Specialist Mathematics Units 1 & 2

Chapter 13 — Loci, conics and the absolute value function

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Theory

A **locus** is a set of points singled out by a geometric rule: it is the collection of *all* points that satisfy a stated condition, and *only* those points. Familiar curves arise this way. A circle is the locus of points a fixed distance from a centre; a perpendicular bisector is the locus of points equidistant from two fixed points. The power of coordinate geometry is that such a rule can be turned into an **equation**: if $P=(x,y)$ is a *typical* point of the locus, writing the geometric condition in terms of x and y and simplifying produces the equation satisfied by exactly the points of the locus. Three of the most important loci — the **parabola**, the **circle** and the **ellipse** — are studied here. They are all **conics** (curves obtained by slicing a cone), and each has a clean defining condition from which its equation is derived below.

The method. To find the equation of a locus:

1. let $P=(x,y)$ be a general point of the locus;
2. translate the defining condition into an equation in x and y, usually with the distance formula $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$;
3. simplify to a clean Cartesian equation, squaring to remove square roots where needed.

The parabola as a locus.

Fix a point S, the **focus**, and a line d not through S, the **directrix**. The **parabola** is the locus of points equidistant from the focus and the directrix:

$$PS=PM,$$

where M is the foot of the perpendicular from P to d, so PM is the distance from P to the directrix. To derive its equation in the simplest position, place the focus at $S=(0,a)$ with $a>0$ and take the directrix to be the horizontal line $y=-a$. For a point $P=(x,y)$,

$$PS=\sqrt{x^2+(y-a)^2}, PM=|y-(-a)|=|y+a|.$$

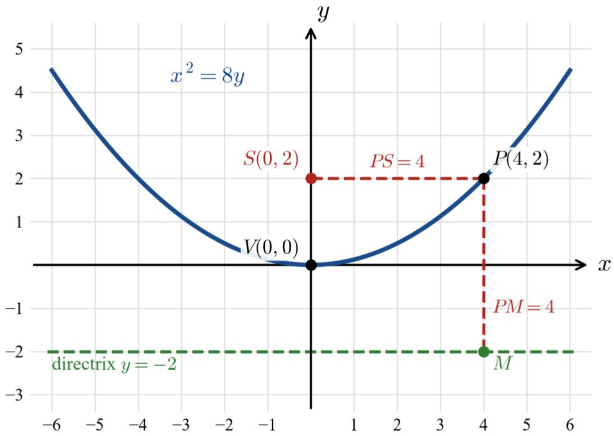
The condition $PS=PM$, squared, gives

$$x^2+(y-a)^2=(y+a)^2.$$

Expanding both sides, $x^2+y^2-2ay+a^2=y^2+2ay+a^2$; the y^2 and a^2 terms cancel, leaving

$$x^2=4ay.$$

This is the standard equation of a parabola with **vertex** at the origin. Its features read directly from the derivation: the **axis of symmetry** is the y-axis ($x=0$), the **vertex** is $(0,0)$, the **focus** is $(0,a)$, the **directrix** is $y=-a$, and the **focal length** (vertex to focus) is a. The curve opens *towards* the focus — here upward. For example, with $a=2$ the parabola $x^2=8y$ has focus $(0,2)$ and directrix $y=-2$: the point $P=(4,2)$ on it is distance 4 from the focus and distance 4 from the directrix, as the definition requires.



The same derivation, with the focus and directrix moved, gives the other standard orientations. All four have vertex at the origin and focal length $a > 0$:

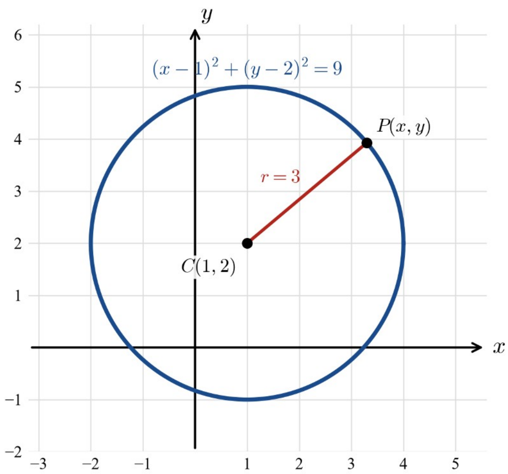
Equation	Opens	Axis	Focus	Directrix
$x^2 = 4ay$	up	$x = 0$	$(0, a)$	$y = -a$
$x^2 = -4ay$	down	$x = 0$	$(0, -a)$	$y = a$
$y^2 = 4ax$	right	$y = 0$	$(a, 0)$	$x = -a$
$y^2 = -4ax$	left	$y = 0$	$(-a, 0)$	$x = a$

A parabola with vertex moved to (h, k) is obtained by replacing x with $x - h$ and y with $y - k$; for instance $(x - h)^2 = 4a(y - k)$ is the upward parabola with vertex (h, k), focus $(h, k + a)$ and directrix $y = k - a$.

The circle as a locus.

A **circle** is the locus of points at a fixed distance $r > 0$ (the **radius**) from a fixed point $C = (h, k)$ (the **centre**). For $P = (x, y)$ the condition $PC = r$ reads $\sqrt{(x-h)^2 + (y-k)^2} = r$, and squaring gives the standard form

$$(x - h)^2 + (y - k)^2 = r^2.$$



Expanding removes the brackets:

$$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0,$$

which has the shape of the **general form**

$$x^2 + y^2 + Dx + Ey + F = 0,$$

where the coefficients of x^2 and y^2 are equal and there is no xy term. Given an equation in general form, its centre and radius are recovered by **completing the square** in x and in y. Matching coefficients, $D=-2h$ and $E=-2k$, so the centre is $(-D/2,-E/2)$, and the radius satisfies

$$r^2=(D/2)^2+(E/2)^2-F.$$

The equation represents a genuine circle only when this right-hand side is **positive**; if it is zero the locus is the single point C, and if it is negative there are no real points at all.

The ellipse as a locus.

An **ellipse** is the locus of points the *sum* of whose distances from two fixed points F_1 and F_2, the **foci**, is a constant. Write the two foci as $F_1=(-c,0)$ and $F_2=(c,0)$ with $c>0$, and let the constant sum be $2a$, where necessarily $a>c$ (the sum of two sides of the triangle F_1PF_2 exceeds the third side $F_1F_2=2c$). The defining condition on $P=(x,y)$ is

$$PF_1+PF_2=2a\text{, that is }\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a.$$

Move the second radical to the right and square:

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2.$$

Since $(x+c)^2-(x-c)^2=4cx$, this simplifies to $4cx=4a^2-4a\sqrt{(x-c)^2+y^2}$, i.e. $a\sqrt{(x-c)^2+y^2}=a^2-cx$. Squaring again,

$$a^2\left[(x-c)^2+y^2\right]=(a^2-cx)^2,$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2.$$

