Specialist Mathematics Units 3 & 4

Chapter 2 — Vectors

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Significant effort has been made to ensure this content is legal. Please send any areas of concern to admin@vwx.au with appropriate document/legal references so errors can be verified and changes be made.

Contents

Theory

Scalars and vectors.

Many quantities are described completely by a single number together with a unit: the mass of a body, the time shown on a clock, the temperature of a room, the length of a rod. Such a quantity is called a **scalar**. Other quantities need a *direction* as well as a size before they are fully specified: the displacement of a hiker, the velocity of a car, a force pushing on a crate. A quantity that has both a **magnitude** (a size) and a **direction** is called a **vector**.

Directed line segments.

A vector can be pictured as a **directed line segment** — an arrow. The length of the arrow represents the magnitude of the vector and the way it points represents the direction. The **tail** is the starting point and the **head** (the end carrying the arrowhead) is the finishing point. The vector from a point A to a point B is written $\overrightarrow{AB}$, with A the tail and B the head. A vector may also be named by a single bold letter, a.

Component (column) form.

Placing a vector on a set of axes gives a convenient way of recording it with numbers. If a vector moves a_1 units in the x-direction and a_2 units in the y-direction, it is written in **component form** (or **column form**) as

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The numbers a_1 and a_2 are the **components** of a. Working in components turns every geometric operation on vectors into simple arithmetic. (The *magnitude* of a vector — its actual length — is found with Pythagoras' theorem and is developed in a later section, together with unit vectors and the scalar product.)

Equal vectors.

Because a vector carries only magnitude and direction, it is **free**: it may be slid to any position in the plane, provided its length and direction are unchanged. Two vectors are **equal** when they have the same magnitude and the same direction, even if they start at different points. In component form this says that two vectors are equal exactly when their corresponding components are equal:

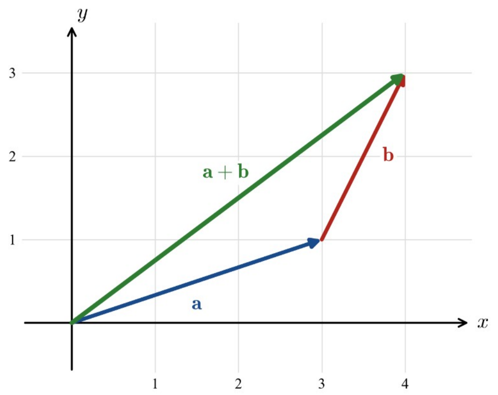
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ if and only if } a_1 = b_1 \text{ and } a_2 = b_2.$$

The zero vector.

The **zero vector**, written $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, has zero magnitude and no defined direction. It is the vector that begins and ends at the same point, so $\overrightarrow{AA} = 0$.

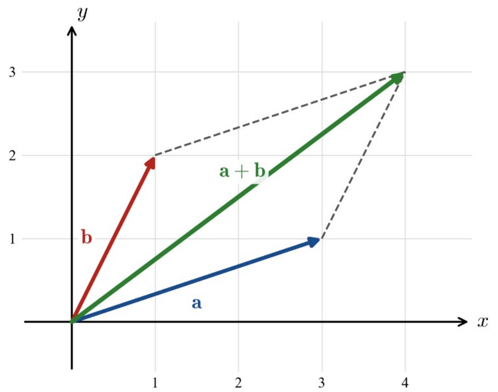
Adding vectors — the triangle law.

To add two vectors, draw them **head to tail**: draw a, then draw b starting from the head of a. The **sum** $a + b$, also called the **resultant**, is the vector from the tail of a to the head of b. This is the **triangle law** of addition.



The parallelogram law.

Equivalently, draw a and b from a common tail and complete the parallelogram; the sum $a+b$ is the diagonal drawn from that common point. This is the **parallelogram law**. It makes clear that the order of addition does not matter, so $a+b=b+a$.



In component form, vectors are added by adding corresponding components:

$$a+b=\begin{pmatrix}a_1\\a_2\end{pmatrix}+\begin{pmatrix}b_1\\b_2\end{pmatrix}=\begin{pmatrix}a_1+b_1\\a_2+b_2\end{pmatrix}.$$

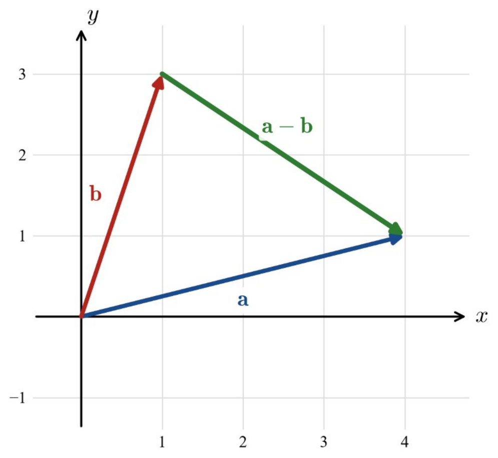
Addition is commutative and associative, and the zero vector leaves any vector unchanged: $a+0=a$.

The negative of a vector and subtraction.

The **negative** of a, written $-a$, has the same magnitude as a but points in the opposite direction. Reversing an arrow reverses the vector, so $\overrightarrow{BA}=-\overrightarrow{AB}$, and adding a vector to its negative gives the zero vector, $a+(-a)=0$. **Subtraction** is then defined as adding the negative:

$$a-b=a+(-b)=\begin{pmatrix}a_1-b_1\\a_2-b_2\end{pmatrix}.$$

Geometrically, if a and b are drawn from a common tail, then $a-b$ is the vector from the head of b to the head of a.

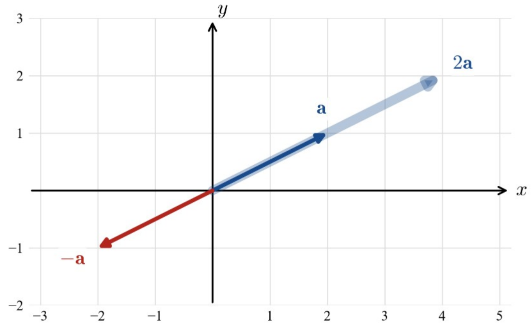


Scalar multiplication.

Multiplying a vector a by a real number k (a **scalar**) gives the vector $k\,a$, found by multiplying each component by k:

$$k\,a = k\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k\,a_1 \\ k\,a_2 \end{pmatrix}.$$

Geometrically, $k\,a$ is a **scaled** in length by the factor $|k|$: it points the same way as a when $k>0$ and the opposite way when $k<0$, while $k=0$ gives the zero vector. For example $2\,a$ is twice as long as a in the same direction, and $-a$ is the same length in the reverse direction.



Two non-zero vectors are **parallel** exactly when one is a scalar multiple of the other: a is parallel to b when $b=k\,a$ for some non-zero scalar k. They point the same way if $k>0$ and opposite ways if $k<0$.

Vector algebra.

Scalar multiplication combines with addition through the **distributive laws**

$$k(a+b)=k\,a+k\,b,(k+l)a=k\,a+l\,a,$$

while scaling twice in succession obeys the **associative law**

$$k(l\,a)=(k\,l)a.$$

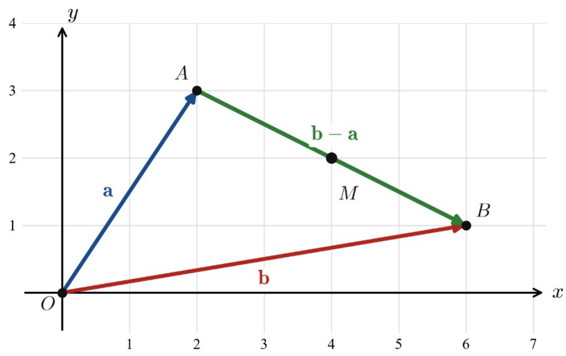
Together these let vector expressions be expanded and simplified just like ordinary algebra, and like terms are collected by treating each distinct vector as a separate variable.

Position vectors.

Choose a fixed origin O. The **position vector** of a point A is the vector from the origin to that point, $\overrightarrow{OA}=a$; likewise $\overrightarrow{OB}=b$. Any vector joining two points can then be written using position vectors. Travelling from A to O and then from O to B,

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=-a+b=b-a.$$

So the vector from A to B is *the position vector of the head minus the position vector of the tail.*



The midpoint of a segment.

Let M be the **midpoint** of AB. Then $\overrightarrow{AM} = 1/2\,\overrightarrow{AB} = 1/2(b - a)$, and adding this to $\overrightarrow{OA}$ gives the position vector of M:

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = a + 1/2(b - a) = 1/2(a + b).$$

The position vector of the midpoint of AB is the **average** of the position vectors of A and B. These position-vector methods are the foundation for the vector proofs of geometric results met later in the chapter.

Worked examples

Example 1 — scaffolded

Given $a = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, find $a + b$, $a - b$ and $2a + 3b$.

Working	Comment
$a + b = \begin{pmatrix} 3 + (-1) \\ 1 + 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$	Add corresponding components.
$a - b = \begin{pmatrix} 3 - (-1) \\ 1 - 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$	Subtract corresponding components; note $3 - (-1) = 4$.
$2a = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, 3b = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$	Multiply each component by the scalar first.
$2a + 3b = \begin{pmatrix} 6 + (-3) \\ 2 + 12 \end{pmatrix} = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$	Then add the two results component by component.

Example 2 — scaffolded

The points A and B have position vectors $a = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ relative to an origin O. Find $\overrightarrow{AB}$ and the position vector of the midpoint M of AB.

Working	Comment
$\overrightarrow{AB} = b - a = \begin{pmatrix} 6 - 2 \\ 5 - (-1) \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$	Head minus tail: position vector of B minus that of A.
$\overrightarrow{OM} = 1/2(a + b) = 1/2\begin{pmatrix} 2 + 6 \\ -1 + 5 \end{pmatrix} = 1/2\begin{pmatrix} 8 \\ 4 \end{pmatrix}$	The midpoint is the average of the two position vectors.
$\overrightarrow{OM} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$	Multiply each component by the scalar $1/2$.

Example 3 — unscaffolded

Simplify $3(a+2b)-2(a-b)$.

Expanding each bracket with the distributive law and then collecting like terms,

$$3(a+2b)-2(a-b)=3a+6b-2a+2b=(3-2)a+(6+2)b.$$

$\therefore 3(a+2b)-2(a-b)=a+8b$.

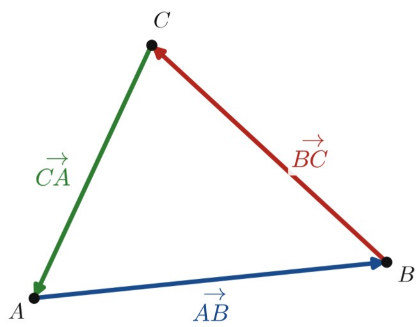
Example 4 — unscaffolded

The points A, B and C have position vectors a, b and c relative to an origin O. Show that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$, and interpret the result.

Write each side of the triangle as head minus tail: $\overrightarrow{AB}=b-a$, $\overrightarrow{BC}=c-b$ and $\overrightarrow{CA}=a-c$. Adding these three vectors,

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=(b-a)+(c-b)+(a-c)=0,$$

since each position vector cancels with its negative. Geometrically the three displacements carry a point once around the triangle ABC and back to its start, a net displacement of zero.



$\therefore \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$.

Practice questions

Scaffolded

Q1. Given $a=\begin{pmatrix}5\\2\end{pmatrix}$ and $b=\begin{pmatrix}-3\\4\end{pmatrix}$, find each of the following. (a) $a+b$ (b) $a-b$ (c) $b-a$ (d) $3a$

Q2. Given $u=\begin{pmatrix}-2\\6\end{pmatrix}$ and $v=\begin{pmatrix}4\\-2\end{pmatrix}$, find each of the following. (a) $2u$ (b) $1/2v$ (c) $u+v$ (d) $2u-3v$

Q3. Let $a=\begin{pmatrix}2\\-5\end{pmatrix}$. (a) Write down $-a$. (b) Find $3a$. (c) Find the scalar k for which $\begin{pmatrix}-6\\15\end{pmatrix}=ka$, and state whether the two vectors point in the same or opposite directions. (d) Evaluate $2a+(-2a)$ and name the resulting vector.

Q4. The points A and B have position vectors $a=\begin{pmatrix}1\\3\end{pmatrix}$ and $b=\begin{pmatrix}7\\-1\end{pmatrix}$ relative to an origin O. (a) Find $\overrightarrow{AB}$. (b) Find $\overrightarrow{BA}$. (c) Find the position vector of the midpoint M of AB. (d) State how $\overrightarrow{BA}$ is related to $\overrightarrow{AB}$.

Q5. Simplify $2(a-3b)+3(b+a)$. (a) Expand $2(a-3b)$. (b) Expand $3(b+a)$. (c) Collect the a terms. (d) Collect the b terms, and state the simplified vector.

Q6. The points A, B and C have position vectors a, b and c relative to an origin O. (a) Express $\overrightarrow{AB}$ in terms of a and b. (b) Express $\overrightarrow{BC}$ in terms of b and c. (c) Express $\overrightarrow{AC}$ in terms of a and c. (d) Hence verify that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.

Unscaffolded

Q7. Given $a=\begin{pmatrix}4\\-3\end{pmatrix}$ and $b=\begin{pmatrix}-2\\5\end{pmatrix}$, find $3a-2b$.

Q8. Given $a=\begin{pmatrix}1\\2\end{pmatrix}$, $b=\begin{pmatrix}-3\\0\end{pmatrix}$ and $c=\begin{pmatrix}4\\-5\end{pmatrix}$, find $a+b+c$.

Q9. A vector $p=\begin{pmatrix}6\\-2\end{pmatrix}$. Find the vector q for which $p+q=0$.

Q10. Simplify $4(a+b)-(3a+2b)$.

Q11. Determine whether $u=\begin{pmatrix}-4\\6\end{pmatrix}$ and $v=\begin{pmatrix}6\\-9\end{pmatrix}$ are parallel. If they are, state the scalar k for which $v=ku$.

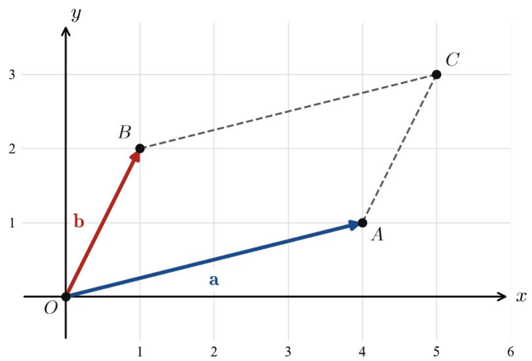
Q12. The points A and B have position vectors $a=\begin{pmatrix}-3\\2\end{pmatrix}$ and $b=\begin{pmatrix}5\\8\end{pmatrix}$. Find $\overrightarrow{AB}$ and the position vector of the midpoint of AB.

Q13. Given $a=\begin{pmatrix}2\\4\end{pmatrix}$ and $b=\begin{pmatrix}-2\\10\end{pmatrix}$, find the vector x that satisfies $a+2x=b$.

Q14. Given $a=\begin{pmatrix}1\\2\end{pmatrix}$, $b=\begin{pmatrix}3\\-1\end{pmatrix}$ and $c=\begin{pmatrix}5\\3\end{pmatrix}$, find scalars m and n such that $ma+nb=c$.

Q15. M is the midpoint of AB. The point A has position vector $\begin{pmatrix}2\\-3\end{pmatrix}$ and M has position vector $\begin{pmatrix}5\\1\end{pmatrix}$. Find the position vector of B.

Q16. $OACB$ is a parallelogram with $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$. Express each of $\overrightarrow{OC}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of a and b.



Q17. The points A, B and C have position vectors $a=\begin{pmatrix}0\\1\end{pmatrix}$, $b=\begin{pmatrix}2\\5\end{pmatrix}$ and $c=\begin{pmatrix}5\\11\end{pmatrix}$. By finding $\overrightarrow{AB}$ and $\overrightarrow{AC}$, show that A, B and C are collinear.

Q18. The points A and B have position vectors a and b relative to an origin O. M is the midpoint of AB and N is the midpoint of OA. Show that $\overrightarrow{NM}=1/2b$.

Answers

Q1. (a) $\begin{pmatrix}2\\6\end{pmatrix}$; (b) $\begin{pmatrix}8\\-2\end{pmatrix}$; (c) $\begin{pmatrix}-8\\2\end{pmatrix}$; (d) $\begin{pmatrix}15\\6\end{pmatrix}$. **Q2.** (a) $\begin{pmatrix}-4\\12\end{pmatrix}$; (b) $\begin{pmatrix}2\\-1\end{pmatrix}$; (c) $\begin{pmatrix}2\\4\end{pmatrix}$; (d) $\begin{pmatrix}-16\\18\end{pmatrix}$. **Q3.** (a) $\begin{pmatrix}-2\\5\end{pmatrix}$; (b) $\begin{pmatrix}6\\-15\end{pmatrix}$; (c) $k=-3$, opposite directions; (d) 0, the zero vector. **Q4.** (a) $\begin{pmatrix}6\\-4\end{pmatrix}$; (b) $\begin{pmatrix}-6\\4\end{pmatrix}$; (c) $\begin{pmatrix}4\\1\end{pmatrix}$; (d) $\overrightarrow{BA}=-\overrightarrow{AB}$. **Q5.** $5a-3b$. **Q6.** (a) $b-a$; (b) $c-b$; (c) $c-a$; (d) the sum is $c-a=\overrightarrow{AC}$. **Q7.** $\begin{pmatrix}16\\-19\end{pmatrix}$. **Q8.** $\begin{pmatrix}2\\-3\end{pmatrix}$. **Q9.** $q=\begin{pmatrix}-6\\2\end{pmatrix}$. **Q10.** $a+2b$. **Q11.** Parallel, with $k=-3/2$. **Q12.** $\overrightarrow{AB}=\begin{pmatrix}8\\6\end{pmatrix}$; midpoint $\begin{pmatrix}1\\5\end{pmatrix}$. **Q13.** $x=\begin{pmatrix}-2\\3\end{pmatrix}$. **Q14.** $m=2$, $n=1$. **Q15.** $\begin{pmatrix}8\\5\end{pmatrix}$. **Q16.** $\overrightarrow{OC}=a+b$; $\overrightarrow{AB}=b-a$; $\overrightarrow{AC}=b$. **Q17.** $\overrightarrow{AB}=\begin{pmatrix}2\\4\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}5\\10\end{pmatrix}=5/2\overrightarrow{AB}$, so the points are collinear. **Q18.** $\overrightarrow{NM}=1/2b$.

