Specialist Mathematics Units 3 & 4

Chapter 4 — Complex numbers

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Contents

Theory

The imaginary unit.

Some quadratic equations have no real solutions. The equation $x^2 = -1$, for instance, cannot be satisfied by any real number, because the square of a real number is never negative. To extend the number system so that such equations can be solved, we introduce a new number called the **imaginary unit**, written i, with the defining property

$$i^2 = -1.$$

So i is a square root of -1. Every rule of ordinary algebra is kept, and each time i^2 appears it is replaced by -1.

Complex numbers.

A **complex number** is any number that can be written in the **Cartesian form** (also called **rectangular form**)

$$z = a + bi,$$

where a and b are real numbers. The real number a is the **real part** of z and the real number b is the **imaginary part**, written

$$\mathrm{Re}(z) = a, \mathrm{Im}(z) = b.$$

Note that the imaginary part is the real coefficient b, not bi. The complete collection of complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

Every real number a is also a complex number, namely $a + 0i$, so the real numbers $\mathbb{R}$ sit inside $\mathbb{C}$. A complex number of the form bi (with $a = 0$ and $b \neq 0$) is called **purely imaginary**. For example, $z = 3 - 5i$ has $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = -5$, while $z = 7$ is real and $z = 4i$ is purely imaginary.

Equality of complex numbers.

Two complex numbers are equal exactly when their real parts are equal **and** their imaginary parts are equal. That is, for real a, b, c, d,

$$a + bi = c + di \Leftrightarrow a = c \text{ and } b = d.$$

A single equation between complex numbers therefore carries two real equations, one for each part. In particular $a + bi = 0$ forces $a = 0$ and $b = 0$.

Addition and subtraction.

Complex numbers are added and subtracted by combining the real parts and the imaginary parts separately:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

The result is again written in Cartesian form. For example, $(2 + 3i) + (4 - i) = 6 + 2i$.

Multiplication by a real scalar.

To multiply a complex number by a real number k, multiply each part by k:

$$k(a + bi) = ka + kbi.$$

This is exactly how a real multiple distributes, and it lets us form real combinations such as $2z - 3w$.

Multiplication of complex numbers.

Two complex numbers are multiplied by expanding the brackets in the usual way and then replacing i^2 by -1:

$$(a+bi)(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i.$$

There is no need to memorise the final formula — simply expand and use $i^2=-1$. For example,

$$(2+3i)(4-i)=8-2i+12i-3i^2=8+10i+3=11+10i.$$

Powers of the imaginary unit.

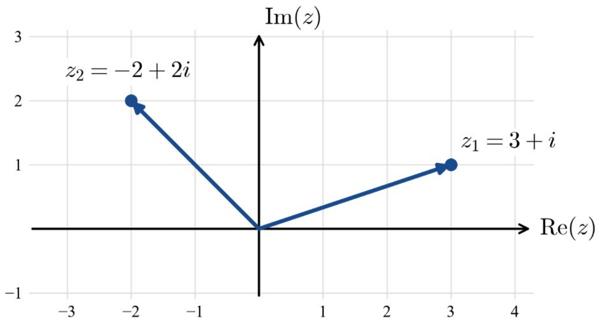
Repeatedly using $i^2=-1$ shows that the powers of i cycle through four values:

$$i^1=i, i^2=-1, i^3=i^2\cdot i=-i, i^4=(i^2)^2=1,$$

and then the pattern repeats, since $i^5=i^4\cdot i=i$, and so on. To evaluate a higher power i^n, divide the exponent by 4 and keep the remainder: if $n=4q+r$ with remainder $r\in\{0,1,2,3\}$, then $i^n=i^r$. For example, $i^{30}=i^{28}\cdot i^2=(i^4)^7\cdot i^2=i^2=-1$.

The Argand diagram.

Because a complex number $z=a+bi$ is fixed by the two real numbers a and b, it can be represented as the point with coordinates (a,b) on a plane. When a Cartesian plane is used this way it is called an **Argand diagram** (or the **complex plane**): the horizontal axis is the **real axis** and the vertical axis is the **imaginary axis**. The real part is read along the real axis and the imaginary part along the imaginary axis. Each complex number can equally be pictured as the arrow, or **position vector**, from the origin O to the point (a,b).



The diagram above shows $z_1=3+i$, a point in the first quadrant, and $z_2=-2+2i$, a point in the second quadrant, each drawn both as a point and as a position vector from O. This picture is developed much further in later sections, where the length and direction of the position vector are used to describe a complex number; here it is enough to be able to plot a complex number and read off its two parts.

Worked examples

Example 1 — scaffolded

Let $z=4-3i$ and $w=-2+5i$. Find (a) $z+w$, (b) $z-w$, and (c) $2z+3w$, giving each answer in the form $a+bi$.

Working	Comment
$z=4-3i$: $\text{Re}(z)=4$, $\text{Im}(z)=-3$; $w=-2+5i$: $\text{Re}(w)=-2$, $\text{Im}(w)=5$.	Identify the real and imaginary parts first.
$z+w=(4+(-2))+(-3+5)i=2+2i.$	Add the real parts and the imaginary parts separately.
$z-w=(4-(-2))+(-3-5)i=6-8i.$	Subtract part by part; watch the double sign $4-(-2)=6$.
$2z=8-6i$ and $3w=-6+15i$.	A real scalar multiplies each part.

Working	Comment
$2z+3w=(8-6)+(-6+15)i=2+9i$.	Add the two results part by part.

Example 2 — scaffolded

Simplify (a) $(2+3i)(1-2i)$, (b) $(1+2i)^2$, and (c) i^{23}, giving each answer in Cartesian form.

Working	Comment
$(2+3i)(1-2i)=2-4i+3i-6i^2$.	Expand the brackets term by term.
$=2-i-6(-1)=2-i+6=8-i$.	Replace i^2 with -1, then collect real and imaginary parts.
$(1+2i)^2=1^2+2(1)(2i)+(2i)^2=1+4i+4i^2$.	Use the perfect-square expansion.
$=1+4i-4=-3+4i$.	Since $4i^2=-4$.
$i^{23}=i^{20}\cdot i^3=(i^4)^5\cdot i^3=1\cdot i^3=-i$.	$23=4\times5+3$, so $i^{23}=i^3=-i$.

Example 3 — unscaffolded

Find the real numbers x and y for which $(x+yi)(1+i)=3+5i$.

Expanding the left-hand side and using $i^2=-1$,

$$(x+yi)(1+i)=x+xi+yi+yi^2=(x-y)+(x+y)i.$$

Two complex numbers are equal only when their real and imaginary parts match, so

$$x-y=3 \text{ and } x+y=5.$$

Adding the equations gives $2x=8$, so $x=4$; then $y=5-x=1$. Checking, $(4+i)(1+i)=4+4i+i+i^2=3+5i$, as required.

$\therefore x=4$ and $y=1$.

Example 4 — unscaffolded

Show that $z=1+2i$ satisfies the equation $z^2-2z+5=0$.

First square z:

$$z^2=(1+2i)^2=1+4i+4i^2=1+4i-4=-3+4i.$$

The real multiple $2z=2+4i$. Substituting into the left-hand side,

$$z^2-2z+5=(-3+4i)-(2+4i)+5=(-3-2+5)+(4-4)i=0+0i.$$

Because both the real and imaginary parts are zero, the left-hand side equals 0.

$\therefore z=1+2i$ is a solution of $z^2-2z+5=0$.

Practice questions

Scaffolded

Q1. Let $z=5+2i$ and $w=3-4i$. (a) State $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. (b) Find $z+w$. (c) Find $z-w$. (d) Find $3z-2w$.

Q2. Expand and simplify $(4+3i)(2-5i)$. (a) Expand the brackets term by term. (b) Replace i^2 with -1. (c) Collect the real and imaginary parts to give the answer in the form $a+bi$.

Q3. Simplify the following. (a) Expand $(4+i)^2$. (b) Simplify i^6. (c) Simplify i^{13}.

Q4. Find the real numbers x and y for which $(x+yi)+(3-2i)=7+i$. (a) Equate the real parts. (b) Equate the imaginary parts. (c) Solve for x and y.

Q5. On an Argand diagram plot each of the following, and state the quadrant (or axis) in which it lies. (a) $z_1=3+2i$. (b) $z_2=-4+i$. (c) $z_3=-2-2i$. (d) $z_4=2-3i$.

Q6. Let $z=3-2i$. (a) Find $2z$. (b) Find z^2. (c) Hence find z^2+4z.

Unscaffolded

Q7. Given $z=6-2i$ and $w=-1+5i$, find $z+w$.

Q8. Given $z=6-2i$ and $w=-1+5i$, find $z-w$.

Q9. Given $z=6-2i$ and $w=-1+5i$, find $3z+2w$.

Q10. Simplify $(4-3i)-(6-8i)$.

Q11. Expand and simplify $(3+2i)(4+5i)$.

Q12. Expand and simplify $(2-5i)(3-i)$.

Q13. Expand and simplify $(3-4i)^2$.

Q14. Expand and simplify $(1+i)^3$.

Q15. Simplify i^{35}.

Q16. Simplify $i^{18}+i^7$.

Q17. Find the real numbers x and y for which $(x+yi)(2-i)=5+5i$.

Q18. Show that $z=2-i$ satisfies the equation $z^2-4z+5=0$.

Answers

Q1. (a) $\mathrm{Re}(z)=5$, $\mathrm{Im}(z)=2$; (b) $8-2i$; (c) $2+6i$; (d) $9+14i$. **Q2.** $23-14i$. **Q3.** (a) $15+8i$; (b) -1; (c) i. **Q4.** $x=4$, $y=3$. **Q5.** (a) first quadrant; (b) second quadrant; (c) third quadrant; (d) fourth quadrant. **Q6.** (a) $6-4i$; (b) $5-12i$; (c) $17-20i$. **Q7.** $5+3i$. **Q8.** $7-7i$. **Q9.** $16+4i$. **Q10.** $-2+5i$. **Q11.** $2+23i$. **Q12.** $1-17i$. **Q13.** $-7-24i$. **Q14.** $-2+2i$. **Q15.** $-i$. **Q16.** $-1-i$. **Q17.** $x=1$, $y=3$. **Q18.** Verified: $z^2-4z+5=0$ (see solution).

Fully-worked solutions

Q1. With $z=5+2i$, the real part is $\mathrm{Re}(z)=5$ and the imaginary part is $\mathrm{Im}(z)=2$. Adding part by part, $z+w=(5+3)+(2+(-4))i=8-2i$. Subtracting, $z-w=(5-3)+(2-(-4))i=2+6i$. For the combination, $3z=15+6i$ and $2w=6-8i$, so $3z-2w=(15-6)+(6-(-8))i=9+14i$.

Q2. Expanding term by term, $(4+3i)(2-5i)=8-20i+6i-15i^2$. Replacing i^2 with -1 gives $8-14i-15(-1)=8-14i+15=23-14i$.

Q3. (a) $(4+i)^2=16+8i+i^2=16+8i-1=15+8i$. (b) $i^6=i^4\cdot i^2=1\cdot(-1)=-1$. (c) $i^{13}=i^{12}\cdot i=(i^4)^3\cdot i=i$.

Q4. Adding, $(x+yi)+(3-2i)=(x+3)+(y-2)i$. Equating this to $7+i$: the real parts give $x+3=7$, so $x=4$; the imaginary parts give $y-2=1$, so $y=3$.

Q5. Each $z=a+bi$ is plotted as the point (a,b). (a) $z_1=3+2i\to(3,2)$: both parts positive, so the **first** quadrant. (b) $z_2=-4+i\to(-4,1)$: real part negative, imaginary part positive, so the **second** quadrant. (c) $z_3=-2-2i\to(-2,-2)$: both parts negative, so the **third** quadrant. (d) $z_4=2-3i\to(2,-3)$: real part positive, imaginary part negative, so the **fourth** quadrant.

Q6. With $z=3-2i$: the real multiple $2z=6-4i$. The square is $z^2=(3-2i)^2=9-12i+4i^2=9-12i-4=5-12i$. Hence $z^2+4z=(5-12i)+(12-8i)=17-20i$.

Q7. $z+w=(6+(-1))+(-2+5)i=5+3i$.

Q8. $z-w=(6-(-1))+(-2-5)i=7-7i$.

Q9. $3z=18-6i$ and $2w=-2+10i$, so $3z+2w=(18+(-2))+(-6+10)i=16+4i$.

Q10. $(4-3i)-(6-8i)=(4-6)+(-3-(-8))i=-2+5i$.

Q11. $(3+2i)(4+5i)=12+15i+8i+10i^2=12+23i-10=2+23i$.

Q12. $(2-5i)(3-i)=6-2i-15i+5i^2=6-17i-5=1-17i$.

Q13. $(3-4i)^2=9-24i+16i^2=9-24i-16=-7-24i$.

Q14. First $(1+i)^2=1+2i+i^2=2i$. Then $(1+i)^3=(1+i)^2(1+i)=2i(1+i)=2i+2i^2=-2+2i$.

Q15. Since $35=4\times8+3$, $i^{35}=(i^4)^8\cdot i^3=i^3=-i$.

Q16. $i^{18}=(i^4)^4\cdot i^2=i^2=-1$ and $i^7=(i^4)^1\cdot i^3=i^3=-i$, so $i^{18}+i^7=-1-i$.

Q17. Expanding, $(x+yi)(2-i)=2x-xi+2yi-yi^2=(2x+y)+(-x+2y)i$. Equating real and imaginary parts to $5+5i$ gives the simultaneous equations $2x+y=5$ and $-x+2y=5$. From the first, $y=5-2x$; substituting, $-x+2(5-2x)=5$, so $-5x+10=5$ and $x=1$. Then $y=5-2(1)=3$. Checking, $(1+3i)(2-i)=2-i+6i-3i^2=2+5i+3=5+5i$.

Q18. Squaring, $z^2=(2-i)^2=4-4i+i^2=4-4i-1=3-4i$. The real multiple $4z=8-4i$. Substituting into the left-hand side,

$$z^2-4z+5=(3-4i)-(8-4i)+5=(3-8+5)+(-4+4)i=0.$$

Since both parts are zero, $z=2-i$ satisfies $z^2-4z+5=0$.

Theory

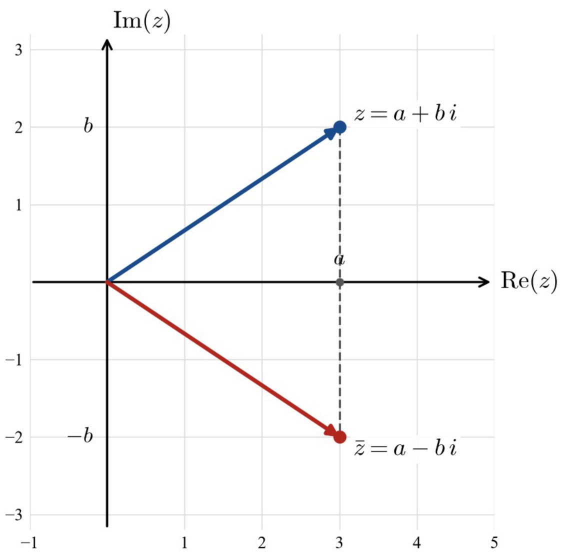
An earlier section introduced complex numbers in **Cartesian form** $z = a + bi$, where $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$ are real and i is the imaginary unit, and showed how to plot z as the point (a, b) on an **Argand diagram** (the complex plane, with a horizontal real axis and a vertical imaginary axis). This section introduces a partner for each complex number — its **conjugate** — and uses it to measure the *size* of a complex number and to carry out **division**.

The complex conjugate.

The **conjugate** of $z = a + bi$ is the complex number

$$\bar{z} = a - bi,$$

obtained by reversing the sign of the imaginary part. Geometrically, the point $(a, -b)$ is the **reflection of z in the real axis**, so z and $\bar{z}$ sit at the same distance from the origin, one directly above the other. Reflecting twice returns the original point, which matches the algebra: $\overline{(\bar{z})} = z$.



Sum and difference with the conjugate.

Adding a complex number to its conjugate cancels the imaginary parts, while subtracting cancels the real parts:

$$z + \bar{z} = (a + bi) + (a - bi) = 2a = 2\,\mathrm{Re}(z),$$

$$z - \bar{z} = (a + bi) - (a - bi) = 2bi = 2i\,\mathrm{Im}(z).$$

So $z + \bar{z}$ is always **real** and $z - \bar{z}$ is always **imaginary**. In particular $z = \bar{z}$ exactly when $b = 0$ (that is, when z is real), and $z = -\bar{z}$ exactly when $a = 0$ (when z is purely imaginary).

Conjugate of a sum and of a product.

Conjugation behaves well with the arithmetic already met: the conjugate of a sum, difference or product is obtained by conjugating each part,

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

These rules extend to any number of terms or factors; for a product this gives $\overline{z^n} = (\bar{z})^n$.

The modulus.

The **modulus** of $z=a+b\,i$, written $|z|$, is its distance from the origin on the Argand diagram. By Pythagoras' theorem,

$$|z|=\sqrt{a^2+b^2}.$$

The modulus is a non-negative real number, and $|z|=0$ only when $z=0$. Multiplying a complex number by its conjugate collapses the imaginary part and produces exactly this squared distance:

$$z\,\acute{z}=(a+b\,i)(a-b\,i)=a^2-(b\,i)^2=a^2+b^2=|z|^2.$$

So $z\,\acute{z}=|z|^2$ is always a **non-negative real number**. This single identity is the key to dividing complex numbers.

Dividing complex numbers.

A quotient $\dfrac{z_1}{z_2}$ (with $z_2\neq 0$) is written in Cartesian form by **realising the denominator**: multiply the numerator and denominator by the conjugate of the denominator. Because $z_2\,\acute{z}_2=|z_2|^2$ is real, the new denominator is a real number and the quotient splits cleanly into real and imaginary parts:

$$\frac{z_1}{z_2}=\frac{z_1}{z_2}\times\frac{\acute{z}_2}{\acute{z}_2}=\frac{z_1\,\acute{z}_2}{z_2\,\acute{z}_2}=\frac{z_1\,\acute{z}_2}{|z_2|^2}.$$

For example, with $z_2=2+i$ the denominator becomes $z_2\,\acute{z}_2=2^2+1^2=5$. The same idea gives the **reciprocal** of a non-zero z: $\dfrac{1}{z}=\dfrac{\acute{z}}{|z|^2}$.

Modulus of a product and of a quotient.

Moduli multiply and divide term by term:

$$|z_1\,z_2|=|z_1|\,|z_2|,\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\;(z_2\neq 0).$$

These let the size of a product or quotient be found without expanding first — a shortcut used throughout the work on complex numbers, and developed further using polar form in a later section.

Worked examples

Example 1 — scaffolded

For $z=3-4\,i$, write down $\acute{z}$ and evaluate $z+\acute{z}$, $z-\acute{z}$ and $z\,\acute{z}$, then state $|z|$ and check that $z\,\acute{z}=|z|^2$.

Working	Comment
$a=3, b=-4$, so $\acute{z}=3+4\,i$.	Reverse the sign of the imaginary part only.
$z+\acute{z}=(3-4\,i)+(3+4\,i)=6=2\,\mathrm{Re}(z)$.	The imaginary parts cancel, leaving $2\,a$.
$z-\acute{z}=(3-4\,i)-(3+4\,i)=-8\,i=2\,i\,\mathrm{Im}(z)$.	The real parts cancel, leaving $2\,b\,i$ with $b=-4$.
$z\,\acute{z}=(3-4\,i)(3+4\,i)=9-(4\,i)^2=9+16=25$.	Difference of two squares; $-(4\,i)^2=+16$.
$\|z\|=\sqrt{3^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5$.	Modulus $\|z\|=\sqrt{a^2+b^2}$.
$\|z\|^2=5^2=25=z\,\acute{z}$.	Confirms the identity $z\,\acute{z}=\|z\|^2$.

Example 2 — scaffolded

Express $\dfrac{4+7\,i}{2+i}$ in Cartesian form $a+b\,i$.

Working	Comment
The conjugate of the denominator $2+i$ is $2-i$.	Realise the denominator using its conjugate.
$\frac{4+7i}{2+i}=\frac{4+7i}{2+i}\times\frac{2-i}{2-i}=\frac{(4+7i)(2-i)}{(2+i)(2-i)}$.	Multiply top and bottom by $2-i$.
Denominator: $(2+i)(2-i)=2^2+1^2=5$.	$z_2\acute{z}_2=\lvert z_2\rvert^2$ is real.
Numerator: $(4+7i)(2-i)=8-4i+14i-7i^2=8+10i+7=15+1($.	Expand and use $i^2=-1$.
$\frac{15+10i}{5}=\frac{15}{5}+\frac{10}{5}i=3+2i$.	Divide each part by the real denominator.

Example 3 — unscaffolded

Given $z_1=1+2i$ and $z_2=3-i$, verify that $|z_1z_2|=|z_1||z_2|$.

First form the product: $z_1z_2=(1+2i)(3-i)=3-i+6i-2i^2=3+5i+2=5+5i$, so

$$|z_1z_2|=\sqrt{5^2+5^2}=\sqrt{50}=5\sqrt{2}.$$

Separately, $|z_1|=\sqrt{1^2+2^2}=\sqrt{5}$ and $|z_2|=\sqrt{3^2+(-1)^2}=\sqrt{10}$, so $|z_1||z_2|=\sqrt{5}\times\sqrt{10}=\sqrt{50}=5\sqrt{2}$.

$\therefore|z_1z_2|=5\sqrt{2}=|z_1||z_2|$, as the product rule for moduli predicts.

Example 4 — unscaffolded

Express $z=\frac{2+11i}{3+4i}$ in Cartesian form and find $|z|$, checking the result with the quotient rule $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$.

Realising the denominator with its conjugate $3-4i$, and using $(3+4i)(3-4i)=3^2+4^2=25$,

$$z=\frac{2+11i}{3+4i}\times\frac{3-4i}{3-4i}=\frac{(2+11i)(3-4i)}{25}=\frac{6-8i+33i-44i^2}{25}=\frac{50+25i}{25}=2+i.$$

Hence $|z|=\sqrt{2^2+1^2}=\sqrt{5}$. The quotient rule agrees: with $|2+11i|=\sqrt{4+121}=\sqrt{125}=5\sqrt{5}$ and $|3+4i|=\sqrt{9+16}=5$,

$$\left|\frac{2+11i}{3+4i}\right|=\frac{5\sqrt{5}}{5}=\sqrt{5}.$$

$\therefore z=2+i$ with $|z|=\sqrt{5}$.

Practice questions

Scaffolded

Q1. For $z=5+2i$: (a) write down $\acute{z}$; (b) find $z+\acute{z}$ and $z-\acute{z}$; (c) find $z\acute{z}$; (d) state $|z|$ and check that $z\acute{z}=|z|^2$.

Q2. For $z=-5+12i$: (a) write down $\acute{z}$; (b) find $z+\acute{z}$ and $z-\acute{z}$; (c) find $z\acute{z}$; (d) state $|z|$.

Q3. Express $\frac{5}{1+2i}$ in Cartesian form. (a) State the conjugate of the denominator. (b) Multiply numerator and denominator by this conjugate. (c) Simplify the denominator to a real number. (d) Write the answer in the form $a+bi$.

Q4. Express $\frac{3+2i}{1-i}$ in Cartesian form. (a) State the conjugate of the denominator. (b) Multiply numerator and denominator by this conjugate. (c) Simplify the numerator and the denominator. (d) Write the answer in the form $a+bi$.

Q5. Let $z_1=4+i$ and $z_2=2+2i$. (a) Find the product $z_1 z_2$. (b) Find $|z_1 z_2|$. (c) Find $|z_1|$ and $|z_2|$. (d) Verify that $|z_1 z_2|=|z_1||z_2|$.

Q6. Let $z_1=2-i$ and $z_2=3+2i$. (a) Find z_1+z_2 and its conjugate $\overline{z_1+z_2}$. (b) Find $\acute{z}_1+\acute{z}_2$ and confirm that $\overline{z_1+z_2}=\acute{z}_1+\acute{z}_2$. (c) Find $z_1 z_2$ and its conjugate $\overline{z_1 z_2}$. (d) Find $\acute{z}_1 \acute{z}_2$ and confirm that $\overline{z_1 z_2}=\acute{z}_1 \acute{z}_2$.

Unscaffolded

Q7. For $z=-7+24i$, write down $\acute{z}$ and find $|z|$.

Q8. Express $\frac{7+i}{2+i}$ in Cartesian form.

Q9. Express the reciprocal $\frac{1}{4-3i}$ in Cartesian form.

Q10. Express $\frac{2+3i}{4i}$ in Cartesian form.

Q11. Express $\frac{5+5i}{3-4i}$ in Cartesian form.

Q12. Let $z_1=4+3i$ and $z_2=1+3i$. Find $\left|\frac{z_1}{z_2}\right|$ exactly, using the quotient rule.

Q13. For $z=-1+5i$, evaluate $z\acute{z}$ and hence state $|z|$.

Q14. Find the complex number z for which $(2+i)z=5-5i$.

Q15. Let $z_1=3+i$ and $z_2=-2+5i$. Verify that $\overline{z_1 z_2}=\acute{z}_1 \acute{z}_2$.

Q16. The complex number $z=2+bi$, where b is a positive real number, has $|z|=\sqrt{13}$. Find b, and hence write down z and $\acute{z}$.

Q17. Express $\frac{(1+i)(3-2i)}{2+i}$ in Cartesian form.

Q18. The complex number $z=a+bi$ and its conjugate $\acute{z}$ are plotted on an Argand diagram. (a) Describe the geometric relationship between the points z and $\acute{z}$. (b) For which complex numbers is $z=\acute{z}$? (c) For which complex numbers is $z=-\acute{z}$?