The $-2a^2cx$ terms cancel; gathering the rest,

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2).$$

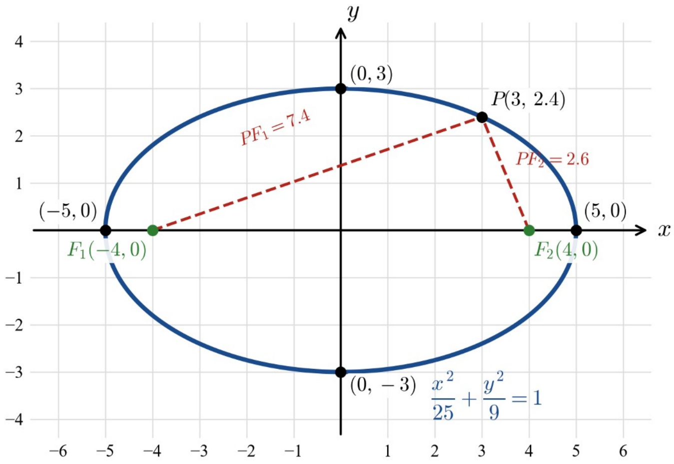
Because $a>c$, the quantity a^2-c^2 is positive; set $b^2=a^2-c^2$ (so $0<b<a$). Dividing through by a^2b^2 gives the standard equation of an ellipse centred at the origin:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Here a is the **semi-major axis** and b the **semi-minor axis**. Setting $y=0$ gives the **vertices** $(\pm a,0)$, the endpoints of the **major axis** (length $2a$); setting $x=0$ gives the endpoints of the **minor axis** $(0,\pm b)$ (length $2b$). The **foci** lie on the major axis, and rearranging $b^2=a^2-c^2$ recovers them through

$$c^2=a^2-b^2,$$

so the foci are $(\pm c,0)$ with $c=\sqrt{a^2-b^2}$. The centre is the midpoint of the foci, here the origin.



The relationship $c^2=a^2-b^2$ is the ellipse's own version of a right-angle relationship, and it settles the orientation automatically. In $\frac{x^2}{p}+\frac{y^2}{q}=1$ the *larger* denominator sits under the variable along which the major axis lies. If $p>q$ the major axis is horizontal, $a^2=p$, $b^2=q$ and the foci are $(\pm c,0)$; if $q>p$ the major axis is vertical, $a^2=q$, $b^2=p$ and the foci are $(0,\pm c)$, with $c^2=a^2-b^2$ (the larger minus the smaller) in either case. When $p=q$ the two axes are equal, $c=0$, the two foci coincide at the centre and the curve is the circle $x^2+y^2=p$. As with the parabola and circle, replacing x by $x-h$ and y by $y-k$ **translates** the whole ellipse so that its centre moves to (h,k):

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

Worked examples

Example 1 — scaffolded

A parabola has focus $S=(0,3)$ and directrix the line $y=-3$. (a) Use the focus–directrix definition to find its equation. (b) State its vertex, axis of symmetry and focal length.

Working	Comment
Let $P=(x,y)$ be a general point of the locus, with foot M on the directrix. The defining condition is $PS=PM$.	A parabola is the locus of points equidistant from focus and directrix.
$PS=\sqrt{x^2+(y-3)^2}$ and $PM=\lvert y+3\rvert$, the distance from P to $y=-3$.	Distance to the focus; vertical distance to the horizontal directrix.
$x^2+(y-3)^2=(y+3)^2$.	Square the condition $PS=PM$.
$x^2+y^2-6y+9=y^2+6y+9$, so $x^2=12y$.	Expand; the y^2 and constant terms cancel, leaving $x^2=12y$.
Comparing with $x^2=4ay$: $4a=12$, so $a=3$.	The standard upward form identifies the focal length a.
Vertex $(0,0)$; axis the y-axis, $x=0$; focal length 3.	Read directly from the standard form.

Example 2 — scaffolded

Show that $x^2+y^2-10x+6y+9=0$ is the equation of a circle, and find its centre and radius.

Working	Comment
Group the x- and y-terms: $(x^2-10x)+(y^2+6y)=-9$.	Move the constant to the right; the x^2, y^2 coefficients are equal, as a circle requires.

Working	Comment
$x^2-10x=(x-5)^2-25$.	Complete the square: half of -10 is -5, and $(-5)^2=25$.
$y^2+6y=(y+3)^2-9$.	Half of 6 is 3, and $3^2=9$.
$(x-5)^2-25+(y+3)^2-9=-9$.	Substitute both completed squares back in.
$(x-5)^2+(y+3)^2=25$.	Carry the constants across: $-9+25+9=25>0$, a genuine circle.
Centre $(5,-3)$, radius $\sqrt{25}=5$.	Compare with $(x-h)^2+(y-k)^2=r^2$.

Example 3 — unscaffolded

For the ellipse $\dfrac{x^2}{40}+\dfrac{y^2}{15}=1$, state the centre, the vertices and the endpoints of the minor axis, and find the coordinates of the foci.

The equation is already in standard form with centre the origin. Since $40>15$, the larger denominator lies under x^2, so the major axis is along the x-axis with $a^2=40$ and $b^2=15$; hence $a=\sqrt{40}=2\sqrt{10}$ and $b=\sqrt{15}$. The vertices, the endpoints of the major axis, are therefore $(\pm 2\sqrt{10},0)$, and the endpoints of the minor axis are $(0,\pm\sqrt{15})$. For the foci, use $c^2=a^2-b^2$:

$$c^2=40-15=25, c=5,$$

and because the major axis is horizontal the foci lie on the x-axis at $(\pm 5,0)$.

∴ centre $(0,0)$; vertices $(\pm 2\sqrt{10},0)$; minor-axis endpoints $(0,\pm\sqrt{15})$; foci $(\pm 5,0)$.

Example 4 — unscaffolded

Find the equation of the locus of all points equidistant from the point $A=(3,1)$ and the line $y=-3$, and identify the curve completely.

Let $P=(x,y)$ be a point of the locus. Its distance from A is $\sqrt{(x-3)^2+(y-1)^2}$, and its distance from the line $y=-3$ is $|y+3|$. Equidistance, squared, gives

$$(x-3)^2+(y-1)^2=(y+3)^2.$$

Expanding the two y-brackets, $(x-3)^2+y^2-2y+1=y^2+6y+9$; the y^2 terms cancel and

$$(x-3)^2=8y+8=8(y+1).$$

This is a parabola in translated standard form $(x-h)^2=4a(y-k)$ with $h=3$, $k=-1$ and $4a=8$, so $a=2$. Its vertex is $(3,-1)$ and it opens upward; the focus is $(3,-1+2)=(3,1)=A$ and the directrix is $y=-1-2=-3$ — consistent with the point and line we started from.

∴ the locus is the parabola $(x-3)^2=8(y+1)$, with vertex $(3,-1)$, focus $(3,1)$ and directrix $y=-3$.

Practice questions

Scaffolded

Q1. A parabola has focus $(0,1)$ and directrix $y=-1$. (a) State, in words, the condition a point $P=(x,y)$ must satisfy to lie on the parabola. (b) Write the condition $PS=PM$ as an equation and square it. (c) Simplify to show the equation is $x^2=4y$. (d) State the vertex and the focal length.

Q2. Consider $x^2+y^2+6x-8y+9=0$. (a) Group the x-terms and the y-terms, moving the constant to the right. (b) Complete the square in x and in y. (c) Write the equation in the form $(x-h)^2+(y-k)^2=r^2$. (d) State the centre and radius.

Q3. The parabola $y^2=-12x$ has vertex at the origin. (a) Compare with $y^2=-4ax$ to find a. (b) State the axis of symmetry and the direction of opening. (c) State the focus. (d) State the directrix.

Q4. For the ellipse $\frac{x^2}{64}+\frac{y^2}{28}=1$: (a) state a^2 and b^2, and hence a and b; (b) state the vertices; (c) find c using $c^2=a^2-b^2$, and state the foci; (d) state the endpoints of the minor axis.

Q5. A circle has a diameter with endpoints $A=(-1,2)$ and $B=(5,-6)$. (a) Find the centre as the midpoint of AB. (b) Find the radius as half the length AB. (c) Write the equation of the circle. (d) Verify that A lies on the circle.

