Specialist Mathematics Units 3 & 4

Chapter 1 — Logic and proof

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Chapter 1 Logic and proof — 1A Direct proof, cases, contradiction and contrapositive

Theory

Statements.

A **statement** (or proposition) is a sentence that is either true or false, but not both. “7 is prime” is a true statement and “6 is prime” is a false statement. A sentence such as “n is even” is not a statement until n is specified, but “for every integer n, $n^2 \geq 0$” is. To **prove** a statement is to establish, by a sequence of logically valid steps built from definitions and results already known, that it is true.

Implications.

Most theorems have the form of an **implication** $P \Rightarrow Q$, read “if P then Q”. Here P is the **hypothesis** (what we assume) and Q is the **conclusion** (what we must reach). The implication $P \Rightarrow Q$ asserts only that whenever P is true, Q is also true; it says nothing about what happens when P is false. Writing $\neg P$ for the negation of P (“not P”), three related statements are worth naming:

- the **converse** $Q \Rightarrow P$,
- the **contrapositive** $\neg Q \Rightarrow \neg P$,
- the **negation** of the implication, which asserts that P is true while Q is false.

A statement and its contrapositive are **logically equivalent** — each is true exactly when the other is — whereas a statement and its converse are not. This equivalence is what makes proof by contrapositive valid.

Definitions used throughout.

Proof rests on precise definitions. For integers a, b and n:

- n is **even** if $n=2k$ for some integer k, and **odd** if $n=2k+1$ for some integer k; every integer is exactly one of these.
- a **divides** b, written $a \,/\, b$, if $b=ak$ for some integer k; we then call a a factor of b.
- a real number is **rational** if it can be written $\dfrac{p}{q}$ with p, q integers and $q \neq 0$, and **irrational** if it cannot.

Direct proof.

A **direct proof** of $P \Rightarrow Q$ assumes P and reaches Q through a chain of implications

$$P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow \cdots \Rightarrow Q,$$

each link justified by a definition, an algebraic step or an earlier result. For example, to prove “if n is odd then n^2 is odd” we start from the definition of odd, expand the square, and re-express the result in the form $2(\text{integer})+1$.

Proof by cases.

When no single argument covers every possibility, we split the situation into a finite list of **cases** that together **exhaust** all possibilities, and prove the conclusion in each. Splitting an integer by parity gives the two cases n even and n odd; splitting by the remainder on division by 3 gives the three cases $n=3k$, $n=3k+1$ and $n=3k+2$ — the **residues** modulo 3. Because the cases leave nothing out, proving the conclusion in every case proves it in general.

Proof by contradiction.

To prove a statement S by **contradiction**, assume instead that S is false and reason until two conclusions collide, so that some statement is forced to be both true and false. A correct chain of reasoning cannot produce an impossibility, so the assumption must be wrong and S is true. Two classic results proved this way are the irrationality of $\sqrt{2}$ and the fact that there is no largest prime.

Proof by contrapositive.

Because $P \Rightarrow Q$ and its contrapositive $\neg Q \Rightarrow \neg P$ are logically equivalent, we may prove whichever is easier. The contrapositive is convenient when negating the conclusion gives a more workable starting point than the hypothesis. For instance "if n^2 is even then n is even" is awkward to attack head-on, but its contrapositive "if n is odd then n^2 is odd" is a routine direct proof.

Biconditional statements: necessary and sufficient conditions.

Some theorems assert that two statements are equivalent. The **biconditional** $P \Leftrightarrow Q$, read "P **if and only if** Q" (often abbreviated "iff"), asserts that the implication $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are *both* true. To prove a biconditional we therefore prove the **two directions separately** — the forward implication $P \Rightarrow Q$ and the backward implication $Q \Rightarrow P$ — each by whichever technique suits it; the two halves may well use different methods. The language of **necessary and sufficient conditions** names these directions. We say Q is a **necessary** condition for P when $P \Rightarrow Q$, because P cannot hold unless Q does; and Q is a **sufficient** condition for P when $Q \Rightarrow P$, because the truth of Q guarantees P. A biconditional $P \Leftrightarrow Q$ therefore says precisely that each of P and Q is **necessary and sufficient** for the other.

Choosing a technique.

Try a direct proof first. If the hypothesis is hard to use but the negation of the conclusion is concrete, reach for the contrapositive. If assuming the statement false gives a firm foothold — a fraction in lowest terms, or a supposed largest element — use contradiction. If the integers behave differently according to their parity or remainder, break into cases. To prove an *if and only if* statement, prove each of the two directions in turn, choosing a technique for each. Whichever technique is used, every proof should close with a sentence stating that the required result has been reached.

Worked examples

Example 1 — scaffolded

Prove that if n is an odd integer, then n^2 is odd.

Working	Comment
Assume n is odd. By definition $n=2k+1$ for some integer k.	A direct proof begins by assuming the hypothesis and applying the definition.
$n^2=(2k+1)^2=4k^2+4k+1$.	Expand the square.
$=2(2k^2+2k)+1$.	Group the terms to expose a factor of 2; here $2k^2+2k$ is an integer.
Writing $m=2k^2+2k$ gives $n^2=2m+1$ with m an integer.	This matches the definition of an odd number, $2(\text{integer})+1$.
Hence n^2 is odd, as required.	Concluding sentence.

Example 2 — scaffolded

Prove that for every integer n, the number n^3-n is divisible by 3.

Working	Comment
Every integer leaves remainder 0, 1 or 2 on division by 3, so $n=3k$, $n=3k+1$ or $n=3k+2$ for some integer k .	Split into cases by residue modulo 3; these three cases exhaust all integers.
Case $n=3k$: $n^3-n=27k^3-3k=3(9k^3-k)$.	Substitute and factor out 3.
Case $n=3k+1$: $n^3-n=27k^3+27k^2+9k+1-(3k+1)=3(9k^3+9k^2+$	Expand $(3k+1)^3$, subtract n, factor out 3.

Working	Comment
.	
Case $n=3k+2$: $n^3-n=27k^3+54k^2+36k+8-(3k+2)=3(9k^3+18k$	Expand $(3k+2)^3$, subtract n, factor out 3.
.	
In every case $n^3-n=3\times(\text{integer})$, so $3/n^3-n$.	Each case produces a multiple of 3.
Since the three cases cover all integers, $3/n^3-n$ for every integer n. This completes the proof.	The cases are exhaustive; concluding sentence.

Example 3 — unscaffolded

Prove that $\sqrt{2}$ is irrational.

Suppose, for contradiction, that $\sqrt{2}$ is rational. Then $\sqrt{2}=\frac{p}{q}$ for some integers p,q with $q\neq 0$, and we may take the fraction in lowest terms, so that p and q have no common factor greater than 1. Squaring gives $2=\frac{p^2}{q^2}$, hence

$$p^2=2q^2.$$

Thus p^2 is even; and since the square of an odd number is odd (Example 1), p itself must be even. Write $p=2m$ for some integer m. Substituting, $4m^2=2q^2$, so $q^2=2m^2$, which shows q^2 is even and therefore q is even as well. But then p and q share the common factor 2, contradicting the assumption that the fraction was in lowest terms. This contradiction shows the supposition was false, so $\sqrt{2}$ is irrational, as required.

Example 4 — unscaffolded

Prove that for every integer n, if n^2 is even then n is even.

We prove the logically equivalent contrapositive: if n is odd then n^2 is odd. Assume n is odd, so $n=2k+1$ for some integer k. Then

$$n^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1,$$

which is of the form $2(\text{integer})+1$ and so is odd. This proves the contrapositive, and because a statement and its contrapositive are logically equivalent, the original statement holds: if n^2 is even then n is even. Hence the result holds.

Example 5 — unscaffolded

Prove that for integers a and b, the product ab is odd **if and only if** a and b are **both** odd. Hence describe the relationship between the two conditions in the language of necessary and sufficient conditions.

This is a biconditional, and the two directions are naturally proved by different techniques.

Backward (both odd $\Rightarrow ab$ odd). Assume a and b are both odd, so $a=2s+1$ and $b=2t+1$ for some integers s,t. Then

$$ab=(2s+1)(2t+1)=4st+2s+2t+1=2(2st+s+t)+1,$$

which has the form $2(\text{integer})+1$, so ab is odd.

Forward (ab odd $\Rightarrow$ both odd). We prove the equivalent contrapositive: if a or b is even, then ab is even. Suppose, without loss of generality, that a is even, so $a=2s$ for some integer s. Then $ab=2sb=2(sb)$ is even. Hence if ab is odd, neither a nor b can be even, so both must be odd.

Both directions hold, so ab is odd if and only if a and b are both odd. Equivalently, "a and b are both odd" is a **necessary and sufficient** condition for "ab is odd", as required.

Practice questions

Scaffolded

Q1. Prove that if m and n are odd integers, then $m+n$ is even. (a) Using the definition of odd, write $m=2k+1$ and $n=2\ell+1$. (b) Add and simplify $m+n$. (c) Exhibit a factor of 2 and state the conclusion.

Q2. Let a, b, c be integers. Prove that if a / b and a / c, then $a / (b+c)$. (a) Use the definition of "divides" to write $b=as$ and $c=at$. (b) Add to express $b+c$ with a factor of a. (c) State why $a / (b+c)$.

Q3. Prove that for every integer n, the number n^2+n is even. (a) Split into the cases n even and n odd. (b) In each case substitute and factor out 2. (c) Conclude that n^2+n is even in both cases.

Q4. Prove that the square of any integer leaves remainder 0 or 1 on division by 3 (equivalently, no perfect square has the form $3k+2$). (a) Write n as $3k$, $3k+1$ or $3k+2$. (b) Square each and find the remainder on division by 3. (c) State which remainders occur, and conclude.

Q5. Prove that the sum of a rational number and an irrational number is irrational. (a) Let r be rational and x irrational, and assume for contradiction that $r+x$ is rational. (b) Solve for x and explain why x would then be rational (a difference of two rationals). (c) State the contradiction and the conclusion.

Q6. Prove that for every integer n, if n^2 is odd then n is odd. (a) State the contrapositive of the implication. (b) Prove the contrapositive by a direct argument. (c) Conclude the original statement.

Unscaffolded

Q7. Prove that the sum of two even integers is even.

Q8. Prove that if n is an odd integer, then n^2-1 is divisible by 8.

Q9. Let a, b, c be integers. Prove that if a / b, then a / bc.

Q10. Let a, b, c be integers. Prove that if a / b and b / c, then a / c.

Q11. Prove that for every integer n, the number n^2 leaves remainder 0 or 1 on division by 4.

Q12. Prove by cases that for every integer n, the number n^2+1 is not divisible by 3.

Q13. Prove that for every integer n, the product $n(n+1)$ of two consecutive integers is even.

Q14. Prove that $\sqrt{3}$ is irrational.

Q15. Prove that there is no largest prime number.

Q16. Prove that $\log_2 3$ is irrational.

Q17. Prove that for every integer n, if n^2 is divisible by 3 then n is divisible by 3.

Q18. Prove that for every integer n, if n^3 is even then n is even.

Q19. Prove that for every integer n, n^2 is even **if and only if** n is even.

Q20. For an integer n, prove that the condition "$6 / n$" is **sufficient but not necessary** for "n is even". State whether "n is even" is a necessary condition for "$6 / n$".

Answers

Q1. Proof — $m+n=2(k+\ell+1)$, even. See solutions. **Q2.** Proof — $b+c=a(s+t)$. See solutions. **Q3.** Proof by cases — even: $n^2+n=2(2k^2+k)$; odd: $n^2+n=2(2k^2+3k+1)$. See solutions. **Q4.** Remainders 0 or 1 only, never 2 ; e.g. $5^2=25=3\times 8+1$. See solutions. **Q5.** Proof by contradiction — $x=(r+x)-r$ would be rational. See solutions. **Q6.** Contrapositive "if n is even then n^2 is even". See solutions. **Q7.** Proof — sum $=2(k+\ell)$. See solutions. **Q8.** Proof — $n^2-1=4k(k+1)$ and $k(k+1)$ is even, so $8 / n^2-1$. See solutions. **Q9.** Proof — $bc=a(sc)$. See solutions. **Q10.** Proof — $c=a(st)$. See solutions. **Q11.** Remainder 0 when n is even, 1 when n is odd. See solutions. **Q12.** Proof by cases — n^2 has remainder 0 or 1 on division by 3, so n^2+1 has remainder 1 or 2, never 0. See solutions. **Q13.** Proof — one of the consecutive integers n, $n+1$ is even. See solutions. **Q14.** Proof by contradiction (as for $\sqrt{2}$); here $3 / p^2$ forces $3 / p$. See solutions. **Q15.** Proof by contradiction — Euclid's $N=p_1 p_2 \cdots p_n+1$. See solutions. **Q16.** Proof by contradiction — $2^p=3^q$ makes an even number equal an odd number. See solutions. **Q17.** Contrapositive "if 3 does not divide n, then 3 does not divide n^2". See solutions. **Q18.** Contrapositive "if n is odd then n^3 is odd". See solutions. **Q19.** Biconditional — forward (n^2 even $\Rightarrow n$ even) by contrapositive; backward (n even $\Rightarrow n^2$ even) directly. See solutions. **Q20.** $6 / n \Rightarrow n$ even (sufficient); $n=2$ is even but 6 does not divide 2 (not necessary). Since $6 / n \Rightarrow n$ even, "n is even" *is* necessary for "$6 / n$". See solutions.

Fully-worked solutions

Q1. Assume m and n are odd. By definition $m=2k+1$ and $n=2\ell+1$ for some integers k,ℓ. Adding,

$$m+n=(2k+1)+(2\ell+1)=2k+2\ell+2=2(k+\ell+1).$$

Since $k+\ell+1$ is an integer, $m+n$ is even, as required.

Q2. Assume a / b and a / c. By the definition of "divides", $b=as$ and $c=at$ for some integers s,t. Then

$$b+c=as+at=a(s+t),$$

and $s+t$ is an integer, so $a / (b+c)$, as required.

Q3. Split by parity. Case n even, $n=2k$: $n^2+n=4k^2+2k=2(2k^2+k)$, which is even. Case n odd, $n=2k+1$: $n^2+n=(4k^2+4k+1)+(2k+1)=4k^2+6k+2=2(2k^2+3k+1)$, which is even. In both cases n^2+n is twice an integer, so it is even for every integer n. This completes the proof.

Q4. Every integer is $n=3k$, $3k+1$ or $3k+2$ for some integer k.

- $n=3k$: $n^2=9k^2=3(3k^2)$, remainder 0.
- $n=3k+1$: $n^2=9k^2+6k+1=3(3k^2+2k)+1$, remainder 1.
- $n=3k+2$: $n^2=9k^2+12k+4=3(3k^2+4k+1)+1$, remainder 1.

So a perfect square leaves remainder 0 or 1 on division by 3, never 2; in particular no perfect square has the form $3k+2$. Hence the result holds.

Q5. Let r be rational and x irrational, and suppose for contradiction that $s=r+x$ is rational. Then $x=s-r$. The difference of two rational numbers is rational: if $s=\frac{a}{b}$ and $r=\frac{c}{d}$ with $b,d\neq 0$, then

$$s-r=\frac{ad-bc}{bd},$$

a ratio of integers with non-zero denominator. So x would be rational, contradicting that x is irrational. Therefore $r+x$ is irrational, as required.

Q6. The contrapositive of "if n^2 is odd then n is odd" is "if n is even then n^2 is even". Assume n is even, so $n=2k$. Then $n^2=4k^2=2(2k^2)$, which is even. This proves the contrapositive, so the original statement holds: if n^2 is odd then n is odd. Hence the result holds.

Q7. Let the two even integers be $m=2k$ and $n=2\ell$ for integers k,ℓ. Then

$$m+n=2k+2\ell=2(k+\ell),$$

twice an integer, so $m+n$ is even, as required.

Q8. Assume n is odd, so $n=2k+1$ for some integer k. Then

$$n^2-1=(2k+1)^2-1=4k^2+4k=4k(k+1).$$

Among the consecutive integers k and $k+1$ one is even, so their product $k(k+1)$ is even; write $k(k+1)=2j$ for some integer j. Then $n^2-1=4\times 2j=8j$, so $8 / n^2-1$, as required.

Q9. Assume a / b, so $b=as$ for some integer s. Then for any integer c,

$$bc=(as)c=a(sc),$$

and sc is an integer, so a / bc, as required.

Q10. Assume a / b and b / c, so $b=as$ and $c=bt$ for some integers s,t. Substituting the first into the second,

$$c=(as)t=a(st),$$

and st is an integer, so a / c, as required.

Q11. Split by parity.

- $n=2k$: $n^2=4k^2=4(k^2)$, remainder 0 on division by 4.
- $n=2k+1$: $n^2=4k^2+4k+1=4(k^2+k)+1$, remainder 1.

So n^2 leaves remainder 0 or 1 on division by 4 for every integer n. This completes the proof.

Q12. Write $n=3k$, $3k+1$ or $3k+2$.

- $n=3k$: $n^2+1=9k^2+1=3(3k^2)+1$, remainder 1.
- $n=3k+1$: $n^2+1=9k^2+6k+2=3(3k^2+2k)+2$, remainder 2.
- $n=3k+2$: $n^2+1=9k^2+12k+5=3(3k^2+4k+1)+2$, remainder 2.

In every case the remainder is 1 or 2, never 0, so 3 never divides n^2+1. Hence the result holds.

Q13. Split by parity. If n is even, $n=2k$, then $n(n+1)=2k(n+1)$ is even. If n is odd, $n=2k+1$, then $n+1=2k+2=2(k+1)$ is even, so $n(n+1)=n\cdot 2(k+1)=2n(k+1)$ is even. In both cases $n(n+1)$ is twice an integer, so it is even, as required.

Q14. Suppose for contradiction that $\sqrt{3}$ is rational, say $\sqrt{3}=\frac{p}{q}$ in lowest terms, so p and q have no common factor greater than 1. Squaring, $3=\frac{p^2}{q^2}$, hence

$$p^2=3q^2,$$

so $3 / p^2$. Now $3 / p^2$ forces $3 / p$: if instead 3 did not divide p then $p=3k+1$ or $p=3k+2$, and squaring leaves remainder 1 (not 0) on division by 3. Write $p=3m$; then $9m^2=3q^2$, so $q^2=3m^2$ and hence $3 / q$ by the same reasoning. But then 3 divides both p and q, contradicting lowest terms. Therefore $\sqrt{3}$ is irrational, as required.

Q15. Suppose for contradiction that there is a largest prime. Then the primes form a finite list $p_1 < p_2 < \cdots < p_n$ with p_n the largest. Consider

$$N = p_1 p_2 \cdots p_n + 1.$$

Since $N > 1$, it has at least one prime factor q. If q were one of $p_1, \ldots, p_n$, then q would divide both $p_1 p_2 \cdots p_n$ and N, and so would divide their difference $N - p_1 p_2 \cdots p_n = 1$; but no prime divides 1. Hence q is a prime not in the list, and in particular $q > p_n$. This contradicts p_n being the largest prime. Therefore there is no largest prime number, as required.

Q16. Suppose for contradiction that $\log_2 3$ is rational. Since $\log_2 3 > 0$, we may write $\log_2 3 = \frac{p}{q}$ with p, q positive integers. Then $2^{p/q} = 3$, and raising both sides to the power q,

$$2^p = 3^q.$$

The left side 2^p is even, because $p \geq 1$; the right side 3^q is odd, being a product of odd numbers. An even number cannot equal an odd number, a contradiction. Therefore $\log_2 3$ is irrational, as required.

Q17. We prove the contrapositive: if 3 does not divide n, then 3 does not divide n^2. Assume 3 does not divide n; then $n = 3k+1$ or $n = 3k+2$ for some integer k. If $n = 3k+1$, then $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, remainder 1. If $n = 3k+2$, then $n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, remainder 1. In both cases n^2 leaves remainder 1 on division by 3, so 3 does not divide n^2. This proves the contrapositive, so the original statement holds: if $3 \mid n^2$ then $3 \mid n$. Hence the result holds.

Q18. We prove the contrapositive: if n is odd then n^3 is odd. Assume n is odd, so $n = 2k+1$ for some integer k. Then

$$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1,$$

which is of the form $2(\text{integer}) + 1$ and so is odd. This proves the contrapositive, so the original statement holds: if n^3 is even then n is even. Hence the result holds.

Q19. This is a biconditional, proved in two directions. *Backward (n even $\Rightarrow n^2$ even).* Assume n is even, $n = 2k$. Then $n^2 = 4k^2 = 2(2k^2)$ is even. *Forward (n^2 even $\Rightarrow n$ even).* We prove the contrapositive "if n is odd then n^2 is odd". Assume n is odd, $n = 2k+1$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd, so if n^2 is even then n is even. Both directions hold, hence n^2 is even if and only if n is even, as required.

Q20. *Sufficient:* assume $6 \mid n$, so $n = 6k = 2(3k)$ for some integer k, which is even; hence $6 \mid n$ guarantees n is even, so "$6 \mid n$" is a sufficient condition for "n is even". *Not necessary:* the integer $n = 2$ is even, yet 6 does not divide 2, so an even integer need not be a multiple of 6; thus "$6 \mid n$" is not a necessary condition for "n is even". Finally, since $6 \mid n \Rightarrow n$ is even, "$6 \mid n$" cannot hold unless "n is even" does, so "n is even" **is** a necessary condition for "$6 \mid n$". Hence the result holds.