Fully-worked solutions

Q1. Add and subtract corresponding components, and multiply each component by the scalar. $a+b=\begin{pmatrix}5+(-3)\\2+4\end{pmatrix}=\begin{pmatrix}2\\6\end{pmatrix}$; $a-b=\begin{pmatrix}5-(-3)\\2-4\end{pmatrix}=\begin{pmatrix}8\\-2\end{pmatrix}$; $b-a=\begin{pmatrix}-3-5\\4-2\end{pmatrix}=\begin{pmatrix}-8\\2\end{pmatrix}$; $3a=\begin{pmatrix}3\times5\\3\times2\end{pmatrix}=\begin{pmatrix}15\\6\end{pmatrix}$. Notice that $b-a=-(a-b)$, as expected.

Q2. $2u=\begin{pmatrix}2\times(-2)\\2\times6\end{pmatrix}=\begin{pmatrix}-4\\12\end{pmatrix}$; $1/2v=\begin{pmatrix}1/2\times4\\1/2\times(-2)\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}$; $u+v=\begin{pmatrix}-2+4\\6+(-2)\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$. For the last part, first scale: $2u=\begin{pmatrix}-4\\12\end{pmatrix}$ and $3v=\begin{pmatrix}12\\-6\end{pmatrix}$, so $2u-3v=\begin{pmatrix}-4-12\\12-(-6)\end{pmatrix}=\begin{pmatrix}-16\\18\end{pmatrix}$.

Q3. (a) $-a=\begin{pmatrix}-2\\5\end{pmatrix}$. (b) $3a=\begin{pmatrix}6\\-15\end{pmatrix}$. (c) Comparing components, $\begin{pmatrix}-6\\15\end{pmatrix}=k\begin{pmatrix}2\\-5\end{pmatrix}$ needs $2k=-6$ and $-5k=15$, both giving $k=-3$. Since k is negative, the two vectors point in opposite directions. (d) $2a+(-2a)=\begin{pmatrix}4\\-10\end{pmatrix}+\begin{pmatrix}-4\\10\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=0$, the zero vector.

Q4. (a) $\overrightarrow{AB}=b-a=\begin{pmatrix}7-1\\-1-3\end{pmatrix}=\begin{pmatrix}6\\-4\end{pmatrix}$. (b) $\overrightarrow{BA}=a-b=\begin{pmatrix}1-7\\3-(-1)\end{pmatrix}=\begin{pmatrix}-6\\4\end{pmatrix}$. (c) $\overrightarrow{OM}=1/2(a+b)=1/2\begin{pmatrix}8\\2\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$. (d) $\overrightarrow{BA}=-\overrightarrow{AB}$: the two vectors have equal length but opposite direction.

Q5. Expanding, $2(a-3b)=2a-6b$ and $3(b+a)=3b+3a$. Adding and collecting like terms,

$$2a-6b+3a+3b=(2+3)a+(-6+3)b=5a-3b.$$

Q6. Writing each vector as head minus tail, $\overrightarrow{AB}=b-a$, $\overrightarrow{BC}=c-b$ and $\overrightarrow{AC}=c-a$. Then

$$\overrightarrow{AB}+\overrightarrow{BC}=(b-a)+(c-b)=c-a=\overrightarrow{AC},$$

which confirms the triangle law: going from A to B and then B to C is the same as going directly from A to C.

Q7. First scale each vector: $3a=\begin{pmatrix}12\\-9\end{pmatrix}$ and $2b=\begin{pmatrix}-4\\10\end{pmatrix}$. Then $3a-2b=\begin{pmatrix}12-(-4)\\-9-10\end{pmatrix}=\begin{pmatrix}16\\-19\end{pmatrix}$.

Q8. Add the three vectors component by component: $a+b+c=\begin{pmatrix}1+(-3)+4\\2+0+(-5)\end{pmatrix}=\begin{pmatrix}2\\-3\end{pmatrix}$.

Q9. From $p+q=0$, the vector q is the negative of p: $q=-p=\begin{pmatrix}-6\\2\end{pmatrix}$.

Q10. Expanding, $4(a+b)=4a+4b$, so

$$4a+4b-3a-2b=(4-3)a+(4-2)b=a+2b.$$

Q11. Seek a scalar k with $v=ku$. The first components give $6=-4k$, so $k=-3/2$; the second components give $-9=6k$, again $k=-3/2$. Because a single value of k satisfies both components, $v=-3/2u$ and the vectors are parallel (pointing in opposite directions, since $k<0$).

Q12. $\overrightarrow{AB}=b-a=\begin{pmatrix}5-(-3)\\8-2\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}$. The midpoint has position vector $1/2(a+b)=1/2\begin{pmatrix}2\\10\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}$.

Q13. Rearranging $a+2x=b$ gives $2x=b-a$, so $x=1/2(b-a)$. Now $b-a=\begin{pmatrix}-2-2\\10-4\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}$, hence $x=1/2\begin{pmatrix}-4\\6\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}$.

Q14. Requiring $ma+nb=c$ and equating components gives the system

$$m+3n=5,2m-n=3.$$

From the second equation $n=2m-3$. Substituting into the first, $m+3(2m-3)=5$, so $7m-9=5$, giving $m=2$ and then $n=2(2)-3=1$. Checking, $2a+b=\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}=c$.

Q15. Let b be the position vector of B. Since M is the midpoint of AB, $1/2(a+b)$ equals the position vector of M, so $a+b=2\times\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}10\\2\end{pmatrix}$. Hence $b=\begin{pmatrix}10\\2\end{pmatrix}-\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}8\\5\end{pmatrix}$.

Q16. In the parallelogram $OACB$ the diagonal from O is $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{OB}=a+b$ (the parallelogram law). The other diagonal is $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=b-a$. Finally $\overrightarrow{AC}$ is the side from A to C; since $OACB$ is a parallelogram this side is equal to $\overrightarrow{OB}$, so $\overrightarrow{AC}=b$. (Equivalently, $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(a+b)-a=b$.)

Q17. Using head minus tail, $\overrightarrow{AB}=b-a=\begin{pmatrix}2-0\\5-1\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$ and $\overrightarrow{AC}=c-a=\begin{pmatrix}5-0\\11-1\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix}$. Since $\overrightarrow{AC}=5/2\begin{pmatrix}2\\4\end{pmatrix}=5/2\overrightarrow{AB}$, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. They share the common point A, so A, B and C lie on the same straight line; that is, the points are collinear.

Q18. Take position vectors relative to O. As N is the midpoint of OA, its position vector is $\overrightarrow{ON}=1/2a$, and as M is the midpoint of AB, $\overrightarrow{OM}=1/2(a+b)$. Therefore

$$\overrightarrow{NM}=\overrightarrow{OM}-\overrightarrow{ON}=1/2(a+b)-1/2a=1/2b.$$

This shows $\overrightarrow{NM}$ is parallel to $\overrightarrow{OB}$ and half its length — a vector form of the midpoint (mid-segment) theorem.

Theory

An earlier section introduced vectors as directed quantities that can be added, subtracted and multiplied by a scalar, and represented the position of a point by its **position vector** from the origin O. This section gives every vector a set of **numbers** — its components along fixed reference directions — so that all of those operations, together with lengths and directions, can be carried out by arithmetic.

Orthogonal unit vectors.

A **unit vector** is a vector of magnitude 1. Along the positive direction of each coordinate axis we place one standard unit vector. In two dimensions,

$$i = \text{the unit vector along the positive } x\text{-axis}, \; j = \text{the unit vector along the positive } y\text{-axis}.$$

These two vectors are **orthogonal** (perpendicular) and each has length 1, so they are called **orthogonal unit vectors**. In three dimensions we add a third axis, the z-axis, perpendicular to both of the others, with unit vector k along its positive direction. The three vectors i, j, k are mutually perpendicular and form a **right-handed** set: turning i towards j advances a right-hand screw along k.

Component (ijk) form and column form.

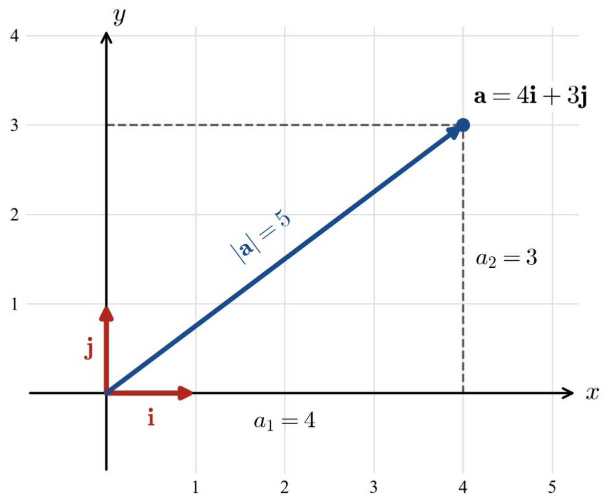
Because i and j point along the axes, every two-dimensional vector is the sum of a multiple of i and a multiple of j:

$$a = a_1 i + a_2 j.$$

The numbers a_1 and a_2 are the **components** of a. The same vector may be written in **column form** $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. In three dimensions,

$$a = a_1 i + a_2 j + a_3 k = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Two vectors are equal exactly when **all** of their corresponding components are equal. The diagram below shows $a = 4i + 3j$: starting at O you move 4 units in the direction of i and then 3 units in the direction of j to reach the tip.



Adding or subtracting vectors, and multiplying by a scalar, is done **component by component**: if $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$ then

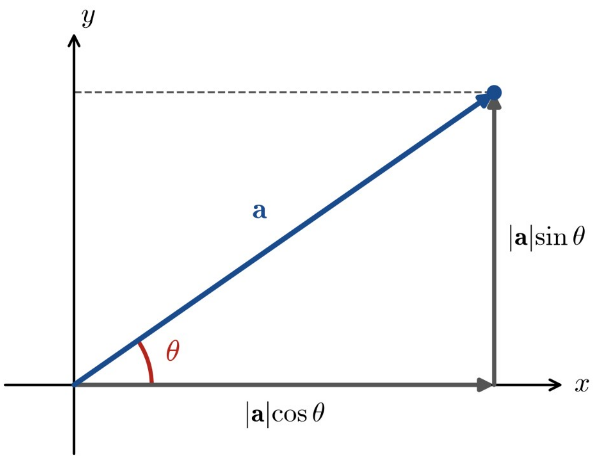
$$a + b = (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k, \lambda a = \lambda a_1 i + \lambda a_2 j + \lambda a_3 k.$$

Resolution into rectangular components.

Reading the process backwards, a single vector can be **resolved** into a part along the x-axis and a part along the y-axis — its **rectangular components**. If a has magnitude $|a|$ and makes an angle θ with the positive direction of the x-axis, then from the right-angled triangle formed by the vector and its shadow on the x-axis,

$$a=|a|\cos\theta\, i+|a|\sin\theta\, j.$$

The horizontal component is $|a|\cos\theta$ and the vertical component is $|a|\sin\theta$.



Magnitude.

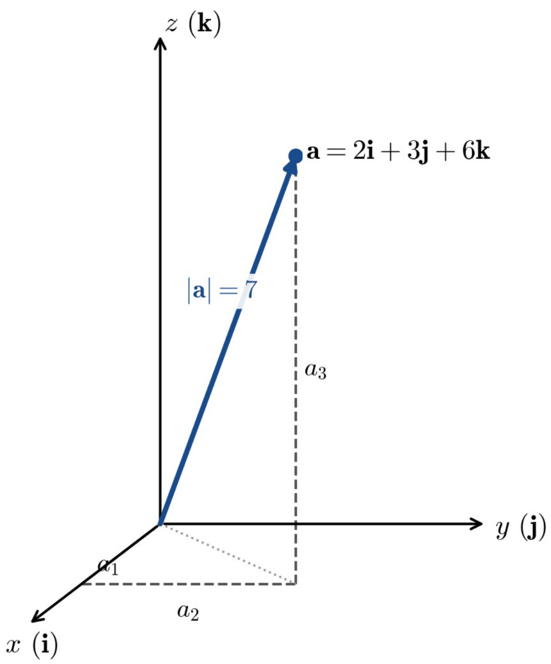
The **magnitude** (or length) of a vector is the distance from its tail to its tip. For a two-dimensional vector, Pythagoras' theorem applied to the rectangle of components gives

$$|a|=\sqrt{a_1^2+a_2^2}.$$

In three dimensions Pythagoras is applied twice — first in the base plane, then up to the tip — giving

$$|a|=\sqrt{a_1^2+a_2^2+a_3^2}.$$

The magnitude is a non-negative real number, and $|a|=0$ only for the zero vector 0. The figure below shows $a=2i+3j+6k$, whose magnitude is $\sqrt{2^2+3^2+6^2}=\sqrt{49}=7$.



Unit vectors.

The unit vector in the direction of a non-zero vector a is written $\hat{a}$ and is found by dividing a by its own length:

$$\hat{a}=\frac{a}{|a|}.$$

This has magnitude 1 and points the same way as a, because dividing by the positive scalar $|a|$ scales the length to 1 without changing the direction. Reversing the idea, a vector of a **chosen magnitude m in the direction of a** is

$$m\hat{a}=\frac{m}{|a|}a.$$

Using a negative value of m produces a vector of magnitude $|m|$ pointing in the **opposite** direction.

Distance between two points.

Let A and B have position vectors a and b. The vector from A to B is

$$\overrightarrow{AB}=b-a,$$

and the **distance** between the points is its magnitude,

$$AB=|b-a|=\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

in three dimensions, and the corresponding two-term expression in two dimensions. This is the vector form of the distance formula.

Direction of a two-dimensional vector.

The direction of a two-dimensional vector is described by the angle θ it makes with the positive direction of the x-axis, measured anticlockwise, with $0°\le\theta<360°$. Comparing $a=a_1 i+a_2 j$ with the resolved form gives

$$\cos\theta=\frac{a_1}{|a|},\sin\theta=\frac{a_2}{|a|},\tan\theta=\frac{a_2}{a_1}(a_1\neq 0).$$

Since $\tan\theta$ alone cannot tell the two directions θ and $\theta+180°$ apart, the safe method is to find the acute **reference angle**

$$\alpha=\tan^{-1}\left|\frac{a_2}{a_1}\right|(0°<\alpha<90°),$$

then read θ from the quadrant fixed by the signs of a_1 and a_2: in the first quadrant $\theta=\alpha$; in the second, $\theta=180°-\alpha$; in the third, $\theta=180°+\alpha$; and in the fourth, $\theta=360°-\alpha$. The acute angle a vector makes with an axis is the reference angle to that axis. (Directions in three dimensions are treated in a later chapter.)

Worked examples

Example 1 — scaffolded

For $a=3i-4j$, find $|a|$, the unit vector $\hat{a}$, and a vector of magnitude 15 in the direction of a.

Working	Comment
$a=3i-4j$, so $a_1=3$ and $a_2=-4$.	Read off the components.
$\lvert a\rvert=\sqrt{a_1^2+a_2^2}=\sqrt{3^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5.$	Magnitude from Pythagoras.
$\hat{a}=\frac{a}{\lvert a\rvert}=\frac{1}{5}(3i-4j)=\frac{3}{5}i-\frac{4}{5}j.$	Divide the vector by its magnitude.

Working	Comment
$15\hat{a} = \frac{15}{5}(3i - 4j) = 3(3i - 4j) = 9i - 12j.$	Scale the unit vector to length 15.
$\lvert 9i - 12j \rvert = \sqrt{81 + 144} = \sqrt{225} = 15$, as required.	The magnitude check confirms the length is 15.

Example 2 — scaffolded

The points A and B have position vectors $a = i + j + 2k$ and $b = 3i + 4j + 8k$. Find $\overrightarrow{AB}$ in component form, the distance AB, and a unit vector in the direction of $\overrightarrow{AB}$.

Working	Comment
$a = i + j + 2k, b = 3i + 4j + 8k.$	Write down the two position vectors.
$\overrightarrow{AB} = b - a = (3 - 1)i + (4 - 1)j + (8 - 2)k = 2i + 3j + 6$	Subtract componentwise: tip minus tail.
$AB = \lvert \overrightarrow{AB} \rvert = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7.$	The distance is the magnitude of $\overrightarrow{AB}$.
$\frac{\overrightarrow{AB}}{AB} = \frac{1}{7}(2i + 3j + 6k) = \frac{2}{7}i + \frac{3}{7}j + \frac{6}{7}k.$	Divide by the length for a unit vector along $\overrightarrow{AB}$.

Example 3 — unscaffolded

A vector p has magnitude 20 and makes an angle of $60°$ with the positive direction of the x-axis, measured anticlockwise. Express p in component form.

Resolving into rectangular components, $p = \lvert p \rvert \cos\theta\, i + \lvert p \rvert \sin\theta\, j$ with $\lvert p \rvert = 20$ and $\theta = 60°$. Using the exact values $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$,

$$p = 20\cos 60°\, i + 20\sin 60°\, j = 20 \cdot \frac{1}{2} i + 20 \cdot \frac{\sqrt{3}}{2} j = 10i + 10\sqrt{3}\, j.$$

$\therefore p = 10i + 10\sqrt{3}\, j.$

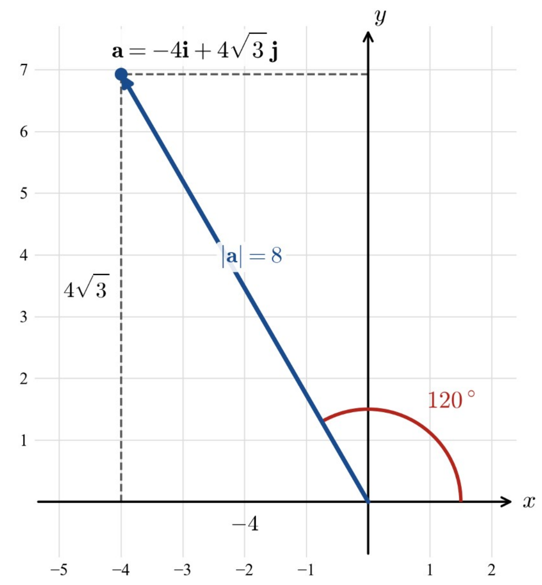
Example 4 — unscaffolded

Find the magnitude of $a = -4i + 4\sqrt{3}\, j$ and the angle it makes with the positive direction of the x-axis, measured anticlockwise.

