Specialist Mathematics Units 3 & 4

Chapter 3 — Vector equations of lines and planes

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Contents

Theory

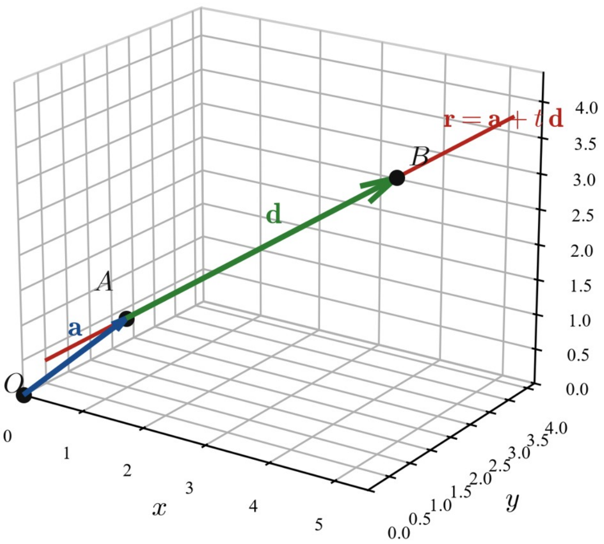
A straight line is fixed as soon as we know **one point on it** and **the direction in which it runs**. Vectors record both pieces of information at once, and this leads to a single compact description of a line that works equally well in two and three dimensions — where the familiar $y=mx+c$ no longer applies.

The vector equation of a line.

Let A be a known point on a line, with position vector $a=\overrightarrow{OA}$, and let d be a non-zero **direction vector** parallel to the line. If R is any other point on the line, with position vector $r=\overrightarrow{OR}$, then $\overrightarrow{AR}$ is parallel to d, so $\overrightarrow{AR}=td$ for some real number t. Since $r=a+\overrightarrow{AR}$,

$$r=a+td,\, t\in\mathbb{R}.$$

This is the **vector equation** of the line. The number t is the **parameter**: as t runs through all real values, the tip of r traces out the whole line. Putting $t=0$ returns the point A, positive values of t move along the line in the direction of d, and negative values move the opposite way.



The equation is not unique. Any point on the line may be used for a, and any non-zero scalar multiple of d points along the same line, so a single line has infinitely many vector equations. For example $d=2i+4j$ and $d=i+2j$ describe the same direction; it is often neatest to divide out any common factor.

The line through two points.

If a line passes through two points A and B with position vectors a and b, a direction vector is the displacement from one to the other,

$$d=\overrightarrow{AB}=b-a.$$

Taking a as the known point gives the vector equation

$$r=a+t(b-a).$$

Here $t=0$ gives A and $t=1$ gives B, so the segment AB corresponds to $0\le t\le 1$ while all other values of t give points on the line beyond the two given points.

Parametric equations.

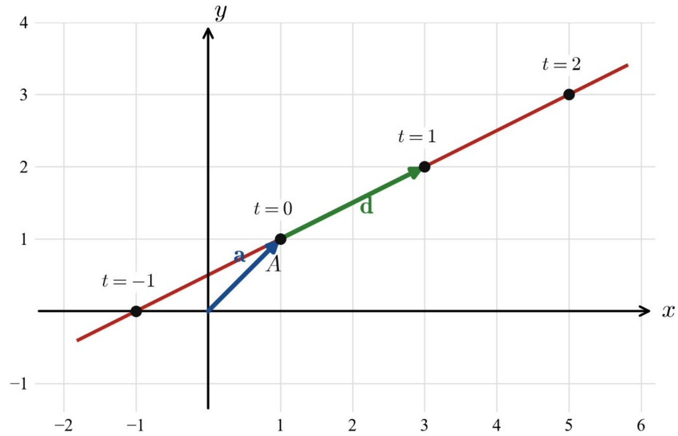
Writing the position vector in components as $r=x\,i+y\,j+z\,k$, with $a=a_1 i+a_2 j+a_3 k$ and $d=d_1 i+d_2 j+d_3 k$, the vector equation can be read one row at a time:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}+t\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Equating components gives the **parametric equations** of the line,

$$x=a_1+t\,d_1,\, y=a_2+t\,d_2,\, z=a_3+t\,d_3,$$

which express each coordinate as a linear function of the parameter t. In two dimensions the third equation is simply absent. The diagram below shows a two-dimensional line $r=a+t\,d$ with its point A (at $t=0$), its position vector a and direction vector d, and the points reached at $t=-1,1,2$.



The Cartesian equation.

Eliminating the parameter t from the parametric equations leaves a relation between the coordinates alone — the **Cartesian equation** of the line.

In two dimensions, solve one parametric equation for t and substitute into the other. From $x=a_1+t\,d_1$ we get $t=\dfrac{x-a_1}{d_1}$ (provided $d_1\neq 0$), and substituting into $y=a_2+t\,d_2$ recovers the straight-line form $y=m\,x+c$ with gradient $m=\dfrac{d_2}{d_1}$. So the gradient of the line is just the ratio of the components of its direction vector.

In three dimensions, making t the subject of each parametric equation gives the **symmetric form**

$$\frac{x-a_1}{d_1}=\frac{y-a_2}{d_2}=\frac{z-a_3}{d_3},$$

valid when no component of d is zero. Each expression equals t, so the three are equal to one another. If one component of d is zero — say $d_2=0$ — then that coordinate is constant ($y=a_2$) and only the remaining two ratios are set equal.

Testing whether a point lies on a line.

A point P lies on the line $r=a+t\,d$ exactly when there is a **single** value of t that reproduces every coordinate of P at once. In practice, substitute the coordinates of P into the parametric equations, solve each for t, and compare: if all components give the **same** t, the point is on the line; if any two disagree, it is not. The same idea shows that three points are **collinear** when the displacement between one pair is a scalar multiple of the displacement between another pair.

The vector form generalises the study of straight lines to three dimensions and is the starting point for finding where lines meet, the angle between them, and — in later sections — the equations of planes.

Worked examples

Example 1 — scaffolded

Find a vector equation and the parametric equations of the line through the point $A(1,2,-1)$ that is parallel to $d=2i-j+3k$, and hence write down one other point on the line.

Working	Comment
$a=i+2j-k$	Position vector of the known point A.
$r=(i+2j-k)+t(2i-j+3k)$	Substitute a and d into $r=a+td$.
$x=1+2t, y=2-t, z=-1+3t$	Equate components to read off the parametric equations.
At $t=1$: $x=3, y=1, z=2$	Any value of t gives a point; here $t=1$.
A second point on the line is $(3,1,2)$.	State the coordinates reached.

Example 2 — scaffolded

Find a vector equation, the parametric equations, and the Cartesian equation of the line through the points $A(2,1)$ and $B(5,7)$.

Working	Comment
$d=b-a=(5-2)i+(7-1)j=3i+6j$	Direction vector is the displacement $\overrightarrow{AB}$.
$r=(2i+j)+t(3i+6j)$	Use $a=2i+j$ as the known point.
$x=2+3t, y=1+6t$	Parametric equations from the components.
$t=\frac{x-2}{3}$, so $y=1+6\cdot\frac{x-2}{3}=1+2(x-2)$	Make t the subject of the x-equation and substitute.
$y=2x-3$	Simplify to Cartesian form; gradient $\frac{d_2}{d_1}=\frac{6}{3}=2$.

Example 3 — unscaffolded

Find a vector equation, the parametric equations, and the symmetric Cartesian form of the line through the points $A(3,-1,2)$ and $B(1,3,5)$.

A direction vector is the displacement from A to B,

$$d=b-a=(1-3)i+(3-(-1))j+(5-2)k=-2i+4j+3k.$$

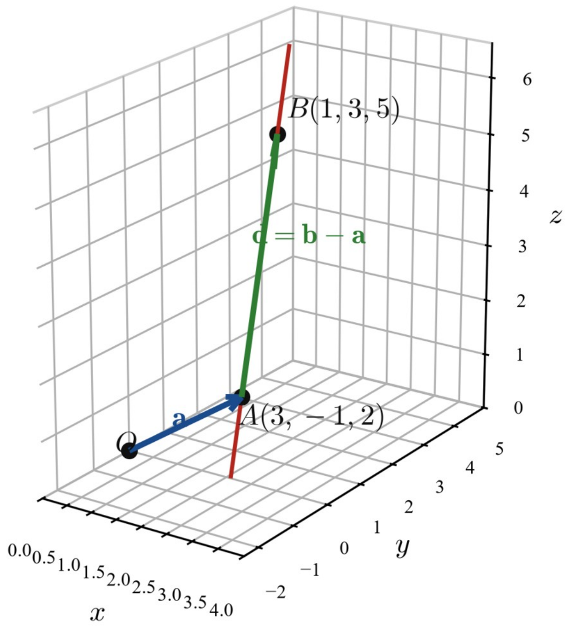
Taking $a=3i-j+2k$ as the known point, the vector equation is

$$r=(3i-j+2k)+t(-2i+4j+3k).$$

Equating components gives the parametric equations

$$x=3-2t, y=-1+4t, z=2+3t.$$

Making t the subject of each and setting the results equal gives the symmetric form.



$$\therefore \frac{x-3}{-2} = \frac{y+1}{4} = \frac{z-2}{3}.$$

Example 4 — unscaffolded

A line has vector equation $r = (i + 2j - k) + t(2i - j + 3k)$. Determine whether the points $P(5, 0, 5)$ and $Q(3, 1, 1)$ lie on the line.

The parametric equations are $x = 1 + 2t$, $y = 2 - t$ and $z = -1 + 3t$. A point lies on the line only if one value of t satisfies all three at once.

For $P(5, 0, 5)$: from x, $5 = 1 + 2t$ gives $t = 2$; from y, $0 = 2 - t$ gives $t = 2$; from z, $5 = -1 + 3t$ gives $t = 2$. All three agree, so P lies on the line (at $t = 2$).

For $Q(3, 1, 1)$: from x, $3 = 1 + 2t$ gives $t = 1$; from y, $1 = 2 - t$ gives $t = 1$; but from z, $1 = -1 + 3t$ gives $t = \frac{2}{3}$. The z-value disagrees with the other two, so no single t works.

$\therefore$ P lies on the line (at $t = 2$), but Q does not.

Practice questions

Scaffolded

Q1. Consider the line through $A(2, -3, 1)$ parallel to $d = i + 4j - 2k$. (a) Write down the position vector a of A. (b) Write a vector equation of the line. (c) State the parametric equations. (d) Find the coordinates of the point reached when $t = 2$.

Q2. Consider the line through $A(1, 2)$ and $B(4, 8)$. (a) Find the direction vector $d = b - a$. (b) Write a vector equation of the line. (c) State the parametric equations. (d) Eliminate t to find the Cartesian equation $y = mx + c$.

Q3. Consider the line through $A(4, 1, -2)$ and $B(6, 5, 1)$. (a) Find the direction vector d. (b) Write a vector equation of the line. (c) State the parametric equations. (d) Write the symmetric Cartesian form.

Q4. A line has vector equation $r = (2i + j - 3k) + t(i - 2j + k)$. (a) State the parametric equations. (b) Show that $P(4, -3, -1)$ lies on the line and give the value of t. (c) Determine whether $Q(0, 5, -4)$ lies on the line.

Q5. A line has parametric equations $x=3-t$ and $y=2+2t$. (a) Write down the position vector a and a direction vector d. (b) Eliminate t to find the Cartesian equation. (c) Find the coordinates of the point where the line crosses the x-axis.

Q6. A line has vector equation $r=(i-j+2k)+t(3i+j-k)$. (a) Find the point on the line when $t=0$. (b) Find the point on the line when $t=-1$. (c) Determine whether $R(7,1,0)$ lies on the line.

Unscaffolded

Q7. Find a vector equation of the line through $A(0,3,-2)$ that is parallel to $d=2i+j+5k$.

Q8. Find a vector equation and the Cartesian equation of the line through $A(2,5)$ and $B(6,-3)$.

Q9. Find the parametric equations of the line through $A(-1,4,2)$ and $B(3,4,-1)$, and describe what is special about this line.

Q10. Write the symmetric Cartesian form of the line through $A(2,-1,3)$ with direction vector $d=3i+2j-k$.

Q11. Determine whether the point $P(7,-4,9)$ lies on the line $r=(i+2j+3k)+t(2i-2j+2k)$.

Q12. A line has vector equation $r=(i-4j)+t(2i+j)$. Find the coordinates of the points where it crosses (a) the x-axis and (b) the y-axis.

Q13. Find the coordinates of the point where the line $r=(2i+3j-k)+t(i-j+2k)$ crosses the plane $z=5$.

Q14. The points $A(1,0,-2)$ and $B(4,6,4)$ are given. Find the position vector of the midpoint M of AB, and show that M lies on the line $r=a+t(b-a)$, stating the value of t.

Q15. Find a vector equation of the line through $A(5,-2,1)$ that is parallel to the line $r=(3i+j-2k)+s(4i-j+2k)$.

Q16. A line has Cartesian equations $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-4}{5}$. Write down a point on the line and a direction vector, and hence give a vector equation of the line.

Q17. Show that the points $A(1,2,3)$, $B(3,5,4)$ and $C(7,11,6)$ are collinear.

Q18. Find the point of intersection of the lines $r=j+t(i+j)$ and $r=(4i+j)+s(-i+j)$.

Answers

Q1. (a) $a=2i-3j+k$; (b) $r=(2i-3j+k)+t(i+4j-2k)$; (c) $x=2+t, y=-3+4t, z=1-2t$; (d) $(4,5,-3)$. **Q2.** (a) $d=3i+6j$; (b) $r=(i+2j)+t(3i+6j)$; (c) $x=1+3t, y=2+6t$; (d) $y=2x$. **Q3.** (a) $d=2i+4j+3k$; (b) $r=(4i+j-2k)+t(2i+4j+3k)$; (c) $x=4+2t, y=1+4t, z=-2+3t$; (d) $\frac{x-4}{2}=\frac{y-1}{4}=\frac{z+2}{3}$. **Q4.** (a) $x=2+t, y=1-2t, z=-3+t$; (b) P lies on the line at $t=2$; (c) Q does not lie on the line. **Q5.** (a) $a=3i+2j$, $d=-i+2j$; (b) $y=-2x+8$; (c) $(4,0)$. **Q6.** (a) $(1,-1,2)$; (b) $(-2,-2,3)$; (c) R lies on the line at $t=2$. **Q7.** $r=(3j-2k)+t(2i+j+5k)$. **Q8.** $r=(2i+5j)+t(4i-8j)$; $y=-2x+9$. **Q9.** $x=-1+4t, y=4, z=2-3t$; the y-coordinate is constant, so the line lies in the plane $y=4$ and is parallel to the xz-plane. **Q10.** $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-1}$. **Q11.** Yes; P lies on the line at $t=3$. **Q12.** (a) $(9,0)$; (b) $\left(0,-\frac{9}{2}\right)$. **Q13.** $(5,0,5)$. **Q14.** $M=\frac{5}{2}i+3j+k$; M lies on the line at $t=\frac{1}{2}$. **Q15.** $r=(5i-2j+k)+t(4i-j+2k)$. **Q16.** Point $(1,-3,4)$, direction $2i-j+5k$; $r=(i-3j+4k)+t(2i-j+5k)$. **Q17.** $\overrightarrow{AC}=3\overrightarrow{AB}$, so A, B, C are collinear. **Q18.** $(2,3)$.