Theory

Open sentences and predicates.

An equation or inequality that contains a variable, such as $x^2=4$ or $x>3$, is neither true nor false on its own: its truth depends on what the variable stands for. Such a statement is called an **open sentence** or a **predicate**, and is written $P(x)$. Two things must be settled before $P(x)$ has a truth value: the value substituted for x, and the **domain** — the set of values x is allowed to take (sometimes called the *universe*). We name the usual number domains $\mathbb{N}$ (the positive integers), $\mathbb{Z}$ (all integers), $\mathbb{Q}$ (the rationals) and $\mathbb{R}$ (the real numbers).

The universal quantifier.

To claim that a predicate holds for *every* value in the domain we attach the symbol $\forall$, read "for all" (or "for every", "for each"). For example,

$$\forall x \in \mathbb{R}, x^2 \geq 0$$

reads "for every real number x, $x^2 \geq 0$" — a true statement. A statement of the form $\forall x\, P(x)$ is a **universal statement**. It is true only when $P(x)$ is true for *every* x in the domain, and it becomes false the moment a single value makes $P(x)$ false.

The existential quantifier.

To claim that a predicate holds for *at least one* value we attach the symbol $\exists$, read "there exists" (or "there is at least one", "for some"). For example,

$$\exists x \in \mathbb{R}, x^2 = 2$$

reads "there exists a real number x with $x^2=2$" — true, since $x=\sqrt{2}$ works. A statement $\exists x\, P(x)$ is an **existential statement**. It is true when at least one x makes $P(x)$ true, and false only when every x in the domain fails.

Translating between words and symbols.

The quantifier and the domain must both be stated, because changing either can change the truth value.

Words	Symbols
Every integer n satisfies $n \leq n^2$.	$\forall n \in \mathbb{Z}, n \leq n^2$
Some real number squares to give 2.	$\exists x \in \mathbb{R}, x^2 = 2$
No real number has a negative square.	$\forall x \in \mathbb{R}, x^2 \geq 0$
For every real x there is a real y with $x+y=0$.	$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x+y=0$

The last line uses **two quantifiers**. Here the order matters: $\forall x\, \exists y$ ("each x has its own y") is a weaker claim than $\exists y\, \forall x$ ("one y works for all x"). Reading a chain of quantifiers from left to right, each later choice may depend on the earlier ones.

Negating quantified statements.

The negation of a statement asserts exactly the opposite. To deny that *all* values are P, it is enough to point to *one* that is not; to deny that *some* value is P, we must say that *every* value fails. In symbols,

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x), \neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x).$$

So negation **swaps the quantifier** ($\forall \leftrightarrow \exists$) and **negates the predicate**. For a statement with two quantifiers we apply the rule from the outside in, swapping each quantifier in turn and finally negating the innermost predicate, while the *order* of the quantifiers is preserved:

$$\neg(\forall x \exists y P(x,y)) \equiv \exists x \forall y \neg P(x,y).$$

When the predicate is an implication, recall that the negation of "if A then B" is "A and not B": that is, $\neg(A \Rightarrow B) \equiv A$ and $\neg B$.

Counterexamples.

A **counterexample** to a universal statement $\forall x P(x)$ is a single value $x=a$ in the domain for which $P(a)$ is false. Because the negation of $\forall x P(x)$ is $\exists x \neg P(x)$, exhibiting one counterexample *proves the negation* and therefore disproves the universal statement outright — **one counterexample is always enough**.

The reverse does not hold. Checking that $P(a)$ is true for one value, for ten values, or for a million values does **not** prove $\forall x P(x)$, because the statement makes a claim about *every* member of the domain, and an infinite domain can never be exhausted by examples. Confirming cases can only make a universal statement look plausible; it can never make it certain.

Forming and testing conjectures.

A pattern spotted in a few cases suggests a **conjecture** — a statement believed true but not yet proved. Testing further cases is worthwhile: if any case fails, that case *is* a counterexample and the conjecture is false. But if every tested case passes, the conjecture is still only plausible; a general argument (a proof) is needed to be sure. Universal and existential statements behave in opposite ways: a universal statement is *settled false* by one counterexample but *settled true* only by proof, whereas an existential statement is *settled true* by one example (a **witness**) but *settled false* only by ruling out the whole domain.

Worked examples

Example 1 — scaffolded

Show that the following statement is false: for all real numbers x, if $x^2 > x$ then $x > 1$.

Working	Comment
In symbols the claim is $\forall x \in \mathbb{R}, (x^2 > x \Rightarrow x > 1)$.	The claim is universal, so a single counterexample will disprove it.
Its negation is $\exists x \in \mathbb{R}$ with $x^2 > x$ and $x \le 1$.	Negating $A \Rightarrow B$ gives "A and $\neg B$"; here $\neg(x > 1)$ is $x \le 1$.
Look for such an x among the negatives; try $x = -2$.	If $x < 0$ then $x^2 > 0 > x$, so the hypothesis is easy to satisfy while $x \le 1$.
Test the hypothesis: $x^2 = (-2)^2 = 4$, and $4 > -2$, so $x^2 > x$ is true.	The "if" part holds at $x = -2$.
Test the conclusion: $x = -2$ is not greater than 1, so $x > 1$ is false.	With a true hypothesis and a false conclusion the implication is false here.
So $x = -2$ satisfies the negation, and one counterexample disproves the whole statement.	A single failing value settles a universal claim.

$\therefore$ the statement is false, disproved by the counterexample $x = -2$, as required.

Example 2 — scaffolded

Consider the statement S: "for every integer m there exists an integer n such that $n > m$." Write S in quantifier notation, form its negation $\neg S$, and determine which of S and $\neg S$ is true.

Working	Comment
In symbols, $S: \forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, n > m$.	Two quantifiers: "for every m", then "there exists n".

Working	Comment
S is true: given any integer m, choose $n=m+1$, an integer with $n>m$.	To establish an $\exists$ claim, exhibit one value that works; here it may depend on m.
Negate from the outside: $\neg S: \exists m \in \mathbb{Z}, \neg(\exists n \in \mathbb{Z}, n>m)$.	Swap the leading $\forall$ to $\exists$ and carry $\neg$ inward.
The inner part becomes $\forall n \in \mathbb{Z}, n \leq m$.	Swap $\exists$ to $\forall$ and negate $n>m$ to $n \leq m$.
Hence $\neg S: \exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, n \leq m$.	Every $\forall$ became $\exists$ and vice versa; the order is unchanged.
$\neg S$ claims some integer is greater than or equal to *every* integer — a largest integer. None exists, since $m+1>m$ always.	A statement and its negation always take opposite truth values.

$\therefore S$ is true and its negation $\neg S$ is false. This completes the argument.

Example 3 — unscaffolded

Show that the following claim is false: for every positive integer n, the number n^2-n+11 is prime.

Testing the first few positive integers, the values $n=1,2,\ldots,10$ give $11,13,17,23,31,41,53,67,83,101$, and every one of these is prime, so the claim looks convincing. However, no finite list of successes can prove a statement about *all* positive integers. Testing $n=11$,

$$11^2-11+11=121=11\times 11,$$

which has 11 as a factor other than 1 and itself, so 121 is composite, not prime. Thus $n=11$ is a counterexample.

Therefore the claim is false; the single value $n=11$ disproves it, however many earlier cases succeed.

Example 4 — unscaffolded

Noticing that $2^2-1=3$, $2^3-1=7$ and $2^5-1=31$ are all prime, a student conjectures that 2^p-1 is prime for every prime number p. Investigate this conjecture.

The listed cases $p=2,3,5$ support the conjecture, and the next prime $p=7$ also works, since $2^7-1=127$ is prime. So far the pattern is persuasive. But these are only finitely many cases, and a conjecture about all primes cannot be proved by examples alone. Testing the next prime, $p=11$,

$$2^{11}-1=2047=23\times 89,$$

which is a product of two integers greater than 1, so 2047 is not prime. Hence $p=11$ is a counterexample.

Therefore the conjecture is false. A single counterexample, $p=11$, refutes it, whereas the earlier prime cases only made it look likely — one example can never prove a universal statement, but one counterexample can disprove it.

Practice questions

Scaffolded

Q1. Consider the statement “every prime number is odd.” (a) Taking the domain to be the prime numbers, write the statement in quantifier notation. (b) Write its negation in words and in symbols. (c) Find a counterexample. (d) State whether the original statement or its negation is true.

Q2. Consider the statement “for all real numbers x, $x^2 \geq x$.” (a) Write the statement in quantifier notation. (b) Test the value $x=1/2$. (c) Use your result to give a counterexample. (d) Write the negation in symbols and state which statement is true.

Q3. Consider “for every real number x there exists a real number y such that $x+y=0$.” (a) Write the statement in quantifier notation. (b) Decide whether it is true, justifying your answer. (c) Write its negation in quantifier notation. (d) State whether the negation is true or false.

Q4. Consider the claim “for every positive integer n, n^2+n+5 is prime.” (a) Evaluate n^2+n+5 for $n=1,2,3$. (b) Evaluate it for $n=4$. (c) Explain why the $n=4$ value is not prime. (d) State a counterexample and say whether the claim is true or false.

Q5. Consider “for all real numbers a and b, if $a<b$ then $a^2<b^2$.” (a) Write the statement in quantifier notation. (b) Test the values $a=-3$, $b=1$. (c) Give a counterexample. (d) State whether the statement is true or false.

Q6. A student conjectures that “for every positive integer n, n^2-n+1 is odd.” (a) Test the conjecture for $n=1,2,3$. (b) Explain why testing these three values does not prove the conjecture. (c) By writing $n^2-n=n(n-1)$, explain why n^2-n is always even. (d) Hence decide whether the conjecture is true, giving a full justification.

Unscaffolded

Q7. Find a counterexample to disprove the statement “every odd integer is prime.”

Q8. Show that the statement “for all real numbers x, if $x^2>4$ then $x>2$” is false.

Q9. Disprove the claim “the sum of two irrational numbers is always irrational.”

Q10. Show that the statement “for all positive real numbers a and b, $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$” is false.

Q11. Find a counterexample to the claim “for every positive integer n, $2^n>n^2$.”

Q12. Write the statement “there is a smallest positive real number” in quantifier notation, form its negation, and determine which is true.

Q13. Let S be the statement “for every integer m there exists an integer n such that $m+n=5$.” Write S and $\neg S$ in quantifier notation and state which is true.

Q14. Disprove the statement “for all real numbers x and y, $|x+y|=|x|+|y|$.”

Q15. Show that the claim “for every positive integer n, n^2-n+41 is prime” is false.

Q16. Find a counterexample to the statement “for all real numbers x, $x^3>x$.”

Q17. Disprove the claim “for every positive integer n, $n!+1$ is prime.”

Q18. Write the statement “every real number has a multiplicative inverse” in quantifier notation, form its negation, and hence give a counterexample showing the original statement is false.

Answers

Q1. Counterexample $p=2$ (prime but even); the negation "some prime is not odd" is true, so the original statement is false. **Q2.** Counterexample $x=1/2$, since $1/4\ge 1/2$ is false; the statement is false and its negation $\exists x\in\mathbb{R}, x^2<x$ is true. **Q3.** True (take $y=-x$); the negation $\exists x\in\mathbb{R}, \forall y\in\mathbb{R}, x+y\ne 0$ is false. **Q4.** Counterexample $n=4$: $n^2+n+5=25=5^2$, not prime; the claim is false. **Q5.** Counterexample $a=-3$, $b=1$, since $9<1$ is false; the statement is false. **Q6.** True: $n^2-n=n(n-1)$ is even, so n^2-n+1 is odd for every positive integer n (examples alone do not prove it). **Q7.** $n=9=3\times 3$ is odd but not prime; the statement is false. **Q8.** $x=-3$: $9>4$ holds but $-3>2$ is false; the statement is false. **Q9.** $\sqrt{2}+(-\sqrt{2})=0$ is rational; the claim is false. **Q10.** $a=b=1$: $\sqrt{2}\ne 2$; the statement is false. **Q11.** $n=3$: $2^3=8$ and $3^2=9$, so $8>9$ is false; the claim is false. **Q12.** $\exists x\in\mathbb{R}, x>0, \forall y\in\mathbb{R}, y>0\Rightarrow x\le y$; its negation is true, so there is no smallest positive real and the original is false. **Q13.** S: $\forall m\in\mathbb{Z}, \exists n\in\mathbb{Z}, m+n=5$ is true (take $n=5-m$); $\neg S: \exists m\in\mathbb{Z}, \forall n\in\mathbb{Z}, m+n\ne 5$ is false. **Q14.** $x=1$, $y=-1$: $|0|=0$ but $|1|+|-1|=2$; the statement is false. **Q15.** $n=41$: $41^2-41+41=1681=41^2$, not prime; the claim is false. **Q16.** $x=1/2$: $x^3=1/8$ and $1/8>1/2$ is false; the statement is false. **Q17.** $n=4$: $4!+1=25=5^2$, not prime; the claim is false. **Q18.** $\forall x\in\mathbb{R}, \exists y\in\mathbb{R}, xy=1$; its negation $\exists x\in\mathbb{R}, \forall y\in\mathbb{R}, xy\ne 1$ holds at $x=0$, so the original is false.

Fully-worked solutions

Q1. With the domain taken to be the prime numbers, the statement is $\forall p, p\text{ is odd}$. Its negation swaps the quantifier and negates the predicate, giving $\exists p, p\text{ is even}$ — in words, "some prime number is not odd." The prime $p=2$ is even, so $p=2$ is a counterexample. Hence the original statement is false and its negation is true.

Q2. In symbols the statement is $\forall x\in\mathbb{R}, x^2\ge x$. Testing $x=1/2$ gives $x^2=1/4$, and $1/4\ge 1/2$ is false. So $x=1/2$ is a counterexample, and the statement is false. Its negation is $\exists x\in\mathbb{R}, x^2<x$, which is true (for instance at $x=1/2$).

Q3. In symbols the statement is $\forall x\in\mathbb{R}, \exists y\in\mathbb{R}, x+y=0$. It is true: given any real x, the real number $y=-x$ satisfies $x+y=x+(-x)=0$, so a suitable y always exists. Negating from the outside gives $\neg S: \exists x\in\mathbb{R}, \forall y\in\mathbb{R}, x+y\ne 0$, which claims some real x has no additive inverse. Since $y=-x$ always works, this negation is false.

Q4. For $n=1,2,3$: n^2+n+5 equals $7, 11, 17$, each of which is prime, so the claim looks plausible. For $n=4$: $4^2+4+5=16+4+5=25$. Now $25=5\times 5$, so it has the factor 5 besides 1 and itself and is therefore not prime. Hence $n=4$ is a counterexample and the claim is false.

Q5. In symbols the statement is $\forall a, b\in\mathbb{R}, (a<b\Rightarrow a^2<b^2)$. Test $a=-3$, $b=1$: the hypothesis $a<b$ holds because $-3<1$, but $a^2=9$ and $b^2=1$, so $a^2<b^2$ would require $9<1$, which is false. With a true hypothesis and a false conclusion the implication fails, so $a=-3$, $b=1$ is a counterexample and the statement is false.

Q6. For $n=1,2,3$: n^2-n+1 equals $1, 3, 7$, all odd, so the conjecture is supported. Testing three values does not prove it, because the conjecture claims something for *every* positive integer, and three cases leave infinitely many untested. To settle it, write $n^2-n=n(n-1)$, the product of the consecutive integers n and $n-1$. One of any two consecutive integers is even, so their product $n(n-1)$ is even. Adding 1 to an even number gives an odd number, so n^2-n+1 is odd for every positive integer n. Hence the conjecture is true, established by a general argument rather than by examples.

Q7. The statement "every odd integer is prime" is universal, so one counterexample disproves it. Take $n=9$: it is odd, but $9=3\times 3$ has the factor 3 apart from 1 and itself, so 9 is not prime. Therefore $n=9$ is a counterexample and the statement is false.

Q8. In symbols the claim is $\forall x\in\mathbb{R}, (x^2>4\Rightarrow x>2)$; its negation seeks an x with $x^2>4$ and $x\le 2$. Take $x=-3$: then $x^2=9>4$, so the hypothesis holds, but $x=-3$ is not greater than 2, so the conclusion fails. Thus $x=-3$ is a counterexample and the statement is false.

Q9. The claim is universal over pairs of irrational numbers, so one pair suffices to disprove it. Both $\sqrt{2}$ and $-\sqrt{2}$ are irrational, yet their sum is $\sqrt{2}+(-\sqrt{2})=0$, which is rational. Therefore this pair is a counterexample and the claim is false.

Q10. Take $a=1$ and $b=1$, both positive. Then $\sqrt{a+b}=\sqrt{1+1}=\sqrt{2}$, whereas $\sqrt{a}+\sqrt{b}=\sqrt{1}+\sqrt{1}=2$. Since $\sqrt{2}\approx 1.41\neq 2$, the two sides are unequal, so $a=b=1$ is a counterexample and the statement is false.

Q11. Test $n=3$: $2^n=2^3=8$ and $n^2=3^2=9$, so the claimed inequality $2^n>n^2$ becomes $8>9$, which is false. Hence $n=3$ is a counterexample and the claim is false. (The inequality also fails at $n=2$ and $n=4$, though it does hold for all $n\geq 5$.)

Q12. Writing the domain as the positive reals, the statement is

$$\exists\, x\in\mathbb{R}, x>0,\ \forall\, y\in\mathbb{R}, (y>0\Rightarrow x\leq y),$$

"there is a positive real x that is less than or equal to every positive real y." Its negation swaps both quantifiers and negates the predicate:

$$\forall\, x\in\mathbb{R}, x>0, \exists\, y\in\mathbb{R}, y>0, y<x.$$

The negation is true: given any positive real x, the number $y=x/2$ is positive and satisfies $y<x$, so there is always a smaller positive real. Hence there is no smallest positive real number, and the original statement is false.

Q13. In symbols S: $\forall\, m\in\mathbb{Z}, \exists\, n\in\mathbb{Z}, m+n=5$. This is true: for any integer m, take $n=5-m$, which is an integer and gives $m+n=m+(5-m)=5$. Negating from the outside,

$$\neg S: \exists\, m\in\mathbb{Z},\ \forall\, n\in\mathbb{Z}, m+n\neq 5,$$

which claims some integer m can never be completed to 5. Since $n=5-m$ always works, $\neg S$ is false. Therefore S is true and $\neg S$ is false.

Q14. In symbols the statement is $\forall\, x, y\in\mathbb{R}, |x+y|=|x|+|y|$. Take $x=1$ and $y=-1$: then $|x+y|=|1+(-1)|=|0|=0$, while $|x|+|y|=|1|+|-1|=1+1=2$. Since $0\neq 2$, the pair $x=1$, $y=-1$ is a counterexample and the statement is false.

Q15. For $n=1,2,\ldots,40$ the value n^2-n+41 turns out to be prime, which makes the claim look very convincing. But testing $n=41$,

$$41^2-41+41=41^2=1681=41\times 41,$$

which has the factor 41 apart from 1 and itself, so it is not prime. Hence $n=41$ is a counterexample and the claim is false, no matter how many earlier cases succeed.

Q16. Test $x=1/2$: then $x^3=(1/2)^3=1/8$, and the claimed inequality $x^3>x$ becomes $1/8>1/2$, which is false. Hence $x=1/2$ is a counterexample and the statement is false. (It also fails for every $x\leq -1$, for example $x=-2$ gives $-8>-2$, which is false.)

Q17. For $n=1,2,3$: $n!+1$ equals $2,3,7$, all prime. Testing $n=4$: $4!+1=24+1=25=5^2$, which has the factor 5 apart from 1 and itself and so is not prime. Hence $n=4$ is a counterexample and the claim is false.

Q18. Taking the domain to be the real numbers, "every real number has a multiplicative inverse" is

$$\forall\, x\in\mathbb{R}, \exists\, y\in\mathbb{R}, xy=1.$$

Its negation swaps both quantifiers and negates the predicate:

$$\exists\, x\in\mathbb{R},\ \forall\, y\in\mathbb{R}, xy\neq 1.$$

The negation holds at $x=0$: for every real y, $0\cdot y=0\neq 1$, so 0 has no multiplicative inverse. Therefore $x=0$ is a counterexample and the original statement is false.

Theory

An inequality such as $A \geq B$ is *proved*, not merely checked on a few examples: a single case that works tells us nothing about all the others. The main tool of this section is the **"consider the difference" method**, which rests on the most useful single fact in the subject — a real square is never negative.

The non-negativity of a square.

For every real number t,

$$t^2 \geq 0 \text{, and } t^2 = 0 \text{ only when } t = 0.$$

Every proof below is built on this observation. Because a square can never be negative, any expression we manage to write *as* a square, or as a sum of squares, is automatically ≥ 0.

The "consider the difference" method.

To prove $A \geq B$, form the difference $A - B$ and show that $A - B \geq 0$; adding B to both sides then gives $A \geq B$. In practice we rewrite $A - B$ as a square, or a sum of squares, so that its non-negativity is visible at a glance. Equality in $A \geq B$ holds exactly when $A - B = 0$, that is, when every square in the rewriting is zero — this is how the **equality condition** is found.

The prototype is the difference $a^2 + b^2 - 2ab$. Since

$$a^2 + b^2 - 2ab = (a - b)^2 \geq 0,$$

we obtain $a^2 + b^2 \geq 2ab$ for all real a and b, with equality if and only if $a = b$. This result is used constantly.