Answers

Q1. $\acute{z}=5-2i$; $z+\acute{z}=10$, $z-\acute{z}=4i$; $z\acute{z}=29$; $|z|=\sqrt{29}$ (and $|z|^2=29$). **Q2.** $\acute{z}=-5-12i$; $z+\acute{z}=-10$, $z-\acute{z}=24i$; $z\acute{z}=169$; $|z|=13$. **Q3.** $z=1-2i$. **Q4.** $z=\frac{1}{2}+\frac{5}{2}i$. **Q5.** $z_1 z_2=6+10i$; $|z_1 z_2|=2\sqrt{34}$; $|z_1|=\sqrt{17}$, $|z_2|=2\sqrt{2}$; $\sqrt{17}\times 2\sqrt{2}=2\sqrt{34}$. **Q6.** $z_1+z_2=5+i$, $\overline{z_1+z_2}=5-i=\acute{z}_1+\acute{z}_2$; $z_1 z_2=8+i$, $\overline{z_1 z_2}=8-i=\acute{z}_1\acute{z}_2$. **Q7.** $\acute{z}=-7-24i$; $|z|=25$. **Q8.** $z=3-i$. **Q9.** $z=\frac{4}{25}+\frac{3}{25}i$. **Q10.** $z=\frac{3}{4}-\frac{1}{2}i$. **Q11.** $z=-\frac{1}{5}+\frac{7}{5}i$. **Q12.** $\left|\frac{z_1}{z_2}\right|=\frac{\sqrt{10}}{2}$. **Q13.** $z\acute{z}=26$; $|z|=\sqrt{26}$. **Q14.** $z=1-3i$. **Q15.** $z_1 z_2=-11+13i$, so $\overline{z_1 z_2}=-11-13i$; and $\acute{z}_1\acute{z}_2=(3-i)(-2-5i)=-11-13i$. **Q16.** $b=3$; $z=2+3i$, $\acute{z}=2-3i$. **Q17.** $z=\frac{11}{5}-\frac{3}{5}i$. **Q18.** (a) $\acute{z}$ is the reflection of z in the real axis. (b) $z=\acute{z}$ when z is real ($b=0$). (c) $z=-\acute{z}$ when z is purely imaginary ($a=0$, including $z=0$).

Fully-worked solutions

Q1. $z=5+2i$ has $a=5$, $b=2$, so $\acute{z}=5-2i$. Then $z+\acute{z}=(5+2i)+(5-2i)=10=2\,\mathrm{Re}(z)$ and $z-\acute{z}=(5+2i)-(5-2i)=4i=2i\,\mathrm{Im}(z)$. The product is $z\acute{z}=(5+2i)(5-2i)=5^2+2^2=25+4=29$, and $|z|=\sqrt{5^2+2^2}=\sqrt{29}$. Since $|z|^2=29=z\acute{z}$, the identity $z\acute{z}=|z|^2$ holds.

Q2. $z=-5+12i$ has $\acute{z}=-5-12i$. Then $z+\acute{z}=-10=2\,\mathrm{Re}(z)$ and $z-\acute{z}=24i=2i\,\mathrm{Im}(z)$. The product is $z\acute{z}=(-5)^2+12^2=25+144=169$, so $|z|=\sqrt{169}=13$ (consistent with $z\acute{z}=|z|^2=169$).

Q3. The conjugate of $1+2i$ is $1-2i$. Multiplying top and bottom by it,

$$\frac{5}{1+2i}=\frac{5}{1+2i}\times\frac{1-2i}{1-2i}=\frac{5(1-2i)}{1^2+2^2}=\frac{5(1-2i)}{5}=1-2i.$$

Q4. The conjugate of $1-i$ is $1+i$, and $(1-i)(1+i)=1^2+1^2=2$. The numerator is $(3+2i)(1+i)=3+3i+2i+2i^2=3+5i-2=1+5i$. Hence

$$\frac{3+2i}{1-i}=\frac{1+5i}{2}=\frac{1}{2}+\frac{5}{2}i.$$

Q5. $z_1 z_2=(4+i)(2+2i)=8+8i+2i+2i^2=8+10i-2=6+10i$, so $|z_1 z_2|=\sqrt{6^2+10^2}=\sqrt{136}=2\sqrt{34}$. Separately $|z_1|=\sqrt{4^2+1^2}=\sqrt{17}$ and $|z_2|=\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2}$, so $|z_1||z_2|=\sqrt{17}\times 2\sqrt{2}=2\sqrt{34}=|z_1 z_2|$.

Q6. $z_1+z_2=(2-i)+(3+2i)=5+i$, so $\overline{z_1+z_2}=5-i$. Also $\acute{z}_1=2+i$ and $\acute{z}_2=3-2i$, giving $\acute{z}_1+\acute{z}_2=5-i$; the two agree. For the product, $z_1 z_2=(2-i)(3+2i)=6+4i-3i-2i^2=6+i+2=8+i$, so $\overline{z_1 z_2}=8-i$; and $\acute{z}_1\acute{z}_2=(2+i)(3-2i)=6-4i+3i-2i^2=6-i+2=8-i$, which again agrees.

Q7. $z=-7+24i$ has $\acute{z}=-7-24i$ and $|z|=\sqrt{(-7)^2+24^2}=\sqrt{49+576}=\sqrt{625}=25$.

Q8. The conjugate of $2+i$ is $2-i$, and $(2+i)(2-i)=2^2+1^2=5$. The numerator is $(7+i)(2-i)=14-7i+2i-i^2=14-5i+1=15-5i$. Hence

$$\frac{7+i}{2+i}=\frac{15-5i}{5}=3-i.$$

Q9. Using $\frac{1}{z}=\frac{\acute{z}}{|z|^2}$ with $z=4-3i$, the conjugate is $4+3i$ and $|z|^2=4^2+(-3)^2=25$, so

$$\frac{1}{4-3i}=\frac{4+3i}{25}=\frac{4}{25}+\frac{3}{25}i.$$

Q10. The conjugate of $4i$ is $-4i$, and $(4i)(-4i)=-16i^2=16$. The numerator is $(2+3i)(-4i)=-8i-12i^2=12-8i$. Hence

$$\frac{2+3i}{4i}=\frac{12-8i}{16}=\frac{3}{4}-\frac{1}{2}i.$$

Q11. The conjugate of $3-4i$ is $3+4i$, and $(3-4i)(3+4i)=3^2+4^2=25$. The numerator is $(5+5i)(3+4i)=15+20i+15i+20i^2=15+35i-20=-5+35i$. Hence

$$\frac{5+5i}{3-4i}=\frac{-5+35i}{25}=-\frac{1}{5}+\frac{7}{5}i.$$

Q12. By the quotient rule $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$. Here $|z_1|=|4+3i|=\sqrt{16+9}=5$ and $|z_2|=|1+3i|=\sqrt{1+9}=\sqrt{10}$, so

$$\left|\frac{z_1}{z_2}\right|=\frac{5}{\sqrt{10}}=\frac{5\sqrt{10}}{10}=\frac{\sqrt{10}}{2}.$$

Q13. $z\bar{z}=|z|^2=(-1)^2+5^2=1+25=26$ (directly, $(-1+5i)(-1-5i)=1-25i^2=26$). Hence $|z|=\sqrt{26}$.

Q14. Dividing both sides by $2+i$, $z=\frac{5-5i}{2+i}$. The conjugate of $2+i$ is $2-i$ with $(2+i)(2-i)=5$, and the numerator is $(5-5i)(2-i)=10-5i-10i+5i^2=10-15i-5=5-15i$. Hence

$$z=\frac{5-15i}{5}=1-3i.$$

(Check: $(2+i)(1-3i)=2-6i+i-3i^2=5-5i$.)

Q15. $z_1z_2=(3+i)(-2+5i)=-6+15i-2i+5i^2=-6+13i-5=-11+13i$, so $\overline{z_1z_2}=-11-13i$. Also $\bar{z}_1=3-i$ and $\bar{z}_2=-2-5i$, so $\bar{z}_1\bar{z}_2=(3-i)(-2-5i)=-6-15i+2i+5i^2=-6-13i-5=-11-13i$. The two results agree, confirming $\overline{z_1z_2}=\bar{z}_1\bar{z}_2$.

Q16. $|z|^2=2^2+b^2=4+b^2$, and $|z|=\sqrt{13}$ gives $4+b^2=13$, so $b^2=9$. As $b>0$, $b=3$. Therefore $z=2+3i$ and $\bar{z}=2-3i$.

Q17. First expand the numerator: $(1+i)(3-2i)=3-2i+3i-2i^2=3+i+2=5+i$. Then realise the denominator with the conjugate $2-i$, where $(2+i)(2-i)=5$, and $(5+i)(2-i)=10-5i+2i-i^2=10-3i+1=11-3i$. Hence

$$\frac{(1+i)(3-2i)}{2+i}=\frac{5+i}{2+i}=\frac{11-3i}{5}=\frac{11}{5}-\frac{3}{5}i.$$

Q18. (a) $\bar{z}=a-bi$ is the reflection of $z=a+bi$ in the real axis: the two points share the same real part a and have opposite imaginary parts, so they lie symmetrically above and below the horizontal axis. (b) $z=\bar{z}$ means $a+bi=a-bi$, i.e. $2bi=0$, so $b=0$: this holds exactly for the **real** numbers, which lie on the real axis and are their own reflection. (c) $z=-\bar{z}$ means $a+bi=-a+bi$, i.e. $2a=0$, so $a=0$: this holds exactly for the **purely imaginary** numbers $z=bi$ (including $z=0$), which lie on the imaginary axis.

Theory

Every complex number $z = a + b\,i$ (with a, b real and i the imaginary unit) can be plotted as the point (a, b) on an **Argand diagram**: a Cartesian plane whose horizontal axis is the **real axis** and whose vertical axis is the **imaginary axis**. The real part $a = \mathrm{Re}(z)$ is measured along the real axis and the imaginary part $b = \mathrm{Im}(z)$ along the imaginary axis. So far a complex number has been written in **Cartesian form** $a + b\,i$. This section develops a second description — **polar form** — that records instead *how far* the point is from the origin and *in what direction*.

Modulus.

The **modulus** of $z = a + b\,i$ is its distance from the origin O on the Argand diagram. By Pythagoras' theorem,

$$r = |z| = \sqrt{a^2 + b^2}.$$

The modulus is a non-negative real number, and $|z| = 0$ only when $z = 0$. For example, the distance of $z = a + b\,i$ from O is the length of the vector drawn from O to the point (a, b).

Argument.

The **argument** of a non-zero z, written $arg(z)$, is the angle the vector from O to (a, b) makes with the **positive real axis**, measured **anticlockwise as positive** and clockwise as negative. Writing θ for an argument of z, the right-angled triangle formed by the point and its projection onto the real axis gives

$$\cos\theta = \frac{a}{r}, \sin\theta = \frac{b}{r}, \tan\theta = \frac{b}{a} (a \neq 0).$$

Because a full turn of 2π returns to the same direction, the argument is only fixed up to whole turns: if θ is an argument of z then so is $\theta + 2\pi k$ for every integer k. The argument of $z = 0$ is undefined.

Principal argument.

To make the argument unique we choose the one value in a single revolution. The **principal argument**, written $Arg(z)$, is the argument that satisfies

$$-\pi < Arg(z) \leq \pi.$$

So points above the real axis have a positive principal argument (up to π on the negative real axis), and points below it have a negative principal argument (down to, but not including, $-\pi$).

Finding the argument by quadrant.

The equation $\tan\theta = \frac{b}{a}$ alone does not fix the quadrant, because angles that differ by π share the same tangent. The safe method is to plot z, find the acute **reference angle**

$$\alpha = \tan^{-1}\left|\frac{b}{a}\right| (0 < \alpha < \pi/2),$$

and then read the principal argument from the quadrant in which z lies.

Quadrant of z	signs of (a, b)	$Arg(z)$
first	$a > 0, b > 0$	α
second	$a < 0, b > 0$	$\pi - \alpha$
third	$a < 0, b < 0$	$-(\pi - \alpha)$
fourth	$a > 0, b < 0$	$-\alpha$

For the numbers lying **on** an axis the argument is read directly: a positive real number has $Arg(z)=0$; a negative real number has $Arg(z)=\pi$; a positive imaginary number has $Arg(z)=\frac{\pi}{2}$; and a negative imaginary number has $Arg(z)=-\frac{\pi}{2}$.

Polar form.

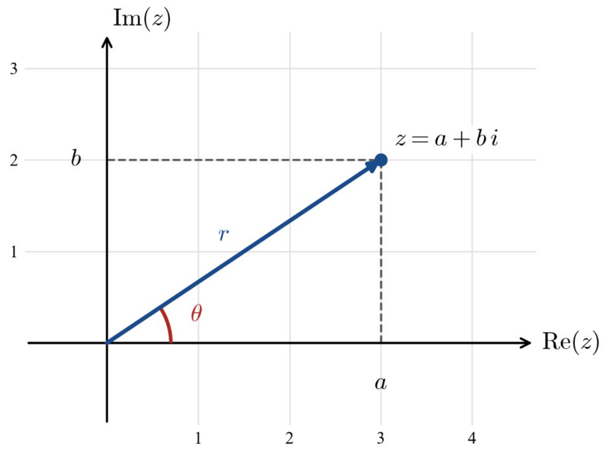
Since $a=r\cos\theta$ and $b=r\sin\theta$, every complex number can be written as

$$z=r\cos\theta+i\,r\sin\theta=r(\cos\theta+i\sin\theta).$$

The abbreviation $cis\,\theta$ stands for $\cos\theta+i\sin\theta$, so the **polar form** (also called **modulus–argument form**) is

$$z=r\,cis\,\theta=r(\cos\theta+i\sin\theta),$$

where $r=|z|\geq 0$ and θ is any argument of z; we usually quote the principal argument. This diagram shows the three quantities: the point z, the modulus r as the length of the vector from O, and the argument θ as the arc turned from the positive real axis.



Converting between the forms.

To go from **Cartesian to polar**: compute $r=\sqrt{a^2+b^2}$, find the reference angle α, and read $Arg(z)$ from the quadrant. To go from **polar to Cartesian**: evaluate $a=r\cos\theta$ and $b=r\sin\theta$. Standard angles give exact surd values, so answers stay exact rather than decimal.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Reading these both ways, and using the symmetry of $\cos$ and $\sin$, gives the exact value for every multiple of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Equal complex numbers.

Two non-zero numbers written in polar form are equal exactly when they have the **same modulus** and their **arguments differ by a whole number of turns**: $r_1\,cis\,\theta_1=r_2\,cis\,\theta_2$ if and only if $r_1=r_2$ and $\theta_1=\theta_2+2\pi k$ for some integer k. Quoting the principal argument removes this ambiguity.

Polar form is especially powerful for multiplying, dividing and taking powers of complex numbers — the moduli multiply or divide while the arguments add or subtract — and those operations are developed in a later section. Here the focus is on the modulus, the argument, and converting between Cartesian and polar form.

Worked examples

Example 1 — scaffolded

Express $z=1+\sqrt{3}\,i$ in polar form $r\,cis\,\theta$, using the principal argument.

Working	Comment
$a=1\,,b=\sqrt{3}$; both positive, so z is in the first quadrant.	Identify the real and imaginary parts and the quadrant first.
$r=\sqrt{a^2+b^2}=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$.	Modulus $r=\sqrt{a^2+b^2}$.
$\tan\alpha=\left\lvert\frac{b}{a}\right\rvert=\frac{\sqrt{3}}{1}=\sqrt{3}$, so $\alpha=\frac{\pi}{3}$.	Reference angle α is acute; $\tan\frac{\pi}{3}=\sqrt{3}$.
First quadrant, so $Arg(z)=\alpha=\frac{\pi}{3}$.	In quadrant 1 the argument equals the reference angle.
$z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\,cis\frac{\pi}{3}$.	Substitute r and θ into $r\,cis\,\theta$.

Example 2 — scaffolded

Express $z=-2-2i$ in polar form, using the principal argument.

Working	Comment
$a=-2\,,b=-2$; both negative, so z is in the third quadrant.	Sketching the sign pattern fixes the quadrant.
$r=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$.	Simplify the surd: $\sqrt{8}=2\sqrt{2}$.
$\tan\alpha=\left\lvert\frac{-2}{-2}\right\rvert=1$, so $\alpha=\frac{\pi}{4}$.	Reference angle from the acute triangle.
Third quadrant, so $Arg(z)=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{4}\right)=-\frac{3\pi}{4}$.	Below the real axis the principal argument is negative, here between $-\pi$ and $-\frac{\pi}{2}$.
$z=2\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right)$.	Polar form with $r=2\sqrt{2}$.

Example 3 — unscaffolded

Express $z=4\,cis\frac{5\pi}{6}$ in Cartesian form $a+bi$.

Here $r=4$ and $\theta=\frac{5\pi}{6}$, so $a=r\cos\theta$ and $b=r\sin\theta$. From the exact values $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6}=\frac{1}{2}$,

$$z=4\left(-\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)=4\cdot\left(-\frac{\sqrt{3}}{2}\right)+4\cdot\frac{1}{2}i=-2\sqrt{3}+2i.$$

$\therefore z=-2\sqrt{3}+2i$, a point in the second quadrant.

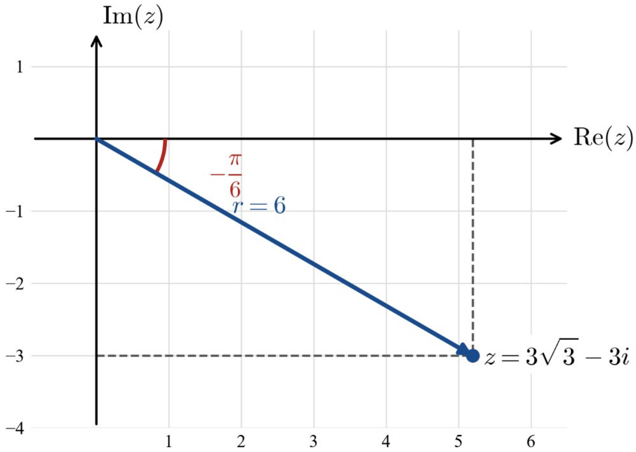
Example 4 — unscaffolded

Plot $z=3\sqrt{3}-3i$ on an Argand diagram and express it in polar form, using the principal argument.

The real part $a=3\sqrt{3}>0$ and the imaginary part $b=-3<0$, so z lies in the fourth quadrant. Its modulus is

$$r=\sqrt{(3\sqrt{3})^2+(-3)^2}=\sqrt{27+9}=\sqrt{36}=6.$$

The reference angle satisfies $\tan\alpha=\left|\frac{-3}{3\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$, so $\alpha=\frac{\pi}{6}$. In the fourth quadrant the principal argument is $-\alpha$, hence $Arg(z)=-\frac{\pi}{6}$.



$\therefore z=6\,cis\left(-\frac{\pi}{6}\right)$.

Practice questions

Scaffolded

Q1. Express $z=4+4i$ in polar form, using the principal argument. (a) State the real and imaginary parts and the quadrant of z. (b) Find the modulus r. (c) Find the reference angle, then the principal argument. (d) Write z in the form $r\,cis\,\theta$.

Q2. Express $z=-1+\sqrt{3}i$ in polar form. (a) State the quadrant of z. (b) Find the modulus. (c) Find the reference angle α, then $Arg(z)=\pi-\alpha$. (d) Write z in polar form.

Q3. Express $z=-\sqrt{3}-i$ in polar form. (a) State the quadrant of z. (b) Find the modulus. (c) Find the reference angle, then the principal argument. (d) Write z in polar form.

Q4. Express $z=6\,cis\frac{\pi}{2}$ in Cartesian form. (a) Write down $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$. (b) Find $a=r\cos\theta$. (c) Find $b=r\sin\theta$. (d) State z in the form $a+bi$.

Q5. Express $z=2\,cis\left(-\frac{\pi}{4}\right)$ in Cartesian form. (a) Write down $\cos\left(-\frac{\pi}{4}\right)$ and $\sin\left(-\frac{\pi}{4}\right)$. (b) Find a and b. (c) State z in the form $a+bi$.

Q6. Express $z=5-5\sqrt{3}i$ in polar form. (a) State the quadrant of z. (b) Find the modulus. (c) Find the reference angle, then the principal argument. (d) Write z in polar form.

Unscaffolded

Q7. Express $z=2\sqrt{3}+2i$ in polar form, using the principal argument.

Q8. Express $z=-6$ in polar form.

Q9. Express $z = -7i$ in polar form.

Q10. Express $z = 3 - 3i$ in polar form.

Q11. Express $z = -5 + 5i$ in polar form.

Q12. Express $z = 8\,cis\frac{2\pi}{3}$ in Cartesian form.

Q13. Express $z = \sqrt{2}\,cis\left(-\frac{3\pi}{4}\right)$ in Cartesian form.

Q14. Express $z = 10\,cis\,\pi$ in Cartesian form.

Q15. Express $z = -4\sqrt{3} + 4i$ in polar form.

Q16. Plot $z = -2 - 2\sqrt{3}i$ on an Argand diagram, and state its modulus and principal argument.

Q17. A complex number z has $|z| = 6$ and $Arg(z) = -\frac{5\pi}{6}$. Find z in Cartesian form.

Q18. A complex number z has modulus 4 and one of its arguments is $\frac{7\pi}{6}$. (a) Explain why $\frac{7\pi}{6}$ is not the principal argument. (b) State $Arg(z)$. (c) Write z in the form $r\,cis(Arg(z))$ and in Cartesian form.

Answers

Q1. $r=4\sqrt{2}$, $Arg(z)=\frac{\pi}{4}$; $z=4\sqrt{2}\,cis\frac{\pi}{4}$. **Q2.** $r=2$, $Arg(z)=\frac{2\pi}{3}$; $z=2\,cis\frac{2\pi}{3}$. **Q3.** $r=2$, $Arg(z)=-\frac{5\pi}{6}$; $z=2\,cis\left(-\frac{5\pi}{6}\right)$. **Q4.** $z=6i$. **Q5.** $z=\sqrt{2}-\sqrt{2}i$. **Q6.** $r=10$, $Arg(z)=-\frac{\pi}{3}$; $z=10\,cis\left(-\frac{\pi}{3}\right)$. **Q7.** $z=4\,cis\frac{\pi}{6}$. **Q8.** $z=6\,cis\,\pi$. **Q9.** $z=7\,cis\left(-\frac{\pi}{2}\right)$. **Q10.** $z=3\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$. **Q11.** $z=5\sqrt{2}\,cis\frac{3\pi}{4}$. **Q12.** $z=-4+4\sqrt{3}i$. **Q13.** $z=-1-i$. **Q14.** $z=-10$. **Q15.** $z=8\,cis\frac{5\pi}{6}$. **Q16.** $r=4$, $Arg(z)=-\frac{2\pi}{3}$; $z=4\,cis\left(-\frac{2\pi}{3}\right)$. **Q17.** $z=-3\sqrt{3}-3i$. **Q18.** (a) $\frac{7\pi}{6}>\pi$, so it lies outside $(-\pi,\pi]$; (b) $Arg(z)=-\frac{5\pi}{6}$; (c) $z=4\,cis\left(-\frac{5\pi}{6}\right)=-2\sqrt{3}-2i$.

Fully-worked solutions

Q1. $a=4$, $b=4$, both positive, so z is in the first quadrant. $r=\sqrt{4^2+4^2}=\sqrt{32}=4\sqrt{2}$. The reference angle satisfies $\tan\alpha=\left|\frac{4}{4}\right|=1$, so $\alpha=\frac{\pi}{4}$; in quadrant 1, $Arg(z)=\alpha=\frac{\pi}{4}$. Hence $z=4\sqrt{2}\,cis\frac{\pi}{4}$.

Q2. $a=-1$, $b=\sqrt{3}>0$, so z is in the second quadrant. $r=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{1+3}=2$. The reference angle satisfies $\tan\alpha=\left|\frac{\sqrt{3}}{-1}\right|=\sqrt{3}$, so $\alpha=\frac{\pi}{3}$; in quadrant 2, $Arg(z)=\pi-\alpha=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$. Hence $z=2\,cis\frac{2\pi}{3}$.

Q3. $a=-\sqrt{3}$, $b=-1$, both negative, so z is in the third quadrant. $r=\sqrt{(-\sqrt{3})^2+(-1)^2}=\sqrt{3+1}=2$. The reference angle satisfies $\tan\alpha=\left|\frac{-1}{-\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$, so $\alpha=\frac{\pi}{6}$; in quadrant 3, $Arg(z)=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{6}\right)=-\frac{5\pi}{6}$. Hence $z=2\,cis\left(-\frac{5\pi}{6}\right)$.

Q4. With $r=6$ and $\theta=\frac{\pi}{2}$, use $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$. Then $a=6\times0=0$ and $b=6\times1=6$, so $z=0+6i=6i$. (As expected, a modulus of 6 and argument $\frac{\pi}{2}$ places z on the positive imaginary axis.)

Q5. With $r=2$ and $\theta=-\frac{\pi}{4}$, use $\cos\left(-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$. Then $a=2\times\frac{\sqrt{2}}{2}=\sqrt{2}$ and $b=2\times\left(-\frac{\sqrt{2}}{2}\right)=-\sqrt{2}$, so $z=\sqrt{2}-\sqrt{2}i$.

Q6. $a=5>0$, $b=-5\sqrt{3}<0$, so z is in the fourth quadrant. $r=\sqrt{5^2+(-5\sqrt{3})^2}=\sqrt{25+75}=\sqrt{100}=10$. The reference angle satisfies $\tan\alpha=\left|\frac{-5\sqrt{3}}{5}\right|=\sqrt{3}$, so $\alpha=\frac{\pi}{3}$; in quadrant 4, $Arg(z)=-\alpha=-\frac{\pi}{3}$. Hence $z=10\,cis\left(-\frac{\pi}{3}\right)$.

Q7. $a=2\sqrt{3}>0$, $b=2>0$, so z is in the first quadrant. $r=\sqrt{(2\sqrt{3})^2+2^2}=\sqrt{12+4}=\sqrt{16}=4$. The reference angle satisfies $\tan\alpha=\left|\frac{2}{2\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$, so $\alpha=\frac{\pi}{6}$, and in quadrant 1 this is the principal argument. Hence $z=4\,cis\frac{\pi}{6}$.

Q8. $z=-6$ is a negative real number, lying on the negative real axis. Its modulus is $r=|-6|=6$ and its principal argument is $Arg(z)=\pi$. Hence $z=6\,cis\,\pi$.

Q9. $z=-7i$ is a negative imaginary number, lying on the negative imaginary axis. Its modulus is $r=|-7i|=7$ and its principal argument is $Arg(z)=-\frac{\pi}{2}$. Hence $z=7\,cis\left(-\frac{\pi}{2}\right)$.

Q10. $a=3>0$, $b=-3<0$, so z is in the fourth quadrant. $r=\sqrt{3^2+(-3)^2}=\sqrt{18}=3\sqrt{2}$. The reference angle satisfies $\tan\alpha=\left|\frac{-3}{3}\right|=1$, so $\alpha=\frac{\pi}{4}$; in quadrant 4, $Arg(z)=-\frac{\pi}{4}$. Hence $z=3\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$.