The magnitude is

$$\lvert a \rvert = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{16 + 48} = \sqrt{64} = 8.$$

Because $a_1 = -4 < 0$ and $a_2 = 4\sqrt{3} > 0$, the vector lies in the second quadrant. The reference angle satisfies $\tan\alpha = \left\lvert \frac{4\sqrt{3}}{-4} \right\rvert = \sqrt{3}$, so $\alpha = 60°$. In the second quadrant the required angle is $\theta = 180° - \alpha = 180° - 60° = 120°$.



As a check, $|a|\cos 120° = 8 \cdot \left(-\dfrac{1}{2}\right) = -4$ and $|a|\sin 120° = 8 \cdot \dfrac{\sqrt{3}}{2} = 4\sqrt{3}$, which recovers the components.

$\therefore |a| = 8$ and a makes an angle of $120°$ with the positive x-axis.

Practice questions

Scaffolded

Q1. For $a = 5i + 12j$: (a) find $|a|$; (b) find the unit vector $\hat{a}$; (c) find a vector of magnitude 26 in the direction of a.

Q2. For $b = 2i - 6j + 3k$: (a) find $|b|$; (b) find the unit vector $\hat{b}$; (c) find a vector of magnitude 14 in the direction of b.

Q3. A vector a has magnitude 10 and makes an angle of $45°$ with the positive x-axis. (a) Find the i-component $|a|\cos 45°$. (b) Find the j-component $|a|\sin 45°$. (c) Write a in component form.

Q4. The points A and B have position vectors $2i - j + 3k$ and $6i + 2j + 15k$. (a) Find $\overrightarrow{AB}$ in component form. (b) Find the distance AB. (c) Find a unit vector in the direction of $\overrightarrow{AB}$.

Q5. Find the direction of $c = 4i - 4j$. (a) Find the magnitude $|c|$. (b) Find the reference angle α. (c) State the angle measured anticlockwise from the positive x-axis.

Q6. Find, in component form, a vector v of magnitude 6 that makes an angle of $150°$ with the positive x-axis. (a) Write down $\cos 150°$ and $\sin 150°$. (b) Find the i- and j-components. (c) State v in component form.

Unscaffolded

Q7. Find $|a|$ and $\hat{a}$ for $a = 8i - 15j$.

Q8. Find the magnitude of $v = 6i + 6j - 7k$ and a unit vector in its direction.

Q9. A vector has magnitude 14 and makes an angle of $30°$ with the positive x-axis. Express it in component form.

Q10. Find the distance between the points $A(-1, 4)$ and $B(2, 8)$.

Q11. Find the distance between the points $P(2, 1, -3)$ and $Q(3, 5, 5)$.

Q12. Find the vector of magnitude 10 in the direction of $d = 3i - 4j$.

Q13. Find the vector of magnitude 12 in the direction of $i - 2j + 2k$.

Q14. Find the magnitude of $b = -5i + 5\sqrt{3}\,j$ and the angle it makes with the positive x-axis, measured anticlockwise.

Q15. Find the magnitude of $c = -2\sqrt{3}i - 2\,j$ and the angle it makes with the positive x-axis, measured anticlockwise.

Q16. The points A and B have position vectors $i + 2\,j - 2k$ and $4i + 6\,j + 10k$. Find $\overrightarrow{AB}$, the distance AB, and a unit vector in the direction of $\overrightarrow{AB}$.

Q17. A displacement of magnitude 16 is directed at $135°$ to the positive x-axis, measured anticlockwise. Express it in component form.

Q18. For $a = 2i - j + 2k$: (a) find $|a|$ and $\hat{a}$; (b) find a vector b of magnitude 9 in the direction **opposite** to a.

Answers

Q1. (a) $|a|=13$; (b) $\hat{a}=\frac{1}{13}(5i+12j)$; (c) $10i+24j$. **Q2.** (a) $|b|=7$; (b) $\hat{b}=\frac{1}{7}(2i-6j+3k)$; (c) $4i-12j+6k$. **Q3.** (a) $5\sqrt{2}$; (b) $5\sqrt{2}$; (c) $a=5\sqrt{2}i+5\sqrt{2}j$. **Q4.** (a) $\overrightarrow{AB}=4i+3j+12k$; (b) $AB=13$; (c) $\frac{1}{13}(4i+3j+12k)$. **Q5.** (a) $|c|=4\sqrt{2}$; (b) $\alpha=45°$; (c) $\theta=315°$. **Q6.** (a) $\cos 150°=-\frac{\sqrt{3}}{2}$, $\sin 150°=\frac{1}{2}$; (b) $-3\sqrt{3}$ and 3; (c) $v=-3\sqrt{3}i+3j$. **Q7.** $|a|=17$, $\hat{a}=\frac{1}{17}(8i-15j)$. **Q8.** $|v|=11$, unit vector $\frac{1}{11}(6i+6j-7k)$. **Q9.** $7\sqrt{3}i+7j$. **Q10.** $AB=5$. **Q11.** $PQ=9$. **Q12.** $6i-8j$. **Q13.** $4i-8j+8k$. **Q14.** $|b|=10$, $\theta=120°$. **Q15.** $|c|=4$, $\theta=210°$. **Q16.** $\overrightarrow{AB}=3i+4j+12k$, $AB=13$, unit vector $\frac{1}{13}(3i+4j+12k)$. **Q17.** $-8\sqrt{2}i+8\sqrt{2}j$. **Q18.** (a) $|a|=3$, $\hat{a}=\frac{1}{3}(2i-j+2k)$; (b) $b=-6i+3j-6k$.

Fully-worked solutions

Q1. $a_1=5$, $a_2=12$. (a) $|a|=\sqrt{5^2+12^2}=\sqrt{25+144}=\sqrt{169}=13$. (b) $\hat{a}=\frac{a}{|a|}=\frac{1}{13}(5i+12j)$. (c) $26\hat{a}=\frac{26}{13}(5i+12j)=2(5i+12j)=10i+24j$; its magnitude is $\sqrt{10^2+24^2}=\sqrt{676}=26$.

Q2. $b=2i-6j+3k$. (a) $|b|=\sqrt{2^2+(-6)^2+3^2}=\sqrt{4+36+9}=\sqrt{49}=7$. (b) $\hat{b}=\frac{1}{7}(2i-6j+3k)$. (c) $14\hat{b}=\frac{14}{7}(2i-6j+3k)=2(2i-6j+3k)=4i-12j+6k$.

Q3. With $|a|=10$ and $\theta=45°$, use $\cos 45°=\sin 45°=\frac{\sqrt{2}}{2}$. (a) $|a|\cos 45°=10\cdot\frac{\sqrt{2}}{2}=5\sqrt{2}$. (b) $|a|\sin 45°=10\cdot\frac{\sqrt{2}}{2}=5\sqrt{2}$. (c) $a=5\sqrt{2}i+5\sqrt{2}j$.

Q4. $a=2i-j+3k$, $b=6i+2j+15k$. (a) $\overrightarrow{AB}=b-a=(6-2)i+(2-(-1))j+(15-3)k=4i+3j+12k$. (b) $AB=\sqrt{4^2+3^2+12^2}=\sqrt{16+9+144}=\sqrt{169}=13$. (c) $\frac{\overrightarrow{AB}}{AB}=\frac{1}{13}(4i+3j+12k)$.

Q5. $c=4i-4j$, so $c_1=4>0$ and $c_2=-4<0$: the vector is in the fourth quadrant. (a) $|c|=\sqrt{4^2+(-4)^2}=\sqrt{32}=4\sqrt{2}$. (b) $\tan\alpha=\left|\frac{-4}{4}\right|=1$, so $\alpha=45°$. (c) In the fourth quadrant, $\theta=360°-\alpha=360°-45°=315°$.

Q6. $|v|=6$, $\theta=150°$. (a) $\cos 150°=-\frac{\sqrt{3}}{2}$ and $\sin 150°=\frac{1}{2}$. (b) i-component $=6\cdot\left(-\frac{\sqrt{3}}{2}\right)=-3\sqrt{3}$; j-component $=6\cdot\frac{1}{2}=3$. (c) $v=-3\sqrt{3}i+3j$.

Q7. $|a|=\sqrt{8^2+(-15)^2}=\sqrt{64+225}=\sqrt{289}=17$, so $\hat{a}=\frac{1}{17}(8i-15j)$.

Q8. $|v|=\sqrt{6^2+6^2+(-7)^2}=\sqrt{36+36+49}=\sqrt{121}=11$, so a unit vector in the direction of v is $\frac{1}{11}(6i+6j-7k)$.

Q9. Using $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$, the vector is $14\cos 30°\, i + 14\sin 30°\, j = 14 \cdot \frac{\sqrt{3}}{2} i + 14 \cdot \frac{1}{2} j = 7\sqrt{3}\, i + 7\, j$.

Q10. $\overrightarrow{AB} = (2-(-1))i + (8-4)j = 3i + 4j$, so $AB = \sqrt{3^2+4^2} = \sqrt{25} = 5$.

Q11. $\overrightarrow{PQ} = (3-2)i + (5-1)j + (5-(-3))k = i + 4j + 8k$, so $PQ = \sqrt{1^2+4^2+8^2} = \sqrt{1+16+64} = \sqrt{81} = 9$.

Q12. $|d| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$. The required vector is $10\hat{d} = \frac{10}{5}(3i-4j) = 2(3i-4j) = 6i - 8j$.

Q13. The direction vector $i - 2j + 2k$ has magnitude $\sqrt{1^2+(-2)^2+2^2} = \sqrt{9} = 3$. The required vector is $\frac{12}{3}(i-2j+2k) = 4(i-2j+2k) = 4i - 8j + 8k$.

Q14. $b = -5i + 5\sqrt{3}\, j$, so $b_1 = -5 < 0$ and $b_2 = 5\sqrt{3} > 0$: second quadrant.
$|b| = \sqrt{(-5)^2 + (5\sqrt{3})^2} = \sqrt{25+75} = \sqrt{100} = 10$. The reference angle has $\tan\alpha = \left|\frac{5\sqrt{3}}{-5}\right| = \sqrt{3}$, so $\alpha = 60°$; in the second quadrant $\theta = 180° - 60° = 120°$.

Q15. $c = -2\sqrt{3}\, i - 2j$, so $c_1 = -2\sqrt{3} < 0$ and $c_2 = -2 < 0$: third quadrant.
$|c| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = \sqrt{16} = 4$. The reference angle has $\tan\alpha = \left|\frac{-2}{-2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$, so $\alpha = 30°$; in the third quadrant $\theta = 180° + 30° = 210°$.

Q16. $a = i + 2j - 2k$, $b = 4i + 6j + 10k$. $\overrightarrow{AB} = b - a = (4-1)i + (6-2)j + (10-(-2))k = 3i + 4j + 12k$;
$AB = \sqrt{3^2+4^2+12^2} = \sqrt{9+16+144} = \sqrt{169} = 13$; and a unit vector in the direction of $\overrightarrow{AB}$ is $\frac{1}{13}(3i+4j+12k)$.

Q17. Using $\cos 135° = -\frac{\sqrt{2}}{2}$ and $\sin 135° = \frac{\sqrt{2}}{2}$, the displacement is
$16\cos 135°\, i + 16\sin 135°\, j = 16 \cdot \left(-\frac{\sqrt{2}}{2}\right) i + 16 \cdot \frac{\sqrt{2}}{2} j = -8\sqrt{2}\, i + 8\sqrt{2}\, j$.

Q18. $a = 2i - j + 2k$. (a) $|a| = \sqrt{2^2+(-1)^2+2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$, so $\hat{a} = \frac{1}{3}(2i - j + 2k)$. (b) The opposite direction reverses the sign, so $b = -9\hat{a} = -\frac{9}{3}(2i - j + 2k) = -3(2i - j + 2k) = -6i + 3j - 6k$, whose magnitude is $\sqrt{(-6)^2+3^2+(-6)^2} = \sqrt{81} = 9$.

Theory

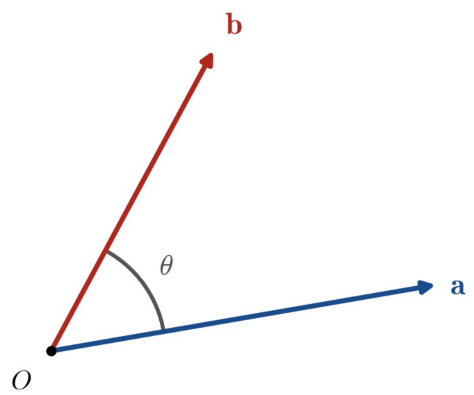
The vector operations so far — addition, subtraction and multiplication by a scalar — each produce a **vector**. This section introduces a way of combining two vectors that produces a **number** (a scalar): the **scalar product**, more often called the **dot product** because of the symbol $a \cdot b$ written between the two vectors. It measures how much the two vectors point in the same direction, and it is the key tool for finding the angle between two vectors and for testing whether they are perpendicular.

Geometric definition.

Place the two non-zero vectors a and b tail to tail and let θ be the angle between them, so that $0° \le \theta \le 180°$. The dot product is

$$a \cdot b = |a||b|\cos\theta.$$

Because $|a|$ and $|b|$ are positive, the sign of $a \cdot b$ is the sign of $\cos\theta$.



Component definition.

If $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, then

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

For two-dimensional vectors the third term is simply absent: $a \cdot b = a_1 b_1 + a_2 b_2$. In words: multiply matching components and add the results. This is usually the quickest way to evaluate a dot product from components.

The two definitions describe the **same number**. This can be shown with the cosine rule, or by expanding $a \cdot b$ with the distributive rule below together with the standard-unit-vector facts

$$i \cdot i = j \cdot j = k \cdot k = 1, i \cdot j = j \cdot k = k \cdot i = 0,$$

which hold because i, j, k are mutually perpendicular unit vectors.

Properties.

For all vectors a, b, c and every scalar k:

- **Commutative:** $a \cdot b = b \cdot a$.
- **Distributive over addition:** $a \cdot (b + c) = a \cdot b + a \cdot c$.
- **Scalar factors:** $(k a) \cdot b = k(a \cdot b) = a \cdot (k b)$.
- **Square of a vector:** $a \cdot a = |a|^2$, since here $\theta = 0°$ and $\cos 0° = 1$. Equivalently $|a| = \sqrt{a \cdot a}$.
- **Zero vector:** $a \cdot 0 = 0$.

These rules let a dot product of sums be expanded just like ordinary algebra; for example $(a+b)\cdot(a+b)=a\cdot a+2a\cdot b+b\cdot b=|a|^2+2a\cdot b+|b|^2$.

The angle between two vectors.

Rearranging the geometric definition gives the angle between two non-zero vectors directly from their components:

$$\cos\theta=\frac{a\cdot b}{|a||b|},0°\le\theta\le180°.$$

Evaluate $a\cdot b$, $|a|$ and $|b|$ from components, substitute, and take the inverse cosine. The sign of $a\cdot b$ classifies the angle at a glance:

$a\cdot b$	angle θ
positive	acute ($0°<\theta<90°$), or $\theta=0°$ (parallel, like directions)
zero	a right angle ($\theta=90°$)
negative	obtuse ($90°<\theta<180°$), or $\theta=180°$ (parallel, opposite directions)

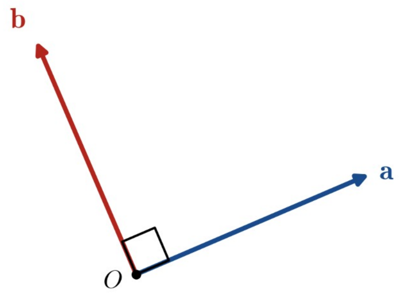
The two extreme cases $\theta=0°$ and $\theta=180°$ arise only for parallel vectors, described below. For any other pair, a positive dot product means an acute angle and a negative dot product an obtuse angle.

Perpendicular vectors.

Two non-zero vectors are **perpendicular** (also called *orthogonal*) exactly when the angle between them is $90°$. Since $\cos 90°=0$, the geometric definition makes the dot product zero, and conversely a zero dot product forces $\cos\theta=0$. Hence

$$a\perp b\Leftrightarrow a\cdot b=0.$$

This is the most-used consequence of the dot product: to test perpendicularity, simply check whether the dot product is zero.



$\mathbf{a}\cdot\mathbf{b}=0$

Parallel vectors.

Two non-zero vectors are **parallel** when one is a scalar multiple of the other, $a=kb$ for some scalar $k\neq0$: they point the same way when $k>0$ (so $\theta=0°$) and opposite ways when $k<0$ (so $\theta=180°$). In either case $\cos\theta=\pm1$, so the dot product takes its extreme value

$$a\cdot b=\pm|a||b|,$$

positive for like directions and negative for opposite directions. In practice the quickest parallel test is to check whether the components are in a common ratio, that is, whether one vector is a scalar multiple of the other.

Finding an unknown.

Because the dot product turns a geometric condition into a single equation, it is ideal for finding an unknown. To make two vectors perpendicular, set $a \cdot b = 0$ and solve for the unknown component; to find an angle, substitute into the cosine formula. Both techniques appear in the worked examples below.

Resolving one vector along and perpendicular to another — the **scalar and vector resolutes** (projections) — is built directly on the dot product and is developed in a later section. Here the focus is the dot product itself, the angle between two vectors, and the perpendicular and parallel tests.