Sums of squares.

Many expressions in several variables are non-negative because they can be rearranged into a **sum of squares**. For example, for all real a, b and c,

$$2(a^2 + b^2 + c^2 - ab - bc - ca) = (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0,$$

so $a^2 + b^2 + c^2 \geq ab + bc + ca$, with equality if and only if $a = b = c$. *Completing the square* is the usual way to uncover such a form.

The AM–GM inequality.

For two non-negative reals a and b, the **arithmetic mean** is at least the **geometric mean**:

$$\frac{a+b}{2} \geq \sqrt{ab} \text{, equivalently } a + b \geq 2\sqrt{ab}.$$

This follows from $(\sqrt{a} - \sqrt{b})^2 \geq 0$, and equality holds if and only if $a = b$. For three non-negative reals a, b and c,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \text{, equivalently } a^3 + b^3 + c^3 \geq 3abc,$$

with equality if and only if $a = b = c$. The AM–GM inequality is a ready-made building block: rather than returning to squares each time, we often apply it directly.

A useful special case: $x + \frac{1}{x} \geq 2$.

For every $x > 0$,

$$x+\frac{1}{x}\geq 2\text{, with equality if and only if } x=1.$$

This is AM–GM applied to x and $\frac{1}{x}$, and also follows from $x+\frac{1}{x}-2=\frac{(x-1)^2}{x}\geq 0$. It is the pattern behind many "minimum value" problems.

Known results as tools.

Once proved, an inequality may be quoted and reused. The most useful are $(a-b)^2\geq 0$ (equivalently $a^2+b^2\geq 2ab$), AM–GM, and $x+\frac{1}{x}\geq 2$. Two facts about manipulating inequalities are needed throughout:

- Multiplying or dividing both sides by a **positive** number preserves the direction of the inequality; by a **negative** number **reverses** it. When the multiplier is a variable of unknown sign, split the argument into cases.
- For $p,q\geq 0$, $p^2\geq q^2$ if and only if $p\geq q$. This lets us compare two non-negative quantities by comparing their squares — helpful when square roots or absolute values appear.

Proof by cases (the sign of a variable).

When a step depends on whether a quantity is positive or negative — for instance dividing by a variable, or removing an absolute value — split the argument into **exhaustive** cases and prove the inequality within each. A typical split is $x>0$ versus $x<0$ (with $x=0$ handled separately if relevant). The statement is proved only once every case is covered.

Chaining inequalities.

If $A\geq B$ and $B\geq C$ then $A\geq C$ (transitivity), and two inequalities in the same direction may be added: $A\geq B$ and $C\geq D$ give $A+C\geq B+D$. Longer results are frequently assembled by chaining several simple ones together, as in $\sqrt{ab}\leq\frac{a+b}{2}\leq\sqrt{\frac{a^2+b^2}{2}}$.

Stating equality.

A complete proof of $A\geq B$ also names the exact conditions under which $A=B$. With the difference method this comes for free: equality holds precisely when every square used is zero. Always finish a proof by stating these conditions.

Worked examples

Example 1 — scaffolded

Prove that for all real numbers a and b, $a^2+b^2\geq 2ab$, and state when equality holds.

Working	Comment
Consider the difference a^2+b^2-2ab.	To prove $A\geq B$ it is enough to show $A-B\geq 0$.
$a^2+b^2-2ab=(a-b)^2$.	Recognise the perfect square — this is the key algebraic step.
$(a-b)^2\geq 0$ for all real a,b, because any real number squared is non-negative.	A square is never negative; this is the foundation of the method.
Therefore $a^2+b^2-2ab\geq 0$, so $a^2+b^2\geq 2ab$.	Add $2ab$ to both sides.
Equality holds exactly when $(a-b)^2=0$, i.e. when $a=b$.	A square is zero only when its base is zero.

Hence $a^2+b^2\geq 2ab$ for all real a and b, with equality if and only if $a=b$, as required.

Example 2 — scaffolded

Prove the **AM–GM inequality for two non-negative reals**: for all $a, b \geq 0$, $\frac{a+b}{2} \geq \sqrt{ab}$, and state when equality holds.

Working	Comment
Since $a, b \geq 0$, both $\sqrt{a}$ and $\sqrt{b}$ are defined real numbers.	AM–GM needs non-negative inputs so that the square roots exist.
Consider $(\sqrt{a} - \sqrt{b})^2 \geq 0$.	Start from a square, which is automatically non-negative.
Expand: $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{a}\sqrt{b} + b = a + b - 2\sqrt{ab}$.	Use $(\sqrt{a})^2 = a$ and $\sqrt{a}\sqrt{b} = \sqrt{ab}$.
So $a + b - 2\sqrt{ab} \geq 0$, giving $a + b \geq 2\sqrt{ab}$.	Rearrange the non-negativity.
Divide both sides by 2: $\frac{a+b}{2} \geq \sqrt{ab}$.	Dividing by the positive number 2 preserves the inequality.
Equality holds when $\sqrt{a} - \sqrt{b} = 0$, i.e. $a = b$.	The square is zero only when $\sqrt{a} = \sqrt{b}$.

Hence $\frac{a+b}{2} \geq \sqrt{ab}$ for all $a, b \geq 0$, with equality if and only if $a = b$. This completes the proof.

Example 3 — unscaffolded

Prove that for every real number $x > 0$, $x + \frac{1}{x} \geq 2$, and state when equality holds.

Consider the difference $x + \frac{1}{x} - 2$. Writing it over the common denominator x,

$$x + \frac{1}{x} - 2 = \frac{x^2 - 2x + 1}{x} = \frac{(x-1)^2}{x}.$$

The numerator $(x-1)^2 \geq 0$ because it is a square, and the denominator $x > 0$ by hypothesis, so the quotient is non-negative:

$$\frac{(x-1)^2}{x} \geq 0 \Rightarrow x + \frac{1}{x} - 2 \geq 0 \Rightarrow x + \frac{1}{x} \geq 2.$$

Equality holds exactly when $(x-1)^2 = 0$, that is when $x = 1$. Hence the result holds for all $x > 0$, with equality if and only if $x = 1$, as required.

Example 4 — unscaffolded

Prove the **triangle inequality** for real numbers: for all real a and b, $|a+b| \leq |a| + |b|$. State when equality holds. *(This proof splits into cases on the sign of ab.)*

Both sides are non-negative, so it is enough to compare their squares and then use the fact that, for $p, q \geq 0$, $p^2 \geq q^2$ implies $p \geq q$. Since $|a+b|^2 = (a+b)^2$, consider the difference of the squares:

$$(|a| + |b|)^2 - |a+b|^2 = (a^2 + 2|a||b| + b^2) - (a^2 + 2ab + b^2) = 2(|a||b| - ab).$$

Because $|a||b| = |ab|$, this equals $2(|ab| - ab)$. We show $|ab| \geq ab$ by cases on the sign of ab.

Case 1: $ab \geq 0$. Then $|ab| = ab$, so $|ab| - ab = 0$.

Case 2: $ab < 0$. Then $|ab| = -ab > 0 > ab$, so $|ab| - ab > 0$.

In both cases $|ab| - ab \ge 0$, hence $(|a|+|b|)^2 - |a+b|^2 = 2(|ab| - ab) \ge 0$, giving $(|a|+|b|)^2 \ge |a+b|^2$. As $|a|+|b| \ge 0$ and $|a+b| \ge 0$, comparing the squares gives $|a+b| \le |a|+|b|$. Equality holds exactly when $|ab| = ab$, i.e. when $ab \ge 0$ — when a and b have the same sign or at least one of them is zero. This completes the proof.

Practice questions

Scaffolded

Q1. Prove that for all real numbers x, $x^2+4 \ge 4x$, and state when equality holds. (a) Write down the difference x^2+4-4x. (b) Express the difference as a perfect square. (c) Explain why this square is ≥ 0. (d) State the inequality and the value of x that gives equality.

Q2. Prove that for all real numbers a and b, $a^2+b^2+2 \ge 2(a+b)$. (a) Form the difference and expand $2(a+b)$. (b) Group the terms and write the difference as a sum of two squares. (c) Explain why the sum of squares is ≥ 0. (d) State when equality holds.

Q3. Prove that for all real numbers $a, b \ge 0$, $a+b \ge 2\sqrt{ab}$. (a) Start from the square $(\sqrt{a}-\sqrt{b})^2 \ge 0$. (b) Expand it. (c) Rearrange to reach the required inequality. (d) State when equality holds.

Q4. Prove that for all real numbers $x>0$, $x+\dfrac{9}{x} \ge 6$. (a) Form the difference $x+\dfrac{9}{x}-6$ and write it over the denominator x. (b) Factorise the numerator as a perfect square. (c) Explain why the quotient is ≥ 0 (use $x>0$). (d) State when equality holds.

Q5. Prove that for all real numbers a and b, $a^2+ab+b^2 \ge 0$. (a) Treating the expression as a quadratic in a, complete the square in the form $\left(a+\dfrac{b}{2}\right)^2+\cdots$. (b) Simplify the remaining term. (c) Explain why the result is ≥ 0. (d) State when equality holds.

Q6. Prove that for all real numbers a, b and c, $a^2+b^2+c^2 \ge ab+bc+ca$. (a) Form the difference $a^2+b^2+c^2-ab-bc-ca$. (b) Show that **twice** the difference equals $(a-b)^2+(b-c)^2+(c-a)^2$. (c) Explain why the difference is ≥ 0. (d) State when equality holds.

Unscaffolded

Q7. Prove that for all real numbers x, $x^4+1 \ge 2x^2$, and state when equality holds.

Q8. Prove that for all real numbers a and b, $a^2+b^2+1 \ge ab+a+b$, and state when equality holds.

Q9. Prove that for all real numbers $a, b>0$, $\dfrac{a}{b}+\dfrac{b}{a} \ge 2$, and state when equality holds.

Q10. Prove that for all real numbers $a, b>0$, $(a+b)\left(\dfrac{1}{a}+\dfrac{1}{b}\right) \ge 4$, and state when equality holds.

Q11. Prove that for all real numbers $a, b, c \ge 0$, $a^3+b^3+c^3 \ge 3abc$, and state when equality holds. You may use the identity

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Q12. Using the result of the previous question, prove the AM–GM inequality for three non-negative reals: for $a, b, c \ge 0$, $\dfrac{a+b+c}{3} \ge \sqrt[3]{abc}$. State when equality holds.

Q13. Prove that for all real numbers x, $\dfrac{x}{x^2+1} \le \dfrac{1}{2}$, and state when equality holds.

Q14. Prove that for all real numbers a and b, $2\left(a^2+b^2\right) \geq (a+b)^2$, and state when equality holds.

Q15. Prove that for all real numbers x and y, $x^2+y^2 \geq 2|x\,y|$, and state when equality holds.

Q16. Prove, by cases on the sign of $a\,b$, that for all real numbers a and b with $a\,b>0$, $\frac{a}{b}+\frac{b}{a} \geq 2$, while for all real a and b with $a\,b<0$, $\frac{a}{b}+\frac{b}{a} \leq -2$. State when equality holds in each case.

Q17. Prove that for all real numbers $a, b>0$,

$$\sqrt{a\,b} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

(the chain: geometric mean ≤ arithmetic mean ≤ root-mean-square). State when equality holds.

Q18. Find, with proof, the minimum value of $x+\frac{16}{x}$ for $x>0$, and the value of x at which it occurs.

Answers

Q1. Proof — $x^2+4-4x=(x-2)^2\ge 0$; equality at $x=2$. **Q2.** Proof — difference $=(a-1)^2+(b-1)^2\ge 0$; equality at $a=b=1$. **Q3.** Proof — from $(\sqrt{a}-\sqrt{b})^2\ge 0$; equality at $a=b$. **Q4.** Proof — $x+\frac{9}{x}-6=\frac{(x-3)^2}{x}\ge 0$; equality at $x=3$. **Q5.** Proof — $a^2+ab+b^2=\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}\ge 0$; equality at $a=b=0$. **Q6.** Proof — $2\times(\text{difference})=(a-b)^2+(b-c)^2+(c-a)^2\ge 0$; equality at $a=b=c$. **Q7.** Proof — $x^4+1-2x^2=(x^2-1)^2\ge 0$; equality at $x=\pm 1$. **Q8.** Proof — $2\times(\text{difference})=(a-b)^2+(a-1)^2+(b-1)^2\ge 0$; equality at $a=b=1$. **Q9.** Proof — $\frac{a}{b}+\frac{b}{a}-2=\frac{(a-b)^2}{ab}\ge 0$; equality at $a=b$. **Q10.** Proof — expands to $2+\frac{a}{b}+\frac{b}{a}\ge 2+2=4$; equality at $a=b$. **Q11.** Proof — factor $=\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\ge 0$; equality at $a=b=c$. **Q12.** Proof — substitute $a=p^3$, $b=q^3$, $c=r^3$ into Q11; equality at $a=b=c$. **Q13.** Proof — $\frac{1}{2}-\frac{x}{x^2+1}=\frac{(x-1)^2}{2(x^2+1)}\ge 0$; equality at $x=1$. **Q14.** Proof — $2(a^2+b^2)-(a+b)^2=(a-b)^2\ge 0$; equality at $a=b$. **Q15.** Proof — from $(|x|-|y|)^2\ge 0$; equality when $|x|=|y|$. **Q16.** Proof by cases; equality at $a=b$ (when $ab>0$) and at $a=-b$ (when $ab<0$). **Q17.** Proof — left inequality is AM–GM (Q3), right inequality from $2(a^2+b^2)\ge(a+b)^2$; equality throughout at $a=b$. **Q18.** Minimum value 8, attained at $x=4$.

Fully-worked solutions

Q1. Consider the difference x^2+4-4x. This factorises as a perfect square:

$$x^2+4-4x=x^2-4x+4=(x-2)^2.$$

Since a square is non-negative, $(x-2)^2\ge 0$, so $x^2+4-4x\ge 0$ and therefore $x^2+4\ge 4x$. Equality holds exactly when $(x-2)^2=0$, i.e. when $x=2$. Hence the result holds, with equality only at $x=2$, as required.

Q2. Form the difference and regroup the terms:

$$a^2+b^2+2-2(a+b)=(a^2-2a+1)+(b^2-2b+1)=(a-1)^2+(b-1)^2.$$

Each square is ≥ 0, so their sum is ≥ 0; hence $a^2+b^2+2\ge 2(a+b)$. Equality requires both $(a-1)^2=0$ and $(b-1)^2=0$, that is $a=1$ and $b=1$. This completes the proof.

Q3. Since $a,b\ge 0$, the roots $\sqrt{a}$ and $\sqrt{b}$ are real. Starting from the square $(\sqrt{a}-\sqrt{b})^2\ge 0$ and expanding,

$$a-2\sqrt{ab}+b\ge 0\Rightarrow a+b\ge 2\sqrt{ab}.$$

Equality holds when $\sqrt{a}-\sqrt{b}=0$, i.e. $a=b$. Hence $a+b\ge 2\sqrt{ab}$ for all $a,b\ge 0$, with equality if and only if $a=b$, as required.

Q4. For $x>0$, write the difference over the denominator x:

$$x+\frac{9}{x}-6=\frac{x^2-6x+9}{x}=\frac{(x-3)^2}{x}.$$

The numerator $(x-3)^2\ge 0$ is a square and the denominator $x>0$, so the quotient is ≥ 0; thus $x+\frac{9}{x}\ge 6$. Equality holds when $(x-3)^2=0$, i.e. $x=3$. This completes the proof.

Q5. Treat the expression as a quadratic in a and complete the square:

$$a^2+ab+b^2=\left(a+\frac{b}{2}\right)^2-\frac{b^2}{4}+b^2=\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}.$$

Both terms on the right are squares (up to positive multiples), hence each is ≥ 0, so the sum is ≥ 0. For equality we need $\frac{3b^2}{4}=0$, giving $b=0$, and then $\left(a+\frac{b}{2}\right)^2=a^2=0$, giving $a=0$. So $a^2+ab+b^2\geq 0$ for all real a and b, with equality only at $a=b=0$, as required.

Q6. Multiply the difference by 2 and regroup:

$$2(a^2+b^2+c^2-ab-bc-ca)=(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)=(a-b)^2+(b-c)^2+(c-a)^2.$$

The right-hand side is a sum of squares, hence ≥ 0, so the difference is ≥ 0 and $a^2+b^2+c^2\geq ab+bc+ca$. Equality requires $a-b=b-c=c-a=0$, i.e. $a=b=c$. This completes the proof.

Q7. Consider the difference x^4+1-2x^2. Regarding it as a quadratic in x^2,

$$x^4+1-2x^2=(x^2)^2-2(x^2)+1=(x^2-1)^2\geq 0,$$

so $x^4+1\geq 2x^2$. Equality holds when $x^2-1=0$, i.e. $x=1$ or $x=-1$. Hence the result holds, with equality at $x=\pm 1$, as required.

Q8. Multiply the difference by 2 and split into three squares:

$$2(a^2+b^2+1-ab-a-b)=(a^2-2ab+b^2)+(a^2-2a+1)+(b^2-2b+1)=(a-b)^2+(a-1)^2+(b-1)^2.$$

This is a sum of squares, hence ≥ 0, so the difference is ≥ 0 and $a^2+b^2+1\geq ab+a+b$. Equality requires $a-b=0$, $a-1=0$ and $b-1=0$ simultaneously, i.e. $a=b=1$. This completes the proof.

Q9. For $a,b>0$, write the difference over the common denominator ab:

$$\frac{a}{b}+\frac{b}{a}-2=\frac{a^2+b^2-2ab}{ab}=\frac{(a-b)^2}{ab}.$$

The numerator $(a-b)^2\geq 0$ and the denominator $ab>0$, so the quotient is ≥ 0; hence $\frac{a}{b}+\frac{b}{a}\geq 2$. Equality holds when $(a-b)^2=0$, i.e. $a=b$. This completes the proof.

Q10. For $a,b>0$, expand the product:

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=1+\frac{a}{b}+\frac{b}{a}+1=2+\left(\frac{a}{b}+\frac{b}{a}\right).$$

By the previous question, $\frac{a}{b}+\frac{b}{a}\geq 2$, so the product is $\geq 2+2=4$. Equality holds exactly when $\frac{a}{b}+\frac{b}{a}=2$, i.e. $a=b$. Hence $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$, with equality at $a=b$, as required.

Q11. For $a,b,c\geq 0$, apply the given identity:

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

By Q6, $a^2+b^2+c^2-ab-bc-ca=1/2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\geq 0$, and $a+b+c\geq 0$ since each of a,b,c is non-negative. The product of two non-negative numbers is non-negative, so $a^3+b^3+c^3-3abc\geq 0$, i.e. $a^3+b^3+c^3\geq 3abc$. Equality requires the second factor to vanish, which happens exactly when $a=b=c$ (and this indeed makes the whole difference zero). Hence the result holds, with equality if and only if $a=b=c$, as required.

Q12. Let $p=\sqrt[3]{a}$, $q=\sqrt[3]{b}$ and $r=\sqrt[3]{c}$, so that $p,q,r\geq 0$ and $a=p^3$, $b=q^3$, $c=r^3$. Applying the previous question to p,q,r,

$$p^3+q^3+r^3\geq 3pqr \Rightarrow a+b+c\geq 3\sqrt[3]{abc},$$

since $pqr=\sqrt[3]{a}\sqrt[3]{b}\sqrt[3]{c}=\sqrt[3]{abc}$. Dividing both sides by the positive number 3 gives $\frac{a+b+c}{3}\geq\sqrt[3]{abc}$. Equality holds when $p=q=r$, i.e. $a=b=c$. This completes the proof.

Q13. For all real x, form the difference and write it over the common denominator:

$$\frac{1}{2}-\frac{x}{x^2+1}=\frac{(x^2+1)-2x}{2(x^2+1)}=\frac{(x-1)^2}{2(x^2+1)}.$$

The numerator $(x-1)^2\geq 0$, and the denominator $2(x^2+1)>0$ for every real x (since $x^2+1\geq 1>0$), so the quotient is ≥ 0. Hence $\frac{x}{x^2+1}\leq\frac{1}{2}$, with equality when $(x-1)^2=0$, i.e. $x=1$. This completes the proof.

Q14. Consider the difference of the two sides:

$$2(a^2+b^2)-(a+b)^2=2a^2+2b^2-(a^2+2ab+b^2)=a^2-2ab+b^2=(a-b)^2\geq 0.$$

Hence $2(a^2+b^2)\geq(a+b)^2$, with equality when $a-b=0$, i.e. $a=b$. This completes the proof.

Q15. For all real x and y, note that $|xy|=|x||y|$ and consider the square $(|x|-|y|)^2\geq 0$. Expanding,

$$|x|^2-2|x||y|+|y|^2\geq 0.$$

Since $|x|^2=x^2$ and $|y|^2=y^2$, this reads $x^2+y^2-2|xy|\geq 0$, so $x^2+y^2\geq 2|xy|$. Equality holds when $|x|-|y|=0$, i.e. $|x|=|y|$. This completes the proof.

Q16. We argue by cases on the sign of ab.

Case 1: $ab>0$. Then

$$\frac{a}{b}+\frac{b}{a}-2=\frac{a^2+b^2-2ab}{ab}=\frac{(a-b)^2}{ab},$$

where the numerator $(a-b)^2\geq 0$ and the denominator $ab>0$, so the quotient is ≥ 0. Hence $\frac{a}{b}+\frac{b}{a}\geq 2$, with equality when $(a-b)^2=0$, i.e. $a=b$.