Fully-worked solutions

Q1. The position vector of $A(2,-3,1)$ is $a=2i-3j+k$. With direction $d=i+4j-2k$, the vector equation is

$$r=(2i-3j+k)+t(i+4j-2k).$$

Equating components gives $x=2+t$, $y=-3+4t$ and $z=1-2t$. At $t=2$, $x=2+2=4$, $y=-3+8=5$ and $z=1-4=-3$, so the point is $(4,5,-3)$.

Q2. A direction vector is $d=b-a=(4-1)i+(8-2)j=3i+6j$. Using $a=i+2j$,

$$r=(i+2j)+t(3i+6j),$$

so $x=1+3t$ and $y=2+6t$. From the first, $t=\frac{x-1}{3}$, and substituting into the second, $y=2+6\cdot\frac{x-1}{3}=2+2(x-1)=2x$. Hence the Cartesian equation is $y=2x$. (As a check, $B(4,8)$ satisfies $8=2\times 4$.)

Q3. A direction vector is $d=b-a=(6-4)i+(5-1)j+(1-(-2))k=2i+4j+3k$. Using $a=4i+j-2k$,

$$r=(4i+j-2k)+t(2i+4j+3k),$$

giving $x=4+2t$, $y=1+4t$ and $z=-2+3t$. Making t the subject of each, $t=\frac{x-4}{2}=\frac{y-1}{4}=\frac{z+2}{3}$, which is the symmetric Cartesian form.

Q4. Equating components of r gives $x=2+t$, $y=1-2t$ and $z=-3+t$. For $P(4,-3,-1)$: from x, $4=2+t$ gives $t=2$; from y, $-3=1-2t$ gives $t=2$; from z, $-1=-3+t$ gives $t=2$. All three agree, so P lies on the line at $t=2$. For $Q(0,5,-4)$: from x, $0=2+t$ gives $t=-2$; but from y, $5=1-2t$ gives $t=-2$, while from z, $-4=-3+t$ gives $t=-1$. Since the z-value disagrees, no single t works and Q does not lie on the line.

Q5. Comparing $x=3-t$ and $y=2+2t$ with $x=a_1+td_1$ and $y=a_2+td_2$ gives $a=3i+2j$ and $d=-i+2j$. From the first equation $t=3-x$, and substituting into the second, $y=2+2(3-x)=8-2x$, so the Cartesian equation is $y=-2x+8$. The line crosses the x-axis where $y=0$: then $2+2t=0$ gives $t=-1$, and $x=3-(-1)=4$, so the crossing point is $(4,0)$.

Q6. Equating components gives $x=1+3t$, $y=-1+t$ and $z=2-t$. At $t=0$ the point is $(1,-1,2)$. At $t=-1$, $x=1-3=-2$, $y=-1-1=-2$ and $z=2+1=3$, giving $(-2,-2,3)$. For $R(7,1,0)$: from x, $7=1+3t$ gives $t=2$; from y, $1=-1+t$ gives $t=2$; from z, $0=2-t$ gives $t=2$. All agree, so R lies on the line at $t=2$.

Q7. The position vector of $A(0,3,-2)$ is $a=3j-2k$. With $d=2i+j+5k$, a vector equation is

$$r=(3j-2k)+t(2i+j+5k).$$

Q8. A direction vector is $d=b-a=(6-2)i+(-3-5)j=4i-8j$. Using $a=2i+5j$,

$$r=(2i+5j)+t(4i-8j),$$

so $x=2+4t$ and $y=5-8t$. From the first, $t=\dfrac{x-2}{4}$, and $y=5-8\cdot\dfrac{x-2}{4}=5-2(x-2)=-2x+9$. Hence $y=-2x+9$. (As a check, $B(6,-3)$ satisfies $-3=-12+9$.)

Q9. A direction vector is $d=b-a=(3-(-1))i+(4-4)j+(-1-2)k=4i+0j-3k$. Using $a=-i+4j+2k$, the parametric equations are

$$x=-1+4t,\ y=4,\ z=2-3t.$$

Because the j-component of d is zero, the y-coordinate never changes: every point of the line has $y=4$. The line therefore lies in the plane $y=4$ and runs parallel to the xz-plane.

Q10. With $a=2i-j+3k$ and $d=3i+2j-k$, the parametric equations are $x=2+3t$, $y=-1+2t$ and $z=3-t$. Making t the subject of each and equating gives the symmetric form

$$\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-1}.$$

Q11. The parametric equations are $x=1+2t$, $y=2-2t$ and $z=3+2t$. For $P(7,-4,9)$: from x, $7=1+2t$ gives $t=3$; from y, $-4=2-2t$ gives $t=3$; from z, $9=3+2t$ gives $t=3$. All three agree, so P lies on the line at $t=3$.

Q12. The parametric equations are $x=1+2t$ and $y=-4+t$. The line meets the x-axis where $y=0$: then $-4+t=0$ gives $t=4$, and $x=1+8=9$, so the point is $(9,0)$. It meets the y-axis where $x=0$: then $1+2t=0$ gives $t=-\dfrac{1}{2}$, and $y=-4-\dfrac{1}{2}=-\dfrac{9}{2}$, so the point is $\left(0,-\dfrac{9}{2}\right)$.

Q13. The parametric equations are $x=2+t$, $y=3-t$ and $z=-1+2t$. The line meets the plane $z=5$ where $-1+2t=5$, that is $t=3$. Then $x=2+3=5$ and $y=3-3=0$, so the crossing point is $(5,0,5)$.

Q14. The midpoint of $A(1,0,-2)$ and $B(4,6,4)$ has position vector

$$m=1/2(a+b)=1/2((1+4)i+(0+6)j+(-2+4)k)=5/2i+3j+k,$$

so $M=\left(\dfrac{5}{2},3,1\right)$. A direction vector for the line is $b-a=3i+6j+6k$, giving parametric equations $x=1+3t$, $y=6t$ and $z=-2+6t$. Setting $x=\dfrac{5}{2}$ gives $1+3t=\dfrac{5}{2}$, so $t=\dfrac{1}{2}$; the same $t=\dfrac{1}{2}$ gives $y=3$ and $z=1$, matching M. Hence M lies on the line at $t=\dfrac{1}{2}$ — as expected, since the midpoint is halfway from A ($t=0$) to B ($t=1$).

Q15. Parallel lines share a direction, so a direction vector for the new line is $d=4i-j+2k$, taken from the given line. Using the point $A(5,-2,1)$ with position vector $a=5i-2j+k$,

$$r=(5i-2j+k)+t(4i-j+2k).$$

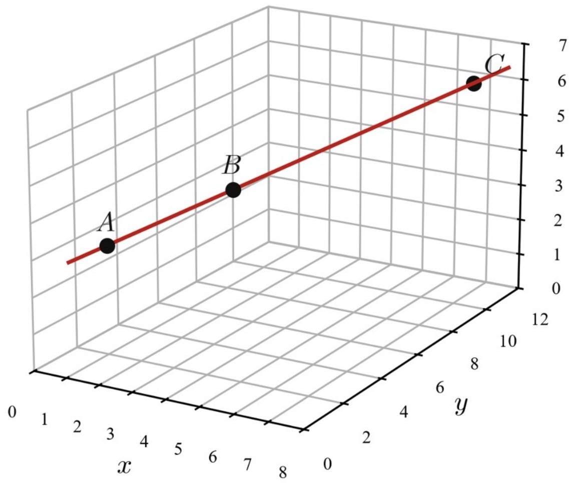
Q16. In the symmetric form $\dfrac{x-a_1}{d_1}=\dfrac{y-a_2}{d_2}=\dfrac{z-a_3}{d_3}$, the numerators give a point and the denominators give a direction vector. Comparing with $\dfrac{x-1}{2}=\dfrac{y+3}{-1}=\dfrac{z-4}{5}$, a point on the line is $(1,-3,4)$ and a direction vector is $d=2i-j+5k$. A vector equation is therefore

$$r=(i-3j+4k)+t(2i-j+5k).$$

Q17. Form two displacement vectors from A:

$$\overrightarrow{AB}=b-a=2i+3j+k,\overrightarrow{AC}=c-a=6i+9j+3k.$$

Since $\overrightarrow{AC}=6i+9j+3k=3(2i+3j+k)=3\overrightarrow{AB}$, the two displacements are parallel and share the point A. Hence A, B and C lie on a single line and are collinear. On the line $r=a+t\overrightarrow{AB}$, the point B occurs at $t=1$ and C at $t=3$.



Q18. Write each line in parametric form. The first, $r=j+t(i+j)$, gives $x=t$ and $y=1+t$. The second, $r=(4i+j)+s(-i+j)$, gives $x=4-s$ and $y=1+s$. At an intersection the coordinates match:

$$t=4-s,1+t=1+s.$$

The second equation gives $t=s$, and substituting into the first gives $s=4-s$, so $2s=4$ and $s=2$; hence $t=2$. Both lines then give $x=2$ and $y=3$, so the point of intersection is $(2,3)$.

Chapter 3 Vector equations of lines and planes — 3B Intersecting, parallel and skew lines

Theory

An earlier section wrote a straight line in space in vector form

$$r = a + s\,d\,,$$

where a is the position vector of a known point on the line, d is a direction vector, and the parameter s ranges over all real numbers. This section asks how **two** such lines are related. In the plane, two distinct straight lines either cross at a single point or are parallel. In three dimensions there is a third possibility: two lines may point in different directions and still never meet. Lines of this kind are called **skew**.

The three cases.

Given the two lines

$$\ell_1 : r = a_1 + s\,d_1\,,\, \ell_2 : r = a_2 + t\,d_2\,,$$

where the parameters s and t are independent (so different letters are used), exactly one of the following holds:

- **Intersecting** — the lines share exactly one point;
- **Parallel** — the direction vectors are scalar multiples of each other. If the lines also share a point they are the **same line** (coincident); otherwise they are parallel and distinct, with no point in common;
- **Skew** — the directions are not parallel, yet the lines have no common point. This can happen only in three dimensions.

The relationship is settled by two questions — are the directions parallel, and do the lines share a point?

directions parallel?	share a point?	relationship
yes	yes	same line (coincident)
yes	no	parallel and distinct
no	yes	intersecting
no	no	skew

Testing for parallel lines.

The directions d_1 and d_2 are parallel when one is a scalar multiple of the other, $d_2 = k\,d_1$; equivalently, their components are in a common ratio. When the directions are parallel, take the known point of one line and test whether it lies on the other by substituting and solving. If it does, every point is shared and the lines coincide; if not, the lines are parallel and distinct.

Finding a point of intersection.

When the directions are **not** parallel, look for values of s and t that give the same point on both lines by setting

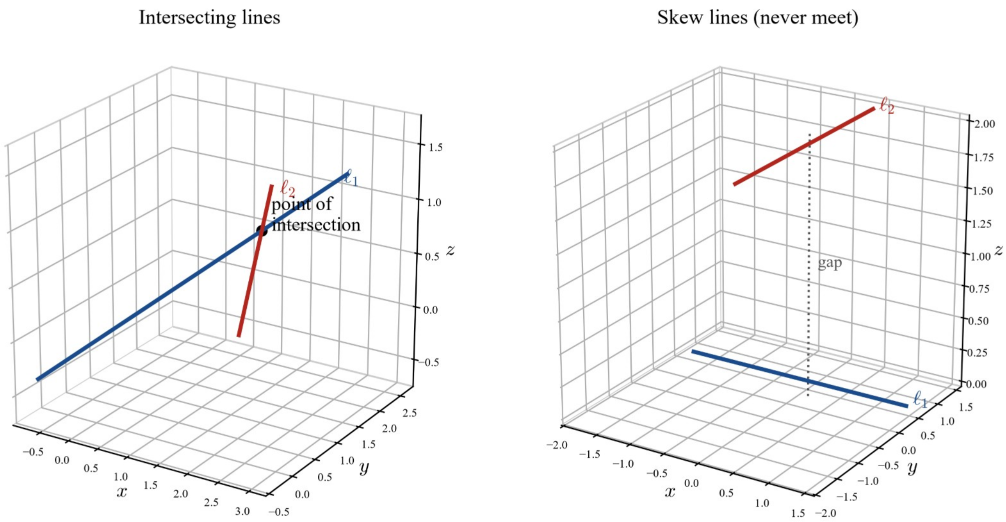
$$a_1 + s\,d_1 = a_2 + t\,d_2\,.$$

Equating the i, j and k components produces **three equations in the two unknowns** s and t. Solve any two of them for s and t, then substitute those values into the remaining equation:

- if the third equation is **satisfied**, the system is consistent, the lines **intersect**, and the point is found by putting s back into ℓ_1 (or t into ℓ_2);
- if the third equation is **not satisfied**, the system is inconsistent, no common point exists, and — since the lines are not parallel — they are **skew**.

Why skew lines need three dimensions.

In two dimensions, equating components gives only two equations in s and t; non-parallel directions make these always solvable, so two non-parallel lines in the plane must meet. Space adds a third component equation, and it is this extra equation that can fail — which is exactly what allows non-parallel lines to miss one another.



The angle between two lines.

Each line sets a direction, and the angle between two lines is measured between those directions. Because two lines meeting at a point actually form two angles that add to $180°$, we quote the **acute** (or right) angle. Using the dot product of the direction vectors,

$$\cos\theta = \frac{|d_1 \cdot d_2|}{|d_1||d_2|}, 0° \le \theta \le 90°.$$

The absolute value in the numerator keeps $\cos\theta \ge 0$, so θ comes out acute (or exactly $90°$). Two special cases follow immediately: the lines are **perpendicular** when $d_1 \cdot d_2 = 0$, and **parallel** when the directions are scalar multiples, giving $\theta = 0°$. Since the angle depends only on the directions, it is defined for skew lines just as for intersecting ones.

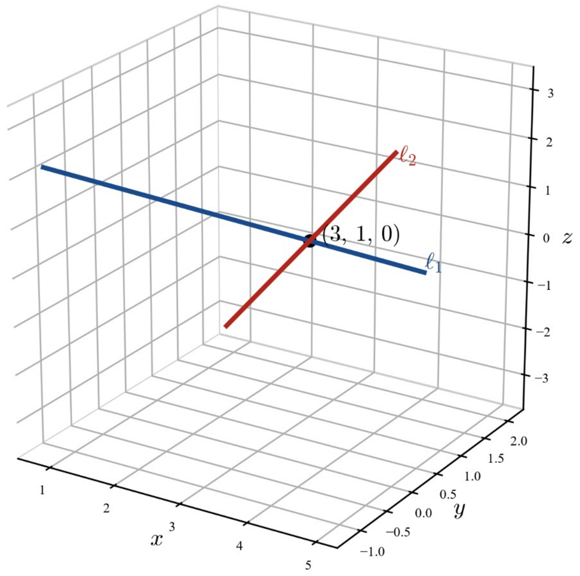
Worked examples

Example 1 — scaffolded

Show that the lines $\ell_1 : r = (i - j + 2k) + s(i + j - k)$ and $\ell_2 : r = (i + 2j - 3k) + t(2i - j + 3k)$ intersect, and find the point of intersection.