Q6. An ellipse has foci $(0,4)$ and $(0,-4)$, and the sum of the distances from any point on it to the two foci is 12. (a) State the value of $2a$, hence a, and state c. (b) Find b^2 from $b^2=a^2-c^2$. (c) Write the equation of the ellipse, taking care over which denominator is larger. (d) State the vertices and the endpoints of the minor axis.

Unscaffolded

Q7. Find the equation of the parabola with focus $(0,-5)$ and directrix $y=5$, deriving it from the focus–directrix definition.

Q8. Find the equation of the parabola with focus $(7,0)$ and directrix $x=-7$.

Q9. Express $x^2+y^2-2x+10y+17=0$ in standard form and state the centre and radius.

Q10. A circle has a diameter with endpoints $(2,-4)$ and $(6,2)$. Find its equation.

Q11. For the ellipse $\frac{x^2}{32}+\frac{y^2}{81}=1$, state the vertices and find the foci.

Q12. For the ellipse $\frac{(x+2)^2}{45}+\frac{(y-3)^2}{20}=1$, state the centre, and find the vertices and the foci.

Q13. Find the equation of the circle with centre $(-2,3)$ that passes through the point $(3,5)$.

Q14. Find the equation of the parabola with vertex $(2,-3)$ and focus $(2,0)$, and state its directrix.

Q15. Find the equation of the locus of points the sum of whose distances from $(0,3)$ and $(0,-3)$ is 10, and identify the curve.

Q16. Express $x^2+y^2-8x-6y+18=0$ in standard form, and state the centre and the exact radius.

Q17. For the parabola $x^2=-6y$, state the vertex, focus and directrix.

Q18. Find the equation of the locus of all points equidistant from the point $(-2,4)$ and the line $x=6$. Identify the curve and state its vertex, focus and directrix.

Answers

Q1. (a) equidistant from focus $(0,1)$ and directrix $y=-1$; (b) $x^2+(y-1)^2=(y+1)^2$; (c) $x^2=4y$; (d) vertex $(0,0)$, focal length 1. **Q2.** (a) $(x^2+6x)+(y^2-8y)=-9$; (c) $(x+3)^2+(y-4)^2=16$; (d) centre $(-3,4)$, radius 4. **Q3.** (a) $a=3$; (b) axis $y=0$, opens left; (c) focus $(-3,0)$; (d) directrix $x=3$. **Q4.** (a) $a^2=64, b^2=28, a=8, b=2\sqrt{7}$; (b) $(\pm 8,0)$; (c) $c=6$, foci $(\pm 6,0)$; (d) $(0,\pm 2\sqrt{7})$. **Q5.** (a) $(2,-2)$; (b) 5; (c) $(x-2)^2+(y+2)^2=25$; (d) $(-1-2)^2+(2+2)^2=9+16=25$. **Q6.** (a) $2a=12, a=6, c=4$; (b) $b^2=20$; (c) $\frac{x^2}{20}+\frac{y^2}{36}=1$; (d) vertices $(0,\pm 6)$, minor-axis endpoints $(\pm 2\sqrt{5},0)$. **Q7.** $x^2=-20y$. **Q8.** $y^2=28x$. **Q9.** $(x-1)^2+(y+5)^2=9$; centre $(1,-5)$, radius 3. **Q10.** $(x-4)^2+(y+1)^2=13$. **Q11.** vertices $(0,\pm 9)$; foci $(0,\pm 7)$. **Q12.** centre $(-2,3)$; vertices $(-2\pm 3\sqrt{5},3)$; foci $(3,3)$ and $(-7,3)$. **Q13.** $(x+2)^2+(y-3)^2=29$. **Q14.** $(x-2)^2=12(y+3)$; directrix $y=-6$. **Q15.** $\frac{x^2}{16}+\frac{y^2}{25}=1$; an ellipse (major axis vertical). **Q16.** $(x-4)^2+(y-3)^2=7$; centre $(4,3)$, radius $\sqrt{7}$. **Q17.** vertex $(0,0)$; focus $(0,-3/2)$; directrix $y=3/2$. **Q18.** $(y-4)^2=-16(x-2)$; a parabola, vertex $(2,4)$, focus $(-2,4)$, directrix $x=6$.

Fully-worked solutions

Q1. (a) P lies on the parabola exactly when it is equidistant from the focus $(0,1)$ and the directrix $y=-1$, i.e. $PS=PM$. (b) With $PS=\sqrt{x^2+(y-1)^2}$ and $PM=|y+1|$, squaring gives $x^2+(y-1)^2=(y+1)^2$. (c) Expanding, $x^2+y^2-2y+1=y^2+2y+1$; the y^2 and constant terms cancel, leaving $x^2=4y$. (d) In $x^2=4ay$ we have $4a=4$, so $a=1$: the vertex is $(0,0)$ and the focal length is 1.

Q2. (a) Grouping, $(x^2+6x)+(y^2-8y)=-9$. (b) Completing each square, $x^2+6x=(x+3)^2-9$ and $y^2-8y=(y-4)^2-16$. (c) Substituting, $(x+3)^2-9+(y-4)^2-16=-9$, so $(x+3)^2+(y-4)^2=-9+9+16=16$. (d) Hence the centre is $(-3,4)$ and the radius is $\sqrt{16}=4$.

Q3. (a) Comparing $y^2=-12x$ with $y^2=-4ax$ gives $4a=12$, so $a=3$. (b) The axis of symmetry is the x-axis, $y=0$, and because of the minus sign the parabola opens to the left. (c) The focus of $y^2=-4ax$ is $(-a,0)=(-3,0)$. (d) The directrix is $x=a$, that is $x=3$.

Q4. (a) The denominators give $a^2=64$ and $b^2=28$, so $a=8$ and $b=\sqrt{28}=2\sqrt{7}$; since $64>28$ the major axis is horizontal. (b) The vertices are $(\pm 8,0)$. (c) From $c^2=a^2-b^2=64-28=36$, $c=6$, so the foci are $(\pm 6,0)$. (d) The endpoints of the minor axis are $(0,\pm 2\sqrt{7})$.

Q5. (a) The centre is the midpoint of AB: $(-1+5/2, 2+(-6)/2)=(2,-2)$. (b) The length $AB=\sqrt{(5-(-1))^2+(-6-2)^2}=\sqrt{36+64}=\sqrt{100}=10$, so the radius is $r=1/2(10)=5$. (c) The equation is $(x-2)^2+(y+2)^2=25$. (d) Substituting A: $(-1-2)^2+(2+2)^2=9+16=25$, so A lies on the circle.

Q6. (a) The constant sum equals $2a$, so $2a=12$ and $a=6$; the foci are $(0,\pm 4)$, so $c=4$. (b) $b^2=a^2-c^2=36-16=20$. (c) The foci lie on the y-axis, so the major axis is vertical and the larger denominator, $a^2=36$, belongs under y^2: $\frac{x^2}{20}+\frac{y^2}{36}=1$. (d) The vertices (ends of the major axis) are $(0,\pm 6)$, and the endpoints of the minor axis are $(\pm\sqrt{20},0)=(\pm 2\sqrt{5},0)$.

Q7. Let $P=(x,y)$. Equidistance from $(0,-5)$ and $y=5$ gives $\sqrt{x^2+(y+5)^2}=|y-5|$; squaring, $x^2+(y+5)^2=(y-5)^2$. Then $x^2+y^2+10y+25=y^2-10y+25$, so $x^2=-20y$. (This is $x^2=-4ay$ with $a=5$, opening downward, as the focus below the directrix requires.)