Case 2: $ab<0$. Then

$$\frac{a}{b}+\frac{b}{a}+2=\frac{a^2+b^2+2ab}{ab}=\frac{(a+b)^2}{ab},$$

where the numerator $(a+b)^2\geq 0$ but the denominator $ab<0$, so the quotient is ≤ 0. Hence $\frac{a}{b}+\frac{b}{a}+2\leq 0$, that is $\frac{a}{b}+\frac{b}{a}\leq -2$, with equality when $(a+b)^2=0$, i.e. $a=-b$.

These two cases cover every pair of real numbers with $ab\neq 0$. This completes the proof.

Q17. Let $a,b>0$. The **left** inequality $\sqrt{ab}\leq\frac{a+b}{2}$ is the AM–GM inequality proved in Q3, with equality if and only if $a=b$. For the **right** inequality, Q14 gives $2(a^2+b^2)\geq(a+b)^2$; dividing by 4,

$$\frac{a^2+b^2}{2} \ge \left(\frac{a+b}{2}\right)^2.$$

Both sides are non-negative, so taking square roots preserves the inequality: $\sqrt{\frac{a^2+b^2}{2}} \ge \frac{a+b}{2}$, with equality if and only if $a=b$. Chaining the two results,

$$\sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}},$$

with equality throughout if and only if $a=b$. This completes the proof.

Q18. For $x>0$, consider the difference between $x+\frac{16}{x}$ and the claimed minimum 8:

$$x+\frac{16}{x}-8=\frac{x^2-8x+16}{x}=\frac{(x-4)^2}{x}.$$

The numerator $(x-4)^2 \ge 0$ and the denominator $x>0$, so the quotient is ≥ 0; hence $x+\frac{16}{x} \ge 8$ for all $x>0$. Equality holds when $(x-4)^2=0$, i.e. $x=4$, and there $x+\frac{16}{x}=4+4=8$. Since the expression is never less than 8 and attains the value 8 at $x=4$, its minimum value is 8, occurring at $x=4$. This completes the proof.

Theory

A **finite series** is the sum of the first n terms of a sequence $t_1, t_2, \dots, t_n$. Using **sigma notation** this **partial sum** is written

$$S_n = \sum_{k=1}^{n} t_k = t_1 + t_2 + \cdots + t_n,$$

where k is the index of summation, 1 is the lower limit and n the upper limit. To **prove a closed form** for S_n is to show that the whole sum equals a single expression in n containing no "$\cdots$". One powerful way to do this — a direct alternative to proof by mathematical induction — is to write each term as a **difference of consecutive values of one function**, so that almost everything cancels. Such a sum is called a **telescoping sum**, because it collapses like a folding telescope.

The telescoping principle.

Suppose the general term can be written as

$$t_k = f(k) - f(k+1)$$

for some function f. Then

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} \left(f(k) - f(k+1)\right) = f(1) - f(n+1).$$

To see why, write the sum out term by term:

$$\sum_{k=1}^{n} \left(f(k) - f(k+1)\right) = \left(f(1) - f(2)\right) + \left(f(2) - f(3)\right) + \cdots + \left(f(n) - f(n+1)\right).$$

Each of the values $f(2), f(3), \dots, f(n)$ appears once with a plus sign and once with a minus sign, so all of these interior terms cancel in pairs. Only the opening $f(1)$ and the closing $-f(n+1)$ survive, giving $f(1) - f(n+1)$.

The same argument run the other way gives the **mirror image**: if instead $t_k = g(k+1) - g(k)$, then

$$\sum_{k=1}^{n} \left(g(k+1) - g(k)\right) = g(n+1) - g(1).$$

Both forms are used constantly; the only thing to watch is which end survives.

Revealing the telescoping form with partial fractions.

A term rarely arrives already written as a difference. When t_k is a rational expression whose denominator factorises, **partial fractions** are the tool for hunting out the hidden difference. The key example is

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

which is exactly $f(k) - f(k+1)$ with $f(k) = \frac{1}{k}$. To find such a decomposition, write $\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$, clear denominators to get $1 = A(k+1) + Bk$, and substitute convenient values: $k = 0$ gives $A = 1$ and $k = -1$ gives $B = -1$. More generally, for two linear factors with the same coefficient of k, say $ak + b$ and $ak + c$ with $a > 0$ and $c > b$,

$$\frac{1}{(ak+b)(ak+c)} = \frac{1}{c-b}\left(\frac{1}{ak+b} - \frac{1}{ak+c}\right),$$

so a constant factor of $\frac{1}{c-b}$ is pulled out in front. The bracket telescopes provided the two factors are a whole number of index steps apart — that is, provided $c-b=ma$ for some positive integer m. Replacing k by $k+m$ then turns $ak+b$ into $ak+c$, so the bracket is $f(k)-f(k+m)$ with $f(k)=\frac{1}{ak+b}$; the key example above is $a=1$, $b=0$, $c=1$, giving $m=1$ and the standard form $f(k)-f(k+1)$. This condition must be checked, not assumed. In

$$\frac{1}{2k(2k+1)}=\frac{1}{2k}-\frac{1}{2k+1}$$

the factors differ by 1 while the index step is $a=2$, so the $\frac{1}{2k+1}$ subtracted at index k is never reproduced at any later index: no cancellation occurs and the method does not apply.

Wider gaps.

If the two factors differ by more than one step — for instance $\frac{1}{k(k+2)}$ — the decomposition

$$\frac{1}{k(k+2)}=\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)$$

telescopes with a **gap of two**. Writing $t_k=1/2\big(h(k)-h(k+2)\big)$ with $h(k)=\frac{1}{k}$, the terms $h(3),h(4),\ldots,h(n)$ still cancel, but now the first **two** and the last **two** survive:

$$\sum_{k=1}^{n}\big(h(k)-h(k+2)\big)=h(1)+h(2)-h(n+1)-h(n+2).$$

In general a gap of m leaves m uncancelled terms at each end.

Limit of the sum.

Because the closed form is a single expression in n, we may ask what happens as $n\to\infty$. If $f(n+1)\to L$, then

$$\sum_{k=1}^{\infty}t_k=\lim_{n\to\infty}\big(f(1)-f(n+1)\big)=f(1)-L.$$

When this limit exists the infinite series **converges** to that value; when $f(n+1)$ grows without bound the series **diverges** and no finite limit exists. Telescoping is one of the few techniques that yields the exact value of an infinite sum rather than an approximation.

Partial products.

Just as the **partial sum** $S_n=\sum_{k=1}^{n}t_k$ adds the first n terms of a sequence, the **partial product** multiplies them. Using **product notation** — a capital Greek pi $\prod$ in place of the sigma —

$$P_n=\prod_{k=1}^{n}t_k=t_1\times t_2\times\cdots\times t_n.$$

A product **telescopes** when each factor is a **ratio of consecutive values** of one function. If

$$t_k=\frac{f(k+1)}{f(k)},$$

then writing the product out,

$$\prod_{k=1}^{n}\frac{f(k+1)}{f(k)}=\frac{f(2)}{f(1)}\times\frac{f(3)}{f(2)}\times\cdots\times\frac{f(n+1)}{f(n)}=\frac{f(n+1)}{f(1)},$$

because every interior value $f(2),f(3),\ldots,f(n)$ occurs once in a numerator and once in a denominator, and so cancels. Only the final numerator $f(n+1)$ and the first denominator $f(1)$ survive. This is the multiplicative twin of the telescoping sum: differences that cancel become factors that cancel. As with sums, once the closed form P_n is known we may let $n\to\infty$ to read off the **limiting behaviour** of the partial products: they may approach a finite value — as in Example 5 below — or grow without bound.

Worked examples

Example 1 — scaffolded

Prove that

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}\text{ for every positive integer } n,$$

and find the limit of the sum as $n\to\infty$.

Working	Comment
Let $t_k=\frac{1}{k(k+1)}$ and seek A,B with $\frac{1}{k(k+1)}=\frac{A}{k}+\frac{B}{k+1}$.	Set up partial fractions to expose a difference.
$1=A(k+1)+Bk$. Put $k=0$: $A=1$. Put $k=-1$: $-B=1$, so $B=-1$.	Clear denominators, then substitute values that zero a factor.
$\therefore t_k=\frac{1}{k}-\frac{1}{k+1}=f(k)-f(k+1)$ with $f(k)=\frac{1}{k}$.	The term is now a difference of consecutive values of f.
$S_n=(1-1/2)+(1/2-1/3)+\cdots+(1/n-1/n+1)$.	Write the sum out; interior fractions appear twice with opposite signs.
Interior terms cancel, leaving $S_n=f(1)-f(n+1)=1-\frac{1}{n+1}$.	Apply the telescoping principle.
$1-\frac{1}{n+1}=\frac{(n+1)-1}{n+1}=\frac{n}{n+1}$.	Combine over a common denominator.
As $n\to\infty$, $\frac{1}{n+1}\to 0$, so $S_n\to 1$.	The surviving end term $f(n+1)\to 0$.

Hence $\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$ for every positive integer n, and the series converges to 1 as $n\to\infty$, as required.

Example 2 — scaffolded

Prove that

$$\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1},$$

and hence state the value of the infinite series.

Working	Comment
Let $t_k=\dfrac{1}{(2k-1)(2k+1)}$; write $\dfrac{1}{(2k-1)(2k+1)}=\dfrac{A}{2k-1}+\dfrac{B}{2k+1}$.	The denominator is a product of two linear factors.
$1=A(2k+1)+B(2k-1)$. Put $k=1/2$: $2A=1$, $A=1/2$. Put $k=-1/2$: $-2B=1$, $B=-1/2$.	Substitute the values that zero each factor.
$\therefore t_k=\dfrac{1}{2}\left(\dfrac{1}{2k-1}-\dfrac{1}{2k+1}\right)$.	Replacing k by $k+1$ turns $2k-1$ into $2k+1$, so the bracket is $f(k)-f(k+1)$.
$S_n=\dfrac{1}{2}[(1-1/3)+(1/3-1/5)+\cdots+(1/2n-1-1/2n+$.	Factor $1/2$ out; the bracket telescopes.
$=\dfrac{1}{2}\left(1-\dfrac{1}{2n+1}\right)$.	Only the first and last fractions remain.
$=\dfrac{1}{2}\cdot\dfrac{2n}{2n+1}=\dfrac{n}{2n+1}$.	Simplify using $1-1/2n+1=2n/2n+1$.
As $n\to\infty$, $\dfrac{1}{2n+1}\to 0$, so $S_n\to\dfrac{1}{2}$.	Take the limit of the closed form.

Therefore the partial sum equals $\dfrac{n}{2n+1}$, and the infinite series converges to $\dfrac{1}{2}$. This completes the proof.

Example 3 — unscaffolded

Prove that

$$\sum_{k=1}^{n}\frac{1}{k(k+2)}=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)},$$

and determine the value of the series as $n\to\infty$.

Here $a=1$ and the two factors differ by 2, so they are two index steps apart and partial fractions give a gap-two telescoping form. Writing $\dfrac{1}{k(k+2)}=\dfrac{A}{k}+\dfrac{B}{k+2}$ and clearing denominators, $1=A(k+2)+Bk$; putting $k=0$ gives $A=1/2$ and $k=-2$ gives $B=-1/2$, so

$$\frac{1}{k(k+2)}=\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right).$$

With $h(k)=\dfrac{1}{k}$ the sum is

$$S_n=\frac{1}{2}\sum_{k=1}^{n}\left(h(k)-h(k+2)\right)=\frac{1}{2}\left(h(1)+h(2)-h(n+1)-h(n+2)\right),$$

because every interior value $h(3),h(4),\ldots,h(n)$ cancels, leaving the first two and the last two terms. Substituting $h(1)=1$ and $h(2)=1/2$,

$$S_n=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}.$$

As $n\to\infty$ both $\dfrac{1}{2(n+1)}$ and $\dfrac{1}{2(n+2)}$ tend to 0, so the series converges to $\dfrac{3}{4}$. Hence the result holds.

Example 4 — unscaffolded

Prove that

$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2},$$

and find the limit of the sum as $n \to \infty$.

Rather than solving for partial fractions, notice that the numerator is exactly the difference of the two squared factors in the denominator:

$$(k+1)^2 - k^2 = (k^2+2k+1) - k^2 = 2k+1.$$

Therefore

$$\frac{2k+1}{k^2(k+1)^2} = \frac{(k+1)^2 - k^2}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2},$$

which is $f(k) - f(k+1)$ with $f(k) = \frac{1}{k^2}$. By the telescoping principle,

$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = f(1) - f(n+1) = \frac{1}{1^2} - \frac{1}{(n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

As $n \to \infty$, $\frac{1}{(n+1)^2} \to 0$, so the sum tends to 1. This completes the proof.

Example 5 — unscaffolded

Prove that

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n} \text{ for every integer } n \geq 2,$$

and find the limit of the product as $n \to \infty$.

Factor each term so that it becomes a product of two ratios. Since $1 - \frac{1}{k^2} = \frac{k^2-1}{k^2} = \frac{(k-1)(k+1)}{k^2}$, we may write

$$1 - \frac{1}{k^2} = \frac{k-1}{k} \times \frac{k+1}{k}.$$

The whole product therefore splits into two telescoping products,

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \left(\prod_{k=2}^{n} \frac{k-1}{k}\right)\left(\prod_{k=2}^{n} \frac{k+1}{k}\right).$$

The first is $\frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{n-1}{n} = \frac{1}{n}$, in which every interior value cancels and only the first numerator 1 and the last denominator n remain; the second is $\frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{n+1}{n} = \frac{n+1}{2}$, leaving the last numerator $n+1$ over the first denominator 2. Multiplying,

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{1}{n} \times \frac{n+1}{2} = \frac{n+1}{2n}.$$

As $n \to \infty$, $\frac{n+1}{2n} \to \frac{1}{2}$, so the partial products approach $\frac{1}{2}$. This completes the proof.

Practice questions

Scaffolded

Q1. Prove that $\sum_{k=1}^{n} \frac{1}{(3k-1)(3k+2)} = \frac{1}{6} - \frac{1}{3(3n+2)}$, and find the limit as $n \to \infty$. (a) Find constants A, B with $\frac{1}{(3k-1)(3k+2)} = \frac{A}{3k-1} + \frac{B}{3k+2}$. (b) Hence write the general term as $f(k) - f(k+1)$, stating f. (c) Write out the telescoping sum and cancel to obtain the closed form. (d) State the limit of S_n as $n \to \infty$.

Q2. Prove that $\sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)} = \frac{1}{6} - \frac{1}{2(2n+3)}$, and find its limit. (a) Decompose $\frac{1}{(2k+1)(2k+3)}$ into partial fractions. (b) Identify f so that the term is $f(k) - f(k+1)$. (c) Telescope the sum and simplify. (d) State the value of the infinite series.

Q3. Prove that $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$. (a) Show that $\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right)$. (b) Identify $f(k) = \frac{1}{k(k+1)}$ so that the term is $1/2(f(k) - f(k+1))$. (c) Telescope and simplify to the closed form. (d) Find the limit of S_n as $n \to \infty$.

Q4. Prove that $\sum_{k=1}^{n} \log_e\left(\frac{k+1}{k}\right) = \log_e(n+1)$. (a) Use a logarithm law to write the term as $g(k+1) - g(k)$, stating g. (b) Telescope the sum using the mirror-image principle. (c) Explain why the series has no finite limit as $n \to \infty$.

Q5. Prove that $\sum_{k=1}^{n} \frac{1}{\sqrt{k+1} + \sqrt{k}} = \sqrt{n+1} - 1$. (a) Rationalise the term to show it equals $\sqrt{k+1} - \sqrt{k}$. (b) Telescope the sum. (c) State, with a reason, whether the series converges as $n \to \infty$.

Q6. Prove that $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$. (a) Show that $k \cdot k! = (k+1)! - k!$. (b) Telescope the sum using the mirror-image principle. (c) State whether the sum approaches a finite limit as $n \to \infty$.

Unscaffolded

Q7. Prove that $\sum_{k=1}^{n} \frac{1}{(4k-3)(4k+1)} = \frac{n}{4n+1}$, and hence find the value of the series as $n \to \infty$.

Q8. Prove that $\sum_{k=1}^{n} \frac{3}{k(k+3)} = \frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$, and find its limit.

Q9. Prove that $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$, and find the value of the series as $n \to \infty$.

Q10. Prove that $\sum_{k=1}^{n} \frac{1}{(3k+1)(3k+4)} = \frac{n}{4(3n+4)}$, and state its limit.

Q11. Prove that $\sum_{k=1}^{n} \frac{4}{(2k-1)(2k+1)} = \frac{4n}{2n+1}$, and find the value of the infinite series.

Q12. Prove that $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{1}{2} - \frac{1}{n+2}$, and hence find its limit.

Q13. Prove that $\sum_{k=1}^{n} \frac{1}{\sqrt{2k+1}+\sqrt{2k-1}} = \frac{\sqrt{2n+1}-1}{2}$. Does the series converge as $n \to \infty$? Justify your answer.

Q14. Prove that $\sum_{k=1}^{n} \log_e\left(1 - \frac{1}{(k+1)^2}\right) = \log_e\left(\frac{n+2}{2(n+1)}\right)$, and find the value of the series as $n \to \infty$.

Q15. Prove that $\sum_{k=1}^{n} \frac{6}{(3k-2)(3k+1)} = \frac{6n}{3n+1}$, and state its limit.

Q16. Prove that $\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$, and find the value of the infinite series.

Q17. Prove that $\sum_{k=1}^{n} \frac{5}{(5k-4)(5k+1)} = \frac{5n}{5n+1}$, and find its limit.

Q18. Prove that $\sum_{k=1}^{n} \frac{3k^2+3k+1}{k^3(k+1)^3} = 1 - \frac{1}{(n+1)^3}$, and find the limit of the sum as $n \to \infty$.

Q19. Prove that $\prod_{k=1}^{n} \frac{k}{k+1} = \frac{1}{n+1}$, and find the limit of the product as $n \to \infty$.

Q20. Prove that $\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = n+1$, and describe the behaviour of the partial products as $n \to \infty$.

Answers

Q1. $S_n=\frac{1}{6}-\frac{1}{3(3n+2)}$; limit $\frac{1}{6}$. Proof — see solutions. **Q2.** $S_n=\frac{1}{6}-\frac{1}{2(2n+3)}$; limit $\frac{1}{6}$. Proof — see solutions. **Q3.** $S_n=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$; limit $\frac{1}{4}$. Proof — see solutions. **Q4.** $S_n=\log_e(n+1)$; no finite limit — the series diverges to ∞. Proof — see solutions. **Q5.** $S_n=\sqrt{n+1}-1$; no finite limit — the series diverges. Proof — see solutions. **Q6.** $S_n=(n+1)!-1$; no finite limit — the series diverges. Proof — see solutions. **Q7.** $S_n=\frac{n}{4n+1}$; limit $\frac{1}{4}$. **Q8.** $S_n=\frac{11}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}$; limit $\frac{11}{6}$. **Q9.** $S_n=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$; limit $\frac{3}{4}$. **Q10.** $S_n=\frac{n}{4(3n+4)}$; limit $\frac{1}{12}$. **Q11.** $S_n=\frac{4n}{2n+1}$; limit 2. **Q12.** $S_n=\frac{1}{2}-\frac{1}{n+2}$; limit $\frac{1}{2}$. **Q13.** $S_n=\frac{\sqrt{2n+1}-1}{2}$; no finite limit — the series diverges. **Q14.** $S_n=\log_e\left(\frac{n+2}{2(n+1)}\right)$; limit $-\log_e 2$. **Q15.** $S_n=\frac{6n}{3n+1}$; limit 2. **Q16.** $S_n=1-\frac{1}{(n+1)!}$; limit 1. **Q17.** $S_n=\frac{5n}{5n+1}$; limit 1. **Q18.** $S_n=1-\frac{1}{(n+1)^3}$; limit 1. **Q19.** $P_n=\frac{1}{n+1}$; the partial products approach 0. Proof — see solutions. **Q20.** $P_n=n+1$; the partial products grow without bound (no finite limit). Proof — see solutions.

Fully-worked solutions

Q1. Write $\frac{1}{(3k-1)(3k+2)}=\frac{A}{3k-1}+\frac{B}{3k+2}$. Clearing denominators gives $1=A(3k+2)+B(3k-1)$. Putting $k=1/3$ gives $1=3A$, so $A=1/3$; putting $k=-2/3$ gives $1=-3B$, so $B=-1/3$. Hence

$$t_k=\frac{1}{3}\left(\frac{1}{3k-1}-\frac{1}{3k+2}\right).$$

Since $3(k+1)-1=3k+2$, this is $f(k)-f(k+1)$ with $f(k)=\frac{1}{3(3k-1)}$, so by telescoping

$$S_n=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)=\frac{1}{6}-\frac{1}{3(3n+2)}.$$

As $n\to\infty$, $\frac{1}{3(3n+2)}\to 0$, so $S_n\to\frac{1}{6}$. This completes the proof.