Q11. $a=-5<0$, $b=5>0$, so z is in the second quadrant. $r=\sqrt{(-5)^2+5^2}=\sqrt{50}=5\sqrt{2}$. The reference angle satisfies $\tan\alpha=\left|\frac{5}{-5}\right|=1$, so $\alpha=\frac{\pi}{4}$; in quadrant 2, $Arg(z)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$. Hence $z=5\sqrt{2}\,cis\frac{3\pi}{4}$.

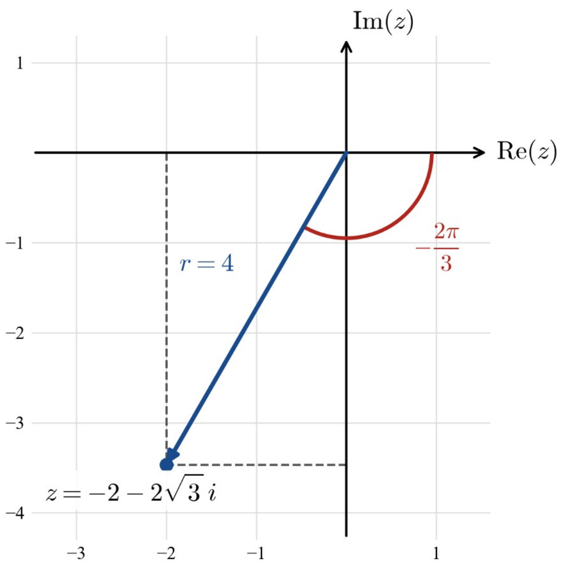
Q12. With $r=8$ and $\theta=\frac{2\pi}{3}$, use $\cos\frac{2\pi}{3}=-\frac{1}{2}$ and $\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$. Then $a=8\times\left(-\frac{1}{2}\right)=-4$ and $b=8\times\frac{\sqrt{3}}{2}=4\sqrt{3}$, so $z=-4+4\sqrt{3}i$.

Q13. With $r=\sqrt{2}$ and $\theta=-\frac{3\pi}{4}$, use $\cos\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}$. Then $a=\sqrt{2}\times\left(-\frac{\sqrt{2}}{2}\right)=-1$ and $b=\sqrt{2}\times\left(-\frac{\sqrt{2}}{2}\right)=-1$, so $z=-1-i$.

Q14. With $r=10$ and $\theta=\pi$, use $\cos\pi=-1$ and $\sin\pi=0$. Then $a=10\times(-1)=-10$ and $b=10\times 0=0$, so $z=-10$.

Q15. $a=-4\sqrt{3}<0$, $b=4>0$, so z is in the second quadrant. $r=\sqrt{(-4\sqrt{3})^2+4^2}=\sqrt{48+16}=\sqrt{64}=8$. The reference angle satisfies $\tan\alpha=\left|\frac{4}{-4\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$, so $\alpha=\frac{\pi}{6}$; in quadrant 2, $Arg(z)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$. Hence $z=8\,cis\frac{5\pi}{6}$.

Q16. $a=-2<0$, $b=-2\sqrt{3}<0$, so z is in the third quadrant. $r=\sqrt{(-2)^2+(-2\sqrt{3})^2}=\sqrt{4+12}=\sqrt{16}=4$. The reference angle satisfies $\tan\alpha=\left|\frac{-2\sqrt{3}}{-2}\right|=\sqrt{3}$, so $\alpha=\frac{\pi}{3}$; in quadrant 3, $Arg(z)=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{3}\right)=-\frac{2\pi}{3}$. The Argand diagram below shows the point, the modulus $r=4$ and the argument arc.



Hence $z=4\,cis\left(-\frac{2\pi}{3}\right)$.

Q17. Using $z=r(\cos\theta+i\sin\theta)$ with $r=6$ and $\theta=-\frac{5\pi}{6}$: $\cos\left(-\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{5\pi}{6}\right)=-\frac{1}{2}$. Then

$$z=6\left(-\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}\right)=6\cdot\left(-\frac{\sqrt{3}}{2}\right)+6\cdot\left(-\frac{1}{2}\right)i=-3\sqrt{3}-3i.$$

Q18. (a) The principal argument must satisfy $-\pi<Arg(z)\leq\pi$. Since $\frac{7\pi}{6}>\pi$, the value $\frac{7\pi}{6}$ lies outside this interval and so is not the principal argument. (b) Subtracting one full turn, $\frac{7\pi}{6}-2\pi=\frac{7\pi-12\pi}{6}=-\frac{5\pi}{6}$, which lies in $(-\pi,\pi]$, so $Arg(z)=-\frac{5\pi}{6}$. (c) With $r=4$, $z=4\,cis\left(-\frac{5\pi}{6}\right)$. Using $\cos\left(-\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{5\pi}{6}\right)=-\frac{1}{2}$,

$$z=4\left(-\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}\right)=-2\sqrt{3}-2i.$$

Theory

An earlier section wrote a complex number in **polar form** (also called modulus–argument form)

$$z = r\,cis\,\theta = r(\cos\theta + i\sin\theta),$$

where $r = |z| \ge 0$ is the modulus and $\theta = arg(z)$ is an argument, with the **principal argument** $Arg(z)$ chosen in $-\pi < Arg(z) \le \pi$. Cartesian form $a + bi$ is convenient for adding and subtracting, but for **multiplying, dividing and taking powers** polar form is far simpler: as this section shows, the moduli multiply or divide while the arguments add or subtract.

Multiplication.

Take $z_1 = r_1\,cis\,\theta_1$ and $z_2 = r_2\,cis\,\theta_2$. Multiplying the two brackets,

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2).$$

Expanding and collecting real and imaginary parts (recall $i^2 = -1$),

$$z_1 z_2 = r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The two brackets are exactly the compound-angle expansions of $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$, so

$$z_1 z_2 = r_1 r_2\,cis(\theta_1 + \theta_2).$$

In words: **to multiply, multiply the moduli and add the arguments.**

Division.

The reciprocal of $z_2 = r_2\,cis\,\theta_2$ (with $z_2 \ne 0$) is $\dfrac{1}{z_2} = \dfrac{1}{r_2}\,cis(-\theta_2)$, because their product is $r_2 \cdot \dfrac{1}{r_2}\,cis(\theta_2 - \theta_2) = 1\,cis\,0 = 1$. Multiplying z_1 by this reciprocal gives

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2).$$

In words: **to divide, divide the moduli and subtract the arguments.**

Returning to the principal range.

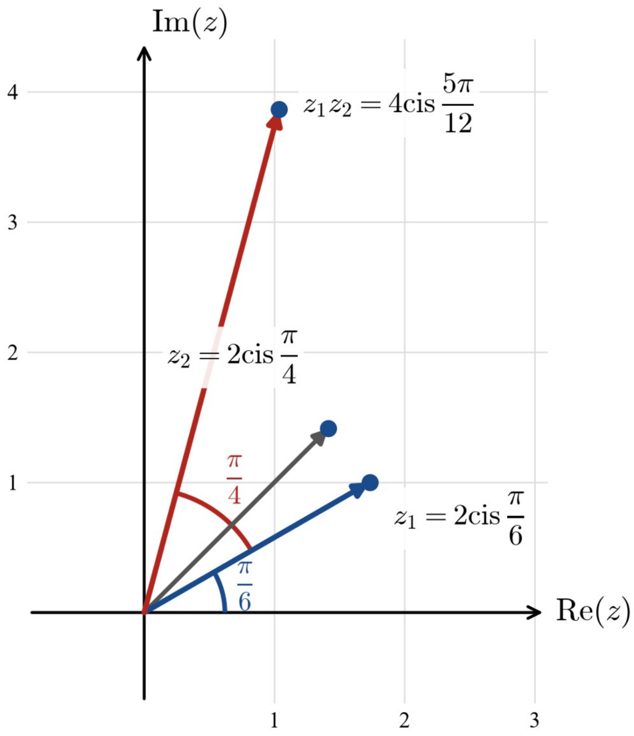
Adding or subtracting two arguments can produce an angle outside $(-\pi, \pi]$. Since adding or subtracting a whole turn 2π does not change the direction, add or subtract 2π as needed to bring the argument back into the principal range before quoting the answer. For example, an argument of $\dfrac{19\pi}{12}$ exceeds π, so subtract one turn: $\dfrac{19\pi}{12} - 2\pi = -\dfrac{5\pi}{12}$.

Geometric interpretation: rotation and scaling.

The multiplication rule gives multiplication a vivid geometric meaning. Multiplying a complex number z by $w = r\,cis\,\phi$ produces a new point whose modulus is r times that of z and whose argument is ϕ more than that of z. So **multiplying by $w = r\,cis\,\phi$ rotates the point z anticlockwise about the origin through the angle ϕ and scales its distance from the origin by the factor r.** Two special cases are worth noting:

- if $|w| = 1$ (that is $w = cis\,\phi$) the multiplication is a **pure rotation** through ϕ, with no change in modulus;
- since $i = cis\,\dfrac{\pi}{2}$, multiplying by i rotates a point $\dfrac{\pi}{2}$ (that is 90°) anticlockwise about the origin.

The diagram below shows the product of $z_1 = 2\,cis\,\frac{\pi}{6}$ and $z_2 = 2\,cis\,\frac{\pi}{4}$. The moduli multiply to $2 \times 2 = 4$ and the arguments add to $\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$, giving $z_1 z_2 = 4\,cis\,\frac{5\pi}{12}$.



Powers by repeated multiplication.

Applying the multiplication rule to $z = r\,cis\,\theta$ multiplied by itself,

$$z^2 = z \cdot z = r \cdot r\,cis(\theta + \theta) = r^2\,cis(2\theta),$$

and then multiplying by z once more,

$$z^3 = z^2 \cdot z = r^2 \cdot r\,cis(2\theta + \theta) = r^3\,cis(3\theta).$$

Continuing this way, each extra factor of z multiplies the modulus by r and adds θ to the argument, so a positive integer power raises the modulus to that power and multiplies the argument by it. Geometrically each factor repeats the same rotation-and-scale, so the points $z, z^2, z^3, \ldots$ spiral out (or in) turning by a constant angle θ at each step. This lets us compute small powers directly; the general result is stated as **De Moivre's theorem** in a later section. As always, reduce the final argument to the principal range.

Worked examples

Example 1 — scaffolded

Given $z_1 = 3\,cis\,\frac{\pi}{4}$ and $z_2 = 2\,cis\,\frac{\pi}{6}$, find $z_1 z_2$ in polar form, using the principal argument.

Working	Comment
$r_1 r_2 = 3 \times 2 = 6.$	To multiply, multiply the moduli.
$\theta_1 + \theta_2 = \frac{\pi}{4} + \frac{\pi}{6} = \frac{3\pi}{12} + \frac{2\pi}{12} = \frac{5\pi}{12}.$	Add the arguments over a common denominator.
$-\pi < \frac{5\pi}{12} \le \pi$, so the argument is already principal.	Check the sum lies in $(-\pi, \pi]$.

Working	Comment
$z_1 z_2 = 6\,cis\,\frac{5\pi}{12}$.	Combine the new modulus and argument.

Example 2 — scaffolded

Given $z_1 = 10\,cis\,\frac{5\pi}{6}$ and $z_2 = 2\,cis\left(-\frac{3\pi}{4}\right)$, find $\frac{z_1}{z_2}$ in polar form, using the principal argument.

Working	Comment
$\frac{r_1}{r_2} = \frac{10}{2} = 5$.	To divide, divide the moduli.
$\theta_1 - \theta_2 = \frac{5\pi}{6} - \left(-\frac{3\pi}{4}\right) = \frac{5\pi}{6} + \frac{3\pi}{4} = \frac{10\pi}{12} + \frac{9\pi}{12} = \frac{19\pi}{12}$.	Subtract the arguments; two negatives make a plus.
$\frac{19\pi}{12} > \pi$, so subtract one turn: $\frac{19\pi}{12} - 2\pi = \frac{19\pi}{12} - \frac{24\pi}{12} = -\frac{5\pi}{12}$.	Bring the argument into the principal range.
$\frac{z_1}{z_2} = 5\,cis\left(-\frac{5\pi}{12}\right)$.	Combine the new modulus and principal argument.

Example 3 — unscaffolded

Given $z_1 = 4\,cis\,\frac{2\pi}{3}$ and $z_2 = 3\,cis\,\frac{3\pi}{4}$, find $z_1 z_2$ in polar form, using the principal argument.

Multiplying the moduli gives $r_1 r_2 = 4 \times 3 = 12$, and adding the arguments gives

$$\theta_1 + \theta_2 = \frac{2\pi}{3} + \frac{3\pi}{4} = \frac{8\pi}{12} + \frac{9\pi}{12} = \frac{17\pi}{12}.$$

Since $\frac{17\pi}{12} > \pi$, this angle is outside the principal range, so subtract one full turn:

$$\frac{17\pi}{12} - 2\pi = \frac{17\pi}{12} - \frac{24\pi}{12} = -\frac{7\pi}{12},$$

which lies in $(-\pi, \pi]$.

$\therefore z_1 z_2 = 12\,cis\left(-\frac{7\pi}{12}\right)$.

Example 4 — unscaffolded

Let $z = 1 + i$. Use polar form to find z^3 by repeated multiplication, and give the answer in both polar and Cartesian form.

From an earlier section, $z = 1 + i$ has modulus $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and principal argument $\frac{\pi}{4}$, so $z = \sqrt{2}\,cis\,\frac{\pi}{4}$. Squaring by the multiplication rule (modulus squared, argument doubled),

$$z^2 = (\sqrt{2})^2\,cis\left(2 \times \frac{\pi}{4}\right) = 2\,cis\,\frac{\pi}{2},$$

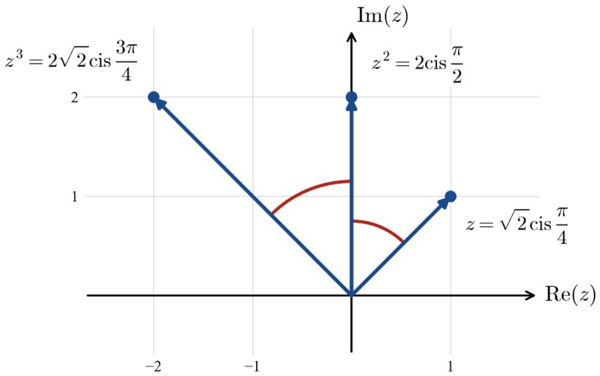
then multiplying by z once more,

$$z^3 = z^2 \cdot z = 2 \times \sqrt{2}\, cis\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = 2\sqrt{2}\, cis\frac{3\pi}{4}.$$

The argument $\frac{3\pi}{4}$ is already principal. Converting to Cartesian with $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$,

$$z^3 = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right) = -2 + 2i.$$

The diagram shows z, z^2 and z^3: each step turns a further $\frac{\pi}{4}$ anticlockwise and stretches the modulus by $\sqrt{2}$, so the points spiral outwards.



$\therefore z^3 = 2\sqrt{2}\, cis\frac{3\pi}{4} = -2 + 2i.$

Practice questions

Scaffolded

Q1. Given $z_1 = 5\, cis\frac{\pi}{3}$ and $z_2 = 3\, cis\frac{\pi}{4}$, find $z_1 z_2$ in polar form. (a) Multiply the moduli. (b) Add the arguments over a common denominator. (c) Check the argument lies in $(-\pi, \pi]$. (d) Write $z_1 z_2$ in the form $r\, cis\, \theta$.

Q2. Given $z_1 = 12\, cis\frac{3\pi}{4}$ and $z_2 = 3\, cis\frac{\pi}{3}$, find $\frac{z_1}{z_2}$ in polar form. (a) Divide the moduli. (b) Subtract the arguments. (c) Check the argument is principal. (d) Write $\frac{z_1}{z_2}$ in polar form.

Q3. Given $z_1 = 2\, cis\frac{5\pi}{6}$ and $z_2 = 5\, cis\frac{3\pi}{4}$, find $z_1 z_2$ in polar form. (a) Multiply the moduli. (b) Add the arguments. (c) The sum exceeds π; subtract 2π to reach the principal argument. (d) Write $z_1 z_2$ in polar form.

Q4. Given $z_1 = 8\, cis\left(-\frac{5\pi}{6}\right)$ and $z_2 = 2\, cis\frac{2\pi}{3}$, find $\frac{z_1}{z_2}$ in polar form. (a) Divide the moduli. (b) Subtract the arguments. (c) The result is below $-\pi$; add 2π to reach the principal argument. (d) Write $\frac{z_1}{z_2}$ in polar form.

Q5. Let $z = 2\, cis\frac{\pi}{6}$. Find z^3 by repeated multiplication. (a) Find $z^2 = z \cdot z$ (square the modulus, double the argument). (b) Find $z^3 = z^2 \cdot z$. (c) Check the argument is principal, and write z^3 in polar form.

Q6. Given $z_1 = 2\,cis\frac{5\pi}{12}$ and $z_2 = 4\,cis\frac{\pi}{4}$, find $z_1 z_2$ and give the answer in Cartesian form. (a) Multiply the moduli and add the arguments. (b) Write the product in polar form. (c) Using $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, convert to $a+bi$.

Unscaffolded

Q7. Given $z_1 = 6\,cis\frac{\pi}{8}$ and $z_2 = 2\,cis\frac{\pi}{8}$, find $z_1 z_2$ in polar form.

Q8. Given $z_1 = 20\,cis\frac{2\pi}{3}$ and $z_2 = 4\,cis\frac{\pi}{6}$, find $\frac{z_1}{z_2}$ in polar form.

Q9. Given $z_1 = 4\,cis\frac{5\pi}{6}$ and $z_2 = 3\,cis\frac{5\pi}{6}$, find $z_1 z_2$ in polar form, using the principal argument.

Q10. Given $z_1 = 6\,cis\left(-\frac{3\pi}{4}\right)$ and $z_2 = 2\,cis\frac{2\pi}{3}$, find $\frac{z_1}{z_2}$ in polar form, using the principal argument.

Q11. Let $z = 5\,cis\frac{3\pi}{8}$. Find z^2 in polar form.

Q12. Let $z = 2\,cis\frac{2\pi}{3}$. Find z^3 in polar form, using the principal argument, and state what kind of number it is.

Q13. Let $z = 2\,cis\frac{\pi}{3}$. Find z^4, giving the answer in Cartesian form.

Q14. Given $z_1 = 6\,cis\frac{3\pi}{4}$, $z_2 = 2\,cis\frac{\pi}{3}$ and $z_3 = 3\,cis\frac{\pi}{6}$, find $\frac{z_1 z_2}{z_3}$ in polar form.

Q15. The point $z = 3\,cis\frac{\pi}{5}$ is multiplied by $w = cis\frac{3\pi}{10}$. Describe the geometric effect on z, and state the resulting number in polar form.

Q16. Express i in polar form. Hence describe the geometric effect of multiplying any complex number by i, and find iz in polar form when $z = 4\,cis\frac{\pi}{6}$.

Q17. A complex number z_1 has $|z_1| = 5$ and $Arg(z_1) = \frac{2\pi}{3}$; a complex number z_2 has $|z_2| = 2$ and $Arg(z_2) = \frac{3\pi}{4}$. Find $|z_1 z_2|$ and $Arg(z_1 z_2)$.

Q18. Let $z = 2\,cis\frac{\pi}{4}$. Find the smallest positive integer n for which z^n is a real number, and state the value of z^n for that n.

Answers

Q1. $z_1 z_2 = 15\,cis\frac{7\pi}{12}$. **Q2.** $\frac{z_1}{z_2} = 4\,cis\frac{5\pi}{12}$. **Q3.** $z_1 z_2 = 10\,cis\left(-\frac{5\pi}{12}\right)$. **Q4.** $\frac{z_1}{z_2} = 4\,cis\frac{\pi}{2}(=4i)$. **Q5.** $z^3 = 8\,cis\frac{\pi}{2}(=8i)$. **Q6.** $z_1 z_2 = 8\,cis\frac{2\pi}{3} = -4 + 4\sqrt{3}i$. **Q7.** $z_1 z_2 = 12\,cis\frac{\pi}{4}$. **Q8.** $\frac{z_1}{z_2} = 5\,cis\frac{\pi}{2}(=5i)$. **Q9.** $z_1 z_2 = 12\,cis\left(-\frac{\pi}{3}\right)$. **Q10.** $\frac{z_1}{z_2} = 3\,cis\frac{7\pi}{12}$. **Q11.** $z^2 = 25\,cis\frac{3\pi}{4}$. **Q12.** $z^3 = 8\,cis\,0 = 8$, a positive real number. **Q13.** $z^4 = 16\,cis\left(-\frac{2\pi}{3}\right) = -8 - 8\sqrt{3}i$. **Q14.** $\frac{z_1 z_2}{z_3} = 4\,cis\frac{11\pi}{12}$. **Q15.** A pure anticlockwise rotation of $\frac{3\pi}{10}$ about O (modulus unchanged); result $3\,cis\frac{\pi}{2}(=3i)$. **Q16.** $i = cis\frac{\pi}{2}$; multiplying by i rotates a point $\frac{\pi}{2}$ anticlockwise about O with no change in modulus; $iz = 4\,cis\frac{2\pi}{3}$. **Q17.** $|z_1 z_2| = 10$, $Arg(z_1 z_2) = -\frac{7\pi}{12}$. **Q18.** $n=4$; $z^4 = 16\,cis\,\pi = -16$ (a negative real number).

Fully-worked solutions

Q1. Multiply the moduli: $r_1 r_2 = 5 \times 3 = 15$. Add the arguments: $\frac{\pi}{3} + \frac{\pi}{4} = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{7\pi}{12}$, which satisfies $-\pi < \frac{7\pi}{12} \le \pi$. Hence $z_1 z_2 = 15\,cis\frac{7\pi}{12}$.

Q2. Divide the moduli: $\frac{r_1}{r_2} = \frac{12}{3} = 4$. Subtract the arguments: $\frac{3\pi}{4} - \frac{\pi}{3} = \frac{9\pi}{12} - \frac{4\pi}{12} = \frac{5\pi}{12}$, already principal. Hence $\frac{z_1}{z_2} = 4\,cis\frac{5\pi}{12}$.

Q3. Multiply the moduli: $2 \times 5 = 10$. Add the arguments: $\frac{5\pi}{6} + \frac{3\pi}{4} = \frac{10\pi}{12} + \frac{9\pi}{12} = \frac{19\pi}{12}$. Since $\frac{19\pi}{12} > \pi$, subtract one turn: $\frac{19\pi}{12} - 2\pi = -\frac{5\pi}{12}$. Hence $z_1 z_2 = 10\,cis\left(-\frac{5\pi}{12}\right)$.

Q4. Divide the moduli: $\frac{8}{2} = 4$. Subtract the arguments: $-\frac{5\pi}{6} - \frac{2\pi}{3} = -\frac{5\pi}{6} - \frac{4\pi}{6} = -\frac{9\pi}{6} = -\frac{3\pi}{2}$. Since $-\frac{3\pi}{2} \le -\pi$, add one turn: $-\frac{3\pi}{2} + 2\pi = \frac{\pi}{2}$. Hence $\frac{z_1}{z_2} = 4\,cis\frac{\pi}{2}$, which equals $4i$.

Q5. Squaring: $z^2 = 2^2\,cis\left(2 \times \frac{\pi}{6}\right) = 4\,cis\frac{\pi}{3}$. Multiplying by z again: $z^3 = z^2 \cdot z = 4 \times 2\,cis\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = 8\,cis\frac{\pi}{2}$, and $\frac{\pi}{2}$ is principal. Hence $z^3 = 8\,cis\frac{\pi}{2} = 8i$.

Q6. Multiply the moduli: $2 \times 4 = 8$. Add the arguments: $\frac{5\pi}{12} + \frac{\pi}{4} = \frac{5\pi}{12} + \frac{3\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$, so $z_1 z_2 = 8\,cis\frac{2\pi}{3}$. Converting with $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $z_1 z_2 = 8\left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right) = -4 + 4\sqrt{3}i$.

Q7. Multiply the moduli: $6 \times 2 = 12$. Add the arguments: $\frac{\pi}{8} + \frac{\pi}{8} = \frac{2\pi}{8} = \frac{\pi}{4}$, which is principal. Hence $z_1 z_2 = 12\,cis\frac{\pi}{4}$.

Q8. Divide the moduli: $\frac{20}{4}=5$. Subtract the arguments: $\frac{2\pi}{3}-\frac{\pi}{6}=\frac{4\pi}{6}-\frac{\pi}{6}=\frac{3\pi}{6}=\frac{\pi}{2}$, which is principal. Hence $\frac{z_1}{z_2}=5\,cis\frac{\pi}{2}=5i$.

Q9. Multiply the moduli: $4\times3=12$. Add the arguments: $\frac{5\pi}{6}+\frac{5\pi}{6}=\frac{10\pi}{6}=\frac{5\pi}{3}$. Since $\frac{5\pi}{3}>\pi$, subtract one turn: $\frac{5\pi}{3}-2\pi=-\frac{\pi}{3}$. Hence $z_1z_2=12\,cis\left(-\frac{\pi}{3}\right)$.

Q10. Divide the moduli: $\frac{6}{2}=3$. Subtract the arguments: $-\frac{3\pi}{4}-\frac{2\pi}{3}=-\frac{9\pi}{12}-\frac{8\pi}{12}=-\frac{17\pi}{12}$. Since $-\frac{17\pi}{12}\le-\pi$, add one turn: $-\frac{17\pi}{12}+2\pi=-\frac{17\pi}{12}+\frac{24\pi}{12}=\frac{7\pi}{12}$. Hence $\frac{z_1}{z_2}=3\,cis\frac{7\pi}{12}$.

Q11. Squaring raises the modulus to the power 2 and doubles the argument: $z^2=5^2\,cis\left(2\times\frac{3\pi}{8}\right)=25\,cis\frac{3\pi}{4}$, and $\frac{3\pi}{4}$ is principal. Hence $z^2=25\,cis\frac{3\pi}{4}$.

Q12. Cubing raises the modulus to the power 3 and triples the argument: $z^3=2^3\,cis\left(3\times\frac{2\pi}{3}\right)=8\,cis(2\pi)$. Subtracting one turn, $2\pi-2\pi=0$, so $z^3=8\,cis\,0=8(\cos0+i\sin0)=8$, a positive real number.