Q8. Let $P=(x,y)$. Equidistance from $(7,0)$ and $x=-7$ gives $\sqrt{(x-7)^2+y^2}=|x+7|$; squaring, $(x-7)^2+y^2=(x+7)^2$. Hence $y^2=(x+7)^2-(x-7)^2=28x$, i.e. $y^2=28x$ (a rightward parabola with $a=7$).

Q9. Grouping and completing the square, $(x^2-2x)+(y^2+10y)=-17$ becomes $(x-1)^2-1+(y+5)^2-25=-17$, so $(x-1)^2+(y+5)^2=-17+1+25=9$. The centre is $(1,-5)$ and the radius is $\sqrt{9}=3$.

Q10. The centre is the midpoint $(2+6/2,-4+2/2)=(4,-1)$. The diameter length is $\sqrt{(6-2)^2+(2-(-4))^2}=\sqrt{16+36}=\sqrt{52}=2\sqrt{13}$, so the radius is $r=\sqrt{13}$ and $r^2=13$. The equation is $(x-4)^2+(y+1)^2=13$.

Q11. The larger denominator, 81, sits under y^2, so the major axis is vertical with $a^2=81$ and $b^2=32$; hence $a=9$ and the vertices are $(0,\pm 9)$. From $c^2=a^2-b^2=81-32=49$, $c=7$, and the foci lie on the y-axis at $(0,\pm 7)$.

Q12. The equation is a translated ellipse with centre $(-2,3)$. Under the shift, $a^2=45$ and $b^2=20$ with $45>20$, so the major axis is horizontal; $a=\sqrt{45}=3\sqrt{5}$, so the vertices are $(-2\pm 3\sqrt{5},3)$. From $c^2=a^2-b^2=45-20=25$, $c=5$, so the foci are 5 units left and right of the centre: $(-2+5,3)=(3,3)$ and $(-2-5,3)=(-7,3)$.

Q13. The radius is the distance from the centre $(-2,3)$ to $(3,5)$: $r=\sqrt{(3-(-2))^2+(5-3)^2}=\sqrt{25+4}=\sqrt{29}$. Hence $r^2=29$ and the circle is $(x+2)^2+(y-3)^2=29$.

Q14. The vertex $(2,-3)$ and focus $(2,0)$ share the vertical line $x=2$, and the focus is above the vertex, so the parabola opens upward with focal length $a=0-(-3)=3$. In translated form $(x-h)^2=4a(y-k)$ with $(h,k)=(2,-3)$ and $4a=12$,

$$(x-2)^2=12(y+3).$$

The directrix lies $a=3$ below the vertex: $y=-3-3=-6$.

Q15. Let $P=(x,y)$. The condition is $\sqrt{x^2+(y-3)^2}+\sqrt{x^2+(y+3)^2}=10$, an ellipse with foci $(0,\pm 3)$ and constant sum $2a=10$, so $a=5$ and $c=3$. Then $b^2=a^2-c^2=25-9=16$. The foci are on the y-axis, so the major axis is vertical and $a^2=25$ sits under y^2:

$$\frac{x^2}{16}+\frac{y^2}{25}=1.$$

The locus is an ellipse with vertices $(0,\pm 5)$ and minor-axis endpoints $(\pm 4,0)$.

Q16. Completing the square, $(x^2-8x)+(y^2-6y)=-18$ becomes $(x-4)^2-16+(y-3)^2-9=-18$, so $(x-4)^2+(y-3)^2=-18+16+9=7$. The centre is $(4,3)$ and the radius is $\sqrt{7}$ (an exact surd, since 7 is not a perfect square).

Q17. Writing $x^2=-6y$ as $x^2=-4ay$ gives $4a=6$, so $a=3/2$. The vertex is $(0,0)$; the minus sign means the parabola opens downward, so the focus is $(0,-3/2)$ and the directrix is $y=3/2$.

Q18. Let $P=(x,y)$. Equidistance from $(-2,4)$ and the line $x=6$ gives $\sqrt{(x+2)^2+(y-4)^2}=|x-6|$; squaring, $(x+2)^2+(y-4)^2=(x-6)^2$. Expanding the two x-brackets, $x^2+4x+4+(y-4)^2=x^2-12x+36$; the x^2 terms cancel and

$$(y-4)^2=-16x+32=-16(x-2).$$

This is a parabola in the form $(y-k)^2=-4a(x-h)$ with $(h,k)=(2,4)$ and $4a=16$, so $a=4$; it opens to the left. The vertex is $(2,4)$, the focus is 4 units left at $(-2,4)$, and the directrix is 4 units right at $x=6$ — matching the given point and line.

Theory

Section 13A obtained the parabola, circle and ellipse as loci. The remaining conic is the **hyperbola**, whose defining condition swaps the ellipse's *sum* of focal distances for a *difference*. After deriving it we turn to two new ways of describing every conic: a **parametric** description, in which x and y are each given as functions of a third variable, and a brief look at **polar** coordinates, in which a point is fixed by a distance and an angle rather than by two perpendicular displacements.

The hyperbola as a locus.

A **hyperbola** is the locus of points the *difference* of whose distances from two fixed points F_1 and F_2, the **foci**, has a constant magnitude. Place the foci at $F_1=(-c,0)$ and $F_2=(c,0)$ with $c>0$, and let the constant difference be $2a$, where now $0<a<c$ (the difference of two sides of the triangle F_1PF_2 is *less* than the third side $F_1F_2=2c$, so $2a<2c$). Taking the branch nearer F_2, the defining condition on $P=(x,y)$ is

$$PF_1-PF_2=2a\text{, that is }\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a.$$

Move the second radical across and square:

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2.$$

Since $(x+c)^2-(x-c)^2=4cx$, this reduces to $4cx-4a^2=4a\sqrt{(x-c)^2+y^2}$, i.e. $cx-a^2=a\sqrt{(x-c)^2+y^2}$. Squaring again,

$$c^2x^2-2a^2cx+a^4=a^2\left[(x-c)^2+y^2\right]=a^2x^2-2a^2cx+a^2c^2+a^2y^2.$$

The $-2a^2cx$ terms cancel; gathering the rest,

$$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2).$$

Because $c>a$, the quantity c^2-a^2 is positive; set $b^2=c^2-a^2$ (so $b>0$). Dividing through by a^2b^2 gives the standard equation of a hyperbola centred at the origin:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The single sign change from the ellipse's $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the whole difference between the two curves, yet it transforms a bounded oval into an unbounded pair of branches.

Centre, vertices and foci.

The equation is unchanged when $x\to -x$ or $y\to -y$, so the curve is symmetric in both axes and hence in the origin, which is its **centre**. Setting $y=0$ gives $x^2=a^2$, so the curve crosses the x-axis at the **vertices** $(\pm a,0)$; the segment joining them is the **transverse axis**, of length $2a$. Setting $x=0$ gives $-y^2/b^2=1$, which has no real solution — the hyperbola never meets the y-axis. Rearranging the defining relation $b^2=c^2-a^2$ recovers the foci through

$$c^2=a^2+b^2,$$

so the **foci** are $(\pm c,0)$ with $c=\sqrt{a^2+b^2}$. Note the contrast with the ellipse, where $c^2=a^2-b^2$: for a hyperbola the foci lie *beyond* the vertices ($c>a$), and a need not exceed b.

Asymptotes.