Worked examples

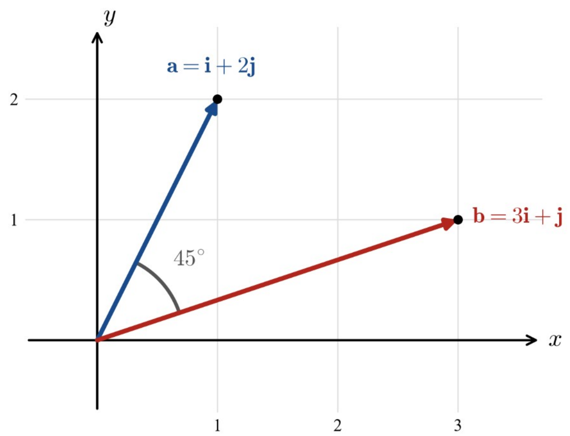
Example 1 — scaffolded

Given $a = 2i + 3j - k$ and $b = i + 4j + 2k$, find (a) $a \cdot b$ and (b) $a \cdot a$, and hence state $|a|$.

Working	Comment
$a \cdot b = (2)(1) + (3)(4) + (-1)(2)$	Multiply matching components and add.
$= 2 + 12 - 2 = 12$	The result is a scalar, not a vector.
$a \cdot a = (2)(2) + (3)(3) + (-1)(-1) = 4 + 9 + 1 = 14$	Same rule with b replaced by a.
$\lvert a\rvert^2 = a \cdot a = 14$, so $\lvert a\rvert = \sqrt{14}$	Uses $a \cdot a = \lvert a\rvert^2$.

Example 2 — scaffolded

Find the angle θ between $a = i + 2j$ and $b = 3i + j$.



Working	Comment
$a \cdot b = (1)(3) + (2)(1) = 5$	Dot product from components.
$\lvert a\rvert = \sqrt{1^2 + 2^2} = \sqrt{5}$, $\lvert b\rvert = \sqrt{3^2 + 1^2} = \sqrt{10}$	Magnitude of each vector.
$\cos\theta = \dfrac{a \cdot b}{\lvert a\rvert\lvert b\rvert} = \dfrac{5}{\sqrt{5}\sqrt{10}} = \dfrac{5}{\sqrt{50}}$	Substitute into the angle formula.
$= \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$	Simplify, since $\sqrt{50} = 5\sqrt{2}$.
$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 45°$	The unique angle in $0° \le \theta \le 180°$.

Example 3 — unscaffolded

Find the value of m for which $a = 3i + mj - 2k$ and $b = 2i + 4j + mk$ are perpendicular.

Two non-zero vectors are perpendicular exactly when their dot product is zero, so the condition is $a \cdot b = 0$. From components,

$$a \cdot b = (3)(2) + (m)(4) + (-2)(m) = 6 + 4m - 2m = 6 + 2m.$$

Setting this equal to zero gives $6 + 2m = 0$, so $m = -3$. Substituting $m = -3$ gives $a = 3i - 3j - 2k$ and $b = 2i + 4j - 3k$, whose dot product is $6 - 12 + 6 = 0$, confirming the vectors are perpendicular.

$\therefore m = -3$.

Example 4 — unscaffolded

The vectors a and b satisfy $|a| = 5$, $|b| = 8$, and the angle between them is $30°$. Find (a) $a \cdot b$ and (b) $a \cdot (a + b)$.

Here the components are unknown, so use the geometric definition $a \cdot b = |a||b|\cos\theta$. With $\theta = 30°$ and $\cos 30° = \frac{\sqrt{3}}{2}$,

$$a \cdot b = 5 \times 8 \times \frac{\sqrt{3}}{2} = 20\sqrt{3}.$$

For part (b), expand using the distributive rule and $a \cdot a = |a|^2$:

$$a \cdot (a + b) = a \cdot a + a \cdot b = |a|^2 + a \cdot b = 25 + 20\sqrt{3}.$$

$\therefore a \cdot b = 20\sqrt{3}$ and $a \cdot (a + b) = 25 + 20\sqrt{3}$.

Practice questions

Scaffolded

Q1. Let $a = 4i + j - 2k$ and $b = 3i - 2j + k$. (a) Find $a \cdot b$. (b) Find $a \cdot a$. (c) Hence state $|a|$.

Q2. Find the angle θ between $a = 2i + j$ and $b = i + 3j$. (a) Find $a \cdot b$. (b) Find $|a|$ and $|b|$. (c) Find $\cos\theta$. (d) State θ.

Q3. Find the value of t for which $a = ti + 3j + 2k$ and $b = 2i - 4j + 5k$ are perpendicular. (a) Write $a \cdot b$ in terms of t. (b) Set the dot product to zero and solve for t.

Q4. The vectors a and b have $|a| = 6$, $|b| = 10$, and the angle between them is $60°$. (a) Find $a \cdot b$. (b) Write down $a \cdot a$. (c) Find $(2a) \cdot b$.

Q5. Determine whether $a = 2i - 3j + k$ and $b = -4i + 6j - 2k$ are parallel, perpendicular, or neither. (a) Is b a scalar multiple of a? (b) Confirm your answer by comparing $a \cdot b$ with $|a||b|$.

Q6. The vectors a and b have $|a| = 3$, $|b| = 5$ and $a \cdot b = -6$. (a) Expand $(a + b) \cdot (a + b)$. (b) Hence find $|a + b|$.

Unscaffolded

Q7. Given $a = 5i - 2j + 4k$ and $b = i + 3j - 2k$, find $a \cdot b$ and state whether the angle between a and b is acute, right, or obtuse.

Q8. Find the angle between $a = 2i + 2j + k$ and $b = i - 2j + 2k$.

Q9. Find the values of λ for which $a = \lambda i - 4j$ and $b = \lambda i + j$ are perpendicular.

Q10. Show that $a = 6i - 3j$ and $b = 2i + 4j$ are perpendicular.

Q11. The points $A(1, 0, 2)$, $B(3, 2, 1)$ and $C(3, -1, 4)$ are given. Find the size of angle BAC.

Q12. Find the angle between $a = i + j + k$ and $b = i + j$, to the nearest degree.

Q13. The vectors a and b satisfy $|a| = 2$, $|b| = 3$ and $a \cdot b = 1$. Find $(2a - b) \cdot (a + 3b)$.

Q14. Find the value of t for which $a = 2i + tj + 3k$ is perpendicular to $b = 4i + j - 2k$.

Q15. The vectors a and b have $|a|=4$, $|b|=7$ and the angle between them is $120°$. Find (a) $a \cdot b$ and (b) $|a+b|$.

Q16. Find the angle between $a=3i+j$ and $b=i-2j$, to the nearest degree, and state whether it is acute or obtuse.

Q17. Find the value of s for which $a=si-6j$ is parallel to $b=3i+9j$.

Q18. The triangle ABC has vertices $A(1,2,1)$, $B(4,3,3)$ and $C(2,3,-1)$. Show that the triangle is right-angled, and state at which vertex the right angle occurs.

Answers

Q1. (a) $a\cdot b=8$; (b) $a\cdot a=21$; (c) $|a|=\sqrt{21}$. **Q2.** $a\cdot b=5$, $|a|=\sqrt{5}$, $|b|=\sqrt{10}$, $\cos\theta=\frac{1}{\sqrt{2}}$; $\theta=45°$. **Q3.** $t=1$. **Q4.** (a) $a\cdot b=30$; (b) $a\cdot a=36$; (c) $(2a)\cdot b=60$. **Q5.** $b=-2a$, so the vectors are parallel (opposite directions); $a\cdot b=-28=-|a||b|$. **Q6.** (a) $|a+b|^2=22$; (b) $|a+b|=\sqrt{22}$. **Q7.** $a\cdot b=-9$; the angle is obtuse. **Q8.** $\theta=90°$ (the vectors are perpendicular). **Q9.** $\lambda=2$ or $\lambda=-2$. **Q10.** $a\cdot b=0$, so the vectors are perpendicular. **Q11.** $\angle BAC=90°$. **Q12.** $\theta\approx 35°$. **Q13.** $(2a-b)\cdot(a+3b)=-14$. **Q14.** $t=-2$. **Q15.** (a) $a\cdot b=-14$; (b) $|a+b|=\sqrt{37}$. **Q16.** $\theta\approx 82°$; the angle is acute. **Q17.** $s=-2$. **Q18.** Right-angled at A.

Fully-worked solutions

Q1. Using the component form, $a\cdot b=(4)(3)+(1)(-2)+(-2)(1)=12-2-2=8$. With b replaced by a, $a\cdot a=(4)(4)+(1)(1)+(-2)(-2)=16+1+4=21$. Since $a\cdot a=|a|^2$, it follows that $|a|^2=21$ and so $|a|=\sqrt{21}$.

Q2. The dot product is $a\cdot b=(2)(1)+(1)(3)=5$. The magnitudes are $|a|=\sqrt{2^2+1^2}=\sqrt{5}$ and $|b|=\sqrt{1^2+3^2}=\sqrt{10}$. Then

$$\cos\theta=\frac{a\cdot b}{|a||b|}=\frac{5}{\sqrt{5}\sqrt{10}}=\frac{5}{\sqrt{50}}=\frac{5}{5\sqrt{2}}=\frac{1}{\sqrt{2}},$$

so $\theta=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=45°$.

Q3. In terms of t, $a\cdot b=(t)(2)+(3)(-4)+(2)(5)=2t-12+10=2t-2$. The vectors are perpendicular when $a\cdot b=0$, that is $2t-2=0$, giving $t=1$.

Q4. Using the geometric definition with $\cos 60°=\frac{1}{2}$, $a\cdot b=|a||b|\cos 60°=6\times 10\times\frac{1}{2}=30$. Also $a\cdot a=|a|^2=6^2=36$. By the scalar-factor rule, $(2a)\cdot b=2(a\cdot b)=2\times 30=60$.

Q5. Comparing components, $b=-4i+6j-2k=-2(2i-3j+k)=-2a$, so b is a scalar multiple of a and the vectors are **parallel** (pointing in opposite directions, since the scalar is negative). As a check, $a\cdot b=(2)(-4)+(-3)(6)+(1)(-2)=-8-18-2=-28$, while $|a|=\sqrt{14}$ and $|b|=\sqrt{56}=2\sqrt{14}$, so $|a||b|=2\times 14=28$. Thus $a\cdot b=-|a||b|$, which corresponds to $\theta=180°$, confirming the vectors are parallel and opposite.

Q6. Expanding with the distributive rule,

$$(a+b)\cdot(a+b)=a\cdot a+2a\cdot b+b\cdot b=|a|^2+2a\cdot b+|b|^2.$$

Substituting $|a|=3$, $|b|=5$ and $a\cdot b=-6$ gives $9+2(-6)+25=9-12+25=22$. Since $(a+b)\cdot(a+b)=|a+b|^2$, we have $|a+b|=\sqrt{22}$.

Q7. The dot product is $a\cdot b=(5)(1)+(-2)(3)+(4)(-2)=5-6-8=-9$. Because the dot product is negative, $\cos\theta<0$ and so $\theta>90°$. The vectors are not parallel — b is not a scalar multiple of a — so $\theta\neq 180°$, and the angle between them is **obtuse**.

Q8. The dot product is $a\cdot b=(2)(1)+(2)(-2)+(1)(2)=2-4+2=0$, so the vectors are perpendicular and $\theta=90°$. (As a check on the magnitudes, $|a|=\sqrt{4+4+1}=3$ and $|b|=\sqrt{1+4+4}=3$, giving $\cos\theta=\frac{0}{3\times 3}=0$.)

Q9. The dot product is $a\cdot b=(\lambda)(\lambda)+(-4)(1)=\lambda^2-4$. The vectors are perpendicular when $\lambda^2-4=0$, that is $\lambda^2=4$, so $\lambda=2$ or $\lambda=-2$.

Q10. The dot product is $a \cdot b=(6)(2)+(-3)(4)=12-12=0$. A zero dot product between two non-zero vectors means the angle between them is $90°$, so a and b are perpendicular.

Q11. The angle BAC is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ drawn from A. Now $\overrightarrow{AB}=B-A=2i+2j-k$ and $\overrightarrow{AC}=C-A=2i-j+2k$. Their dot product is $(2)(2)+(2)(-1)+(-1)(2)=4-2-2=0$, so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular and $\angle BAC=90°$.

Q12. The dot product is $a \cdot b=(1)(1)+(1)(1)+(1)(0)=2$, and the magnitudes are $|a|=\sqrt{3}$ and $|b|=\sqrt{2}$. Then

$$\cos\theta=\frac{2}{\sqrt{3}\sqrt{2}}=\frac{2}{\sqrt{6}}\approx 0.8165,$$

so $\theta=\cos^{-1}(0.8165)\approx 35.26°$, which is $35°$ to the nearest degree.

Q13. Expanding with the distributive rule and $a \cdot b=b \cdot a$,

$$(2a-b)\cdot(a+3b)=2a\cdot a+6a\cdot b-b\cdot a-3b\cdot b=2|a|^2+5a\cdot b-3|b|^2.$$

Substituting $|a|=2$, $|b|=3$ and $a \cdot b=1$ gives $2(4)+5(1)-3(9)=8+5-27=-14$.

Q14. In terms of t, $a \cdot b=(2)(4)+(t)(1)+(3)(-2)=8+t-6=t+2$. The vectors are perpendicular when $t+2=0$, so $t=-2$.

Q15. Using $\cos 120°=-\frac{1}{2}$, $a \cdot b=|a||b|\cos 120°=4\times 7\times\left(-\frac{1}{2}\right)=-14$. For the magnitude of the sum,

$$|a+b|^2=|a|^2+2a\cdot b+|b|^2=16+2(-14)+49=16-28+49=37,$$

so $|a+b|=\sqrt{37}$.

Q16. The dot product is $a \cdot b=(3)(1)+(1)(-2)=1$, and the magnitudes are $|a|=\sqrt{10}$ and $|b|=\sqrt{5}$. Then

$$\cos\theta=\frac{1}{\sqrt{10}\sqrt{5}}=\frac{1}{\sqrt{50}}=\frac{1}{5\sqrt{2}}\approx 0.1414,$$

so $\theta=\cos^{-1}(0.1414)\approx 81.87°$, which is $82°$ to the nearest degree. This angle lies between $0°$ and $90°$, so it is **acute**, as the positive dot product indicates.

Q17. For a to be parallel to b, one vector must be a scalar multiple of the other, so the components are in a common ratio: $\frac{s}{3}=\frac{-6}{9}$. Hence $s=3\times\left(-\frac{6}{9}\right)=3\times\left(-\frac{2}{3}\right)=-2$. Then $a=-2i-6j=-\frac{2}{3}b$, confirming the vectors are parallel.

Q18. Form the vectors along two sides from each vertex and test their dot products for a right angle. From A, $\overrightarrow{AB}=B-A=3i+j+2k$ and $\overrightarrow{AC}=C-A=i+j-2k$, with $\overrightarrow{AB}\cdot\overrightarrow{AC}=(3)(1)+(1)(1)+(2)(-2)=3+1-4=0$. Since this dot product is zero, $\overrightarrow{AB}\perp\overrightarrow{AC}$, so the triangle is right-angled at A. (The angles at B and C are not right angles: for instance $\overrightarrow{BA}\cdot\overrightarrow{BC}=(-3i-j-2k)\cdot(-2i+0j-4k)=6+0+8=14\neq 0$.)

Theory

An earlier section developed the **scalar (dot) product** of two vectors: for $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$,

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 = |a||b| \cos\theta ,$$

where θ is the angle between the two vectors. This section uses the dot product to answer a geometric question: *how much of* a *points along the direction of* b*?* The answer splits a into a part **parallel** to b and a part **perpendicular** to it.

The unit vector sets the direction.

Only the *direction* of b matters, so we work with the unit vector

$$\hat{b} = \frac{b}{|b|} ,$$

which points the same way as b but has length 1. Because $\hat{b}$ has unit length, $\hat{b} \cdot \hat{b} = 1$ and $b \cdot \hat{b} = |b|$.

Scalar resolute (scalar projection).

The **scalar resolute** of a in the direction of b is the signed length of the shadow that a casts on the line of b:

$$a \cdot \hat{b} = \frac{a \cdot b}{|b|} = |a| \cos\theta .$$

It is a **number, not a vector**. Its sign records the angle θ between a and b:

angle θ	scalar resolute $a \cdot \hat{b}$
acute $\left(0 \le \theta < \frac{\pi}{2}\right)$	positive
right $\left(\theta = \frac{\pi}{2}\right)$	zero
obtuse $\left(\frac{\pi}{2} < \theta \le \pi\right)$	negative

A zero scalar resolute therefore means a and b are perpendicular; a negative value means the shadow falls on the *opposite* side, so a leans away from b.

Vector resolute (vector projection).

Attaching the direction $\hat{b}$ to this signed length gives the **vector resolute** of a in the direction of b — the part of a that is parallel to b:

$$a_{/\!/} = (a \cdot \hat{b})\hat{b} = \frac{a \cdot b}{|b|^2} b = \frac{a \cdot b}{b \cdot b} b .$$

The two right-hand forms are equal because $\hat{b} = \frac{b}{|b|}$ and $|b|^2 = b \cdot b$; the second form avoids surds and is usually the quicker one to compute with. By construction $a_{/\!/}$ is a scalar multiple of b, so it is parallel to b.

Resolving a into parallel and perpendicular components.

Whatever is left of a after removing the parallel part is the **perpendicular component**

$$a_{\perp} = a - a_{/\!/} ,$$

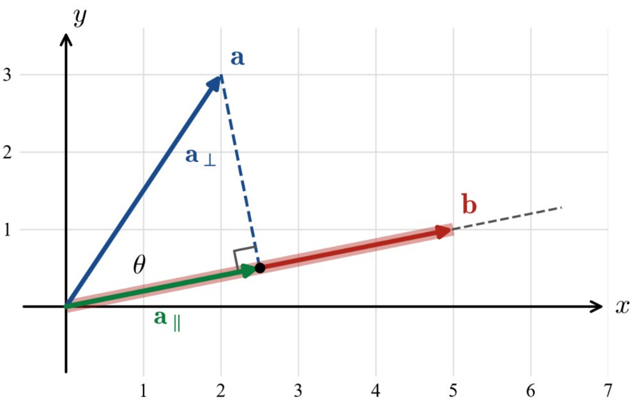
so that a is rebuilt as the sum of the two pieces,

$$a = a_{/\!/} + a_{\perp}.$$

The perpendicular component really is perpendicular to b:

$$a_{\perp} \cdot b = \left(a - \frac{a \cdot b}{b \cdot b} b \right) \cdot b = a \cdot b - \frac{a \cdot b}{b \cdot b} (b \cdot b) = a \cdot b - a \cdot b = 0.$$

This identity holds for *every* choice of a and non-zero b, and evaluating $a_{\perp} \cdot b$ is the standard way to check a resolution: if the answer is not 0, an arithmetic slip has been made. The diagram shows the construction — $a_{/\!/}$ lies along the line of b, and $a_{\perp}$ drops from the tip of $a_{/\!/}$ to the tip of a, meeting that line at a right angle.



