}{dx^2}=3\times 3e^{3x}=9e^{3x}$. At $x=0$, $e^0=1$, so the value is 9.

**Q3.** $\frac{dy}{dx}=-4\sin 4x$ (chain rule on $\cos 4x$). Differentiating again, $\frac{d^2y}{dx^2}=-4(4\cos 4x)=-16\cos 4x$. Since $y=\cos 4x$, this is $\frac{d^2y}{dx^2}=-16y$.

**Q4.** Product rule with $u=x$, $v=e^{2x}$: $\frac{dy}{dx}=1\cdot e^{2x}+x\cdot 2e^{2x}=e^{2x}(1+2x)$. Differentiate again with the product rule on $e^{2x}$ and $(1+2x)$: $\frac{d^2y}{dx^2}=2e^{2x}(1+2x)+e^{2x}(2)=e^{2x}(4x+4)=4e^{2x}(x+1)$. At $x=0$ the value is $4e^0(1)=4$.

**Q5.** From the standard result, $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-1/2}$. Differentiating with the chain rule,

$$\frac{d^2y}{dx^2}=-1/2(1-x^2)^{-3/2}\cdot(-2x)=\frac{x}{(1-x^2)^{3/2}}.$$

At $x=\frac{1}{2}$: $1-x^2=\frac{3}{4}$, so $(1-x^2)^{3/2}=\left(\frac{3}{4}\right)^{3/2}=\frac{3\sqrt{3}}{8}$, and $\frac{d^2y}{dx^2}=\frac{1/2}{3\sqrt{3}/8}=\frac{1}{2}\cdot\frac{8}{3\sqrt{3}}=\frac{4}{3\sqrt{3}}=\frac{4\sqrt{3}}{9}$.

**Q6.** $f'(x)=3x^2-12x+9$ and $f''(x)=6x-12$. Then $f''(x)>0$ means $6x-12>0$, i.e. $x>2$; and $f''(x)<0$ means $x<2$. So the gradient $f'(x)$ is increasing for $x>2$ and decreasing for $x<2$.

**Q7.** $f'(x)=12x^2-14x+2$, so $f''(x)=24x-14$. At $x=2$, $f''(2)=48-14=34$.

**Q8.** Write $y = x - 4x^{-2}$. Then $\frac{dy}{dx} = 1 + 8x^{-3}$ and $\frac{d^2y}{dx^2} = -24x^{-4} = -\frac{24}{x^4}$.

**Q9.** $\frac{dy}{dx} = -2e^{-2x}$ and $\frac{d^2y}{dx^2} = (-2)(-2)e^{-2x} = 4e^{-2x}$. Since $y = e^{-2x}$, this is $4y$.

**Q10.** $\frac{dy}{dx} = 6\cos 3x - 3\sin 3x$. Differentiating again, $\frac{d^2y}{dx^2} = -18\sin 3x - 9\cos 3x$. Factorising, $-18\sin 3x - 9\cos 3x = -9(2\sin 3x + \cos 3x) = -9y$.

**Q11.** Product rule with $u = x^2$, $v = e^{-x}$: $\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)$. Differentiate again:

$\frac{d^2y}{dx^2} = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 4x + 2)$. At $x = 0$ the value is $e^0(2) = 2$.

**Q12.** Quotient rule with $u = x$, $v = x + 1$: $\frac{dy}{dx} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} = (x+1)^{-2}$. Differentiate again with the chain rule: $\frac{d^2y}{dx^2} = -2(x+1)^{-3} = -\frac{2}{(x+1)^3}$.

**Q13.** From the standard result, $\frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1}$. Differentiating with the chain rule,

$\frac{d^2y}{dx^2} = -(1+x^2)^{-2} \cdot 2x = -\frac{2x}{(1+x^2)^2}$. At $x = 1$: $\frac{d^2y}{dx^2} = -\frac{2}{(2)^2} = -\frac{1}{2}$.

**Q14.** Product rule: $\frac{dy}{dx} = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$. Differentiate again:

$\frac{d^2y}{dx^2} = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = e^x(2\cos x) = 2e^x \cos x$.

**Q15.** Chain rule: $\frac{dy}{dx} = 4(2x-1)^3 \cdot 2 = 8(2x-1)^3$. Differentiate again: $\frac{d^2y}{dx^2} = 8 \cdot 3(2x-1)^2 \cdot 2 = 48(2x-1)^2$.

At $x = 1$, $\frac{d^2y}{dx^2} = 48(1)^2 = 48$.

**Q16.** From the standard result, $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} = -(1-x^2)^{-1/2}$. Differentiating with the chain rule,

$$\frac{d^2y}{dx^2} = -(-1/2)(1-x^2)^{-3/2} \cdot (-2x) = -\frac{x}{(1-x^2)^{3/2}}.$$

At $x = \frac{1}{2}$, as in Q5 the denominator is $\frac{3\sqrt{3}}{8}$, so the value is $-\frac{4\sqrt{3}}{9}$.

**Q17.** Displacement $x(t) = t^3 - 6t^2 + 5t$. Velocity is $\dot{x} = 3t^2 - 12t + 5$, and acceleration is the second derivative $\ddot{x} = 6t - 12$. (a) $\ddot{x} = 6t - 12$. (b) At $t = 1$, $\ddot{x} = 6 - 12 = -6\ \text{m/s}^2$. (c) Setting $6t - 12 = 0$ gives $t = 2$ s.

**Q18.** $f'(x) = 4x^3 - 12x$ and $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$. Then $f''(x) > 0$ when $x^2 > 1$, i.e. $x < -1$ or $x > 1$; and $f''(x) < 0$ when $x^2 < 1$, i.e. $-1 < x < 1$. So the gradient $f'(x)$ is increasing for $x < -1$ and for $x > 1$, and decreasing for $-1 < x < 1$. (The gradient changes from decreasing to increasing at $x = \pm 1$, where $f'' = 0$; the meaning of such points is taken up in a later section.)

## Theory

**The second derivative and the shape of a curve.**

In an earlier section the **second derivative** $f''(x)$ — also written $\frac{d^2y}{dx^2}$ — was introduced as the derivative of the gradient function $f'(x)$. It measures the rate at which the gradient itself is changing. Where the first derivative describes whether a curve is rising or falling, the second derivative describes the way the curve **bends**. This section uses the sign of $f''$ to describe that bending — its **concavity** — and to locate the points where the bending reverses.

**Concavity.**

A curve is **concave up** on an interval if it bends upwards there, like the inside of a bowl. Equivalently, the gradient $f'(x)$ is **increasing** as $x$ increases, so the tangent lines lie **below** the curve. Since $f''$ is the derivative of $f'$, a **positive** $f''$ forces the gradient to increase. A curve is **concave down** if it bends the other way, like an arch: the gradient is **decreasing** and the tangents lie **above** the curve, which is what a **negative** $f''$ forces.

$$f''(x)>0 \text{ on an interval} \Rightarrow \text{the graph is concave up there},$$

$$f''(x)<0 \text{ on an interval} \Rightarrow \text{the graph is concave down there}.$$

Concavity is governed by the sign of $f''$, not of $f'$. A curve can be increasing yet concave down, or decreasing yet concave up — rising and bending are separate ideas.

These arrows point one way only. Read backwards they weaken: a curve that is concave up on an interval has $f''\geq 0$ there, not necessarily $f''>0$, because $f''$ can vanish at an isolated point without the bending being interrupted. The standard case is $y=x^4$, which is concave up across the whole of $\mathbb{R}$ even though $f''(0)=0$.

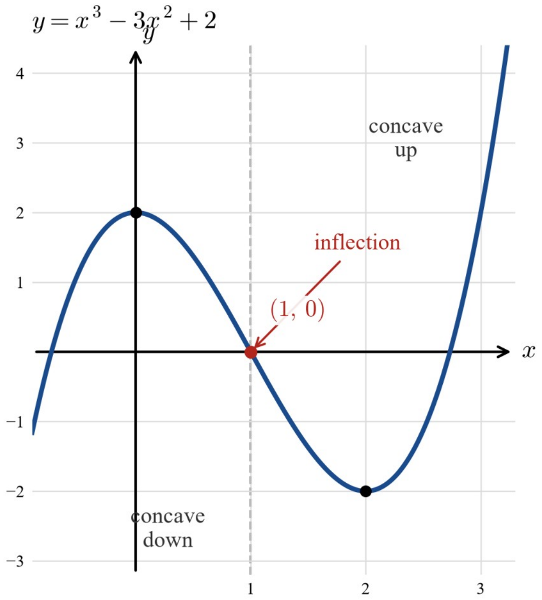
**Points of inflection.**

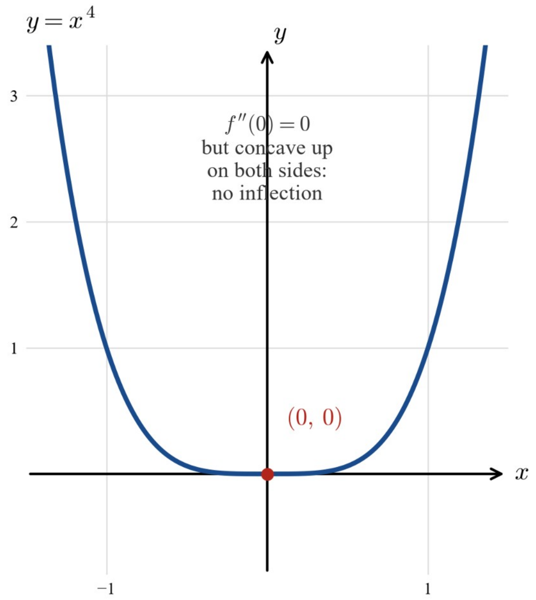
A **point of inflection** is a point on the curve where the **concavity changes** — from concave up to concave down, or from concave down to concave up. Just to one side of such a point the curve is concave up, so $f''\geq 0$ there; just to the other it is concave down, so $f''\leq 0$. Since $f''$ is continuous for the functions in this course, it cannot pass between the two without taking the value zero at the point itself:

$$\text{at a point of inflection}, f''(x)=0.$$

**Warning — $f''=0$ is necessary but not sufficient.**

Solving $f''(x)=0$ only locates the **candidates**. A candidate is a genuine point of inflection **only if** $f''$ **actually changes sign** as $x$ passes through it. Return to $y=x^4$, which has $f''(x)=12x^2$, so $f''(0)=0$; but $12x^2>0$ on **both** sides of $x=0$, so the curve is concave up on either side and the concavity never changes — the origin is a minimum, **not** a point of inflection. Always test the sign of $f''$ on each side of a candidate before concluding.




The left-hand graph above is $y = x^3 - 3x^2 + 2$: it is concave down for $x < 1$ and concave up for $x > 1$, so the concavity changes at $(1, 0)$, which is a point of inflection. The right-hand graph is the trap $y = x^4$, where $f''(0) = 0$ yet the curve is concave up on both sides, so there is no inflection at the origin.

**Finding and classifying a point of inflection.**

1. Compute $f''(x)$ and solve $f''(x) = 0$ to find the candidate $x$-values.
2. For each candidate, test the sign of $f''$ just to the left and just to the right.
3. If the sign changes, there is a point of inflection; substitute the $x$-value back into $f$ (not $f''$) to find the $y$-coordinate.

**Stationary points of inflection.**

At every point of inflection the tangent line crosses the curve. If in addition that tangent is **horizontal** — that is, $f'(x) = 0$ as well as $f''(x) = 0$ with a sign change — the point is a **stationary point of inflection**. If instead $f'(x) \neq 0$ there, the inflection is **non-stationary** (an ordinary, or "sloping", point of inflection). For example $y = x^3$ has $f'(0) = 0$ and $f''(0) = 0$ with a sign change, so the origin is a stationary point of inflection.

**The second-derivative test.**

Concavity also gives a quick way to classify a stationary point. Suppose $f'(a) = 0$.

- If $f''(a) < 0$, the curve is concave down at $x = a$, so the stationary point is a **local maximum**.
- If $f''(a) > 0$, the curve is concave up at $x = a$, so the stationary point is a **local minimum**.
- If $f''(a) = 0$, the test is **inconclusive** — the point may be a maximum, a minimum, or a stationary point of inflection — and you must fall back on a sign test of $f'$ (or check whether $f''$ changes sign).

$$f'(a) = 0, f''(a) < 0 \Rightarrow \text{local maximum}, f'(a) = 0, f''(a) > 0 \Rightarrow \text{local minimum}.$$

**Sketching with concavity and inflection.**

Putting these together gives a complete picture of a curve. Use $f'(x) = 0$ to find the stationary points and the second-derivative test to classify them; use $f''(x) = 0$ with a sign change to find the inflection points and the intervals of concavity; then mark the axis intercepts and draw a smooth curve that bends the correct way — concave up or concave down — on each interval.

## Worked examples

### *Example 1 — scaffolded*

Find the intervals of concavity and any point of inflection of $f(x) = x^3 - 6x^2 + 9x + 1$.

| Working | Comment |
|---|---|
| $f'(x) = 3x^2 - 12x + 9$. | Differentiate once to get the gradient function. |
| $f''(x) = 6x - 12$. | Differentiate again for the second derivative. |
| $6x - 12 = 0 \Rightarrow x = 2$. | Solve $f''(x) = 0$ to find the candidate. |
| For $x < 2$: $f'' < 0$ (e.g. $f''(0) = -12$), concave down. For $x > 2$: $f'' > 0$ (e.g. $f''(3) = 6$), concave up. | Test the sign of $f''$ on each side; the sign changes. |
| $f(2) = 8 - 24 + 18 + 1 = 3$. | The sign change confirms an inflection; substitute into $f$ for the $y$-value. |
| Point of inflection $(2, 3)$; concave down for $x < 2$, concave up for $x > 2$. | State the point and the intervals of concavity. |

### *Example 2 — scaffolded*

Use the second-derivative test to find and classify the stationary points of $f(x) = 2x^3 - 3x^2 - 12x + 4$.

| Working | Comment |
|---|---|
| $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$. | Differentiate and factorise. |
| $f'(x) = 0 \Rightarrow x = 2 \text{ or } x = -1$. | Stationary points where the gradient is zero. |
| $f''(x) = 12x - 6$. | Second derivative for the test. |
| $f''(-1) = -18 < 0$, so $x = -1$ gives a local maximum. | $f'' < 0$: concave down $\Rightarrow$ maximum. |
| $f(-1) = -2 - 3 + 12 + 4 = 11$, so the maximum is $(-1, 11)$. | Substitute into $f$ for the $y$-value. |
| $f''(2) = 18 > 0$, so $x = 2$ gives a local minimum. | $f'' > 0$: concave up $\Rightarrow$ minimum. |
| $f(2) = 16 - 12 - 24 + 4 = -16$, so the minimum is $(2, -16)$. | Substitute into $f$ for the $y$-value. |

### *Example 3 — unscaffolded*

Show that $f(x) = x^3 - 3x^2 + 3x + 2$ has a stationary point of inflection, and find it.

Differentiating, $f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$. Setting $f'(x) = 0$ gives $x = 1$ as the only stationary point (a repeated root). The second derivative is $f''(x) = 6x - 6 = 6(x-1)$, and $f''(1) = 0$, so the second-derivative test is inconclusive and we must test the sign of $f''$ directly. For $x < 1$, $f'' < 0$ (concave down); for $x > 1$, $f'' > 0$ (concave up). The concavity changes at $x = 1$, so this is a point of inflection, and because $f'(1) = 0$ the tangent there is horizontal, making it a stationary point of inflection. Its height is $f(1) = 1 - 3 + 3 + 2 = 3$.

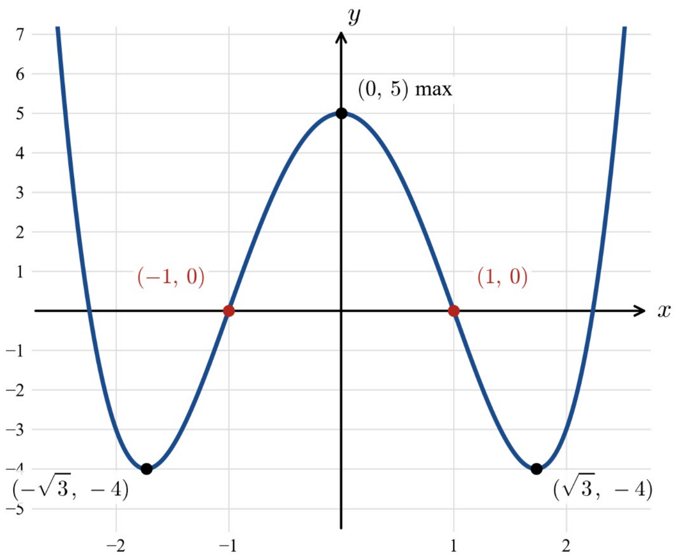
$\therefore$ $(1, 3)$ is a stationary point of inflection.