Solving the standard equation for y gives $y^2=\frac{b^2}{a^2}\left(x^2-a^2\right)$, so on the upper half of the right-hand branch

$$y=\frac{b}{a}\sqrt{x^2-a^2}.$$

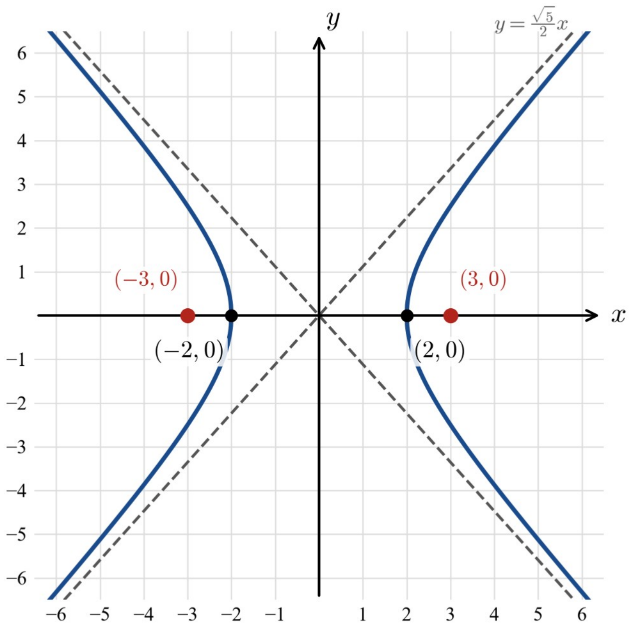
Compare this with the straight line $y=\frac{b}{a}x$. Their vertical separation is

$$\frac{b}{a}x-\frac{b}{a}\sqrt{x^2-a^2}=\frac{b}{a}\cdot\frac{x^2-\left(x^2-a^2\right)}{x+\sqrt{x^2-a^2}}=\frac{b}{a}\cdot\frac{a^2}{x+\sqrt{x^2-a^2}},$$

after rationalising the numerator. As $x\to\infty$ the denominator grows without bound, so this gap shrinks to 0: the branch approaches the line ever more closely without ever reaching it. By the symmetry of the curve the same holds for $y=-\frac{b}{a}x$, so the hyperbola has the two **asymptotes**

$$y=\pm\frac{b}{a}x.$$

A quick way to recover them: replacing the 1 on the right of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ by 0 factorises as $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=0$, giving $y=\pm\frac{b}{a}x$ directly. The asymptotes cross at the centre and frame the branches, which gives a reliable **recipe for sketching a hyperbola**: mark the centre; draw the *guide rectangle* bounded by $x=\pm a$ and $y=\pm b$; extend its diagonals, which are the asymptotes $y=\pm\frac{b}{a}x$; mark the vertices $(\pm a,0)$, and the foci $(\pm c,0)$ if they are wanted; then draw each branch out from its vertex, opening away from the centre and closing on the asymptotes without ever meeting them.



The diagram shows $\frac{x^2}{4}-\frac{y^2}{5}=1$, for which $a=2$, $b=\sqrt{5}$ and $c=\sqrt{4+5}=3$: vertices $(\pm 2,0)$, foci $(\pm 3,0)$, and asymptotes $y=\pm\frac{\sqrt{5}}{2}x$.

As with the other conics, replacing x by $x-h$ and y by $y-k$ **translates** the whole hyperbola so that its centre moves to (h,k):

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,$$

with vertices $(h\pm a, k)$, foci $(h\pm c, k)$ and asymptotes $y-k=\pm\frac{b}{a}(x-h)$. If instead the y-term is the positive one, $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the branches open up and down: the vertices are $(0,\pm a)$, the foci $(0,\pm c)$ and the asymptotes $y=\pm\frac{a}{b}x$.

The rectangular hyperbola.

When $a=b$ the asymptotes $y=\pm x$ are perpendicular, and the hyperbola is called **rectangular** (or equilateral); its equation reduces to $x^2-y^2=a^2$. Rotating the axes through 45° turns this into an especially simple form. Under the rotation $x=\frac{X-Y}{\sqrt{2}}$, $y=\frac{X+Y}{\sqrt{2}}$ the product becomes

$$xy=\frac{(X-Y)(X+Y)}{2}=\frac{X^2-Y^2}{2},$$

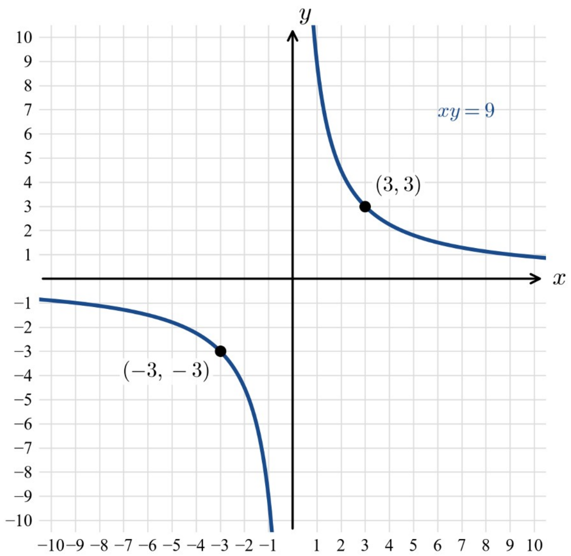
so a rectangular hyperbola whose asymptotes are the *coordinate axes themselves* has the tidy equation

$$xy=c^2$$

for a positive constant, here written c^2. Its asymptotes are the lines $x=0$ and $y=0$; by symmetry its vertices lie on $y=x$, where $x^2=c^2$ gives (c,c) and $(-c,-c)$. This is exactly the reciprocal function $y=\frac{c^2}{x}$ met in Methods — now recognised as a hyperbola. It is neatly described in parametric form by

$$x=ct,\ y=\frac{c}{t}\ (t\neq 0),$$

since then $xy=ct\cdot\frac{c}{t}=c^2$ for every non-zero t.



Parametric form of a curve.

So far each conic has been given by a single Cartesian equation relating x and y. An alternative is to give a **parametric** description: two equations

$$x=f(t),\ y=g(t),$$

expressing each coordinate as a function of a **parameter** t. As t runs through its allowed values the point (x, y) traces the curve, and t can carry extra meaning — often an angle, or (in kinematics) time. To recover the ordinary Cartesian equation we **eliminate the parameter**: combine the two equations so as to remove t, leaving a single relation between x and y. The tool for doing so depends on the functions involved — usually rearranging one equation for t and substituting, or using a Pythagorean identity when the parameter appears through circular functions.

For the circular-function forms the key identity is the Pythagorean relation. Dividing $\cos^2 t+\sin^2 t=1$ by $\cos^2 t$ gives

$$1+\tan^2 t=sec^2 t\text{, equivalently } sec^2 t-\tan^2 t=1,$$

which is precisely the combination needed to produce a *difference* of squares for the hyperbola. The standard parametrisations of the relations met in this chapter, with the identity or substitution used to eliminate t, are collected below. The simplest of them is the **line**: starting from the point (x_1, y_1) and stepping in a fixed direction (λ, μ), the point $(x_1+\lambda t, y_1+\mu t)$ sweeps out the whole line as t runs over $\mathbb{R}$, so t measures how far along that direction the point has moved.