Working	Comment
$d_1 = i + j - k$ and $d_2 = 2i - j + 3k$ are not scalar multiples, so the lines are not parallel.	A common point may exist; test for one.
Set $a_1 + s d_1 = a_2 + t d_2$: $(1+s)i + (-1+s)j + (2-s)k = (1+2t)i + (2-t)j + (-$.	Equate the two position vectors.
Equate components: $i : s - 2t = 0; j : s + t = 3; k : s + 3t = 5.$	Three equations in the two unknowns s, t.
From i, $s = 2t$; substitute into j: $2t + t = 3$, so $t = 1$ and $s = 2$.	Solve two of the equations.
Check in k: $s + 3t = 2 + 3(1) = 5$, which is satisfied, so the lines intersect.	The third equation is consistent.

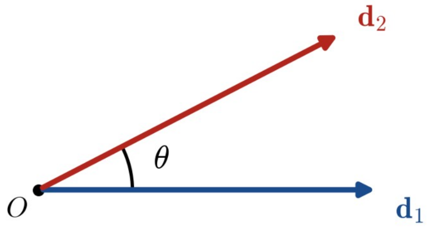
Working	Comment
Put $s=2$ into ℓ_1: $r=3i+j$, the point $(3,1,0)$.	Substitute back into either line.



Example 2 — scaffolded

Find the acute angle between the lines $\ell_1: r=(i+3j+2k)+s(2i+2j+k)$ and $\ell_2: r=(4i+j+2k)+t(i+2j+2k)$, to the nearest degree.

Working	Comment
$d_1=2i+2j+k$ and $d_2=i+2j+2k$.	The angle depends only on the directions.
$d_1\cdot d_2=(2)(1)+(2)(2)+(1)(2)=8$.	Dot product from components.
$\lvert d_1\rvert=\sqrt{2^2+2^2+1^2}=3$, $\lvert d_2\rvert=\sqrt{1^2+2^2+2^2}=3$.	Magnitude of each direction.
$\cos\theta=\dfrac{\lvert d_1\cdot d_2\rvert}{\lvert d_1\rvert\lvert d_2\rvert}=\dfrac{8}{3\times 3}=\dfrac{8}{9}$.	The absolute value gives the acute angle.
$\theta=\cos^{-1}\left(\dfrac{8}{9}\right)\approx 27°$.	The acute angle between the lines.



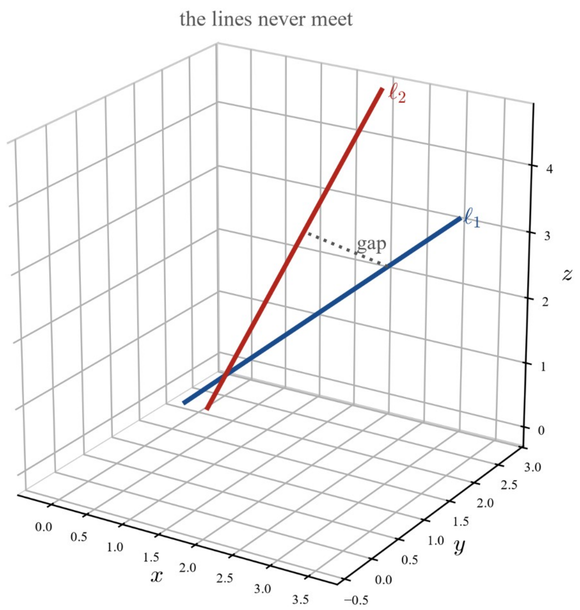
Example 3 — unscaffolded

Determine whether the lines $\ell_1: r=(i+2j+k)+s(i+k)$ and $\ell_2: r=(2i+j+3k)+t(j+k)$ are intersecting, parallel or skew.

The direction vectors $d_1 = i + k$ and $d_2 = j + k$ are not scalar multiples of each other, so the lines are not parallel; they are therefore either intersecting or skew. Setting the position vectors equal and comparing components,

$$i: 1 + s = 2, \; j: 2 = 1 + t, \; k: 1 + s = 3 + t.$$

The first equation gives $s = 1$ and the second gives $t = 1$. Substituting into the third, the left side is $1 + 1 = 2$ while the right side is $3 + 1 = 4$; since $2 \neq 4$, no pair (s, t) satisfies all three equations. The lines have no common point, and as they are not parallel they must be skew.



∴ the lines are skew.

Example 4 — unscaffolded

Classify the lines $\ell_1 : r = (i + 2j + 3k) + s(2i - j + 3k)$ and $\ell_2 : r = (4i + 5k) + t(4i - 2j + 6k)$.

The directions are $d_1 = 2i - j + 3k$ and $d_2 = 4i - 2j + 6k$. Since $d_2 = 2d_1$, the direction vectors are parallel, so the two lines are either the same line or parallel and distinct. To decide, test whether the point $(4, 0, 5)$ of ℓ_2 lies on ℓ_1 by requiring

$$i + 2j + 3k + s(2i - j + 3k) = 4i + 5k.$$

The i component gives $1 + 2s = 4$, so $s = 3/2$, while the j component gives $2 - s = 0$, so $s = 2$. These values disagree, so no single s places the point on ℓ_1.

∴ the lines are parallel and distinct (not the same line).

Practice questions

Scaffolded

Q1. Show that the lines $\ell_1 : r = (2i + j) + s(i + 2j + k)$ and $\ell_2 : r = (2i + 4j - k) + t(i - j + 2k)$ intersect, and find the point. (a) Write the component equations for x, y and z. (b) Using the x and y equations, solve for s and t. (c) Check that the z equation is satisfied. (d) State the point of intersection.

Q2. Two lines are given by $\ell_1 : r = (i + j + k) + s(i + 2j - k)$ and $\ell_2 : r = (3i + 5j - k) + t(2i + 4j - 2k)$. (a) Show that the direction vectors are parallel. (b) Test whether the point $(3, 5, -1)$ of ℓ_2 lies on ℓ_1. (c) Hence state whether the lines are parallel and distinct, or the same line.

Q3. Two lines are given by $\ell_1 : r = (i + j + 2k) + s(2i + j)$ and $\ell_2 : r = (i + 4j + k) + t(i + k)$. (a) Explain why the lines are not parallel. (b) From the y and z equations, find s and t. (c) Show that the x equation is not satisfied, and classify the lines.

Q4. Find the acute angle between the lines $\ell_1 : r = (i + k) + s(3i + 4j)$ and $\ell_2 : r = (2j + k) + t(-4j + 3k)$. (a) Write down d_1 and d_2 and find $d_1 \cdot d_2$. (b) Find $|d_1|$ and $|d_2|$. (c) Find $\cos\theta$ (using the absolute value) and hence θ to the nearest degree.

Q5. Two lines are given by $\ell_1 : r = (i + k) + s(i + j + k)$ and $\ell_2 : r = (i + 2j + 2k) + t(i - j)$. (a) Check that the directions are not parallel. (b) Set up the three component equations. (c) Solve two of them and check the third to classify the lines. (d) If they intersect, find the point.

Q6. Two lines are given by $\ell_1 : r = (i + j + k) + s(i + 2j + 2k)$ and $\ell_2 : r = (4i + j + 4k) + t(2i - 2j + k)$. (a) Show that the lines intersect, and find the point. (b) Find $d_1 \cdot d_2$ and hence the angle between the lines.

Unscaffolded

Q7. Show that the lines $\ell_1 : r = (2i - j + k) + s(i + j + 2k)$ and $\ell_2 : r = (i - j + 4k) + t(2i + j - k)$ intersect, and find the point of intersection.

Q8. Determine whether the lines $\ell_1 : r = i + s(i + 2j + k)$ and $\ell_2 : r = (j + 2k) + t(2i + j + k)$ are intersecting, parallel or skew.

Q9. Show that the lines $\ell_1 : r = (i + 2j) + s(3i - j + 2k)$ and $\ell_2 : r = (j + k) + t(6i - 2j + 4k)$ are parallel, and determine whether they are the same line.

Q10. Find the acute angle, to the nearest degree, between the lines $\ell_1 : r = (i + j + k) + s(i + 2j + 2k)$ and $\ell_2 : r = (3i - k) + t(2i - j - 2k)$.

Q11. Determine whether the lines $\ell_1 : r = (2i + j + 3k) + s(i - j + k)$ and $\ell_2 : r = (i + 2k) + t(2i + j - k)$ are intersecting, parallel or skew.

Q12. The lines $\ell_1 : r = (i + j + 2k) + s(2i + j + k)$ and $\ell_2 : r = (4i + aj + 2k) + t(i + 2j - k)$ intersect. Find the value of a and the point of intersection.

Q13. Find the exact acute angle between the lines $\ell_1 : r = (i + k) + s(i - j + 2k)$ and $\ell_2 : r = 2j + t(2i + j + k)$.

Q14. Show that the lines $\ell_1 : r = (2i + j + 3k) + s(i - 2j + k)$ and $\ell_2 : r = (4i - 3j + 5k) + t(2i - 4j + 2k)$ are the same line.

Q15. The lines $\ell_1 : r = (i + 2j + k) + s(i + k)$ and $\ell_2 : r = (2i + j + k) + t(j + k)$ intersect. Find the point of intersection and the acute angle between the lines.

Q16. Show that the lines $\ell_1 : r = (i + j) + s(2i + j + 2k)$ and $\ell_2 : r = 3k + t(i + 2j - 2k)$ are perpendicular.

Q17. Show that the lines $\ell_1 : r = (3i + j + 5k) + s(i - j + 2k)$ and $\ell_2 : r = (-i + j + 4k) + t(3i + j - k)$ intersect, and find the point.

Q18. The lines $\ell_1 : r = (i + j + k) + s(i + 2j + k)$ and $\ell_2 : r = (pi + 5j) + t(2i - j + k)$ intersect. Find the value of p and the point of intersection.

Answers

Q1. $s=1$, $t=1$; the lines intersect at $(3,3,1)$. **Q2.** $d_2=2d_1$ and $(3,5,-1)$ lies on ℓ_1 (at $s=2$); the lines are the same line (coincident). **Q3.** Not parallel; $s=3$ and $t=1$ satisfy the y and z equations but not the x equation, so the lines are skew. **Q4.** $d_1 \cdot d_2=-16$, $|d_1|=|d_2|=5$, $\cos\theta=\frac{16}{25}$; $\theta\approx 50°$. **Q5.** Not parallel; $s=1$, $t=1$ are consistent, so the lines intersect at $(2,1,2)$. **Q6.** Intersect at $(2,3,3)$; $d_1 \cdot d_2=0$, so $\theta=90°$ (perpendicular). **Q7.** Intersect at $(3,0,3)$ (at $s=1$, $t=1$). **Q8.** Skew. **Q9.** $d_2=2d_1$ (parallel) and $(0,1,1)$ does not lie on ℓ_1; parallel and distinct. **Q10.** $\cos\theta=\frac{4}{9}$; $\theta\approx 64°$. **Q11.** Skew. **Q12.** $a=4$; the lines meet at $(3,2,3)$. **Q13.** $\cos\theta=\frac{1}{2}$, so $\theta=60°$. **Q14.** $d_2=2d_1$ and $(4,-3,5)$ lies on ℓ_1 (at $s=2$); the lines are the same line. **Q15.** Intersect at $(2,2,2)$; $\theta=60°$. **Q16.** $d_1 \cdot d_2=0$, so the lines are perpendicular ($\theta=90°$). **Q17.** Intersect at $(2,2,3)$ (at $s=-1$, $t=1$). **Q18.** $p=-2$; the lines meet at $(2,3,2)$.

Fully-worked solutions

Q1. The component equations are, from ℓ_1, $x=2+s$, $y=1+2s$, $z=s$, and from ℓ_2, $x=2+t$, $y=4-t$, $z=-1+2t$. Equating,

$$x: 2+s=2+t \Rightarrow s=t;\ y: 1+2s=4-t \Rightarrow 2s+t=3;\ z: s=-1+2t \Rightarrow s-2t=-1.$$

Using $s=t$ in $2s+t=3$ gives $3s=3$, so $s=t=1$. The z equation holds, since $s-2t=1-2=-1$, so the lines intersect. Putting $s=1$ into ℓ_1, $r=3i+3j+k$, the point $(3,3,1)$.

Q2. The directions are $d_1=i+2j-k$ and $d_2=2i+4j-2k=2d_1$, so the lines are parallel or identical. Test the point $(3,5,-1)$ of ℓ_2 on ℓ_1: the x component $1+s=3$ gives $s=2$; then $1+2(2)=5$ matches the y-coordinate and $1-2=-1$ matches the z-coordinate. As $s=2$ reproduces the point in every component, $(3,5,-1)$ lies on ℓ_1, so the two lines are the **same line**.

Q3. The directions $d_1=2i+j$ and $d_2=i+k$ are not scalar multiples, so the lines are not parallel. Comparing components, $y: 1+s=4$ gives $s=3$, and $z: 2=1+t$ gives $t=1$. The x equation requires $1+2s=1+t$, that is $2s-t=0$; but $2(3)-1=5\neq 0$. No pair (s,t) satisfies all three equations, so the lines do not meet; being non-parallel, they are **skew**.

Q4. With $d_1=3i+4j$ and $d_2=-4j+3k$, the dot product is $d_1 \cdot d_2=(3)(0)+(4)(-4)+(0)(3)=-16$, and the magnitudes are $|d_1|=\sqrt{9+16}=5$ and $|d_2|=\sqrt{16+9}=5$. Taking the absolute value for the acute angle,

$$\cos\theta=\frac{|-16|}{5\times 5}=\frac{16}{25}=0.64,$$

so $\theta=\cos^{-1}(0.64)\approx 50°$.

Q5. The directions $d_1=i+j+k$ and $d_2=i-j$ are not scalar multiples, so the lines are not parallel. Equating components,

$$x: 1+s=1+t \Rightarrow s=t;\ y: s=2-t \Rightarrow s+t=2;\ z: 1+s=2 \Rightarrow s=1.$$

From z, $s=1$; then $s=t$ gives $t=1$, and the y equation holds since $1+1=2$. The lines intersect; putting $s=1$ into ℓ_1, $r=2i+j+2k$, the point $(2,1,2)$.

Q6. (a) The directions $d_1=i+2j+2k$ and $d_2=2i-2j+k$ are not parallel. Equating components,

$$x: 1+s=4+2t \Rightarrow s-2t=3;\ y: 1+2s=1-2t \Rightarrow s+t=0;\ z: 1+2s=4+t \Rightarrow 2s-t=3.$$

From y, $s=-t$; substituting into x, $-t-2t=3$, so $t=-1$ and $s=1$. The z equation holds, since $2(1)-(-1)=3$. The lines meet; with $s=1$ in ℓ_1, $r=2i+3j+3k$, the point $(2,3,3)$. (b) $d_1 \cdot d_2=(1)(2)+(2)(-2)+(2)(1)=0$, so the directions are perpendicular and $\theta=90°$.