### *Example 4 — unscaffolded*

Sketch $y = x^4 - 6x^2 + 5$, showing all stationary points and points of inflection.

The gradient is $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$, which is zero at $x = 0$ and $x = \pm\sqrt{3}$, and the second derivative is $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$. Applying the second-derivative test: $f''(0) = -12 < 0$, so $(0, 5)$ is a local maximum; $f''(\pm\sqrt{3}) = 12(3) - 12 = 24 > 0$, so $(\sqrt{3}, -4)$ and $(-\sqrt{3}, -4)$ are local minima, using $f(\pm\sqrt{3}) = 9 - 18 + 5 = -4$.

For the inflection points, $f''(x)=0$ gives $x^2=1$, so $x=\pm 1$. Testing the sign of $f''$: concave up for $x<-1$, concave down for $-1<x<1$, and concave up for $x>1$. The concavity changes at both candidates, so $(-1,0)$ and $(1,0)$ are points of inflection, using $f(\pm 1)=1-6+5=0$. The curve is symmetric in the $y$-axis.




$\therefore$ the curve has a local maximum at $(0,5)$, local minima at $(\pm\sqrt{3},-4)$, and points of inflection at $(\pm 1,0)$.

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## Practice questions

### *Scaffolded*

**Q1.** Find the intervals of concavity and the point of inflection of $f(x)=x^3-3x^2-9x+2$. (a) Find $f''(x)$. (b) Solve $f''(x)=0$. (c) Test the sign of $f''$ on each side of the candidate. (d) State the point of inflection and the intervals of concavity.

**Q2.** Use the second-derivative test to find and classify the stationary points of $f(x)=x^3-3x^2+4$. (a) Find $f'(x)$ and solve $f'(x)=0$. (b) Find $f''(x)$. (c) Evaluate $f''$ at each stationary point. (d) Classify each stationary point and state its coordinates.

**Q3.** Find the points of inflection and the intervals of concavity of $f(x)=x^4-6x^2+2$. (a) Find $f''(x)$. (b) Solve $f''(x)=0$. (c) Test the sign of $f''$ in each interval. (d) State the inflection points and where the curve is concave up and concave down.

**Q4.** Show that $f(x)=x^3-6x^2+12x-3$ has a stationary point of inflection, and find it. (a) Find $f'(x)$ and solve $f'(x)=0$. (b) Find $f''(x)$ and evaluate it at the stationary point. (c) Confirm that $f''$ changes sign there. (d) State the stationary point of inflection.

**Q5.** For $f(x)=(x-1)^4+2$, show that although $f''=0$ at $x=1$ there is no point of inflection there. (a) Find $f''(x)$. (b) Solve $f''(x)=0$. (c) Test the sign of $f''$ on each side of $x=1$. (d) Explain why there is no inflection, and state what kind of point $(1,2)$ is.

**Q6.** For $f(x)=x^3-3x$, find and classify the stationary points, find the point of inflection, and sketch the curve. (a) Find $f'(x)$ and solve $f'(x)=0$. (b) Use the second-derivative test to classify the stationary points. (c) Find the point of inflection. (d) Sketch the curve, labelling these features.

### *Unscaffolded*

**Q7.** Find the point of inflection of $f(x)=x^3+3x^2-2$ and state the intervals of concavity.

**Q8.** Use the second-derivative test to find and classify the stationary points of $f(x)=x^3-6x^2+5$.

**Q9.** Find the points of inflection of $f(x)=x^4-6x^2$ and the intervals on which the curve is concave up.

**Q10.** Show that $f(x)=-x^3+3x^2-3x$ has a stationary point of inflection, and find it.

**Q11.** Show that $f(x)=(x-3)^4$ has $f''=0$ at $x=3$ but no point of inflection there. What kind of point is $(3,0)$?

**Q12.** For $f(x)=x^3-6x^2+9x$, find and classify the stationary points and find the point of inflection.

**Q13.** The second derivative of a function is $f''(x)=6x-12$. (a) State the intervals on which the graph is concave up and concave down. (b) Give the $x$-coordinate of the point of inflection.

**Q14.** Both $f(x)=x^4$ and $g(x)=x^3$ satisfy $f''(0)=0$ and $g''(0)=0$. Show that only one of them has a point of inflection at the origin, stating which, and classify the point $(0,0)$ on each curve.

**Q15.** Find the point of inflection of $f(x)=xe^x$ and state the intervals of concavity, giving the $y$-coordinate as an exact value.

**Q16.** For $f(x)=3x^4-4x^3$: (a) find the stationary points and classify them; (b) find all points of inflection, stating which (if any) is a stationary point of inflection.

**Q17.** The curve $y=x^3+kx^2+3x$ has a point of inflection at $x=2$. Find the value of $k$.

**Q18.** The second derivative of a function is $f''(x)=x^2(x-2)$. Find every $x$-value where $f''(x)=0$, and determine at which of them the graph has a point of inflection. Justify your answer.

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## Answers

**Q1.** Point of inflection $(1,-9)$; concave down for $x<1$, concave up for $x>1$. **Q2.** Local maximum $(0,4)$; local minimum $(2,0)$. **Q3.** Inflections $(-1,-3)$ and $(1,-3)$; concave up for $x<-1$ and $x>1$, concave down for $-1<x<1$. **Q4.** Stationary point of inflection $(2,5)$. **Q5.** $f''(x)=12(x-1)^2>0$ for every $x\neq1$, so $f''$ does not change sign at $x=1$; there is no inflection and $(1,2)$ is a local minimum. **Q6.** Local maximum $(-1,2)$, local minimum $(1,-2)$, point of inflection $(0,0)$. **Q7.** Point of inflection $(-1,0)$; concave down for $x<-1$, concave up for $x>-1$. **Q8.** Local maximum $(0,5)$; local minimum $(4,-27)$. **Q9.** Inflections $(-1,-5)$ and $(1,-5)$; concave up for $x<-1$ and $x>1$. **Q10.** Stationary point of inflection $(1,-1)$. **Q11.** $f''(x)=12(x-3)^2>0$ for every $x\neq3$, so there is no sign change and no inflection; $(3,0)$ is a local minimum. **Q12.** Local maximum $(1,4)$, local minimum $(3,0)$, point of inflection $(2,2)$. **Q13.** (a) Concave down for $x<2$, concave up for $x>2$; (b) $x=2$. **Q14.** $g(x)=x^3$ has a stationary point of inflection at $(0,0)$; $f(x)=x^4$ has a local minimum at $(0,0)$, not an inflection. **Q15.** Point of inflection $\left(-2,-\frac{2}{e^2}\right)$; concave down for $x<-2$, concave up for $x>-2$. **Q16.** (a) Local minimum $(1,-1)$; at $(0,0)$ the second-derivative test is inconclusive. (b) Inflections $(0,0)$ — a stationary point of inflection — and $\left(\frac{2}{3},-\frac{16}{27}\right)$, which is non-stationary. **Q17.** $k=-6$. **Q18.** $f''=0$ at $x=0$ and $x=2$; only $x=2$ gives a point of inflection.

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## Fully-worked solutions

**Q1.** $f'(x)=3x^2-6x-9$ and $f''(x)=6x-6$. Solving $6x-6=0$ gives the candidate $x=1$. For $x<1$, $f''<0$ (for instance $f''(0)=-6$), so the curve is concave down; for $x>1$, $f''>0$ (for instance $f''(2)=6$), so it is concave up. The sign changes, so $x=1$ is a point of inflection, with $f(1)=1-3-9+2=-9$. Hence the point of inflection is $(1,-9)$; the curve is concave down for $x<1$ and concave up for $x>1$.

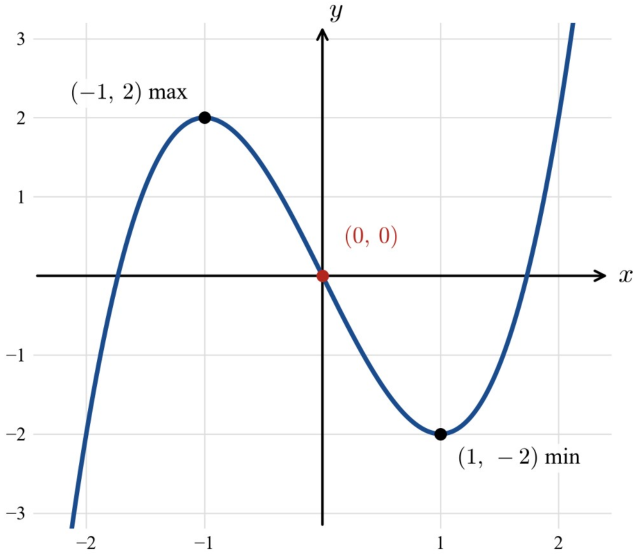
**Q2.** $f'(x)=3x^2-6x=3x(x-2)$, so the stationary points are at $x=0$ and $x=2$. The second derivative is $f''(x)=6x-6$. Since $f''(0)=-6<0$, the point $x=0$ is a local maximum, with $f(0)=4$, giving $(0,4)$. Since $f''(2)=6>0$, the point $x=2$ is a local minimum, with $f(2)=8-12+4=0$, giving $(2,0)$.

**Q3.** $f'(x)=4x^3-12x$ and $f''(x)=12x^2-12=12(x^2-1)$. Solving $12(x^2-1)=0$ gives the candidates $x=\pm1$. Testing the sign of $f''$: for $x<-1$, $f''>0$ (e.g. $f''(-2)=36$), concave up; for $-1<x<1$, $f''<0$ (e.g. $f''(0)=-12$), concave down; for $x>1$, $f''>0$ (e.g. $f''(2)=36$), concave up. The sign changes at both candidates, so both are points of inflection, with $f(\pm1)=1-6+2=-3$. Hence the inflections are $(-1,-3)$ and $(1,-3)$; the curve is concave up for $x<-1$ and for $x>1$, and concave down for $-1<x<1$.

**Q4.** $f'(x)=3x^2-12x+12=3(x-2)^2$, so the only stationary point is $x=2$ (a repeated root). The second derivative is $f''(x)=6x-12=6(x-2)$, and $f''(2)=0$, so the second-derivative test is inconclusive. Testing the sign: for $x<2$, $f''<0$ (concave down); for $x>2$, $f''>0$ (concave up). The concavity changes, so $x=2$ is a point of inflection, and since $f'(2)=0$ the tangent is horizontal. With $f(2)=8-24+24-3=5$, the stationary point of inflection is $(2,5)$.

**Q5.** $f'(x)=4(x-1)^3$ and $f''(x)=12(x-1)^2$. Solving $12(x-1)^2=0$ gives the candidate $x=1$. However $f''(x)=12(x-1)^2>0$ for **every** $x\neq1$, so $f''$ is positive on both sides of $x=1$ and does not change sign — the curve is concave up on either side of $x=1$, and so concave up throughout, the isolated zero of $f''$ not interrupting it. Therefore there is no point of inflection at $x=1$. Because $f'(1)=0$ and the curve is concave up (indeed $f(x)\geq2$ for all $x$), the point $(1,2)$ is a local minimum. This shows that $f''=0$ alone does not guarantee an inflection.

**Q6.** $f'(x)=3x^2-3=3(x^2-1)$, so the stationary points are at $x=\pm1$. The second derivative is $f''(x)=6x$. Since $f''(-1)=-6<0$, $x=-1$ is a local maximum with $f(-1)=-1+3=2$, giving $(-1,2)$; since $f''(1)=6>0$, $x=1$ is a local minimum with $f(1)=1-3=-2$, giving $(1,-2)$. For the inflection, $f''(x)=6x=0$ gives $x=0$; $f''$ changes from negative to positive there, so $(0,0)$ is a point of inflection. The curve is concave down for $x<0$ and concave up for $x>0$.




**Q7.** $f'(x)=3x^2+6x$ and $f''(x)=6x+6$. Solving $6x+6=0$ gives $x=-1$. For $x<-1$, $f''<0$ (concave down); for $x>-1$, $f''>0$ (concave up); the sign changes, so $x=-1$ is a point of inflection, with $f(-1)=-1+3-2=0$. Hence the inflection is $(-1,0)$, the curve concave down for $x<-1$ and concave up for $x>-1$.

**Q8.** $f'(x)=3x^2-12x=3x(x-4)$, so the stationary points are at $x=0$ and $x=4$. The second derivative is $f''(x)=6x-12$. Since $f''(0)=-12<0$, $x=0$ is a local maximum with $f(0)=5$, giving $(0,5)$; since $f''(4)=12>0$, $x=4$ is a local minimum with $f(4)=64-96+5=-27$, giving $(4,-27)$.

**Q9.** $f'(x)=4x^3-12x$ and $f''(x)=12x^2-12$. Solving $12x^2-12=0$ gives $x=\pm1$. As in Q3 the sign of $f''$ is positive, negative, positive across $x=-1$ and $x=1$, so both are points of inflection, with $f(\pm1)=1-6=-5$. Hence the inflections are $(-1,-5)$ and $(1,-5)$, and the curve is concave up for $x<-1$ and for $x>1$.

**Q10.** $f'(x)=-3x^2+6x-3=-3(x-1)^2$, so the only stationary point is $x=1$ (a repeated root). The second derivative is $f''(x)=-6x+6=-6(x-1)$, and $f''(1)=0$, so the test is inconclusive. Testing the sign: for $x<1$, $f''>0$ (concave up); for $x>1$, $f''<0$ (concave down). The concavity changes, so $x=1$ is a point of inflection, and since $f'(1)=0$ it is stationary. With $f(1)=-1+3-3=-1$, the stationary point of inflection is $(1,-1)$.

**Q11.** $f'(x)=4(x-3)^3$ and $f''(x)=12(x-3)^2$. Solving $12(x-3)^2=0$ gives $x=3$. But $f''(x)=12(x-3)^2>0$ for every $x\neq3$, so $f''$ does not change sign at $x=3$ — the curve is concave up on both sides — and there is no point of inflection. Since $f'(3)=0$ and the curve is concave up (with $f(x)\geq0$), the point $(3,0)$ is a local minimum.

**Q12.** $f'(x)=3x^2-12x+9=3(x-1)(x-3)$, so the stationary points are at $x=1$ and $x=3$. The second derivative is $f''(x)=6x-12$. Since $f''(1)=-6<0$, $x=1$ is a local maximum with $f(1)=1-6+9=4$, giving $(1,4)$; since $f''(3)=6>0$, $x=3$ is a local minimum with $f(3)=27-54+27=0$, giving $(3,0)$. For the inflection, $f''(x)=6x-12=0$ gives $x=2$; $f''$ changes from negative to positive there, so $(2,2)$ is a point of inflection, using $f(2)=8-24+18=2$.

**Q13.** (a) The graph is concave up where $f''(x)>0$, that is $6x-12>0$, so $x>2$; and concave down where $6x-12<0$, so $x<2$. (b) Since $f''$ changes sign at $x=2$ (from negative to positive), the point of inflection occurs at $x=2$.

**Q14.** For $g(x)=x^3$: $g'(x)=3x^2$ and $g''(x)=6x$. At $x=0$ both are zero, and $g''$ changes from negative (for $x<0$) to positive (for $x>0$), so the concavity changes and $(0,0)$ is a point of inflection; because $g'(0)=0$ it is a stationary point of inflection. For $f(x)=x^4$: $f''(x)=12x^2$, which is zero at $x=0$ but positive on both sides, so $f''$ does not change sign and there is no inflection — $(0,0)$ is a local minimum. Hence only $y=x^3$ has a point of inflection at the origin.

**Q15.** By the product rule, $f'(x)=e^x(x+1)$ and $f''(x)=e^x(x+2)$. Since $e^x>0$ for all $x$, solving $f''(x)=0$ requires $x+2=0$, so $x=-2$. The factor $e^x$ is always positive, so $f''$ takes the sign of $x+2$: negative for $x<-2$ (concave down) and positive for $x>-2$ (concave up). The sign changes, so $x=-2$ is a point of inflection, with $f(-2)=-2e^{-2}=-\frac{2}{e^2}$. Hence the inflection is $\left(-2,-\frac{2}{e^2}\right)$; the curve is concave down for $x<-2$ and concave up for $x>-2$.