Distance to a line (a closest-point idea).

Because $a_{\perp}$ meets the line of b at right angles, its magnitude $|a_{\perp}|$ is the **shortest distance** from the tip of a to that line, and the foot of the perpendicular locates the closest point on the line to the tip of a. The same idea returns for the shortest distance from a point to a line in a later chapter.

Rectangular components are resolutes.

Writing $a = a_1 i + a_2 j + a_3 k$ is already an act of resolution. Because i, j and k are mutually perpendicular *unit* vectors, $\hat{i} = i$ and

$$a \cdot i = a_1, a \cdot j = a_2, a \cdot k = a_3,$$

so each coefficient is the scalar resolute of a in that axis direction, and

$$a = (a \cdot i) i + (a \cdot j) j + (a \cdot k) k.$$

Resolving in the direction of a general b is the same operation with $\hat{b}$ in place of i: it measures a against a direction of our own choosing rather than against the axes.

This works for any perpendicular pair. If b and c are non-zero perpendicular vectors in the plane, then resolving a along each of them rebuilds a,

$$a = (a \cdot \hat{b}) \hat{b} + (a \cdot \hat{c}) \hat{c},$$

because the first term is $a_{/\!/}$ and the second must then be $a_{\perp}$, which is parallel to c. The pair b, c acts as a rotated set of axes.

Worked examples

Example 1 — scaffolded

Find the scalar resolute of $a = 4i + 3j$ in the direction of $b = 2i - j$.

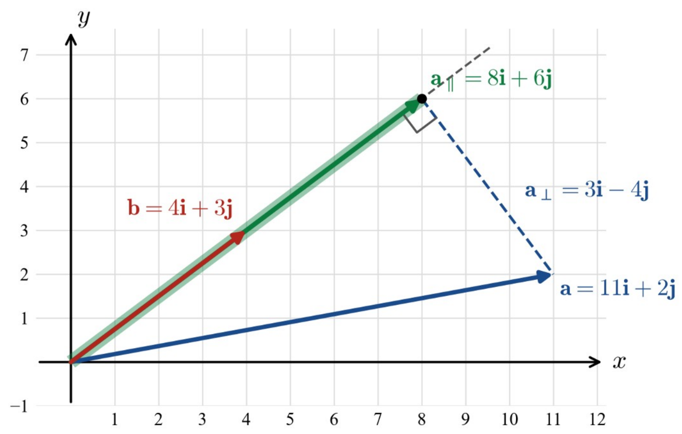
Working	Comment
The scalar resolute is $a \cdot \hat{b}$, so first find $\hat{b}$.	The direction is set by b, through its unit vector.
$\lvert b\rvert=\sqrt{2^2+(-1)^2}=\sqrt{4+1}=\sqrt{5}$.	Magnitude of b.
$\hat{b}=\frac{1}{\sqrt{5}}(2i-j)$.	Unit vector in the direction of b.
$a \cdot b=(4)(2)+(3)(-1)=8-3=5$.	Dot product from components (an earlier section).
$a \cdot \hat{b}=\frac{a \cdot b}{\lvert b\rvert}=\frac{5}{\sqrt{5}}=\sqrt{5}$.	Divide by $\lvert b\rvert$; rationalise $\frac{5}{\sqrt{5}}=\sqrt{5}$.

The scalar resolute is positive, so the angle between a and b is acute.

Example 2 — scaffolded

Resolve $a=11i+2j$ into components parallel and perpendicular to $b=4i+3j$.

Working	Comment
$\lvert b\rvert=\sqrt{4^2+3^2}=\sqrt{25}=5$, so $\hat{b}=\frac{1}{5}(4i+3j)$.	Unit vector in the direction of b.
$a \cdot b=(11)(4)+(2)(3)=44+6=50$.	Dot product.
$a \cdot \hat{b}=\frac{50}{5}=10$.	Scalar resolute: the signed length of the parallel part.
$a_{/\!/}=(a \cdot \hat{b})\hat{b}=10 \cdot \frac{1}{5}(4i+3j)=2(4i+3j)=8i+6j$.	Vector resolute: scalar resolute times $\hat{b}$.
$a_{\perp}=a-a_{/\!/}=(11-8)i+(2-6)j=3i-4j$.	The perpendicular part is what remains.
$a_{\perp} \cdot b=(3)(4)+(-4)(3)=12-12=0$, and $a_{/\!/}+a_{\perp}=11i+2j=a$.	Check: $a_{\perp}$ is perpendicular to b and the parts rebuild a.



Example 3 — unscaffolded

The vectors $b=3i+4j$ and $c=-4i+3j$ are perpendicular. Find the vector resolutes of $a=10i+5j$ in the direction of b and in the direction of c, and show that together they rebuild a.

The two directions really are perpendicular, since $b \cdot c=(3)(-4)+(4)(3)=-12+12=0$. Both have $b \cdot b=3^2+4^2=25$ and $c \cdot c=(-4)^2+3^2=25$, so $\lvert b\rvert=\lvert c\rvert=5$. Resolving along b, where $a \cdot b=(10)(3)+(5)(4)=50$,

$$\frac{a \cdot b}{b \cdot b}b=\frac{50}{25}(3i+4j)=2(3i+4j)=6i+8j,$$

and resolving along c, where $a \cdot c = (10)(-4) + (5)(3) = -25$,

$$\frac{a \cdot c}{c \cdot c} c = \frac{-25}{25}(-4i + 3j) = -(-4i + 3j) = 4i - 3j.$$

The scalar resolutes are $\frac{50}{5} = 10$ and $\frac{-25}{5} = -5$; the negative one says a leans away from c, which is why that vector resolute points opposite to c. Adding the two pieces,

$$(6i + 8j) + (4i - 3j) = 10i + 5j = a.$$

$\therefore$ the vector resolutes are $6i + 8j$ along b and $4i - 3j$ along c, and their sum is a — the perpendicular pair b, c resolves a just as i, j do.

Example 4 — unscaffolded

Resolve $a = 5i - 3j + 2k$ into components parallel and perpendicular to $b = 2i - 2j + k$, and verify that the perpendicular component is orthogonal to b.

Here $b \cdot b = 2^2 + (-2)^2 + 1^2 = 9$, so $|b| = 3$ and $\hat{b} = \frac{1}{3}(2i - 2j + k)$. The dot product is

$$a \cdot b = (5)(2) + (-3)(-2) + (2)(1) = 10 + 6 + 2 = 18,$$

so the scalar resolute is $a \cdot \hat{b} = \frac{18}{3} = 6$. Using the surd-free form for the vector resolute,

$$a_{/\!/} = \frac{a \cdot b}{b \cdot b} b = \frac{18}{9}(2i - 2j + k) = 2(2i - 2j + k) = 4i - 4j + 2k.$$

The perpendicular component is

$$a_{\perp} = a - a_{/\!/} = (5 - 4)i + (-3 + 4)j + (2 - 2)k = i + j.$$

Checking orthogonality, $a_{\perp} \cdot b = (1)(2) + (1)(-2) + (0)(1) = 2 - 2 = 0$, and the two parts rebuild a since $(4i - 4j + 2k) + (i + j) = 5i - 3j + 2k$.

$\therefore a = (4i - 4j + 2k) + (i + j)$, with the perpendicular component $i + j$ orthogonal to b.

Practice questions

Scaffolded

Q1. Find the scalar resolute of $a = 2i + 4j$ in the direction of $b = 4i + 3j$. (a) Find $|b|$ and hence $\hat{b}$. (b) Find $a \cdot b$. (c) State the scalar resolute $\frac{a \cdot b}{|b|}$.

Q2. Find the scalar resolute of $a = 5i + j$ in the direction of $b = 3i + 3j$. (a) Find $|b|$. (b) Find $a \cdot b$. (c) State the scalar resolute, giving an exact surd.

Q3. Find the vector resolute of $a = 8i + j$ in the direction of $b = i + 2j$. (a) Find $b \cdot b$. (b) Find $a \cdot b$. (c) State the vector resolute $\frac{a \cdot b}{b \cdot b} b$.

Q4. Resolve $a = 4i + 7j$ into components parallel and perpendicular to $b = 2i + j$. (a) Find the scalar resolute of a in the direction of b. (b) Find the parallel component $a_{/\!/}$. (c) Find the perpendicular component $a_{\perp} = a - a_{/\!/}$. (d) Verify that $a_{\perp} \cdot b = 0$.

Q5. Find the scalar resolute of $a=2i+j+3k$ in the direction of $b=2i+2j+k$. (a) Find $|b|$ and hence $\hat{b}$. (b) Find $a \cdot b$. (c) State the scalar resolute.

Q6. Resolve $a=4i+2j+5k$ into components parallel and perpendicular to $b=i+2j+2k$. (a) Find the scalar resolute of a in the direction of b. (b) Find the parallel component $a_{\parallel}$. (c) Find the perpendicular component $a_{\perp}$. (d) Verify that $a_{\perp} \cdot b=0$.

Unscaffolded

Q7. Find the scalar resolute of $a=6i+2j$ in the direction of $b=3i+4j$.

Q8. Find the scalar resolute of $a=4i+5j$ in the direction of $b=i+j$, giving an exact answer.

Q9. Find the vector resolute of $a=7i+6j$ in the direction of $b=2i+j$.

Q10. Find the component of $a=3i+4j$ that is perpendicular to $b=2i+j$.

Q11. Resolve $a=5i+5j$ into components parallel and perpendicular to $b=i-3j$. Comment on the sign of the scalar resolute.

Q12. Find the scalar resolute of $a=i+5j+3k$ in the direction of $b=2i+2j+k$.

Q13. Find the scalar and vector resolutes of $a=3i+4j-k$ in the direction of $b=2i-2j-2k$, and state what the results tell you about a and b.

Q14. Find the vector resolute of $a=8i+6j+k$ in the direction of $b=2i+2j-k$.

Q15. Resolve $a=5i+4k$ into components parallel and perpendicular to $b=2i+j+2k$.

Q16. Find the value of m for which the scalar resolute of $a=mi+3j$ in the direction of $b=4i-3j$ is 3.

Q17. For $a=3i+j$ and $b=i+j$: (a) find the vector resolute of a in the direction of b; (b) find the perpendicular component $a_{\perp}$; (c) hence find the shortest distance from the tip of a to the line through the origin in the direction of b.

Q18. Let b be any non-zero vector. (a) Show that, for every vector a, the vector $a-(a \cdot \hat{b})\hat{b}$ is perpendicular to b. (b) Verify the result for $a=4i-3j$ and $b=i-2j$ by resolving a into its parallel and perpendicular components.

Answers

Q1. 4. **Q2.** $3\sqrt{2}$. **Q3.** $2i+4j$. **Q4.** $a_{/\!/}=6i+3j$, $a_{\perp}=-2i+4j$ (scalar resolute $3\sqrt{5}$). **Q5.** 3. **Q6.** $a_{/\!/}=2i+4j+4k$, $a_{\perp}=2i-2j+k$ (scalar resolute 6). **Q7.** $\frac{26}{5}$. **Q8.** $\frac{9\sqrt{2}}{2}$. **Q9.** $8i+4j$. **Q10.** $-i+2j$. **Q11.** $a_{/\!/}=-i+3j$, $a_{\perp}=6i+2j$; scalar resolute $-\sqrt{10}$ is negative, so the angle between a and b is obtuse. **Q12.** 5. **Q13.** Scalar resolute 0 and vector resolute 0; a is perpendicular to b. **Q14.** $6i+6j-3k$. **Q15.** $a_{/\!/}=4i+2j+4k$, $a_{\perp}=i-2j$. **Q16.** $m=6$. **Q17.** (a) $2i+2j$; (b) $i-j$; (c) $\sqrt{2}$. **Q18.** (a) Proof — $b\cdot\left(a-(a\cdot\hat{b})\hat{b}\right)=a\cdot b-a\cdot b=0$. (b) $a_{/\!/}=2i-4j$, $a_{\perp}=2i+j$, and $a_{\perp}\cdot b=0$.

Fully-worked solutions

Q1. $|b|=\sqrt{4^2+3^2}=\sqrt{25}=5$, so $\hat{b}=\frac{1}{5}(4i+3j)$. The dot product is $a\cdot b=(2)(4)+(4)(3)=8+12=20$, so the scalar resolute is $\frac{a\cdot b}{|b|}=\frac{20}{5}=4$.

Q2. $|b|=\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$. The dot product is $a\cdot b=(5)(3)+(1)(3)=15+3=18$, so the scalar resolute is $\frac{18}{3\sqrt{2}}=\frac{6}{\sqrt{2}}=\frac{6\sqrt{2}}{2}=3\sqrt{2}$.

Q3. $b\cdot b=1^2+2^2=5$ and $a\cdot b=(8)(1)+(1)(2)=8+2=10$. The vector resolute is $\frac{a\cdot b}{b\cdot b}b=\frac{10}{5}(i+2j)=2(i+2j)=2i+4j$.

Q4. $|b|=\sqrt{2^2+1^2}=\sqrt{5}$ and $a\cdot b=(4)(2)+(7)(1)=8+7=15$. The scalar resolute is $\frac{15}{\sqrt{5}}=3\sqrt{5}$. The parallel component is

$$a_{/\!/}=\frac{a\cdot b}{b\cdot b}b=\frac{15}{5}(2i+j)=3(2i+j)=6i+3j,$$

and the perpendicular component is $a_{\perp}=a-a_{/\!/}=(4-6)i+(7-3)j=-2i+4j$. Check: $a_{\perp}\cdot b=(-2)(2)+(4)(1)=-4+4=0$, so $a_{\perp}$ is perpendicular to b.

Q5. $|b|=\sqrt{2^2+2^2+1^2}=\sqrt{9}=3$, so $\hat{b}=\frac{1}{3}(2i+2j+k)$. The dot product is $a\cdot b=(2)(2)+(1)(2)+(3)(1)=4+2+3=9$, so the scalar resolute is $\frac{9}{3}=3$.

Q6. $b\cdot b=1^2+2^2+2^2=9$, so $|b|=3$. The dot product is $a\cdot b=(4)(1)+(2)(2)+(5)(2)=4+4+10=18$, giving a scalar resolute $\frac{18}{3}=6$. The parallel component is

$$a_{/\!/}=\frac{18}{9}(i+2j+2k)=2(i+2j+2k)=2i+4j+4k,$$

and $a_{\perp}=a-a_{/\!/}=(4-2)i+(2-4)j+(5-4)k=2i-2j+k$. Check: $a_{\perp}\cdot b=(2)(1)+(-2)(2)+(1)(2)=2-4+2=0$.

Q7. $|b|=\sqrt{3^2+4^2}=\sqrt{25}=5$ and $a\cdot b=(6)(3)+(2)(4)=18+8=26$, so the scalar resolute is $\frac{26}{5}$.

Q8. $|b|=\sqrt{1^2+1^2}=\sqrt{2}$ and $a\cdot b=(4)(1)+(5)(1)=9$, so the scalar resolute is $\dfrac{9}{\sqrt{2}}=\dfrac{9\sqrt{2}}{2}$.

Q9. $b\cdot b=2^2+1^2=5$ and $a\cdot b=(7)(2)+(6)(1)=14+6=20$. The vector resolute is $\dfrac{20}{5}(2i+j)=4(2i+j)=8i+4j$.

Q10. $b\cdot b=2^2+1^2=5$ and $a\cdot b=(3)(2)+(4)(1)=6+4=10$, so $a_{/\!/}=\dfrac{10}{5}(2i+j)=4i+2j$. The perpendicular component is $a_{\perp}=a-a_{/\!/}=(3-4)i+(4-2)j=-i+2j$. Check: $a_{\perp}\cdot b=(-1)(2)+(2)(1)=-2+2=0$.

Q11. $b\cdot b=1^2+(-3)^2=10$, so $|b|=\sqrt{10}$, and $a\cdot b=(5)(1)+(5)(-3)=5-15=-10$. The scalar resolute is $\dfrac{-10}{\sqrt{10}}=-\sqrt{10}$; being negative, the angle between a and b is obtuse. The parallel component is

$$a_{/\!/}=\frac{-10}{10}(i-3j)=-(i-3j)=-i+3j,$$

which points opposite to b, and $a_{\perp}=a-a_{/\!/}=(5+1)i+(5-3)j=6i+2j$. Check: $a_{\perp}\cdot b=(6)(1)+(2)(-3)=6-6=0$.

Q12. $|b|=\sqrt{2^2+2^2+1^2}=\sqrt{9}=3$ and $a\cdot b=(1)(2)+(5)(2)+(3)(1)=2+10+3=15$, so the scalar resolute is $\dfrac{15}{3}=5$.

Q13. The dot product is $a\cdot b=(3)(2)+(4)(-2)+(-1)(-2)=6-8+2=0$. Hence the scalar resolute $\dfrac{a\cdot b}{|b|}=0$, and the vector resolute $\dfrac{a\cdot b}{b\cdot b}b=0$, the zero vector. A zero scalar resolute means a has no component along b, so a is perpendicular to b.

Q14. $b\cdot b=2^2+2^2+(-1)^2=9$ and $a\cdot b=(8)(2)+(6)(2)+(1)(-1)=16+12-1=27$. The vector resolute is $\dfrac{27}{9}(2i+2j-k)=3(2i+2j-k)=6i+6j-3k$.