Q13. $z^4=2^4\,cis\left(4\times\frac{\pi}{3}\right)=16\,cis\frac{4\pi}{3}$. Since $\frac{4\pi}{3}>\pi$, subtract one turn: $\frac{4\pi}{3}-2\pi=-\frac{2\pi}{3}$, so $z^4=16\,cis\left(-\frac{2\pi}{3}\right)$. Converting with $\cos\left(-\frac{2\pi}{3}\right)=-\frac{1}{2}$ and $\sin\left(-\frac{2\pi}{3}\right)=-\frac{\sqrt{3}}{2}$, $z^4=16\left(-\frac{1}{2}-i\cdot\frac{\sqrt{3}}{2}\right)=-8-8\sqrt{3}i$.

Q14. The moduli combine as $\frac{6\times2}{3}=4$, and the arguments as $\frac{3\pi}{4}+\frac{\pi}{3}-\frac{\pi}{6}=\frac{9\pi}{12}+\frac{4\pi}{12}-\frac{2\pi}{12}=\frac{11\pi}{12}$, which is principal. Hence $\frac{z_1z_2}{z_3}=4\,cis\frac{11\pi}{12}$.

Q15. Since $|w|=1$, multiplying by $w=cis\frac{3\pi}{10}$ is a **pure rotation**: it turns z anticlockwise about the origin through $\frac{3\pi}{10}$ and leaves its modulus unchanged. The result has modulus $3\times1=3$ and argument $\frac{\pi}{5}+\frac{3\pi}{10}=\frac{2\pi}{10}+\frac{3\pi}{10}=\frac{5\pi}{10}=\frac{\pi}{2}$. Hence the result is $3\,cis\frac{\pi}{2}=3i$.

Q16. Since $|i|=1$ and $Arg(i)=\frac{\pi}{2}$, we have $i=cis\frac{\pi}{2}$. Multiplying a number by i therefore adds $\frac{\pi}{2}$ to its argument and leaves its modulus unchanged, that is, it rotates the point $\frac{\pi}{2}$ (or 90°) anticlockwise about the origin. For $z=4\,cis\frac{\pi}{6}$, $iz=4\,cis\left(\frac{\pi}{6}+\frac{\pi}{2}\right)=4\,cis\frac{2\pi}{3}$ (the argument $\frac{2\pi}{3}$ is principal).

Q17. The modulus of a product is the product of the moduli: $|z_1z_2|=5\times2=10$. An argument of the product is the sum of the arguments: $\frac{2\pi}{3}+\frac{3\pi}{4}=\frac{8\pi}{12}+\frac{9\pi}{12}=\frac{17\pi}{12}$. As $\frac{17\pi}{12}>\pi$, subtract one turn: $\frac{17\pi}{12}-2\pi=-\frac{7\pi}{12}$, which is principal. Hence $|z_1z_2|=10$ and $Arg(z_1z_2)=-\frac{7\pi}{12}$.

Q18. By repeated multiplication, $z^n = 2^n\, cis\,\frac{n\pi}{4}$. This is real exactly when its argument $\frac{n\pi}{4}$ is a whole multiple of π, that is when $\frac{n}{4}$ is an integer, so n must be a multiple of 4; the smallest positive such n is $n=4$. Then $z^4 = 2^4\, cis\left(4 \times \frac{\pi}{4}\right) = 16\, cis\,\pi = 16(\cos\pi + i\sin\pi) = -16$, a negative real number.

Theory

Over the real numbers a quadratic equation $az^2+bz+c=0$ has no solution when its graph fails to cross the horizontal axis: the formula produces the square root of a negative number, which is not real. Now that the imaginary unit i with $i^2=-1$ is available, that obstruction disappears. **Every** quadratic equation can be solved, and its solutions are complex numbers. The two familiar techniques — **completing the square** and the **quadratic formula** — carry across unchanged; the only new ingredient is that square roots of negative numbers are now permitted. The modulus, conjugate and Cartesian arithmetic developed in earlier sections are used freely here.

Square roots of a negative real number.

For a positive real number k,

$$\sqrt{-k}=\sqrt{k}\,i\,,\text{because}\left(\sqrt{k}\,i\right)^2=k\,i^2=-k\,.$$

For instance $\sqrt{-9}=3i$, $\sqrt{-16}=4i$ and $\sqrt{-5}=\sqrt{5}\,i$. The number $-\sqrt{k}\,i$ is also a square root of $-k$, so a negative real has two square roots, $\pm\sqrt{k}\,i$, just as a positive real has two square roots $\pm\sqrt{k}$.

The discriminant and the nature of the roots.

For a quadratic $az^2+bz+c=0$ with **real** coefficients a,b,c (and $a\neq 0$), the **discriminant** is

$$\Delta=b^2-4ac\,.$$

Its sign classifies the roots exactly as before, with the negative case now interpreted over $\mathbb{C}$:

Δ	Nature of the roots
$\Delta>0$	two distinct real roots
$\Delta=0$	one repeated real root
$\Delta<0$	two distinct complex (non-real) roots

When $\Delta<0$ the two roots are not real; as we shall see, they form a **conjugate pair**.

Solving by completing the square.

The idea is to rewrite the quadratic so that the variable appears only inside a perfect square, then take square roots. For a monic quadratic $z^2+pz+q=0$,

$$z^2+pz+q=\left(z+\frac{p}{2}\right)^2-\frac{p^2}{4}+q\,,$$

so the equation becomes $\left(z+\frac{p}{2}\right)^2=\frac{p^2}{4}-q$. If the right-hand side is negative, its square roots are imaginary and the solutions are complex. For example, $z^2=-4$ gives $z=\pm\sqrt{-4}=\pm 2i$. A non-monic quadratic is first divided through by its leading coefficient.

The quadratic formula over $\mathbb{C}$.

Completing the square on the general quadratic $az^2+bz+c=0$ produces the usual formula

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{\Delta}}{2a}\,.$$

When the coefficients are real and $\Delta<0$, write $\sqrt{\Delta}=\sqrt{-\Delta}\,i$ (with $-\Delta>0$), so

$$z=\frac{-b\pm\sqrt{-\Delta}\,i}{2a}=-\frac{b}{2a}\pm\frac{\sqrt{4ac-b^2}}{2a}i.$$

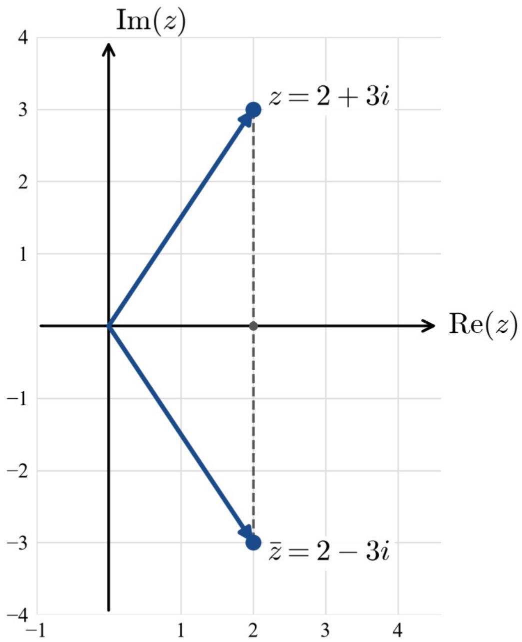
The two solutions share the real part $-\dfrac{b}{2a}$ and have opposite imaginary parts.

The conjugate root theorem.

Reading off the formula above, when a, b, c are real and $\Delta<0$ the roots are

$$z=p+qi \text{ and } \bar{z}=p-qi,\; p=-\frac{b}{2a},\; q=\frac{\sqrt{4ac-b^2}}{2a}.$$

Each is the **conjugate** of the other. This is the **conjugate root theorem** (for quadratics): if a quadratic has real coefficients and one non-real root $z=p+qi$, then its conjugate $\bar{z}=p-qi$ is the other root. On an Argand diagram the two roots are mirror images in the real axis.



The theorem also runs the other way: to have real coefficients, a quadratic with a non-real root **must** have the conjugate as its second root. (The same conjugate-pairing result holds for polynomials of higher degree with real coefficients, developed in a later section.)

Sum and product of the roots.

If α and β are the roots of $az^2+bz+c=0$, then $az^2+bz+c=a(z-\alpha)(z-\beta)$. Expanding the right-hand side and comparing coefficients gives

$$\alpha+\beta=-\frac{b}{a},\; \alpha\beta=\frac{c}{a}.$$

Dividing a quadratic by a shows that a **monic** quadratic with roots α and β is

$$z^2-(\alpha+\beta)z+\alpha\beta=0.$$

These relations give a quick way to build a quadratic from its roots, and a quick check on any solution. For a conjugate pair $\alpha=p+qi$, $\beta=p-qi$,

$$\alpha+\beta=2p\,(\text{real}),\; \alpha\beta=(p+qi)(p-qi)=p^2+q^2\,(\text{real}),$$

so the resulting quadratic $z^2-2pz+(p^2+q^2)=0$ has real coefficients — consistent with the conjugate root theorem. Note that $\alpha\beta=p^2+q^2=|\alpha|^2$, the square of the modulus.

Quadratics with non-real coefficients.

If the coefficients a,b,c are themselves complex, the quadratic formula still applies — it is an identity in any field where $2a\neq 0$ — but two features change. First, the discriminant $\Delta=b^2-4ac$ is generally a complex number, so its sign no longer classifies the roots and $\sqrt{\Delta}$ means a complex square root. To find $\sqrt{c+di}$ we seek a complex number $u+vi$ with $(u+vi)^2=c+di$; expanding and equating real and imaginary parts gives

$$u^2-v^2=c,2uv=d,$$

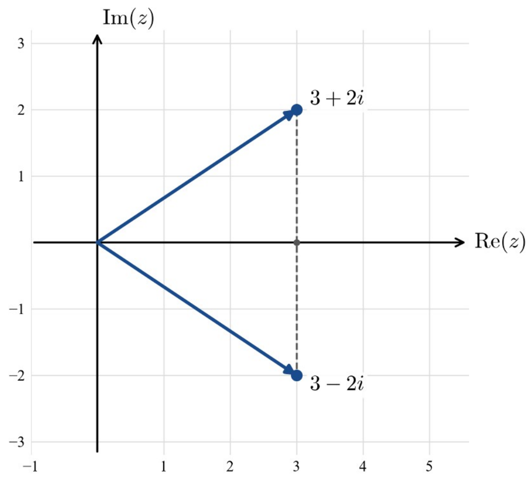
a pair of equations solved for real u,v. Second, because the coefficients are not all real, the conjugate root theorem **does not** apply: the two roots of such a quadratic need not be conjugates of one another.

Worked examples

Example 1 — scaffolded

Solve $z^2-6z+13=0$ over $\mathbb{C}$ by completing the square.

Working	Comment
$z^2-6z=(z-3)^2-9$.	Halve the coefficient of z; here $\frac{-6}{2}=-3$.
$(z-3)^2-9+13=0$, so $(z-3)^2+4=0$.	Substitute back into the equation and simplify the constant.
$(z-3)^2=-4$.	Isolate the perfect square.
$z-3=\pm\sqrt{-4}=\pm 2i$.	Square root of a negative real: $\sqrt{-4}=2i$.
$z=3+2i$ or $z=3-2i$.	A conjugate pair, as the coefficients are real.



Example 2 — scaffolded

Solve $2z^2+2z+5=0$ over $\mathbb{C}$ using the quadratic formula.

Working	Comment
$a=2,b=2,c=5$; $\Delta=2^2-4(2)(5)=4-40=-36$.	Compute the discriminant first; $\Delta<0$ signals complex roots.
$\sqrt{\Delta}=\sqrt{-36}=6i$.	Write the square root of the negative discriminant using i.

Working	Comment
$z=\frac{-2\pm 6i}{2(2)}=\frac{-2\pm 6i}{4}$.	Substitute into $z=\frac{-b\pm\sqrt{\Delta}}{2a}$.
$z=-\frac{1}{2}\pm\frac{3}{2}i$.	Divide real and imaginary parts by 4 and simplify.
Check: sum $=-1=-\frac{b}{a}$, product $=\frac{1}{4}+\frac{9}{4}=\frac{5}{2}=\frac{c}{a}$.	Vieta's relations confirm the answer.

Example 3 — unscaffolded

A quadratic equation with real coefficients has $z=4-i$ as one solution. Find a monic quadratic equation that it satisfies.

Because the coefficients are real, the conjugate root theorem guarantees that the conjugate $\overline{4-i}=4+i$ is the second root. The sum and product of the two roots are

$$\alpha+\beta=(4-i)+(4+i)=8,\alpha\beta=(4-i)(4+i)=4^2-i^2=16+1=17.$$

A monic quadratic with these roots is $z^2-(\alpha+\beta)z+\alpha\beta=0$, that is $z^2-8z+17=0$. As a check, completing the square gives $(z-4)^2+1=0$, so $(z-4)^2=-1$ and $z=4\pm i$, as required.

$\therefore z^2-8z+17=0$.

Example 4 — unscaffolded

Solve $z^2-(3+i)z+(2+2i)=0$ over $\mathbb{C}$.

The coefficients are not all real, so the conjugate root theorem does not apply; we use the quadratic formula with $a=1$, $b=-(3+i)$ and $c=2+2i$. The discriminant is

$$\Delta=b^2-4ac=(3+i)^2-4(2+2i)=(8+6i)-(8+8i)=-2i.$$

This is complex, so we find $\sqrt{-2i}$ by seeking $u+vi$ with $(u+vi)^2=-2i$. Equating real and imaginary parts gives $u^2-v^2=0$ and $2uv=-2$. The first forces $u=\pm v$, and since $uv=-1<0$ the signs are opposite; taking $u=1$, $v=-1$ gives $\sqrt{-2i}=\pm(1-i)$ (indeed $(1-i)^2=1-2i+i^2=-2i$). Then

$$z=\frac{(3+i)\pm(1-i)}{2}.$$

The plus sign gives $z=\frac{4}{2}=2$, and the minus sign gives $z=\frac{2+2i}{2}=1+i$. These roots are not conjugates, as expected for non-real coefficients.

$\therefore z=2$ or $z=1+i$.

Practice questions

Scaffolded

Q1. Solve $z^2-4z+5=0$ over $\mathbb{C}$ by completing the square. (a) Write z^2-4z in the form $(z-h)^2-h^2$. (b) Substitute back and rearrange to $(z-h)^2=k$. (c) Take square roots, expressing $\sqrt{k}$ in terms of i. (d) State the two solutions.

Q2. Solve $z^2+2z+10=0$ using the quadratic formula. (a) State a,b,c and compute the discriminant $\Delta=b^2-4ac$. (b) Since $\Delta<0$, write $\sqrt{\Delta}$ in terms of i. (c) Apply the formula $z=\frac{-b\pm\sqrt{\Delta}}{2a}$. (d) State the two solutions and confirm they are conjugates.

Q3. Consider $3z^2-2z+1=0$. (a) Compute the discriminant. (b) State the nature of the roots (real or complex; distinct or repeated). (c) Solve the equation using the quadratic formula. (d) Write the roots as a conjugate pair $p\pm qi$.

Q4. A quadratic with real coefficients has $z=-2+3i$ as one root. (a) State the other root. (b) Find the sum of the roots. (c) Find the product of the roots. (d) Write down a monic quadratic equation with these roots.

Q5. The quadratic $2z^2+6z+5=0$ has roots α and β. (a) Without solving, state $\alpha+\beta$ and $\alpha\beta$. (b) Solve the equation. (c) Verify that your solutions have the sum and product found in (a).

Q6. Solve $z^2-(3+2i)z+6i=0$ using the quadratic formula. (a) Identify a,b,c and compute the discriminant $\Delta=b^2-4ac$. (b) Find $\sqrt{\Delta}$ by writing $\sqrt{\Delta}=u+vi$ and solving $u^2-v^2=\mathrm{Re}(\Delta)$, $2uv=\mathrm{Im}(\Delta)$. (c) Apply the quadratic formula. (d) State the two solutions. (The coefficients are not all real, so the roots need not be conjugates.)

Unscaffolded

Q7. Solve $z^2+25=0$ over $\mathbb{C}$.

Q8. Solve $z^2-2z+2=0$ over $\mathbb{C}$.

Q9. Solve $z^2+6z+34=0$ over $\mathbb{C}$.

Q10. Solve $z^2-10z+29=0$ over $\mathbb{C}$.

Q11. Solve $4z^2+4z+5=0$ over $\mathbb{C}$.

Q12. Solve $3z^2+2z+4=0$ over $\mathbb{C}$, giving the roots in exact form.

Q13. Find a monic quadratic equation with real coefficients that has $z=1+4i$ as a root.

Q14. A quadratic with real coefficients has leading coefficient 2 and $z=-3+i$ as a root. Find the quadratic.

Q15. The equation $z^2+bz+c=0$ has real coefficients and one solution $z=2-5i$. Find b and c.

Q16. The quadratic $2z^2-3z+5=0$ has roots α and β. State the sum and product of the roots, then solve the equation and confirm your solutions agree.

Q17. Solve $z^2-(4+2i)z+(2+4i)=0$ over $\mathbb{C}$.

Q18. Solve $z^2-(2+i)z+(-1+7i)=0$ over $\mathbb{C}$.

Answers

Q1. $z=2\pm i$. **Q2.** $z=-1\pm 3i$. **Q3.** $\Delta=-8$; two distinct complex (conjugate) roots; $z=\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$. **Q4.** other root $-2-3i$; sum -4, product 13; $z^2+4z+13=0$. **Q5.** $\alpha+\beta=-3$, $\alpha\beta=\frac{5}{2}$; $z=-\frac{3}{2}\pm\frac{1}{2}i$. **Q6.** $\Delta=5-12i$, $\sqrt{\Delta}=\pm(3-2i)$; $z=3$ or $z=2i$. **Q7.** $z=\pm 5i$. **Q8.** $z=1\pm i$. **Q9.** $z=-3\pm 5i$. **Q10.** $z=5\pm 2i$. **Q11.** $z=-\frac{1}{2}\pm i$. **Q12.** $z=-\frac{1}{3}\pm\frac{\sqrt{11}}{3}i$. **Q13.** $z^2-2z+17=0$. **Q14.** $2z^2+12z+20=0$. **Q15.** $b=-4$, $c=29$. **Q16.** $\alpha+\beta=\frac{3}{2}$, $\alpha\beta=\frac{5}{2}$; $z=\frac{3}{4}\pm\frac{\sqrt{31}}{4}i$. **Q17.** $z=3+i$ or $z=1+i$. **Q18.** $z=3-i$ or $z=-1+2i$.

Fully-worked solutions

Q1. Completing the square, $z^2-4z=(z-2)^2-4$, so the equation becomes $(z-2)^2-4+5=0$, that is $(z-2)^2+1=0$. Hence $(z-2)^2=-1$, and taking square roots $z-2=\pm\sqrt{-1}=\pm i$. Therefore $z=2\pm i$. (Check: $\Delta=(-4)^2-4(1)(5)=-4<0$, so complex conjugate roots are expected.)

Q2. Here $a=1$, $b=2$, $c=10$, so $\Delta=2^2-4(1)(10)=4-40=-36$. Since $\Delta<0$, $\sqrt{\Delta}=\sqrt{-36}=6i$. The quadratic formula gives

$$z=\frac{-2\pm 6i}{2(1)}=\frac{-2\pm 6i}{2}=-1\pm 3i.$$

The two solutions $-1+3i$ and $-1-3i$ are conjugates, consistent with the real coefficients.

Q3. With $a=3$, $b=-2$, $c=1$, the discriminant is $\Delta=(-2)^2-4(3)(1)=4-12=-8$. Because $\Delta<0$, the equation has two distinct complex roots forming a conjugate pair. Now $\sqrt{-8}=2\sqrt{2}i$, so

$$z=\frac{-(-2)\pm 2\sqrt{2}i}{2(3)}=\frac{2\pm 2\sqrt{2}i}{6}=\frac{1}{3}\pm\frac{\sqrt{2}}{3}i.$$

So $p=\frac{1}{3}$ and $q=\frac{\sqrt{2}}{3}$, giving the pair $z=\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$.

Q4. (a) The coefficients are real, so by the conjugate root theorem the other root is the conjugate $\overline{-2+3i}=-2-3i$. (b) The sum of the roots is $(-2+3i)+(-2-3i)=-4$. (c) The product is $(-2+3i)(-2-3i)=(-2)^2-(3i)^2=4+9=13$. (d) A monic quadratic with these roots is $z^2-(\text{sum})z+(\text{product})=z^2-(-4)z+13=z^2+4z+13=0$.

Q5. (a) For $2z^2+6z+5=0$, Vieta's relations give $\alpha+\beta=-\frac{b}{a}=-\frac{6}{2}=-3$ and $\alpha\beta=\frac{c}{a}=\frac{5}{2}$. (b) The discriminant is $\Delta=6^2-4(2)(5)=36-40=-4$, so $\sqrt{\Delta}=2i$ and

$$z=\frac{-6\pm 2i}{2(2)}=\frac{-6\pm 2i}{4}=-\frac{3}{2}\pm\frac{1}{2}i.$$

(c) The sum is $\left(-\frac{3}{2}+\frac{1}{2}i\right)+\left(-\frac{3}{2}-\frac{1}{2}i\right)=-3$, and the product is $\left(-\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{9}{4}+\frac{1}{4}=\frac{10}{4}=\frac{5}{2}$, matching part (a).

Q6. (a) With $a=1$, $b=-(3+2i)$, $c=6i$,

$$\Delta=b^2-4ac=(3+2i)^2-4(6i)=(9+12i+4i^2)-24i=(5+12i)-24i=5-12i.$$

(b) Seek $\sqrt{\Delta}=u+v\,i$ with $(u+v\,i)^2=5-12\,i$, i.e. $u^2-v^2=5$ and $2\,u\,v=-12$, so $u\,v=-6$. Taking $u=3$, $v=-2$ satisfies both ($9-4=5$ and $2(3)(-2)=-12$), so $\sqrt{\Delta}=\pm(3-2\,i)$. (c) The quadratic formula gives

$$z=\frac{(3+2\,i)\pm(3-2\,i)}{2}.$$

(d) The plus sign gives $z=\frac{6}{2}=3$; the minus sign gives $z=\frac{4\,i}{2}=2\,i$. So $z=3$ or $z=2\,i$ — not a conjugate pair, since the coefficients are not all real.

Q7. $z^2+25=0$ gives $z^2=-25$, so $z=\pm\sqrt{-25}=\pm5\,i$.

Q8. Completing the square, $z^2-2\,z+2=(z-1)^2-1+2=(z-1)^2+1$, so $(z-1)^2=-1$ and $z-1=\pm i$. Hence $z=1\pm i$. (Equivalently $\Delta=4-8=-4$.)

Q9. Completing the square, $z^2+6\,z+34=(z+3)^2-9+34=(z+3)^2+25$, so $(z+3)^2=-25$ and $z+3=\pm5\,i$. Hence $z=-3\pm5\,i$. (Here $\Delta=36-136=-100$.)

Q10. Completing the square, $z^2-10\,z+29=(z-5)^2-25+29=(z-5)^2+4$, so $(z-5)^2=-4$ and $z-5=\pm2\,i$. Hence $z=5\pm2\,i$. (Here $\Delta=100-116=-16$.)

Q11. With $a=4$, $b=4$, $c=5$, $\Delta=4^2-4(4)(5)=16-80=-64$, so $\sqrt{\Delta}=8\,i$. Then

$$z=\frac{-4\pm8\,i}{2(4)}=\frac{-4\pm8\,i}{8}=-\frac{1}{2}\pm i.$$

Q12. With $a=3$, $b=2$, $c=4$, $\Delta=2^2-4(3)(4)=4-48=-44$. Now $\sqrt{-44}=\sqrt{4\cdot 11}\,i=2\sqrt{11}\,i$, so

$$z=\frac{-2\pm2\sqrt{11}\,i}{2(3)}=\frac{-2\pm2\sqrt{11}\,i}{6}=-\frac{1}{3}\pm\frac{\sqrt{11}}{3}\,i.$$

Q13. Since the coefficients are real, the conjugate $\overline{1+4\,i}=1-4\,i$ is the other root. The sum is $(1+4\,i)+(1-4\,i)=2$ and the product is $(1+4\,i)(1-4\,i)=1^2+4^2=17$. A monic quadratic with these roots is $z^2-2\,z+17=0$.

Q14. The real coefficients force the conjugate $-3-i$ to be the second root. A monic quadratic with roots $-3\pm i$ has sum $(-3+i)+(-3-i)=-6$ and product $(-3+i)(-3-i)=9+1=10$, giving $z^2+6\,z+10$. Multiplying by the required leading coefficient 2,

$$2(z^2+6\,z+10)=2\,z^2+12\,z+20=0.$$

Q15. The coefficients are real, so the conjugate $\overline{2-5\,i}=2+5\,i$ is the other root. Comparing $z^2+b\,z+c=0$ with $z^2-(\alpha+\beta)z+\alpha\,\beta=0$: $b=-(\alpha+\beta)=-((2-5\,i)+(2+5\,i))=-4$, and $c=\alpha\,\beta=(2-5\,i)(2+5\,i)=2^2+5^2=29$. Hence $b=-4$ and $c=29$.

Q16. By Vieta's relations, $\alpha+\beta=-\frac{-3}{2}=\frac{3}{2}$ and $\alpha\,\beta=\frac{5}{2}$. To solve, $\Delta=(-3)^2-4(2)(5)=9-40=-31$, so $\sqrt{\Delta}=\sqrt{31}\,i$ and

$$z=\frac{3\pm\sqrt{31}\,i}{2(2)}=\frac{3\pm\sqrt{31}\,i}{4}=\frac{3}{4}\pm\frac{\sqrt{31}}{4}\,i.$$

Their sum is $\frac{3}{4}+\frac{3}{4}=\frac{3}{2}$ and their product is $\left(\frac{3}{4}\right)^2+\left(\frac{\sqrt{31}}{4}\right)^2=\frac{9}{16}+\frac{31}{16}=\frac{40}{16}=\frac{5}{2}$, confirming the values above.

Q17. With $a=1$, $b=-(4+2\,i)$, $c=2+4\,i$, the discriminant is

$$\Delta=(4+2i)^2-4(2+4i)=(16+16i+4i^2)-8-16i=(12+16i)-8-16i=4.$$

Here Δ is a positive real, so $\sqrt{\Delta}=2$ and

$$z=\frac{(4+2i)\pm 2}{2}.$$

The plus sign gives $z=\frac{6+2i}{2}=3+i$; the minus sign gives $z=\frac{2+2i}{2}=1+i$. So $z=3+i$ or $z=1+i$.