**Q16.** (a) $f'(x)=12x^3-12x^2=12x^2(x-1)$, so the stationary points are at $x=0$ (a repeated root) and $x=1$. The second derivative is $f''(x)=36x^2-24x=12x(3x-2)$. Since $f''(1)=12>0$, $x=1$ is a local minimum with $f(1)=3-4=-1$, giving $(1,-1)$. At $x=0$, $f''(0)=0$, so the test is inconclusive. (b) Solving $f''(x)=12x(3x-2)=0$ gives $x=0$ and $x=\frac{2}{3}$. Both are sign changes of $f''$, so both are points of inflection. At $x=0$, $f'(0)=0$ as well, so $(0,0)$ is a **stationary** point of inflection. At $x=\frac{2}{3}$, $f'\left(\frac{2}{3}\right)\neq 0$, so it is non-stationary, with $f\left(\frac{2}{3}\right)=3\cdot\frac{16}{81}-4\cdot\frac{8}{27}=\frac{16}{27}-\frac{32}{27}=-\frac{16}{27}$, giving $\left(\frac{2}{3},-\frac{16}{27}\right)$.

**Q17.** For $y=x^3+kx^2+3x$, $\frac{dy}{dx}=3x^2+2kx+3$ and $\frac{d^2y}{dx^2}=6x+2k$. A point of inflection requires $\frac{d^2y}{dx^2}=0$ at $x=2$, so $6(2)+2k=0$, giving $12+2k=0$ and $k=-6$. (With $k=-6$, $\frac{d^2y}{dx^2}=6x-12$ does change sign at $x=2$, confirming a genuine inflection.)

**Q18.** Setting $f''(x)=x^2(x-2)=0$ gives $x=0$ (a repeated root) and $x=2$. At $x=2$ the factor $(x-2)$ changes sign while $x^2>0$, so $f''$ changes sign — from negative to positive — and there is a point of inflection. At $x=0$ the factor $x^2>0$ on both sides, so it does not change sign, and $(x-2)<0$ near $x=0$, so $f''$ stays negative on both sides — no sign change, and therefore no inflection. Hence $f''=0$ at $x=0$ and $x=2$, but the graph has a point of inflection only at $x=2$.

### Theory

In many situations two or more quantities change together as time passes, and they are linked by an equation coming from the geometry or physics of the problem. As a circular ripple spreads, its radius $r$ and its area $A$ both grow, and they are tied together by $A=\pi r^2$. If we know how fast one quantity is changing, the equation lets us find how fast the other is changing. A problem of this type — where we relate the **rate of change** of one quantity to the rate of change of another — is called a **related rates** problem, and the tool that connects the rates is the **chain rule**.

**The chain rule links the rates.**

Every quantity in the problem is really a function of time $t$, so its rate of change is its derivative with respect to $t$. If the area of a circle depends on its radius, and the radius depends on time, then the chain rule gives the rate at which the area changes:

$$\frac{dA}{dt}=\frac{dA}{dr}\cdot\frac{dr}{dt}.$$

The middle factor $\frac{dA}{dr}$ comes from the relation $A=\pi r^2$, namely $\frac{dA}{dr}=2\pi r$, while $\frac{dr}{dt}$ is the given rate at which the radius grows. The same pattern relates a volume to a radius,

$$\frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt},$$

or any pair of linked quantities. Each "$\frac{d\,\square}{dt}$" is read as *the rate of change of* $\square$ *per unit time*.

**The method.**

Almost every related rates problem is solved by the same four steps.

1. **Identify** the changing quantities. Write down the rate you are given and the rate you are asked to find, each as a derivative with respect to $t$ (for example, "given $\frac{dr}{dt}=0.5$, find $\frac{dA}{dt}$").
2. **Relate** the quantities with an equation. This comes from a known formula (area, volume, surface area) or from the geometry of the situation (Pythagoras' theorem, similar triangles).
3. **Differentiate** both sides of that equation with respect to $t$, using the chain rule. Every variable is treated as a function of $t$, so differentiating $r^2$ gives $2r\frac{dr}{dt}$, not simply $2r$.
4. **Substitute** the known values and the given rate *at the instant of interest*, then **solve** for the required rate.

**Substitute only after differentiating.**

The most common error is to substitute a value that is *changing* before differentiating. A quantity that varies with time must be kept as a variable while you differentiate; only once the derivative relation is written down do you put in the value it takes at the particular instant. A fixed quantity — such as the length of a ladder or the dimensions of a fixed tank — may of course be substituted at any stage, because its rate of change is zero.

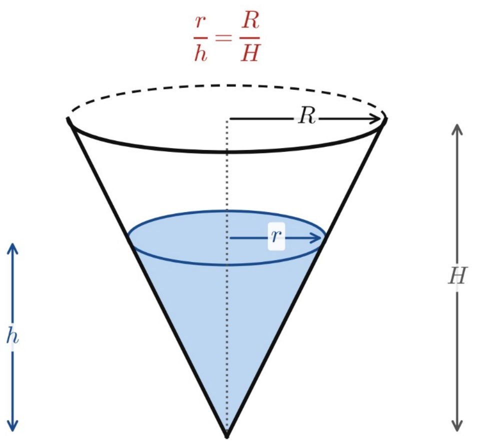
**Units and the instant of evaluation.**

A rate carries the units of *the quantity divided by time*. If a radius is measured in centimetres and time in seconds, then $\frac{dr}{dt}$ is in $\mathrm{cm/s}$, an area rate $\frac{dA}{dt}$ is in $\mathrm{cm^2/s}$, and a volume rate $\frac{dV}{dt}$ is in $\mathrm{cm^3/s}$. Always state the units with the answer. Because the rate usually depends on the current size of the object, a related rates answer is the value *at one instant* — the moment named in the question — not a constant.

**Reducing to one variable: the cone.**

Sometimes the natural formula involves more variables than you have rates for. For a right circular cone the volume is $V=\frac{1}{3}\pi r^2 h$, which contains both the surface radius $r$ and the depth $h$. When liquid sits in an inverted cone of fixed shape, the surface radius and the depth stay in the **same ratio** as the full radius $R$ and full height $H$, because the liquid surface and the rim cut out **similar triangles**:

$$\frac{r}{h}=\frac{R}{H}.$$




This lets you replace $r$ by a multiple of $h$ (or the reverse), so that $V$ is written in terms of a single variable before you differentiate. For a cone whose full radius is half its full height, $r=\frac{h}{2}$, and

$$V=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi h^3}{12}, \text{so } \frac{dV}{dt}=\frac{\pi h^2}{4}\cdot\frac{dh}{dt}.$$

**Differentiating a constraint: the ladder.**

When the relation is a constraint rather than a formula for one quantity, we differentiate the whole equation with respect to $t$. A ladder of fixed length $L$ leaning against a vertical wall has its foot a distance $x$ from the wall and its top a height $y$ up the wall, with $x^2+y^2=L^2$. Differentiating each term with respect to $t$ (the constant $L^2$ has derivative $0$) gives

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0,$$

which rearranges to $\frac{dy}{dt}=-\frac{x}{y}\cdot\frac{dx}{dt}$. The negative sign shows the top slides **down** ($y$ decreasing) while the foot is pulled **out** ($x$ increasing) — the signs of the rates carry the direction of motion, so read them carefully.

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## Worked examples

### *Example 1 — scaffolded*

A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant $0.5\,\mathrm{m/s}$. Find the rate at which the area enclosed by the ripple is increasing at the instant the radius is $3\,\mathrm{m}$.

| Working | Comment |
|---|---|
| Given $\frac{dr}{dt}=0.5$; find $\frac{dA}{dt}$ when $r=3$. | Identify the known rate and the required rate. |

| Working | Comment |
|---|---|
| $A=\pi r^2$. | The relation between area and radius. |
| $\dfrac{dA}{dt}=\dfrac{dA}{dr}\cdot\dfrac{dr}{dt}=2\pi r\cdot\dfrac{dr}{dt}$. | Differentiate with respect to $t$ using the chain rule. |
| $\dfrac{dA}{dt}=2\pi(3)(0.5)=3\pi$. | Substitute $r=3$ and $\dfrac{dr}{dt}=0.5$ *after* differentiating. |
| $\dfrac{dA}{dt}=3\pi\,\mathrm{m}^2/\mathrm{s}\approx 9.42\,\mathrm{m}^2/\mathrm{s}$. | State the answer with area-rate units. |

### *Example 2 — scaffolded*

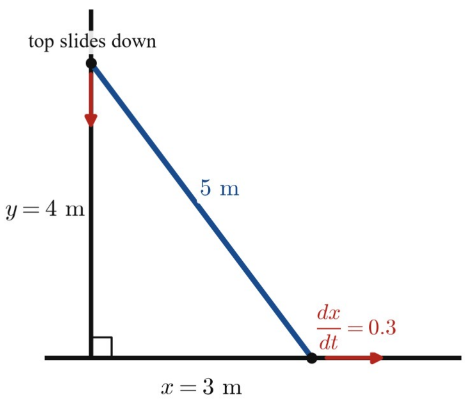
Air is pumped into a spherical balloon at $100\,\mathrm{cm}^3/\mathrm{s}$. Find the rate at which the radius is increasing at the instant the radius is $5\,\mathrm{cm}$.

| Working | Comment |
|---|---|
| Given $\dfrac{dV}{dt}=100$; find $\dfrac{dr}{dt}$ when $r=5$. | The volume rate is known; the radius rate is wanted. |
| $V=\dfrac{4}{3}\pi r^3$. | Volume of a sphere in terms of its radius. |
| $\dfrac{dV}{dt}=4\pi r^2\cdot\dfrac{dr}{dt}$. | Differentiate with respect to $t$; $\dfrac{dV}{dr}=4\pi r^2$. |
| $100=4\pi(5)^2\dfrac{dr}{dt}=100\pi\dfrac{dr}{dt}$. | Substitute the known rate and $r=5$. |
| $\dfrac{dr}{dt}=\dfrac{100}{100\pi}=\dfrac{1}{\pi}\,\mathrm{cm}/\mathrm{s}\approx 0.318\,\mathrm{cm}/\mathrm{s}$. | Solve for the required rate; units $\mathrm{cm}/\mathrm{s}$. |

### *Example 3 — unscaffolded*

A ladder $5\,\mathrm{m}$ long rests against a vertical wall. The foot of the ladder is pulled horizontally away from the wall at $0.3\,\mathrm{m}/\mathrm{s}$. Find the rate at which the top of the ladder is sliding down the wall at the instant the foot is $3\,\mathrm{m}$ from the wall.

Let $x$ be the distance of the foot from the wall and $y$ the height of the top up the wall. The ladder length is fixed, so by Pythagoras' theorem $x^2+y^2=5^2=25$. At the instant $x=3$, $y=\sqrt{25-9}=4$.




Differentiating the constraint with respect to $t$,

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0\Rightarrow\frac{dy}{dt}=-\frac{x}{y}\cdot\frac{dx}{dt}.$$

Substituting $x=3$, $y=4$ and $\frac{dx}{dt}=0.3$,

$$\frac{dy}{dt}=-\frac{3}{4}(0.3)=-0.225\,\mathrm{m/s}.$$

The negative sign confirms the top is moving down.

∴ the top of the ladder is sliding down the wall at $0.225\,\mathrm{m/s}$ at that instant.

### *Example 4 — unscaffolded*

Water is poured into an inverted right circular cone (apex at the bottom) of height $12\,\mathrm{m}$ and top radius $6\,\mathrm{m}$ at a rate of $3\,\mathrm{m}^3/\mathrm{min}$. Find the rate at which the depth of water is rising at the instant the depth is $4\,\mathrm{m}$.

Let the water have depth $h$ and surface radius $r$. Because the water forms a cone similar to the whole cone, $\frac{r}{h}=\frac{6}{12}=\frac{1}{2}$, so $r=\frac{h}{2}$. The volume of water is

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi h^3}{12}.$$

Differentiating with respect to $t$,

$$\frac{dV}{dt}=\frac{\pi}{12}\cdot 3h^2\cdot\frac{dh}{dt}=\frac{\pi h^2}{4}\cdot\frac{dh}{dt}.$$

The tank is filling, so $\frac{dV}{dt}=3$. Substituting this and $h=4$,

$$3=\frac{\pi(4)^2}{4}\cdot\frac{dh}{dt}=4\pi\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=\frac{3}{4\pi}.$$

∴ the depth is rising at $\frac{3}{4\pi}\,\mathrm{m/min}\approx 0.239\,\mathrm{m/min}$ at that instant.

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## Practice questions

### *Scaffolded*

**Q1.** The radius of a circular oil slick increases at $2\,\mathrm{cm/s}$. Find the rate at which its area is increasing when the radius is $10\,\mathrm{cm}$. (a) Write down the relation between the area $A$ and the radius $r$. (b) Differentiate it with respect to $t$ to relate $\frac{dA}{dt}$ and $\frac{dr}{dt}$. (c) Substitute $r=10$ and $\frac{dr}{dt}=2$. (d) State the answer with units.

**Q2.** The radius of a spherical balloon increases at $0.5\,\mathrm{cm/s}$. Find the rate at which its volume is increasing when the radius is $6\,\mathrm{cm}$. (a) Write down $V$ in terms of $r$. (b) Differentiate with respect to $t$. (c) Substitute $r=6$ and $\frac{dr}{dt}=0.5$. (d) State the answer with units.

**Q3.** The area enclosed by a spreading circular ripple increases at $8\pi\,\mathrm{cm}^2/\mathrm{s}$. Find the rate at which the radius is increasing when the radius is $4\,\mathrm{cm}$. (a) Write down the relation between $A$ and $r$, and differentiate with respect to $t$. (b) Substitute $\frac{dA}{dt}=8\pi$ and $r=4$. (c) Solve for $\frac{dr}{dt}$ and state the units.

**Q4.** The edge length of a cube increases at $0.1\,\mathrm{cm/s}$. Find the rate at which its volume is increasing when the edge is $5\,\mathrm{cm}$. (a) Write the volume $V$ in terms of the edge length $x$. (b) Differentiate with respect to $t$. (c) Substitute $x=5$ and $\frac{dx}{dt}=0.1$, and state the answer with units.

**Q5.** A ladder $10\,\mathrm{m}$ long leans against a vertical wall. Its foot slides away from the wall at $0.5\,\mathrm{m/s}$. Find the rate at which the top slides down when the foot is $6\,\mathrm{m}$ from the wall. (a) Write the relation between $x$ (foot distance) and $y$ (top height). (b) Find $y$ when $x=6$. (c) Differentiate the relation with respect to $t$ and make $\frac{dy}{dt}$ the subject. (d) Substitute the values and state the answer with units.

**Q6.** Water is poured into an inverted cone of height $9\,\mathrm{m}$ and top radius $3\,\mathrm{m}$ at $5\,\mathrm{m}^3/\mathrm{min}$. Find the rate at which the depth is rising when the depth is $6\,\mathrm{m}$. (a) Use similar triangles to write $r$ in terms of $h$. (b) Write $V$ in terms of $h$ only, and differentiate with respect to $t$. (c) Substitute $\frac{dV}{dt}=5$ and $h=6$, and state the answer with units.

***Unscaffolded***

**Q7.** The radius of a circle increases at $3\,\mathrm{mm/s}$. Find the rate at which its area is increasing when the radius is $8\,\mathrm{mm}$.

**Q8.** Air is pumped into a spherical balloon at $50\,\mathrm{cm}^3/\mathrm{s}$. Find the rate at which the radius is increasing when the radius is $10\,\mathrm{cm}$.

**Q9.** The radius of a sphere increases at $0.4\,\mathrm{cm/s}$. Find the rate at which its surface area $S=4\pi r^2$ is increasing when the radius is $15\,\mathrm{cm}$.

**Q10.** A ladder $13\,\mathrm{m}$ long rests against a vertical wall. Its foot is pulled away from the wall at $0.2\,\mathrm{m/s}$. Find the rate at which the top slides down the wall when the foot is $5\,\mathrm{m}$ from the wall.

**Q11.** Water is poured into an inverted cone of height $8\,\mathrm{m}$ and top radius $4\,\mathrm{m}$ at $2\,\mathrm{m}^3/\mathrm{min}$. Find the rate at which the depth is rising when the depth is $2\,\mathrm{m}$.