Q2. Write $\frac{1}{(2k+1)(2k+3)}=\frac{A}{2k+1}+\frac{B}{2k+3}$, giving $1=A(2k+3)+B(2k+1)$. Putting $k=-1/2$ gives $1=2A$, so $A=1/2$; putting $k=-3/2$ gives $1=-2B$, so $B=-1/2$. Hence

$$t_k=\frac{1}{2}\left(\frac{1}{2k+1}-\frac{1}{2k+3}\right),$$

which is $f(k)-f(k+1)$ with $f(k)=\frac{1}{2(2k+1)}$ because $2(k+1)+1=2k+3$. Telescoping,

$$S_n=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)=\frac{1}{6}-\frac{1}{2(2n+3)}.$$

As $n\to\infty$ the second term tends to 0, so $S_n\to\frac{1}{6}$. Hence the result holds.

Q3. Using the difference of the two denominators,

$$\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}=\frac{(k+2)-k}{k(k+1)(k+2)}=\frac{2}{k(k+1)(k+2)},$$

so $\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\left(\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\right)$. This is $1/2(f(k)-f(k+1))$ with $f(k)=\frac{1}{k(k+1)}$, since $f(k+1)=\frac{1}{(k+1)(k+2)}$. Telescoping,

$$S_n=\frac{1}{2}(f(1)-f(n+1))=\frac{1}{2}\left(\frac{1}{1\cdot 2}-\frac{1}{(n+1)(n+2)}\right)=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

As $n\to\infty$, $\frac{1}{2(n+1)(n+2)}\to 0$, so $S_n\to\frac{1}{4}$. This completes the proof.

Q4. By the logarithm law $\log_e\left(\frac{k+1}{k}\right)=\log_e(k+1)-\log_e k$. With $g(k)=\log_e k$ this is the mirror form $g(k+1)-g(k)$, so

$$S_n=\sum_{k=1}^{n}(g(k+1)-g(k))=g(n+1)-g(1)=\log_e(n+1)-\log_e 1=\log_e(n+1),$$

since $\log_e 1=0$. As $n\to\infty$, $\log_e(n+1)$ increases without bound, so the series has no finite limit; it diverges to ∞. Hence the result holds.

Q5. Rationalising the denominator,

$$\frac{1}{\sqrt{k+1}+\sqrt{k}}=\frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k+1}+\sqrt{k})(\sqrt{k+1}-\sqrt{k})}=\frac{\sqrt{k+1}-\sqrt{k}}{(k+1)-k}=\sqrt{k+1}-\sqrt{k}.$$

With $g(k)=\sqrt{k}$ this is the mirror form $g(k+1)-g(k)$, so

$$S_n=g(n+1)-g(1)=\sqrt{n+1}-\sqrt{1}=\sqrt{n+1}-1.$$

As $n\to\infty$, $\sqrt{n+1}\to\infty$, so S_n increases without bound and the series diverges; it has no finite limit. This completes the proof.

Q6. Factorising the right-hand side, $(k+1)!-k!=k!((k+1)-1)=k!\cdot k=k\cdot k!$, so $k\cdot k!=(k+1)!-k!$. With $g(k)=k!$ this is the mirror form $g(k+1)-g(k)$, hence

$$S_n=g(n+1)-g(1)=(n+1)!-1!=(n+1)!-1.$$

As $n\to\infty$, $(n+1)!$ increases without bound, so the series diverges and has no finite limit. Hence the result holds.

Q7. Write $\frac{1}{(4k-3)(4k+1)}=\frac{A}{4k-3}+\frac{B}{4k+1}$, giving $1=A(4k+1)+B(4k-3)$. Putting $k=3/4$ gives $1=4A$, so $A=1/4$; putting $k=-1/4$ gives $1=-4B$, so $B=-1/4$. Thus $t_k=\frac{1}{4}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$, which is $f(k)-f(k+1)$ with $f(k)=\frac{1}{4(4k-3)}$ because $4(k+1)-3=4k+1$. Telescoping,

$$S_n=\frac{1}{4}\left(1-\frac{1}{4n+1}\right)=\frac{1}{4}\cdot\frac{4n}{4n+1}=\frac{n}{4n+1}.$$

As $n \to \infty$, $\dfrac{1}{4n+1} \to 0$, so $S_n \to \dfrac{1}{4}$. This completes the proof.

Q8. Since $\dfrac{1}{k(k+3)}=\dfrac{1}{3}\left(\dfrac{1}{k}-\dfrac{1}{k+3}\right)$, we have $\dfrac{3}{k(k+3)}=\dfrac{1}{k}-\dfrac{1}{k+3}$, a gap-three telescoping form. With $h(k)=\dfrac{1}{k}$,

$$S_n=\sum_{k=1}^{n}(h(k)-h(k+3))=h(1)+h(2)+h(3)-h(n+1)-h(n+2)-h(n+3),$$

because a gap of 3 leaves the first three and last three terms uncancelled. Substituting $h(1)+h(2)+h(3)=1+1/2+1/3=11/6$,

$$S_n=\frac{11}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}.$$

As $n \to \infty$ the three trailing fractions tend to 0, so $S_n \to \dfrac{11}{6}$. Hence the result holds.

Q9. For $k \geq 2$, $k^2-1=(k-1)(k+1)$, and $\dfrac{1}{(k-1)(k+1)}=\dfrac{1}{2}\left(\dfrac{1}{k-1}-\dfrac{1}{k+1}\right)$, a gap-two form. With $h(k)=\dfrac{1}{k-1}$ (so $h(k+2)=\dfrac{1}{k+1}$), summing from $k=2$ to n leaves the first two and the last two terms:

$$S_n=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}.$$

As $n \to \infty$ both trailing terms tend to 0, so $S_n \to \dfrac{3}{4}$. This completes the proof.

Q10. Write $\dfrac{1}{(3k+1)(3k+4)}=\dfrac{A}{3k+1}+\dfrac{B}{3k+4}$, giving $1=A(3k+4)+B(3k+1)$. Putting $k=-1/3$ gives $1=3A$, so $A=1/3$; putting $k=-4/3$ gives $1=-3B$, so $B=-1/3$. Thus $t_k=\dfrac{1}{3}\left(\dfrac{1}{3k+1}-\dfrac{1}{3k+4}\right)$, which is $f(k)-f(k+1)$ with $f(k)=\dfrac{1}{3(3k+1)}$ because $3(k+1)+1=3k+4$. Telescoping,

$$S_n=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3n+4}\right)=\frac{1}{12}-\frac{1}{3(3n+4)}=\frac{n}{4(3n+4)}.$$

As $n \to \infty$, $S_n \to \dfrac{1}{12}$. Hence the result holds.

Q11. From the standard decomposition $\dfrac{1}{(2k-1)(2k+1)}=\dfrac{1}{2}\left(\dfrac{1}{2k-1}-\dfrac{1}{2k+1}\right)$, multiplying by 4 gives $t_k=2\left(\dfrac{1}{2k-1}-\dfrac{1}{2k+1}\right)$, which is $f(k)-f(k+1)$ with $f(k)=\dfrac{2}{2k-1}$. Telescoping,

$$S_n=2\left(1-\frac{1}{2n+1}\right)=2\cdot\frac{2n}{2n+1}=\frac{4n}{2n+1}.$$

As $n \to \infty$, $\dfrac{1}{2n+1} \to 0$, so $S_n \to 2$. This completes the proof.

Q12. Here $\dfrac{1}{(k+1)(k+2)}=\dfrac{1}{k+1}-\dfrac{1}{k+2}$, which is $f(k)-f(k+1)$ with $f(k)=\dfrac{1}{k+1}$. Telescoping,

$$S_n = f(1) - f(n+1) = \frac{1}{2} - \frac{1}{n+2}.$$

As $n \to \infty$, $\frac{1}{n+2} \to 0$, so $S_n \to \frac{1}{2}$. Hence the result holds.

Q13. Rationalising, and using $(\sqrt{2k+1})^2 - (\sqrt{2k-1})^2 = (2k+1) - (2k-1) = 2$,

$$\frac{1}{\sqrt{2k+1}+\sqrt{2k-1}} = \frac{\sqrt{2k+1}-\sqrt{2k-1}}{(2k+1)-(2k-1)} = \frac{1}{2}(\sqrt{2k+1}-\sqrt{2k-1}).$$

With $g(k) = \sqrt{2k-1}$ (so $g(k+1) = \sqrt{2k+1}$) this is $1/2(g(k+1) - g(k))$, so

$$S_n = \frac{1}{2}(g(n+1) - g(1)) = \frac{1}{2}(\sqrt{2n+1} - \sqrt{1}) = \frac{\sqrt{2n+1}-1}{2}.$$

As $n \to \infty$, $\sqrt{2n+1} \to \infty$, so S_n grows without bound: the series does **not** converge and has no finite limit. This completes the proof.

Q14. First $1 - \frac{1}{(k+1)^2} = \frac{(k+1)^2 - 1}{(k+1)^2} = \frac{k(k+2)}{(k+1)^2}$, so by logarithm laws

$$\log_e\left(1 - \frac{1}{(k+1)^2}\right) = \log_e\left(\frac{k}{k+1}\right) + \log_e\left(\frac{k+2}{k+1}\right) = [\log_e(k+2) - \log_e(k+1)] - [\log_e(k+1) - \log_e k].$$

With $g(k) = \log_e(k+1) - \log_e k$ this is the mirror form $g(k+1) - g(k)$, so

$$S_n = g(n+1) - g(1) = [\log_e(n+2) - \log_e(n+1)] - [\log_e 2 - \log_e 1] = \log_e\left(\frac{n+2}{n+1}\right) - \log_e 2 = \log_e\left(\frac{n+2}{2(n+1)}\right).$$

As $n \to \infty$, $\frac{n+2}{2(n+1)} \to \frac{1}{2}$, so $S_n \to \log_e\left(\frac{1}{2}\right) = -\log_e 2$. Hence the result holds.

Q15. From $\frac{1}{(3k-2)(3k+1)} = \frac{1}{3}\left(\frac{1}{3k-2} - \frac{1}{3k+1}\right)$, multiplying by 6 gives $t_k = 2\left(\frac{1}{3k-2} - \frac{1}{3k+1}\right)$, which is $f(k) - f(k+1)$ with $f(k) = \frac{2}{3k-2}$ because $3(k+1) - 2 = 3k+1$. Telescoping,

$$S_n = 2\left(1 - \frac{1}{3n+1}\right) = 2 \cdot \frac{3n}{3n+1} = \frac{6n}{3n+1}.$$

As $n \to \infty$, $S_n \to 2$. This completes the proof.

Q16. Since $k = (k+1) - 1$,

$$\frac{k}{(k+1)!} = \frac{(k+1)-1}{(k+1)!} = \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!},$$

using $\frac{k+1}{(k+1)!} = \frac{1}{k!}$. This is $f(k) - f(k+1)$ with $f(k) = \frac{1}{k!}$, so telescoping gives

$$S_n = f(1) - f(n+1) = \frac{1}{1!} - \frac{1}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

As $n \to \infty$, $(n+1)! \to \infty$, so $\frac{1}{(n+1)!} \to 0$ and $S_n \to 1$. Hence the result holds.

Q17. From $\frac{1}{(5k-4)(5k+1)}=\frac{1}{5}\left(\frac{1}{5k-4}-\frac{1}{5k+1}\right)$, multiplying by 5 gives $t_k=\frac{1}{5k-4}-\frac{1}{5k+1}$, which is $f(k)-f(k+1)$ with $f(k)=\frac{1}{5k-4}$ because $5(k+1)-4=5k+1$. Telescoping,

$$S_n=1-\frac{1}{5n+1}=\frac{5n}{5n+1}.$$

As $n\to\infty$, $\frac{1}{5n+1}\to 0$, so $S_n\to 1$. This completes the proof.

Q18. Expanding the difference of cubes, $(k+1)^3-k^3=3k^2+3k+1$, so

$$\frac{3k^2+3k+1}{k^3(k+1)^3}=\frac{(k+1)^3-k^3}{k^3(k+1)^3}=\frac{1}{k^3}-\frac{1}{(k+1)^3},$$

which is $f(k)-f(k+1)$ with $f(k)=\frac{1}{k^3}$. Telescoping,

$$S_n=f(1)-f(n+1)=\frac{1}{1^3}-\frac{1}{(n+1)^3}=1-\frac{1}{(n+1)^3}.$$

As $n\to\infty$, $\frac{1}{(n+1)^3}\to 0$, so $S_n\to 1$. Hence the result holds.

Q19. Each factor is already a ratio of consecutive values of $f(k)=k$: writing $\frac{k}{k+1}=\frac{f(k)}{f(k+1)}$ with $f(k)=k$, the product telescopes as

$$\prod_{k=1}^{n}\frac{k}{k+1}=\frac{1}{2}\times\frac{2}{3}\times\cdots\times\frac{n}{n+1}=\frac{1}{n+1},$$

since every interior value cancels, leaving the first numerator 1 and the last denominator $n+1$. As $n\to\infty$, $\frac{1}{n+1}\to 0$, so the partial products approach 0. This completes the proof.

Q20. Here $1+\frac{1}{k}=\frac{k+1}{k}=\frac{f(k+1)}{f(k)}$ with $f(k)=k$, so the product telescopes as

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)=\frac{2}{1}\times\frac{3}{2}\times\cdots\times\frac{n+1}{n}=\frac{n+1}{1}=n+1,$$

the last numerator $n+1$ over the first denominator 1. As $n\to\infty$, $n+1$ grows without bound, so the partial products have no finite limit. Hence the result holds.

Theory

Many statements in mathematics are claims about **every** positive integer n: a formula for the sum of a series, a divisibility property, or an inequality that is supposed to hold from some point on. Checking a claim for $n=1,2,3,\ldots$ can make it *look* true, but no amount of checking proves a statement about infinitely many integers. **Proof by mathematical induction** is the technique that turns such evidence into a proof.

The domino picture.

Imagine an endless line of dominoes. If you know two things — that the **first** domino falls, and that **whenever one domino falls it knocks over the next** — then you can be certain that **every** domino falls. Induction works the same way. The "first domino" is the smallest case; the "each knocks over the next" is an implication that carries the truth of one case forward to the case after it.

The principle of mathematical induction.

Let $P(n)$ be a statement about an integer n, and let n_0 be the starting value (usually $n_0=1$). If

1. **(Base case)** $P(n_0)$ is true, and
2. **(Inductive step)** for every integer $k\geq n_0$, the truth of $P(k)$ implies the truth of $P(k+1)$, that is $P(k)\Rightarrow P(k+1)$,

then $P(n)$ is true for **all** integers $n\geq n_0$.

Structure of an induction proof.

Every proof by induction has the same four parts, and it is worth writing each one explicitly.

- **Name the statement.** Write "Let $P(n)$ be the statement …" so that $P(1)$, $P(k)$ and $P(k+1)$ have a clear meaning.
- **Base case.** Substitute the smallest value $n=n_0$ and show the statement holds there — for an identity or an inequality by evaluating both sides, for a divisibility claim by checking the value is a multiple. Do not skip this: the inductive step alone proves nothing.
- **Inductive step.** *Assume* $P(k)$ is true for some integer $k\geq n_0$ — this assumption is the **inductive hypothesis** — and use it to *prove* $P(k+1)$. The whole argument lives in getting from $P(k)$ to $P(k+1)$.
- **Conclusion.** State that, since $P(n_0)$ holds and $P(k)\Rightarrow P(k+1)$, by the principle of mathematical induction $P(n)$ is true for all integers $n\geq n_0$.

The single most important habit is to **write the inductive hypothesis down in full** and to make clear the exact moment you use it — the step is only valid because $P(k)$ was assumed.

Application 1: closed forms for sums of a finite series.

Suppose $P(n)$ claims that a sum $S_n=a_1+a_2+\ldots+a_n$ equals some closed formula. Writing $S_n=\sum_{r=1}^{n} a_r$, the key algebraic fact is that the sum to $k+1$ terms is the sum to k terms plus one more term:

$$S_{k+1}=S_k+a_{k+1}.$$

In the inductive step you replace S_k by the formula (using the hypothesis), add the $(k+1)$-th term, and then **factorise and simplify until the result is exactly the formula with n replaced by $k+1$**. Standard results proved this way include

$$\sum_{r=1}^{n} r=\frac{n(n+1)}{2}, \sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}, \sum_{r=1}^{n} r^3=\left(\frac{n(n+1)}{2}\right)^2,$$

as well as the sum of the first n odd numbers, $\sum_{r=1}^{n}(2r-1)=n^2$, and geometric-type sums such as $\sum_{r=1}^{n} 2^{r-1}=2^n-1$.

Application 2: divisibility.

To prove that an integer d divides an expression $f(n)$ for all $n\geq 1$, written $d/f(n)$, set $f(n)$ up as the statement $P(n)$. The base case checks that $f(n_0)$ is a multiple of d. For the step, assume $f(k)$ is divisible by d — say $f(k)=dm$ for some integer m — and show $f(k+1)$ is also a multiple of d. A reliable method is to relate $f(k+1)$ to $f(k)$: often

$$f(k+1)=(\text{a multiple of } d)+(\text{a multiple of } f(k)),$$

so that the hypothesis makes the whole expression divisible by d. Because m is only required to be an integer, keep it as a symbol rather than a number.

Application 3: inequalities.

Induction also proves inequalities such as $2^n>n^2$ for $n\geq 5$ or $n!\geq 2^{n-1}$ for $n\geq 1$. Two features are common. First, the starting value n_0 is often larger than 1, because the inequality may fail for small n (for instance $2^n>n^2$ is false at $n=2,3,4$, even though it happens to be true at $n=1$); the base case must be taken at the start of the claimed range — the smallest n_0 from which the statement holds for *every* integer $n\geq n_0$, which need not be the first value that satisfies it. Second, the step usually needs an **auxiliary inequality**: from $P(k)$ you reach some intermediate expression, and you must then argue that this intermediate expression is itself ≥ (or >) the target for $P(k+1)$. That auxiliary comparison is frequently settled by rewriting a difference in terms of a perfect square — which is never negative — and then bounding it below using $k\geq n_0$.

The worked examples below include a proof from each of these families: two scaffolded — a sum and a divisibility proof — and two unscaffolded, a sum and an inequality.

Worked examples

Example 1 — scaffolded

Use mathematical induction to prove that

$$\sum_{r=1}^{n} r=\frac{n(n+1)}{2} \text{ for all integers } n\geq 1.$$

Let $P(n)$ be this statement, and write $S_n=\sum_{r=1}^{n} r=1+2+\ldots+n$, so $P(n)$ claims $S_n=\frac{n(n+1)}{2}$.

Working	Comment
Base case $n=1$: $S_1=1$ and $\frac{1(1+1)}{2}=1$.	Substitute $n=1$ into both sides; they agree, so $P(1)$ is true.
Assume $S_k=\frac{k(k+1)}{2}$ for some integer $k\geq 1$.	This is the inductive hypothesis — assume $P(k)$.
$S_{k+1}=S_k+(k+1)$	The sum to $k+1$ terms is the sum to k terms plus the next term.
$=\frac{k(k+1)}{2}+(k+1)$	Replace S_k using the hypothesis — this is where $P(k)$ is used.

Working	Comment
$=(k+1)\left(\frac{k}{2}+1\right)=(k+1)\cdot\frac{k+2}{2}$	Take out the common factor $(k+1)$ and simplify the bracket.
$=\frac{(k+1)(k+2)}{2}=\frac{(k+1)((k+1)+1)}{2}$	This is the formula with n replaced by $k+1$, i.e. $P(k+1)$.
So $P(k)\Rightarrow P(k+1)$.	The step is complete.

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for every $k\geq1$, by the principle of mathematical induction $P(n)$ is true for all integers $n\geq1$, as required.

Example 2 — scaffolded

Use mathematical induction to prove that $3^{2n}-1$ is divisible by 8 for all integers $n\geq1$; that is, $8\,/\,3^{2n}-1$.

Let $P(n)$ be this statement and write $f(n)=3^{2n}-1$.

Working	Comment
Base case $n=1$: $f(1)=3^2-1=9-1=8=8\times1$.	A multiple of 8, so $P(1)$ is true.
Assume $f(k)=3^{2k}-1=8m$ for some integer m, $k\geq1$.	The inductive hypothesis; rearranged, $3^{2k}=8m+1$.
$f(k+1)=3^{2(k+1)}-1=3^{2k+2}-1=9\cdot3^{2k}-1$	Use $3^{2k+2}=3^2\cdot3^{2k}=9\cdot3^{2k}$.
$=9(8m+1)-1$	Substitute $3^{2k}=8m+1$ from the hypothesis.
$=72m+9-1=72m+8=8(9m+1)$	Expand and factor out 8; $9m+1$ is an integer.
So $f(k+1)$ is a multiple of 8, i.e. $P(k)\Rightarrow P(k+1)$.	$P(k+1)$ holds.

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for every $k\geq1$, by the principle of mathematical induction $8\,/\,3^{2n}-1$ for all integers $n\geq1$. This completes the proof.

Example 3 — unscaffolded

Use mathematical induction to prove that

$$\sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}\text{ for all integers } n\geq1.$$

Let $P(n)$ be this statement, with $S_n=\sum_{r=1}^{n} r^2$.

Base case. For $n=1$, $S_1=1^2=1$ and $\frac{1(1+1)(2\cdot1+1)}{6}=\frac{1\cdot2\cdot3}{6}=1$. The two sides agree, so $P(1)$ is true.