Q7. The directions $i+j+2k$ and $2i+j-k$ are not parallel. Equating components,

$$x:2+s=1+2t \Rightarrow s-2t=-1;\ y:-1+s=-1+t \Rightarrow s=t;\ z:1+2s=4-t \Rightarrow 2s+t=3.$$

From y, $s=t$; then x gives $t-2t=-1$, so $t=1$ and $s=1$. The z equation holds, since $2(1)+1=3$, so the lines intersect. With $s=1$ in ℓ_1, $r=3i+3k$, the point $(3,0,3)$.

Q8. The directions $d_1=i+2j+k$ and $d_2=2i+j+k$ are not scalar multiples, so the lines are not parallel. Equating components,

$$x:1+s=2t \Rightarrow s-2t=-1;\ y:2s=1+t \Rightarrow 2s-t=1;\ z:s=2+t \Rightarrow s-t=2.$$

From the x and y equations, $s=2t-1$ gives $2(2t-1)-t=1$, so $3t=3$, $t=1$ and $s=1$. Substituting into the z equation, $s-t=1-1=0$, but it requires $s-t=2$; since $0\neq 2$ the system is inconsistent. The lines have no common point and are not parallel, so they are **skew**.

Q9. The directions $d_1=3i-j+2k$ and $d_2=6i-2j+4k=2d_1$ are parallel, so the lines are parallel or identical. Test the point $(0,1,1)$ of ℓ_2 on ℓ_1: the x component $1+3s=0$ gives $s=-1/3$, while the y component $2-s=1$ gives $s=1$. These disagree, so the point does not lie on ℓ_1; the lines are **parallel and distinct**.

Q10. With $d_1=i+2j+2k$ and $d_2=2i-j-2k$,

$$d_1 \cdot d_2=(1)(2)+(2)(-1)+(2)(-2)=2-2-4=-4,$$

and $|d_1|=\sqrt{1+4+4}=3$, $|d_2|=\sqrt{4+1+4}=3$. Then $\cos\theta=\dfrac{|-4|}{3\times 3}=\dfrac{4}{9}\approx 0.4444$, giving $\theta\approx 64°$.

Q11. The directions $d_1=i-j+k$ and $d_2=2i+j-k$ are not scalar multiples, so the lines are not parallel. Equating components,

$$x:2+s=1+2t \Rightarrow s-2t=-1;\ y:1-s=t \Rightarrow s+t=1;\ z:3+s=2-t \Rightarrow s+t=-1.$$

The y and z equations demand $s+t=1$ and $s+t=-1$ simultaneously, which is impossible. The system is inconsistent, so the lines have no common point; being non-parallel, they are **skew**.

Q12. The directions $d_1=2i+j+k$ and $d_2=i+2j-k$ are not parallel. Two of the component equations do not involve a:

$$x:1+2s=4+t \Rightarrow 2s-t=3;\ z:2+s=2-t \Rightarrow s+t=0.$$

Adding, $3s=3$, so $s=1$ and $t=-1$. The y equation then fixes a: $1+s=a+2t$ gives $1+1=a+2(-1)$, so $a=4$. The point of intersection, from ℓ_1 at $s=1$, is $r=3i+2j+3k$, that is $(3,2,3)$.

Q13. With $d_1=i-j+2k$ and $d_2=2i+j+k$,

$$d_1 \cdot d_2=(1)(2)+(-1)(1)+(2)(1)=3,$$

and $|d_1|=\sqrt{1+1+4}=\sqrt{6}$, $|d_2|=\sqrt{4+1+1}=\sqrt{6}$. Then $\cos\theta=\dfrac{|3|}{\sqrt{6}\sqrt{6}}=\dfrac{3}{6}=\dfrac{1}{2}$, so $\theta=60°$.

Q14. The directions $d_1=i-2j+k$ and $d_2=2i-4j+2k=2d_1$ are parallel, so the lines are parallel or identical. Test $(4,-3,5)$ on ℓ_1: the x component $2+s=4$ gives $s=2$; then $1-2(2)=-3$ matches the y-coordinate and $3+2=5$ matches the z-coordinate. Every component agrees at $s=2$, so $(4,-3,5)$ lies on ℓ_1 and the two lines are the **same line**.

Q15. The directions $d_1 = i + k$ and $d_2 = j + k$ are not parallel. Equating components,

$$x : 1 + s = 2 \Rightarrow s = 1 ; \; y : 2 = 1 + t \Rightarrow t = 1 ; \; z : 1 + s = 1 + t \Rightarrow s = t ,$$

which holds since $s = t = 1$. The lines intersect; with $s = 1$ in ℓ_1, $r = 2i + 2j + 2k$, the point $(2, 2, 2)$. For the angle, $d_1 \cdot d_2 = (1)(0) + (0)(1) + (1)(1) = 1$ and $|d_1| = |d_2| = \sqrt{2}$, so $\cos\theta = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$ and $\theta = 60°$.

Q16. The directions are $d_1 = 2i + j + 2k$ and $d_2 = i + 2j - 2k$. Their dot product is

$$d_1 \cdot d_2 = (2)(1) + (1)(2) + (2)(-2) = 2 + 2 - 4 = 0 .$$

A zero dot product makes $\cos\theta = 0$, so $\theta = 90°$ and the lines are perpendicular.

Q17. The directions $d_1 = i - j + 2k$ and $d_2 = 3i + j - k$ are not parallel. Equating components,

$$x : 3 + s = -1 + 3t \Rightarrow s - 3t = -4 ; \; y : 1 - s = 1 + t \Rightarrow s + t = 0 ; \; z : 5 + 2s = 4 - t \Rightarrow 2s + t = -1 .$$

From y, $s = -t$; substituting into x, $-t - 3t = -4$, so $t = 1$ and $s = -1$. The z equation holds, since $2(-1) + 1 = -1$. The lines intersect; with $s = -1$ in ℓ_1, $r = 2i + 2j + 3k$, the point $(2, 2, 3)$.

Q18. The directions $d_1 = i + 2j + k$ and $d_2 = 2i - j + k$ are not parallel. The two component equations that do not involve p are

$$y : 1 + 2s = 5 - t \Rightarrow 2s + t = 4 ; \; z : 1 + s = t \Rightarrow s - t = -1 .$$

Adding, $3s = 3$, so $s = 1$ and $t = 2$. The x equation then fixes p: $1 + s = p + 2t$ gives $1 + 1 = p + 2(2)$, so $p = -2$. The point of intersection, from ℓ_1 at $s = 1$, is $r = 2i + 3j + 2k$, that is $(2, 3, 2)$.

Theory

The vector operations met so far combine two vectors in ways that either produce another vector — addition, subtraction, multiplication by a scalar — or, in the case of the scalar (dot) product from an earlier chapter, produce a single number. This section introduces a different way of combining two **three-dimensional** vectors that produces a **new vector**: the **vector product**, almost always called the **cross product** because of the symbol $a \times b$ written between the two vectors. Its defining feature is direction — $a \times b$ is perpendicular to a and to b at the same time — which makes it the natural tool for finding a vector at right angles to two given vectors and for measuring the area of the figure they span. Unlike the dot product, the cross product is defined only for vectors in three dimensions.

Determinant form.

If $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, the cross product is defined by the symbolic determinant

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

with the components of a on the middle row and those of b on the bottom row. It is expanded along the top row, each entry multiplying the 2×2 determinant left after deleting its own row and column, where $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr$. The middle term carries a leading minus sign.

Component formula.

Expanding the determinant gives the component formula

$$a \times b = (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k.$$

In practice it is quickest to work from the determinant: to find the i-component cover the i-column and cross-multiply the block that remains, $a_2 b_3 - a_3 b_2$; do the same under the (negated) j-column and the k-column. The result is a **vector**, not a number — this is the first contrast with the dot product.

The standard unit vectors.

Because i, j, k are a right-handed set of mutually perpendicular unit vectors, the determinant gives the cyclic pattern

$$i \times j = k, \; j \times k = i, \; k \times i = j,$$

and reversing any pair reverses the sign,

$$j \times i = -k, \; k \times j = -i, \; i \times k = -j.$$

A unit vector crossed with itself gives the zero vector: $i \times i = j \times j = k \times k = 0$. Reading the cyclic order $i \to j \to k \to i$ forwards gives the positive results; going backwards introduces the minus sign.

Properties.

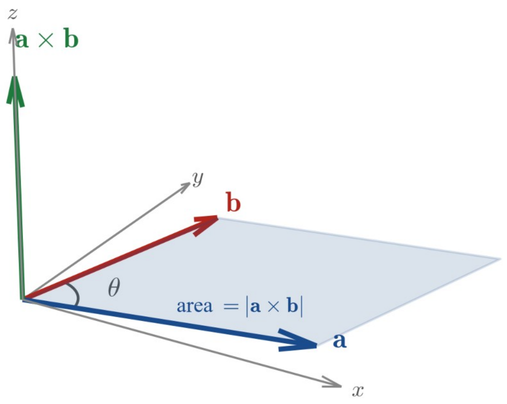
For all vectors a, b, c and every scalar k:

- **Anticommutative:** $b \times a = -(a \times b)$. The order matters — swapping the vectors reverses the result.
- **A vector with itself:** $a \times a = 0$, and $a \times 0 = 0$.
- **Distributive over addition:** $a \times (b + c) = a \times b + a \times c$.
- **Scalar factors:** $(ka) \times b = k(a \times b) = a \times (kb)$.

The cross product is **not** commutative, so care is needed to keep the vectors in the given order.

Perpendicularity and the right-hand rule.

The key geometric fact is that $a \times b$ is **perpendicular to both** a and b, and hence perpendicular (normal) to the whole plane containing them. Of the two opposite directions perpendicular to that plane, the one taken by $a \times b$ is fixed by the **right-hand rule**: point the fingers of the right hand along a and curl them towards b; the extended thumb then points along $a \times b$. Since $b \times a = -(a \times b)$, reversing the order points the result the opposite way. Perpendicularity is easily checked with the dot product: $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$.

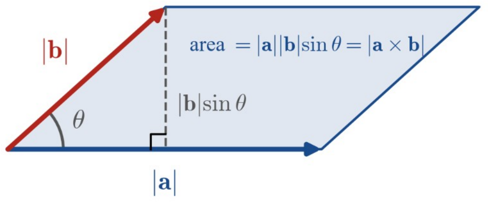


Magnitude, and the area of a parallelogram.

The length of the cross product is

$$|a \times b| = |a||b| \sin\theta,$$

where θ is the angle between a and b, taken with $0° \le \theta \le 180°$ so that $\sin\theta \ge 0$. Geometrically $|a||b|\sin\theta$ is exactly the **area of the parallelogram** with a and b as adjacent sides: $|a|$ is the base and $|b|\sin\theta$ the perpendicular height.



The triangle with a and b as two of its sides is half of that parallelogram, so

$$\text{area of parallelogram} = |a \times b|, \text{area of triangle} = 1/2|a \times b|.$$

Two special cases are worth noting. If a and b are **parallel** then $\theta = 0°$ or $180°$, so $\sin\theta = 0$ and $a \times b = 0$ — a zero cross product is the test for parallel vectors, exactly as a zero dot product tests for perpendicular ones. If a and b are perpendicular then $\theta = 90°$ and $|a \times b| = |a||b|$. The dot and cross products are complementary:

	scalar (dot) product $a \cdot b$	vector (cross) product $a \times b$
result	a scalar	a vector
size	$\|a\|\|b\|\cos\theta$	magnitude $\|a\|\|b\|\sin\theta$
equals zero when	$a \perp b$	$a \parallel b$

Finding a vector perpendicular to two vectors.

Because $a \times b$ is perpendicular to both a and b, computing the cross product is the standard way to produce a vector at right angles to two given non-parallel vectors. For example, a vector perpendicular to the plane through three points A, B, C is found as $\overrightarrow{AB} \times \overrightarrow{AC}$. To obtain a **unit** vector perpendicular to both, divide by the magnitude,

$$\pm \frac{a \times b}{|a \times b|},$$

the two signs giving the two opposite perpendicular directions. These uses of the cross product — the normal to a plane, and the area of a triangle or parallelogram — are developed further in the sections on planes; here the focus is the cross product itself, its determinant and component forms, and its magnitude.

Worked examples

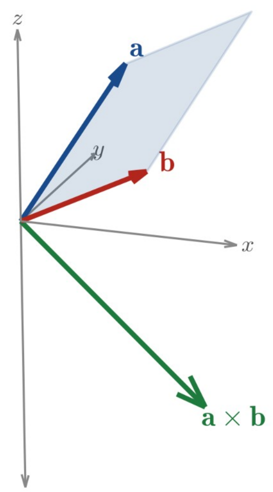
Example 1 — scaffolded

Given $a = 2i + 3j + k$ and $b = i - j + 2k$, find $a \times b$ using the determinant form, and verify that the result is perpendicular to a.

Working	Comment
$a \times b = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix}$	Components of a on the middle row, b on the bottom row.
i-component: $(3)(2) - (1)(-1) = 6 + 1 = 7$	Cover the i-column and cross-multiply the block that remains.
j-component: $-[(2)(2) - (1)(1)] = -(4 - 1) = -3$	Cover the j-column; keep the leading minus sign.
k-component: $(2)(-1) - (3)(1) = -2 - 3 = -5$	Cover the k-column.
$a \times b = 7i - 3j - 5k$	Collect the three components; the result is a vector.
Check: $a \cdot (a \times b) = (2)(7) + (3)(-3) + (1)(-5) = 14 - 9 - 5 =$	A zero dot product confirms $a \times b \perp a$.

Example 2 — scaffolded

Find the area of the parallelogram with adjacent sides $a = i + 2j + 2k$ and $b = 2i + k$, and hence the area of the triangle having a and b as two of its sides.



Working	Comment
$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix}$	Here $b = 2i + 0j + k$.
i-component: $(2)(1) - (2)(0) = 2$	Cover the i-column.
j-component: $-[(1)(1) - (2)(2)] = -(1-4) = 3$	Cover the j-column; the minus sign turns -3 into $+3$.
k-component: $(1)(0) - (2)(2) = -4$	Cover the k-column.
$a \times b = 2i + 3j - 4k$	Collect the components.
$\lvert a \times b \rvert = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{4+9+16} = \sqrt{29}$	Area of the parallelogram $= \lvert a \times b \rvert$.
Triangle area $= \frac{1}{2}\sqrt{29}$	The triangle is half of the parallelogram.

Example 3 — unscaffolded

Find a unit vector perpendicular to both $a = i - j + 2k$ and $b = 3i + j - k$.

The cross product is perpendicular to both vectors, so compute it first:

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = [(-1)(-1) - (2)(1)]i - [(1)(-1) - (2)(3)]j + [(1)(1) - (-1)(3)]k.$$

Evaluating each bracket gives $(1-2)i - (-1-6)j + (1+3)k = -i + 7j + 4k$. Its magnitude is

$$\lvert a \times b \rvert = \sqrt{(-1)^2 + 7^2 + 4^2} = \sqrt{1+49+16} = \sqrt{66}.$$

As a check, $a \cdot (a \times b) = (1)(-1) + (-1)(7) + (2)(4) = -1 - 7 + 8 = 0$, confirming the result is perpendicular to a. Dividing by the magnitude gives a unit vector.

$\therefore$ a unit vector perpendicular to both is $\frac{1}{\sqrt{66}}(-i + 7j + 4k)$, or its negative.