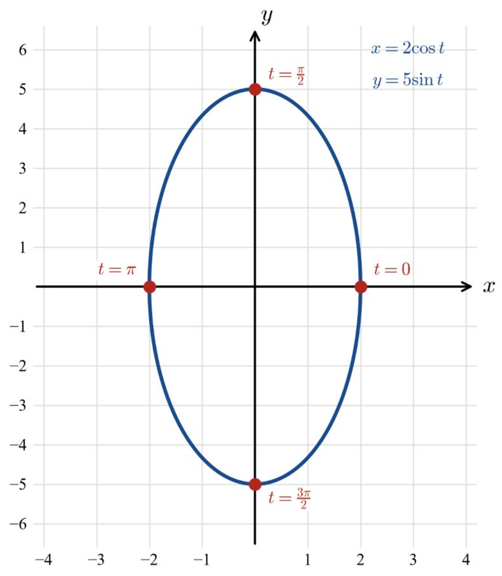
Curve	Parametric form	Eliminate t using	Cartesian equation
Line	$x=x_1+\lambda t, y=y_1+\mu t$	$t=\frac{x-x_1}{\lambda}(\lambda\neq 0)$	$y-y_1=\frac{\mu}{\lambda}(x-x_1)$
Parabola	$x=at^2, y=2at$	$t=\frac{y}{2a}$	$y^2=4ax$
Circle	$x=r\cos t, y=r\sin t$	$\cos^2 t+\sin^2 t=1$	$x^2+y^2=r^2$
Ellipse	$x=a\cos t, y=b\sin t$	$\cos^2 t+\sin^2 t=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
Hyperbola	$x=a\,sec\,t, y=b\tan t$	$sec^2 t-\tan^2 t=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Rectangular hyperbola	$x=ct, y=\frac{c}{t}$	multiply x and y	$xy=c^2$

Take the ellipse as a worked instance. From $x=a\cos t$ and $y=b\sin t$,

$$\cos t=\frac{x}{a}, \sin t=\frac{y}{b},$$

and substituting into $\cos^2 t+\sin^2 t=1$ gives $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. For the hyperbola, $sec\,t=\frac{x}{a}$ and $\tan t=\frac{y}{b}$ substituted into $sec^2 t-\tan^2 t=1$ give $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

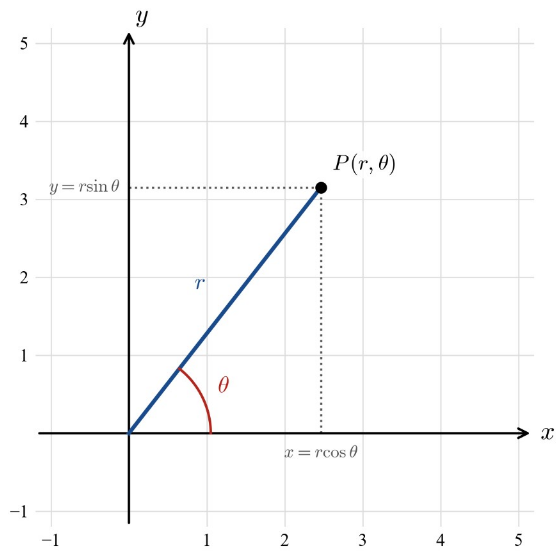
Eliminating the parameter can *lose information*, so the Cartesian equation is sometimes only part of the story. The **range of the parameter** may trace just a portion of the full curve: for the hyperbola, restricting t to $(-\pi/2, \pi/2)$ makes $sec\,t\geq 1$, so $x\geq a$ and only the *right-hand* branch is drawn, even though the Cartesian equation admits both branches. It is always worth checking which values the parametric point actually takes.



The figure traces the ellipse $x = 2\cos t$, $y = 5\sin t$, whose Cartesian equation is $\frac{x^2}{4} + \frac{y^2}{25} = 1$; the marked points show where the parameter takes the values $t = 0, \pi/2, \pi, 3\pi/2$, sweeping the curve anticlockwise from $(2, 0)$.

Polar coordinates — a first look.

A point in the plane can be fixed not only by its perpendicular displacements (x, y) but by how *far* it lies from a fixed origin and in what *direction*. Choose an origin O, called the **pole**, and a ray from it, the **polar axis** (taken along the positive x-axis). A point P then has **polar coordinates** (r, θ), where r is its distance OP and θ is the angle the ray OP makes with the polar axis, measured anticlockwise.



Dropping a perpendicular from P to the polar axis forms a right-angled triangle, from which the conversion between the two systems is immediate:

$$x = r\cos\theta, \; y = r\sin\theta,$$

and, in reverse,

$$r^2 = x^2 + y^2, \; \tan\theta = \frac{y}{x} \, (x \neq 0),$$

with the quadrant of P deciding which value of θ to take. (Polar coordinates are not unique: adding 2π to θ returns the same point, and the pole itself has $r = 0$ for every θ.) A **polar equation** relates r and θ; some curves are far simpler to state this way. The equation $r = k$ (constant) is the **circle** of radius k centred at the pole, and $\theta = \alpha$

(constant) is the **ray** from the pole at angle α — a half-line, not a full line, because r is a distance and so cannot be negative; the whole line through the pole at angle α needs the second ray $\theta=\alpha+\pi$ as well. (It is the ray $arg(z)=\alpha$ of Section 11C, with the pole itself now included, since $r=0$ is allowed here.) A relation mixing r and θ is converted to Cartesian form with the conversion identities above; multiplying by r to create the groups r^2, $r\cos\theta$ and $r\sin\theta$ is the standard first move. For example, multiplying $r=2a\cos\theta$ by r gives $r^2=2ar\cos\theta$, that is $x^2+y^2=2ax$, or $(x-a)^2+y^2=a^2$: a circle of radius a through the pole, centred at $(a,0)$.

Polar forms of the standard relations.

The substitution runs the other way just as well: replacing x by $r\cos\theta$ and y by $r\sin\theta$ turns any Cartesian equation into a polar one, so every relation of this chapter has a polar form alongside its Cartesian and parametric ones. Take the parabola $y^2=4ax$. Substituting gives $r^2\sin^2\theta=4ar\cos\theta$, and dividing by r (the pole lies on the curve, at $r=0$) leaves

$$r\sin^2\theta=4a\cos\theta\text{, that is } r=\frac{4a\cos\theta}{\sin^2\theta}\ (\theta\neq0,\pi).$$

The same one-line substitution gives the rest.

Curve	Cartesian form	Polar form
Line through the pole	$y=x\tan\alpha\ (\alpha\neq\pi/2)$	$\theta=\alpha$ and $\theta=\alpha+\pi$ (one ray each)
Line not through the pole	$px+qy=k$	$r(p\cos\theta+q\sin\theta)=k$
Circle, centred at the pole	$x^2+y^2=k^2$	$r=k$
Circle through the pole	$(x-a)^2+y^2=a^2$	$r=2a\cos\theta$
Parabola	$y^2=4ax$	$r\sin^2\theta=4a\cos\theta$
Ellipse	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$	$r^2\left(\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}\right)=1$
Hyperbola	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$	$r^2\left(\dfrac{\cos^2\theta}{a^2}-\dfrac{\sin^2\theta}{b^2}\right)=1$

The last row carries a caution: the equation makes r^2 the reciprocal of the bracket, so a point exists in the direction θ only when that bracket is positive. A polar equation can therefore describe *no* point at all in some directions — for the hyperbola, precisely the directions in which the curve has no branch.

Worked examples

Example 1 — scaffolded

For the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{5}=1$, state the centre and vertices, find the equations of the asymptotes, find the coordinates of the foci, and sketch the curve.

Working	Comment
The equation is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ with $a^2=4$ and $b^2=5$, so $a=2$ and $b=\sqrt{5}$.	Read a^2,b^2 from the denominators; $a,b>0$.
Centre $(0,0)$; setting $y=0$ gives $x^2=4$, so the vertices are $(\pm2,0)$.	The curve is centred at the origin and crosses the x-axis at $\pm a$.