Q18. With $a=1$, $b=-(2+i)$, $c=-1+7i$, the discriminant is

$$\Delta=(2+i)^2-4(-1+7i)=(4+4i+i^2)+4-28i=(3+4i)+4-28i=7-24i.$$

Seek $\sqrt{\Delta}=u+vi$ with $u^2-v^2=7$ and $2uv=-24$, so $uv=-12$. Taking $u=4$, $v=-3$ works ($16-9=7$ and $2(4)(-3)=-24$), so $\sqrt{\Delta}=\pm(4-3i)$. Then

$$z=\frac{(2+i)\pm(4-3i)}{2}.$$

The plus sign gives $z=\frac{6-2i}{2}=3-i$; the minus sign gives $z=\frac{-2+4i}{2}=-1+2i$. So $z=3-i$ or $z=-1+2i$.

Theory

A **polynomial in** z of degree n is an expression

$$P(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0, a_n\neq 0,$$

whose coefficients $a_0, a_1, \ldots, a_n$ are numbers and n is a non-negative integer. A **zero** (or **root**) of P is a value $z=\alpha$ for which $P(\alpha)=0$; solving the equation $P(z)=0$ means finding all such α. Over the real numbers a polynomial need not factorise completely: z^2+1 has no real zero, because no real number squares to -1. An earlier section showed that *every* quadratic can nonetheless be solved once we allow complex numbers. This section extends that idea to polynomials of any degree.

Zeros and factors.

The **factor theorem** carries over from real to complex polynomials unchanged:

$$(z-\alpha)\text{ is a factor of } P(z) \Leftrightarrow P(\alpha)=0.$$

So a zero and a linear factor are two sides of the same coin — knowing one gives the other. If $P(z)=(z-\alpha)Q(z)$, then the remaining zeros of P are exactly the zeros of the quotient $Q(z)$, a polynomial of degree one lower. This "peel off a factor, then work on what is left" idea drives every method below.

The fundamental theorem of algebra.

The key fact that makes the complex numbers so powerful for algebra is the **fundamental theorem of algebra**:

> every polynomial of degree $n\geq 1$ with complex (in particular real) coefficients has at least one zero in $\mathbb{C}$.

Applying the factor theorem repeatedly then unlocks the whole polynomial: pull out one zero α_1 to write $P(z)=(z-\alpha_1)Q_1(z)$; the quotient Q_1 again has a zero α_2; and so on. After n steps,

$$P(z)=a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n).$$

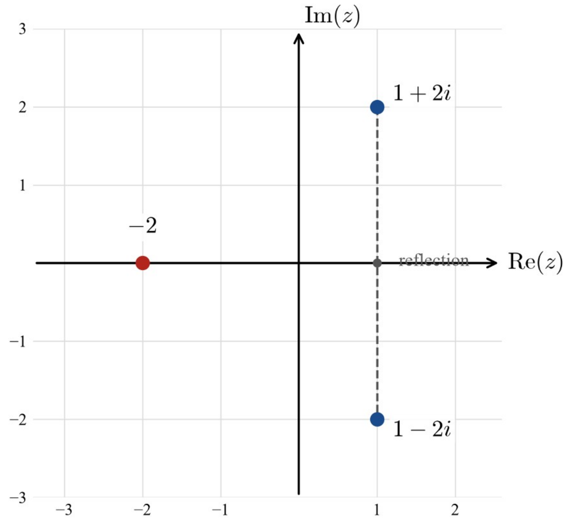
So a polynomial of degree n has **exactly n zeros in $\mathbb{C}$, counted with multiplicity**, and factorises completely into n linear factors over $\mathbb{C}$. The **multiplicity** of a zero is the number of times its factor $(z-\alpha)$ appears; a zero of multiplicity 2 is a *repeated* (double) zero and is counted twice in the tally of n. For instance a cubic always has three zeros over $\mathbb{C}$ and a quartic always has four, once repeats are counted.

The conjugate root theorem.

When the coefficients are **real** the non-real zeros are tightly linked. Recall that complex conjugation, $\overline{a+bi}=a-bi$, respects addition and multiplication: $\overline{w_1+w_2}=\overline{w_1}+\overline{w_2}$ and $\overline{w_1 w_2}=\overline{w_1}\,\overline{w_2}$, and it fixes every real number. Taking the conjugate of $P(z)=0$ term by term, with each real coefficient unchanged, gives $P(\overline{z})=\overline{P(z)}=\overline{0}=0$. This is the **conjugate root theorem**:

> if a polynomial has **real coefficients** and $\alpha=a+bi$ (with $b\neq 0$) is a zero, then its conjugate $\overline{\alpha}=a-bi$ is also a zero.

Geometrically the non-real zeros of a real polynomial occur in pairs placed symmetrically about the real axis. The diagram shows the three zeros of the real cubic z^3+z+10: the real zero -2 sits on the real axis, while the non-real zeros $1+2i$ and $1-2i$ are mirror images of each other in it.



Two consequences follow. First, because non-real zeros are paired, a real polynomial of **odd** degree must have at least one **real** zero. Second, multiplying a conjugate pair of factors clears the imaginary parts:

$$(z-\alpha)(z-\overline{\alpha})=z^2-(\alpha+\overline{\alpha})z+\alpha\,\overline{\alpha}=z^2-2\,\mathrm{Re}(\alpha)z+|\alpha|^2,$$

a **real quadratic** whose discriminant is negative, so it cannot be broken down further over the reals — it is **irreducible over** $\mathbb{R}$. Hence every polynomial with real coefficients factorises into **real linear factors and real irreducible quadratic factors**, and each such quadratic splits into a conjugate pair of linear factors only when we pass to $\mathbb{C}$.

A word of caution: the conjugate root theorem needs the coefficients to be *real*. A polynomial with non-real coefficients, such as $z-i$, can have a lone non-real zero whose conjugate is not a zero.

Solving a cubic or quartic given one non-real zero.

Suppose a polynomial with real coefficients is known to have the non-real zero α. Then $\overline{\alpha}$ is a zero too, and the pair supplies the real quadratic factor $z^2-2\,\mathrm{Re}(\alpha)z+|\alpha|^2$. Dividing $P(z)$ by this quadratic (or comparing coefficients) leaves a factor of degree two lower, which is then solved directly. For a cubic the leftover factor is linear, giving the one remaining real zero; for a quartic it is a quadratic, solved by factorising or by completing the square as in an earlier section.

Finding a real zero by inspection.

If no zero is given, a real cubic can often be started by spotting a real zero. Any *integer* zero of a polynomial with integer coefficients must divide the constant term, so testing the divisors of a_0 (both signs) is a quick search. Once a real zero α is found, divide out $(z-\alpha)$ and solve the remaining quadratic over $\mathbb{C}$. Equations of the special form $z^n=w$, whose solutions are the nth roots of a complex number, are handled with polar form and De Moivre's theorem in a later section.

Worked examples

Example 1 — scaffolded

The polynomial $P(z)$ has real coefficients, and $z=3+2i$ is one of its zeros. Write down a second zero of $P(z)$, and hence find a quadratic factor of $P(z)$ with real coefficients.

Working	Comment
P has real coefficients, so non-real zeros come in conjugate pairs.	Conditions for the conjugate root theorem are met.
Since $3+2i$ is a zero, its conjugate $3-2i$ is also a zero.	The conjugate of $a+bi$ is $a-bi$.

Working	Comment
Each zero α gives a factor $(z-\alpha)$, so $(z-(3+2i))$ and $(z-(3-2i))$ are factors.	Factor theorem in reverse.
Product $= z^2 - [(3+2i)+(3-2i)]z + (3+2i)(3-2i) = z^2 - 6z$ - .	Sum $= 2\,\mathrm{Re}(3+2i) = 6$; product $= \lvert 3+2i \rvert^2 = 9+4 = 13$.
$z^2 - 6z + 13$ is a real quadratic factor of $P(z)$.	Discriminant $36 - 52 = -16 < 0$: irreducible over $\mathbb{R}$.

Example 2 — scaffolded

Solve $z^3 - 7z^2 + 17z - 15 = 0$ over $\mathbb{C}$, given that $z = 2 + i$ is one solution.

Working	Comment
The coefficients are real, so the conjugate $2 - i$ is also a solution.	Conjugate root theorem.
The pair gives the real quadratic factor $(z-(2+i))(z-(2-i)) = z^2 - 4z + 5$.	Sum $= 4$; product $= \lvert 2+i \rvert^2 = 5$.
A cubic has three zeros, so $z^3 - 7z^2 + 17z - 15 = (z^2 - 4z + 5)(z - k)$ for some real k.	The leftover factor is linear.
Constant terms: $5 \times (-k) = -15$, so $k = 3$.	Check the z^2 term: $-4 - k = -7$.
The three solutions are $z = 2+i, 2-i, 3$.	Two non-real (a conjugate pair) and one real.

Example 3 — unscaffolded

Solve $z^3 - 5z^2 + 17z - 13 = 0$ over $\mathbb{C}$.

The coefficients are real integers, so any integer zero divides the constant term 13; the candidates are $\pm 1, \pm 13$. Testing $z = 1$ gives $1 - 5 + 17 - 13 = 0$, so $z = 1$ is a zero and $(z - 1)$ is a factor. Dividing,

$$z^3 - 5z^2 + 17z - 13 = (z-1)(z^2 - 4z + 13).$$

The quadratic factor has discriminant $16 - 52 = -36 < 0$, so it is solved over $\mathbb{C}$ by completing the square:

$$z^2 - 4z + 13 = (z-2)^2 + 9 = 0 \quad (z-2)^2 = -9 \quad z - 2 = \pm 3i.$$

$\therefore z = 1, 2+3i$ or $2-3i$; the non-real zeros form a conjugate pair, exactly as the conjugate root theorem requires.

Example 4 — unscaffolded

Given that $z = 1 + 2i$ is a zero of $P(z) = z^4 - 3z^3 + z^2 + 7z - 30$, find all zeros of $P(z)$ and factorise it fully over $\mathbb{C}$.

Because P has real coefficients, the conjugate $1 - 2i$ is also a zero. The pair supplies the real quadratic factor

$$(z-(1+2i))(z-(1-2i)) = z^2 - 2z + 5,$$

using sum $= 2$ and product $= \lvert 1+2i \rvert^2 = 5$. A quartic has four zeros, so the remaining factor is a quadratic; dividing gives

$$P(z) = (z^2 - 2z + 5)(z^2 - z - 6) = (z^2 - 2z + 5)(z-3)(z+2),$$

since $z^2 - z - 6 = (z-3)(z+2)$ has the real zeros $z = 3$ and $z = -2$.

$\therefore$ the zeros are $1+2i, 1-2i, 3, -2$, and $P(z) = (z-(1+2i))(z-(1-2i))(z-3)(z+2)$.

Practice questions

Scaffolded

Q1. The polynomial $P(z)$ has real coefficients and $z=4-i$ is a zero. (a) Write down a second zero of $P(z)$. (b) Find the sum and the product of this conjugate pair. (c) Hence write down a real quadratic factor of $P(z)$.

Q2. A polynomial with real coefficients has $z=-2+3i$ as a zero. (a) State the conjugate zero. (b) Find a monic quadratic factor with real coefficients.

Q3. Solve $z^3-8z^2+25z-26=0$ over $\mathbb{C}$, given that $z=3+2i$ is one solution. (a) Write down another solution. (b) Find the real quadratic factor coming from this pair. (c) Find the remaining linear factor. (d) State all three solutions.

Q4. Solve $z^3+z^2+4z+4=0$ over $\mathbb{C}$. (a) Show that $z=-1$ is a solution. (b) Factor out $(z+1)$ to find the quadratic factor. (c) Solve the quadratic factor over $\mathbb{C}$. (d) State all three solutions.

Q5. Solve $z^4-8z^3+24z^2-32z+15=0$ over $\mathbb{C}$, given that $z=2+i$ is one solution. (a) Write down another solution. (b) Find the real quadratic factor from this pair. (c) Divide to find the other quadratic factor. (d) Solve it and state all four solutions.

Q6. Find a monic polynomial of least degree, with real coefficients, that has $z=5$ and $z=1-i$ as zeros. (a) Explain why $z=1+i$ must also be a zero. (b) Find the real quadratic factor from the conjugate pair. (c) Multiply by the factor for $z=5$ and expand.

Unscaffolded

Q7. A polynomial with real coefficients has $z=2-5i$ as a zero. Write down another zero and a real quadratic factor.

Q8. Solve $z^3-2z^2+9z-18=0$ over $\mathbb{C}$.

Q9. Solve $z^3-3z^2+z+5=0$ over $\mathbb{C}$, given that $z=2+i$ is one solution.

Q10. Solve $z^4-5z^2-36=0$ over $\mathbb{C}$.

Q11. Solve $z^4+4=0$ over $\mathbb{C}$, and hence factorise z^4+4 into two real quadratic factors.

Q12. A polynomial with real coefficients has $z=1-\sqrt{3}i$ as a zero. Find a monic real quadratic factor.

Q13. Solve $z^3+8=0$ over $\mathbb{C}$.

Q14. The polynomial $P(z)=z^3+az^2+bz+10$ has real coefficients and $z=1-3i$ is a zero. Find the real numbers a and b, and the real zero of $P(z)$.

Q15. Solve $z^4-2z^3+6z^2-8z+8=0$ over $\mathbb{C}$, given that $z=2i$ is one solution.

Q16. Solve $z^4-6z^3+17z^2-28z+20=0$ over $\mathbb{C}$, given that $z=1+2i$ is a solution and that the equation has a repeated real root.

Q17. Let $P(z)=z^4+5z^2+4$. (a) Factorise $P(z)$ fully into linear factors over $\mathbb{C}$. (b) State the four zeros of $P(z)$.

Q18. $P(z)$ is a polynomial of degree 5 with real coefficients. Three of its zeros are $z=2$, $z=3-i$ and $z=4+2i$. (a) Write down the other two zeros. (b) Explain why $P(z)$ can have no further zeros. (c) Given that $P(z)$ is monic, write $P(z)$ as a product of one real linear factor and two real irreducible quadratic factors.

Answers

Q1. (a) $4+i$; (b) sum 8, product 17; (c) $z^2-8z+17$. **Q2.** (a) $-2-3i$; (b) $z^2+4z+13$. **Q3.** $z=3+2i, 3-2i, 2$; $z^3-8z^2+25z-26=(z^2-6z+13)(z-2)$. **Q4.** $z=-1, 2i, -2i$; $(z+1)(z^2+4)$. **Q5.** $z=2+i, 2-i, 1, 3$; $(z^2-4z+5)(z-1)(z-3)$. **Q6.** $P(z)=(z-5)(z^2-2z+2)=z^3-7z^2+12z-10$. **Q7.** $2+5i$; real quadratic factor $z^2-4z+29$. **Q8.** $z=2, 3i, -3i$. **Q9.** $z=2+i, 2-i, -1$. **Q10.** $z=3, -3, 2i, -2i$. **Q11.** $z=1+i, 1-i, -1+i, -1-i$; $z^4+4=(z^2-2z+2)(z^2+2z+2)$. **Q12.** z^2-2z+4. **Q13.** $z=-2, 1+\sqrt{3}i, 1-\sqrt{3}i$. **Q14.** $a=-1, b=8$; real zero $z=-1$. **Q15.** $z=2i, -2i, 1+i, 1-i$. **Q16.** $z=1+2i, 1-2i, 2$ (with $z=2$ a repeated root). **Q17.** (a) $(z-i)(z+i)(z-2i)(z+2i)$; (b) $z=i, -i, 2i, -2i$. **Q18.** (a) $3+i$ and $4-2i$; (b) a degree-5 polynomial has exactly five zeros counted with multiplicity, and these five are all distinct; (c) $P(z)=(z-2)(z^2-6z+10)(z^2-8z+20)$.

Fully-worked solutions

Q1. P has real coefficients, so by the conjugate root theorem the conjugate of $4-i$, namely $4+i$, is also a zero. Their sum is $(4-i)+(4+i)=8$ and their product is $(4-i)(4+i)=16+1=17$ (this is $|4-i|^2$). Hence a real quadratic factor is $z^2-8z+17$.

Q2. The conjugate zero is $-2-3i$. A monic real quadratic factor is

$$(z-(-2+3i))(z-(-2-3i))=z^2-2\,\mathrm{Re}(-2+3i)\,z+|-2+3i|^2=z^2+4z+13.$$

Q3. The coefficients are real, so $3-2i$ is also a solution. The pair gives the real quadratic factor $(z-(3+2i))(z-(3-2i))=z^2-6z+13$ (sum 6, product $9+4=13$). Writing $z^3-8z^2+25z-26=(z^2-6z+13)(z-k)$ and comparing constant terms, $13(-k)=-26$, so $k=2$ (check the z^2 term: $-6-k=-8$). The three solutions are $z=3+2i, 3-2i, 2$.

Q4. Substituting $z=-1$: $(-1)^3+(-1)^2+4(-1)+4=-1+1-4+4=0$, so $z=-1$ is a solution and $(z+1)$ is a factor. Dividing, $z^3+z^2+4z+4=(z+1)(z^2+4)$. Then $z^2+4=0$ gives $z^2=-4$, so $z=\pm 2i$. The three solutions are $z=-1, 2i, -2i$.

Q5. Real coefficients give the conjugate solution $2-i$, and the pair gives the factor $(z-(2+i))(z-(2-i))=z^2-4z+5$ (sum 4, product 5). Dividing the quartic,

$$z^4-8z^3+24z^2-32z+15=(z^2-4z+5)(z^2-4z+3).$$

The second factor is $z^2-4z+3=(z-1)(z-3)$, with real zeros $z=1$ and $z=3$. The four solutions are $z=2+i, 2-i, 1, 3$.

Q6. As P has real coefficients and $1-i$ is a zero, its conjugate $1+i$ must also be a zero. The conjugate pair gives the real quadratic factor $(z-(1-i))(z-(1+i))=z^2-2z+2$ (sum 2, product $1+1=2$). Multiplying by the factor $(z-5)$ for the real zero,

$$P(z)=(z-5)(z^2-2z+2)=z^3-2z^2+2z-5z^2+10z-10=z^3-7z^2+12z-10.$$

This monic cubic has least degree 3: the real zero forces one linear factor and the non-real zero forces a conjugate pair, hence two more.

Q7. By the conjugate root theorem $2+5i$ is also a zero. The pair gives

$$(z-(2-5i))(z-(2+5i))=z^2-4z+|2-5i|^2=z^2-4z+29,$$

since $|2-5i|^2=4+25=29$.

Q8. The coefficients are real integers, so any integer zero divides 18. Testing $z=2$: $8-8+18-18=0$, so $z=2$ is a zero and $(z-2)$ is a factor. Dividing, $z^3-2z^2+9z-18=(z-2)(z^2+9)$. Then $z^2+9=0$ gives $z^2=-9$, so $z=\pm 3i$. The solutions are $z=2, 3i, -3i$.

Q9. Real coefficients give the conjugate solution $2-i$, and the pair gives the factor $(z-(2+i))(z-(2-i))=z^2-4z+5$. Writing $z^3-3z^2+z+5=(z^2-4z+5)(z-k)$ and comparing constant terms, $5(-k)=5$, so $k=-1$ (check the z^2 term: $-4-k=-3$). The solutions are $z=2+i, 2-i, -1$.

Q10. Treat the equation as a quadratic in z^2. With $w=z^2$, $w^2-5w-36=0$, so $(w-9)(w+4)=0$ and $w=9$ or $w=-4$. Then $z^2=9$ gives $z=\pm 3$, and $z^2=-4$ gives $z=\pm 2i$. The four solutions are $z=3, -3, 2i, -2i$.

Q11. There is no real solution, since $z^4+4\ge 4>0$ for real z, so build a difference of two squares to factorise over $\mathbb{R}$ first:

$$z^4+4=z^4+4z^2+4-4z^2=(z^2+2)^2-(2z)^2=(z^2-2z+2)(z^2+2z+2).$$

Solving each real quadratic by completing the square, $z^2-2z+2=(z-1)^2+1=0$ gives $z=1\pm i$, and $z^2+2z+2=(z+1)^2+1=0$ gives $z=-1\pm i$. The four solutions are $z=1+i, 1-i, -1+i, -1-i$, and the real factorisation is $z^4+4=(z^2-2z+2)(z^2+2z+2)$.

Q12. The conjugate zero is $1+\sqrt{3}i$. The monic real quadratic factor is

$$(z-(1-\sqrt{3}i))(z-(1+\sqrt{3}i))=z^2-2z+|1-\sqrt{3}i|^2=z^2-2z+4,$$

since $|1-\sqrt{3}i|^2=1+3=4$.

Q13. Factor the sum of cubes: $z^3+8=(z+2)(z^2-2z+4)$. Then $z+2=0$ gives $z=-2$, and $z^2-2z+4=(z-1)^2+3=0$ gives $(z-1)^2=-3$, so $z-1=\pm\sqrt{3}i$. The solutions are $z=-2, 1+\sqrt{3}i, 1-\sqrt{3}i$.

Q14. Since P has real coefficients and $1-3i$ is a zero, the conjugate $1+3i$ is also a zero. The pair gives the real quadratic factor $(z-(1-3i))(z-(1+3i))=z^2-2z+10$ (sum 2, product $1+9=10$). As P is cubic and monic with constant term 10, write $P(z)=(z^2-2z+10)(z-k)$; comparing constant terms, $10(-k)=10$, so $k=-1$ and the real zero is $z=-1$. Expanding,

$$P(z)=(z^2-2z+10)(z+1)=z^3-z^2+8z+10,$$

so $a=-1$ and $b=8$.

Q15. Real coefficients and the zero $2i$ give the conjugate zero $-2i$, and the pair gives the factor $(z-2i)(z+2i)=z^2+4$. Dividing,

$$z^4-2z^3+6z^2-8z+8=(z^2+4)(z^2-2z+2).$$

The second factor $z^2-2z+2=(z-1)^2+1=0$ gives $z=1\pm i$. The four solutions are $z=2i, -2i, 1+i, 1-i$.

Q16. Real coefficients and the zero $1+2i$ give the conjugate zero $1-2i$, and the pair gives the factor $(z-(1+2i))(z-(1-2i))=z^2-2z+5$ (sum 2, product $1+4=5$). Dividing,

$$z^4-6z^3+17z^2-28z+20=(z^2-2z+5)(z^2-4z+4).$$

The second factor is $z^2-4z+4=(z-2)^2$, so $z=2$ is a repeated (double) root. The solutions are $z=1+2i, 1-2i$ and $z=2$ (twice); counted with multiplicity these four zeros match the degree, as the fundamental theorem of algebra requires.

Q17. (a) Treating P as a quadratic in z^2, $z^4+5z^2+4=(z^2+1)(z^2+4)$. Over $\mathbb{C}$ each factor splits: $z^2+1=(z-i)(z+i)$ and $z^2+4=(z-2i)(z+2i)$, so

$$P(z) = (z - i)(z + i)(z - 2i)(z + 2i).$$

(b) The four zeros are $z = i, -i, 2i, -2i$, all purely imaginary and forming two conjugate pairs.

Q18. (a) P has real coefficients, so the conjugates of the non-real zeros are also zeros: $3 + i$ and $4 - 2i$. (b) By the fundamental theorem of algebra a degree-5 polynomial has exactly five zeros counted with multiplicity. The five values $2, 3 - i, 3 + i, 4 - 2i, 4 + 2i$ are distinct and already number five, so there can be no further zeros. (c) The real zero gives the linear factor $(z - 2)$; the pair $3 \pm i$ gives $z^2 - 6z + 10$ (sum 6, product $9 + 1 = 10$); the pair $4 \pm 2i$ gives $z^2 - 8z + 20$ (sum 8, product $16 + 4 = 20$). Hence, since P is monic,

$$P(z) = (z - 2)(z^2 - 6z + 10)(z^2 - 8z + 20).$$

Theory

An earlier section showed that multiplying two complex numbers in polar form **multiplies the moduli and adds the arguments**:

$$r_1\,cis\,\theta_1 \times r_2\,cis\,\theta_2 = r_1 r_2\,cis(\theta_1+\theta_2).$$

Applying this repeatedly to a single number $z = r\,cis\,\theta$ gives a compact rule for integer powers, and reversing that rule leads to a method for extracting roots.

De Moivre's theorem.

For every positive integer n, and also for $n=0$ and every negative integer n provided $z \neq 0$,

$$(r\,cis\,\theta)^n = r^n\,cis(n\theta),$$

that is, $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$. In words: to raise a complex number in polar form to an integer power, **raise the modulus to that power and multiply the argument by it**. Here i is the imaginary unit and $r = |z| \geq 0$.

Proof for positive integers (mathematical induction).

Let $P(n)$ be the statement $(r\,cis\,\theta)^n = r^n\,cis(n\theta)$.

Base case. $P(1)$ reads $(r\,cis\,\theta)^1 = r^1\,cis(1\cdot\theta)$, which is true.

Inductive step. Assume $P(k)$ holds for some positive integer k, so that $(r\,cis\,\theta)^k = r^k\,cis(k\theta)$. Then

$$(r\,cis\,\theta)^{k+1} = (r\,cis\,\theta)^k \times (r\,cis\,\theta) = r^k\,cis(k\theta) \times r\,cis\,\theta.$$

Using the multiplication rule (moduli multiply, arguments add),

$$= r^k \cdot r\,cis(k\theta+\theta) = r^{k+1}\,cis((k+1)\theta),$$

which is $P(k+1)$. Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$, by the principle of mathematical induction $P(n)$ holds for **all positive integers** n.

Zero and negative integers.

Both of these cases need $z \neq 0$, that is $r > 0$, since z^0 and negative powers of z are defined only for a non-zero base. For $n=0$, $z^0 = 1 = r^0\,cis\,0$. For a negative integer $n = -m$ with $m > 0$, using the division rule $\dfrac{1}{r\,cis\,\theta} = \dfrac{1}{r}\,cis(-\theta)$,

$$(r\,cis\,\theta)^{-m} = \frac{1}{(r\,cis\,\theta)^m} = \frac{1}{r^m\,cis(m\theta)} = r^{-m}\,cis(-m\theta).$$

So De Moivre's theorem holds for **every** integer n — positive, zero or negative — with $z \neq 0$ required whenever $n \leq 0$.

Computing an integer power.