Example 4 — unscaffolded

The vectors a and b have $\lvert a \rvert = 6$, $\lvert b \rvert = 4$, and the angle between them is $30°$. Find (a) $\lvert a \times b \rvert$ and (b) the area of the triangle with a and b as two of its sides, and state the value of $\lvert b \times a \rvert$.

Here the components are unknown, so use the magnitude formula $\lvert a \times b \rvert = \lvert a \rvert \lvert b \rvert \sin\theta$. With $\theta = 30°$ and $\sin 30° = \frac{1}{2}$

,

$$\lvert a \times b \rvert = 6 \times 4 \times \frac{1}{2} = 12.$$

The triangle is half the parallelogram of area $\lvert a \times b \rvert$, so its area is $\frac{1}{2} \times 12 = 6$. Finally, since $b \times a = -(a \times b)$ the two products have the same length, so $\lvert b \times a \rvert = 12$.

$\therefore \lvert a \times b \rvert = 12$, the triangle has area 6, and $\lvert b \times a \rvert = 12$.

Practice questions

Scaffolded

Q1. Given $a = i + 2j + 3k$ and $b = 2i + j - k$, find $a \times b$ using the determinant form. (a) Find the i-component. (b) Find the j-component (remember the leading minus sign). (c) Find the k-component. (d) Write $a \times b$ in component form.

Q2. Let $a = 2i + j - k$ and $b = i + 3j + 2k$. (a) Find $a \times b$. (b) Show that $a \cdot (a \times b) = 0$. (c) Show that $b \cdot (a \times b) = 0$, and state what parts (b) and (c) tell you about $a \times b$.

Q3. Find the area of the parallelogram with adjacent sides $a = 2i - j + 3k$ and $b = i + 2j + k$. (a) Find $a \times b$. (b) Find $|a \times b|$. (c) State the area of the parallelogram. (d) State the area of the triangle with a and b as two sides.

Q4. The points $A(1, 0, 1)$, $B(3, 1, 2)$ and $C(2, 3, 0)$ are the vertices of a triangle. (a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (c) Find $|\overrightarrow{AB} \times \overrightarrow{AC}|$. (d) Hence find the area of triangle ABC.

Q5. Find a unit vector perpendicular to both $a = 2i + j + k$ and $b = i - j + 3k$. (a) Find $a \times b$. (b) Find $|a \times b|$. (c) Write down a unit vector perpendicular to both a and b.

Q6. The vectors a and b have $|a| = 5$, $|b| = 8$ and the angle between them is $45°$. (a) Write down the formula for $|a \times b|$. (b) Find $|a \times b|$. (c) Find the area of the triangle having a and b as two of its sides.

Unscaffolded

Q7. Given $a = 3i - j + 2k$ and $b = 2i + 4j - k$, find $a \times b$.

Q8. For $a = i - 2k$ and $b = 3i + 2j + k$, find $a \times b$ and $b \times a$, and state how the two results are related.

Q9. Evaluate $(i + 2j) \times (3j + k)$.

Q10. Find the area of the parallelogram with adjacent sides $a = 4i - k$ and $b = i + 2j + 3k$.

Q11. Find the area of the triangle with vertices $A(2, 1, 0)$, $B(4, 2, 1)$ and $C(1, 3, 2)$.

Q12. Find a vector perpendicular to both $a = i + j - k$ and $b = 2i - j + k$, and simplify it as far as possible.

Q13. Find a unit vector perpendicular to both $a = i + 2j + 2k$ and $b = 2i + 2j + k$.

Q14. The vectors a and b have $|a| = 3$, $|b| = 10$ and the angle between them is $60°$. Find $|a \times b|$.

Q15. Show that $a = 2i - 4j + 6k$ and $b = 3i - 6j + 9k$ are parallel.

Q16. Find the value of m for which $a = 2i + mj + 3k$ and $b = 4i + 2j + 6k$ are parallel.

Q17. The points $A(1, 1, 0)$, $B(2, 0, 3)$ and $C(0, 2, 1)$ lie in a plane. Find a vector perpendicular to this plane.

Q18. The vectors a and b have $|a| = 4$, $|b| = 5$ and $|a \times b| = 10$. Find the acute angle between a and b.

Answers

Q1. $a\times b=-5i+7j-3k$. **Q2.** $a\times b=5i-5j+5k$; $a\cdot(a\times b)=0$ and $b\cdot(a\times b)=0$, so $a\times b$ is perpendicular to both a and b. **Q3.** $a\times b=-7i+j+5k$; $|a\times b|=5\sqrt{3}$; parallelogram area $5\sqrt{3}$; triangle area $\frac{5\sqrt{3}}{2}$. **Q4.** $\overrightarrow{AB}=2i+j+k$, $\overrightarrow{AC}=i+3j-k$; $\overrightarrow{AB}\times\overrightarrow{AC}=-4i+3j+5k$; $|\overrightarrow{AB}\times\overrightarrow{AC}|=5\sqrt{2}$; area of triangle $=\frac{5\sqrt{2}}{2}$. **Q5.** $a\times b=4i-5j-3k$, $|a\times b|=5\sqrt{2}$; unit vector $\pm\frac{1}{5\sqrt{2}}(4i-5j-3k)$. **Q6.** $|a\times b|=20\sqrt{2}$; triangle area $10\sqrt{2}$. **Q7.** $a\times b=-7i+7j+14k$. **Q8.** $a\times b=4i-7j+2k$, $b\times a=-4i+7j-2k$; they are negatives, $b\times a=-(a\times b)$. **Q9.** $2i-j+3k$. **Q10.** $|a\times b|=\sqrt{237}$; area $\sqrt{237}$. **Q11.** $\frac{5\sqrt{2}}{2}$. **Q12.** $a\times b=-3j-3k=-3(j+k)$; a simplest perpendicular vector is $j+k$. **Q13.** $\pm\frac{1}{\sqrt{17}}(-2i+3j-2k)$. **Q14.** $15\sqrt{3}$. **Q15.** $a\times b=0$ (indeed $b=3/2\,a$), so a and b are parallel. **Q16.** $m=1$. **Q17.** $-4i-4j$ (any non-zero multiple of $i+j$). **Q18.** $30°$.

Fully-worked solutions

Q1. Setting $a=i+2j+3k$ on the middle row and $b=2i+j-k$ on the bottom row,

$$a\times b=\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 1 & -1\end{vmatrix}.$$

The i-component is $(2)(-1)-(3)(1)=-2-3=-5$; the j-component is $-[(1)(-1)-(3)(2)]=-(-1-6)=7$; the k-component is $(1)(1)-(2)(2)=1-4=-3$. Hence $a\times b=-5i+7j-3k$.

Q2. With $a=2i+j-k$ and $b=i+3j+2k$, the i-component is $(1)(2)-(-1)(3)=2+3=5$; the j-component is $-[(2)(2)-(-1)(1)]=-(4+1)=-5$; the k-component is $(2)(3)-(1)(1)=6-1=5$. So $a\times b=5i-5j+5k$. Then $a\cdot(a\times b)=(2)(5)+(1)(-5)+(-1)(5)=10-5-5=0$ and $b\cdot(a\times b)=(1)(5)+(3)(-5)+(2)(5)=5-15+10=0$. Both dot products are zero, which shows that $a\times b$ is perpendicular to both a and b.

Q3. With $a=2i-j+3k$ and $b=i+2j+k$, the i-component is $(-1)(1)-(3)(2)=-1-6=-7$; the j-component is $-[(2)(1)-(3)(1)]=-(2-3)=1$; the k-component is $(2)(2)-(-1)(1)=4+1=5$. So $a\times b=-7i+j+5k$ and

$$|a\times b|=\sqrt{(-7)^2+1^2+5^2}=\sqrt{49+1+25}=\sqrt{75}=5\sqrt{3}.$$

The parallelogram area equals $|a\times b|=5\sqrt{3}$, and the triangle area is half of this, $\frac{5\sqrt{3}}{2}$.

Q4. The side vectors from A are $\overrightarrow{AB}=B-A=2i+j+k$ and $\overrightarrow{AC}=C-A=i+3j-k$. Their cross product has i-component $(1)(-1)-(1)(3)=-4$, j-component $-[(2)(-1)-(1)(1)]=-(-3)=3$, and k-component $(2)(3)-(1)(1)=5$, so $\overrightarrow{AB}\times\overrightarrow{AC}=-4i+3j+5k$. Then

$$|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{(-4)^2+3^2+5^2}=\sqrt{16+9+25}=\sqrt{50}=5\sqrt{2}.$$

The area of a triangle is half the cross-product magnitude, so triangle ABC has area $\frac{1}{2}\times5\sqrt{2}=\frac{5\sqrt{2}}{2}$.

Q5. With $a=2i+j+k$ and $b=i-j+3k$, the i-component is $(1)(3)-(1)(-1)=3+1=4$; the j-component is $-[(2)(3)-(1)(1)]=-(6-1)=-5$; the k-component is $(2)(-1)-(1)(1)=-2-1=-3$. So $a\times b=4i-5j-3k$ with

$$|a\times b|=\sqrt{4^2+(-5)^2+(-3)^2}=\sqrt{16+25+9}=\sqrt{50}=5\sqrt{2}.$$

Dividing by the magnitude gives a unit vector perpendicular to both, $\pm\frac{1}{5\sqrt{2}}(4i-5j-3k)$.

Q6. Using $|a\times b|=|a||b|\sin\theta$ with $\sin 45°=\frac{\sqrt{2}}{2}$,

$$|a\times b|=5\times 8\times\frac{\sqrt{2}}{2}=20\sqrt{2}.$$

The triangle has half this area, $\frac{1}{2}\times 20\sqrt{2}=10\sqrt{2}$.

Q7. With $a=3i-j+2k$ and $b=2i+4j-k$, the i-component is $(-1)(-1)-(2)(4)=1-8=-7$; the j-component is $-[(3)(-1)-(2)(2)]=-(-3-4)=7$; the k-component is $(3)(4)-(-1)(2)=12+2=14$. Hence $a\times b=-7i+7j+14k$.

Q8. Writing $a=i+0j-2k$ and $b=3i+2j+k$, the cross product has i-component $(0)(1)-(-2)(2)=4$, j-component $-[(1)(1)-(-2)(3)]=-(1+6)=-7$, and k-component $(1)(2)-(0)(3)=2$, so $a\times b=4i-7j+2k$. Reversing the order reverses every sign, giving $b\times a=-4i+7j-2k$. The two results are negatives of each other: $b\times a=-(a\times b)$, illustrating that the cross product is anticommutative.

Q9. Write $u=i+2j=i+2j+0k$ and $v=3j+k=0i+3j+k$. Then the i-component is $(2)(1)-(0)(3)=2$, the j-component is $-[(1)(1)-(0)(0)]=-1$, and the k-component is $(1)(3)-(2)(0)=3$, so $(i+2j)\times(3j+k)=2i-j+3k$.

Q10. With $a=4i+0j-k$ and $b=i+2j+3k$, the i-component is $(0)(3)-(-1)(2)=2$; the j-component is $-[(4)(3)-(-1)(1)]=-(12+1)=-13$; the k-component is $(4)(2)-(0)(1)=8$. So $a\times b=2i-13j+8k$ and the parallelogram area is

$$|a\times b|=\sqrt{2^2+(-13)^2+8^2}=\sqrt{4+169+64}=\sqrt{237}.$$

Q11. The side vectors from A are $\overrightarrow{AB}=B-A=2i+j+k$ and $\overrightarrow{AC}=C-A=-i+2j+2k$. Their cross product has i-component $(1)(2)-(1)(2)=0$, j-component $-[(2)(2)-(1)(-1)]=-(4+1)=-5$, and k-component $(2)(2)-(1)(-1)=4+1=5$, so $\overrightarrow{AB}\times\overrightarrow{AC}=-5j+5k$ with magnitude $\sqrt{0^2+(-5)^2+5^2}=\sqrt{50}=5\sqrt{2}$. The area of the triangle is half of this, $\frac{5\sqrt{2}}{2}$.

Q12. With $a=i+j-k$ and $b=2i-j+k$, the i-component is $(1)(1)-(-1)(-1)=1-1=0$; the j-component is $-[(1)(1)-(-1)(2)]=-(1+2)=-3$; the k-component is $(1)(-1)-(1)(2)=-1-2=-3$. So $a\times b=-3j-3k=-3(j+k)$. Any non-zero multiple of this is also perpendicular to both, so a simplest perpendicular vector is $j+k$.

Q13. With $a=i+2j+2k$ and $b=2i+2j+k$, the i-component is $(2)(1)-(2)(2)=2-4=-2$; the j-component is $-[(1)(1)-(2)(2)]=-(1-4)=3$; the k-component is $(1)(2)-(2)(2)=2-4=-2$. So $a\times b=-2i+3j-2k$ with $|a\times b|=\sqrt{(-2)^2+3^2+(-2)^2}=\sqrt{4+9+4}=\sqrt{17}$. A unit vector perpendicular to both is therefore $\pm\frac{1}{\sqrt{17}}(-2i+3j-2k)$.

Q14. Using $|a\times b|=|a||b|\sin\theta$ with $\sin 60°=\frac{\sqrt{3}}{2}$,

$$|a \times b| = 3 \times 10 \times \frac{\sqrt{3}}{2} = 15\sqrt{3}.$$

Q15. With $a = 2i - 4j + 6k$ and $b = 3i - 6j + 9k$, the i-component is $(-4)(9) - (6)(-6) = -36 + 36 = 0$; the j-component is $-[(2)(9) - (6)(3)] = -(18 - 18) = 0$; the k-component is $(2)(-6) - (-4)(3) = -12 + 12 = 0$. So $a \times b = 0$, and a zero cross product means the vectors are parallel. (Indeed $b = 3/2\,a$.)

Q16. With $a = 2i + mj + 3k$ and $b = 4i + 2j + 6k$, the cross product is

$$a \times b = (6m - 6)i - (12 - 12)j + (4 - 4m)k = (6m - 6)i + (4 - 4m)k.$$

The vectors are parallel exactly when $a \times b = 0$, which needs $6m - 6 = 0$ and $4 - 4m = 0$; both give $m = 1$. (Then $a = 2i + j + 3k$ and $b = 2a$.)

Q17. A vector perpendicular to the plane is perpendicular to any two vectors lying in it, such as $\overrightarrow{AB} = B - A = i - j + 3k$ and $\overrightarrow{AC} = C - A = -i + j + k$. Their cross product has i-component $(-1)(1) - (3)(1) = -4$, j-component $-[(1)(1) - (3)(-1)] = -(1 + 3) = -4$, and k-component $(1)(1) - (-1)(-1) = 1 - 1 = 0$, so $\overrightarrow{AB} \times \overrightarrow{AC} = -4i - 4j$. This is perpendicular to the plane; any non-zero multiple, such as $i + j$, serves equally well.

Q18. Rearranging $|a \times b| = |a||b| \sin\theta$ for $\sin\theta$,

$$\sin\theta = \frac{|a \times b|}{|a||b|} = \frac{10}{4 \times 5} = \frac{10}{20} = \frac{1}{2}.$$

The acute angle with $\sin\theta = \frac{1}{2}$ is $\theta = 30°$.