Working	Comment
Asymptotes $y=\pm\frac{b}{a}x=\pm\frac{\sqrt{5}}{2}x$.	The asymptotes are $y=\pm\frac{b}{a}x$ for this orientation.
For the foci, $c^2=a^2+b^2=4+5=9$, so $c=3$.	A hyperbola uses $c^2=a^2+b^2$ (a sum), unlike the ellipse.
The foci lie on the x-axis at $(\pm 3,0)$.	The foci sit beyond the vertices, since $c>a$.
Sketch: draw the guide rectangle $x=\pm 2$, $y=\pm\sqrt{5}$, extend its diagonals as the asymptotes, mark the vertices $(\pm 2,0)$ and foci $(\pm 3,0)$, then draw a branch out from each vertex closing on the asymptotes.	The sketching recipe: rectangle, then asymptotes, then vertices, then branches. The result is the diagram in the theory above.

Example 2 — scaffolded

A curve is given parametrically by $x=2\cos\theta$ and $y=5\sin\theta$. Eliminate the parameter to find its Cartesian equation, and identify the curve, stating its axes.

Working	Comment
$\cos\theta=\frac{x}{2}$ and $\sin\theta=\frac{y}{5}$.	Rearrange each equation to isolate a circular function.
$\cos^2\theta+\sin^2\theta=1$, so $\left(\frac{x}{2}\right)^2+\left(\frac{y}{5}\right)^2=1$.	Substitute into the Pythagorean identity to remove θ.
$\frac{x^2}{4}+\frac{y^2}{25}=1$.	Square each bracket; this is the Cartesian equation.
Both terms are added, so the curve is an **ellipse** centred at the origin.	The + sign distinguishes an ellipse from a hyperbola.
Since $25>4$, the major axis is vertical: semi-major axis 5 along the y-axis, semi-minor axis 2 along the x-axis.	The larger denominator lies under y^2, so the major axis is vertical.

Example 3 — unscaffolded

The point P is described by $x=3\sec t$ and $y=2\tan t$. Show that P lies on a hyperbola, write down the hyperbola's equation, and state its vertices, asymptotes and foci.

From the parametric equations, $\sec t=\frac{x}{3}$ and $\tan t=\frac{y}{2}$. Dividing $\cos^2 t+\sin^2 t=1$ by $\cos^2 t$ gives the identity $\sec^2 t-\tan^2 t=1$, and substituting,

$$\left(\frac{x}{3}\right)^2-\left(\frac{y}{2}\right)^2=1\text{, that is }\frac{x^2}{9}-\frac{y^2}{4}=1.$$

This is a hyperbola centred at the origin with $a^2=9$ and $b^2=4$, so $a=3$ and $b=2$. The vertices are $(\pm 3,0)$ and the asymptotes are $y=\pm\frac{b}{a}x=\pm\frac{2}{3}x$. For the foci, $c^2=a^2+b^2=9+4=13$, so $c=\sqrt{13}$ and the foci are $(\pm\sqrt{13},0)$.

$\therefore$ P lies on $\frac{x^2}{9}-\frac{y^2}{4}=1$, with vertices $(\pm 3,0)$, asymptotes $y=\pm\frac{2}{3}x$ and foci $(\pm\sqrt{13},0)$.

Example 4 — unscaffolded

(a) Find the Cartesian coordinates of the point with polar coordinates $(4,2\pi/3)$. (b) Express the polar equation $r=6\cos\theta$ in Cartesian form and identify the curve.

(b) Using $x=r\cos\theta$ and $y=r\sin\theta$ with $r=4$ and $\theta=2\pi/3$,

$$x=4\cos\frac{2\pi}{3}=4(-1/2)=-2,\ y=4\sin\frac{2\pi}{3}=4(\sqrt{3}/2)=2\sqrt{3}.$$

So the point is $(-2, 2\sqrt{3})$.

(c) Multiplying $r = 6\cos\theta$ by r produces the convertible groups $r^2 = 6r\cos\theta$. Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, this becomes $x^2 + y^2 = 6x$. Completing the square in x,

$$x^2 - 6x + y^2 = 0 \Rightarrow (x-3)^2 + y^2 = 9.$$

This is a circle of radius 3 centred at $(3, 0)$ — a circle that passes through the pole.

$\therefore$ (a) $(-2, 2\sqrt{3})$; (b) $(x-3)^2 + y^2 = 9$, a circle of centre $(3, 0)$ and radius 3.

Practice questions

Scaffolded

Q1. For the hyperbola $\dfrac{x^2}{36} - \dfrac{y^2}{13} = 1$: (a) state a^2 and b^2, and hence a and b; (b) state the vertices; (c) find the equations of the asymptotes; (d) find c using $c^2 = a^2 + b^2$, and state the foci; (e) sketch the hyperbola, drawing the guide rectangle and asymptotes first and marking the vertices and foci.

Q2. A curve is given by $x = 4t^2$ and $y = 8t$. (a) Make t the subject of the equation for y. (b) Substitute this into $x = 4t^2$. (c) Simplify to a Cartesian equation of the form $y^2 = 4ax$. (d) Identify the curve and state the value of a.

Q3. A curve is given by $x = 3\cos t$ and $y = 3\sin t$. (a) Write $\cos t$ and $\sin t$ in terms of x and y. (b) State the identity connecting $\cos^2 t$ and $\sin^2 t$. (c) Substitute to eliminate t. (d) State the Cartesian equation and identify the curve.

Q4. Consider the rectangular hyperbola $xy = 36$. (a) Writing $xy = c^2$, state the value of c. (b) State the equations of the two asymptotes. (c) State the coordinates of the two vertices. (d) Find the value of y when $x = 4$, and confirm the point lies on the curve.

Q5. Consider the polar equation $r = 8\cos\theta$. (a) Multiply both sides by r. (b) Replace r^2 and $r\cos\theta$ by their Cartesian equivalents. (c) Complete the square in x. (d) State the centre and radius of the resulting circle.

Q6. A hyperbola centred at the origin has vertices $(\pm 5, 0)$ and one focus at $(7, 0)$. (a) State the value of a. (b) State the value of c. (c) Find b^2 using $b^2 = c^2 - a^2$. (d) Write the equation of the hyperbola and state its asymptotes.

Unscaffolded

Q7. For the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$, state the vertices, find the equations of the asymptotes, find the coordinates of the foci, and sketch the curve.

Q8. A curve is given by $x = 5\,sec\,t$ and $y = 4\tan t$. Eliminate the parameter to find its Cartesian equation, and state the asymptotes and the foci of the resulting hyperbola. State also which branch is traced when t is restricted to $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Q9. A curve is given by $x = 7\cos t$ and $y = 2\sin t$. Find its Cartesian equation, identify the curve, find the coordinates of its foci, and write its polar form. Sketch the curve, marking the points at which $t = 0$ and $t = \dfrac{\pi}{2}$.

Q10. For the rectangular hyperbola $xy = 25$, state the asymptotes, find the coordinates of the points where the line $y = x$ meets the curve, give a parametric representation of the curve, and sketch it.

Q11. Express the polar equation $r = 10\sin\theta$ in Cartesian form, identify the curve completely, and sketch it.

Q12. (a) Find the Cartesian coordinates of the point with polar coordinates $\left(6, \frac{5\pi}{6}\right)$. (b) A line is given parametrically by $x = 1 + 2t$ and $y = -2 + 3t$. Eliminate the parameter to find its Cartesian equation, and hence express the line in polar form.