Inductive step. Assume $S_k=\frac{k(k+1)(2k+1)}{6}$ for some integer $k\geq1$. Then

$$S_{k+1}=S_k+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2.$$

Taking out the common factor $\frac{k+1}{6}$,

$$S_{k+1}=\frac{k+1}{6}[k(2k+1)+6(k+1)]=\frac{k+1}{6}[2k^2+7k+6]=\frac{(k+1)(k+2)(2k+3)}{6},$$

since $2k^2+7k+6=(k+2)(2k+3)$. Writing $k+2=(k+1)+1$ and $2k+3=2(k+1)+1$, this is

$$S_{k+1}=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},$$

which is exactly $P(k+1)$. Hence $P(k)\Rightarrow P(k+1)$.

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for every $k\geq 1$, by the principle of mathematical induction the formula holds for all integers $n\geq 1$. This completes the proof.

Example 4 — unscaffolded

Use mathematical induction to prove that $2^n>n^2$ for all integers $n\geq 5$.

Let $P(n)$ be the statement $2^n>n^2$.

Base case. For $n=5$, $2^5=32$ and $5^2=25$, and $32>25$, so $P(5)$ is true. (The starting value is 5 because $2^n>n^2$ fails at $n=2,3,4$.)

Inductive step. Assume $2^k>k^2$ for some integer $k\geq 5$. Multiplying both sides by 2,

$$2^{k+1}=2\cdot 2^k>2k^2.$$

It remains to show $2k^2\geq (k+1)^2$. Consider the difference

$$2k^2-(k+1)^2=2k^2-k^2-2k-1=k^2-2k-1=(k-1)^2-2.$$

For $k\geq 5$ we have $(k-1)^2\geq 4^2=16>2$, so $(k-1)^2-2>0$ and therefore $2k^2>(k+1)^2$. Combining the two inequalities,

$$2^{k+1}>2k^2>(k+1)^2,$$

so $2^{k+1}>(k+1)^2$, which is $P(k+1)$. Hence $P(k)\Rightarrow P(k+1)$.

Since $P(5)$ is true and $P(k)\Rightarrow P(k+1)$ for every $k\geq 5$, by the principle of mathematical induction $2^n>n^2$ for all integers $n\geq 5$. Hence the result holds.

Practice questions

Scaffolded

Q1. Use mathematical induction to prove that $\sum_{r=1}^{n}(2r-1)=n^2$ for all integers $n\geq 1$. (a) Verify the base case $P(1)$. (b) Write down the inductive hypothesis $P(k)$. (c) Assuming $P(k)$, show that the sum to $k+1$ terms equals $(k+1)^2$. (d) State the conclusion.

Q2. Use mathematical induction to prove that $\sum_{r=1}^{n}r^3=\left(\frac{n(n+1)}{2}\right)^2$ for all integers $n\geq 1$. (a) Verify $P(1)$. (b) State the inductive hypothesis. (c) Add $(k+1)^3$ to the hypothesis and simplify to the formula at $n=k+1$. (d) Conclude.

Q3. Use mathematical induction to prove that $\sum_{r=1}^{n}2^{r-1}=2^n-1$ for all integers $n\geq 1$. (a) Verify $P(1)$. (b) State the inductive hypothesis. (c) Add the $(k+1)$-th term 2^k and simplify to $2^{k+1}-1$. (d) Conclude.

Q4. Use mathematical induction to prove that $6\,/\,n^3+5n$ for all integers $n\geq 1$. (a) Verify the base case. (b) State the inductive hypothesis in the form $f(k)=6m$. (c) Expand $f(k+1)$, and use that $k(k+1)$ is even to show $f(k+1)$ is a multiple of 6. (d) Conclude.

Q5. Use mathematical induction to prove that $3 / n^3 - n$ for all integers $n \geq 1$. (a) Verify the base case. (b) State the inductive hypothesis. (c) Show that $f(k+1) - f(k)$ is a multiple of 3, hence $f(k+1)$ is too. (d) Conclude.

Q6. Use mathematical induction to prove that $n! \geq 2^{n-1}$ for all integers $n \geq 1$. (a) Verify the base case. (b) State the inductive hypothesis. (c) Using $(k+1)! = (k+1) \cdot k!$ and $k+1 \geq 2$, show $(k+1)! \geq 2^k$. (d) Conclude.

Unscaffolded

Q7. Use mathematical induction to prove that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all integers $n \geq 1$.

Q8. Use mathematical induction to prove that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

Q9. Use mathematical induction to prove that $\sum_{r=1}^{n} (3r-2) = \frac{n(3n-1)}{2}$ for all integers $n \geq 1$.

Q10. Use mathematical induction to prove that $\sum_{r=1}^{n} r 2^r = (n-1)2^{n+1} + 2$ for all integers $n \geq 1$.

Q11. Use mathematical induction to prove that $\sum_{r=1}^{n} 5^{r-1} = \frac{5^n - 1}{4}$ for all integers $n \geq 1$.

Q12. Use mathematical induction to prove that $9 / 4^n + 6n - 1$ for all integers $n \geq 1$.

Q13. Use mathematical induction to prove that $7 / 8^n - 1$ for all integers $n \geq 1$.

Q14. Use mathematical induction to prove that $6 / 7^n - 1$ for all integers $n \geq 1$.

Q15. Use mathematical induction to prove that $5 / n^5 - n$ for all integers $n \geq 1$.

Q16. Use mathematical induction to prove that $3^n > n^3$ for all integers $n \geq 4$.

Q17. Use mathematical induction to prove that $(1+x)^n \geq 1 + nx$ for all integers $n \geq 1$, where x is a fixed real number with $x \geq -1$.

Q18. Use mathematical induction to prove that $n! > 2^n$ for all integers $n \geq 4$.

Answers

Q1. Proof by induction; base $P(1)$: $1=1^2$; step reaches $(k+1)^2$. See solutions. **Q2.** Proof by induction; base $P(1)$: $1=1$; step reaches $((k+1)(k+2)/2)^2$. See solutions. **Q3.** Proof by induction; base $P(1)$: $1=2^1-1$; step reaches $2^{k+1}-1$. See solutions. **Q4.** Proof by induction; base $f(1)=6$; step: $f(k+1)=6m+3k(k+1)+6$. See solutions. **Q5.** Proof by induction; base $f(1)=0$; step: $f(k+1)-f(k)=3(k^2+k)$. See solutions. **Q6.** Proof by induction; base $1!=1\geq 2^0$; step uses $k+1\geq 2$. See solutions. **Q7.** Proof by induction; base $P(1)$: $2=2$; step reaches $(k+1)(k+2)(k+3)/3$. See solutions. **Q8.** Proof by induction; base $P(1)$: $1/2=1/2$; step reaches $k+1/k+2$. See solutions. **Q9.** Proof by induction; base $P(1)$: $1=1$; step reaches $(k+1)(3k+2)/2$. See solutions. **Q10.** Proof by induction; base $P(1)$: $2=2$; step reaches $k\,2^{k+2}+2$. See solutions. **Q11.** Proof by induction; base $P(1)$: $1=1$; step reaches $5^{k+1}-1/4$. See solutions. **Q12.** Proof by induction; base $f(1)=9$; step: $f(k+1)=9(4m-2k+1)$. See solutions. **Q13.** Proof by induction; base $f(1)=7$; step: $f(k+1)=7(8m+1)$. See solutions. **Q14.** Proof by induction; base $f(1)=6$; step: $f(k+1)=6(7m+1)$. See solutions. **Q15.** Proof by induction; base $f(1)=0$; step: $f(k+1)=f(k)+5(k^4+2k^3+2k^2+k)$. See solutions. **Q16.** Proof by induction; base $3^4=81>64$; step uses $3k^3\geq(k+1)^3$ for $k\geq 4$. See solutions. **Q17.** Proof by induction (Bernoulli's inequality); base $n=1$ gives equality; step uses $1+x\geq 0$. See solutions. **Q18.** Proof by induction; base $4!=24>16$; step uses $k+1>2$. See solutions.

Fully-worked solutions

Q1. Let $P(n)$ be the statement $S_n=n^2$, where $S_n=\sum_{r=1}^{n}(2r-1)$. *Base case* $n=1$: $S_1=2(1)-1=1$ and $1^2=1$, so $P(1)$ holds. *Inductive step*: assume $S_k=k^2$ for some integer $k\geq 1$. The $(k+1)$-th term is $2(k+1)-1=2k+1$, so

$$S_{k+1}=S_k+(2k+1)=k^2+2k+1=(k+1)^2,$$

which is $P(k+1)$. Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$, by induction $\sum_{r=1}^{n}(2r-1)=n^2$ for all integers $n\geq 1$. This completes the proof.

Q2. Let $P(n)$ be the statement $S_n=\left(\frac{n(n+1)}{2}\right)^2$, where $S_n=\sum_{r=1}^{n}r^3$. *Base case* $n=1$: $S_1=1^3=1$ and $\left(\frac{1\cdot 2}{2}\right)^2=1$, so $P(1)$ holds. *Inductive step*: assume $S_k=\left(\frac{k(k+1)}{2}\right)^2=\frac{k^2(k+1)^2}{4}$. Then

$$S_{k+1}=S_k+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3=(k+1)^2\left[\frac{k^2}{4}+(k+1)\right]=(k+1)^2\cdot\frac{k^2+4k+4}{4}.$$

Since $k^2+4k+4=(k+2)^2$,

$$S_{k+1}=\frac{(k+1)^2(k+2)^2}{4}=\left(\frac{(k+1)(k+2)}{2}\right)^2=\left(\frac{(k+1)((k+1)+1)}{2}\right)^2,$$

which is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction the formula holds for all integers $n\geq 1$. This completes the proof.

Q3. Let $P(n)$ be the statement $S_n=2^n-1$, where $S_n=\sum_{r=1}^{n}2^{r-1}$. *Base case* $n=1$: $S_1=2^0=1$ and $2^1-1=1$, so $P(1)$ holds. *Inductive step*: assume $S_k=2^k-1$. The $(k+1)$-th term is $2^{(k+1)-1}=2^k$, so

$$S_{k+1}=S_k+2^k=(2^k-1)+2^k=2\cdot 2^k-1=2^{k+1}-1,$$

which is $P(k+1)$. Since $P(1)$ holds and $P(k) \Rightarrow P(k+1)$, by induction $\sum_{r=1}^{n} 2^{r-1} = 2^n - 1$ for all integers $n \geq 1$. This completes the proof.

Q4. Let $P(n)$ be the statement $6 / f(n)$, where $f(n) = n^3 + 5n$. *Base case* $n=1$: $f(1) = 1+5 = 6 = 6 \times 1$, so $P(1)$ holds. *Inductive step*: assume $f(k) = k^3 + 5k = 6m$ for some integer m. Then

$$f(k+1) = (k+1)^3 + 5(k+1) = k^3 + 3k^2 + 3k + 1 + 5k + 5 = (k^3 + 5k) + 3k^2 + 3k + 6.$$

So $f(k+1) = 6m + 3k(k+1) + 6$. Because $k(k+1)$ is a product of two consecutive integers it is even, say $k(k+1) = 2t$; then $3k(k+1) = 6t$ and

$$f(k+1) = 6m + 6t + 6 = 6(m+t+1),$$

a multiple of 6, which is $P(k+1)$. Since $P(1)$ holds and $P(k) \Rightarrow P(k+1)$, by induction $6 / n^3 + 5n$ for all integers $n \geq 1$. This completes the proof.

Q5. Let $P(n)$ be the statement $3 / f(n)$, where $f(n) = n^3 - n$. *Base case* $n=1$: $f(1) = 1 - 1 = 0 = 3 \times 0$, so $P(1)$ holds. *Inductive step*: assume $f(k) = k^3 - k = 3m$ for some integer m. Then

$$f(k+1) = (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3k^2 + 3k,$$

so $f(k+1) = 3m + 3(k^2 + k) = 3(m + k^2 + k)$, a multiple of 3, which is $P(k+1)$. Since $P(1)$ holds and $P(k) \Rightarrow P(k+1)$, by induction $3 / n^3 - n$ for all integers $n \geq 1$. This completes the proof.

Q6. Let $P(n)$ be the statement $n! \geq 2^{n-1}$. *Base case* $n=1$: $1! = 1$ and $2^0 = 1$, so $1! \geq 2^0$ and $P(1)$ holds. *Inductive step*: assume $k! \geq 2^{k-1}$ for some integer $k \geq 1$. Then

$$(k+1)! = (k+1) \cdot k! \geq (k+1) \cdot 2^{k-1}.$$

Since $k \geq 1$, we have $k+1 \geq 2$, so

$$(k+1) \cdot 2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k = 2^{(k+1)-1}.$$

Hence $(k+1)! \geq 2^{(k+1)-1}$, which is $P(k+1)$. Since $P(1)$ holds and $P(k) \Rightarrow P(k+1)$, by induction $n! \geq 2^{n-1}$ for all integers $n \geq 1$. This completes the proof.

Q7. Let $P(n)$ be the statement $S_n = \frac{n(n+1)(n+2)}{3}$, where $S_n = \sum_{r=1}^{n} r(r+1)$. *Base case* $n=1$: $S_1 = 1 \cdot 2 = 2$ and $\frac{1 \cdot 2 \cdot 3}{3} = 2$, so $P(1)$ holds. *Inductive step*: assume $S_k = \frac{k(k+1)(k+2)}{3}$. The $(k+1)$-th term is $(k+1)(k+2)$, so

$$S_{k+1} = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = (k+1)(k+2)\left[\frac{k}{3} + 1\right] = (k+1)(k+2) \cdot \frac{k+3}{3}.$$

Thus $S_{k+1} = \frac{(k+1)(k+2)(k+3)}{3}$, which is $P(k+1)$. Since $P(1)$ holds and $P(k) \Rightarrow P(k+1)$, by induction the formula holds for all integers $n \geq 1$. This completes the proof.

Q8. Let $P(n)$ be the statement $S_n = \frac{n}{n+1}$, where $S_n = \sum_{r=1}^{n} \frac{1}{r(r+1)}$. *Base case* $n=1$: $S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$ and $\frac{1}{1+1} = \frac{1}{2}$, so $P(1)$ holds. *Inductive step*: assume $S_k = \frac{k}{k+1}$. The $(k+1)$-th term is $\frac{1}{(k+1)(k+2)}$, so

$$S_{k+1}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}.$$

This is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction the formula holds for all integers $n\geq 1$. This completes the proof.

Q9. Let $P(n)$ be the statement $S_n=\frac{n(3n-1)}{2}$, where $S_n=\sum_{r=1}^{n}(3r-2)$. *Base case* $n=1$: $S_1=3(1)-2=1$ and $\frac{1(3-1)}{2}=1$, so $P(1)$ holds. *Inductive step*: assume $S_k=\frac{k(3k-1)}{2}$. The $(k+1)$-th term is $3(k+1)-2=3k+1$, so

$$S_{k+1}=\frac{k(3k-1)}{2}+(3k+1)=\frac{k(3k-1)+2(3k+1)}{2}=\frac{3k^2+5k+2}{2}=\frac{(k+1)(3k+2)}{2}.$$

Since $3k+2=3(k+1)-1$, this is $\frac{(k+1)(3(k+1)-1)}{2}=P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction the formula holds for all integers $n\geq 1$. This completes the proof.

Q10. Let $P(n)$ be the statement $S_n=(n-1)2^{n+1}+2$, where $S_n=\sum_{r=1}^{n} r\,2^r$. *Base case* $n=1$: $S_1=1\cdot 2^1=2$ and $(1-1)2^2+2=0+2=2$, so $P(1)$ holds. *Inductive step*: assume $S_k=(k-1)2^{k+1}+2$. The $(k+1)$-th term is $(k+1)2^{k+1}$, so

$$S_{k+1}=(k-1)2^{k+1}+2+(k+1)2^{k+1}=[(k-1)+(k+1)]2^{k+1}+2=2k\cdot 2^{k+1}+2=k\,2^{k+2}+2.$$

Since $k\,2^{k+2}+2=((k+1)-1)2^{(k+1)+1}+2$, this is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction the formula holds for all integers $n\geq 1$. This completes the proof.

Q11. Let $P(n)$ be the statement $S_n=\frac{5^n-1}{4}$, where $S_n=\sum_{r=1}^{n}5^{r-1}$. *Base case* $n=1$: $S_1=5^0=1$ and $\frac{5^1-1}{4}=\frac{4}{4}=1$, so $P(1)$ holds. *Inductive step*: assume $S_k=\frac{5^k-1}{4}$. The $(k+1)$-th term is $5^{(k+1)-1}=5^k$, so

$$S_{k+1}=\frac{5^k-1}{4}+5^k=\frac{5^k-1+4\cdot 5^k}{4}=\frac{5\cdot 5^k-1}{4}=\frac{5^{k+1}-1}{4},$$

which is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction the formula holds for all integers $n\geq 1$. This completes the proof.

Q12. Let $P(n)$ be the statement $9/f(n)$, where $f(n)=4^n+6n-1$. *Base case* $n=1$: $f(1)=4+6-1=9=9\times 1$, so $P(1)$ holds. *Inductive step*: assume $f(k)=4^k+6k-1=9m$ for some integer m; rearranged, $4^k=9m-6k+1$. Then

$$f(k+1)=4^{k+1}+6(k+1)-1=4\cdot 4^k+6k+5=4(9m-6k+1)+6k+5.$$

Expanding, $f(k+1)=36m-24k+4+6k+5=36m-18k+9=9(4m-2k+1)$, a multiple of 9, which is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction $9/4^n+6n-1$ for all integers $n\geq 1$. This completes the proof.

Q13. Let $P(n)$ be the statement $7/f(n)$, where $f(n)=8^n-1$. *Base case* $n=1$: $f(1)=8-1=7=7\times 1$, so $P(1)$ holds. *Inductive step*: assume $f(k)=8^k-1=7m$ for some integer m; so $8^k=7m+1$. Then

$$f(k+1)=8^{k+1}-1=8\cdot 8^k-1=8(7m+1)-1=56m+7=7(8m+1),$$

a multiple of 7, which is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction $7/8^n-1$ for all integers $n\geq1$. This completes the proof.

Q14. Let $P(n)$ be the statement $6/f(n)$, where $f(n)=7^n-1$. *Base case* $n=1$: $f(1)=7-1=6=6\times1$, so $P(1)$ holds. *Inductive step*: assume $f(k)=7^k-1=6m$ for some integer m; so $7^k=6m+1$. Then

$$f(k+1)=7^{k+1}-1=7\cdot7^k-1=7(6m+1)-1=42m+6=6(7m+1),$$

a multiple of 6, which is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction $6/7^n-1$ for all integers $n\geq1$. This completes the proof.

Q15. Let $P(n)$ be the statement $5/f(n)$, where $f(n)=n^5-n$. *Base case* $n=1$: $f(1)=1-1=0=5\times0$, so $P(1)$ holds. *Inductive step*: assume $f(k)=k^5-k=5m$ for some integer m. Using $(k+1)^5=k^5+5k^4+10k^3+10k^2+5k+1$,

$$f(k+1)=(k+1)^5-(k+1)=k^5+5k^4+10k^3+10k^2+5k+1-k-1=(k^5-k)+5k^4+10k^3+10k^2+5k.$$

So $f(k+1)=5m+5(k^4+2k^3+2k^2+k)=5(m+k^4+2k^3+2k^2+k)$, a multiple of 5, which is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction $5/n^5-n$ for all integers $n\geq1$. This completes the proof.

Q16. Let $P(n)$ be the statement $3^n>n^3$. *Base case* $n=4$: $3^4=81$ and $4^3=64$, and $81>64$, so $P(4)$ holds. *Inductive step*: assume $3^k>k^3$ for some integer $k\geq4$. Multiplying by 3,

$$3^{k+1}=3\cdot3^k>3k^3.$$

It remains to show $3k^3\geq(k+1)^3$, equivalently $3\geq\left(1+\frac{1}{k}\right)^3$. For $k\geq4$, $1+\frac{1}{k}\leq1+\frac{1}{4}=\frac{5}{4}$, so $\left(1+\frac{1}{k}\right)^3\leq\left(\frac{5}{4}\right)^3=\frac{125}{64}<3$. Hence $3k^3\geq(k+1)^3$, and combining,

$$3^{k+1}>3k^3\geq(k+1)^3,$$

so $3^{k+1}>(k+1)^3$, which is $P(k+1)$. Since $P(4)$ holds and $P(k)\Rightarrow P(k+1)$, by induction $3^n>n^3$ for all integers $n\geq4$. This completes the proof.

Q17. Let $P(n)$ be the statement $(1+x)^n\geq1+nx$, where $x\geq-1$ is fixed. Note that $x\geq-1$ gives $1+x\geq0$. *Base case* $n=1$: $(1+x)^1=1+x=1+1\cdot x$, so the two sides are equal and the inequality holds; $P(1)$ is true. *Inductive step*: assume $(1+x)^k\geq1+kx$ for some integer $k\geq1$. Multiplying both sides by $1+x$, which is non-negative so the inequality direction is preserved,

$$(1+x)^{k+1}=(1+x)^k(1+x)\geq(1+kx)(1+x)=1+(k+1)x+kx^2.$$

Since $kx^2\geq0$, we have $1+(k+1)x+kx^2\geq1+(k+1)x$, so

$$(1+x)^{k+1}\geq1+(k+1)x,$$

which is $P(k+1)$. Since $P(1)$ holds and $P(k)\Rightarrow P(k+1)$, by induction $(1+x)^n\geq1+nx$ for all integers $n\geq1$. This completes the proof.