**Q12.** The edge length of a cube increases at $0.3\,\mathrm{cm/s}$. Find the rate at which its total surface area is increasing when the edge is $4\,\mathrm{cm}$.

**Q13.** A stone dropped into a pond creates a circular ripple whose enclosed area increases at $12\pi\,\mathrm{cm}^2/\mathrm{s}$. Find the rate at which the radius is increasing when the radius is $3\,\mathrm{cm}$.

**Q14.** A conical tank (apex at the bottom) of height $6\,\mathrm{m}$ and top radius $2\,\mathrm{m}$ is draining at $0.5\,\mathrm{m}^3/\mathrm{min}$. Find the rate at which the water depth is falling when the depth is $3\,\mathrm{m}$.

**Q15.** A ladder $5\,\mathrm{m}$ long rests against a vertical wall; its foot slides away at $0.4\,\mathrm{m/s}$. At the instant the foot is $3\,\mathrm{m}$ from the wall, find the rate at which the area of the triangle formed by the ladder, the wall and the ground is changing.

**Q16.** A spherical snowball melts so that its volume decreases at $8\,\mathrm{cm}^3/\mathrm{min}$. Find the rate at which its radius is decreasing at the instant the radius is $4\,\mathrm{cm}$.

**Q17.** A point moves along the parabola $y=x^2$ so that its $x$-coordinate increases at $3$ units per second. Find the rate at which its $y$-coordinate is changing when $x=2$.

**Q18.** A child $1.5\,\mathrm{m}$ tall walks directly away from the base of a $4.5\,\mathrm{m}$ tall lamp post at $1.2\,\mathrm{m/s}$. Let $x$ be the child's distance from the post and $s$ the length of the child's shadow. Find (a) the rate at which the shadow is lengthening, and (b) the speed at which the tip of the shadow is moving along the ground.

### Answers

**Q1.** $\frac{dA}{dt}=40\pi\,\text{cm}^2/\text{s}\approx125.7\,\text{cm}^2/\text{s}$. **Q2.** $\frac{dV}{dt}=72\pi\,\text{cm}^3/\text{s}\approx226.2\,\text{cm}^3/\text{s}$. **Q3.** $\frac{dr}{dt}=1\,\text{cm}/\text{s}$. **Q4.** $\frac{dV}{dt}=7.5\,\text{cm}^3/\text{s}$. **Q5.** $\frac{dy}{dt}=-0.375\,\text{m}/\text{s}$ (the top slides down at $0.375\,\text{m}/\text{s}$). **Q6.** $\frac{dh}{dt}=\frac{5}{4\pi}\,\text{m}/\text{min}\approx0.398\,\text{m}/\text{min}$. **Q7.** $\frac{dA}{dt}=48\pi\,\text{mm}^2/\text{s}\approx150.8\,\text{mm}^2/\text{s}$. **Q8.** $\frac{dr}{dt}=\frac{1}{8\pi}\,\text{cm}/\text{s}\approx0.0398\,\text{cm}/\text{s}$. **Q9.** $\frac{dS}{dt}=48\pi\,\text{cm}^2/\text{s}\approx150.8\,\text{cm}^2/\text{s}$. **Q10.** $\frac{dy}{dt}=-\frac{1}{12}\,\text{m}/\text{s}$ (the top slides down at $\frac{1}{12}\,\text{m}/\text{s}\approx0.083\,\text{m}/\text{s}$). **Q11.** $\frac{dh}{dt}=\frac{2}{\pi}\,\text{m}/\text{min}\approx0.637\,\text{m}/\text{min}$. **Q12.** $\frac{dS}{dt}=14.4\,\text{cm}^2/\text{s}$. **Q13.** $\frac{dr}{dt}=2\,\text{cm}/\text{s}$. **Q14.** $\frac{dh}{dt}=-\frac{1}{2\pi}\,\text{m}/\text{min}$ (the depth falls at $\frac{1}{2\pi}\,\text{m}/\text{min}\approx0.159\,\text{m}/\text{min}$). **Q15.** $\frac{dA}{dt}=0.35\,\text{m}^2/\text{s}$ (the area is increasing). **Q16.** $\frac{dr}{dt}=-\frac{1}{8\pi}\,\text{cm}/\text{min}$ (the radius shrinks at $\frac{1}{8\pi}\,\text{cm}/\text{min}\approx0.0398\,\text{cm}/\text{min}$). **Q17.** $\frac{dy}{dt}=12$ units per second. **Q18.** (a) $\frac{ds}{dt}=0.6\,\text{m}/\text{s}$; (b) the tip moves at $1.8\,\text{m}/\text{s}$.

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### Fully-worked solutions

**Q1.** Given $\frac{dr}{dt}=2$, find $\frac{dA}{dt}$ when $r=10$. The relation is $A=\pi r^2$, so differentiating with respect to $t$,

$$\frac{dA}{dt}=2\pi r\frac{dr}{dt}=2\pi(10)(2)=40\pi.$$

Hence $\frac{dA}{dt}=40\pi\,\text{cm}^2/\text{s}\approx125.7\,\text{cm}^2/\text{s}$.

**Q2.** Given $\frac{dV}{dt}$ is required and $\frac{dr}{dt}=0.5$. With $V=\frac{4}{3}\pi r^3$,

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}=4\pi(6)^2(0.5)=4\pi(36)(0.5)=72\pi.$$

Hence $\frac{dV}{dt}=72\pi\,\text{cm}^3/\text{s}\approx226.2\,\text{cm}^3/\text{s}$.

**Q3.** From $A=\pi r^2$, $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$. Substituting $\frac{dA}{dt}=8\pi$ and $r=4$,

$$8\pi=2\pi(4)\frac{dr}{dt}=8\pi\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=1.$$

Hence the radius increases at $1\,\text{cm}/\text{s}$.

**Q4.** With edge $x$, the volume is $V=x^3$, so $\frac{dV}{dt}=3x^2\frac{dx}{dt}$. Substituting $x=5$ and $\frac{dx}{dt}=0.1$,

$$\frac{dV}{dt}=3(5)^2(0.1)=3(25)(0.1)=7.5.$$

Hence $\frac{dV}{dt}=7.5\,\text{cm}^3/\text{s}$.

**Q5.** Let $x$ be the foot distance and $y$ the top height, with $x^2+y^2=10^2=100$. When $x=6$, $y=\sqrt{100-36}=8$. Differentiating the relation with respect to $t$,

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0\Rightarrow\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}=-\frac{6}{8}(0.5)=-0.375.$$

Hence $\frac{dy}{dt}=-0.375\,\mathrm{m/s}$; the top slides down at $0.375\,\mathrm{m/s}$.

**Q6.** By similar triangles $\frac{r}{h}=\frac{3}{9}=\frac{1}{3}$, so $r=\frac{h}{3}$ and

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h=\frac{\pi h^3}{27}.$$

Differentiating, $\frac{dV}{dt}=\frac{\pi h^2}{9}\frac{dh}{dt}$. Substituting $\frac{dV}{dt}=5$ and $h=6$,

$$5=\frac{\pi(6)^2}{9}\frac{dh}{dt}=4\pi\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=\frac{5}{4\pi}.$$

Hence the depth rises at $\frac{5}{4\pi}\,\mathrm{m/min}\approx0.398\,\mathrm{m/min}$.

**Q7.** From $A=\pi r^2$, $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$. With $\frac{dr}{dt}=3$ and $r=8$,

$$\frac{dA}{dt}=2\pi(8)(3)=48\pi.$$

Hence $\frac{dA}{dt}=48\pi\,\mathrm{mm^2/s}\approx150.8\,\mathrm{mm^2/s}$.

**Q8.** From $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$. Substituting $\frac{dV}{dt}=50$ and $r=10$,

$$50=4\pi(10)^2\frac{dr}{dt}=400\pi\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=\frac{50}{400\pi}=\frac{1}{8\pi}.$$

Hence $\frac{dr}{dt}=\frac{1}{8\pi}\,\mathrm{cm/s}\approx0.0398\,\mathrm{cm/s}$.

**Q9.** From $S=4\pi r^2$, $\frac{dS}{dt}=8\pi r\frac{dr}{dt}$. With $\frac{dr}{dt}=0.4$ and $r=15$,

$$\frac{dS}{dt}=8\pi(15)(0.4)=48\pi.$$

Hence $\frac{dS}{dt}=48\pi\,\mathrm{cm^2/s}\approx150.8\,\mathrm{cm^2/s}$.

**Q10.** With $x^2+y^2=13^2=169$, at $x=5$ we have $y=\sqrt{169-25}=12$. Differentiating,

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0\Rightarrow\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}=-\frac{5}{12}(0.2)=-\frac{1}{12}.$$

Hence $\frac{dy}{dt}=-\frac{1}{12}\,\mathrm{m/s}$; the top slides down at $\frac{1}{12}\,\mathrm{m/s}\approx0.083\,\mathrm{m/s}$.

**Q11.** By similar triangles $\frac{r}{h}=\frac{4}{8}=\frac{1}{2}$, so $r=\frac{h}{2}$ and $V=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi h^3}{12}$. Differentiating, $\frac{dV}{dt}=\frac{\pi h^2}{4}\frac{dh}{dt}$. With $\frac{dV}{dt}=2$ and $h=2$,

$$2=\frac{\pi(2)^2}{4}\frac{dh}{dt}=\pi\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=\frac{2}{\pi}.$$

Hence the depth rises at $\frac{2}{\pi}\,\mathrm{m/min}\approx 0.637\,\mathrm{m/min}$.

**Q12.** A cube of edge $x$ has total surface area $S=6x^2$, so $\frac{dS}{dt}=12x\frac{dx}{dt}$. With $\frac{dx}{dt}=0.3$ and $x=4$,

$$\frac{dS}{dt}=12(4)(0.3)=14.4.$$

Hence $\frac{dS}{dt}=14.4\,\mathrm{cm^2/s}$.

**Q13.** From $A=\pi r^2$, $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$. Substituting $\frac{dA}{dt}=12\pi$ and $r=3$,

$$12\pi=2\pi(3)\frac{dr}{dt}=6\pi\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=2.$$

Hence the radius increases at $2\,\mathrm{cm/s}$.

**Q14.** By similar triangles $\frac{r}{h}=\frac{2}{6}=\frac{1}{3}$, so $r=\frac{h}{3}$ and $V=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h=\frac{\pi h^3}{27}$. Differentiating, $\frac{dV}{dt}=\frac{\pi h^2}{9}\frac{dh}{dt}$. Draining means $\frac{dV}{dt}=-0.5$; with $h=3$,

$$-0.5=\frac{\pi(3)^2}{9}\frac{dh}{dt}=\pi\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=-\frac{0.5}{\pi}=-\frac{1}{2\pi}.$$

Hence the depth falls at $\frac{1}{2\pi}\,\mathrm{m/min}\approx 0.159\,\mathrm{m/min}$.

**Q15.** Let $x$ be the foot distance and $y$ the top height, with $x^2+y^2=25$. When $x=3$, $y=4$. First find $\frac{dy}{dt}$: from $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$,

$$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}=-\frac{3}{4}(0.4)=-0.3.$$

The triangle has area $A=\frac{1}{2}xy$, so by the product rule

$$\frac{dA}{dt}=\frac{1}{2}\left(y\frac{dx}{dt}+x\frac{dy}{dt}\right)=\frac{1}{2}\left(4(0.4)+3(-0.3)\right)=\frac{1}{2}(1.6-0.9)=0.35.$$

Hence $\frac{dA}{dt}=0.35\,\mathrm{m^2/s}$; the area is increasing.

**Q16.** From $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$. Melting gives $\frac{dV}{dt}=-8$; with $r=4$,

$$-8=4\pi(4)^2\frac{dr}{dt}=64\pi\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=-\frac{8}{64\pi}=-\frac{1}{8\pi}.$$

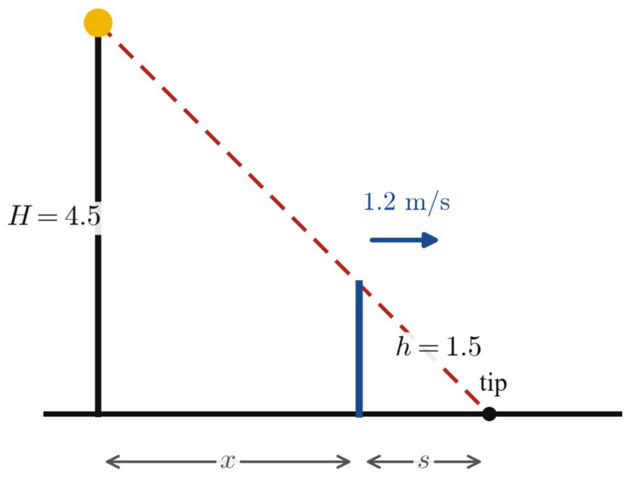
Hence $\frac{dr}{dt}=-\frac{1}{8\pi}\,\mathrm{cm/min}$; the radius shrinks at $\frac{1}{8\pi}\,\mathrm{cm/min}\approx 0.0398\,\mathrm{cm/min}$.

**Q17.** With $y=x^2$, differentiating with respect to $t$ gives $\frac{dy}{dt}=2x\frac{dx}{dt}$. Substituting $x=2$ and $\frac{dx}{dt}=3$,

$$\frac{dy}{dt}=2(2)(3)=12.$$

Hence the $y$-coordinate increases at 12 units per second.

**Q18.** The lamp top, the child's head and the shadow tip are collinear, giving two similar triangles: the large one of height $H=4.5$ across the base $x+s$, and the small one of height $h=1.5$ across the base $s$.




Hence

$$\frac{H}{x+s}=\frac{h}{s}\Rightarrow 1.5(x+s)=4.5s\Rightarrow 1.5x=3s\Rightarrow s=\frac{x}{2}.$$

(a) Differentiating $s=\frac{x}{2}$ with respect to $t$, $\frac{ds}{dt}=\frac{1}{2}\frac{dx}{dt}=\frac{1}{2}(1.2)=0.6$, so the shadow lengthens at $0.6\,\mathrm{m/s}$. (b) The tip is at distance $x+s=x+\frac{x}{2}=\frac{3x}{2}$ from the post, so its speed is

$$\frac{d}{dt}\left(\frac{3x}{2}\right)=\frac{3}{2}\frac{dx}{dt}=\frac{3}{2}(1.2)=1.8.$$

Hence the tip of the shadow moves along the ground at $1.8\,\mathrm{m/s}$. (Notice the tip speed does not depend on $x$, so it is the same at every instant.)

### Theory

A **rational function** is a quotient of two polynomials,

$$y = f(x) = \frac{P(x)}{Q(x)}, Q(x) \neq 0,$$

where $P$ and $Q$ have no common polynomial factor. Its natural domain is every real number **except** the zeros of the denominator, since division by zero is undefined: the domain is $\{x : Q(x) \neq 0\}$. The graph of a rational function is governed by two kinds of behaviour — what happens **near** the excluded $x$-values, and what happens **far out** as $x \to \pm\infty$. Both are described by **asymptotes**: straight lines that the curve approaches arbitrarily closely. A curve can never cross a **vertical** asymptote, since the function is undefined there, but it may well cross a **horizontal** or **oblique** one; what makes a line an asymptote is that the gap between line and curve tends to $0$, not that the curve stays off the line. This section combines that asymptotic analysis with differential calculus — using $f'(x)$ for **stationary points** and $f''(x)$ for **points of inflection** — to produce a complete sketch.

**Axis intercepts.**

The intercepts are found exactly as for any curve. The $y$**-intercept** is $f(0) = \frac{P(0)}{Q(0)}$, provided $Q(0) \neq 0$. The $x$-**intercepts** are the values where $y = 0$; a fraction is zero only when its numerator is zero, so they are the solutions of $P(x) = 0$ that are **not** also zeros of $Q$.

**Vertical asymptotes.**

Let $x = a$ be a zero of $Q$ that is not a zero of $P$. As $x$ approaches $a$ the denominator approaches $0$ while the numerator approaches the non-zero value $P(a)$, so the quotient grows without bound: $x = a$ is a **vertical asymptote**. To decide *which way* each branch goes, examine the **sign** of $f(x)$ just either side of $a$. For a simple (single) zero of $Q$ the denominator changes sign as $x$ passes $a$, so the two one-sided behaviours are opposite: on one side $y \to +\infty$ and on the other $y \to -\infty$. For example, for $y = \frac{1}{x-2}$ the numerator is the positive constant $1$; just to the right of 2 the denominator is a small positive number so $y \to +\infty$, while just to the left it is a small negative number so $y \to -\infty$.