Q15. $b\cdot b=2^2+1^2+2^2=9$ and, with $a=5i+0j+4k$, $a\cdot b=(5)(2)+(0)(1)+(4)(2)=10+0+8=18$. The parallel component is

$$a_{/\!/}=\frac{18}{9}(2i+j+2k)=2(2i+j+2k)=4i+2j+4k,$$

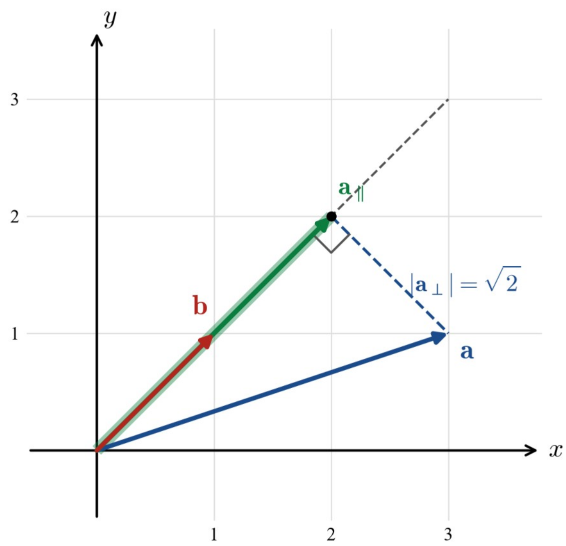
and $a_{\perp}=a-a_{/\!/}=(5-4)i+(0-2)j+(4-4)k=i-2j$. Check: $a_{\perp}\cdot b=(1)(2)+(-2)(1)+(0)(2)=2-2+0=0$.

Q16. $|b|=\sqrt{4^2+(-3)^2}=\sqrt{25}=5$ and $a\cdot b=(m)(4)+(3)(-3)=4m-9$, so the scalar resolute is $\dfrac{a\cdot b}{|b|}=\dfrac{4m-9}{5}$. Setting this equal to 3 gives

$$\frac{4m-9}{5}=3 \quad 4m-9=15 \quad 4m=24 \quad m=6.$$

Check: with $m=6$, $a\cdot b=24-9=15$ and $\dfrac{15}{5}=3$, as required.

Q17. $b\cdot b=1^2+1^2=2$ and $a\cdot b=(3)(1)+(1)(1)=4$. (a) The vector resolute is $\dfrac{4}{2}(i+j)=2(i+j)=2i+2j$. (b) $a_{\perp}=a-a_{/\!/}=(3-2)i+(1-2)j=i-j$. (c) The shortest distance is $|a_{\perp}|=\sqrt{1^2+(-1)^2}=\sqrt{2}$.



Q18. (a) Write $\hat{b}=\frac{b}{|b|}$, so that $b\cdot\hat{b}=|b|$ and $a\cdot\hat{b}=\frac{a\cdot b}{|b|}$. Then

$$b\cdot\left(a-(a\cdot\hat{b})\hat{b}\right)=b\cdot a-(a\cdot\hat{b})(b\cdot\hat{b})=a\cdot b-\frac{a\cdot b}{|b|}|b|=a\cdot b-a\cdot b=0,$$

so the vector $a-(a\cdot\hat{b})\hat{b}$ is perpendicular to b for every a. (b) Here $b\cdot b=1^2+(-2)^2=5$ and $a\cdot b=(4)(1)+(-3)(-2)=4+6=10$, so $a_{/\!/}=\frac{10}{5}(i-2j)=2i-4j$ and $a_{\perp}=a-a_{/\!/}=(4-2)i+(-3+4)j=2i+j$. This is the vector in part (a), and indeed $a_{\perp}\cdot b=(2)(1)+(1)(-2)=2-2=0$.

Theory

Two vectors can be **added**, and any vector can be **multiplied by a scalar**. Repeating and combining these operations builds new vectors from old ones. If $a_1, a_2, \dots, a_n$ are vectors and $k_1, k_2, \dots, k_n$ are real numbers, then the vector

$$k_1 a_1 + k_2 a_2 + \cdots + k_n a_n$$

is called a **linear combination** of $a_1, \dots, a_n$, and the numbers k_i are its **coefficients**. For instance $3a - 2b$ is a linear combination of a and b, with coefficients 3 and -2.

Linear dependence and independence.

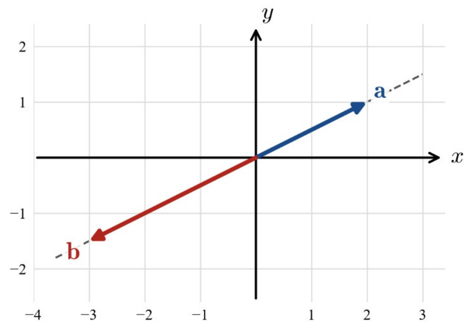
A set of vectors is **linearly dependent** if at least one of them can be written as a linear combination of the others. Equivalently, $a_1, \dots, a_n$ are linearly dependent if there are scalars $k_1, \dots, k_n$, **not all zero**, for which

$$k_1 a_1 + k_2 a_2 + \cdots + k_n a_n = 0.$$

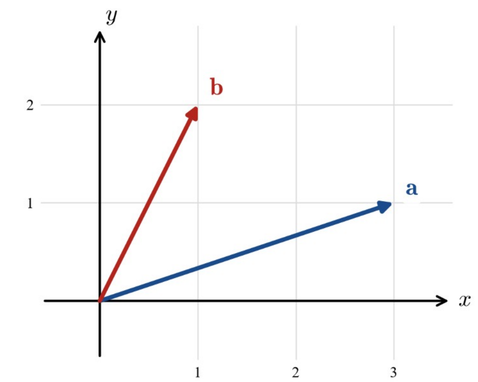
If the *only* choice of scalars producing 0 is $k_1 = k_2 = \cdots = k_n = 0$ (the **trivial** combination), the vectors are **linearly independent**. So "dependent" means some non-trivial combination collapses to the zero vector, while "independent" means no such collapse is possible.

Two vectors: dependence is parallelism.

Two non-zero vectors a and b are linearly dependent if and only if one is a scalar multiple of the other, $b = \lambda a$ for some scalar λ. Geometrically this means they are **parallel** — the same direction when $\lambda > 0$, the opposite direction when $\lambda < 0$. If no such λ exists the vectors point in genuinely different directions and are independent. The diagram below contrasts the two cases: on the left $b = -3/2\,a$ (dependent, parallel), while on the right no single scalar turns a into b (independent).



dependent: $\mathbf{b} = -\frac{3}{2}\mathbf{a}$ (parallel)



independent: not parallel

In components, $a = a_1 i + a_2 j$ and $b = b_1 i + b_2 j$ are parallel exactly when their components are in the same ratio, $\frac{b_1}{a_1} = \frac{b_2}{a_2}$ (equivalently when $a_1 b_2 - a_2 b_1 = 0$). The same idea works for vectors written with i, j, k in three dimensions: a single scalar λ must reproduce **every** component. Note also that the zero vector is a scalar multiple of every vector ($0 = 0a$), so any set containing 0 is automatically linearly dependent.

A basis for the plane.

Two linearly independent vectors a and b in the plane form a **basis**: every vector c in the plane can be written as a linear combination $c = sa + tb$ in exactly one way. To find the scalars s and t, equate the i components and the j components of the two sides; this gives two linear equations solved simultaneously. A consequence is that **any three vectors in the plane are always linearly dependent** — once two independent vectors have been chosen, a third vector must be a combination of them.

Collinearity of points.

Three points A, B and C are **collinear** when they lie on a single straight line. This happens exactly when $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel — that is, when

$$\overrightarrow{AB} = \lambda \overrightarrow{AC}$$

for some scalar λ — because the two displacements start at the common point A. If the position vectors of A, B and C relative to an origin O are a, b and c, then $\overrightarrow{AB} = b - a$ and $\overrightarrow{AC} = c - a$, and the same parallel test is applied. If the two displacements are not parallel, the points form a genuine triangle and are not collinear.

Coplanar vectors.

In three dimensions the same ideas describe flatness. Three vectors are **coplanar** when, drawn from a common point, they lie in one plane; this is exactly the condition that they are linearly dependent, with one of them a linear combination of the other two. Three **independent** vectors in space are not coplanar: they point in three genuinely different directions and, like a basis for the plane, can be combined to reach every vector in space. Testing three vectors for dependence again reduces to writing one as a combination of the other two and checking whether the resulting component equations are consistent.

Worked examples

Example 1 — scaffolded

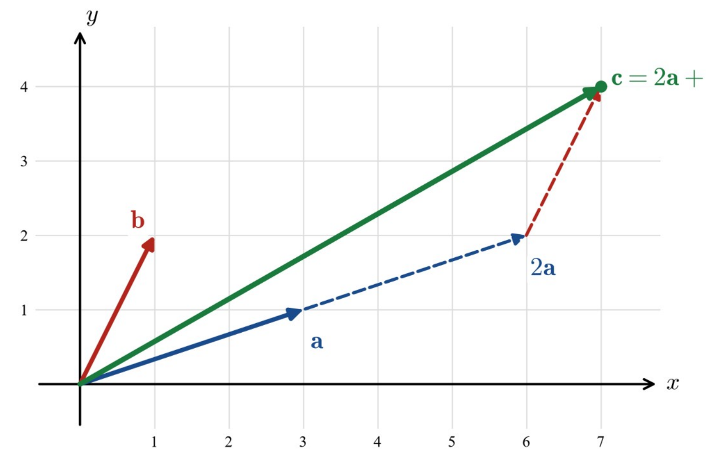
Show that $a = 2i - 3j$ and $b = -6i + 9j$ are linearly dependent, and describe their geometric relationship.

Working	Comment
Seek a scalar λ with $b = \lambda a$.	Two non-zero vectors are dependent exactly when one is a scalar multiple of the other.
Equate the i components: $-6 = 2\lambda$, so $\lambda = -3$.	Match the coefficients of i.
Equate the j components: $9 = -3\lambda$, so $\lambda = -3$.	Match the coefficients of j; the same λ must work.
Both give $\lambda = -3$, so $b = -3a$, i.e. $3a + b = 0$.	A non-trivial combination (scalars 3 and 1, not all zero) equal to 0 confirms dependence.
Since $\lambda = -3 < 0$, a and b are parallel and point in opposite directions.	The sign of λ gives the relative direction.

Example 2 — scaffolded

The vectors $a = 3i + j$ and $b = i + 2j$ are linearly independent. Express $c = 7i + 4j$ as a linear combination of a and b.

Working	Comment
If $b = \lambda a$ then $1 = 3\lambda$ and $2 = \lambda$, which cannot both hold, so a and b are independent.	Independent vectors are not parallel, so they form a basis for the plane.
Write $c = sa + tb$: $7i + 4j = s(3i + j) + t(i + 2j)$.	Introduce unknown scalars s and t.
Equate components: $3s + t = 7$ (from i) and $s + 2t = 4$ (from j).	One equation per component gives a 2×2 system.
From the first, $t = 7 - 3s$; substitute: $s + 2(7 - 3s) = 4$, so $-5s = -10$ and $s = 2$.	Solve by substitution.
Then $t = 7 - 3(2) = 1$, so $c = 2a + b$.	Check: $2(3i + j) + (i + 2j) = 7i + 4j$, as required.



Example 3 — unscaffolded

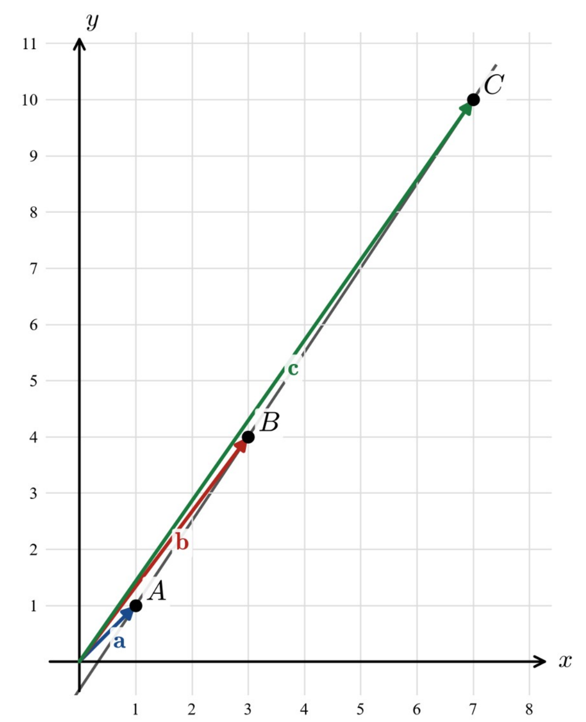
The points $A=(1,1)$, $B=(3,4)$ and $C=(7,10)$ have position vectors a, b and c. Show that A, B and C are collinear.

The two displacements from A are

$$\overrightarrow{AB}=b-a=(3i+4j)-(i+j)=2i+3j,$$

$$\overrightarrow{AC}=c-a=(7i+10j)-(i+j)=6i+9j.$$

Since $6i+9j=3(2i+3j)$, we have $\overrightarrow{AC}=3\overrightarrow{AB}$, equivalently $\overrightarrow{AB}=1/3\overrightarrow{AC}$. The two vectors are parallel and both begin at A, so the three points lie on one straight line.



$\therefore A$, B and C are collinear, with $\overrightarrow{AC}=3\overrightarrow{AB}$.

Example 4 — unscaffolded

Determine whether $a=i+2j+3k$, $b=2i+j+k$ and $c=4i+5j+7k$ are linearly dependent. If they are, express c as a linear combination of a and b.

First, a and b are independent: if $b=\lambda a$ then $2=\lambda$ from the i components but $1=2\lambda$ from the j components, which is impossible. So the plane they span is well defined, and the three vectors are dependent precisely when c lies in it. Seek scalars s and t with $c=sa+tb$. Matching the three components,

$$s+2t=4, 2s+t=5, 3s+t=7.$$

The first two equations give $s=2$ and $t=1$, and these also satisfy the third, since $3(2)+1=7$. Hence $c=2a+b$, so $2a+b-c=0$ is a non-trivial combination equal to 0.

$\therefore$ the three vectors are linearly dependent (geometrically, coplanar), with $c=2a+b$.

Practice questions

Scaffolded

Q1. Determine whether $a=4i-6j$ and $b=-2i+3j$ are linearly dependent. (a) Seek λ with $a=\lambda b$ from the i components. (b) Check that the j components give the same λ. (c) State whether the vectors are dependent or independent, and describe them geometrically.

Q2. The vectors $a=i+2j$ and $b=3i-j$ are independent. Express $c=5i+3j$ as a linear combination of them. (a) Write the vector equation $c=sa+tb$. (b) Equate components to form two equations in s and t. (c) Solve for s and t. (d) State c as a linear combination of a and b.

Q3. Consider the points $A=(2,1)$, $B=(4,5)$ and $C=(8,13)$. (a) Find $\overrightarrow{AB}$. (b) Find $\overrightarrow{AC}$. (c) Show that $\overrightarrow{AC}=\lambda\overrightarrow{AB}$ for some scalar λ, and hence decide whether A, B and C are collinear.

Q4. The vectors $a=2i+j$, $b=i+3j$ and $c=8i+9j$ all lie in the plane. (a) Explain why any three vectors in the plane must be linearly dependent. (b) Set up $c=sa+tb$ and equate components. (c) Solve for s and t, and write down the dependence relation.

Q5. Determine whether $a=i-2j+4k$ and $b=-3i+6j-12k$ are linearly dependent. (a) Seek λ with $b=\lambda a$ from the i components. (b) Check that the j and k components give the same λ. (c) State the conclusion.

Q6. Find the value of m for which $a=2i+mj$ and $b=3i+12j$ are parallel. (a) Write the condition for a and b to be parallel. (b) Form and solve an equation for m.

Unscaffolded

Q7. Determine whether $a=6i-4j$ and $b=-9i+6j$ are linearly dependent, and describe them geometrically.

Q8. Determine whether $a=3i+5j$ and $b=6i+9j$ are linearly independent.

Q9. Express $c=3i+8j$ as a linear combination of $a=2i+3j$ and $b=3i+j$.

Q10. Show that the points $A=(1,2)$, $B=(4,3)$ and $C=(10,5)$ are collinear.

Q11. Determine whether the points $A=(0,1)$, $B=(2,5)$ and $C=(3,8)$ are collinear.

Q12. Find the value of p for which $A=(1,2)$, $B=(3,6)$ and $C=(5,p)$ are collinear.

Q13. Determine whether $a=i+2j-k$, $b=2i-j+3k$ and $c=4i+3j+k$ are linearly dependent. If so, express c in terms of a and b.

Q14. Determine whether $a=i+k$, $b=j+k$ and $c=i+j$ are linearly independent (that is, not coplanar).

Q15. Express $c=-4i+11j$ as a linear combination of $a=i+4j$ and $b=2i-j$.

Q16. The points A, B and C have position vectors $a=i-2j$, $b=4i+4j$ and $c=6i+8j$. Show that A, B and C are collinear and find the ratio $\overrightarrow{AB}:\overrightarrow{BC}$.

Q17. Show that the points $A=(2,1,3)$, $B=(4,4,5)$ and $C=(8,10,9)$ are collinear.

Q18. The vectors $a=i+2j+3k$, $b=2i+j+k$ and $c=i-4j+\lambda k$ are given. Find the value of λ for which the three vectors are linearly dependent, and give the dependence relation.

Answers

Q1. Dependent; $a=-2b$, so a and b are parallel (opposite directions). **Q2.** $s=2$, $t=1$; $c=2a+b$. **Q3.** $\overrightarrow{AB}=2i+4j$, $\overrightarrow{AC}=6i+12j=3\overrightarrow{AB}$; collinear. **Q4.** $s=3$, $t=2$; $c=3a+2b$ (the three are dependent). **Q5.** Dependent; $b=-3a$ (parallel, opposite directions). **Q6.** $m=8$. **Q7.** Dependent; $b=-3/2a$ (parallel, opposite directions). **Q8.** Independent. **Q9.** $s=3$, $t=-1$; $c=3a-b$. **Q10.** Collinear ($\overrightarrow{AC}=3\overrightarrow{AB}$). **Q11.** Not collinear. **Q12.** $p=10$. **Q13.** Dependent (coplanar); $c=2a+b$. **Q14.** Independent (not coplanar). **Q15.** $s=2$, $t=-3$; $c=2a-3b$. **Q16.** Collinear; $\overrightarrow{AB}:\overrightarrow{BC}=3:2$. **Q17.** Collinear ($\overrightarrow{AC}=3\overrightarrow{AB}$). **Q18.** $\lambda=-7$; $c=-3a+2b$.