To evaluate z^n when z is given as $a+bi$: write $z = r\,cis\,\theta$, apply De Moivre to obtain $r^n\,cis(n\theta)$, reduce $n\theta$ to a principal value by adding or subtracting whole turns of 2π, and convert back to Cartesian form if an $a+bi$ answer is required. A high power that would be very tedious to expand by the binomial theorem collapses to a single line.

The n-th roots of a complex number.

An n**-th root** of a complex number w is a solution z of $z^n = w$. Take $w \neq 0$ (the excluded case is trivial: $z^n = 0$ forces $z = 0$, so 0 has the single n-th root 0) and write $w = r\,cis\,\theta$ with $r > 0$; seek $z = s\,cis\,\phi$. By De Moivre, $z^n = s^n cis(n\phi)$, so $z^n = w$ requires

$$s^n = r \text{ and } n\phi = \theta + 2k\pi\,(k \text{ an integer}).$$

The first equation gives $s = r^{1/n}$, the positive real n-th root of the modulus. The second gives $\phi = \dfrac{\theta + 2k\pi}{n}$. Taking $k = 0, 1, 2, \ldots, n-1$ yields n different arguments within one revolution; any other integer k simply repeats one of these (its argument differs by a whole number of turns). Hence a non-zero complex number has exactly n **distinct** n**-th roots**,

$$z_k = r^{1/n} cis\left(\frac{\theta + 2k\pi}{n}\right), k = 0, 1, 2, \ldots, n-1.$$

Geometry of the roots.

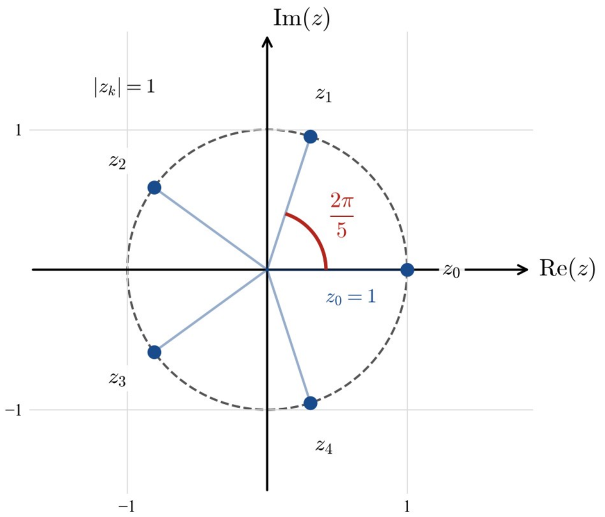
Every root has the same modulus $r^{1/n}$, so all n roots lie on a **circle of radius** $r^{1/n}$ centred at the origin. Consecutive arguments differ by $\dfrac{2\pi}{n}$, so the roots are **equally spaced** around that circle: they are the vertices of a regular n-sided polygon. In practice, find one root, then step the argument on by $\dfrac{2\pi}{n}$ repeatedly to generate the rest.

Roots of unity.

The n-th roots of 1 are the n**-th roots of unity**. Since $1 = cis\,0$ has $r = 1$ and $\theta = 0$,

$$z_k = cis\left(\frac{2k\pi}{n}\right), k = 0, 1, \ldots, n-1.$$

They lie on the **unit circle** (radius $r^{1/n} = 1$), equally spaced by $\dfrac{2\pi}{n}$, and one root is always $z_0 = 1$. The diagram below shows the five fifth roots of unity — the vertices of a regular pentagon, each neighbouring pair subtending $\dfrac{2\pi}{5}$ at the origin.



For $n \geq 2$ the roots of unity are placed symmetrically about the origin and **sum to zero**. Once the roots of unity are known, the n-th roots of any other number $w = r\,cis\,\theta$ are obtained by scaling the circle to radius $r^{1/n}$ and rotating so the first root sits at argument $\dfrac{\theta}{n}$.

Worked examples

Example 1 — scaffolded

Use De Moivre's theorem to evaluate $(1+i)^{10}$, giving the answer in Cartesian form.

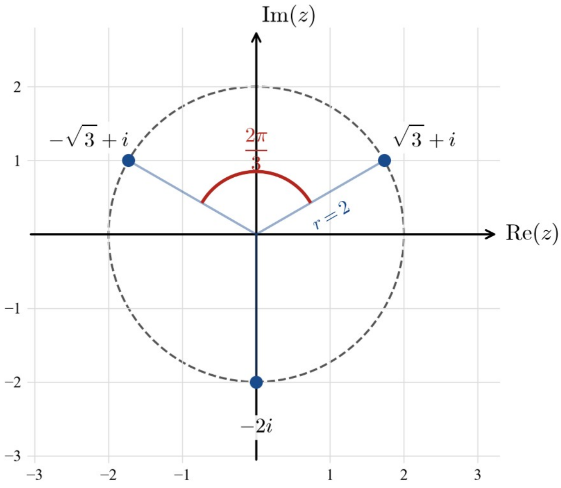
Working	Comment
$r=\sqrt{1^2+1^2}=\sqrt{2}$; $a=b=1>0$, so z is in quadrant 1 and $Arg(z)=\frac{\pi}{4}$. Thus $1+i=\sqrt{2}\,cis\,\frac{\pi}{4}$.	First write the base in polar form.
$(1+i)^{10}=(\sqrt{2})^{10}\,cis\left(10\times\frac{\pi}{4}\right)$.	Apply De Moivre: modulus to the power, argument times the power.
$(\sqrt{2})^{10}=(2^{1/2})^{10}=2^5=32$ and $10\times\frac{\pi}{4}=\frac{5\pi}{2}$.	Simplify the modulus and the new argument.
$\frac{5\pi}{2}-2\pi=\frac{\pi}{2}$, so $(1+i)^{10}=32\,cis\,\frac{\pi}{2}$.	Subtract one full turn to reach a principal argument.
$32\,cis\,\frac{\pi}{2}=32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=32(0+i)=32i$.	Convert back to Cartesian form.

Example 2 — scaffolded

Find the three cube roots of $8i$, giving each in Cartesian form, and show them on an Argand diagram.

Working	Comment
$8i=8\,cis\,\frac{\pi}{2}$, so $r=8$, $\theta=\frac{\pi}{2}$ and $n=3$.	Write w in polar form.
$r^{1/n}=8^{1/3}=2$, so $z_k=2\,cis\left(\frac{\pi/2+2k\pi}{3}\right)$, $k=0,1,2$.	Each root has modulus 2; the arguments step by $\frac{2\pi}{3}$.
$k=0$: $z_0=2\,cis\,\frac{\pi}{6}=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=\sqrt{3}+i$.	$\frac{\pi/2}{3}=\frac{\pi}{6}$.
$k=1$: $z_1=2\,cis\,\frac{5\pi}{6}=2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-\sqrt{3}+i$.	$\frac{\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6}$.
$k=2$: $z_2=2\,cis\left(-\frac{\pi}{2}\right)=2(0-i)=-2i$.	$\frac{5\pi}{6}+\frac{2\pi}{3}=\frac{3\pi}{2}$, i.e. $-\frac{\pi}{2}$.

The three roots have modulus 2 and arguments $\frac{\pi}{6},\frac{5\pi}{6},-\frac{\pi}{2}$, equally spaced by $\frac{2\pi}{3}$ on the circle of radius 2.



Example 3 — unscaffolded

Use De Moivre's theorem to evaluate $(\sqrt{3}-i)^7$, giving the answer in Cartesian form.

The base has modulus $r=\sqrt{(\sqrt{3})^2+(-1)^2}=\sqrt{3+1}=2$. With $a=\sqrt{3}>0$ and $b=-1<0$ it lies in the fourth quadrant, reference angle $\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$, so $Arg=-\frac{\pi}{6}$ and $\sqrt{3}-i=2\,cis\left(-\frac{\pi}{6}\right)$. By De Moivre's theorem,

$$(\sqrt{3}-i)^7=2^7\,cis\left(7\times\left(-\frac{\pi}{6}\right)\right)=128\,cis\left(-\frac{7\pi}{6}\right).$$

Adding one full turn, $-\frac{7\pi}{6}+2\pi=\frac{5\pi}{6}$, so

$$(\sqrt{3}-i)^7=128\,cis\frac{5\pi}{6}=128\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-64\sqrt{3}+64i.$$

$\therefore(\sqrt{3}-i)^7=-64\sqrt{3}+64i.$

Example 4 — unscaffolded

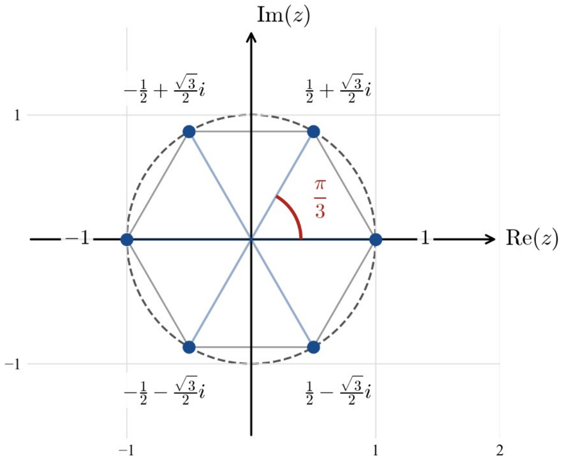
Solve $z^6=1$ over the complex numbers, giving the roots in Cartesian form, and show them on an Argand diagram.

The solutions are the sixth roots of unity. Writing $1=cis\,0$, the roots are $z_k=cis\frac{2k\pi}{6}=cis\frac{k\pi}{3}$ for $k=0,1,\ldots,5$, all of modulus 1 and spaced by $\frac{2\pi}{6}=\frac{\pi}{3}$:

$$z_0=cis\,0=1,\ z_1=cis\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ z_2=cis\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,$$

$$z_3=cis\,\pi=-1,\ z_4=cis\left(-\frac{2\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ z_5=cis\left(-\frac{\pi}{3}\right)=\frac{1}{2}-\frac{\sqrt{3}}{2}i.$$

(For $k=3,4,5$ the arguments $\pi,\frac{4\pi}{3},\frac{5\pi}{3}$ have been quoted as the principal values $\pi,-\frac{2\pi}{3},-\frac{\pi}{3}$.) The six roots are the vertices of a regular hexagon on the unit circle.



$\therefore$ the six solutions are $1, -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Practice questions

Scaffolded

Q1. Use De Moivre's theorem to evaluate $(1+\sqrt{3}i)^6$. (a) Write $1+\sqrt{3}i$ in polar form. (b) Apply De Moivre's theorem to find the modulus and argument of the power. (c) State $(1+\sqrt{3}i)^6$ in Cartesian form.

Q2. Use De Moivre's theorem to evaluate $(-1+i)^8$. (a) Write $-1+i$ in polar form. (b) Apply De Moivre's theorem. (c) State the answer in Cartesian form.

Q3. Use De Moivre's theorem to evaluate $(1-i)^6$. (a) Write $1-i$ in polar form. (b) Apply De Moivre's theorem, then reduce the argument to a principal value. (c) State the answer in Cartesian form.

Q4. Find the two square roots of $2i$. (a) Write $2i$ in polar form. (b) Write $z_k = 2^{1/2} cis\left(\frac{\pi/2 + 2k\pi}{2}\right)$ for $k = 0, 1$. (c) Evaluate each root in Cartesian form.

Q5. Find the three cube roots of -27. (a) Write -27 in polar form. (b) Write the roots $z_k = 3\,cis\left(\frac{\pi + 2k\pi}{3}\right)$, $k = 0, 1, 2$. (c) Evaluate each root in Cartesian form.

Q6. Find the fourth roots of unity by solving $z^4 = 1$. (a) Write 1 in polar form and state the modulus of each root. (b) Write $z_k = cis\left(\frac{2k\pi}{4}\right)$, $k = 0, 1, 2, 3$. (c) State the four roots in Cartesian form.

Unscaffolded

Q7. Evaluate $(\sqrt{3}+i)^5$, giving the answer in Cartesian form.

Q8. Evaluate $(1+i)^{12}$.

Q9. Evaluate $(-\sqrt{3}+i)^6$.

Q10. Evaluate $(1+i)^{-6}$, giving the answer in Cartesian form.

Q11. Evaluate $(2-2i)^5$, giving the answer in Cartesian form.

Q12. Evaluate $(\sqrt{3}-i)^8$, giving the answer in Cartesian form.

Q13. Find the two square roots of -9.

Q14. Find the three cube roots of $27i$, giving each in Cartesian form.

Q15. Find the four fourth roots of -16, giving each in Cartesian form.

Q16. Solve $z^3=1$ over the complex numbers, giving the cube roots of unity in Cartesian form, and show that they sum to zero.

Q17. Solve $z^5=32$ over the complex numbers, leaving each root in polar form.

Q18. Let $z=cis\,\theta=\cos\theta+i\sin\theta$. (a) Expand $(\cos\theta+i\sin\theta)^3$ using the binomial theorem (with $i^2=-1$), collecting real and imaginary parts. (b) De Moivre's theorem gives $(\cos\theta+i\sin\theta)^3=\cos3\theta+i\sin3\theta$. By equating real and imaginary parts, deduce expressions for $\cos3\theta$ and $\sin3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Answers

Q1. $(1+\sqrt{3}i)^6=64$. **Q2.** $(-1+i)^8=16$. **Q3.** $(1-i)^6=8i$. **Q4.** $1+i$ and $-1-i$. **Q5.** $-3, \frac{3}{2}+\frac{3\sqrt{3}}{2}i, \frac{3}{2}-\frac{3\sqrt{3}}{2}i$. **Q6.** $1, i, -1, -i$. **Q7.** $-16\sqrt{3}+16i$. **Q8.** -64. **Q9.** -64. **Q10.** $\frac{1}{8}i$. **Q11.** $-128+128i$. **Q12.** $-128+128\sqrt{3}i$. **Q13.** $3i$ and $-3i$. **Q14.** $\frac{3\sqrt{3}}{2}+\frac{3}{2}i, -\frac{3\sqrt{3}}{2}+\frac{3}{2}i, -3i$. **Q15.** $\sqrt{2}+\sqrt{2}i, -\sqrt{2}+\sqrt{2}i, -\sqrt{2}-\sqrt{2}i, \sqrt{2}-\sqrt{2}i$. **Q16.** $1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$; their sum is 0. **Q17.** $2\,cis\,0, 2\,cis\left(\pm\frac{2\pi}{5}\right), 2\,cis\left(\pm\frac{4\pi}{5}\right)$. **Q18.** $\cos 3\theta=4\cos^3\theta-3\cos\theta$; $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

Fully-worked solutions

Q1. $1+\sqrt{3}i$ has $r=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{4}=2$ and, being in quadrant 1 with $\tan\alpha=\sqrt{3}$, argument $\frac{\pi}{3}$; so $1+\sqrt{3}i=2\,cis\frac{\pi}{3}$. By De Moivre,

$$(1+\sqrt{3}i)^6=2^6\,cis\left(6\times\frac{\pi}{3}\right)=64\,cis(2\pi)=64\,cis\,0=64.$$

Q2. $-1+i$ has $r=\sqrt{(-1)^2+1^2}=\sqrt{2}$ and, in quadrant 2 with reference angle $\frac{\pi}{4}$, argument $\frac{3\pi}{4}$; so $-1+i=\sqrt{2}\,cis\frac{3\pi}{4}$. Then

$$(-1+i)^8=(\sqrt{2})^8\,cis\left(8\times\frac{3\pi}{4}\right)=2^4\,cis(6\pi)=16\,cis\,0=16.$$

Q3. $1-i$ has $r=\sqrt{1^2+(-1)^2}=\sqrt{2}$ and, in quadrant 4 with reference angle $\frac{\pi}{4}$, argument $-\frac{\pi}{4}$; so $1-i=\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$. Then

$$(1-i)^6=(\sqrt{2})^6\,cis\left(6\times\left(-\frac{\pi}{4}\right)\right)=8\,cis\left(-\frac{3\pi}{2}\right).$$

Adding one full turn, $-\frac{3\pi}{2}+2\pi=\frac{\pi}{2}$, so $(1-i)^6=8\,cis\frac{\pi}{2}=8i$.

Q4. $2i=2\,cis\frac{\pi}{2}$, so the square roots have modulus $2^{1/2}=\sqrt{2}$ and arguments $\frac{\pi/2+2k\pi}{2}$. For $k=0$: $z_0=\sqrt{2}\,cis\frac{\pi}{4}=\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=1+i$. For $k=1$: the argument is $\frac{\pi/2+2\pi}{2}=\frac{5\pi}{4}$, i.e. $-\frac{3\pi}{4}$, so $z_1=\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right)=\sqrt{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)=-1-i$. The two square roots are $1+i$ and $-1-i$ (opposite ends of a diameter).

Q5. $-27=27\,cis\,\pi$, so the cube roots have modulus $27^{1/3}=3$ and arguments $\frac{\pi+2k\pi}{3}$. For $k=0$: $z_0=3\,cis\frac{\pi}{3}=3\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=\frac{3}{2}+\frac{3\sqrt{3}}{2}i$. For $k=1$: $z_1=3\,cis\,\pi=-3$. For $k=2$: the argument is $\frac{\pi+4\pi}{3}=\frac{5\pi}{3}$, i.e. $-\frac{\pi}{3}$, so $z_2=3\,cis\left(-\frac{\pi}{3}\right)=\frac{3}{2}-\frac{3\sqrt{3}}{2}i$.

Q6. $1 = cis\,0$, so each fourth root has modulus $1^{1/4} = 1$ and argument $\dfrac{2k\pi}{4} = \dfrac{k\pi}{2}$. For $k = 0, 1, 2, 3$ these are $cis\,0 = 1$, $cis\,\dfrac{\pi}{2} = i$, $cis\,\pi = -1$ and $cis\left(-\dfrac{\pi}{2}\right) = -i$ (taking $\dfrac{3\pi}{2}$ as the principal value $-\dfrac{\pi}{2}$). Hence the fourth roots of unity are $1, i, -1, -i$.

Q7. $\sqrt{3} + i = 2\,cis\,\dfrac{\pi}{6}$ (quadrant 1, $r = 2$). Then

$$(\sqrt{3}+i)^5 = 2^5\,cis\left(5 \times \frac{\pi}{6}\right) = 32\,cis\,\frac{5\pi}{6} = 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -16\sqrt{3} + 16i\,.$$

Q8. $1 + i = \sqrt{2}\,cis\,\dfrac{\pi}{4}$. Then

$$(1+i)^{12} = (\sqrt{2})^{12}\,cis\left(12 \times \frac{\pi}{4}\right) = 2^6\,cis(3\pi) = 64\,cis\,\pi = -64\,,$$

since 3π and π differ by a full turn.

Q9. $-\sqrt{3} + i = 2\,cis\,\dfrac{5\pi}{6}$ (quadrant 2, $r = 2$). Then

$$(-\sqrt{3}+i)^6 = 2^6\,cis\left(6 \times \frac{5\pi}{6}\right) = 64\,cis(5\pi) = 64\,cis\,\pi = -64\,,$$

reducing 5π by two full turns to π.

Q10. $1 + i = \sqrt{2}\,cis\,\dfrac{\pi}{4}$, and De Moivre applies to the negative power:

$$(1+i)^{-6} = (\sqrt{2})^{-6}\,cis\left(-6 \times \frac{\pi}{4}\right) = \frac{1}{8}\,cis\left(-\frac{3\pi}{2}\right).$$

Since $-\dfrac{3\pi}{2} + 2\pi = \dfrac{\pi}{2}$, this is $\dfrac{1}{8}\,cis\,\dfrac{\pi}{2} = \dfrac{1}{8}i$.

Q11. $2 - 2i$ has $r = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$ and argument $-\dfrac{\pi}{4}$ (quadrant 4), so $2 - 2i = 2\sqrt{2}\,cis\left(-\dfrac{\pi}{4}\right)$. Then, with $2\sqrt{2} = 2^{3/2}$ giving $(2\sqrt{2})^5 = 2^{15/2} = 128\sqrt{2}$,

$$(2-2i)^5 = (2\sqrt{2})^5\,cis\left(5 \times \left(-\frac{\pi}{4}\right)\right) = 128\sqrt{2}\,cis\left(-\frac{5\pi}{4}\right).$$

Adding one full turn, $-\dfrac{5\pi}{4} + 2\pi = \dfrac{3\pi}{4}$, so

$$(2-2i)^5 = 128\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 128(-1+i) = -128 + 128i\,.$$

Q12. $\sqrt{3} - i = 2\,cis\left(-\dfrac{\pi}{6}\right)$ (quadrant 4, $r = 2$). Then

$$(\sqrt{3}-i)^8 = 2^8\,cis\left(8 \times \left(-\frac{\pi}{6}\right)\right) = 256\,cis\left(-\frac{4\pi}{3}\right).$$

Adding one full turn, $-\frac{4\pi}{3}+2\pi=\frac{2\pi}{3}$, so

$$(\sqrt{3}-i)^8=256\,cis\frac{2\pi}{3}=256\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-128+128\sqrt{3}i.$$

Q13. $-9=9\,cis\,\pi$, so the square roots have modulus $9^{1/2}=3$ and arguments $\frac{\pi+2k\pi}{2}$. For $k=0$: $z_0=3\,cis\frac{\pi}{2}=3i$. For $k=1$: the argument is $\frac{3\pi}{2}$, i.e. $-\frac{\pi}{2}$, so $z_1=3\,cis\left(-\frac{\pi}{2}\right)=-3i$. The square roots of -9 are $\pm3i$.

Q14. $27i=27\,cis\frac{\pi}{2}$, so the cube roots have modulus $27^{1/3}=3$ and arguments $\frac{\pi/2+2k\pi}{3}$. For $k=0$: $z_0=3\,cis\frac{\pi}{6}=3\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=\frac{3\sqrt{3}}{2}+\frac{3}{2}i$. For $k=1$: $z_1=3\,cis\frac{5\pi}{6}=3\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-\frac{3\sqrt{3}}{2}+\frac{3}{2}i$. For $k=2$: the argument is $\frac{\pi/2+4\pi}{3}=\frac{3\pi}{2}$, i.e. $-\frac{\pi}{2}$, so $z_2=3\,cis\left(-\frac{\pi}{2}\right)=-3i$.

Q15. $-16=16\,cis\,\pi$, so the fourth roots have modulus $16^{1/4}=2$ and arguments $\frac{\pi+2k\pi}{4}$. For $k=0$: $z_0=2\,cis\frac{\pi}{4}=\sqrt{2}+\sqrt{2}i$. For $k=1$: $z_1=2\,cis\frac{3\pi}{4}=-\sqrt{2}+\sqrt{2}i$. For $k=2$: argument $\frac{5\pi}{4}$, i.e. $-\frac{3\pi}{4}$, so $z_2=2\,cis\left(-\frac{3\pi}{4}\right)=-\sqrt{2}-\sqrt{2}i$. For $k=3$: argument $\frac{7\pi}{4}$, i.e. $-\frac{\pi}{4}$, so $z_3=2\,cis\left(-\frac{\pi}{4}\right)=\sqrt{2}-\sqrt{2}i$. The four roots are the vertices of a square on the circle of radius 2.

Q16. $1=cis\,0$, so the cube roots of unity have modulus 1 and arguments $\frac{2k\pi}{3}$. For $k=0,1,2$:

$$z_0=cis\,0=1,\ z_1=cis\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ z_2=cis\left(-\frac{2\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i.$$

Their sum is

$$1+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=1-\frac{1}{2}-\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{\sqrt{3}}{2}i=0,$$

so the three cube roots of unity sum to zero.

Q17. $32=32\,cis\,0$, so each fifth root has modulus $32^{1/5}=2$ and argument $\frac{2k\pi}{5}$, $k=0,1,2,3,4$. Taking principal values, the roots are

$$2\,cis\,0,\ 2\,cis\frac{2\pi}{5},\ 2\,cis\frac{4\pi}{5},\ 2\,cis\left(-\frac{4\pi}{5}\right),\ 2\,cis\left(-\frac{2\pi}{5}\right),$$

i.e. $2\,cis\,0$ together with the conjugate pairs $2\,cis\left(\pm\frac{2\pi}{5}\right)$ and $2\,cis\left(\pm\frac{4\pi}{5}\right)$. (These angles are not standard, so the polar form is left as the exact answer.)

Q18. (a) Expanding by the binomial theorem, with $i^2=-1$, $i^3=-i$,

$$(\cos\theta+i\sin\theta)^3=\cos^3\theta+3\cos^2\theta(i\sin\theta)+3\cos\theta(i\sin\theta)^2+(i\sin\theta)^3$$

$$=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta=(\cos^3\theta-3\cos\theta\sin^2\theta)+i(3\cos^2\theta\sin\theta-\sin^3\theta).$$

(b) De Moivre gives $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$. Equating real parts and using $\sin^2\theta = 1 - \cos^2\theta$,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos\theta.$$

Equating imaginary parts and using $\cos^2\theta = 1 - \sin^2\theta$,

$$\sin 3\theta = 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Theory

An earlier section plotted a single complex number $z = x + yi$ as the point (x, y) on an **Argand diagram**, whose horizontal axis is the real axis and whose vertical axis is the imaginary axis. This section studies whole **sets** of complex numbers. A **complex relation** is an equation or inequality that z must satisfy; the set of every point that satisfies it is a subset of the Argand plane. An **equation** (using =) usually describes a *curve* — a line or a circle, called a **locus** — while an **inequality** (using $<, \leq, >, \geq$) usually describes a *region* with area.

The standard method is always the same: write $z = x + yi$, translate the relation into an ordinary Cartesian condition on x and y, and sketch the resulting curve or region. Two drawing conventions are used throughout. A boundary that **is** part of the set (from =, ≤ or ≥) is drawn as a **solid** line or curve; a boundary that is **excluded** (from < or >, or an undefined point) is drawn **dashed**. The region belonging to the set is **shaded**.

Circles and discs.