**Behaviour as $x \to \pm\infty$: horizontal and oblique asymptotes.**

Far from the origin the graph is controlled by the highest-degree terms of $P$ and $Q$. Comparing the degrees $\deg P$ and $\deg Q$ gives three cases. (A numerator whose degree exceeds the denominator's by two or more has no straight-line asymptote at all; that case is not required here.)

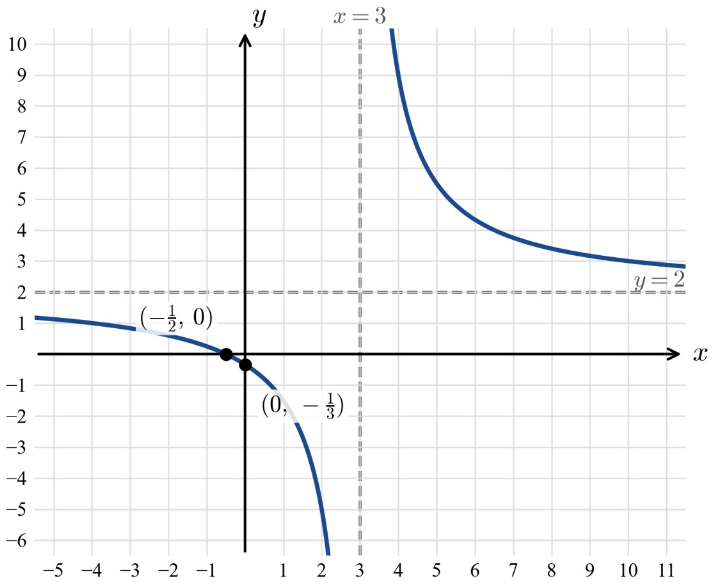
| Degrees | End behaviour | Asymptote |
|---|---|---|
| $\deg P < \deg Q$ | $y \to 0$ | horizontal asymptote $y = 0$ |
| $\deg P = \deg Q$ | $y \to \dfrac{\text{lead. coeff. of } P}{\text{lead. coeff. of } Q}$ | horizontal asymptote $y = \dfrac{a_n}{b_n}$ |
| $\deg P = \deg Q + 1$ | $y \approx mx + c$ | oblique (slant) asymptote $y = mx + c$ |

When the numerator's degree is exactly one more than the denominator's, carry out **polynomial division** to write

$$\frac{P(x)}{Q(x)} = (mx + c) + \frac{R(x)}{Q(x)},$$

where the remainder $R$ has degree less than $Q$. The remainder term $\frac{R(x)}{Q(x)} \to 0$ as $x \to \pm\infty$, so the curve approaches the line $y = mx + c$: this is the **oblique asymptote**. (If instead $deg\,P = deg\,Q$ the same division leaves a constant quotient, recovering the horizontal asymptote in the middle row.)

The diagram below shows $y = \frac{2x+1}{x-3}$. Here $Q(x) = x - 3$ vanishes at $x = 3$ (a vertical asymptote), and since $deg\,P = deg\,Q = 1$ with equal leading coefficients, division gives $\frac{2x+1}{x-3} = 2 + \frac{7}{x-3}$, so $y \to 2$: the horizontal asymptote is $y = 2$. The intercepts are $(-1/2, 0)$ and $(0, -1/3)$.




**Simple quotient functions.**

A **simple quotient function** is the reciprocal of a polynomial, $y = \frac{1}{g(x)}$. Its features follow directly from those of $g$:

- wherever $g(x) = 0$ there is a **vertical asymptote**;
- since the numerator has degree $0 < deg\,g$, the line $y = 0$ is a **horizontal asymptote** (the curve flattens towards the $x$-axis far out);
- $y$ is large in size where $g$ is small in size, and small where $g$ is large. The stationary points of $\frac{1}{g}$ are exactly those of $g$, and each one **reverses its nature**: a **local minimum of** $g$ gives a **local maximum of** $\frac{1}{g}$, and a local maximum of $g$ gives a local minimum. This holds on either kind of branch, whichever sign $g$ takes there;
- $\frac{1}{g}$ is never zero, so a reciprocal has **no $x$-intercepts**.

**Stationary points and points of inflection.**

Asymptotes and intercepts fix the skeleton of the graph; calculus fixes the turning behaviour between them. Differentiate with the **quotient rule** and solve $f'(x) = 0$ for the **stationary points**, then classify each one either by the sign of $f'$ on either side or by the **second derivative**: $f''(x) > 0$ gives a local minimum and $f''(x) < 0$ a local maximum. **Points of inflection**, where the concavity changes, occur where $f''(x) = 0$ **and** $f''$ changes sign. Because a rational function can have several branches separated by vertical asymptotes, always test candidate points against the branch they lie on.

**A sketching procedure.**

1. State the **domain** (exclude the zeros of $Q$).
2. Find the **axis intercepts** ($P(x)=0$ and $f(0)$).
3. Find the **vertical asymptotes** and the sign behaviour on each side.
4. Compare degrees to find the **horizontal or oblique asymptote** (use division if needed).
5. Use $f'$ for **stationary points** and $f''$ for **points of inflection**, and classify them.
6. Assemble the branches, drawing every asymptote dashed and marking the key points.

## Worked examples

### *Example 1 — scaffolded*

Sketch $y=\frac{x-4}{x+1}$, stating the equations of the asymptotes and the coordinates of the axis intercepts.

| Working | Comment |
|---|---|
| Domain: $x+1\neq 0$, so $x\neq -1$. | Exclude the zero of the denominator. |
| $Q(x)=x+1=0$ at $x=-1$, and $P(-1)=-5\neq 0$, so $x=-1$ is a vertical asymptote. | A vertical asymptote sits at a zero of $Q$ that is not a zero of $P$. |
| $deg\,P=deg\,Q=1$ with leading coefficients 1 and 1, so as $x\to\pm\infty$, $y\to\frac{1}{1}=1$: horizontal asymptote $y=1$. | Equal degrees give a horizontal asymptote equal to the ratio of leading coefficients. |
| $x$-intercept: $P=0$ at $x=4$, giving $(4,0)$. $y$-intercept: $f(0)=\frac{-4}{1}=-4$, giving $(0,-4)$. | Numerator zero for the $x$-intercept; put $x=0$ for the $y$-intercept. |
| $f'(x)=\frac{(x+1)-(x-4)}{(x+1)^2}=\frac{5}{(x+1)^2}>0$, so there are no stationary points; the curve increases on each branch. | Quotient rule; a derivative of constant sign means no turning points. |
| Sign behaviour: writing $y=1-\frac{5}{x+1}$, as $x\to -1^-$, $y\to +\infty$, and as $x\to -1^+$, $y\to -\infty$. | Rewrite in divided form to read off each side of the asymptote. |

The graph has two increasing branches: the left branch (for $x<-1$) rises from $y=1$ toward $+\infty$, and the right branch (for $x>-1$) rises from $-\infty$ back toward $y=1$, passing through $(4,0)$ and $(0,-4)$.

### *Example 2 — scaffolded*

Sketch $y=x+\frac{1}{x}$, stating the asymptotes and the coordinates and nature of any stationary points.

| Working | Comment |
|---|---|
| Combine: $y=\frac{x^2+1}{x}$. Domain $x\neq 0$; the numerator $x^2+1\neq 0$ there, so $x=0$ is a vertical asymptote. | Rewrite as a single fraction to expose $P$ and $Q$. |
| $deg\,P=2=deg\,Q+1$, so there is an oblique asymptote. The split $y=x+\frac{1}{x}$ already isolates the quotient $x$; since $\frac{1}{x}\to 0$ as $x\to\pm\infty$, the oblique asymptote is $y=x$. | Degree of numerator one more than denominator gives a slant asymptote equal to the polynomial quotient. |
| Intercepts: $x^2+1>0$ so there is no $x$-intercept, and $x=0$ is excluded so there is no $y$-intercept. | Check both intercepts; here the curve meets neither axis. |

| Working | Comment |
|---|---|
| $f'(x)=1-\frac{1}{x^2}=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$. | Stationary points where $f'(x)=0$. |
| $f(1)=2$ and $f(-1)=-2$. With $f''(x)=\frac{2}{x^3}$: $f''(1)=2>0$ (minimum) and $f''(-1)=-2<0$ (maximum). | Evaluate the function, then use $f''$ to classify. |
| Stationary points: $(1,2)$ is a local minimum and $(-1,-2)$ is a local maximum. | State coordinates and nature. |

The curve is odd, with one branch in the first quadrant (above the line $y=x$, turning at the minimum $(1,2)$) and one in the third quadrant (below $y=x$, turning at the maximum $(-1,-2)$).

### *Example 3 — unscaffolded*

Sketch $y=\frac{x^2+3}{x-1}$, showing all asymptotes, the $y$-intercept, and the coordinates and nature of the stationary points.

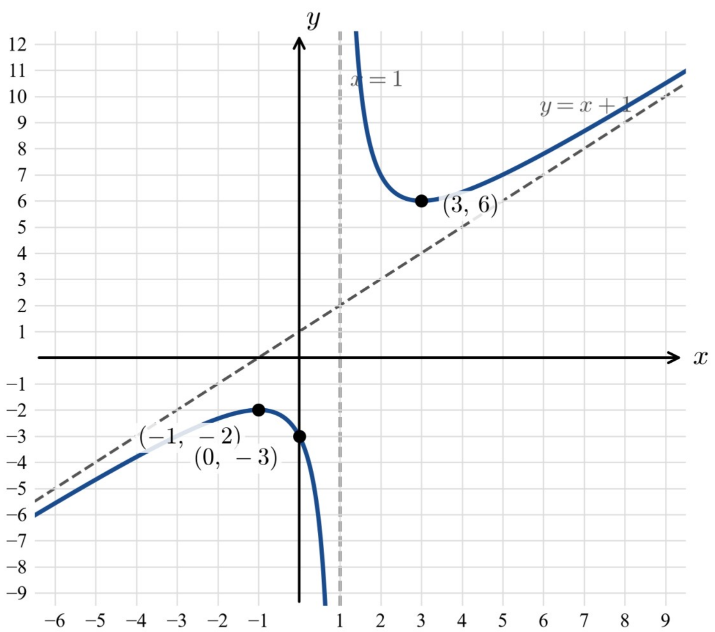
The domain is $x\neq 1$. There is no $x$-intercept because $x^2+3>0$ for all $x$; the $y$-intercept is $f(0)=\frac{3}{-1}=-3$. Since $Q(1)=0$ while $P(1)=4\neq 0$, the line $x=1$ is a vertical asymptote. The numerator has degree one more than the denominator, so polynomial division is used:

$$\frac{x^2+3}{x-1}=(x+1)+\frac{4}{x-1}.$$

The remainder term $\frac{4}{x-1}\to 0$ as $x\to\pm\infty$, so the oblique asymptote is $y=x+1$. For the turning points, differentiate:

$$f'(x)=\frac{2x(x-1)-(x^2+3)}{(x-1)^2}=\frac{x^2-2x-3}{(x-1)^2}=\frac{(x-3)(x+1)}{(x-1)^2}.$$

So $f'(x)=0$ at $x=3$ and $x=-1$, giving $f(3)=\frac{12}{2}=6$ and $f(-1)=\frac{4}{-2}=-2$. Since $f''(x)=\frac{8}{(x-1)^3}$, we have $f''(3)=1>0$ (minimum) and $f''(-1)=-1<0$ (maximum).



∴ the curve has a vertical asymptote $x=1$, an oblique asymptote $y=x+1$, $y$-intercept $(0,-3)$, a local maximum at $(-1,-2)$ and a local minimum at $(3,6)$.

### *Example 4 — unscaffolded*

Sketch $y=\dfrac{x}{x^2+1}$, showing its asymptotic behaviour, the stationary points and the points of inflection.

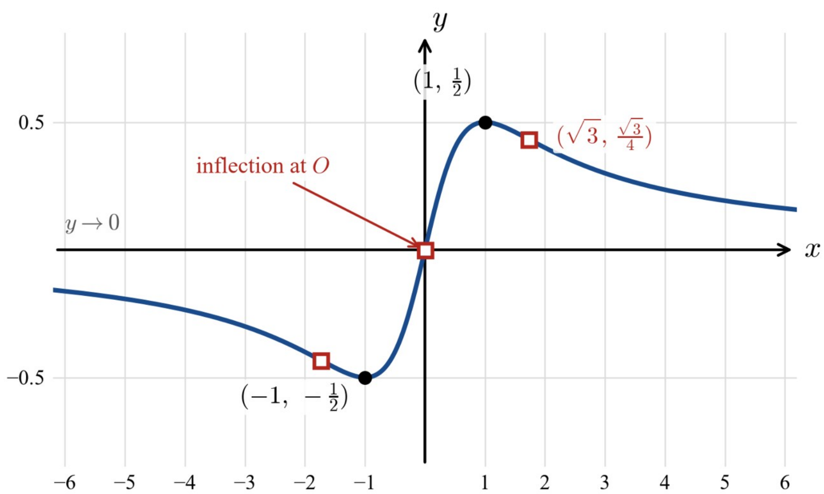
The denominator $x^2+1$ is never zero, so the domain is all real $x$ and there is **no vertical asymptote**. The only intercept is the origin, since $x=0$ makes both $x=0$ and $f(0)=0$. As $deg\,P=1<2=deg\,Q$, the curve flattens onto the horizontal asymptote $y=0$ as $x\to\pm\infty$. The function is odd, so the graph has point symmetry about the origin. Differentiating,

$$f'(x)=\frac{(x^2+1)-x(2x)}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2},$$

which is zero at $x=\pm1$, giving $f(1)=\dfrac{1}{2}$ and $f(-1)=-\dfrac{1}{2}$. The second derivative is

$$f''(x)=\frac{2x(x^2-3)}{(x^2+1)^3},$$

so $f''(1)<0$ (the point $(1,1/2)$ is a local maximum) and $f''(-1)>0$ (the point $(-1,-1/2)$ is a local minimum). Setting $f''(x)=0$ gives $x=0$ and $x=\pm\sqrt{3}$, and $f''$ changes sign at each, so there are three points of inflection: $(0,0)$ and $\left(\pm\sqrt{3},\pm\dfrac{\sqrt{3}}{4}\right)$. Notice that the curve **crosses** its horizontal asymptote $y=0$ at the origin: an asymptote governs the behaviour far out, and need not be a line the curve avoids.




∴ the curve is a bounded odd function with horizontal asymptote $y=0$, a maximum at $(1,1/2)$, a minimum at $(-1,-1/2)$, and inflection points at $(0,0)$ and $(\pm\sqrt{3},\pm\sqrt{3}/4)$.

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## Practice questions

### *Scaffolded*

**Q1.** Consider $y=\dfrac{x+2}{x-1}$. (a) State the domain and the equation of the vertical asymptote. (b) Use the degrees to find the horizontal asymptote. (c) Find the axis intercepts. (d) Show that $f'(x)=\dfrac{-3}{(x-1)^2}$ and hence explain why there are no stationary points.

**Q2.** Consider the simple quotient function $y=\frac{1}{x^2-1}$. (a) State the equations of the two vertical asymptotes. (b) State the horizontal asymptote and the $y$-intercept. (c) Explain why the graph is symmetric in the $y$-axis. (d) Find $f'(x)$, hence the coordinates and nature of the one stationary point.

**Q3.** Consider $y=x+\frac{4}{x}$. (a) Write $y$ as a single fraction and state the vertical asymptote. (b) State the oblique asymptote. (c) Find $f'(x)$ and solve $f'(x)=0$. (d) Find the coordinates and nature of the stationary points.

**Q4.** Consider $y=\frac{3x}{x-2}$. (a) State the vertical asymptote. (b) Find the horizontal asymptote. (c) Find the axis intercepts. (d) Find $f'(x)$ and state whether the curve has any stationary points.

**Q5.** Consider $y=\frac{x^2}{x^2+1}$. (a) Explain why there is no vertical asymptote, and state the horizontal asymptote. (b) Find the coordinates and nature of the stationary point. (c) Given $f''(x)=\frac{2(1-3x^2)}{(x^2+1)^3}$, find the points of inflection. (d) State the range of the function.

**Q6.** Consider $y=\frac{x-1}{x+2}$. (a) State the vertical asymptote. (b) Find the horizontal asymptote. (c) Find the axis intercepts. (d) Find $f'(x)$ and describe how the curve behaves on each branch.