Fully-worked solutions

Q1. Seek λ with $a=\lambda b$. The i components give $4=-2\lambda$, so $\lambda=-2$; the j components give $-6=3\lambda$, so $\lambda=-2$ as well. The same value works, so $a=-2b$ and the vectors are linearly dependent. Because $\lambda=-2<0$, they are parallel but point in opposite directions.

Q2. Write $c=sa+tb$: $5i+3j=s(i+2j)+t(3i-j)$. Equating components, $s+3t=5$ (from i) and $2s-t=3$ (from j). From the second, $t=2s-3$; substituting into the first, $s+3(2s-3)=5$, so $7s=14$ and $s=2$. Then $t=2(2)-3=1$. Hence $c=2a+b$.

Q3. $\overrightarrow{AB}=(4-2)i+(5-1)j=2i+4j$ and $\overrightarrow{AC}=(8-2)i+(13-1)j=6i+12j$. Since $6i+12j=3(2i+4j)$, we have $\overrightarrow{AC}=3\overrightarrow{AB}$ (so $\lambda=3$). The two vectors are parallel and both start at A, so A, B and C lie on one straight line: they are collinear.

Q4. (a) Two independent (non-parallel) vectors already span the whole plane, so every plane vector is a linear combination of them; a third vector must therefore be such a combination, making any three plane vectors linearly dependent. (b) Here a and b are independent ($b\neq\lambda a$), so write $c=sa+tb$: $8i+9j=s(2i+j)+t(i+3j)$, giving $2s+t=8$ and $s+3t=9$. (c) From the first, $t=8-2s$; substituting, $s+3(8-2s)=9$, so $-5s=-15$ and $s=3$, then $t=8-6=2$. Thus $c=3a+2b$, i.e. $3a+2b-c=0$.

Q5. Seek λ with $b=\lambda a$. The i components give $-3=\lambda$; the j components give $6=-2\lambda$, so $\lambda=-3$; the k components give $-12=4\lambda$, so $\lambda=-3$. All three agree, so $b=-3a$ and the vectors are linearly dependent (parallel, pointing in opposite directions).

Q6. Two vectors are parallel when one is a scalar multiple of the other, equivalently when their components are in the same ratio: $\dfrac{2}{3}=\dfrac{m}{12}$. Cross-multiplying, $3m=24$, so $m=8$. (Then $a=2/3b$.)

Q7. Seek λ with $b=\lambda a$. The i components give $-9=6\lambda$, so $\lambda=-3/2$; the j components give $6=-4\lambda$, so $\lambda=-3/2$. The same value works, so $b=-3/2a$: the vectors are linearly dependent, parallel and pointing in opposite directions.

Q8. If $b=\lambda a$ then the i components give $6=3\lambda$, so $\lambda=2$, while the j components give $9=5\lambda$, so $\lambda=9/5$. These disagree, so no single scalar works: a and b are not parallel and hence linearly independent.

Q9. Write $c=sa+tb$: $3i+8j=s(2i+3j)+t(3i+j)$, giving $2s+3t=3$ and $3s+t=8$. From the second, $t=8-3s$; substituting, $2s+3(8-3s)=3$, so $-7s=-21$ and $s=3$, then $t=8-9=-1$. Hence $c=3a-b$.

Q10. $\overrightarrow{AB}=(4-1)i+(3-2)j=3i+j$ and $\overrightarrow{AC}=(10-1)i+(5-2)j=9i+3j=3(3i+j)=3\overrightarrow{AB}$. The vectors are parallel and share A, so A, B and C are collinear.

Q11. $\overrightarrow{AB}=2i+4j$ and $\overrightarrow{AC}=3i+7j$. If $\overrightarrow{AC}=\lambda\overrightarrow{AB}$ then $3=2\lambda$ (so $\lambda=3/2$) and $7=4\lambda$ (so $\lambda=7/4$). These differ, so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel and the points are not collinear.

Q12. $\overrightarrow{AB}=(3-1)i+(6-2)j=2i+4j$ and $\overrightarrow{AC}=(5-1)i+(p-2)j=4i+(p-2)j$. For collinearity, $\overrightarrow{AC}=\lambda\overrightarrow{AB}$: the i components give $4=2\lambda$, so $\lambda=2$, and then the j components require $p-2=4\lambda=8$, so $p=10$.

Q13. First, a and b are independent (no single λ gives $b=\lambda a$). Seek s and t with $c=sa+tb$. Matching components,

$$s+2t=4,2s-t=3,-s+3t=1.$$

The first two give $s=2$ and $t=1$; these satisfy the third, since $-2+3=1$. So $c=2a+b$, and $2a+b-c=0$ shows the three vectors are linearly dependent (coplanar).

Q14. Suppose $\alpha a+\beta b+\gamma c=0$. Comparing components gives $\alpha+\gamma=0$ (from i), $\beta+\gamma=0$ (from j) and $\alpha+\beta=0$ (from k). The first two give $\alpha=-\gamma$ and $\beta=-\gamma$; the third then gives $-\gamma-\gamma=0$, so $\gamma=0$, and hence $\alpha=\beta=0$. The only combination equal to 0 is the trivial one, so the vectors are linearly independent — they are not coplanar.

Q15. Write $c=sa+tb$: $-4i+11j=s(i+4j)+t(2i-j)$, giving $s+2t=-4$ and $4s-t=11$. From the second, $t=4s-11$; substituting, $s+2(4s-11)=-4$, so $9s=18$ and $s=2$, then $t=8-11=-3$. Hence $c=2a-3b$.

Q16. $\overrightarrow{AB}=b-a=(4-1)i+(4-(-2))j=3i+6j$ and $\overrightarrow{AC}=c-a=(6-1)i+(8-(-2))j=5i+10j$. Since $5i+10j=5/3(3i+6j)=5/3\overrightarrow{AB}$, the vectors are parallel and share A, so A, B and C are collinear. Also $\overrightarrow{BC}=c-b=2i+4j=2/3(3i+6j)=2/3\overrightarrow{AB}$, so $\overrightarrow{AB}:\overrightarrow{BC}=3:2$.

Q17. $\overrightarrow{AB}=(4-2)i+(4-1)j+(5-3)k=2i+3j+2k$ and $\overrightarrow{AC}=(8-2)i+(10-1)j+(9-3)k=6i+9j+6k=3(2i+3j+2k)=3\overrightarrow{AB}$. The vectors are parallel and share A, so the three points are collinear.

Q18. Because a and b are independent, the three vectors are dependent exactly when c is a linear combination $sa+tb$. The i and j components give $s+2t=1$ and $2s+t=-4$; solving, $s=-3$ and $t=2$. The k component of $sa+tb$ is $3s+t=3(-3)+2=-7$, so the vectors are dependent only when $\lambda=-7$. With that value, $c=-3a+2b$.

Theory

A geometric fact — that two lines meet at right angles, that a point bisects a segment, that three lines pass through one point — can often be proved most cleanly by turning the figure into vectors and calculating. The idea is to name every important point by a **position vector**, rewrite the property to be proved as a vector equation, and then settle it with the algebra of vectors and the scalar (dot) product. Because the position vectors are kept general, the argument holds for *every* figure of the given type, not merely for one drawn example.

Position vectors.

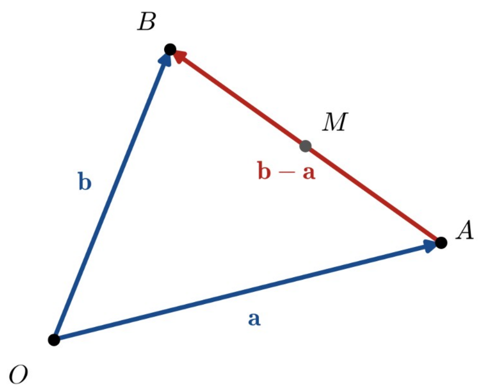
Choose a fixed point O as **origin**. Each point A is then named by its **position vector** $a=\overrightarrow{OA}$, the vector from O to A. Points are written with capital letters $A, B, C, \ldots$ and their position vectors with the matching bold lower-case letters $a, b, c, \ldots$

Displacement between two points.

The vector joining A to B is the difference of their position vectors,

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=b-a,$$

since $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$. The figure below shows the two position vectors a and b and the displacement $b-a$ from A to B, with M the midpoint of AB.



Midpoints and division of a segment.

The midpoint M of AB has position vector $1/2(a+b)$, the average of the two ends. More generally the point P dividing AB so that $AP:PB=t:(1-t)$ has position vector

$$\overrightarrow{OP}=(1-t)a+tb=a+t(b-a),$$

and $t=1/2$ recovers the midpoint. A point P lies on the line through A and B exactly when $\overrightarrow{AP}=\lambda\overrightarrow{AB}$ for some scalar λ.

Parallel vectors and collinearity.

Two non-zero vectors are **parallel** when one is a scalar multiple of the other, $u=\lambda v$. Three points A, B, C are **collinear** when $\overrightarrow{AC}=\lambda\overrightarrow{AB}$ for some scalar λ; matching this with the section formula above, C divides AB in the ratio $AC:CB=\lambda:(1-\lambda)$, and $\lambda=1/2$ gives the midpoint of AB.

The scalar (dot) product as a proof tool.

The scalar product satisfies $a\cdot b=|a||b|\cos\theta$, where θ is the angle between the vectors. Three consequences make it the key to metric proofs.

- **Length from the dot product:** $a \cdot a = |a|^2$, so a squared length is a dot product and can be expanded algebraically.
- **Perpendicularity test:** for non-zero vectors, $a \cdot b = 0$ if and only if $a \perp b$ — the angle between them is a right angle.
- **Algebra of the dot product:** it is commutative, $a \cdot b = b \cdot a$, distributive over addition, $a \cdot (b+c) = a \cdot b + a \cdot c$, and scalars pull out, $(\lambda a) \cdot b = \lambda(a \cdot b)$.

Together these let a product of vector brackets be multiplied out exactly like an ordinary product. For instance,

$$(a+b) \cdot (a-b) = a \cdot a - a \cdot b + b \cdot a - b \cdot b = |a|^2 - |b|^2,$$

because the two middle terms cancel. Many of the proofs below are a single expansion of this kind.

A method for vector proofs.

Most of the results that follow use the same plan.

1. **Choose an origin** at a convenient point of the figure — often a vertex or a centre — so that as many position vectors as possible are simple.
2. **Assign position vectors** to the remaining points, and record any imposed condition as a vector equation: equal lengths as $|a| = |b|$, a right angle as $a \cdot b = 0$, a point on a circle of radius r as $|a| = r$.
3. **Translate the claim** into a vector statement: *perpendicular* means a dot product is 0; *bisect* means two midpoints coincide; *parallel and half the length* means one displacement is a scalar multiple of another; *concurrent* means one point lies on several lines.
4. **Compute** with the algebra of vectors and the dot product until the required statement appears.
5. **State the conclusion in words.**

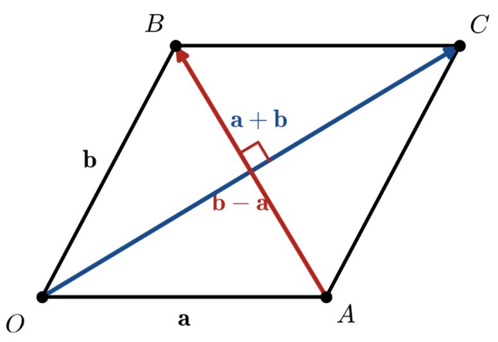
The vector (cross) product and the vector equations of lines and planes give further proof techniques and are developed in a later chapter; here every result is reached with position vectors and the scalar product alone.

Worked examples

Example 1 — scaffolded

Prove that the diagonals of a rhombus are perpendicular.

Take the rhombus $OACB$ with O at the origin, $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. The diagonals are OC and AB.



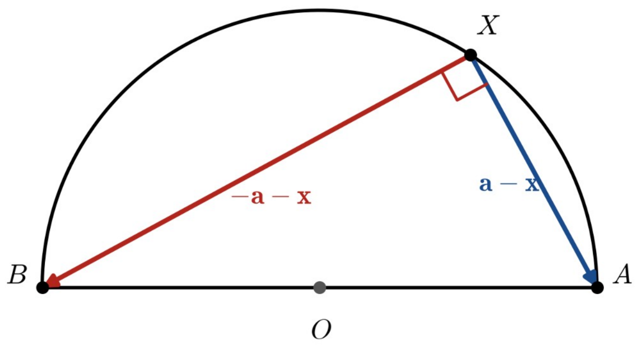
Working	Comment
Let $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$, with $\|a\| = \|b\|$.	Put the origin at a vertex; all sides of a rhombus are equal, so the two edge vectors have equal length.
The fourth vertex C has $\overrightarrow{OC} = a + b$.	$OACB$ is a parallelogram, so $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$.
The diagonals are $\overrightarrow{OC} = a + b$ and $\overrightarrow{AB} = b - a$.	One diagonal runs O to C, the other A to B; $\overrightarrow{AB} = b - a$.

Working	Comment
$\overrightarrow{OC}\cdot\overrightarrow{AB}=(a+b)\cdot(b-a)$.	Test perpendicularity with the dot product.
$=a\cdot b-a\cdot a+b\cdot b-b\cdot a=\lvert b\rvert^2-\lvert a\rvert^2$.	Expand; the cross terms cancel and $v\cdot v=\lvert v\rvert^2$.
$=0$, since $\lvert a\rvert=\lvert b\rvert$.	The equal-side condition makes the product vanish.
A zero scalar product of two non-zero diagonals means they meet at right angles. Hence the diagonals of a rhombus are perpendicular, as required.	Concluding sentence.

Example 2 — scaffolded

Prove that the angle subtended by a diameter of a circle at a point on the circle is a right angle.

Let the circle have centre O and radius r, with diameter AB and X any point on the circle. The claim is that $\angle AXB$ is a right angle.



Working	Comment
Place the centre O at the origin. Let A be one end of the diameter, $\overrightarrow{OA}=a$; the other end is $\overrightarrow{OB}=-a$.	The centre bisects the diameter, so its ends are a and $-a$; here $\lvert a\rvert=r$.
Let X be any point on the circle, $\overrightarrow{OX}=x$, so $\lvert x\rvert=r$.	Every point of the circle is a distance r from the centre.
The chords from X are $\overrightarrow{XA}=a-x$ and $\overrightarrow{XB}=-a-x$.	$\overrightarrow{XA}=\overrightarrow{OA}-\overrightarrow{OX}$, and likewise for $\overrightarrow{XB}$.
$\overrightarrow{XA}\cdot\overrightarrow{XB}=(a-x)\cdot(-a-x)$.	The angle at X is the angle between these two chords.
$=-a\cdot a-a\cdot x+x\cdot a+x\cdot x=\lvert x\rvert^2-\lvert a\rvert^2$.	Expand; the middle terms cancel and $v\cdot v=\lvert v\rvert^2$.
$=r^2-r^2=0$.	Both X and A lie on the circle of radius r.
Since $\overrightarrow{XA}\cdot\overrightarrow{XB}=0$, the chords are perpendicular. Hence the angle in a semicircle is a right angle, as required.	Concluding sentence.

Example 3 — unscaffolded

Prove the **parallelogram law**: for any two vectors a and b,

$$\lvert a+b\rvert^2+\lvert a-b\rvert^2=2\left(\lvert a\rvert^2+\lvert b\rvert^2\right).$$

Write each squared length as a dot product and expand, using $\lvert v\rvert^2=v\cdot v$ and the distributive law:

$$\lvert a+b\rvert^2=(a+b)\cdot(a+b)=\lvert a\rvert^2+2a\cdot b+\lvert b\rvert^2,$$

$$\lvert a-b\rvert^2=(a-b)\cdot(a-b)=\lvert a\rvert^2-2a\cdot b+\lvert b\rvert^2.$$

Adding the two lines, the terms $\pm 2a\cdot b$ cancel:

$$\lvert a+b\rvert^2+\lvert a-b\rvert^2=2\lvert a\rvert^2+2\lvert b\rvert^2=2\left(\lvert a\rvert^2+\lvert b\rvert^2\right).$$

In a parallelogram with edge vectors a and b the diagonals are $a+b$ and $a-b$, so this identity says the sum of the squares of the diagonals equals the sum of the squares of the four sides. Hence the parallelogram law holds for all vectors a and b, as required.

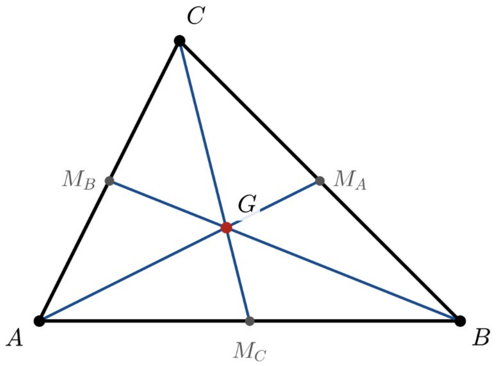
Example 4 — unscaffolded

Prove that the three medians of a triangle are **concurrent**, meeting at the centroid G whose position vector is $1/3(a+b+c)$.

Let the triangle have vertices A, B, C with position vectors a, b, c relative to any origin O. A **median** joins a vertex to the midpoint of the opposite side, so the midpoints

$$m_A=1/2(b+c), m_B=1/2(a+c), m_C=1/2(a+b)$$

are the far ends of the medians from A, B and C.



Consider the point G with position vector $g=1/3(a+b+c)$. The point two-thirds of the way from A to m_A has position vector

$$a+2/3(m_A-a)=a+2/3(1/2(b+c)-a)=a-2/3a+1/3(b+c)=1/3(a+b+c)=g,$$

so G lies on the median from A, dividing it in the ratio $AG:GM_A=2:1$. The expression $1/3(a+b+c)$ is symmetric in a, b and c, so the identical calculation on the median from B gives

$$b+2/3(m_B-b)=1/3(a+b+c)=g,$$

and likewise for the median from C. All three medians therefore pass through the single point G. Hence the medians of a triangle are concurrent, meeting at the centroid $1/3(a+b+c)$, as required.