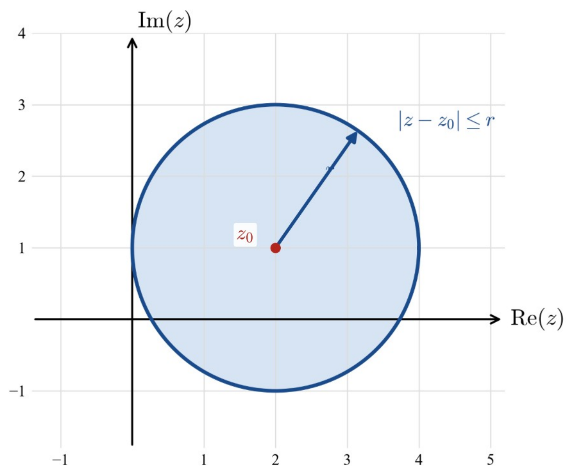
The modulus $|z - z_0|$ is the **distance** between the points z and z_0. Hence, for a fixed complex number z_0 and a positive real number r,

$$|z - z_0| = r$$

is the set of all points a fixed distance r from z_0 — a **circle**, centre z_0, radius r. Writing $z = x + yi$ and $z_0 = a + bi$, the distance is $|(x - a) + (y - b)i|$, so squaring both sides gives the Cartesian equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

The related inequalities describe **regions**: $|z - z_0| \leq r$ is the **closed disc** (interior together with its boundary circle), $|z - z_0| < r$ is the **open disc** (boundary dashed and excluded), and $|z - z_0| \geq r$ or $> r$ is the **exterior** of the circle.

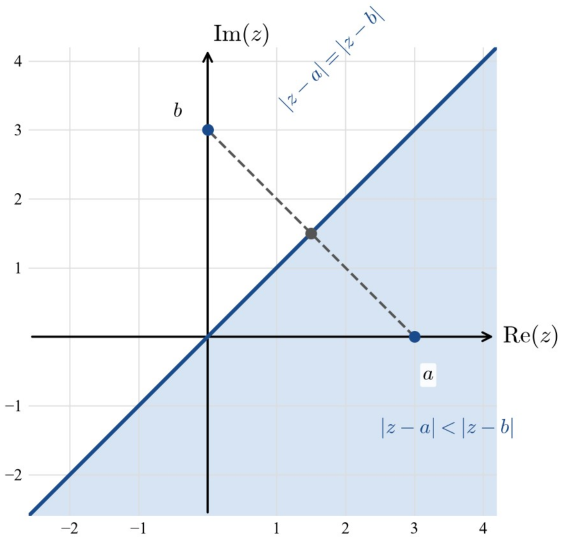


Perpendicular bisectors and half-planes.

For two fixed distinct points a and b, the relation

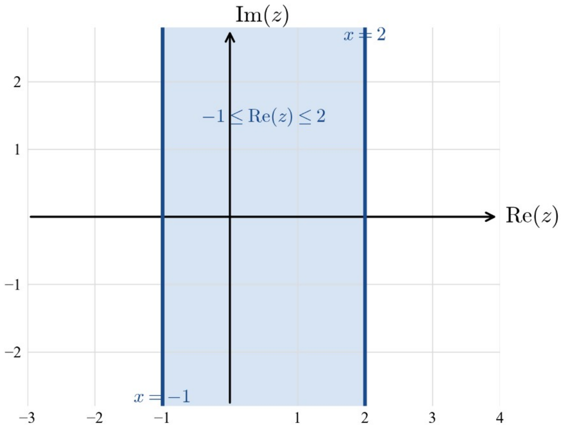
$$|z - a| = |z - b|$$

says that z is **equidistant** from a and b. The set of all such points is the **perpendicular bisector** of the line segment joining a and b. To find its equation, put $z = x + yi$ and square both sides: because $|z - a|^2 = |z - b|^2$ expands with matching x^2 and y^2 terms on each side, these cancel and a **straight line** remains. The inequalities give **half-planes**: $|z - a| < |z - b|$ is the half-plane of points **closer to** a (the side of the bisector containing a), and $|z - a| > |z - b|$ is the other half-plane, closer to b.



Vertical and horizontal lines, and strips.

Since $\mathrm{Re}(z)=x$ and $\mathrm{Im}(z)=y$, conditions on the real and imaginary parts are simply conditions on x and y. So $\mathrm{Re}(z)=k$ is the **vertical line** $x=k$, and $\mathrm{Im}(z)=k$ is the **horizontal line** $y=k$. The corresponding inequalities are half-planes: $\mathrm{Re}(z)\geq k$ is everything on or to the right of $x=k$; $\mathrm{Im}(z)<k$ is everything below $y=k$. Combining two such conditions gives a **strip**: for example $c\leq\mathrm{Re}(z)\leq d$ is the vertical strip between the lines $x=c$ and $x=d$, and pairing a real-part strip with an imaginary-part strip produces a **rectangle**.

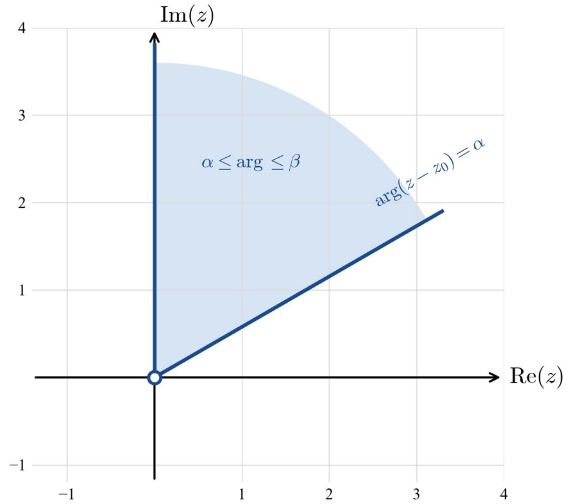


Rays and sectors.

The quantity $arg(z-z_0)$ is the angle that the vector **from** z_0 **to** z makes with the positive real direction. Therefore

$$arg(z-z_0)=\alpha$$

is a **ray** (a half-line) that begins at z_0 and points in the direction α. The starting point z_0 is **not** included, because $arg(0)$ is undefined, so its endpoint is drawn as an open dot. Every point on the ray has the form $z=z_0+t(\cos\alpha+i\sin\alpha)$ for a real $t>0$, so in Cartesian terms, with $z_0=a+bi$, $x=a+t\cos\alpha$ and $y=b+t\sin\alpha$. When $\cos\alpha\neq 0$ this says the ray lies on the line through (a,b) of gradient $\tan\alpha$, but only the **half** of that line on the side matching the direction α — the part with $x>a$ if $\cos\alpha>0$, and the part with $x<a$ if $\cos\alpha<0$. When $\alpha=\pm\frac{\pi}{2}$ the direction is vertical, $\tan\alpha$ is undefined and the ray has no gradient: it is instead the **vertical half-line** $x=a$, taken upwards ($y>b$) for $\alpha=\frac{\pi}{2}$ and downwards ($y<b$) for $\alpha=-\frac{\pi}{2}$. An inequality such as $\alpha\leq arg(z-z_0)\leq\beta$ describes a **sector**: the infinite wedge between the two rays at angles α and β, with the vertex z_0 itself excluded.



Intersections of regions.

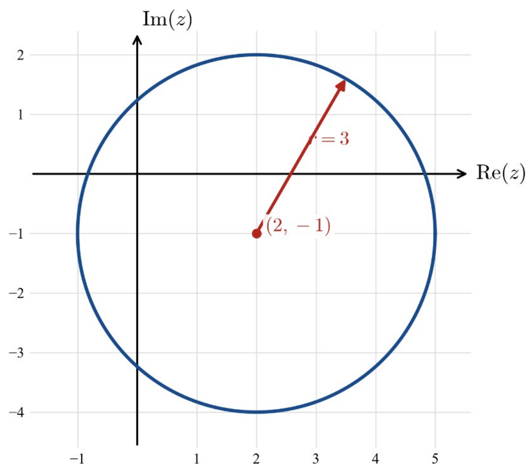
The word **and** — equivalently the set intersection ∩ — means a point must satisfy **every** relation at once. To sketch such a compound set, draw each region separately and keep only the **overlap**. This is how richer shapes are built: a half-disc (a disc intersected with a half-plane through its centre), a **circular segment** (a disc cut by a line), an **annulus** or ring ($r_1 \le |z| \le r_2$, the region between two concentric circles), a rectangle, or a **lens** (the overlap of two discs). The **corner points** of a compound region are found by solving the equations of the boundary curves simultaneously.

Worked examples

Example 1 — scaffolded

Sketch the set $\{z : |z-(2-i)|=3\}$ on an Argand diagram and give its Cartesian equation.

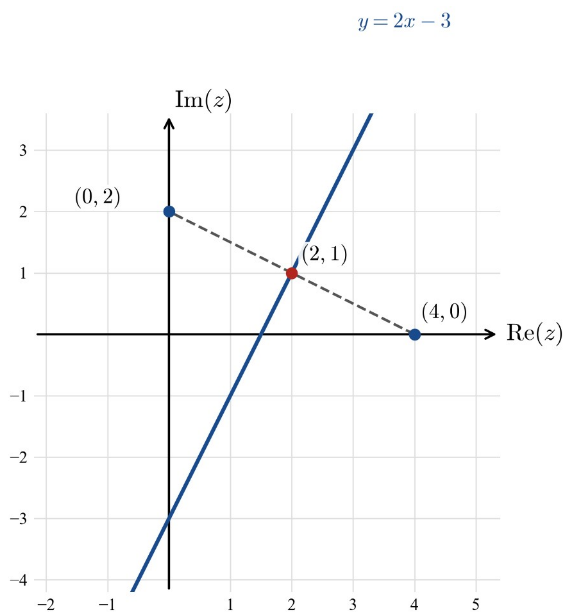
Working	Comment
Here $z_0 = 2 - i$, so the centre is $(2, -1)$, and $r = 3$.	$\lvert z - z_0\rvert = r$ is a circle of centre z_0 and radius r.
Let $z = x + yi$. Then $z - z_0 = (x-2)+(y+1)i$, so $\lvert(x-2)+(y+1)i\rvert = 3$.	Substitute and collect the real and imaginary parts of $z - z_0$.
Squaring: $(x-2)^2+(y+1)^2=9$.	Distance$^2 = r^2$ removes the modulus.
So the set is the circle of centre $(2, -1)$ and radius 3, drawn as a solid curve.	Every point of an = relation is included, so the curve is solid.



Example 2 — scaffolded

Describe and sketch the set $\{z : |z-4| = |z-2i|\}$.

Working	Comment
z is equidistant from $a=4$, the point $(4,0)$, and $b=2i$, the point $(0,2)$.	$\|z-a\|=\|z-b\|$ is the perpendicular bisector of segment ab.
Let $z=x+yi$ and square: $(x-4)^2+y^2=x^2+(y-2)^2$.	Use $\|z-a\|^2=\|z-b\|^2$.
$x^2-8x+16+y^2=x^2+y^2-4y+4$.	Expand both sides.
The x^2 and y^2 cancel: $-8x+16=-4y+4$, so $y=2x-3$.	A straight line remains — the bisector.
Line of gradient 2 through $(0,-3)$; it is the perpendicular bisector of the segment from $(4,0)$ to $(0,2)$, passing through the midpoint $(2,1)$.	Solid line; the midpoint check $1=2(2)-3$ confirms it.



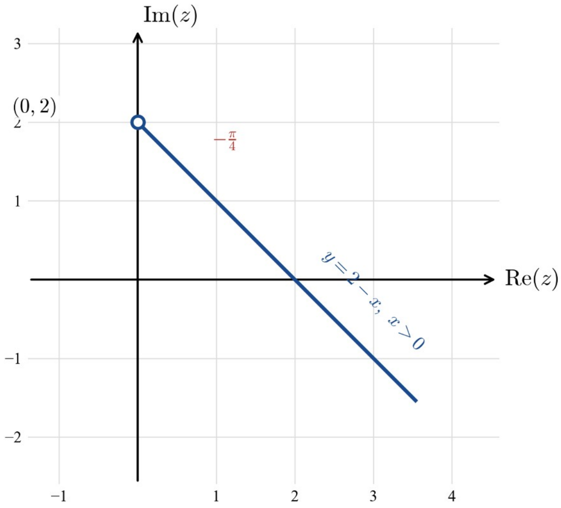
Example 3 — unscaffolded

Sketch the set $\{z: arg(z-2i)=-\frac{\pi}{4}\}$ and describe it in Cartesian form.

Write $z_0=2i$, the point $(0,2)$. The relation fixes the argument of $z-z_0$, so the set is a **ray** starting at $(0,2)$ and pointing in the direction $-\frac{\pi}{4}$. Because $\tan\left(-\frac{\pi}{4}\right)=-1$, the ray lies on the line through $(0,2)$ of gradient -1,

$$y-2=-1(x-0), \text{that is } y=2-x.$$

The direction $-\frac{\pi}{4}$ points to the right and downwards ($\cos\left(-\frac{\pi}{4}\right)>0$ and $\sin\left(-\frac{\pi}{4}\right)<0$), so only the part of the line with $x>0$ (equivalently $y<2$) belongs to the set. The endpoint $(0,2)$ is excluded because $arg(0)$ is undefined, and is drawn as an open dot.



$\therefore$ the set is the open ray $y = 2 - x$, $x > 0$, a half-line from — but not including — the point $(0, 2)$.

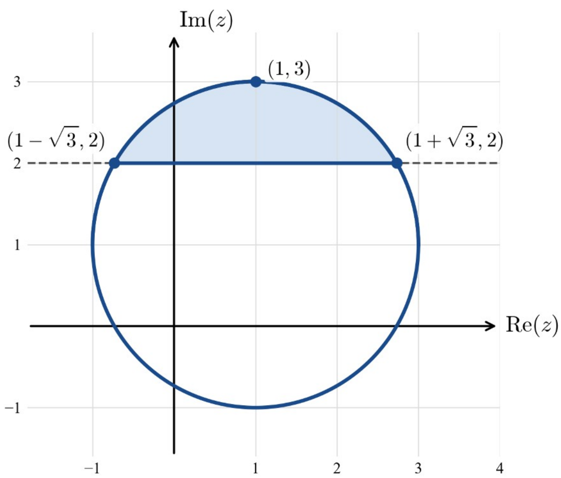
Example 4 — unscaffolded

Sketch and describe the region $\{z : |z - (1+i)| \le 2 \text{ and } \operatorname{Im}(z) \ge 2\}$.

The first relation is a **closed disc** of centre $(1, 1)$ and radius 2, that is $(x-1)^2 + (y-1)^2 \le 4$. The second relation, $\operatorname{Im}(z) \ge 2$, is the closed half-plane on and above the horizontal line $y = 2$. The set is their **intersection**: the part of the disc lying on or above $y = 2$. The line $y = 2$ meets the boundary circle where

$$(x-1)^2 + (2-1)^2 = 4 \Rightarrow (x-1)^2 = 3 \Rightarrow x = 1 \pm \sqrt{3},$$

so the chord runs from $(1-\sqrt{3}, 2)$ to $(1+\sqrt{3}, 2)$, and the highest point of the disc is $(1, 3)$. The overlap is therefore a **circular segment**, bounded below by the chord along $y = 2$ and above by the arc of the circle; both boundaries are solid and included.



$\therefore$ the region is the circular segment of the disc $(x-1)^2 + (y-1)^2 \le 4$ that lies on or above the line $y = 2$, with corner points $(1-\sqrt{3}, 2)$ and $(1+\sqrt{3}, 2)$.

Practice questions

Scaffolded

Q1. Consider the relation $|z - (3+4i)| = 5$. (a) State the centre and radius of this circle. (b) Taking $z = x + yi$, show its Cartesian equation is $(x-3)^2 + (y-4)^2 = 25$. (c) Show that the circle passes through the origin. (d) Sketch the circle, marking the centre and the origin.

Q2. Consider the relation $|z+2-i|<2$. (a) Write it in the form $|z-z_0|<r$ and state z_0 and r. (b) Is the boundary circle part of the set? What does this mean for the sketch? (c) Give the Cartesian description of the region. (d) Sketch the region.

Q3. Consider the relation $|z-1|=|z-5|$. (a) State the two points a and b that z is equidistant from. (b) Find the midpoint of the segment joining them. (c) Show that the locus is the vertical line $x=3$. (d) Sketch the line.

Q4. Consider the region $\{z:-1\leq \mathrm{Re}(z)\leq 2\}$. (a) What line is $\mathrm{Re}(z)=-1$? What line is $\mathrm{Re}(z)=2$? (b) Are these boundary lines included in the set? (c) Describe the region in words. (d) Sketch the region, shading it.

Q5. Consider the relation $arg(z)=\frac{\pi}{3}$. (a) From which point does the ray start, and is that point included? (b) State the gradient of the line on which the ray lies. (c) Give the Cartesian description, including the restriction on x. (d) Sketch the ray.

Q6. Consider the region $\{z:|z|\leq 2 \text{ and } \mathrm{Re}(z)\geq 0\}$. (a) Describe each of the two relations separately. (b) The intersection keeps only points in both. Which half of the disc is it? (c) State the two pieces of its boundary. (d) Sketch the region, shading it.

Unscaffolded

Q7. State the centre, radius and Cartesian equation of $|z-5i|=4$, and sketch it.

Q8. Sketch the region $\{z:|z-3|\leq 3\}$, describe it, and state where its boundary meets the axes.

Q9. Find the centre and radius of the circle $|z+1+2i|=3$, and give its Cartesian equation.

Q10. Describe and sketch the set $\{z:|z-2-2i|=|z+2+2i|\}$.

Q11. Show that $\{z:|z+2i|=|z-4|\}$ is a straight line, and find its equation.

Q12. Sketch the region $\{z:\mathrm{Im}(z)<-1\}$ and describe it.

Q13. Sketch the region $\{z:0\leq \mathrm{Re}(z)\leq 3 \text{ and } -1\leq \mathrm{Im}(z)\leq 2\}$ and describe it.

Q14. Sketch the set $\{z:arg(z-2)=\frac{3\pi}{4}\}$ and give its Cartesian description.

Q15. Sketch the set $\{z:arg(z+i)=\frac{\pi}{6}\}$ and give its Cartesian description.

Q16. Sketch the region $\{z:\frac{\pi}{6}\leq arg(z)\leq \frac{\pi}{3}\}$ and describe it.

Q17. Sketch the region $\{z:1\leq |z|\leq 3\}$ and describe it.

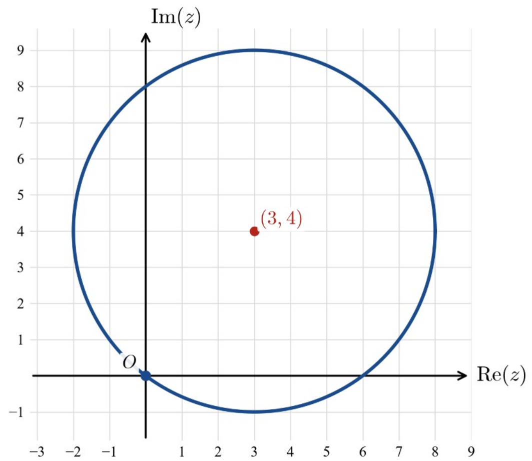
Q18. Sketch the region $\{z:|z|\leq 2 \text{ and } |z-2|\leq 2\}$, find the two points where the boundary circles meet, and describe the region.

Answers

Q1. Centre $(3,4)$, radius 5; $(x-3)^2+(y-4)^2=25$; passes through O since $|3+4i|=5$. **Q2.** $z_0=-2+i$, $r=2$; open disc (boundary excluded), $(x+2)^2+(y-1)^2<4$. **Q3.** Equidistant from $(1,0)$ and $(5,0)$; perpendicular bisector is the vertical line $x=3$. **Q4.** Closed vertical strip between $x=-1$ and $x=2$, both lines included. **Q5.** Ray from O (excluded) along $y=\sqrt{3}x$ with $x>0$. **Q6.** Right half of the closed disc of radius 2: $x^2+y^2\le 4$ with $x\ge 0$. **Q7.** Centre $(0,5)$, radius 4; $x^2+(y-5)^2=16$. **Q8.** Closed disc centre $(3,0)$, radius 3: $(x-3)^2+y^2\le 9$; boundary meets the x-axis at $(0,0)$ and $(6,0)$ and is tangent to the y-axis at $(0,0)$. **Q9.** Centre $(-1,-2)$, radius 3; $(x+1)^2+(y+2)^2=9$. **Q10.** Perpendicular bisector of $(2,2)$ and $(-2,-2)$: the line $y=-x$. **Q11.** Straight line $y=3-2x$ (perpendicular bisector of $(0,-2)$ and $(4,0)$). **Q12.** Open half-plane below the line $y=-1$ (boundary dashed, excluded). **Q13.** Closed rectangle $0\le x\le 3$, $-1\le y\le 2$. **Q14.** Ray from $(2,0)$ (excluded) on $y=2-x$ with $x<2$. **Q15.** Ray from $(0,-1)$ (excluded) on $y=\frac{x}{\sqrt{3}}-1$ with $x>0$. **Q16.** Sector from O (excluded) between the rays $arg(z)=\frac{\pi}{6}$ and $arg(z)=\frac{\pi}{3}$: the first-quadrant wedge between $y=\frac{x}{\sqrt{3}}$ and $y=\sqrt{3}x$. **Q17.** Closed annulus $1\le|z|\le 3$: the ring between the circles of radius 1 and 3, both included. **Q18.** Circles meet at $(1,\sqrt{3})$ and $(1,-\sqrt{3})$; the region is the lens-shaped overlap of the two closed discs.

Fully-worked solutions

Q1. The relation $|z-(3+4i)|=5$ has $z_0=3+4i$ and $r=5$, so the centre is $(3,4)$ and the radius is 5. With $z=x+yi$, $z-z_0=(x-3)+(y-4)i$; squaring $|z-z_0|=5$ gives $(x-3)^2+(y-4)^2=25$. The circle passes through the origin because the distance of $(3,4)$ from $(0,0)$ is $\sqrt{3^2+4^2}=\sqrt{25}=5=r$, so $(0,0)$ lies on it.



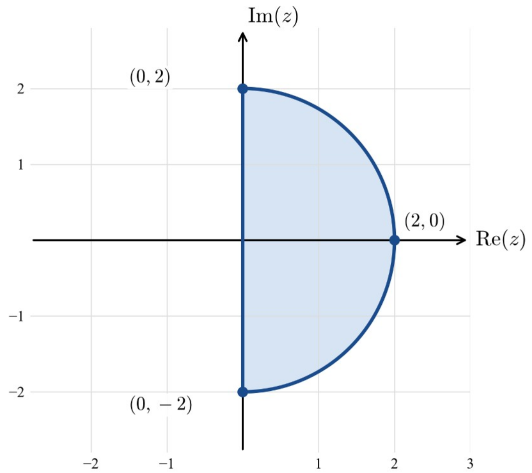
Q2. Rewrite $|z+2-i|=|z-(-2+i)|$, so $z_0=-2+i$ (the point $(-2,1)$) and $r=2$. The inequality is **strict** ($<$), so the boundary circle is **not** part of the set and is drawn dashed; the set is the open disc of interior points only. With $z=x+yi$ the Cartesian description is $(x+2)^2+(y-1)^2<4$: all points less than 2 units from $(-2,1)$.

Q3. Here z is equidistant from $a=1$, the point $(1,0)$, and $b=5$, the point $(5,0)$. Their midpoint is $\left(\frac{1+5}{2},0\right)=(3,0)$. Putting $z=x+yi$ and squaring, $(x-1)^2+y^2=(x-5)^2+y^2$, so $x^2-2x+1=x^2-10x+25$, giving $8x=24$ and $x=3$. As a and b lie on the real axis, the perpendicular bisector is the vertical line $x=3$ through their midpoint.

Q4. The line $\text{Re}(z)=-1$ is the vertical line $x=-1$, and $\text{Re}(z)=2$ is the vertical line $x=2$. Both use $\le$, so both boundary lines are **included** (solid). The region $-1\le\text{Re}(z)\le 2$ places no restriction on $\text{Im}(z)$, so it is the closed vertical **strip** consisting of every point whose real part lies between -1 and 2 inclusive — an infinite band between the lines $x=-1$ and $x=2$.

Q5. The relation $arg(z)=\frac{\pi}{3}$ is $arg(z-0)=\frac{\pi}{3}$, a ray starting at the origin O; the origin itself is **excluded** because $arg(0)$ is undefined. The ray lies on the line through O of gradient $\tan\frac{\pi}{3}=\sqrt{3}$, namely $y=\sqrt{3}x$. Since $\frac{\pi}{3}$ points into the first quadrant ($\cos\frac{\pi}{3}>0$), only the part with $x>0$ (and hence $y>0$) belongs to the set. So the Cartesian description is $y=\sqrt{3}x$, $x>0$, with an open endpoint at O.

Q6. The relation $|z|\le 2$ is the closed disc of centre O and radius 2, that is $x^2+y^2\le 4$; the relation $\text{Re}(z)\ge 0$ is the closed half-plane on and to the right of the imaginary axis, $x\ge 0$. Their intersection keeps points satisfying both, so it is the **right half** of the disc. Its boundary has two pieces: the right-hand semicircle $x^2+y^2=4$ (with $x\ge 0$) and the segment of the imaginary axis from $(0,-2)$ to $(0,2)$; both are solid.



Q7. The relation $|z-5i|=4$ has $z_0=5i$ (the point $(0,5)$) and $r=4$. With $z=x+yi$, $z-5i=x+(y-5)i$, and squaring $|z-5i|=4$ gives $x^2+(y-5)^2=16$: the circle of centre $(0,5)$ and radius 4, drawn solid.

Q8. The relation $|z-3|\le 3$ is the closed disc of centre $(3,0)$ and radius 3, that is $(x-3)^2+y^2\le 9$ — every point on or inside that circle. The boundary meets the x-axis where $y=0$: $(x-3)^2=9$, so $x=0$ or $x=6$, giving $(0,0)$ and $(6,0)$. It meets the y-axis where $x=0$: $9+y^2=9$, so $y=0$ only, meaning the circle just touches the y-axis (is tangent to it) at the origin.

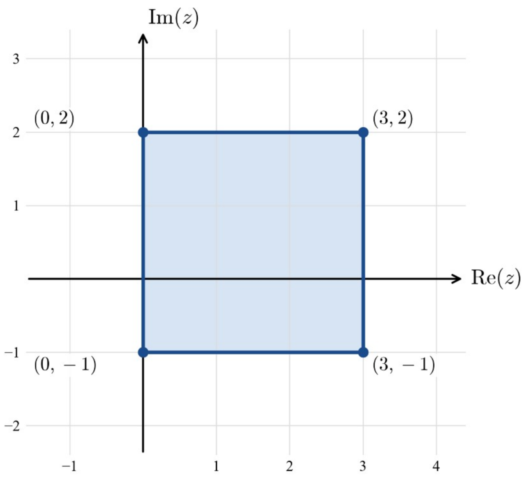
Q9. Rewrite $|z+1+2i|=|z-(-1-2i)|$, so $z_0=-1-2i$ (the point $(-1,-2)$) and $r=3$. With $z=x+yi$, $z-z_0=(x+1)+(y+2)i$, and squaring the relation gives $(x+1)^2+(y+2)^2=9$: the circle of centre $(-1,-2)$ and radius 3.