***Unscaffolded***

**Q7.** Sketch $y=\frac{1}{x+3}$, showing the asymptotes and the $y$-intercept.

**Q8.** Sketch $y=\frac{2x-4}{x+1}$, showing the asymptotes and the axis intercepts.

**Q9.** For $y=x+\frac{9}{x}$, state the asymptotes and find the coordinates and nature of the stationary points.

**Q10.** Sketch $y=\frac{x^2-2x-3}{x-2}$, showing the vertical and oblique asymptotes and the axis intercepts.

**Q11.** For $y=\frac{1}{(x-1)(x+2)}$, state the asymptotes and find the coordinates and nature of the stationary point.

**Q12.** For $y=\frac{x}{(x-1)^2}$, state the asymptotes and find the stationary point and the point of inflection.

**Q13.** Sketch $y=\frac{x^2+1}{x^2-1}$, showing the asymptotes, the $y$-intercept and the stationary point. Comment on the symmetry.

**Q14.** Sketch $y=\frac{x^2-4}{x^2-1}$, showing the asymptotes, the axis intercepts and the stationary point.

**Q15.** Sketch $y=\frac{x^2}{x-1}$, showing the vertical and oblique asymptotes and the coordinates and nature of the stationary points.

**Q16.** For $y=\frac{x^3}{x^2-1}$, state the vertical and oblique asymptotes and find the coordinates and nature of the three stationary points.

**Q17.** Consider $y=\frac{1}{x^2+2x+2}$. (a) By completing the square, show that the curve has no vertical asymptote. (b) Find the coordinates and nature of the stationary point, and the horizontal asymptote.

**Q18.** The curve $y=\frac{ax+b}{x-2}$ has horizontal asymptote $y=3$ and passes through the point $(0,1)$. (a) Use the horizontal asymptote to find $a$. (b) Use the point $(0,1)$ to find $b$. (c) State the vertical asymptote and the $x$-intercept of the resulting curve.

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## Answers

**Q1.** Domain $x \neq 1$; vertical asymptote $x=1$, horizontal asymptote $y=1$; intercepts $(-2,0)$ and $(0,-2)$; $f'(x)=\frac{-3}{(x-1)^2}<0$, so no stationary points. **Q2.** Vertical asymptotes $x=-1$ and $x=1$; horizontal asymptote $y=0$, $y$-intercept $(0,-1)$; even function; $f'(x)=-\frac{2x}{(x^2-1)^2}$, stationary point $(0,-1)$, a local maximum. **Q3.** $y=\frac{x^2+4}{x}$, vertical asymptote $x=0$; oblique asymptote $y=x$; $f'(x)=1-\frac{4}{x^2}$, zero at $x=\pm 2$; $(2,4)$ local minimum, $(-2,-4)$ local maximum. **Q4.** Vertical asymptote $x=2$, horizontal asymptote $y=3$; intercept $(0,0)$ only; $f'(x)=\frac{-6}{(x-2)^2}<0$, no stationary points. **Q5.** No vertical asymptote; horizontal asymptote $y=1$; stationary point $(0,0)$, a local minimum; points of inflection $\left(\pm\frac{\sqrt{3}}{3},\frac{1}{4}\right)$; range $[0,1)$. **Q6.** Vertical asymptote $x=-2$, horizontal asymptote $y=1$; intercepts $(1,0)$ and $\left(0,-\frac{1}{2}\right)$; $f'(x)=\frac{3}{(x+2)^2}>0$, increasing on each branch. **Q7.** Vertical asymptote $x=-3$, horizontal asymptote $y=0$; $y$-intercept $\left(0,\frac{1}{3}\right)$; no stationary points. **Q8.** Vertical asymptote $x=-1$, horizontal asymptote $y=2$; intercepts $(2,0)$ and $(0,-4)$; no stationary points. **Q9.** Vertical asymptote $x=0$, oblique asymptote $y=x$; $(3,6)$ local minimum, $(-3,-6)$ local maximum. **Q10.** Vertical asymptote $x=2$, oblique asymptote $y=x$; intercepts $(-1,0)$, $(3,0)$ and $\left(0,\frac{3}{2}\right)$; no stationary points. **Q11.** Vertical asymptotes $x=-2$ and $x=1$, horizontal asymptote $y=0$; stationary point $\left(-\frac{1}{2},-\frac{4}{9}\right)$, a local maximum. **Q12.** Vertical asymptote $x=1$, horizontal asymptote $y=0$; stationary point $\left(-1,-\frac{1}{4}\right)$, a local minimum; point of inflection $\left(-2,-\frac{2}{9}\right)$. **Q13.** Vertical asymptotes $x=\pm 1$, horizontal asymptote $y=1$; $y$-intercept $(0,-1)$; stationary point $(0,-1)$, a local maximum; even (symmetric in the $y$-axis). **Q14.** Vertical asymptotes $x=\pm 1$, horizontal asymptote $y=1$; intercepts $(\pm 2,0)$ and $(0,4)$; stationary point $(0,4)$, a local minimum. **Q15.** Vertical asymptote $x=1$, oblique asymptote $y=x+1$; $(0,0)$ local maximum, $(2,4)$ local minimum. **Q16.** Vertical asymptotes $x=\pm 1$, oblique asymptote $y=x$; $\left(\sqrt{3},\frac{3\sqrt{3}}{2}\right)$ local minimum, $\left(-\sqrt{3},-\frac{3\sqrt{3}}{2}\right)$ local maximum, $(0,0)$ a stationary point of inflection. **Q17.** (a) $x^2+2x+2=(x+1)^2+1 \geq 1>0$, so the denominator is never zero. (b) Stationary point $(-1,1)$, a local maximum; horizontal asymptote $y=0$. **Q18.** (a) $a=3$; (b) $b=-2$; (c) curve $y=\frac{3x-2}{x-2}$, vertical asymptote $x=2$, $x$-intercept $\left(\frac{2}{3},0\right)$.

## Fully-worked solutions

**Q1.** The denominator vanishes at $x=1$, so the domain is $x \neq 1$ and $x=1$ is a vertical asymptote (the numerator is $3 \neq 0$ there). The degrees are equal (1 and 1) with leading coefficients 1 and 1, so $y \to 1$ as $x \to \pm\infty$: horizontal asymptote $y=1$. Setting $y=0$ gives $x=-2$, the point $(-2,0)$; and $f(0)=\frac{2}{-1}=-2$, the point $(0,-2)$. By the quotient rule,

$$f'(x)=\frac{(x-1)-(x+2)}{(x-1)^2}=\frac{-3}{(x-1)^2},$$

which is negative for every $x \neq 1$, so $f'$ is never zero: there are no stationary points and the curve decreases on each branch.

**Q2.** The denominator factorises as $x^2-1=(x-1)(x+1)$, zero at $x=1$ and $x=-1$, giving vertical asymptotes $x=1$ and $x=-1$. The numerator has smaller degree than the denominator, so $y\to 0$: horizontal asymptote $y=0$. Then $f(0)=\frac{1}{-1}=-1$, the $y$-intercept $(0,-1)$. Replacing $x$ by $-x$ leaves $y$ unchanged (the function is even), so the graph is symmetric in the $y$-axis. Differentiating,

$$f'(x)=-\frac{2x}{(x^2-1)^2},$$

which is zero only at $x=0$. There $f(0)=-1$, so the stationary point is $(0,-1)$. On the middle branch $(-1<x<1)$ the denominator is negative and $y$ is negative, and the branch turns at its highest point, so $(0,-1)$ is a local maximum.

**Q3.** Writing $y=\frac{x^2+4}{x}$ shows the denominator is $x$, so the domain is $x\neq 0$ and $x=0$ is a vertical asymptote (the numerator is $4\neq 0$ there). The numerator's degree is one more than the denominator's, and the split $y=x+\frac{4}{x}$ isolates the quotient $x$; since $\frac{4}{x}\to 0$ as $x\to\pm\infty$, the oblique asymptote is $y=x$. Differentiating,

$$f'(x)=1-\frac{4}{x^2}=0\Rightarrow x^2=4\Rightarrow x=\pm 2.$$

Then $f(2)=2+2=4$ and $f(-2)=-2-2=-4$. With $f''(x)=\frac{8}{x^3}$, $f''(2)=1>0$ so $(2,4)$ is a local minimum, and $f''(-2)=-1<0$ so $(-2,-4)$ is a local maximum.

**Q4.** The denominator is zero at $x=2$, giving the vertical asymptote $x=2$. The degrees are equal with leading coefficients 3 and 1, so $y\to\frac{3}{1}=3$: horizontal asymptote $y=3$. Both intercepts coincide at the origin: $y=0$ requires $3x=0$, i.e. $x=0$, and $f(0)=0$, so $(0,0)$ is the only intercept. By the quotient rule,

$$f'(x)=\frac{3(x-2)-3x}{(x-2)^2}=\frac{-6}{(x-2)^2}<0,$$

so there are no stationary points; the curve decreases on each branch, approaching $y=3$ far out and $\pm\infty$ near $x=2$.

**Q5.** Because $x^2+1\geq 1>0$ for all $x$, the denominator is never zero, so there is no vertical asymptote and the domain is all real $x$. Dividing, $y=\frac{x^2}{x^2+1}=1-\frac{1}{x^2+1}$, and $\frac{1}{x^2+1}\to 0$ as $x\to\pm\infty$, so the horizontal asymptote is $y=1$. Differentiating,

$$f'(x)=\frac{2x(x^2+1)-x^2(2x)}{(x^2+1)^2}=\frac{2x}{(x^2+1)^2},$$

which is zero at $x=0$, where $f(0)=0$. As $f'$ changes from negative to positive there, $(0,0)$ is a local minimum. Setting the given $f''(x)=\frac{2(1-3x^2)}{(x^2+1)^3}=0$ gives $x^2=\frac{1}{3}$, i.e. $x=\pm\frac{\sqrt{3}}{3}$; then $y=1-\frac{1}{1/3+1}=1-\frac{3}{4}=\frac{1}{4}$, so the points of inflection are $\left(\pm\frac{\sqrt{3}}{3},\frac{1}{4}\right)$. Since the minimum value is 0 and $y\to 1$ without reaching it, the range is $[0,1)$.

**Q6.** The denominator is zero at $x=-2$, giving the vertical asymptote $x=-2$. Equal degrees with leading coefficients 1 and 1 give the horizontal asymptote $y=1$. Setting $y=0$ gives $x=1$, the point $(1,0)$, and $f(0)=\frac{-1}{2}$, the point $\left(0,-\frac{1}{2}\right)$. By the quotient rule,

$$f'(x)=\frac{(x+2)-(x-1)}{(x+2)^2}=\frac{3}{(x+2)^2}>0,$$

so there are no stationary points; the curve increases on each branch, rising from $y=1$ toward $+\infty$ on the left branch and from $-\infty$ back toward $y=1$ on the right branch.

**Q7.** The denominator is zero at $x=-3$, so $x=-3$ is a vertical asymptote. The numerator has smaller degree, so $y\to 0$: horizontal asymptote $y=0$. There is no $x$-intercept (the numerator 1 is never zero), and $f(0)=\frac{1}{3}$ gives the $y$-intercept $\left(0,\frac{1}{3}\right)$. Since $f'(x)=-\frac{1}{(x+3)^2}<0$, the curve decreases on each branch: to the right of $x=-3$ it falls from $+\infty$ toward 0, and to the left it falls from 0 toward $-\infty$.

**Q8.** The denominator is zero at $x=-1$, giving the vertical asymptote $x=-1$. Equal degrees with leading coefficients 2 and 1 give the horizontal asymptote $y=\frac{2}{1}=2$. Setting $2x-4=0$ gives the $x$-intercept $(2,0)$, and $f(0)=\frac{-4}{1}=-4$ gives the $y$-intercept $(0,-4)$. Dividing, $y=2-\frac{6}{x+1}$, so $f'(x)=\frac{6}{(x+1)^2}>0$: no stationary points, increasing on each branch.

**Q9.** Writing $y=\frac{x^2+9}{x}$ shows the vertical asymptote $x=0$, and the split $y=x+\frac{9}{x}$ gives the oblique asymptote $y=x$ (since $\frac{9}{x}\to 0$). Then

$$f'(x)=1-\frac{9}{x^2}=0\Rightarrow x^2=9\Rightarrow x=\pm 3,$$

with $f(3)=3+3=6$ and $f(-3)=-3-3=-6$. As $f''(x)=\frac{18}{x^3}$, $f''(3)>0$ so $(3,6)$ is a local minimum, and $f''(-3)<0$ so $(-3,-6)$ is a local maximum.

**Q10.** The denominator is zero at $x=2$, giving the vertical asymptote $x=2$; the numerator there is $4-4-3=-3\neq 0$. The numerator factorises as $x^2-2x-3=(x-3)(x+1)$, so the $x$-intercepts are $(3,0)$ and $(-1,0)$, and $f(0)=\frac{-3}{-2}=\frac{3}{2}$ gives $\left(0,\frac{3}{2}\right)$. Dividing, $\frac{x^2-2x-3}{x-2}=x-\frac{3}{x-2}$, so the oblique asymptote is $y=x$. Then $f'(x)=1+\frac{3}{(x-2)^2}>0$ for all $x\neq 2$, so there are no stationary points; the curve increases on each branch.

**Q11.** The denominator $(x-1)(x+2)$ is zero at $x=1$ and $x=-2$, giving vertical asymptotes $x=1$ and $x=-2$. The numerator has smaller degree, so the horizontal asymptote is $y=0$; there is no $x$-intercept, and $f(0)=\frac{1}{(-1)(2)}=-\frac{1}{2}$. With $g(x)=x^2+x-2$,

$$f'(x)=-\frac{g'(x)}{g(x)^2}=-\frac{2x+1}{(x^2+x-2)^2}=0\Rightarrow x=-1/2.$$

Then $f(-1/2)=\frac{1}{(-3/2)(3/2)}=\frac{1}{-9/4}=-\frac{4}{9}$. This point lies on the middle branch $(-2<x<1)$, where the denominator is negative and the branch turns at its highest point, so $\left(-\frac{1}{2},-\frac{4}{9}\right)$ is a local maximum.

**Q12.** The denominator $(x-1)^2$ is zero at $x=1$, giving the vertical asymptote $x=1$; the numerator has smaller degree, so the horizontal asymptote is $y=0$. Both intercepts are at the origin. Writing $f(x)=x(x-1)^{-2}$,

$$f'(x)=(x-1)^{-2}+x\cdot(-2)(x-1)^{-3}=\frac{(x-1)-2x}{(x-1)^3}=\frac{-(x+1)}{(x-1)^3},$$

so $f'(x)=0$ at $x=-1$, where $f(-1)=\frac{-1}{4}=-\frac{1}{4}$. Since $f''(x)=\frac{2(x+2)}{(x-1)^4}$ is positive at $x=-1$, the point $\left(-1,-\frac{1}{4}\right)$ is a local minimum. Setting $f''(x)=0$ gives $x=-2$, and $f''$ changes sign there, so $\left(-2,-\frac{2}{9}\right)$ is a point of inflection.

**Q13.** The denominator $x^2-1=(x-1)(x+1)$ gives vertical asymptotes $x=\pm1$. Equal degrees with leading coefficients 1 and 1 give the horizontal asymptote $y=1$; also $y=\frac{x^2+1}{x^2-1}=1+\frac{2}{x^2-1}$, which confirms $y\to1$. There is no $x$-intercept ($x^2+1>0$), and $f(0)=\frac{1}{-1}=-1$. Differentiating,

$$f'(x)=\frac{2x(x^2-1)-(x^2+1)(2x)}{(x^2-1)^2}=\frac{-4x}{(x^2-1)^2},$$

which is zero at $x=0$, where $f(0)=-1$; the middle branch turns at its highest point, so $(0,-1)$ is a local maximum. Replacing $x$ by $-x$ leaves $y$ unchanged, so the function is even and the graph is symmetric in the $y$-axis.

**Q14.** The denominator gives vertical asymptotes $x=\pm1$. Equal degrees with leading coefficients 1 and 1 give the horizontal asymptote $y=1$; indeed $y=1-\frac{3}{x^2-1}$. The numerator $x^2-4=(x-2)(x+2)$ gives $x$-intercepts $(\pm2,0)$, and $f(0)=\frac{-4}{-1}=4$ gives $(0,4)$. Differentiating,

$$f'(x)=\frac{2x(x^2-1)-(x^2-4)(2x)}{(x^2-1)^2}=\frac{6x}{(x^2-1)^2},$$

zero at $x=0$, where $f(0)=4$; the middle branch turns at its lowest point, so $(0,4)$ is a local minimum.