Theory

An earlier section described a **line** by a single direction: starting from a known point, one parameter slides a moving point along the line. A **plane** is a flat, two-dimensional surface, so it needs *two* independent directions to move within it — or, more efficiently, a single direction that is **perpendicular** to the whole surface. That perpendicular direction gives the neatest equation of a plane.

The normal vector.

A **normal** to a plane is any non-zero vector n that is perpendicular to the plane — that is, perpendicular to *every* vector lying in it. A plane has infinitely many normals, but they are all parallel: any non-zero scalar multiple of a normal is again a normal. A single point together with a normal direction fixes the plane completely, because exactly one plane passes through a given point at right angles to a given direction.

The vector equation.

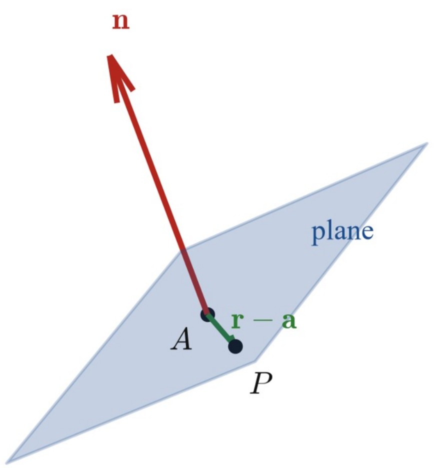
Let A be a known point of the plane, with position vector a, and let n be a normal. A point P with position vector r lies on the plane exactly when the vector $\overrightarrow{AP} = r - a$ lies *in* the plane, and so is perpendicular to n. Using the perpendicularity test from an earlier chapter (a zero dot product),

$$(r - a) \cdot n = 0.$$

Expanding and rearranging gives the **vector equation** of the plane,

$$r \cdot n = a \cdot n.$$

The right-hand side $a \cdot n$ is a fixed number, the same for every point of the plane, so the equation says that the dot product of the position vector with the normal is **constant**.



The Cartesian equation.

Write the normal in components as $n = a\,i + b\,j + c\,k$ and a general point as $r = x\,i + y\,j + z\,k$. Then $r \cdot n = a\,x + b\,y + c\,z$, and writing the constant $a \cdot n = d$ turns the vector equation into the **Cartesian equation**

$$a\,x + b\,y + c\,z = d.$$

The fact worth remembering is that **the coefficients of x, y and z are exactly the components of a normal**: reading off $2x + 3y - z = 5$, a normal is $n = 2i + 3j - k$. The constant d is then found from any one known point on the plane.

A point with a given normal.

If the plane passes through $A(x_0, y_0, z_0)$ with normal $n = a\,i + b\,j + c\,k$, substitute the point to find the constant $d = a\,x_0 + b\,y_0 + c\,z_0 = a\cdot n$, then write $a\,x + b\,y + c\,z = d$.

Three points on the plane.

Three points A, B, C that are not in a straight line determine a plane. The two edge vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ both lie in the plane, so a normal is their **vector product** (from an earlier section):

$$n = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

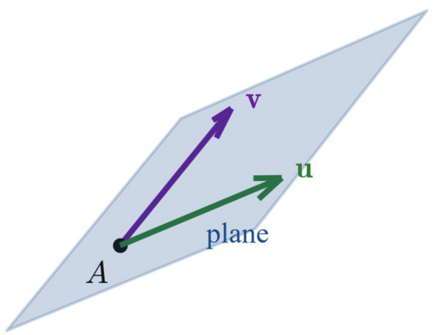
where b_1, b_2, b_3 and c_1, c_2, c_3 are the components of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Once a normal is found, use one of the three points to get the constant d, exactly as above. Any scalar multiple of the normal serves equally well, so it is usual to cancel a common factor to keep the coefficients small.

The parametric (two-parameter) form.

Instead of a normal, a plane can be described by a point and two non-parallel directions lying in it. If a is the position vector of a point on the plane and u, v are two non-parallel vectors in the plane, then every point of the plane is reached by

$$r = a + s\,u + t\,v,$$

where s and t are real parameters. Two parameters are needed because a plane is two-dimensional — compare a line, which needs one. To convert this **parametric form** to Cartesian, take $n = u \times v$ as a normal (it is perpendicular to both u and v) and proceed as before.



Testing whether a point lies in a plane.

A point lies in the plane precisely when its coordinates satisfy the Cartesian equation. Substitute the coordinates into $a\,x + b\,y + c\,z$ and compare with d: equal means the point is on the plane, unequal means it is not. The same test decides whether several given points are **coplanar** — find the plane of three of them, then test the rest.

Converting between the forms.

The three descriptions carry the same information, and moving between them uses only the normal and one point:

- **Vector ↔ Cartesian:** the components of the normal are the coefficients of x, y, z; the constant is $a\cdot n = d$.

- **Parametric → Cartesian:** a normal is $n = u \times v$; the constant comes from the base point a.
- **Cartesian → vector:** read a normal from the coefficients and take any convenient point that satisfies the equation.

Distances from a point to a plane, and angles between planes, build directly on the normal and are developed in a later section. Here the focus is the normal, the three forms of the equation, and moving between them.

Worked examples

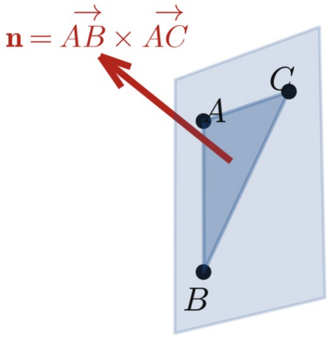
Example 1 — scaffolded

Find the vector equation and the Cartesian equation of the plane through $A(2,-1,3)$ with normal $n = 2i + 3j - k$.

Working	Comment
$a = 2i - j + 3k$, $n = 2i + 3j - k$.	The point gives the position vector a; the normal is given.
$a \cdot n = (2)(2) + (-1)(3) + (3)(-1) = 4 - 3 - 3 = -2$.	The constant on the right of the vector equation.
Vector equation: $r \cdot (2i + 3j - k) = -2$.	Use $r \cdot n = a \cdot n$.
With $r = xi + yj + zk$, $r \cdot n = 2x + 3y - z$.	The coefficients are the components of n.
Cartesian equation: $2x + 3y - z = -2$.	Replace $r \cdot n$ by its component form.
Check A: $2(2) + 3(-1) - 3 = 4 - 3 - 3 = -2$, as required.	The known point must satisfy the equation.

Example 2 — scaffolded

Find the Cartesian equation of the plane through the points $A(1,0,2)$, $B(3,1,1)$ and $C(2,2,3)$.



Working	Comment
$\overrightarrow{AB} = B - A = 2i + j - k$, $\overrightarrow{AC} = C - A = i + 2j + k$.	Two edge vectors that lie in the plane.
$n = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$	A normal is the vector product of the edges.
$= i(1+2) - j(2+1) + k(4-1) = 3i - 3j + 3k$.	Expand the determinant along the top row.
Take $n = i - j + k$ (divide by 3).	Any scalar multiple is also a normal; smaller is tidier.

Working	Comment
$d=a\cdot n=(1)(1)+(0)(-1)+(2)(1)=3.$	Use point A for the constant.
Cartesian equation: $x-y+z=3.$	Check B: $3-1+1=3$; check C: $2-2+3=3.$

Example 3 — unscaffolded

A plane has the parametric equation $r=(2i+k)+s(i+j)+t(i+k)$. Find its Cartesian equation.

The two spanning vectors are $u=i+j$ and $v=i+k$, both lying in the plane, so a normal is their vector product:

$$n=u\times v=\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}=i(1-0)-j(1-0)+k(0-1)=i-j-k.$$

The base point has position vector $a=2i+k$, so

$$d=a\cdot n=(2)(1)+(0)(-1)+(1)(-1)=2-1=1.$$

With $r=xi+yj+zk$ the Cartesian equation is $x-y-z=1$. As a check, the base point $(2,0,1)$ gives $2-0-1=1$.

$\therefore$ the Cartesian equation is $x-y-z=1$.

Example 4 — unscaffolded

The plane Π has Cartesian equation $2x-y+3z=7$. (a) Write down a normal to Π and the vector equation of Π. (b) Determine whether the points $P(1,-2,1)$ and $Q(0,1,3)$ lie in Π.

The coefficients of x, y and z are the components of a normal, so $n=2i-j+3k$, and the vector equation is $r\cdot(2i-j+3k)=7$. To test a point, substitute its coordinates into $2x-y+3z$ and compare with 7. For P,

$$2(1)-(-2)+3(1)=2+2+3=7,$$

which equals the constant, so P lies in Π. For Q,

$$2(0)-(1)+3(3)=0-1+9=8\neq 7,$$

so Q does not lie in Π.

$\therefore n=2i-j+3k$; P lies in Π but Q does not.

Practice questions

Scaffolded

Q1. Find the Cartesian equation of the plane through $A(1,2,-1)$ with normal $n=3i+j+2k$. (a) Find the constant $d=a\cdot n$. (b) Write down the Cartesian equation. (c) Does the point $B(2,-1,1)$ lie in the plane?

Q2. Find the Cartesian equation of the plane through $A(1,0,0)$, $B(0,2,0)$ and $C(0,0,3)$. (a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (b) Find a normal $n=\overrightarrow{AB}\times\overrightarrow{AC}$. (c) Find the constant $d=a\cdot n$. (d) Write down the Cartesian equation.

Q3. A plane has parametric equation $r=(i+2k)+s(i+2j)+t(j+k)$. (a) Write down the two spanning vectors u and v. (b) Find a normal $n=u\times v$. (c) Using the base point, find the constant d. (d) Write down the Cartesian equation.

Q4. A plane has Cartesian equation $x-2y+2z=5$. Test each point. (a) Substitute $P(3,1,2)$. (b) Substitute $Q(1,0,1)$. (c) Substitute $R(5,2,2)$. (d) State which of P, Q, R lie in the plane.

Q5. A plane has Cartesian equation $4x+y-3z=8$. (a) Write down a normal to the plane. (b) Find a point that lies in the plane by setting $y=z=0$. (c) Write down the vector equation of the plane.

Q6. Find the Cartesian equation of the plane through $A(2,1,0)$ that is parallel to the vectors $u = i + k$ and $v = j - k$. (a) Find a normal $n = u \times v$. (b) Find the constant $d = a \cdot n$. (c) Write down the Cartesian equation.

Unscaffolded

Q7. Find the Cartesian equation of the plane through $A(3,-1,2)$ with normal $n = i + 4j - 2k$.

Q8. Find the Cartesian equation of the plane through the points $A(2,0,1)$, $B(1,2,0)$ and $C(3,1,2)$.

Q9. Determine whether the point $P(2,3,-1)$ lies in the plane $3x - 2y + z = 4$.

Q10. A plane has vector equation $r \cdot (2i - j + 2k) = 6$. (a) Write down its Cartesian equation. (b) Does the point $P(1,2,3)$ lie in the plane?

Q11. Find the Cartesian equation of the plane through $A(1,1,2)$ that is parallel to the plane $2x - 3y + z = 5$.

Q12. A plane has parametric equation $r = (2i + j - k) + s(i + 2k) + t(i + j + k)$. Find its Cartesian equation.

Q13. Find the value of k for which the point $(k,2,1)$ lies in the plane $x + 3y - 2z = 9$.

Q14. The points $A(1,2,3)$, $B(2,0,1)$ and $C(-1,1,2)$ lie in a plane. Find its Cartesian equation.

Q15. Find the Cartesian equation of the plane through $A(0,1,4)$ that is perpendicular to the line with direction $3i - j + 2k$.

Q16. Show that the four points $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ and $D(1,1,-1)$ are coplanar.

Q17. A plane meets the coordinate axes at $A(4,0,0)$, $B(0,3,0)$ and $C(0,0,6)$. Find its Cartesian equation.

Q18. The plane Π has equation $2x + y - z = 3$. (a) Write down a normal to Π. (b) The point $P(a,1,2)$ lies in Π; find a. (c) Find the Cartesian equation of the plane parallel to Π that passes through the origin.

Answers

Q1. (a) $d=3$; (b) $3x+y+2z=3$; (c) no ($3(2)-1+2=7\neq 3$). **Q2.** (a) $\overrightarrow{AB}=-i+2j$, $\overrightarrow{AC}=-i+3k$; (b) $n=6i+3j+2k$; (c) $d=6$; (d) $6x+3y+2z=6$. **Q3.** (a) $u=i+2j$, $v=j+k$; (b) $n=2i-j+k$; (c) $d=4$; (d) $2x-y+z=4$. **Q4.** P: 5; Q: 3; R: 5. P and R lie in the plane; Q does not. **Q5.** (a) $n=4i+j-3k$; (b) $(2,0,0)$; (c) $r\cdot(4i+j-3k)=8$. **Q6.** $n=-i+j+k$, $d=-1$; $-x+y+z=-1$, that is $x-y-z=1$. **Q7.** $x+4y-2z=-5$. **Q8.** $x-z=1$. **Q9.** No ($3(2)-2(3)+(-1)=-1\neq 4$). **Q10.** (a) $2x-y+2z=6$; (b) yes ($2-2+6=6$). **Q11.** $2x-3y+z=1$. **Q12.** $2x-y-z=4$. **Q13.** $k=5$. **Q14.** $y-z=-1$. **Q15.** $3x-y+2z=7$. **Q16.** Plane ABC is $x+y+z=1$; D gives $1+1-1=1$, so D lies in it and the four points are coplanar. **Q17.** $3x+4y+2z=12$. **Q18.** (a) $n=2i+j-k$; (b) $a=2$; (c) $2x+y-z=0$.

Fully-worked solutions

Q1. With $a=i+2j-k$ and $n=3i+j+2k$, the constant is

$$d=a\cdot n=(1)(3)+(2)(1)+(-1)(2)=3+2-2=3.$$

Since the coefficients of x, y, z are the components of n, the Cartesian equation is $3x+y+2z=3$. Testing $B(2,-1,1)$: $3(2)+(-1)+2(1)=6-1+2=7$, which is not 3, so B does not lie in the plane.

Q2. The edge vectors are $\overrightarrow{AB}=B-A=-i+2j$ and $\overrightarrow{AC}=C-A=-i+3k$. A normal is

$$n=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3\end{vmatrix}=i(6-0)-j(-3-0)+k(0+2)=6i+3j+2k.$$

Using $A(1,0,0)$, $d=a\cdot n=(1)(6)+0+0=6$, so the Cartesian equation is $6x+3y+2z=6$. (Checks: B gives 6, C gives 6.)

Q3. The spanning vectors are $u=i+2j$ and $v=j+k$. A normal is

$$n=u\times v=\begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 1\end{vmatrix}=i(2-0)-j(1-0)+k(1-0)=2i-j+k.$$

The base point is $a=i+2k$, so $d=a\cdot n=(1)(2)+(0)(-1)+(2)(1)=4$. The Cartesian equation is $2x-y+z=4$.

Q4. Substitute each point into $x-2y+2z$ and compare with 5. For $P(3,1,2)$: $3-2(1)+2(2)=3-2+4=5$, so P lies in the plane. For $Q(1,0,1)$: $1-0+2(1)=3\neq 5$, so Q does not. For $R(5,2,2)$: $5-2(2)+2(2)=5-4+4=5$, so R lies in the plane. Hence P and R lie in the plane, but Q does not.