Q10. Here z is equidistant from $a=2+2i$, the point $(2,2)$, and $b=-2-2i$, the point $(-2,-2)$. Put $z=x+yi$ and square: $(x-2)^2+(y-2)^2=(x+2)^2+(y+2)^2$. Expanding, $-4x-4y+8=4x+4y+8$, so $-8x-8y=0$, giving $y=-x$. The segment from $(2,2)$ to $(-2,-2)$ has gradient 1 and midpoint O, and $y=-x$ passes through O with the perpendicular gradient -1, confirming the perpendicular bisector.

Q11. Here z is equidistant from $a=-2i$, the point $(0,-2)$, and $b=4$, the point $(4,0)$. Put $z=x+yi$ and square: $x^2+(y+2)^2=(x-4)^2+y^2$. Expanding, $x^2+y^2+4y+4=x^2-8x+16+y^2$, so $4y+4=-8x+16$, giving $8x+4y=12$ and hence $y=3-2x$. This is a straight line (gradient -2, through $(0,3)$), the perpendicular bisector of the segment from $(0,-2)$ to $(4,0)$ — its midpoint $(2,-1)$ satisfies $-1=3-2(2)$.

Q12. Since $\text{Im}(z)=y$, the relation $\text{Im}(z)<-1$ is $y<-1$: the open half-plane of all points strictly below the horizontal line $y=-1$. The boundary line $y=-1$ is dashed and excluded, and the region below it (with no restriction on x) is shaded.

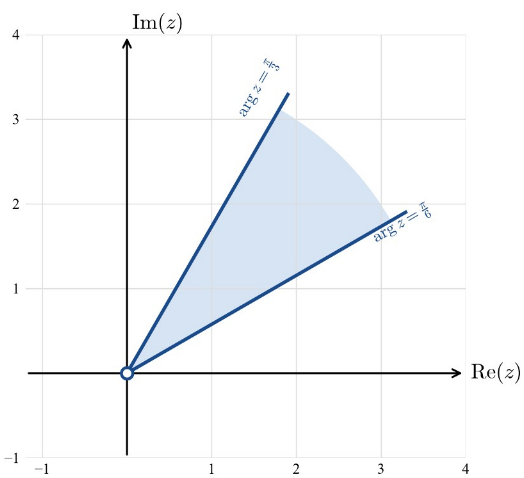
Q13. The pair of conditions $0 \le \mathrm{Re}(z) \le 3$ and $-1 \le \mathrm{Im}(z) \le 2$ becomes $0 \le x \le 3$ and $-1 \le y \le 2$. The first is the vertical strip between $x = 0$ and $x = 3$; the second is the horizontal strip between $y = -1$ and $y = 2$. Their intersection is the closed **rectangle** with corners $(0, -1)$, $(3, -1)$, $(3, 2)$ and $(0, 2)$; all four edges are solid.



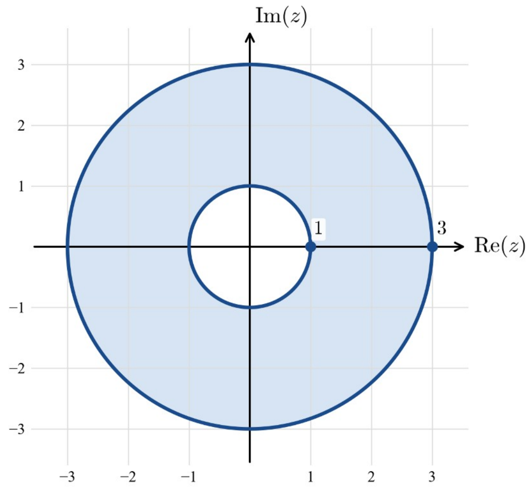
Q14. The set $arg(z-2) = \frac{3\pi}{4}$ is a ray from $z_0 = 2$, the point $(2, 0)$, which is excluded. It lies on the line through $(2, 0)$ of gradient $\tan\frac{3\pi}{4} = -1$: $y - 0 = -1(x-2)$, so $y = 2 - x$. The direction $\frac{3\pi}{4}$ points to the left and upwards ($\cos\frac{3\pi}{4} < 0$, $\sin\frac{3\pi}{4} > 0$), so only the part with $x < 2$ (and $y > 0$) belongs. Cartesian description: $y = 2 - x$, $x < 2$, open at $(2, 0)$.

Q15. The set $arg(z+i) = \frac{\pi}{6}$ is a ray from $z_0 = -i$, the point $(0, -1)$, which is excluded. It lies on the line through $(0, -1)$ of gradient $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$: $y + 1 = \frac{1}{\sqrt{3}}(x - 0)$, so $y = \frac{x}{\sqrt{3}} - 1$. The direction $\frac{\pi}{6}$ has $\cos\frac{\pi}{6} > 0$, so only the part with $x > 0$ belongs. Cartesian description: $y = \frac{x}{\sqrt{3}} - 1$, $x > 0$, open at $(0, -1)$.

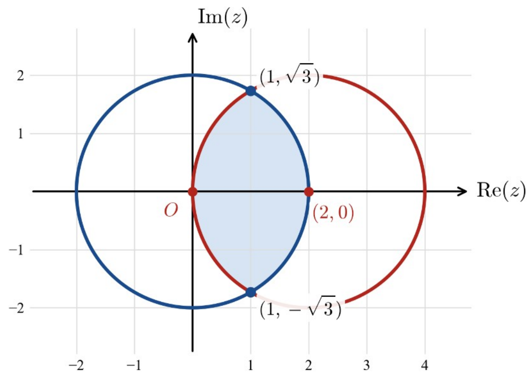
Q16. The relation $\frac{\pi}{6} \le arg(z) \le \frac{\pi}{3}$ is a **sector** of rays from the origin whose argument lies between $\frac{\pi}{6}$ and $\frac{\pi}{3}$. Its two bounding rays are $arg(z) = \frac{\pi}{6}$, lying on $y = \frac{x}{\sqrt{3}}$ (tan $\frac{\pi}{6} = \frac{1}{\sqrt{3}}$), and $arg(z) = \frac{\pi}{3}$, lying on $y = \sqrt{3}x$ (tan $\frac{\pi}{3} = \sqrt{3}$), both with $x > 0$. The set is the first-quadrant wedge between these two rays (both included), with the origin excluded.



Q17. Since $|z|$ is the distance of z from O, the relation $1 \leq |z| \leq 3$ keeps points whose distance from the origin is between 1 and 3 inclusive. This is the closed **annulus** (ring) bounded by the circle $x^2 + y^2 = 1$ on the inside and the circle $x^2 + y^2 = 9$ on the outside; both circles are solid, and the ring between them is shaded.



Q18. The relation $|z| \leq 2$ is the closed disc of centre $(0,0)$, radius 2, and $|z-2| \leq 2$ is the closed disc of centre $(2,0)$, radius 2. The two boundary circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$ meet where the right-hand sides are equal: $x^2 = (x-2)^2$, so $x = 1$; then $y^2 = 4 - 1 = 3$, giving $y = \pm\sqrt{3}$. Hence the circles cross at $(1, \sqrt{3})$ and $(1, -\sqrt{3})$. The intersection of the two discs is the **lens**-shaped region between the two arcs joining these points — every z within 2 units of both $(0,0)$ and $(2,0)$.



Topic 2 Test A — Technology-active

One bound reference and one approved technology (CAS calculator or software) are permitted. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (6 marks)

Let $z=5+i$ and $w=2-3i$.

a. Find $z\,\overline{w}$, expressing your answer in the form $a+bi$. 2 marks

b. Express $\frac{z}{w}$ in the form $a+bi$. 2 marks

c. Find $|z\,w|$, giving an exact value. 2 marks

Question 2 (7 marks)

Let $z=3+\sqrt{3}\,i$ and $w=2-2\sqrt{3}\,i$.

a. Express each of z and w in polar form $r\,cis\,\theta$, using the principal argument. 3 marks

b. Hence express $z\,w$ in polar form. 2 marks

c. Hence express $\frac{z}{w}$ in polar form, using the principal argument. 2 marks

Question 3 (6 marks)

a. Solve $z^2+8z+25=0$ over $\mathbb{C}$ by completing the square, giving the solutions in the form $a+bi$. 3 marks

b. Solve $2z^2-4z+5=0$ over $\mathbb{C}$, giving the solutions in the form $a+bi$. 3 marks

Question 4 (7 marks)

Consider the polynomial $P(z)=z^4-4z^3+24z^2-64z+128$. It is given that $4i$ is a zero of $P(z)$.

a. Write down a second zero of $P(z)$, and hence find a real quadratic factor of $P(z)$. 2 marks

b. Find the other real quadratic factor of $P(z)$. 3 marks

c. Hence solve $P(z)=0$ over $\mathbb{C}$. 2 marks

Question 5 (7 marks)

a. Use De Moivre's theorem to evaluate $(\sqrt{3}+3i)^4$, giving your answer in Cartesian form. 2 marks

b. Find the four solutions of $z^4=-81$, giving each in polar form $r\,cis\,\theta$ with $-\pi<\theta\le\pi$. 3 marks

c. Express each solution in Cartesian form, and describe the geometric figure formed by the four solutions on an Argand diagram. 2 marks

Question 6 (7 marks)

In this question $z=x+yi$, where x and y are real.

a. The relation $|z-3|=|z+i|$ defines a straight line. Show that its Cartesian equation is $y=-3x+4$. 3 marks

b. Sketch, on an Argand diagram, the region $R=\{z:|z-(2+2i)|\le 2\}$, clearly showing its centre and radius. 2 marks

c. Find the maximum value of $|z|$ for $z\in R$, giving an exact value. 2 marks

Test A — answers and solutions

Question 1.

a. The conjugate of $w=2-3i$ is $\overline{w}=2+3i$, so

$$z\overline{w}=(5+i)(2+3i)=10+15i+2i+3i^2=10+17i-3=7+17i.$$

b. Multiply numerator and denominator by $\overline{w}=2+3i$:

$$\frac{z}{w}=\frac{5+i}{2-3i}=\frac{(5+i)(2+3i)}{(2-3i)(2+3i)}=\frac{7+17i}{4+9}=\frac{7+17i}{13}=\frac{7}{13}+\frac{17}{13}i.$$

c. Using $|zw|=|z||w|$ with $|z|=\sqrt{5^2+1^2}=\sqrt{26}$ and $|w|=\sqrt{2^2+(-3)^2}=\sqrt{13}$,

$$|zw|=\sqrt{26}\sqrt{13}=\sqrt{338}=\sqrt{169\times 2}=13\sqrt{2}.$$

(Equivalently, $zw=13-13i$, so $|zw|=\sqrt{13^2+13^2}=13\sqrt{2}$.)

Question 2.

a. For $z=3+\sqrt{3}i$: $r=\sqrt{3^2+(\sqrt{3})^2}=\sqrt{12}=2\sqrt{3}$, and z lies in the first quadrant with $\tan\alpha=\left|\sqrt{3}/3\right|=\frac{1}{\sqrt{3}}$, so $Arg(z)=\frac{\pi}{6}$. Hence $z=2\sqrt{3}\,cis\frac{\pi}{6}$.

For $w=2-2\sqrt{3}i$: $r=\sqrt{2^2+(-2\sqrt{3})^2}=\sqrt{16}=4$, and w lies in the fourth quadrant with $\tan\alpha=\left|-2\sqrt{3}/2\right|=\sqrt{3}$, so $\alpha=\frac{\pi}{3}$ and $Arg(w)=-\frac{\pi}{3}$. Hence $w=4\,cis\left(-\frac{\pi}{3}\right)$.

b. Multiplying moduli and adding arguments,

$$zw=2\sqrt{3}\times 4\,cis\left(\frac{\pi}{6}-\frac{\pi}{3}\right)=8\sqrt{3}\,cis\left(\frac{\pi-2\pi}{6}\right)=8\sqrt{3}\,cis\left(-\frac{\pi}{6}\right).$$

c. Dividing moduli and subtracting arguments,

$$\frac{z}{w}=\frac{2\sqrt{3}}{4}cis\left(\frac{\pi}{6}-\left(-\frac{\pi}{3}\right)\right)=\frac{\sqrt{3}}{2}cis\left(\frac{\pi+2\pi}{6}\right)=\frac{\sqrt{3}}{2}cis\frac{\pi}{2}.$$

Since $\frac{\pi}{2}$ lies in $(-\pi,\pi]$, it is the principal argument.

Question 3.

a. Completing the square,

$$z^2+8z+25=(z+4)^2-16+25=(z+4)^2+9=0,$$

so $(z+4)^2=-9$. Taking square roots, $z+4=\pm 3i$, giving

$$z=-4+3i \text{ or } z=-4-3i.$$

b. By the quadratic formula with $a=2$, $b=-4$, $c=5$, the discriminant is $\Delta=(-4)^2-4(2)(5)=16-40=-24$, so

$$z=\frac{4\pm\sqrt{-24}}{4}=\frac{4\pm 2\sqrt{6}i}{4}=1\pm\frac{\sqrt{6}}{2}i.$$

Hence $z = 1 + \frac{\sqrt{6}}{2} i$ or $z = 1 - \frac{\sqrt{6}}{2} i$.

Question 4.

a. The coefficients of $P(z)$ are real, so by the conjugate root theorem the conjugate $\overline{4i} = -4i$ is also a zero. The corresponding real quadratic factor is

$$(z - 4i)(z + 4i) = z^2 - (4i)^2 = z^2 + 16.$$

b. Dividing $P(z)$ by $z^2 + 16$:

$$z^4 - 4z^3 + 24z^2 - 64z + 128 = (z^2 + 16)(z^2 - 4z + 8).$$

(The quotient $z^2 - 4z + 8$ is obtained by polynomial long division; for example $z^4 \div z^2 = z^2$, then $-4z^3 \div z^2 = -4z$, then the remaining $8z^2 + 128 \div z^2 = 8$, leaving remainder 0.) The other real quadratic factor is $z^2 - 4z + 8$.

c. From part **a** the factor $z^2 + 16 = 0$ gives $z = \pm 4i$. From part **b**, solving $z^2 - 4z + 8 = 0$:

$$z = \frac{4 \pm \sqrt{16 - 32}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i.$$

Hence the four solutions are $z = 4i, -4i, 2 + 2i, 2 - 2i$.

Question 5.

a. For $\sqrt{3} + 3i$: $r = \sqrt{(\sqrt{3})^2 + 3^2} = \sqrt{12} = 2\sqrt{3}$, and the point is in the first quadrant with $\tan\alpha = \frac{3}{\sqrt{3}} = \sqrt{3}$, so $\sqrt{3} + 3i = 2\sqrt{3}\, cis \frac{\pi}{3}$. By De Moivre's theorem,

$$(\sqrt{3} + 3i)^4 = (2\sqrt{3})^4 cis\left(4 \times \frac{\pi}{3}\right) = 144\, cis \frac{4\pi}{3} = 144\, cis\left(-\frac{2\pi}{3}\right),$$

since $\frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$. Hence

$$(\sqrt{3} + 3i)^4 = 144\left(-\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}\right) = -72 - 72\sqrt{3} i.$$

b. Write $-81 = 81\, cis\, \pi$. If $z = r\, cis\, \theta$ then $z^4 = r^4 cis(4\theta) = 81\, cis(\pi + 2\pi k)$, so $r = 81^{1/4} = 3$ and

$$\theta = \frac{\pi + 2\pi k}{4}, k = 0, 1, 2, 3.$$

Taking $k = 0, 1, 2, 3$ and reducing to $(-\pi, \pi]$ gives

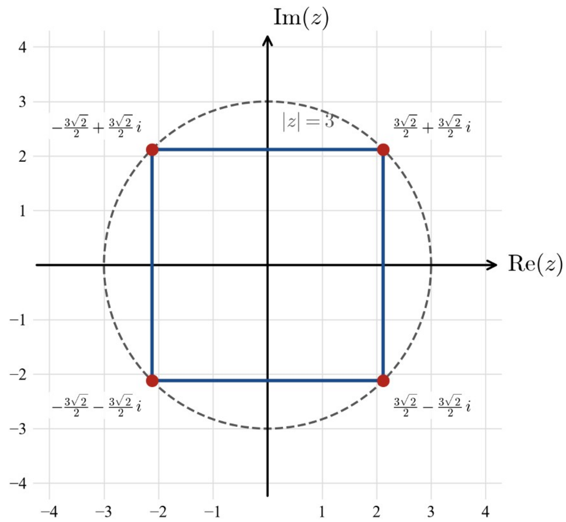
$$3\, cis \frac{\pi}{4}, 3\, cis \frac{3\pi}{4}, 3\, cis\left(-\frac{3\pi}{4}\right), 3\, cis\left(-\frac{\pi}{4}\right).$$

c. Evaluating each, using $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$,

$$3\, cis \frac{\pi}{4} = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i, 3\, cis \frac{3\pi}{4} = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i,$$

$$3\, cis\left(-\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} i, 3\, cis\left(-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} i.$$

The four points all lie on the circle of radius 3 centred at the origin, equally spaced $\frac{\pi}{2}$ apart, so they form the vertices of a **square**.



Question 6.

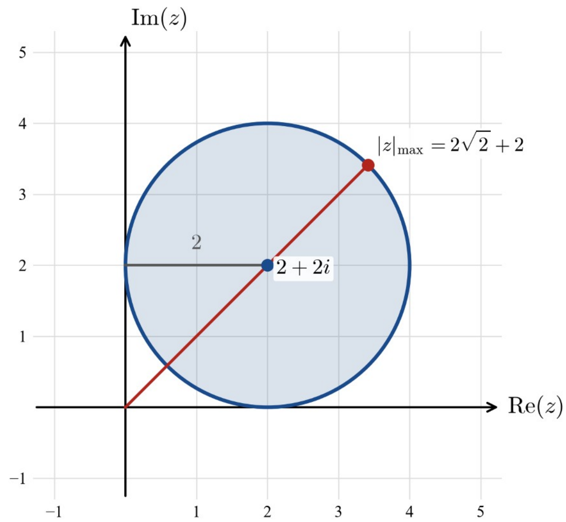
a. With $z=x+yi$, $|z-3|=|z+i|$ becomes $|(x-3)+yi|=|x+(y+1)i|$. Squaring both sides,

$$(x-3)^2+y^2=x^2+(y+1)^2.$$

Expanding, $x^2-6x+9+y^2=x^2+y^2+2y+1$, so $-6x+9=2y+1$. Then $2y=-6x+8$, and dividing by 2 gives

$$y=-3x+4.$$

b. The relation $|z-(2+2i)|\leq 2$ is the closed disc of all points within distance 2 of the centre $2+2i$, i.e. centre $(2,2)$ and radius 2. The region is shaded below.



c. The maximum of $|z|$ is attained at the point of R furthest from the origin, which lies on the line through O and the centre. Since the centre is at distance $|2+2i|=\sqrt{2^2+2^2}=2\sqrt{2}$ from O,

$$|z|_{max}=2\sqrt{2}+2.$$

Topic 2 Test B — Technology-free

Technology-free. No calculator. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

Let $z = 4 - 3i$ and $w = 2 + 5i$.

a. Find $z + w$. 1 mark

b. Find zw, expressing your answer in the form $a + bi$. 2 marks

c. Find $|z|$. 1 mark

d. Express $\frac{z}{w}$ in the form $a + bi$. 3 marks

Question 2 (6 marks)

a. Express $z = -1 - \sqrt{3}i$ in polar form $r\,cis\,\theta$, using the principal argument. 3 marks

b. Express $w = 4\,cis\frac{3\pi}{4}$ in Cartesian form $a + bi$. 3 marks

Question 3 (7 marks)

Let $z = 2\,cis\frac{3\pi}{4}$ and $w = 5\,cis\left(-\frac{\pi}{6}\right)$.

a. Find zw in polar form. 2 marks

b. Find $\frac{z}{w}$ in polar form. 2 marks

c. Find z^3, giving your answer in Cartesian form. 3 marks

Question 4 (6 marks)

Solve each equation over $\mathbb{C}$, giving the solutions in the form $a + bi$.

a. $z^2 - 6z + 25 = 0$. 3 marks

b. $z^2 - 2\sqrt{3}z + 4 = 0$. 3 marks

Question 5 (7 marks)

The polynomial $P(z) = z^3 - 9z^2 + 40z - 100$ has $2 + 4i$ as one of its zeros.

a. Write down another zero of $P(z)$, giving a reason. 2 marks

b. Find the real quadratic factor of $P(z)$ corresponding to the two complex zeros. 2 marks

c. Hence express $P(z)$ as a product of a real linear factor and a real quadratic factor, and state all three zeros of $P(z)$. 3 marks

Question 6 (7 marks)

a. Express -216 in polar form $r\,cis\,\theta$. 1 mark

b. Hence find the three cube roots of -216, giving each in polar form $r\,cis\,\theta$ with $-\pi < \theta \leq \pi$. 3 marks

c. Express each cube root in Cartesian form $a + b\,i$. 3 marks

End of Topic 2 Test B

Test B — answers and solutions

Question 1.

a. $z+w=(4+2)+(-3+5)i=6+2i$.

b. $zw=(4-3i)(2+5i)=8+20i-6i-15i^2=8+14i+15=23+14i$.

c. $|z|=\sqrt{4^2+(-3)^2}=\sqrt{16+9}=\sqrt{25}=5$.

d. Multiply numerator and denominator by $\overline{w}=2-5i$:

$$\frac{z}{w}=\frac{4-3i}{2+5i}=\frac{(4-3i)(2-5i)}{(2+5i)(2-5i)}=\frac{8-20i-6i+15i^2}{4+25}=\frac{8-26i-15}{29}=\frac{-7-26i}{29}.$$

Hence $\frac{z}{w}=-\frac{7}{29}-\frac{26}{29}i$.

Question 2.

a. For $z=-1-\sqrt{3}i$: $r=\sqrt{(-1)^2+(-\sqrt{3})^2}=\sqrt{1+3}=2$. Both parts are negative, so z is in the third quadrant, with reference angle $\tan\alpha=|-\sqrt{3}/-1|=\sqrt{3}$, giving $\alpha=\frac{\pi}{3}$. In the third quadrant $Arg(z)=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{3}\right)=-\frac{2\pi}{3}$. Hence

$$z=2\,cis\left(-\frac{2\pi}{3}\right).$$

b. Using $\cos\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$ and $\sin\frac{3\pi}{4}=\frac{\sqrt{2}}{2}$,

$$w=4\left(-\frac{\sqrt{2}}{2}+i\cdot\frac{\sqrt{2}}{2}\right)=-2\sqrt{2}+2\sqrt{2}i.$$

Question 3.

a. Multiply the moduli and add the arguments:

$$zw=(2)(5)\,cis\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)=10\,cis\left(\frac{9\pi-2\pi}{12}\right)=10\,cis\frac{7\pi}{12}.$$

b. Divide the moduli and subtract the arguments:

$$\frac{z}{w}=\frac{2}{5}cis\left(\frac{3\pi}{4}+\frac{\pi}{6}\right)=\frac{2}{5}cis\left(\frac{9\pi+2\pi}{12}\right)=\frac{2}{5}cis\frac{11\pi}{12}.$$

c. By De Moivre's theorem,

$$z^3=2^3\,cis\left(3\times\frac{3\pi}{4}\right)=8\,cis\frac{9\pi}{4}=8\,cis\frac{\pi}{4},$$

since $\frac{9\pi}{4}-2\pi=\frac{\pi}{4}$. Hence

$$z^3=8\left(\frac{\sqrt{2}}{2}+i\cdot\frac{\sqrt{2}}{2}\right)=4\sqrt{2}+4\sqrt{2}i.$$

Question 4.

a. By the quadratic formula, the discriminant is $\Delta=(-6)^2-4(1)(25)=36-100=-64$, so

$$z=\frac{6\pm\sqrt{-64}}{2}=\frac{6\pm 8i}{2}=3\pm 4i.$$

Hence $z=3+4i$ or $z=3-4i$.

b. The discriminant is $\Delta=(-2\sqrt{3})^2-4(1)(4)=12-16=-4$, so

$$z=\frac{2\sqrt{3}\pm\sqrt{-4}}{2}=\frac{2\sqrt{3}\pm 2i}{2}=\sqrt{3}\pm i.$$

Hence $z=\sqrt{3}+i$ or $z=\sqrt{3}-i$.

Question 5.

a. Since $P(z)$ has real coefficients, the conjugate root theorem guarantees that the conjugate of any non-real zero is also a zero. Therefore $\overline{2+4i}=2-4i$ is another zero of $P(z)$.

b. The quadratic factor with zeros $2+4i$ and $2-4i$ is

$$(z-(2+4i))(z-(2-4i))=(z-2)^2-(4i)^2=z^2-4z+4+16=z^2-4z+20.$$

c. Since $P(z)=(z^2-4z+20)(z-a)$ for some real a, comparing constant terms gives $20\times(-a)=-100$, so $a=5$. Hence

$$P(z)=(z-5)(z^2-4z+20).$$

(Equivalently, the sum of the three zeros equals 9; as $2+4i$ and $2-4i$ sum to 4, the third zero is 5.) The three zeros are $z=5, 2+4i, 2-4i$.

Question 6.

a. -216 is a negative real number, so it has modulus 216 and principal argument π:

$$-216=216\,cis\,\pi.$$

b. If $z=r\,cis\,\theta$ satisfies $z^3=216\,cis(\pi+2\pi k)$, then $r=216^{1/3}=6$ and $\theta=\frac{\pi+2\pi k}{3}$ for $k=0,1,2$. Reducing to $(-\pi,\pi]$ gives

$$6\,cis\frac{\pi}{3}, 6\,cis\,\pi, 6\,cis\left(-\frac{\pi}{3}\right).$$

c. Evaluating each root:

$$6\,cis\frac{\pi}{3}=6\left(\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right)=3+3\sqrt{3}i,$$

$$6\,cis\,\pi=6(-1+0i)=-6,$$

$$6\,cis\left(-\frac{\pi}{3}\right)=6\left(\frac{1}{2}-i\cdot\frac{\sqrt{3}}{2}\right)=3-3\sqrt{3}i.$$

Hence the three cube roots are $3+3\sqrt{3}i, -6, 3-3\sqrt{3}i$.