**Q15.** The denominator is zero at $x=1$, giving the vertical asymptote $x=1$. The numerator's degree is one more than the denominator's, so dividing gives $\frac{x^2}{x-1}=(x+1)+\frac{1}{x-1}$, hence the oblique asymptote is $y=x+1$. Both intercepts are at the origin. Differentiating,

$$f'(x)=\frac{2x(x-1)-x^2}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}=\frac{x(x-2)}{(x-1)^2},$$

so $f'(x)=0$ at $x=0$ and $x=2$, with $f(0)=0$ and $f(2)=4$. Since $f''(x)=\frac{2}{(x-1)^3}$, $f''(0)=-2<0$ so $(0,0)$ is a local maximum, and $f''(2)=2>0$ so $(2,4)$ is a local minimum.

**Q16.** The denominator $x^2-1=(x-1)(x+1)$ gives vertical asymptotes $x=\pm 1$. Dividing, $\frac{x^3}{x^2-1}=x+\frac{x}{x^2-1}$, and the remainder term $\to 0$ as $x\to\pm\infty$, so the oblique asymptote is $y=x$. Differentiating,

$$f'(x)=\frac{3x^2(x^2-1)-x^3(2x)}{(x^2-1)^2}=\frac{x^2(x^2-3)}{(x^2-1)^2},$$

so $f'(x)=0$ at $x=0$ and $x=\pm\sqrt{3}$. Then $f(\sqrt{3})=\frac{3\sqrt{3}}{2}$ and $f(-\sqrt{3})=-\frac{3\sqrt{3}}{2}$. Using $f''(x)=\frac{2x(x^2+3)}{(x^2-1)^3}$: $f''(\sqrt{3})>0$ so $\left(\sqrt{3},\frac{3\sqrt{3}}{2}\right)$ is a local minimum, and $f''(-\sqrt{3})<0$ so $\left(-\sqrt{3},-\frac{3\sqrt{3}}{2}\right)$ is a local maximum. At $x=0$ both $f'(0)=0$ and $f''(0)=0$ with $f''$ changing sign, so $(0,0)$ is a stationary point of inflection.

**Q17.** (a) Completing the square, $x^2+2x+2=(x+1)^2+1\geq 1>0$ for all $x$, so the denominator is never zero and there is no vertical asymptote; the domain is all real $x$. (b) The numerator has smaller degree, so the horizontal asymptote is $y=0$. Differentiating $f(x)=((x+1)^2+1)^{-1}$,

$$f'(x)=-\frac{2(x+1)}{((x+1)^2+1)^2},$$

which is zero at $x=-1$, where $f(-1)=\frac{1}{0+1}=1$. As the denominator $(x+1)^2+1$ is smallest at $x=-1$, the reciprocal is largest there, so $(-1,1)$ is a local maximum.

**Q18.** (a) As $x\to\pm\infty$, $y=\frac{ax+b}{x-2}\to a$ (equal degrees, denominator leading coefficient 1), so the horizontal asymptote is $y=a$. Given this is $y=3$, we have $a=3$. (b) Substituting $(0,1)$: $1=\frac{a(0)+b}{0-2}=\frac{b}{-2}$, so $b=-2$. (c) The curve is therefore $y=\frac{3x-2}{x-2}$. The denominator is zero at $x=2$, giving the vertical asymptote $x=2$, and $y=0$ requires $3x-2=0$, i.e. the $x$-intercept $\left(\frac{2}{3},0\right)$.

### Theory

So far, curves have usually been given **explicitly**: $y$ is written as a formula in $x$, such as $y=x^2-3x$ or $y=\sqrt{25-x^2}$, and $\frac{dy}{dx}$ follows from the standard rules. Many important curves, however, are described by an equation that links $x$ and $y$ together without $y$ being isolated. The circle

$$x^2+y^2=25,$$

the rectangular hyperbola $xy=8$, the ellipse $x^2+xy+y^2=7$ and the **folium of Descartes** $x^3+y^3=9xy$ are all of this kind. Such an equation is said to define $y$ **implicitly** as a function (or several functions) of $x$. We would like the gradient $\frac{dy}{dx}$ without first solving for $y$ — which is often awkward or impossible.

**The key idea.**

Treat $y$ as an (unknown) function of $x$ and differentiate **both sides of the equation with respect to** $x$, keeping the equation balanced. Every term in $x$ alone differentiates as usual. Every term containing $y$ is a function of a function, so the **chain rule** applies: if $u=u(y)$ and $y=y(x)$ then

$$\frac{d}{dx}[u(y)]=\frac{du}{dy}\cdot\frac{dy}{dx}.$$

In particular, for a power of $y$,

$$\frac{d}{dx}[y^n]=ny^{n-1}\frac{dy}{dx}.$$

So $\frac{d}{dx}[y^2]=2y\frac{dy}{dx}$, $\frac{d}{dx}[y^3]=3y^2\frac{dy}{dx}$, $\frac{d}{dx}[\sin y]=\cos y\frac{dy}{dx}$ and $\frac{d}{dx}[\cos y]=-\sin y\frac{dy}{dx}$. The factor $\frac{dy}{dx}$ is produced every time a $y$-term is differentiated — that is the whole of the technique.

**Mixed terms — the product rule.**

A term that contains **both** $x$ and $y$, such as $xy$ or $x^2y$, is a product of two functions of $x$, so the **product rule** is used and the chain rule is applied to the $y$-factor:

$$\frac{d}{dx}[xy]=x\frac{dy}{dx}+y,\ \frac{d}{dx}[x^2y]=x^2\frac{dy}{dx}+2xy.$$

When the $y$-factor is not simply $y$, the chain rule does real work inside the product rule and an extra factor appears beside $\frac{dy}{dx}$:

$$\frac{d}{dx}[xy^2]=2xy\frac{dy}{dx}+y^2,\ \frac{d}{dx}[x\sin y]=x\cos y\frac{dy}{dx}+\sin y.$$

Differentiating $x\cos y+2y=6$, for instance, gives

$$\cos y-x\sin y\frac{dy}{dx}+2\frac{dy}{dx}=0,\ \text{so}\ \frac{dy}{dx}=\frac{\cos y}{x\sin y-2},$$

which is $-\frac{1}{2}$ at the point $(6,0)$ on that curve.

**Solving for the gradient.**

After differentiating, the equation is linear in $\frac{dy}{dx}$. Collect every term that contains $\frac{dy}{dx}$ on one side, factor it out, and divide. The gradient is then expressed **in terms of both $x$ and $y$**. This is expected: for a given $x$-value the relation may have more than one $y$-value, so the curve has several branches, and the value of $y$ selects which branch the gradient is being measured on. On $x^2+y^2=25$, for example, $x=3$ gives $y=4$ on the upper semicircle, where $\frac{dy}{dx}=-\frac{3}{4}$, and $y=-4$ on the lower semicircle, where $\frac{dy}{dx}=\frac{3}{4}$. The four steps are:

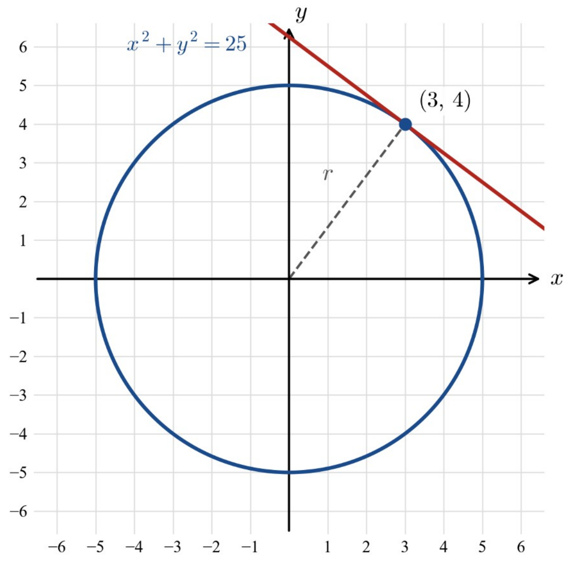
1. differentiate both sides with respect to $x$ (chain rule on $y$-terms, product rule on mixed terms);
2. gather the $\frac{dy}{dx}$ terms on one side and everything else on the other;
3. factor out $\frac{dy}{dx}$;
4. divide to make $\frac{dy}{dx}$ the subject.

**Tangents and normals.**

Once $\frac{dy}{dx}$ is known, the gradient of the curve at a particular point $(x_1, y_1)$ on it is found by substituting **both** coordinates. The **tangent** at that point is

$$y-y_1=m(x-x_1), m=\left.\frac{dy}{dx}\right|_{(x_1,y_1)}.$$

The **normal** is the line through $(x_1, y_1)$ perpendicular to the tangent; provided $m\neq 0$ its gradient is $-\frac{1}{m}$. The diagram shows the circle $x^2+y^2=25$ with its tangent at $(3,4)$: here $\frac{dy}{dx}=-\frac{x}{y}=-\frac{3}{4}$, and the tangent is perpendicular to the radius drawn to that point.



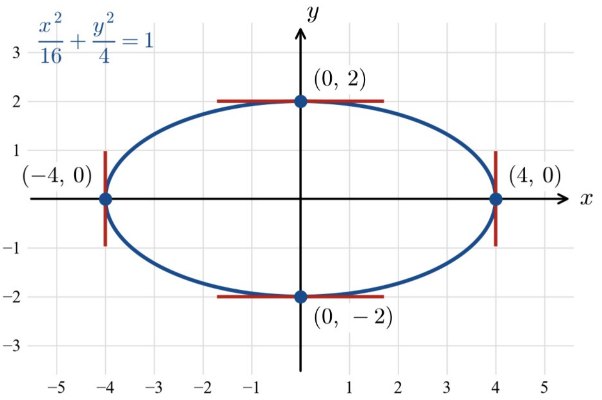

**Horizontal and vertical tangents.**

When $\frac{dy}{dx}$ is written as a quotient $\frac{P(x,y)}{Q(x,y)}$, the tangent is

- **horizontal** where $\frac{dy}{dx}=0$, that is where the **numerator** $P=0$ (and $Q\neq 0$);

- **vertical** where $\dfrac{dy}{dx}$ is **undefined**, that is where the **denominator** $Q=0$ (and $P\neq 0$).

Each condition is solved together with the original equation of the curve. For the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$ the gradient is $\dfrac{dy}{dx}=-\dfrac{x}{4y}$, so the tangent is horizontal at $(0,2)$ and $(0,-2)$ and vertical at $(4,0)$ and $(-4,0)$, as marked below.




**Second derivatives implicitly.**

To obtain $\dfrac{d^2y}{dx^2}$, differentiate the expression for $\dfrac{dy}{dx}$ again with respect to $x$ — using the quotient rule and, once more, the chain rule on every $y$-term — and then **substitute the expression already found for** $\dfrac{dy}{dx}$. For the circle $x^2+y^2=25$, from $\dfrac{dy}{dx}=-\dfrac{x}{y}$,

$$\frac{d^2y}{dx^2}=-\frac{y-x\frac{dy}{dx}}{y^2}=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}=-\frac{y^2+x^2}{y^3}=-\frac{25}{y^3},$$

the last step using $x^2+y^2=25$. The negative sign (for $y>0$) confirms the upper semicircle is concave down.

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## Worked examples

### *Example 1 — scaffolded*

For the circle $x^2+y^2=169$, find $\dfrac{dy}{dx}$ by implicit differentiation, and hence the gradient of the curve at the point $(5,12)$.

| Working | Comment |
| --- | --- |
| $\dfrac{d}{dx}[x^2]+\dfrac{d}{dx}[y^2]=\dfrac{d}{dx}[169]$. | Differentiate both sides with respect to $x$. |
| $2x+2y\dfrac{dy}{dx}=0$. | Chain rule on the $y$-term: $\dfrac{d}{dx}[y^2]=2y\dfrac{dy}{dx}$; the constant gives $0$. |
| $2y\dfrac{dy}{dx}=-2x$. | Move the $x$-term to the right-hand side. |
| $\dfrac{dy}{dx}=-\dfrac{2x}{2y}=-\dfrac{x}{y}$. | Divide by $2y$ and simplify; the gradient depends on both $x$ and $y$. |

| Working | Comment |
|---|---|
| At $(5,12)$: $\frac{dy}{dx}=-\frac{5}{12}$. | Substitute $x=5,\ y=12$. |

### *Example 2 — scaffolded*

The point $(1,2)$ lies on the curve $x^2+xy+y^2=7$. Find $\frac{dy}{dx}$, and hence the equations of the **tangent** and the **normal** to the curve at $(1,2)$.

| Working | Comment |
|---|---|
| $2x+\left(x\frac{dy}{dx}+y\right)+2y\frac{dy}{dx}=0$. | Differentiate term by term; the middle term $xy$ needs the product rule. |
| $(x+2y)\frac{dy}{dx}=-(2x+y)$. | Gather $\frac{dy}{dx}$ terms on the left and factor. |
| $\frac{dy}{dx}=-\frac{2x+y}{x+2y}$. | Divide by $(x+2y)$. |
| At $(1,2)$: $\frac{dy}{dx}=-\frac{2(1)+2}{1+2(2)}=-\frac{4}{5}$. | Substitute the coordinates; this is the tangent gradient $m$. |
| Tangent: $y-2=-\frac{4}{5}(x-1)$, i.e. $4x+5y=14$. | Point–gradient form, then clear fractions. |
| Normal gradient $=-\frac{1}{m}=\frac{5}{4}$; normal: $y-2=\frac{5}{4}(x-1)$, i.e. $5x-4y+3=0$. | The normal is perpendicular, so its gradient is $-\frac{1}{m}$. |

### *Example 3 — unscaffolded*

The folium of Descartes has equation $x^3+y^3=9xy$. Verify that $(2,4)$ lies on the curve, and find the equation of the tangent there.

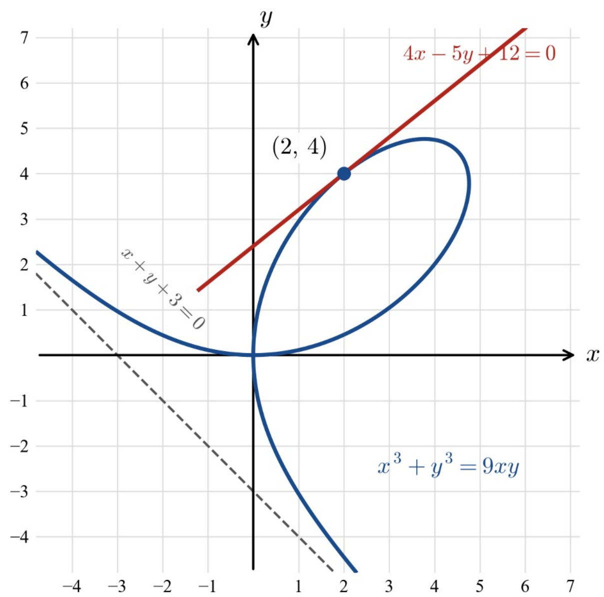
Substituting $(2,4)$: $2^3+4^3=8+64=72$ and $9(2)(4)=72$, so the point lies on the curve. Differentiating both sides with respect to $x$, using the chain rule on $y^3$ and the product rule on $9xy$,

$$3x^2+3y^2\frac{dy}{dx}=9\left(y+x\frac{dy}{dx}\right).$$

Collecting the gradient terms,

$$(3y^2-9x)\frac{dy}{dx}=9y-3x^2,\ \frac{dy}{dx}=\frac{9y-3x^2}{3y^2-9x}=\frac{3y-x^2}{y^2-3x}.$$

At $(2,4)$, $\frac{dy}{dx}=\frac{3(4)-2^2}{4^2-3(2)}=\frac{12-4}{16-6}=\frac{4}{5}$. The tangent is $y-4=\frac{4}{5}(x-2)$; multiplying by 5, $5y-20=4x-8$.




$\therefore$ the tangent to $x^3+y^3=9xy$ at $(2,4)$ is $4x-5y+12=0$.

### *Example 4 — unscaffolded*

For the curve $x^2+y^2=169$, find $\frac{d^2y}{dx^2}$ in terms of $y$, and evaluate it at $(5,12)$.