Q18. Let $P(n)$ be the statement $n!>2^n$. *Base case* $n=4$: $4!=24$ and $2^4=16$, and $24>16$, so $P(4)$ holds. *Inductive step*: assume $k!>2^k$ for some integer $k\geq4$. Then

$$(k+1)!=(k+1)\cdot k!>(k+1)\cdot2^k.$$

Since $k \geq 4$, $k+1 \geq 5 > 2$, so $(k+1) \cdot 2^k > 2 \cdot 2^k = 2^{k+1}$. Hence $(k+1)! > 2^{k+1}$, which is $P(k+1)$. Since $P(4)$ holds and $P(k) \Rightarrow P(k+1)$, by induction $n! > 2^n$ for all integers $n \geq 4$. This completes the proof.

Theory

Computational thinking.

Many Specialist problems are solved not by a single formula but by a **procedure** — a finite list of unambiguous steps that transforms given data into a required result. Designing such a procedure draws on four habits of **computational thinking**:

- **decomposition** — breaking a problem into smaller, self-contained parts (inputs, a process, an output);
- **abstraction** — stripping away detail to keep only what the procedure needs, so the same method serves many particular cases;
- **pattern** — noticing that a step repeats, or that a result matches a known form; and
- **algorithm** — expressing the solution as an ordered sequence of steps that always terminates with the correct answer.

An **algorithm** is a finite, precise, terminating procedure. The same algorithm may be written in English, drawn as a flowchart, or coded for a calculator; to reason about it on paper we use **pseudocode** — structured English that names the steps exactly without tying them to any one programming language.

Pseudocode conventions.

The study design names the elements of algorithm design — sequencing, decision-making and repetition — but prescribes no notation for writing them down, so the conventions used below are those in common use in VCE Mathematics; what matters is that every step be precise and unambiguous. An **assignment** stores a value in a variable and is written with a left arrow: `count ← 0` means "let the variable `count` take the value 0". Reading and reporting values use `input`, `output` and `print`. Blocks are shown by **indentation**. The three control structures are:

```
sequence             one step after another, top to bottom

if condition then       selection (decision)
    ...
else
    ...
end if

for i from a to b do    definite iteration (a fixed number of passes)
    ...
end for

while condition do      indefinite iteration (repeat until a condition fails)
    ...
end while
```

For example, the algorithm below reads a number `n` and outputs the sum $1+2+\cdots+n$:

```
input n
S ← 0
for i from 1 to n do
    S ← S + i
end for
output S
```

The comparison symbols ≤, ≥ and ≠, the multiplication sign ×, and the word `mod` (for the remainder on division — `17 mod 5` is 2) all appear inside pseudocode.

The three building blocks.

Every algorithm is assembled from just three ideas — the elements of **algorithm design**:

- **sequencing**: steps carried out in order, each using the state left by the previous one;
- **selection** (decision): a choice between alternatives, made with `if … then … else`; and
- **iteration** (repetition): repeating a block, either a fixed number of times (`for`) or until a condition changes (`while`).

Choosing between `for` and `while` is a modelling decision. A `for` loop suits a **definite** number of passes known in advance — n steps of Euler's method, the first n terms of a sequence. A `while` loop suits an **indefinite** repetition that continues until some target is met — halving an interval until it is narrow enough, or accumulating terms until a total is exceeded.

Tracing (desk-checking).

To check an algorithm by hand we **trace** it: we step through the instructions keeping a **trace table** whose columns are the variables (and any output) and whose rows record their values after each pass of a loop. Desk-checking exposes off-by-one errors, wrong loop bounds and misordered updates before any code is run, and it is the standard way to answer "what does this algorithm output?".

Algorithms in Specialist Mathematics.

Algorithmic thinking runs through the course. Numerical methods — **Euler's method** for a differential equation, the **bisection** and **Newton** methods for a root, the **trapezium rule** for an integral — are all iterative algorithms. So is generating the terms and partial sums of a **sequence or series**, computing a power of a complex number by **repeated multiplication** (De Moivre in disguise), and finding a greatest common divisor by **Euclid's algorithm**. In each case the skill is the same: decompose the task, express it as pseudocode, and trace it to confirm it does what the mathematics requires.

Worked examples

Example 1 — scaffolded

A differential equation $\frac{dy}{dx} = f(x, y)$ has known initial value $y(x_0) = y_0$. Write an algorithm, in pseudocode, that uses **Euler's method** with step size h to compute an approximation to y after n steps, printing the point reached at each step.

Working	Comment
Inputs: the starting point x_0, the initial value y_0, the step size h, and the number of steps n; the rule $f(x, y)$ is assumed known.	Decomposition and abstraction: identify exactly the data the procedure needs.
Set $x \leftarrow x_0$ and $y \leftarrow y_0$.	Sequencing: establish the starting state before any repetition.
Repeat the update exactly n times, so use a `for` loop `for i from 1 to n`.	Iteration: a definite number of passes is known in advance, so a `for` loop is appropriate.
Inside the loop apply Euler's formula $y \leftarrow y + h \times f(x, y)$, then advance $x \leftarrow x + h$.	Order matters: update y using the *current* x, and only then move x forward.
`print` the point each pass, and `output` the final y.	Report the approximation after every step and at the end.

The completed algorithm:

```
input x0, y0, h, n
x ← x0
y ← y0
for i from 1 to n do
```

```
    y ← y + h × f(x, y)
    x ← x + h
    print x, y
end for
output y
```

For $\frac{dy}{dx}=x+y$ with $y(0)=1$, taking $h=0.5$ and $n=2$, the algorithm prints $(0.5, 1.5)$ then $(1, 2.5)$, giving the estimate $y(1)\approx 2.5$.

Example 2 — scaffolded

The equation $x^2-2=0$ has a root between 1 and 2. Write an algorithm using the **bisection method** that repeatedly halves an interval until its width is at most a tolerance t, then outputs the midpoint as an estimate of the root.

Working	Comment
Inputs: endpoints a, b of an interval on which f changes sign, and a tolerance t.	The sign change guarantees a root lies between a and b.
Repeat *while* the interval is wider than the tolerance: `while (b − a) > t`.	Iteration whose number of passes is not known in advance — a `while` loop, not a `for`.
Each pass, bisect the interval: $m \leftarrow (a+b)/2$.	The midpoint splits the interval into two halves.
Keep the half on which the sign still changes: if $f(a)\times f(m)<0$ then $b \leftarrow m$, else $a \leftarrow m$.	Selection: an `if … then … else` chooses the sub-interval containing the root.
After the loop, `output` the midpoint $(a+b)/2$.	The surviving interval is narrow, so its midpoint approximates the root.

The completed algorithm:

```
input a, b, t
while (b − a) > t do
    m ← (a + b) / 2
    if f(a) × f(m) < 0 then
        b ← m
    else
        a ← m
    end if
end while
output (a + b) / 2
```

Tracing it for $f(x)=x^2-2$ with $a=1$, $b=2$ and $t=0.25$:

Pass	a	b	$b-a$	m	$f(a)\times f(m)$	Action
1	1	2	1	1.5	$(-1)(0.25)=\cdot$	$b \leftarrow 1.5$
2	1	1.5	0.5	1.25	$(-1)(-0.437$	$a \leftarrow 1.25$

Now $b-a=0.25$ is not greater than 0.25, so the loop stops and the algorithm outputs $(1.25+1.5)/2=1.375$ as the estimate of $\sqrt{2}$.

Example 3 — unscaffolded

Desk-check the following implementation of **Euclid's algorithm** for the greatest common divisor, and state its output for the inputs $a=48$, $b=18$.

```
input a, b            (positive integers)
while b ≠ 0 do
    r ← a mod b
```

```
    a ← b
    b ← r
end while
output a
```

Each pass replaces the pair (a, b) by $(b, a \bmod b)$, which does not change the greatest common divisor, and the second entry strictly decreases, so the process must reach 0 and stop. Tracing with $a=48$, $b=18$ (the "mod" column is the remainder $a \bmod b$ computed from the values at the start of the pass):

Pass	a	b	remainder a mod b	new a	new b
1	48	18	12	18	12
2	18	12	6	12	6
3	12	6	0	6	0

After the third pass $b=0$, so the `while` condition fails and the loop ends. The algorithm outputs $a=6$, and indeed $gcd(48, 18)=6$.

Example 4 — unscaffolded

Trace the algorithm below for $n=4$, state its output, and relate the result to a closed form.

```
input n
S ← 0
for r from 1 to n do
    S ← S + r × r
    print r, S
end for
output S
```

The variable S accumulates the sum of the squares $1^2+2^2+\cdots+n^2$; each pass adds the next square and prints the running total.

r	$r \times r$	S after the pass
1	1	1
2	4	5
3	9	14
4	16	30

The output is $S=30$. This matches the closed form for the sum of the first n squares,

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}=\frac{4\cdot5\cdot9}{6}=30.$$

The loop *computes* the sum term by term (the algorithm); *recognising* the closed form is the computational-thinking step of spotting a pattern, and that pattern is what a proof by induction (an earlier section) establishes.

Practice questions

Scaffolded

Q1. Consider the algorithm

```
input n
S ← 0
for i from 1 to n do
```

```
    S ← S + i
end for
output S
```

(a) State the value of S immediately before the loop begins.

(b) Complete a trace table with columns i and S for each pass, taking $n=5$.

(c) State the output.

Q2. Euler's method is applied to $\frac{dy}{dx}=x-y$ with $y(0)=2$, step $h=0.5$, as coded below.

```
x ← 0
y ← 2
for i from 1 to 2 do
    y ← y + 0.5 × (x − y)
    x ← x + 0.5
    print x, y
end for
```

(a) Work through the first pass and state the printed point.

(b) Work through the second pass and state the printed point.

(c) State the algorithm's estimate of $y(1)$.

Q3. Write an algorithm that inputs a positive integer n and outputs $n!=1\times 2\times\cdots\times n$. (a) Choose a variable to hold the running product and state its starting value. (b) Write the `for` loop that multiplies in each factor. (c) State the output when $n=5$.

Q4. Write an algorithm that inputs an integer n and outputs whether it is even or odd. (a) Which quantity, computed with `mod`, decides the answer? (b) Write the `if … then … else` that selects the output. (c) State the output when $n=7$.

Q5. Apply Euclid's algorithm (Example 3) to $a=105$, $b=45$. (a) Give the remainder and the new pair (a,b) after the first pass. (b) Continue the trace until $b=0$. (c) State $gcd(105,45)$.

Q6. Apply the bisection algorithm (Example 2) to $f(x)=x^2-3$ on $[1,2]$ with tolerance $t=0.25$. (a) Give the first midpoint and the decision made. (b) Give the second midpoint and the decision made. (c) State the estimate of $\sqrt{3}$ that is output.

Unscaffolded

Q7. Write an algorithm that inputs the first term a, common ratio r and a positive integer n, and outputs the sum of the first n terms of the geometric sequence $a, ar, ar^2, \ldots$. State the output for $a=1$, $r=1/2$, $n=4$.

Q8. Trace the algorithm below for $a=3$, $n=4$, and state its output.

```
input a, n
p ← 1
for i from 1 to n do
    p ← p × a
end for
output p
```

Q9. Write an algorithm that uses the **trapezium rule** with n strips to approximate $\int_a^b f(x)\,dx$, using $\frac{h}{2}[f(a)+f(b)+2(\text{interior ordinates})]$ with $h=(b-a)/n$. State the estimate for $f(x)=x^2$, $a=0$, $b=1$, $n=4$.

Q10. Write an algorithm for **Newton's method**: given a function f, its derivative f', a starting value x and a number of iterations n, repeatedly apply $x \leftarrow x - \dfrac{f(x)}{f'(x)}$ and output the result. State the value obtained for $f(x) = x^2 - 2$, $f'(x) = 2x$, starting at $x = 1$, after 2 iterations.

Q11. Trace the algorithm below for $n = 6$ and list everything it prints.

```
input n
a ← 0
b ← 1
for i from 1 to n do
    print a
    t ← a + b
    a ← b
    b ← t
end for
```

Q12. Write an algorithm that inputs an integer $n \geq 2$ and outputs whether n is prime, by trial division. Use a Boolean flag together with a loop and a decision. State the output for $n = 13$.

Q13. Write an algorithm that inputs integers n and d (with $d \neq 0$) and outputs whether d divides n. State the output for $n = 18$, $d = 3$.

Q14. Trace the algorithm below for $n = 3$, giving S as an exact fraction, and identify the closed form it is computing.

```
input n
S ← 0
for r from 1 to n do
    S ← S + 1 / (r × (r + 1))
end for
output S
```

Q15. A complex number $z = c + di$ is stored as the pair (c, d), and the product of (c, d) and (e, f) is the pair $(ce - df, cf + de)$. Trace the repeated-multiplication algorithm below, which computes z^n, for $z = 1 + i$ (the pair $(1, 1)$) and $n = 4$, and state z^4.

```
input (c, d), n
p ← (1, 0)
for i from 1 to n do
    p ← p × (c, d)          (complex multiplication)
end for
output p
```

Q16. Trace **Horner's method** below, which evaluates the polynomial with coefficient list $[2, -3, 4, -5]$ (that is, $2x^3 - 3x^2 + 4x - 5$) at $x = 2$, and state the output.

```
input coefficients, x
result ← 0
for each coefficient c in the list, in order do
    result ← result × x + c
end for
output result
```

Q17. For each task below, name the element of computational thinking it best illustrates — **decomposition**, **abstraction**, **pattern** or **algorithm**. (a) Rewriting "estimate $y(1)$ for this particular differential equation" as the general rule $y \leftarrow y + hf(x, y)$ repeated n times. (b) Splitting "solve the problem" into "read the inputs", "run the loop" and "report the output". (c) Noticing that $1 + 3 + 5 + \cdots + (2n - 1)$ always equals n^2. (d) Setting out the exact ordered steps, with their loop bounds, that a calculator could follow without further judgement.

Q18. Write an algorithm that inputs a number `target` and outputs the smallest positive integer n for which the partial sum $1+1/2+1/3+\cdots+1/n$ first exceeds `target`. Explain why a `while` loop, not a `for` loop, is required, and state the value of n output when `target` $=2$.

Answers

Q1. (a) $S=0$. (b) $(i,S)=(1,1),(2,3),(3,6),(4,10),(5,15)$. (c) Output 15. See solutions. **Q2.** (a) $(0.5,1)$. (b) $(1,0.75)$. (c) $y(1)\approx 0.75$. See solutions. **Q3.** Product initialised to 1; loop multiplies in $i=1,\dots,n$; output $5!=120$. See solutions. **Q4.** Decide on n mod 2: output "even" if 0, else "odd"; for $n=7$, output "odd". See solutions. **Q5.** (a) 105 mod $45=15$, new pair $(45,15)$. (b) 45 mod $15=0$, new pair $(15,0)$. (c) $gcd=15$. See solutions. **Q6.** (a) $m=1.5$, and since $f(1)f(1.5)>0$, set $a \leftarrow 1.5$. (b) $m=1.75$, and since $f(1.5)f(1.75)<0$, set $b \leftarrow 1.75$. (c) Estimate 1.625. See solutions. **Q7.** Loop accumulates $a+ar+\cdots+ar^{n-1}$; for the given values, $S=15/8=1.875$. See solutions. **Q8.** p takes values $1,3,9,27,81$; output $3^4=81$. See solutions. **Q9.** $T=11/32=0.34375$. See solutions. **Q10.** $x_1=3/2$, $x_2=17/12\approx 1.4167$. See solutions. **Q11.** Prints $0,1,1,2,3,5$ (the first six Fibonacci numbers). See solutions. **Q12.** Flag `prime` starts true, set false if any d in $2,\dots,n-1$ divides n; for $n=13$, output "prime". See solutions. **Q13.** Output "yes" exactly when n mod $d=0$; for $n=18$, $d=3$, output "yes". See solutions. **Q14.** $S=1/2+1/6+1/12=3/4$; closed form $n/n+1$. See solutions. **Q15.** p passes through $(1,1),(0,2),(-2,2),(-4,0)$; so $z^4=-4$. See solutions. **Q16.** `result` takes values $2,1,6,7$; output 7. See solutions. **Q17.** (a) abstraction; (b) decomposition; (c) pattern; (d) algorithm. See solutions. **Q18.** A `while` loop is needed because n is not known in advance; for `target` $=2$, output $n=4$. See solutions.

Fully-worked solutions

Q1. Before the loop $S \leftarrow 0$, so $S=0$. Each pass adds the current i:

i	1	2	3	4	5
S	1	3	6	10	15

The output is $S=15$, which is $1+2+3+4+5$, as required.

Q2. Start with $x=0$, $y=2$. Pass 1: $y \leftarrow 2+0.5(0-2)=2-1=1$, then $x \leftarrow 0.5$; it prints $(0.5,1)$. Pass 2: $y \leftarrow 1+0.5(0.5-1)=1-0.25=0.75$, then $x \leftarrow 1$; it prints $(1,0.75)$. After two steps the estimate is $y(1)\approx 0.75$.

Q3. Hold the running product in `p`, starting at 1 (the identity for multiplication), and multiply in each factor:

```
input n
p ← 1
for i from 1 to n do
    p ← p × i
end for
output p
```

For $n=5$, `p` passes through $1,1,2,6,24,120$, so the output is $5!=120$.

Q4. The remainder n mod 2 is 0 for an even number and 1 for an odd number, so a single decision suffices:

```
input n
if n mod 2 = 0 then
    output "even"
else
    output "odd"
end if
```

For $n=7$, 7 mod $2=1\neq 0$, so the output is "odd".

Q5. Applying Euclid's algorithm:

Pass	a	b	remainder a mod b	new a	new b
1	105	45	15	45	15

Pass	a	b	remainder a mod b	new a	new b
2	45	15	0	15	0

After the second pass $b=0$, so the loop ends and the output is $a=15$. Hence $gcd(105,45)=15$.

Q6. Here $f(x)=x^2-3$, so $f(1)=-2$, $f(1.5)=-0.75$, $f(1.75)=0.0625$ and $f(2)=1$. Pass 1: $b-a=1>0.25$, so $m=(1+2)/2=1.5$; since $f(1)\times f(1.5)=(-2)(-0.75)=1.5>0$, the sign change is not in $[1,1.5]$, so set $a \leftarrow 1.5$. Pass 2: $b-a=0.5>0.25$, so $m=(1.5+2)/2=1.75$; since $f(1.5)\times f(1.75)=(-0.75)(0.0625)=-0.046875<0$, set $b \leftarrow 1.75$. Now $b-a=0.25$ is not greater than 0.25, so the loop stops and the output is $(1.5+1.75)/2=1.625$, an estimate of $\sqrt{3}\approx 1.732$.

Q7. Accumulate the terms, multiplying the current term by r each pass:

```
input a, r, n
S ← 0
term ← a
for i from 1 to n do
    S ← S + term
    term ← term × r
end for
output S
```

For $a=1$, $r=1/2$, $n=4$ the terms are $1,1/2,1/4,1/8$, so $S=1+1/2+1/4+1/8=15/8=1.875$ (agreeing with $\frac{a(1-r^n)}{1-r}=\frac{1-(1/2)^4}{1-1/2}=\frac{15}{8}$).

Q8. The loop multiplies p by a each pass, so p records the running power of a:

i	1	2	3	4
p	3	9	27	81

Starting from $p=1$, the output is $3^4=81$.

Q9. Add the two end ordinates, then twice each interior ordinate:

```
input a, b, n
h ← (b − a) / n
T ← f(a) + f(b)
for i from 1 to n − 1 do
    T ← T + 2 × f(a + i × h)
end for
T ← (h / 2) × T
output T
```

For $f(x)=x^2$, $a=0$, $b=1$, $n=4$: $h=0.25$, and the ordinates are $f(0)=0$, $f(0.25)=0.0625$, $f(0.5)=0.25$, $f(0.75)=0.5625$, $f(1)=1$. Then

$$T=\frac{0.25}{2}\left[0+1+2(0.0625+0.25+0.5625)\right]=0.125\times 2.75=\frac{11}{32}=0.34375,$$

a slight overestimate of the exact value $\int_0^1 x^2\,dx=1/3$.

Q10. Iterate the Newton update a fixed number of times:

```
input x, n
for i from 1 to n do
    x ← x − f(x) / f '(x)
end for
output x
```

For $f(x)=x^2-2$, $f'(x)=2x$, starting at $x=1$:

$$x_1=1-\frac{1^2-2}{2\cdot 1}=1-\frac{-1}{2}=\frac{3}{2}, x_2=\frac{3}{2}-\frac{(3/2)^2-2}{2\cdot(3/2)}=\frac{3}{2}-\frac{1/4}{3}=\frac{3}{2}-\frac{1}{12}=\frac{17}{12}.$$

So after two iterations the value is $\frac{17}{12}\approx 1.4167$, already close to $\sqrt{2}\approx 1.4142$.

Q11. Each pass prints the current a, then advances the pair (a,b) to $(b,a+b)$:

i	prints	new a	new b
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	5
5	3	5	8
6	5	8	13

It prints $0,1,1,2,3,5$ — the first six Fibonacci numbers.