Q5. The coefficients of x, y, z give the normal $n=4i+j-3k$. Setting $y=z=0$ in $4x+y-3z=8$ gives $4x=8$, so $x=2$ and the point $(2,0,0)$ lies in the plane. The vector equation is $r\cdot(4i+j-3k)=8$.

Q6. A normal is perpendicular to both u and v, so

$$n=u\times v=\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & -1\end{vmatrix}=i(0-1)-j(-1-0)+k(1-0)=-i+j+k.$$

With $A(2,1,0)$, $d=a\cdot n=(2)(-1)+(1)(1)+(0)(1)=-1$, so the Cartesian equation is $-x+y+z=-1$. Multiplying through by -1 gives the tidier form $x-y-z=1$.

Q7. With $a=3i-j+2k$ and $n=i+4j-2k$,

$$d=a\cdot n=(3)(1)+(-1)(4)+(2)(-2)=3-4-4=-5.$$

The Cartesian equation is $x+4y-2z=-5$.

Q8. The edge vectors are $\overrightarrow{AB}=B-A=-i+2j-k$ and $\overrightarrow{AC}=C-A=i+j+k$. A normal is

$$n=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix}=i(2+1)-j(-1+1)+k(-1-2)=3i-3k.$$

Dividing by 3 gives $n=i-k$. Using $A(2,0,1)$, $d=(2)(1)+(0)(0)+(1)(-1)=1$, so the Cartesian equation is $x-z=1$. (Checks: B gives $1-0=1$; C gives $3-2=1$.)

Q9. Substitute $P(2,3,-1)$ into $3x-2y+z$: $3(2)-2(3)+(-1)=6-6-1=-1$. Since $-1\neq 4$, the point P does not lie in the plane.

Q10. The Cartesian equation is obtained by reading the normal components as coefficients: $2x-y+2z=6$. Testing $P(1,2,3)$: $2(1)-2+2(3)=2-2+6=6$, which equals the constant, so P lies in the plane.

Q11. Parallel planes share a normal, so a normal to the required plane is $n=2i-3j+k$, read from $2x-3y+z=5$. Using $A(1,1,2)$, the constant is $d=(1)(2)+(1)(-3)+(2)(1)=2-3+2=1$. Hence the plane is $2x-3y+z=1$.

Q12. The spanning vectors are $u=i+2k$ and $v=i+j+k$, giving normal

$$n=u\times v=\begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix}=i(0-2)-j(1-2)+k(1-0)=-2i+j+k.$$

With base point $a=2i+j-k$, $d=(2)(-2)+(1)(1)+(-1)(1)=-4+1-1=-4$, so the equation is $-2x+y+z=-4$. Multiplying by -1 gives $2x-y-z=4$.

Q13. The point $(k,2,1)$ lies in $x+3y-2z=9$ when $k+3(2)-2(1)=9$, that is $k+6-2=9$, so $k+4=9$ and $k=5$.

Q14. The edge vectors are $\overrightarrow{AB}=B-A=i-2j-2k$ and $\overrightarrow{AC}=C-A=-2i-j-k$. A normal is

$$n=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ 1 & -2 & -2 \\ -2 & -1 & -1 \end{vmatrix}=i(2-2)-j(-1-4)+k(-1-4)=5j-5k.$$

Dividing by 5 gives $n=j-k$. Using $A(1,2,3)$, $d=(1)(0)+(2)(1)+(3)(-1)=2-3=-1$, so the plane is $y-z=-1$. (Checks: B gives $0-1=-1$; C gives $1-2=-1$.)

Q15. A line perpendicular to the plane points along a normal, so $n=3i-j+2k$. Using $A(0,1,4)$, $d=(0)(3)+(1)(-1)+(4)(2)=0-1+8=7$, so the Cartesian equation is $3x-y+2z=7$.

Q16. First find the plane through A, B, C. The edge vectors are $\overrightarrow{AB}=-i+j$ and $\overrightarrow{AC}=-i+k$, so a normal is

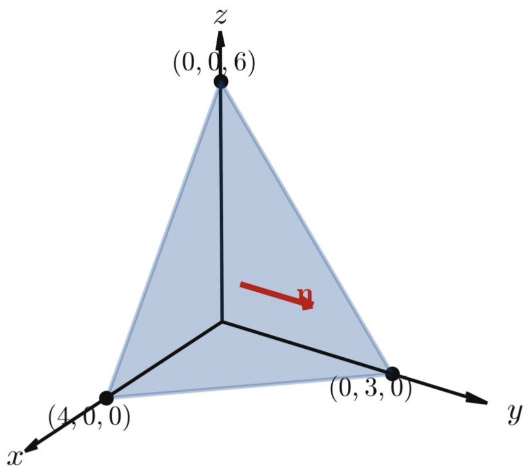
$$n=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}=i(1-0)-j(-1-0)+k(0+1)=i+j+k.$$

Using $A(1,0,0)$, $d=1$, so the plane is $x+y+z=1$. Testing $D(1,1,-1)$: $1+1+(-1)=1$, which satisfies the equation, so D lies in the plane of A, B, C and the four points are coplanar.

Q17. The edge vectors are $\overrightarrow{AB}=B-A=-4i+3j$ and $\overrightarrow{AC}=C-A=-4i+6k$. A normal is

$$n=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ -4 & 3 & 0 \\ -4 & 0 & 6 \end{vmatrix}=i(18-0)-j(-24-0)+k(0+12)=18i+24j+12k.$$

Dividing by the common factor 6 gives $n = 3i + 4j + 2k$. Using $A(4,0,0)$, $d = (4)(3) = 12$, so the Cartesian equation is $3x + 4y + 2z = 12$. (Checks: B gives 12, C gives 12.)



Q18. (a) The coefficients of x, y, z in $2x + y - z = 3$ give the normal $n = 2i + j - k$. (b) The point $P(a,1,2)$ lies in Π when $2a + 1 - 2 = 3$, that is $2a - 1 = 3$, so $2a = 4$ and $a = 2$. (c) A parallel plane has the same normal, so it has the form $2x + y - z = d$; passing through the origin $(0,0,0)$ gives $d = 0$. Hence the plane is $2x + y - z = 0$.

Theory

Earlier sections built the vector language for lines and planes in three dimensions. A **line** through the point with position vector a and parallel to the direction d has vector equation

$$r = a + t\,d, t \in \mathbb{R},$$

with parametric form $x = a_1 + t\,d_1$, $y = a_2 + t\,d_2$, $z = a_3 + t\,d_3$. A **plane** with normal $n = a\,i + b\,j + c\,k$ passing through the point a has vector equation $r \cdot n = a \cdot n$, or in Cartesian form

$$a\,x + b\,y + c\,z = d, d = a \cdot n.$$

This section uses these descriptions, together with the dot product of an earlier chapter and the cross product of an earlier section, to measure the **angles** between lines and planes, the **distance** from a point to a plane, and to locate the **intersections** where a line meets a plane or two planes meet each other.

Angle between two planes.

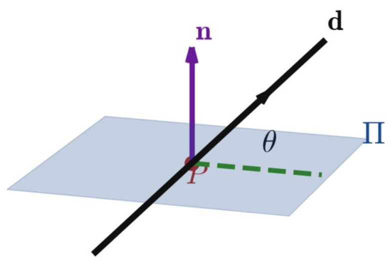
Two planes that are not parallel cut each other along a line, and the angle between the planes is the angle you would measure looking end-on down that line. Because a plane's tilt is carried entirely by its normal, this angle is the same as the angle between the two normals n_1 and n_2. Two lines through a point actually make two supplementary angles; we quote the **acute** one, and taking the absolute value of the dot product guarantees it:

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}, 0° \le \theta \le 90°.$$

Two special cases follow at once. If n_1 and n_2 are parallel the planes are **parallel** ($\theta = 0°$); if $n_1 \cdot n_2 = 0$ the normals are perpendicular, so the planes are **perpendicular** ($\theta = 90°$).

Angle between a line and a plane.

Let a line have direction d and a plane have normal n. The angle θ between the line and the plane is measured between the line and its *shadow* (projection) in the plane, so $0° \le \theta \le 90°$.



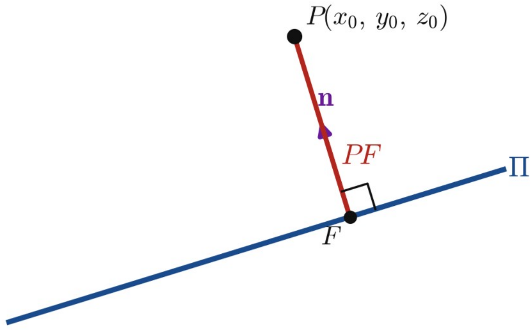
The normal is perpendicular to the plane, so the angle ϕ between d and n and the angle θ between the line and the plane are **complementary**: $\theta = 90° - \phi$. Since $\sin\theta = \sin(90° - \phi) = \cos\phi$, and $\cos\phi$ comes straight from the dot product,

$$\sin\theta=\frac{|d\cdot n|}{|d||n|},0°\le\theta\le 90°.$$

Note the **sine**, not cosine: the line-and-plane formula is the complement of the plane-and-plane formula. If $d\cdot n=0$ the line is **parallel to the plane** ($\theta=0°$); if d is parallel to n the line is **perpendicular to the plane** ($\theta=90°$).

Distance from a point to a plane.

The distance from a point $P(x_0,y_0,z_0)$ to a plane is measured along the normal, from P to the foot F of the perpendicular in the plane.



Let A be any point of the plane, with position vector a, and let p be the position vector of P. The required distance PF is the length of the projection of $\overrightarrow{AP}$ onto the unit normal $\frac{n}{|n|}$, that is $PF=\frac{|\overrightarrow{AP}\cdot n|}{|n|}$. Here $\overrightarrow{AP}\cdot n=(p-a)\cdot n=p\cdot n-a\cdot n$, in which $p\cdot n=ax_0+by_0+cz_0$ and $a\cdot n=d$, because A lies on the plane $ax+by+cz=d$; so the working formula is

$$\text{distance}=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}.$$

In words: substitute the point into $ax+by+cz-d$, take the absolute value, and divide by the length of the normal $\sqrt{a^2+b^2+c^2}$. Two consequences are worth noting: the distance from the **origin** to the plane is $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$, and the distance between two **parallel** planes $ax+by+cz=d_1$ and $ax+by+cz=d_2$ is $\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$, found by taking any point of one plane and measuring to the other.

Intersection of a line and a plane.

To find where the line $r=a+td$ meets the plane $r\cdot n=d$, substitute the line's general point into the plane equation and solve the resulting **single** linear equation for the parameter t:

$$(a+td)\cdot n=d\Rightarrow a\cdot n+t(d\cdot n)=d.$$

The behaviour is governed entirely by $d\cdot n$:

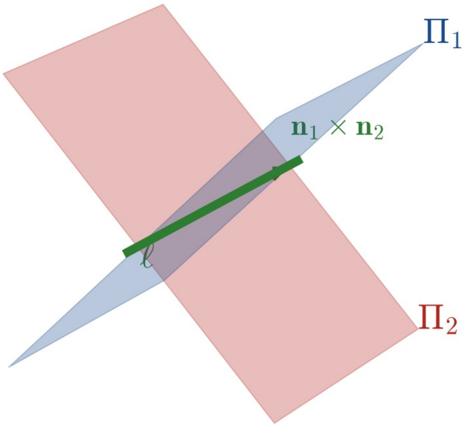
$d\cdot n$	outcome
$\neq 0$	a unique t, giving one point of intersection
$=0$ and $a\cdot n\neq d$	line parallel to the plane, no intersection
$=0$ and $a\cdot n=d$	line lies in the plane (every point satisfies it)

When a unique t exists, substitute it back into $r=a+td$ to read off the point.

Intersection of two planes.

Two non-parallel planes meet in a **line** ℓ. Every point of ℓ lies in both planes, so ℓ is perpendicular to **both** normals; hence its direction is their cross product

$$d = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$



To fix a single point on ℓ, set one convenient coordinate to a value — usually $z = 0$ — and solve the two plane equations as a pair of simultaneous equations in the remaining variables. (Choose a coordinate that actually varies along ℓ; if d has a zero component, set a different coordinate instead.) With a point a and the direction d found, the line of intersection is $r = a + t\,d$. If instead $n_1 \times n_2 = 0$ the normals are parallel, so the planes are parallel: they either coincide or never meet.

Finding where such a line of intersection then pierces a third plane, or the angle it makes with another plane, simply combines the methods above; those mixed problems appear in the questions.

Worked examples

Example 1 — scaffolded

Find the acute angle between the planes $\Pi_1 : x + y = 3$ and $\Pi_2 : x + z = 4$, giving an exact answer.

Working	Comment
$n_1 = i + j = (1, 1, 0)$, $n_2 = i + k = (1, 0, 1)$.	Read each normal off the coefficients of x, y, z.
$n_1 \cdot n_2 = (1)(1) + (1)(0) + (0)(1) = 1$.	Dot product of the normals.
$\lvert n_1 \rvert = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$, $\lvert n_2 \rvert = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$.	Length of each normal.
$\cos\theta = \dfrac{\lvert n_1 \cdot n_2 \rvert}{\lvert n_1 \rvert \lvert n_2 \rvert} = \dfrac{\lvert 1 \rvert}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$.	Angle-between-planes formula, absolute value for the acute angle.
$\theta = \cos^{-1}\left(\dfrac{1}{2}\right) = 60°$.	The unique angle with $0° \le \theta \le 90°$.

Example 2 — scaffolded

Find the distance from the point $P(4,1,4)$ to the plane $2x - y + 2z = 6$.

Working	Comment
$a=2, b=-1, c=2, d=6$; $P=(x_0, y_0, z_0)=(4,1,4)$.	Identify the plane coefficients and the point.
$a x_0 + b y_0 + c z_0 - d = 2(4)-(1)+2(4)-6=8-1+8-$.	Substitute P into $a x + b y + c z - d$.
$\sqrt{a^2+b^2+c^2} = \sqrt{2^2+(-1)^2+2^2} = \sqrt{9} = 3$.	Length of the normal $n=(2,-1,2)$.
distance $= \frac{\lvert 9 \rvert}{3} = 3$.	Absolute value over the normal's length.

Example 3 — unscaffolded

Find the point where the line $r=(i+2j-k)+t(2i+j+3k)$ meets the plane $x+2y-z=8$.

A general point of the line is $(1+2t, 2+t, -1+3t)$. The normal is $n=(1,2,-1)$ and the direction is $d=(2,1,3)$; since $d \cdot n = 2+2-3=1 \neq 0$, the line meets the plane in exactly one point. Substituting the general point into the plane,

$$(1+2t)+2(2+t)-(-1+3t)=1+2t+4+2t+1-3t=6+t.$$

Setting $6+t=8$ gives $t=2$. Substituting $t=2$ back into the line,

$$r=(1+4, 2+2, -1+6)=(5,4,5),$$

and as a check $5+2(4)-5=8$, as required.

$\therefore$ the line meets the plane at $(5,4,5)$.