From Example 1, $\frac{dy}{dx}=-\frac{x}{y}$. Differentiating again with respect to $x$ by the quotient rule, and remembering that $y$ is a function of $x$,

$$\frac{d^2y}{dx^2}=-\frac{\frac{d}{dx}(x)\cdot y-x\cdot\frac{dy}{dx}}{y^2}=-\frac{y-x\frac{dy}{dx}}{y^2}.$$

Substituting $\frac{dy}{dx}=-\frac{x}{y}$,

$$\frac{d^2y}{dx^2}=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}=-\frac{\frac{y^2+x^2}{y}}{y^2}=-\frac{x^2+y^2}{y^3}=-\frac{169}{y^3},$$

since $x^2+y^2=169$ on the curve. At $(5,12)$, $\frac{d^2y}{dx^2}=-\frac{169}{12^3}=-\frac{169}{1728}$.

$\therefore \frac{d^2y}{dx^2}=-\frac{169}{y^3}$, which is $-\frac{169}{1728}$ at $(5,12)$; the negative value shows the curve is concave down there.

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## Practice questions

### *Scaffolded*

**Q1.** The circle $x^2+y^2=100$ passes through $(6,8)$. (a) Differentiate both sides with respect to $x$. (b) Make $\frac{dy}{dx}$ the subject. (c) Find the gradient of the tangent at $(6,8)$. (d) Find the equation of the tangent at $(6,8)$.

**Q2.** A curve has equation $xy=12$. (a) Differentiate both sides, using the product rule on $xy$. (b) Show that $\frac{dy}{dx}=-\frac{y}{x}$. (c) Find the gradient at $(3,4)$. (d) Find the equation of the tangent at $(3,4)$.

**Q3.** The point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ lies on the curve $x\sin y+y=\pi$. (a) Differentiate both sides with respect to $x$, using the product rule on $x\sin y$. (b) Show that $\frac{dy}{dx}=-\frac{\sin y}{x\cos y+1}$. (c) Find the gradient at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$. (d) Find the equation of the tangent at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$.

**Q4.** For the circle $x^2+y^2=100$: (a) State $\frac{dy}{dx}$ (from Q1). (b) Differentiate $\frac{dy}{dx}=-\frac{x}{y}$ again, using the quotient rule. (c) Substitute $\frac{dy}{dx}=-\frac{x}{y}$ and simplify to a single fraction. (d) Hence show $\frac{d^2y}{dx^2}=-\frac{100}{y^3}$, and evaluate it at $(6,8)$.

**Q5.** For the ellipse $x^2+4y^2=100$: (a) Show that $\frac{dy}{dx}=-\frac{x}{4y}$. (b) Find the points where the tangent is **horizontal** (set the numerator to $0$). (c) Find the points where the tangent is **vertical** (set the denominator to $0$).

**Q6.** The point $(1,3)$ lies on the curve $x^2-xy+y^2=7$. (a) Differentiate both sides with respect to $x$. (b) Show that $\frac{dy}{dx}=\frac{2x-y}{x-2y}$. (c) Find the gradient at $(1,3)$. (d) Find the equations of the tangent and the normal at $(1,3)$.

***Unscaffolded***

**Q7.** Find $\frac{dy}{dx}$ for the curve $xy=10$.

**Q8.** Find $\frac{dy}{dx}$ for the curve $x^3+y^3=8$.

**Q9.** Find $\frac{dy}{dx}$ for the curve $3xy^2=x+y$.

**Q10.** Find $\frac{dy}{dx}$ for the curve $x^2\sin y+y=4$.

**Q11.** Find the gradient of the tangent to $x^2+y^2=25$ at the point $(-4,3)$.

**Q12.** Find the equation of the tangent to the ellipse $x^2+4y^2=20$ at $(2,2)$.

**Q13.** Find the equation of the normal to the circle $x^2+y^2=25$ at $(3,4)$, and explain why it passes through the origin.

**Q14.** For the ellipse $4x^2+9y^2=36$, find the points where the tangent is horizontal and the points where it is vertical.

**Q15.** The point $(1,2)$ lies on the curve $x^2-2xy+2y^2=5$. Find $\frac{dy}{dx}$ and the gradient of the curve at $(1,2)$.

**Q16.** For the curve $xy=1$, show that $\frac{d^2y}{dx^2}=\frac{2y}{x^2}$, and hence that $\frac{d^2y}{dx^2}=\frac{2}{x^3}$.

**Q17.** Find the equations of the tangent and the normal to the ellipse $x^2+4y^2=8$ at the point $(2,1)$.

**Q18.** A circle has equation $x^2+y^2=r^2$, where $r>0$. Show that at any point $(x_1,y_1)$ on the circle with $x_1\neq 0$ and $y_1\neq 0$, the tangent is perpendicular to the radius drawn from the origin to that point.

## Answers

**Q1.** (a) $2x+2y\frac{dy}{dx}=0$; (b) $\frac{dy}{dx}=-\frac{x}{y}$; (c) $-\frac{3}{4}$; (d) $3x+4y=50$. **Q2.** (a) $x\frac{dy}{dx}+y=0$; (b) $\frac{dy}{dx}=-\frac{y}{x}$; (c) $-\frac{4}{3}$; (d) $4x+3y=24$. **Q3.** (a) $\sin y+x\cos y\frac{dy}{dx}+\frac{dy}{dx}=0$; (b) $\frac{dy}{dx}=-\frac{\sin y}{x\cos y+1}$; (c) $-1$; (d) $x+y=\pi$. **Q4.** (a) $\frac{dy}{dx}=-\frac{x}{y}$; (b) $\frac{d^2y}{dx^2}=-\frac{y-x\frac{dy}{dx}}{y^2}$; (c) $-\frac{x^2+y^2}{y^3}$; (d) $\frac{d^2y}{dx^2}=-\frac{100}{y^3}$, which is $-\frac{25}{128}$ at $(6,8)$. **Q5.** (a) $\frac{dy}{dx}=-\frac{x}{4y}$; (b) horizontal at $(0,5)$ and $(0,-5)$; (c) vertical at $(10,0)$ and $(-10,0)$. **Q6.** (a) $2x-\left(y+x\frac{dy}{dx}\right)+2y\frac{dy}{dx}=0$; (b) $\frac{dy}{dx}=\frac{2x-y}{x-2y}$; (c) $\frac{1}{5}$; (d) tangent $x-5y+14=0$; normal $5x+y=8$. **Q7.** $\frac{dy}{dx}=-\frac{y}{x}$. **Q8.** $\frac{dy}{dx}=-\frac{x^2}{y^2}$. **Q9.** $\frac{dy}{dx}=\frac{1-3y^2}{6xy-1}$. **Q10.** $\frac{dy}{dx}=-\frac{2x\sin y}{x^2\cos y+1}$. **Q11.** $\frac{4}{3}$. **Q12.** $x+4y=10$. **Q13.** $4x-3y=0$ (that is $y=\frac{4}{3}x$); it passes through $(0,0)$ because the normal is along the radius. **Q14.** Horizontal at $(0,2)$ and $(0,-2)$; vertical at $(3,0)$ and $(-3,0)$. **Q15.** $\frac{dy}{dx}=\frac{x-y}{x-2y}$; at $(1,2)$ the gradient is $\frac{1}{3}$. **Q16.** $\frac{d^2y}{dx^2}=\frac{2y}{x^2}=\frac{2}{x^3}$ (using $y=\frac{1}{x}$). **Q17.** Tangent $x+2y=4$; normal $2x-y=3$. **Q18.** $\frac{dy}{dx}=-\frac{x_1}{y_1}$ and the radius has gradient $\frac{y_1}{x_1}$; their product is $-1$, so they are perpendicular.

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## Fully-worked solutions

**Q1.** Differentiating $x^2+y^2=100$ gives $2x+2y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{x}{y}$. At $(6,8)$ the gradient is $-\frac{6}{8}=-\frac{3}{4}$. The tangent is $y-8=-\frac{3}{4}(x-6)$; multiplying by 4, $4y-32=-3x+18$, so $3x+4y=50$.

**Q2.** Differentiating $xy=12$ with the product rule gives $x\frac{dy}{dx}+y=0$, hence $\frac{dy}{dx}=-\frac{y}{x}$. At $(3,4)$ the gradient is $-\frac{4}{3}$. The tangent is $y-4=-\frac{4}{3}(x-3)$; multiplying by 3, $3y-12=-4x+12$, so $4x+3y=24$.

**Q3.** Differentiating $x\sin y+y=\pi$, using the product rule on $x\sin y$ and the chain rule on $\sin y$,

$$\sin y+x\cos y\frac{dy}{dx}+\frac{dy}{dx}=0\Rightarrow(x\cos y+1)\frac{dy}{dx}=-\sin y\Rightarrow\frac{dy}{dx}=-\frac{\sin y}{x\cos y+1}.$$

At $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$, $\sin y=1$ and $\cos y=0$, so the gradient is $-\frac{1}{0+1}=-1$. The tangent is $y-\frac{\pi}{2}=-\left(x-\frac{\pi}{2}\right)$, that is $x+y=\pi$.

**Q4.** From Q1, $\frac{dy}{dx}=-\frac{x}{y}$. Differentiating again by the quotient rule,

$$\frac{d^2y}{dx^2}=-\frac{y-x\frac{dy}{dx}}{y^2}=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}=-\frac{x^2+y^2}{y^3}=-\frac{100}{y^3},$$

using $x^2+y^2=100$. At $(6,8)$ this is $-\frac{100}{8^3}=-\frac{100}{512}=-\frac{25}{128}$.

**Q5.** Differentiating $x^2+4y^2=100$ gives $2x+8y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{2x}{8y}=-\frac{x}{4y}$. The tangent is **horizontal** when the numerator is zero, i.e. $x=0$; then $4y^2=100$, $y=\pm5$, giving $(0,5)$ and $(0,-5)$. It is **vertical** when the denominator is zero, i.e. $y=0$; then $x^2=100$, $x=\pm10$, giving $(10,0)$ and $(-10,0)$.

**Q6.** Differentiating $x^2-xy+y^2=7$, and using the product rule on $xy$,

$$2x-\left(y+x\frac{dy}{dx}\right)+2y\frac{dy}{dx}=0\Rightarrow(2y-x)\frac{dy}{dx}=y-2x\Rightarrow\frac{dy}{dx}=\frac{y-2x}{2y-x}=\frac{2x-y}{x-2y}.$$

At $(1,3)$ the gradient is $\frac{2(1)-3}{1-2(3)}=\frac{-1}{-5}=\frac{1}{5}$. The tangent is $y-3=\frac{1}{5}(x-1)$; multiplying by 5, $5y-15=x-1$, so $x-5y+14=0$. The normal has gradient $-5$: $y-3=-5(x-1)$, giving $5x+y=8$.

**Q7.** Differentiating $xy=10$ by the product rule gives $x\frac{dy}{dx}+y=0$, so $\frac{dy}{dx}=-\frac{y}{x}$.

**Q8.** Differentiating $x^3+y^3=8$ gives $3x^2+3y^2\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{x^2}{y^2}$.

**Q9.** Differentiating $3xy^2=x+y$, the left-hand side needs the product rule with the chain rule applied to $y^2$:

$$3\left(y^2+2xy\frac{dy}{dx}\right)=1+\frac{dy}{dx}\Rightarrow3y^2+6xy\frac{dy}{dx}=1+\frac{dy}{dx}\Rightarrow(6xy-1)\frac{dy}{dx}=1-3y^2,$$

so $\frac{dy}{dx}=\frac{1-3y^2}{6xy-1}$.

**Q10.** Differentiating $x^2\sin y+y=4$, the first term needs the product rule with the chain rule applied to $\sin y$:

$$2x\sin y+x^2\cos y\frac{dy}{dx}+\frac{dy}{dx}=0\Rightarrow\left(x^2\cos y+1\right)\frac{dy}{dx}=-2x\sin y,$$

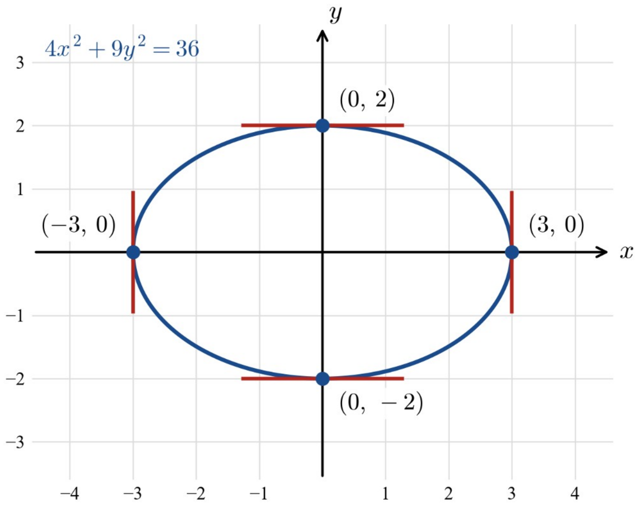
so $\frac{dy}{dx}=-\frac{2x\sin y}{x^2\cos y+1}$.

**Q11.** Differentiating $x^2+y^2=25$ gives $2x+2y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{x}{y}$. At $(-4,3)$ the gradient is $-\frac{-4}{3}=\frac{4}{3}$.

**Q12.** Differentiating $x^2+4y^2=20$ gives $2x+8y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{x}{4y}$. At $(2,2)$ the gradient is $-\frac{2}{8}=-\frac{1}{4}$. The tangent is $y-2=-\frac{1}{4}(x-2)$; multiplying by 4, $4y-8=-x+2$, so $x+4y=10$.

**Q13.** From $\frac{dy}{dx}=-\frac{x}{y}$, the tangent gradient at $(3,4)$ is $-\frac{3}{4}$, so the normal gradient is $\frac{4}{3}$. The normal is $y-4=\frac{4}{3}(x-3)$; multiplying by 3, $3y-12=4x-12$, so $4x-3y=0$, i.e. $y=\frac{4}{3}x$. Since there is no constant term, the line passes through the origin — as expected, because the normal to a circle lies along a radius.

**Q14.** Differentiating $4x^2+9y^2=36$ gives $8x+18y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{8x}{18y}=-\frac{4x}{9y}$. Horizontal tangents occur where $x=0$: then $9y^2=36$, $y=\pm2$, giving $(0,2)$ and $(0,-2)$. Vertical tangents occur where $y=0$: then $4x^2=36$, $x=\pm3$, giving $(3,0)$ and $(-3,0)$.




**Q15.** Differentiating $x^2-2xy+2y^2=5$, using the product rule on $2xy$,

$$2x-2\left(y+x\frac{dy}{dx}\right)+4y\frac{dy}{dx}=0\Rightarrow(4y-2x)\frac{dy}{dx}=2y-2x\Rightarrow\frac{dy}{dx}=\frac{y-x}{2y-x}=\frac{x-y}{x-2y}.$$

At $(1,2)$ the gradient is $\frac{1-2}{1-4}=\frac{-1}{-3}=\frac{1}{3}$.

**Q16.** Differentiating $xy=1$ gives $x\frac{dy}{dx}+y=0$, so $\frac{dy}{dx}=-\frac{y}{x}$. Differentiating again by the quotient rule,

$$\frac{d^2y}{dx^2}=-\frac{\frac{dy}{dx}\cdot x-y}{x^2}=-\frac{\left(-\frac{y}{x}\right)x-y}{x^2}=-\frac{-y-y}{x^2}=\frac{2y}{x^2}.$$

Since $xy=1$ gives $y=\frac{1}{x}$, this is $\frac{2}{x^2}\cdot\frac{1}{x}=\frac{2}{x^3}$.

**Q17.** Differentiating $x^2+4y^2=8$ gives $2x+8y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{x}{4y}$. At $(2,1)$ the gradient is $-\frac{2}{4}=-\frac{1}{2}$. The tangent is $y-1=-\frac{1}{2}(x-2)$; multiplying by 2, $2y-2=-x+2$, so $x+2y=4$. The normal has gradient 2: $y-1=2(x-2)$, giving $2x-y=3$.

**Q18.** Differentiating $x^2+y^2=r^2$ gives $2x+2y\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\frac{x}{y}$. At $(x_1,y_1)$ the tangent gradient is $m_1=-\frac{x_1}{y_1}$. The radius from the origin to $(x_1,y_1)$ has gradient $m_2=\frac{y_1}{x_1}$. Their product is $m_1m_2=-\frac{x_1}{y_1}\cdot\frac{y_1}{x_1}=-1$, so the tangent and the radius are perpendicular.