Q12. Assume n is prime until a divisor is found; test each candidate d from 2 to $n-1$:

```
input n
prime ← true
for d from 2 to n − 1 do
    if n mod d = 0 then
        prime ← false
    end if
end for
if prime = true then
    output "prime"
else
    output "not prime"
end if
```

For $n=13$ none of $d=2,\ldots,12$ divides 13, so `prime` stays true and the output is “prime”. (Only divisors up to $\sqrt{n}$ need be tested, but testing to $n-1$ is correct.)

Q13. Divisibility is decided by a single remainder:

```
input n, d
if n mod d = 0 then
    output "yes"
else
    output "no"
end if
```

For $n=18$, $d=3$: 18 mod $3=0$, so the output is “yes” — 3 divides 18.

Q14. The loop adds $\frac{1}{r(r+1)}$ for $r=1,2,3$:

r	term $\frac{1}{r(r+1)}$	S after the pass
1	$1/2$	$1/2$
2	$1/6$	$2/3$
3	$1/12$	$3/4$

So $S=\frac{6}{12}+\frac{2}{12}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$. This is the telescoping sum with closed form $\frac{n}{n+1}$, which gives $\frac{3}{4}$ at $n=3$.

Q15. Using $(c,d)(e,f)=(ce-df,cf+de)$ with $z=(1,1)$, starting from $p=(1,0)$:

i	p before	$p\times(1,1)$	p after
1	$(1,0)$	$(1\cdot 1-0\cdot 1,1\cdot 1+0\cdot 1)$	$(1,1)$
2	$(1,1)$	$(1-1,1+1)$	$(0,2)$
3	$(0,2)$	$(0-2,0+2)$	$(-2,2)$
4	$(-2,2)$	$(-2-2,-2+2)$	$(-4,0)$

The output is $(-4,0)$, so $z^4=-4$. This agrees with De Moivre's theorem, since $1+i=\sqrt{2}\,cis\frac{\pi}{4}$ gives $(1+i)^4=4\,cis\,\pi=-4$.

Q16. Horner's method rebuilds the polynomial as $((2x-3)x+4)x-5$; each pass does $\text{result}\leftarrow\text{result}\times x+c$:

coefficient c	2	-3	4	-5
`result` after	2	1	6	7

At $x=2$ the output is 7, which checks against $2(8)-3(4)+4(2)-5=16-12+8-5=7$.

Q17. (a) **Abstraction** — the particular problem is replaced by the general update rule that fits every such differential equation. (b) **Decomposition** — one large task is broken into independent sub-tasks (input, process, output). (c) **Pattern** — a repeated numerical regularity is recognised, here $\sum(2r-1)=n^2$. (d) **Algorithm** — the exact ordered steps, with loop bounds, are set down so they can be followed mechanically.

Q18. Because the number of terms needed to pass `target` is not known before running, a `while` loop (which repeats until the condition fails) is required rather than a `for` loop (which needs its count in advance):

```
input target
S ← 0
n ← 0
while S ≤ target do
    n ← n + 1
    S ← S + 1 / n
end while
output n
```

Tracing with `target` $=2$: the running sum is $S=1$ ($n=1$), $3/2$ ($n=2$), $11/6\approx 1.833$ ($n=3$), then $25/12\approx 2.083$ ($n=4$). Only at $n=4$ does S first exceed 2, so the output is $n=4$.

Topic 1 Test A — Technology-active

One bound reference and one approved technology (CAS calculator or software) are permitted. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (8 marks)

Consider the algorithm below, which reads a positive integer `n`.

```
input n
S ← 0
for k from 1 to n do
    if k mod 2 = 0 then
        S ← S + k
    else
        S ← S - k
    end if
end for
output S
```

a. Complete a trace table with columns k and S showing the value of S after each pass of the loop, for $n=6$. 3 marks

b. State the output when $n=6$, and write down a closed form, in terms of n, for the output when n is even. 2 marks

c. Write an algorithm, in pseudocode, that inputs a positive integer n and outputs the number of positive integers that divide n exactly (the number of divisors of n). State the output your algorithm gives when $n=12$. 3 marks

Question 2 (7 marks)

Use mathematical induction to prove that 7^n-2^n is divisible by 5 for all integers $n\geq 1$; that is, $5\,/\,7^n-2^n$. Let $P(n)$ be this statement and write $f(n)=7^n-2^n$.

a. Verify the base case $P(1)$. 1 mark

b. Write down the inductive hypothesis $P(k)$. 1 mark

c. Assuming $P(k)$, prove $P(k+1)$. You may find it useful to write $7^{k+1}=7\cdot 7^k$ and $2^{k+1}=2\cdot 2^k$. 4 marks

d. State the conclusion of the proof. 1 mark

Question 3 (7 marks)

For each positive integer n, let $S_n=\sum_{k=1}^{n}\frac{4}{(2k-1)(2k+3)}$.

a. Show that $\frac{4}{(2k-1)(2k+3)}=\frac{1}{2k-1}-\frac{1}{2k+3}$. 2 marks

b. Hence prove that $S_n=\frac{4}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}$. 3 marks

c. Find the limit of S_n as $n\to\infty$, and hence state the value of the infinite series $\sum_{k=1}^{\infty}\frac{4}{(2k-1)(2k+3)}$. 2 marks

Question 4 (6 marks)

a. A student claims that "for every positive integer n, the value of n^2+n+17 is a prime number." Using technology to assist your search, find a counterexample to this claim, and hence explain why the claim is false. 3 marks

b. Consider the statement T: "$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^2 = x$." Write the negation $\neg T$ in quantifier notation, and determine, with justification, which of T and $\neg T$ is true. 3 marks

Question 5 (6 marks)

a. Show that for every real number $x > 0$, $x^2 + \frac{2}{x} - 3 = \frac{(x-1)^2(x+2)}{x}$. 2 marks

b. Hence prove that $x^2 + \frac{2}{x} \geq 3$ for all $x > 0$, and state the value of x for which equality holds. 3 marks

c. State the minimum value of $x^2 + \frac{2}{x}$ for $x > 0$. 1 mark

Question 6 (6 marks)

Let n be an integer. In this question you will prove that the product $n(n+1)(n+2)$ of three consecutive integers is divisible by 6.

a. Explain why $n(n+1)(n+2)$ is divisible by 2. 2 marks

b. By considering the three cases $n = 3q$, $n = 3q+1$ and $n = 3q+2$ (where q is an integer), prove that $n(n+1)(n+2)$ is divisible by 3. 3 marks

c. Hence explain why $n(n+1)(n+2)$ is divisible by 6. 1 mark

End of Topic 1 Test A

Test A — answers and solutions

Question 1.

a. Each pass adds k when k is even and subtracts k when k is odd. Starting from $S=0$:

k	1	2	3	4	5	6
k even?	no	yes	no	yes	no	yes
S after pass	-1	1	-2	2	-3	3

b. The output is $S=3$. For even n the terms pair as $(-1+2)+(-3+4)+\cdots$, each pair contributing 1, and there are $\frac{n}{2}$ pairs, so the output is $\frac{n}{2}$.

c. Count each d from 1 to n that divides n:

```
input n
count ← 0
for d from 1 to n do
    if n mod d = 0 then
        count ← count + 1
    end if
end for
output count
```

For $n=12$ the divisors are $1,2,3,4,6,12$, so the output is 6.

Question 2.

a. $f(1)=7^1-2^1=7-2=5=5\times 1$, a multiple of 5, so $P(1)$ is true.

b. Assume $f(k)=7^k-2^k=5m$ for some integer m, where $k\geq 1$.

c. Using $7^{k+1}=7\cdot 7^k$ and $2^{k+1}=2\cdot 2^k$,

$$f(k+1)=7^{k+1}-2^{k+1}=7\cdot 7^k-2\cdot 2^k.$$

Split the middle term by adding and subtracting $7\cdot 2^k$:

$$7\cdot 7^k-2\cdot 2^k=7\left(7^k-2^k\right)+(7-2)2^k=7\left(7^k-2^k\right)+5\cdot 2^k.$$

By the hypothesis $7^k-2^k=5m$, so

$$f(k+1)=7(5m)+5\cdot 2^k=5\left(7m+2^k\right).$$

Since $7m+2^k$ is an integer, $f(k+1)$ is a multiple of 5, i.e. $P(k)\Rightarrow P(k+1)$.

d. Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for every $k\geq 1$, by the principle of mathematical induction $5 / 7^n-2^n$ for all integers $n\geq 1$. This completes the proof.

Question 3.

a. Over the common denominator,

$$\frac{1}{2k-1}-\frac{1}{2k+3}=\frac{(2k+3)-(2k-1)}{(2k-1)(2k+3)}=\frac{4}{(2k-1)(2k+3)},$$

as required.

b. With $f(k)=\dfrac{1}{2k-1}$, note that $f(k+2)=\dfrac{1}{2(k+2)-1}=\dfrac{1}{2k+3}$, so the general term is $f(k)-f(k+2)$ — a telescoping sum with a gap of two. Writing the sum out,

$$S_n=(1-1/5)+(1/3-1/7)+(1/5-1/9)+\cdots+(1/2n-1-1/2n+3).$$

Each of $1/5, 1/7, \ldots, 1/2n-1$ appears once subtracted and once added (two steps later) and so cancels; only the first two positive terms and the last two negative terms survive:

$$S_n=1+\frac{1}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}=\frac{4}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}.$$

c. As $n\to\infty$, both $\dfrac{1}{2n+1}\to 0$ and $\dfrac{1}{2n+3}\to 0$, so $S_n\to\dfrac{4}{3}$. Hence the infinite series equals $\dfrac{4}{3}$.

Question 4.

a. Evaluating n^2+n+17 for $n=1,2,3,\ldots$ gives $19, 23, 29, 37, 47, \ldots$, and every value up to $n=15$ (which gives 257) is prime, so the claim looks convincing. However, at $n=16$,

$$16^2+16+17=256+16+17=289=17\times 17,$$

which has the factor 17 apart from 1 and itself and so is not prime. Thus $n=16$ is a counterexample, and one counterexample disproves a universal statement no matter how many earlier cases succeed. Hence the claim is false.

b. The negation is $\neg T: \exists x\in\mathbb{R}, \forall y\in\mathbb{R}, y^2\neq x$. The statement T claims that every real number is the square of some real number; this is false, because for any real y we have $y^2\geq 0$, so a negative number such as $x=-1$ is not the square of any real y. Therefore T is false and $\neg T$ is true, witnessed by $x=-1$.

Question 5.

a. Writing the expression over the denominator x and factorising the numerator,

$$x^2+\frac{2}{x}-3=\frac{x^3-3x+2}{x}, x^3-3x+2=(x-1)(x^2+x-2)=(x-1)^2(x+2),$$

so $x^2+\dfrac{2}{x}-3=\dfrac{(x-1)^2(x+2)}{x}$.

b. For $x>0$ the numerator $(x-1)^2\geq 0$ and $x+2>0$, while the denominator $x>0$, so the quotient is ≥ 0. Hence $x^2+\dfrac{2}{x}-3\geq 0$, that is $x^2+\dfrac{2}{x}\geq 3$. Equality holds exactly when $(x-1)^2=0$, i.e. when $x=1$.

c. Since the expression is never less than 3 and attains 3 at $x=1$, its minimum value is 3.

Question 6.

a. Among any two consecutive integers exactly one is even, so at least one of the consecutive integers n and $n+1$ is even. Hence the product $n(n+1)(n+2)$ contains a factor of 2 and is divisible by 2.

b. Consider the three cases by residue modulo 3.

- If $n=3q$, then the factor n is a multiple of 3.
- If $n=3q+1$, then $n+2=3q+3=3(q+1)$ is a multiple of 3.
- If $n=3q+2$, then $n+1=3q+3=3(q+1)$ is a multiple of 3.

These three cases exhaust all integers, and in each one of the three factors is divisible by 3, so the product $n(n+1)(n+2)$ is divisible by 3.

c. The product is divisible by both 2 and 3. Since 2 and 3 have no common factor greater than 1, the product is divisible by $2 \times 3 = 6$. This completes the proof.

Topic 1 Test B — Technology-free

Technology-free. No calculator. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given. Where more than one mark is available, appropriate working must be shown.

Question 1 (6 marks)

a. Disprove the statement "for all real numbers a and b, if $a^2 = b^2$ then $a = b$" by giving a counterexample. 2 marks

b. Show that the statement "$\forall x \in \mathbb{R}, x^2 \geq x$" is false, but that the statement "$\forall n \in \mathbb{Z}, n^2 \geq n$" is true. 4 marks

Question 2 (6 marks)

You may assume that for an integer p, if p^3 is even then p is even.

Prove, by contradiction, that $\sqrt[3]{2}$ is irrational. 6 marks

Question 3 (7 marks)

Use mathematical induction to prove that $\sum_{r=1}^{n} (2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all integers $n \geq 1$. Let $P(n)$ be this statement and write $S_n = \sum_{r=1}^{n} (2r-1)^2$.

a. Verify the base case $P(1)$. 1 mark

b. Write down the inductive hypothesis $P(k)$. 1 mark

c. Assuming $P(k)$, show that $S_{k+1} = \frac{(k+1)(2k+1)(2k+3)}{3}$. 4 marks

d. State the conclusion of the proof. 1 mark

Question 4 (7 marks)

Let a and b be real numbers.

a. Show that $a^4 + b^4 - a^3 b - a b^3 = (a-b)(a^3 - b^3)$. 2 marks

b. Hence show that $a^4 + b^4 - a^3 b - a b^3 = (a-b)^2 (a^2 + ab + b^2)$. 2 marks

c. By writing $a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 + \frac{3b^2}{4}$, explain why $a^2 + ab + b^2 \geq 0$. 2 marks

d. Hence prove that $a^4 + b^4 \geq a^3 b + a b^3$, and state when equality holds. 1 mark

Question 5 (7 marks)

For each positive integer n, let $S_n = \sum_{k=1}^{n} \frac{5}{(5k-3)(5k+2)}$.

a. Find real numbers A and B such that $\frac{5}{(5k-3)(5k+2)} = \frac{A}{5k-3} + \frac{B}{5k+2}$. 2 marks

b. Hence write the general term as $f(k) - f(k+1)$ for a suitable function f, and prove that $S_n = \frac{1}{2} - \frac{1}{5n+2}$. 3 marks

c. Find the value of the infinite series, that is, the limit of S_n as $n \to \infty$. 2 marks

Question 6 (7 marks)

Consider the algorithm below, in which n is a positive integer.

```
input n
r ← 0
s ← 1
while s ≤ n do
    r ← r + 1
    s ← s + 2 × r + 1
end while
output r
```

a. Complete a trace table with columns r and s recording their values after each pass of the loop, for $n = 10$, and state the output. 3 marks

b. Show that after the pass in which r takes the value m, the variable s equals $(m+1)^2$. Hence explain what the algorithm computes. 2 marks

c. State, with a reason, why a `while` loop rather than a `for` loop is appropriate here, and name the element of computational thinking (decomposition, abstraction, pattern or algorithm) shown in recognising that consecutive squares differ by $2r+1$. 2 marks

End of Topic 1 Test B

Test B — answers and solutions

Question 1.

a. Take $a=1$ and $b=-1$. Then $a^2=1=b^2$, so the hypothesis $a^2=b^2$ holds, but $a=1\neq-1=b$, so the conclusion $a=b$ fails. Hence $a=1$, $b=-1$ is a counterexample and the statement is false.

b. The real statement is false: at $x=1/2$, $x^2=1/4$, and $1/4\geq 1/2$ is false, so $x=1/2$ is a counterexample. The integer statement is true: for any integer n, $n^2-n=n(n-1)$ is a product of two consecutive integers. If $n\geq 1$ then $n\geq 0$ and $n-1\geq 0$, so $n(n-1)\geq 0$; if $n\leq 0$ then $n\leq 0$ and $n-1<0$, so again $n(n-1)\geq 0$. In every case $n^2-n\geq 0$, that is $n^2\geq n$. Hence the statement holds for all integers n. (The two statements differ only in their domain, which shows the domain is essential.)

Question 2.

Suppose, for contradiction, that $\sqrt[3]{2}$ is rational. Then $\sqrt[3]{2}=\frac{p}{q}$ for some integers p, q with $q\neq 0$, and we may take the fraction in lowest terms, so that p and q have no common factor greater than 1. Cubing both sides gives $2=\frac{p^3}{q^3}$, hence

$$p^3=2q^3.$$

Thus p^3 is even, and by the given assumption p is even; write $p=2m$ for some integer m. Substituting, $8m^3=2q^3$, so $q^3=4m^3=2(2m^3)$, which is even; hence q^3 is even and, by the assumption again, q is even. But then p and q share the common factor 2, contradicting the assumption that the fraction was in lowest terms. This contradiction shows the supposition was false, so $\sqrt[3]{2}$ is irrational, as required.

Question 3.

a. For $n=1$, $S_1=(2\cdot 1-1)^2=1$ and $\frac{1(2\cdot 1-1)(2\cdot 1+1)}{3}=\frac{1\cdot 1\cdot 3}{3}=1$. The two sides agree, so $P(1)$ is true.

b. Assume $S_k=\frac{k(2k-1)(2k+1)}{3}$ for some integer $k\geq 1$.

c. The $(k+1)$-th term is $(2(k+1)-1)^2=(2k+1)^2$, so

$$S_{k+1}=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2=\frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}.$$

Now $k(2k-1)+3(2k+1)=2k^2-k+6k+3=2k^2+5k+3=(k+1)(2k+3)$, so

$$S_{k+1}=\frac{(2k+1)(k+1)(2k+3)}{3}=\frac{(k+1)(2k+1)(2k+3)}{3}.$$

Since $2(k+1)-1=2k+1$ and $2(k+1)+1=2k+3$, this is exactly the formula with n replaced by $k+1$, so $P(k)\Rightarrow P(k+1)$.

d. Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for every $k\geq 1$, by the principle of mathematical induction the formula holds for all integers $n\geq 1$. This completes the proof.

Question 4.

a. Grouping in pairs,

$$a^4+b^4-a^3b-ab^3=a^3(a-b)-b^3(a-b)=(a-b)(a^3-b^3).$$

b. Using the factorisation $a^3-b^3=(a-b)(a^2+ab+b^2)$,

$$(a-b)(a^3-b^3)=(a-b)(a-b)(a^2+ab+b^2)=(a-b)^2(a^2+ab+b^2).$$

c. Completing the square in a, $a^2+ab+b^2=\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}$. Both $\left(a+\frac{b}{2}\right)^2\geq 0$ and $\frac{3b^2}{4}\geq 0$ (each is a square, up to a positive multiple), so their sum is ≥ 0; hence $a^2+ab+b^2\geq 0$.

d. By parts **b** and **c**, $a^4+b^4-a^3b-ab^3=(a-b)^2(a^2+ab+b^2)$ is a product of the non-negative number $(a-b)^2$ and the non-negative number a^2+ab+b^2, so it is ≥ 0. Therefore $a^4+b^4\geq a^3b+ab^3$. Equality requires $(a-b)^2=0$, giving $a=b$ (or $a^2+ab+b^2=0$, which forces $a=b=0$); in every case equality holds if and only if $a=b$. This completes the proof.

Question 5.

a. Clearing denominators in $\frac{5}{(5k-3)(5k+2)}=\frac{A}{5k-3}+\frac{B}{5k+2}$ gives $5=A(5k+2)+B(5k-3)$. Putting $k=3/5$: $5=A(3+2)=5A$, so $A=1$. Putting $k=-2/5$: $5=B(-2-3)=-5B$, so $B=-1$. Hence

$$\frac{5}{(5k-3)(5k+2)}=\frac{1}{5k-3}-\frac{1}{5k+2}.$$

b. With $f(k)=\frac{1}{5k-3}$, we have $f(k+1)=\frac{1}{5(k+1)-3}=\frac{1}{5k+2}$, so the general term is $f(k)-f(k+1)$. By the telescoping principle,

$$S_n=f(1)-f(n+1)=\frac{1}{5\cdot 1-3}-\frac{1}{5n+2}=\frac{1}{2}-\frac{1}{5n+2}.$$

c. As $n\to\infty$, $\frac{1}{5n+2}\to 0$, so $S_n\to\frac{1}{2}$. Hence the infinite series equals $\frac{1}{2}$.

Question 6.

a. Start with $r=0$ and $s=1$. Each pass increments r, then updates $s\leftarrow s+2r+1$:

Pass	r	s after pass
1	1	4
2	2	9
3	3	16

After the third pass $s=16$, and $16\leq 10$ is false, so the loop stops. The output is $r=3$.

b. Initially $s=1=(0+1)^2$. Suppose that after the pass setting $r=m-1$ the value is $s=m^2$ (true for the first pass, where $m=1$ and s starts at $1=1^2$). The next pass sets $r=m$ and computes

$$s\leftarrow m^2+2m+1=(m+1)^2,$$

so after the pass setting $r=m$, $s=(m+1)^2$. The loop continues while $s=(r+1)^2\leq n$ and stops at the first r with $(r+1)^2>n$; hence the output r is the largest integer with $r^2\leq n$, that is $r=\lfloor\sqrt{n}\rfloor$, the integer part of $\sqrt{n}$. (For $n=10$, $\lfloor\sqrt{10}\rfloor=3$, agreeing with part **a**.)

c. A `while` loop is appropriate because the number of passes is not known in advance — it depends on the size of $\sqrt{n}$ — whereas a `for` loop requires its number of passes to be fixed beforehand. Recognising that consecutive squares differ by $2r+1$, so each square is obtained from the previous one by adding $2r+1$ (with no multiplication needed), is an instance of **pattern**.