Example 4 — unscaffolded

Find a vector equation of the line of intersection of the planes $\Pi_1 : x+y+z=6$ and $\Pi_2 : x-y+2z=2$.

The normals are $n_1=(1,1,1)$ and $n_2=(1,-1,2)$. The line lies in both planes, so its direction is perpendicular to both normals:

$$d = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = i(2+1) - j(2-1) + k(-1-1) = (3,-1,-2).$$

For one point on the line, set $z=0$: then $x+y=6$ and $x-y=2$. Adding gives $2x=8$, so $x=4$ and $y=2$; the point $(4,2,0)$ lies on both planes. Hence the line of intersection is

$$r=(4,2,0)+s(3,-1,-2), s \in \mathbb{R}.$$

As a check, the general point $(4+3s, 2-s, -2s)$ gives $x+y+z=6$ and $x-y+2z=2$ for every s.

$\therefore r=(4i+2j)+s(3i-j-2k)$.

Practice questions

Scaffolded

Q1. Find the acute angle between the planes $\Pi_1 : x+z=5$ and $\Pi_2 : y+z=2$. (a) Write down the normals n_1 and n_2. (b) Find $n_1 \cdot n_2$, $|n_1|$ and $|n_2|$. (c) Find $\cos\theta$. (d) State θ.

Q2. Find the distance from the point $P(3,3,2)$ to the plane $x+2y+2z=5$. (a) Identify a,b,c,d and the coordinates of P. (b) Evaluate $ax_0+by_0+cz_0-d$. (c) Find $\sqrt{a^2+b^2+c^2}$. (d) State the distance.

Q3. Find the point where the line $r=(2,-1,0)+t(1,2,1)$ meets the plane $x-y+2z=5$. (a) Write down the general point of the line. (b) Substitute it into the plane equation and simplify. (c) Solve for t. (d) State the point of intersection.

Q4. Find a vector equation of the line of intersection of the planes $\Pi_1: x+y+z=4$ and $\Pi_2: 2x-y+z=2$. (a) Write down n_1 and n_2. (b) Find the direction $d=n_1\times n_2$. (c) Setting $z=0$, solve for a point on the line. (d) State the vector equation.

Q5. Find the angle between the line $r=(1,1,0)+t(2,2,1)$ and the plane $x+2y+2z=6$, to the nearest degree. (a) Write down the direction d and the normal n. (b) Find $d\cdot n$, $|d|$ and $|n|$. (c) Find $\sin\theta$. (d) State θ.

Q6. Find the distance from the origin O to the plane $2x+3y+6z=14$. (a) Write down the normal and its length. (b) Evaluate $ax_0+by_0+cz_0-d$ at the origin. (c) State the distance.

Unscaffolded

Q7. Find the acute angle between the planes $2x+2y-z=1$ and $x+2y+2z=5$, to the nearest degree.

Q8. Find the distance from the point $P(2,3,4)$ to the plane $x-2y+2z=1$.

Q9. Find the point where the line $r=(3,1,2)+t(-1,2,1)$ meets the plane $2x+y-z=3$.

Q10. Find a vector equation of the line of intersection of the planes $x+2y-z=6$ and $2x-y+z=2$.

Q11. Find the angle between the line $r=(1,0,1)+t(2,-1,2)$ and the plane $x+y+z=3$, to the nearest degree.

Q12. Show that the planes $3x-y+2z=4$ and $x+y-z=5$ are perpendicular.

Q13. Show that the line $r=(1,1,1)+t(2,1,1)$ is parallel to the plane $x-y-z=4$ and does not lie in it, and hence find the distance between the line and the plane.

Q14. Find the distance between the parallel planes $2x-y+2z=3$ and $2x-y+2z=12$.

Q15. Find the acute angle between the planes $x+2y+2z=1$ and $2x+2y+z=4$, to the nearest degree.

Q16. Find the point where the line $r=(0,1,-2)+t(1,-1,2)$ meets the plane $x+2y+z=2$.

Q17. For the planes $x+y+z=3$ and $2x+y-z=1$: (a) find a vector equation of their line of intersection; (b) find the acute angle between the two planes, to the nearest degree.

Q18. The point $P(4,2,4)$ and the plane $2x-y+2z=5$ are given. Find the foot F of the perpendicular from P to the plane, and hence the distance from P to the plane.

Answers

Q1. $n_1=(1,0,1)$, $n_2=(0,1,1)$; $\cos\theta=\frac{1}{2}$, $\theta=60°$. **Q2.** $\frac{8}{3}$. **Q3.** $t=2$; the point $(4,3,2)$. **Q4.** $r=(2,2,0)+s(2,1,-3)$. **Q5.** $\sin\theta=\frac{8}{9}$, $\theta\approx 63°$. **Q6.** 2. **Q7.** $\cos\theta=\frac{4}{9}$, $\theta\approx 64°$. **Q8.** 1. **Q9.** $t=2$; the point $(1,5,4)$. **Q10.** $r=(2,2,0)+s(1,-3,-5)$. **Q11.** $\sin\theta=\frac{1}{\sqrt{3}}$, $\theta\approx 35°$. **Q12.** $n_1\cdot n_2=0$, so the planes are perpendicular. **Q13.** $d\cdot n=0$ and $(1,1,1)$ is not on the plane; distance $=\frac{5}{\sqrt{3}}=\frac{5\sqrt{3}}{3}$. **Q14.** 3. **Q15.** $\cos\theta=\frac{8}{9}$, $\theta\approx 27°$. **Q16.** $t=2$; the point $(2,-1,2)$. **Q17.** (a) $r=(-2,5,0)+s(-2,3,-1)$; (b) $\theta\approx 62°$. **Q18.** $F=(2,3,2)$; distance $=3$.

Fully-worked solutions

Q1. The normals are $n_1=(1,0,1)$ and $n_2=(0,1,1)$. Then $n_1\cdot n_2=(1)(0)+(0)(1)+(1)(1)=1$, while $|n_1|=\sqrt{1^2+0^2+1^2}=\sqrt{2}$ and $|n_2|=\sqrt{0^2+1^2+1^2}=\sqrt{2}$. Hence

$$\cos\theta=\frac{|n_1\cdot n_2|}{|n_1||n_2|}=\frac{|1|}{\sqrt{2}\sqrt{2}}=\frac{1}{2},$$

so $\theta=\cos^{-1}\left(\frac{1}{2}\right)=60°$.

Q2. Here $a=1$, $b=2$, $c=2$, $d=5$ and $P=(3,3,2)$. Substituting, $a x_0+b y_0+c z_0-d=(3)+2(3)+2(2)-5=3+6+4-5=8$, and $\sqrt{a^2+b^2+c^2}=\sqrt{1+4+4}=\sqrt{9}=3$. Hence the distance is $\frac{|8|}{3}=\frac{8}{3}$.

Q3. A general point of the line is $(2+t,-1+2t,t)$. Substituting into $x-y+2z$,

$$(2+t)-(-1+2t)+2(t)=2+t+1-2t+2t=3+t.$$

Setting $3+t=5$ gives $t=2$, and then $r=(2+2,-1+4,2)=(4,3,2)$. As a check, $4-3+2(2)=5$, as required.

Q4. The normals are $n_1=(1,1,1)$ and $n_2=(2,-1,1)$. The direction of the line is

$$d=n_1\times n_2=\begin{vmatrix} i & j & k\\ 1 & 1 & 1\\ 2 & -1 & 1\end{vmatrix}=i(1+1)-j(1-2)+k(-1-2)=(2,1,-3).$$

Setting $z=0$ gives $x+y=4$ and $2x-y=2$; adding, $3x=6$, so $x=2$ and $y=2$. The point $(2,2,0)$ lies on both planes, so $r=(2,2,0)+s(2,1,-3)$.

Q5. The direction is $d=(2,2,1)$ and the normal is $n=(1,2,2)$. Then $d\cdot n=2+4+2=8$, while $|d|=\sqrt{4+4+1}=3$ and $|n|=\sqrt{1+4+4}=3$. The line-and-plane angle uses the sine:

$$\sin\theta=\frac{|d\cdot n|}{|d||n|}=\frac{8}{3\times 3}=\frac{8}{9},$$

so $\theta=\sin^{-1}\left(\frac{8}{9}\right)\approx 62.73°$, which is $63°$ to the nearest degree.

Q6. The normal is $n=(2,3,6)$ with $|n|=\sqrt{4+9+36}=\sqrt{49}=7$. At the origin, $a x_0+b y_0+c z_0-d=0+0+0-14=-14$, so the distance is $\frac{|-14|}{7}=2$.

Q7. The normals are $n_1=(2,2,-1)$ and $n_2=(1,2,2)$, with $n_1 \cdot n_2=2+4-2=4$ and $|n_1|=|n_2|=\sqrt{4+4+1}=3$. Then

$$\cos\theta=\frac{|4|}{3\times 3}=\frac{4}{9},$$

so $\theta=\cos^{-1}\left(\frac{4}{9}\right)\approx 63.61°$, which is $64°$ to the nearest degree.

Q8. With $a=1$, $b=-2$, $c=2$, $d=1$ and $P=(2,3,4)$, $a x_0+b y_0+c z_0-d=(2)-2(3)+2(4)-1=2-6+8-1=3$, and $\sqrt{1+4+4}=3$. Hence the distance is $\frac{|3|}{3}=1$.

Q9. A general point of the line is $(3-t,1+2t,2+t)$. The direction $d=(-1,2,1)$ and normal $n=(2,1,-1)$ satisfy $d \cdot n=-2+2-1=-1\neq 0$, so there is a unique point. Substituting into $2x+y-z$,

$$2(3-t)+(1+2t)-(2+t)=6-2t+1+2t-2-t=5-t.$$

Setting $5-t=3$ gives $t=2$, so $r=(1,5,4)$. Check: $2(1)+5-4=3$.

Q10. The normals are $n_1=(1,2,-1)$ and $n_2=(2,-1,1)$, so

$$d=n_1\times n_2=\begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}=i(2-1)-j(1+2)+k(-1-4)=(1,-3,-5).$$

Setting $z=0$ gives $x+2y=6$ and $2x-y=2$. From the second, $y=2x-2$; substituting, $x+2(2x-2)=6$, so $5x=10$, $x=2$ and $y=2$. The point $(2,2,0)$ lies on both planes, so $r=(2,2,0)+s(1,-3,-5)$.

Q11. The direction is $d=(2,-1,2)$ and the normal is $n=(1,1,1)$, with $d \cdot n=2-1+2=3$, $|d|=\sqrt{4+1+4}=3$ and $|n|=\sqrt{3}$. Then

$$\sin\theta=\frac{|3|}{3\sqrt{3}}=\frac{1}{\sqrt{3}},$$

so $\theta=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 35.26°$, which is $35°$ to the nearest degree.

Q12. The normals are $n_1=(3,-1,2)$ and $n_2=(1,1,-1)$. Their dot product is $n_1 \cdot n_2=(3)(1)+(-1)(1)+(2)(-1)=3-1-2=0$. Because the normals are perpendicular, the planes are perpendicular (the angle between them is $\cos^{-1}(0)=90°$).

Q13. The direction is $d=(2,1,1)$ and the plane's normal is $n=(1,-1,-1)$. Since $d \cdot n=2-1-1=0$, the direction is perpendicular to the normal, so the line is parallel to the plane. Testing the point $(1,1,1)$ of the line: $1-1-1=-1\neq 4$, so the point is not on the plane and the line does not lie in it. The whole line is therefore a constant distance from the plane; measuring from $(1,1,1)$,

$$\text{distance}=\frac{|(1)-(1)-(1)-4|}{\sqrt{1^2+(-1)^2+(-1)^2}}=\frac{|-5|}{\sqrt{3}}=\frac{5}{\sqrt{3}}=\frac{5\sqrt{3}}{3}.$$

Q14. The planes share the normal $n=(2,-1,2)$ with $|n|=\sqrt{4+1+4}=3$, so they are parallel. A point on the first plane is $(0,-3,0)$, since $2(0)-(-3)+2(0)=3$. Its distance to the second plane $2x-y+2z=12$ is

$$\frac{|2(0)-(-3)+2(0)-12|}{3}=\frac{|-9|}{3}=3.$$

(Equivalently, $\dfrac{|d_1-d_2|}{|n|}=\dfrac{|3-12|}{3}=3$.)

Q15. The normals are $n_1=(1,2,2)$ and $n_2=(2,2,1)$, with $n_1\cdot n_2=2+4+2=8$ and $|n_1|=|n_2|=\sqrt{1+4+4}=3$. Then

$$\cos\theta=\frac{|8|}{3\times 3}=\frac{8}{9},$$

so $\theta=\cos^{-1}\left(\dfrac{8}{9}\right)\approx 27.27°$, which is $27°$ to the nearest degree.

Q16. A general point of the line is $(t,1-t,-2+2t)$. The direction $d=(1,-1,2)$ and normal $n=(1,2,1)$ give $d\cdot n=1-2+2=1\neq 0$, so there is a unique point. Substituting into $x+2y+z$,

$$(t)+2(1-t)+(-2+2t)=t+2-2t-2+2t=t.$$

Setting $t=2$ gives $r=(2,-1,2)$. Check: $2+2(-1)+2=2$.

Q17. (a) The normals are $n_1=(1,1,1)$ and $n_2=(2,1,-1)$, so the direction of the line is

$$d=n_1\times n_2=\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix}=i(-1-1)-j(-1-2)+k(1-2)=(-2,3,-1).$$

Setting $z=0$ gives $x+y=3$ and $2x+y=1$; subtracting, $x=-2$, so $y=5$. The point $(-2,5,0)$ lies on both planes, giving $r=(-2,5,0)+s(-2,3,-1)$. (b) With $n_1\cdot n_2=2+1-1=2$, $|n_1|=\sqrt{3}$ and $|n_2|=\sqrt{6}$,

$$\cos\theta=\frac{|2|}{\sqrt{3}\sqrt{6}}=\frac{2}{\sqrt{18}}=\frac{2}{3\sqrt{2}}=\frac{\sqrt{2}}{3}\approx 0.4714,$$

so $\theta=\cos^{-1}\left(\dfrac{\sqrt{2}}{3}\right)\approx 61.87°$, which is $62°$ to the nearest degree.

Q18. The perpendicular from $P(4,2,4)$ to the plane runs in the direction of the normal $n=(2,-1,2)$, so it is the line $r=(4,2,4)+t(2,-1,2)$. Its general point $(4+2t,2-t,4+2t)$ meets the plane $2x-y+2z=5$ when

$$2(4+2t)-(2-t)+2(4+2t)=8+4t-2+t+8+4t=14+9t=5,$$

so $9t=-9$ and $t=-1$. The foot is $F=(4-2,2+1,4-2)=(2,3,2)$, which satisfies $2(2)-3+2(2)=5$. The distance is the length $|\overrightarrow{PF}|$; since $\overrightarrow{PF}=-n=(-2,1,-2)$,

$$\text{distance}=|\overrightarrow{PF}|=\sqrt{(-2)^2+1^2+(-2)^2}=\sqrt{9}=3.$$

(This agrees with the formula $\dfrac{|2(4)-2+2(4)-5|}{3}=\dfrac{9}{3}=3$.)