Practice questions

Scaffolded

Q1. Prove that the diagonals of a parallelogram bisect each other. Take the parallelogram $OACB$ with O at the origin, $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$. (a) Write down the position vector of the fourth vertex C. (b) Find the position vector of the midpoint of the diagonal OC. (c) Find the position vector of the midpoint of the diagonal AB. (d) Compare the two midpoints and state the conclusion.

Q2. In triangle OAB, let M and N be the midpoints of OA and OB. Prove that MN is parallel to AB and half its length (the midpoint theorem). Take $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$. (a) Write down the position vectors of M and N. (b) Find $\overrightarrow{MN}$. (c) Find $\overrightarrow{AB}$. (d) Compare $\overrightarrow{MN}$ with $\overrightarrow{AB}$ and state the conclusion.

Q3. A parallelogram $OACB$ has $\overrightarrow{OA}=a$, $\overrightarrow{OB}=b$ and equal diagonals, $|\overrightarrow{OC}|=|\overrightarrow{AB}|$. Prove that it is a rectangle. (a) Write $\overrightarrow{OC}$ and $\overrightarrow{AB}$ in terms of a and b. (b) Square the condition $|\overrightarrow{OC}|=|\overrightarrow{AB}|$ and expand both sides. (c) Deduce the value of $a \cdot b$. (d) State what this says about the angle at O, and conclude.

Q4. Let a and b be vectors with $|a|=|b|$. Prove that $a+b$ and $a-b$ are perpendicular. (a) Write down the dot product $(a+b)\cdot(a-b)$ to be evaluated. (b) Expand it using the distributive law. (c) Simplify using $|a|=|b|$. (d) State the conclusion. (This is the algebra behind Example 1.)

Q5. Prove the polarisation identity $a\cdot b=1/4\left(|a+b|^2-|a-b|^2\right)$. (a) Expand $|a+b|^2$ as a dot product. (b) Expand $|a-b|^2$ as a dot product. (c) Subtract the second from the first. (d) Divide by 4 and state the conclusion.

Q6. Triangle OAB is isosceles with $|\overrightarrow{OA}|=|\overrightarrow{OB}|$, and M is the midpoint of the base AB. Prove that OM is perpendicular to AB. Take $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$. (a) Write down the position vector of M, hence $\overrightarrow{OM}$. (b) Write down $\overrightarrow{AB}$. (c) Evaluate $\overrightarrow{OM}\cdot\overrightarrow{AB}$. (d) Use $|a|=|b|$ to simplify, and conclude.

Unscaffolded

Q7. Triangle OAB has a right angle at O. Prove that the midpoint of the hypotenuse AB is equidistant from all three vertices O, A and B.

Q8. Prove that in any parallelogram the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the lengths of the four sides.

Q9. The diagonals of a parallelogram are perpendicular. Prove that the parallelogram is a rhombus.

Q10. The centroid G of triangle ABC has position vector $1/3(a+b+c)$. Prove that G divides each median in the ratio $2:1$, measured from the vertex.

Q11. P and Q are points on a circle with centre O, and M is the midpoint of the chord PQ. Prove that OM is perpendicular to PQ (the perpendicular from the centre bisects the chord).

Q12. Prove that the diagonals of a square are equal in length and perpendicular to each other.

Q13. For a triangle ABC, let $\overrightarrow{AB}=b$ and $\overrightarrow{AC}=c$. By expanding $|\overrightarrow{BC}|^2$ as a dot product, prove the cosine rule

$$|\overrightarrow{BC}|^2=|\overrightarrow{AB}|^2+|\overrightarrow{AC}|^2-2|\overrightarrow{AB}||\overrightarrow{AC}|\cos A,$$

where A is the angle at vertex A.

Q14. Points A, B, C have position vectors a, b, c relative to an origin O, and G is the centroid of the triangle. Prove that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=3\overrightarrow{OG}$.

Q15. Three vectors satisfy $a+b+c=0$ and $|a|=|b|=|c|=r$. Prove that the angle between each pair of them is 120°.

Q16. $ABCD$ is any quadrilateral, and P, Q, R, S are the midpoints of the sides AB, BC, CD, DA respectively. Prove that $PQRS$ is a parallelogram (Varignon's theorem).

Q17. Prove the triangle inequality $|a+b|\le|a|+|b|$ for all vectors a and b. (You may use the result $a\cdot b\le|a||b|$.)

Q18. Triangle ABC is isosceles with $|\overrightarrow{AB}|=|\overrightarrow{AC}|$. Prove that the two medians drawn from B and from C are equal in length.

Answers

Q1. Both diagonals have midpoint $1/2(a+b)$, so they bisect each other — proof in solutions. **Q2.** $\overrightarrow{MN}=1/2(b-a)=1/2\overrightarrow{AB}$, so $MN \parallel AB$ and $|MN|=1/2|AB|$. **Q3.** $|a+b|^2=|b-a|^2$ gives $a \cdot b=0$, so the angle at O is a right angle; a rectangle. **Q4.** $(a+b) \cdot (a-b)=|a|^2-|b|^2=0$; the vectors are perpendicular. **Q5.** Expanding gives $|a+b|^2-|a-b|^2=4a \cdot b$; divide by 4. **Q6.** $\overrightarrow{OM} \cdot \overrightarrow{AB}=1/2(|b|^2-|a|^2)=0$; perpendicular. **Q7.** With $a \cdot b=0$, each of $|MO|^2$, $|MA|^2$, $|MB|^2$ equals $1/4(|a|^2+|b|^2)$. **Q8.** $|a+b|^2+|a-b|^2=2(|a|^2+|b|^2)$, the parallelogram law. **Q9.** $(a+b) \cdot (b-a)=0$ forces $|a|=|b|$, so all sides are equal; a rhombus. **Q10.** $a+2/3(m_A-a)=1/3(a+b+c)=g$, so $AG:GM_A=2:1$ (same on each median). **Q11.** $\overrightarrow{OM} \cdot \overrightarrow{PQ}=1/2(|q|^2-|p|^2)=0$; perpendicular. **Q12.** With $|a|=|b|$ and $a \cdot b=0$, the diagonals $a+b$ and $b-a$ have equal length and dot product 0. **Q13.** $|c-b|^2=|b|^2+|c|^2-2b \cdot c$, with $b \cdot c=|b||c|\cos A$. **Q14.** $\overrightarrow{OG}=1/3(a+b+c)$, so $3\overrightarrow{OG}=a+b+c$. **Q15.** $a \cdot b=b \cdot c=c \cdot a=-1/2r^2$, so $\cos\theta=-1/2$ and each angle is 120°. **Q16.** $\overrightarrow{PQ}=1/2(c-a)=\overrightarrow{SR}$, so opposite sides are equal and parallel; a parallelogram. **Q17.** $|a+b|^2\le(|a|+|b|)^2$ since $a \cdot b\le|a||b|$; take square roots. **Q18.** Each median has squared length $5/4s^2-b \cdot c$ with $s=|\overrightarrow{AB}|=|\overrightarrow{AC}|$; the two are equal.

Fully-worked solutions

Q1. With O at the origin, $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$, the fourth vertex of the parallelogram is C with $\overrightarrow{OC}=a+b$. The midpoint of the diagonal OC has position vector

$$1/2(0+(a+b))=1/2(a+b),$$

and the midpoint of the diagonal AB has position vector

$$1/2(a+b).$$

The two midpoints coincide, so the point $1/2(a+b)$ lies on both diagonals and is the midpoint of each. Hence the diagonals of a parallelogram bisect each other, as required.

Q2. The midpoints are $M=1/2a$ and $N=1/2b$, so

$$\overrightarrow{MN}=N-M=1/2b-1/2a=1/2(b-a).$$

The third side is $\overrightarrow{AB}=b-a$, so $\overrightarrow{MN}=1/2\overrightarrow{AB}$. Being a scalar multiple of $\overrightarrow{AB}$, the vector $\overrightarrow{MN}$ is parallel to AB, and taking lengths gives $|MN|=1/2|AB|$. Hence the segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length, as required.

Q3. The diagonals are $\overrightarrow{OC}=a+b$ and $\overrightarrow{AB}=b-a$. Squaring the condition $|\overrightarrow{OC}|=|\overrightarrow{AB}|$ and writing each square as a dot product,

$$|a+b|^2=|b-a|^2 \Rightarrow |a|^2+2a \cdot b+|b|^2=|a|^2-2a \cdot b+|b|^2.$$

Cancelling $|a|^2+|b|^2$ from both sides leaves $4a \cdot b=0$, so $a \cdot b=0$. The adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are therefore perpendicular, so the parallelogram has a right angle at O. A parallelogram with a right angle is a rectangle. Hence a parallelogram with equal diagonals is a rectangle, as required.

Q4. Expanding the dot product with the distributive law,

$$(a+b) \cdot (a-b)=a \cdot a-a \cdot b+b \cdot a-b \cdot b=|a|^2-|b|^2,$$

since the middle terms cancel. Because $|a|=|b|$, this equals 0. A zero scalar product of two non-zero vectors means they are perpendicular. Hence $a+b$ and $a-b$ are perpendicular, as required.

Q5. Expanding each squared length as a dot product,

$$|a+b|^2=|a|^2+2a\cdot b+|b|^2, |a-b|^2=|a|^2-2a\cdot b+|b|^2.$$

Subtracting the second from the first, the terms $|a|^2$ and $|b|^2$ cancel and the middle terms add:

$$|a+b|^2-|a-b|^2=4a\cdot b.$$

Dividing by 4 gives $a\cdot b=1/4\left(|a+b|^2-|a-b|^2\right)$. Hence the polarisation identity holds, as required.

Q6. The midpoint of the base is $M=1/2(a+b)$, so $\overrightarrow{OM}=1/2(a+b)$, and $\overrightarrow{AB}=b-a$. Then

$$\overrightarrow{OM}\cdot\overrightarrow{AB}=1/2(a+b)\cdot(b-a)=1/2\left(|b|^2-|a|^2\right).$$

Because the triangle is isosceles, $|a|=|b|$, so this is 0. Hence OM is perpendicular to AB; the median from the apex of an isosceles triangle is perpendicular to the base, as required.

Q7. Put the origin at O, with $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$. The right angle at O means $a\cdot b=0$. The midpoint of the hypotenuse is $M=1/2(a+b)$. Its distances to the three vertices satisfy

$$|MO|^2=|1/2(a+b)|^2=1/4\left(|a|^2+2a\cdot b+|b|^2\right)=1/4\left(|a|^2+|b|^2\right),$$

$$|MA|^2=|a-1/2(a+b)|^2=|1/2(a-b)|^2=1/4\left(|a|^2-2a\cdot b+|b|^2\right)=1/4\left(|a|^2+|b|^2\right),$$

$$|MB|^2=|b-1/2(a+b)|^2=|1/2(b-a)|^2=1/4\left(|a|^2+|b|^2\right),$$

each using $a\cdot b=0$. All three are equal, so $|MO|=|MA|=|MB|$. Hence the midpoint of the hypotenuse is equidistant from the three vertices, as required. (It is the centre of the circle through O, A and B, with the hypotenuse as diameter — the converse of Example 2.)

Q8. Let the parallelogram have edge vectors a and b from one vertex. Its diagonals are $a+b$ and $a-b$, and by the parallelogram law (Example 3)

$$|a+b|^2+|a-b|^2=2\left(|a|^2+|b|^2\right)=|a|^2+|a|^2+|b|^2+|b|^2.$$

The right-hand side is the sum of the squares of the four sides, since a parallelogram has two sides of length $|a|$ and two of length $|b|$. Hence the sum of the squares of the diagonals equals the sum of the squares of the four sides, as required.

Q9. Take the parallelogram $OACB$ with $\overrightarrow{OA}=a$, $\overrightarrow{OB}=b$, so the diagonals are $\overrightarrow{OC}=a+b$ and $\overrightarrow{AB}=b-a$. The diagonals being perpendicular gives

$$(a+b)\cdot(b-a)=0\Rightarrow|b|^2-|a|^2=0\Rightarrow|a|=|b|.$$

The adjacent sides are therefore equal, and since opposite sides of a parallelogram are already equal, all four sides are equal. Hence a parallelogram with perpendicular diagonals is a rhombus, as required.

Q10. The median from A ends at $m_A=1/2(b+c)$. The point dividing this median from A in the ratio $2:1$ has position vector

$$a+2/3\left(m_A-a\right)=a+2/3\left(1/2(b+c)-a\right)=1/3(a+b+c)=g,$$

which is exactly the centroid G. So G lies on the median from A with $AG:GM_A=2:1$. Since $1/3(a+b+c)$ is symmetric in the three vertices, the same holds for the medians from B and C. Hence the centroid divides each median in the ratio $2:1$ from the vertex, as required.

Q11. Put the centre O at the origin, so P and Q have position vectors p and q with $|p|=|q|=r$. The midpoint of the chord is $M=1/2(p+q)$, so $\overrightarrow{OM}=1/2(p+q)$ and $\overrightarrow{PQ}=q-p$. Then

$$\overrightarrow{OM}\cdot\overrightarrow{PQ}=1/2(p+q)\cdot(q-p)=1/2\left(|q|^2-|p|^2\right)=1/2\left(r^2-r^2\right)=0.$$

Hence $OM \perp PQ$: the line from the centre to the midpoint of a chord is perpendicular to the chord, so the perpendicular from the centre bisects the chord, as required.

Q12. Take the square $OACB$ with $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$. A square has equal sides and a right angle, so $|a|=|b|$ and $a\cdot b=0$. The diagonals are $\overrightarrow{OC}=a+b$ and $\overrightarrow{AB}=b-a$. For the lengths,

$$|a+b|^2=|a|^2+2a\cdot b+|b|^2=|a|^2+|b|^2, |b-a|^2=|a|^2-2a\cdot b+|b|^2=|a|^2+|b|^2,$$

so $|\overrightarrow{OC}|=|\overrightarrow{AB}|$; the diagonals are equal. For the angle,

$$(a+b)\cdot(b-a)=|b|^2-|a|^2=0,$$

since $|a|=|b|$, so the diagonals are perpendicular. Hence the diagonals of a square are equal in length and perpendicular, as required.

Q13. Since $\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}=c-b$,

$$|\overrightarrow{BC}|^2=(c-b)\cdot(c-b)=|c|^2-2b\cdot c+|b|^2.$$

The angle at A is the angle between $\overrightarrow{AB}=b$ and $\overrightarrow{AC}=c$, so $b\cdot c=|b||c|\cos A$. Substituting and writing $|b|=|\overrightarrow{AB}|$, $|c|=|\overrightarrow{AC}|$,

$$|\overrightarrow{BC}|^2=|\overrightarrow{AB}|^2+|\overrightarrow{AC}|^2-2|\overrightarrow{AB}||\overrightarrow{AC}|\cos A.$$

Hence the cosine rule follows from the scalar product, as required.

Q14. The centroid has position vector $\overrightarrow{OG}=1/3(a+b+c)$. Multiplying by 3,

$$3\overrightarrow{OG}=a+b+c=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}.$$

Hence $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=3\overrightarrow{OG}$, as required.

Q15. From $a+b+c=0$ we have $c=-(a+b)$. Taking the squared length of both sides and using $|a|=|b|=|c|=r$,

$$|c|^2=|a+b|^2\Rightarrow r^2=|a|^2+2a\cdot b+|b|^2=2r^2+2a\cdot b,$$

so $a\cdot b=-1/2r^2$. The condition $a+b+c=0$ is unchanged if the three vectors are permuted, so by the same argument $b\cdot c=c\cdot a=-1/2r^2$. For each pair the angle θ then satisfies

$$\cos\theta=\frac{a\cdot b}{|a||b|}=\frac{-1/2r^2}{r^2}=-1/2,$$

so $\theta=120°$. Hence the angle between each pair of the vectors is 120°, as required.

Q16. Let A, B, C, D have position vectors a, b, c, d. The midpoints of the sides are

$$P=1/2(a+b), Q=1/2(b+c), R=1/2(c+d), S=1/2(d+a).$$

Then

$$\overrightarrow{PQ}=Q-P=1/2(c-a), \overrightarrow{SR}=R-S=1/2(c-a).$$

Since $\overrightarrow{PQ}=\overrightarrow{SR}$, the sides PQ and SR are equal in length and parallel, which makes $PQRS$ a parallelogram. (Both equal $1/2\overrightarrow{AC}$, so each is parallel to the diagonal AC and half its length.) Hence the midpoints of the sides of any quadrilateral form a parallelogram, as required.

Q17. Writing the squared length as a dot product and expanding,

$$|a+b|^2=|a|^2+2a\cdot b+|b|^2.$$

Using $a \cdot b \le |a||b|$,

$$|a+b|^2 \le |a|^2 + 2|a||b| + |b|^2 = (|a|+|b|)^2.$$

Both $|a+b|$ and $|a|+|b|$ are non-negative, so taking square roots preserves the inequality:

$$|a+b| \le |a|+|b|.$$

Hence the triangle inequality holds for all vectors a and b, as required.

Q18. Put the apex A at the origin, with $\overrightarrow{AB}=b$ and $\overrightarrow{AC}=c$, and let $s=|b|=|c|$ be the length of each equal side. The median from B ends at the midpoint of AC, which is $1/2\,c$, so its length satisfies

$$|1/2\,c-b|^2 = 1/4|c|^2 - b \cdot c + |b|^2 = 1/4\,s^2 - b \cdot c + s^2 = 5/4\,s^2 - b \cdot c.$$

The median from C ends at the midpoint of AB, which is $1/2\,b$, so

$$|1/2\,b-c|^2 = 1/4|b|^2 - b \cdot c + |c|^2 = 1/4\,s^2 - b \cdot c + s^2 = 5/4\,s^2 - b \cdot c.$$

The two squared lengths are equal, so the medians themselves are equal. Hence in an isosceles triangle the medians to the two equal sides are equal in length, as required.