Q13. For the hyperbola $\frac{(x-2)^2}{16} - \frac{(y+1)^2}{9} = 1$, state the centre and the vertices, find the equations of the asymptotes, and find the coordinates of the foci.

Q14. A hyperbola centred at the origin has vertices $(0, \pm 4)$ and asymptotes $y = \pm 2x$. Find its equation and the coordinates of its foci.

Q15. A curve is given by $x = 3t^2$ and $y = 6t$. Eliminate the parameter, identify the curve, state its vertex, focus and directrix, and write its polar form.

Q16. A curve is given by $x = 7t$ and $y = \frac{7}{t}$ for $t \neq 0$. Eliminate the parameter and identify the curve, state its asymptotes and vertices, and find the point on the curve where $t = 7$.

Q17. Express the polar equation $r = 2\cos\theta + 4\sin\theta$ in Cartesian form, and identify the curve completely.

Q18. The upper half of the right-hand branch of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a, b > 0)$ is $y = \frac{b}{a}\sqrt{x^2 - a^2}$. (a) By rationalising, show that the vertical gap between the line $y = \frac{b}{a}x$ and this curve equals $\frac{ab}{x + \sqrt{x^2 - a^2}}$, and hence that it tends to 0 as $x \to \infty$. (b) State the asymptotes of the hyperbola, and apply your result to $\frac{x^2}{16} - \frac{y^2}{9} = 1$, giving its asymptotes and foci. (c) Show that the polar form of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is $r^2\left(\frac{\cos^2\theta}{16} - \frac{\sin^2\theta}{9}\right) = 1$, and hence state the directions θ in which the curve has no point at all. Compare your answer with (b).

Answers

Q1. (a) $a^2=36, b^2=13, a=6, b=\sqrt{13}$; (b) $(\pm 6, 0)$; (c) $y=\pm\frac{\sqrt{13}}{6}x$; (d) $c=7$, foci $(\pm 7, 0)$; (e) guide rectangle $x=\pm 6$, $y=\pm\sqrt{13}$, asymptotes along its diagonals, branches opening left and right from $(\pm 6, 0)$ with the foci $(\pm 7, 0)$ just beyond the vertices. **Q2.** (a) $t=\frac{y}{8}$; (c) $y^2=16x$; (d) a parabola with $a=4$ (vertex at the origin, opening right). **Q3.** (c)/(d) $x^2+y^2=9$, a circle centred at the origin of radius 3. **Q4.** (a) $c=6$; (b) $x=0$ and $y=0$; (c) $(6, 6)$ and $(-6, -6)$; (d) $y=9$, and $4\times 9=36$, so $(4, 9)$ lies on the curve. **Q5.** (a) $r^2=8r\cos\theta$; (b) $x^2+y^2=8x$; (c) $(x-4)^2+y^2=16$; (d) centre $(4, 0)$, radius 4. **Q6.** (a) $a=5$; (b) $c=7$; (c) $b^2=24$; (d) $\frac{x^2}{25}-\frac{y^2}{24}=1$, asymptotes $y=\pm\frac{2\sqrt{6}}{5}x$. **Q7.** vertices $(\pm 3, 0)$; asymptotes $y=\pm\frac{\sqrt{7}}{3}x$; foci $(\pm 4, 0)$; sketch: guide rectangle $x=\pm 3$, $y=\pm\sqrt{7}$, asymptotes along its diagonals, a branch opening from each vertex, with the foci $(\pm 4, 0)$ just beyond the vertices. **Q8.** $\frac{x^2}{25}-\frac{y^2}{16}=1$; asymptotes $y=\pm\frac{4}{5}x$; foci $(\pm\sqrt{41}, 0)$; on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ only the right-hand branch is traced. **Q9.** $\frac{x^2}{49}+\frac{y^2}{4}=1$, an ellipse; foci $(\pm 3\sqrt{5}, 0)$; polar form $r^2\left(\frac{\cos^2\theta}{49}+\frac{\sin^2\theta}{4}\right)=1$; sketch: ellipse centred at the origin through $(\pm 7, 0)$ and $(0, \pm 2)$, with $t=0$ at $(7, 0)$ and $t=\frac{\pi}{2}$ at $(0, 2)$. **Q10.** asymptotes $x=0$ and $y=0$; $y=x$ meets the curve at $(5, 5)$ and $(-5, -5)$; e.g. $x=5t, y=\frac{5}{t}$; sketch: one branch in the first quadrant through $(5, 5)$, one in the third through $(-5, -5)$, each approaching both axes. **Q11.** $x^2+(y-5)^2=25$; a circle of centre $(0, 5)$ and radius 5 (through the pole); sketch: circle sitting on the x-axis, touching it at the origin and reaching $(0, 10)$, $(\pm 5, 5)$. **Q12.** (a) $(-3\sqrt{3}, 3)$; (b) $3x-2y=7$, in polar form $r(3\cos\theta-2\sin\theta)=7$. **Q13.** centre $(2, -1)$; vertices $(6, -1)$ and $(-2, -1)$; asymptotes $y+1=\pm\frac{3}{4}(x-2)$; foci $(7, -1)$ and $(-3, -1)$. **Q14.** $\frac{y^2}{16}-\frac{x^2}{4}=1$; foci $(0, \pm 2\sqrt{5})$. **Q15.** $y^2=12x$, a parabola; vertex $(0, 0)$, focus $(3, 0)$, directrix $x=-3$; polar form $r\sin^2\theta=12\cos\theta$, i.e. $r=\frac{12\cos\theta}{\sin^2\theta}$ for $\theta\neq 0, \pi$. **Q16.** $xy=49$, a rectangular hyperbola; asymptotes $x=0$ and $y=0$; vertices $(7, 7)$ and $(-7, -7)$; at $t=7$ the point is $(49, 1)$. **Q17.** $(x-1)^2+(y-2)^2=5$; a circle of centre $(1, 2)$ and radius $\sqrt{5}$. **Q18.** (a) the gap is $\frac{ab}{x+\sqrt{x^2-a^2}}\to 0$ as $x\to\infty$; (b) asymptotes $y=\pm\frac{b}{a}x$; for $\frac{x^2}{16}-\frac{y^2}{9}=1$, asymptotes $y=\pm\frac{3}{4}x$ and foci $(\pm 5, 0)$; (c) no point when $|\tan\theta|\geq\frac{3}{4}$ (including $\theta=\pm\frac{\pi}{2}$) — the directions of the asymptotes of (b) and every steeper direction.

Fully-worked solutions

Q1. The denominators give $a^2=36$ and $b^2=13$, so $a=6$ and $b=\sqrt{13}$. The curve is centred at the origin; setting $y=0$ gives $x^2=36$, so the vertices are $(\pm 6, 0)$. The asymptotes are $y=\pm\frac{b}{a}x=\pm\frac{\sqrt{13}}{6}x$. For the foci, $c^2=a^2+b^2=36+13=49$, so $c=7$ and the foci are $(\pm 7, 0)$. For the sketch, draw the guide rectangle bounded by $x=\pm 6$ and $y=\pm\sqrt{13}\approx\pm 3.6$ and extend its diagonals: these are the asymptotes. Mark the vertices $(\pm 6, 0)$ and the foci $(\pm 7, 0)$, then draw one branch out from each vertex, opening away from the centre and closing on the asymptotes without meeting them.

Q2. (a) From $y=8t$, $t=\frac{y}{8}$. (b) Substituting into $x=4t^2$, $x=4\left(\frac{y}{8}\right)^2=4\cdot\frac{y^2}{64}=\frac{y^2}{16}$. (c) Rearranging, $y^2=16x$. (d) This is the standard rightward parabola $y^2=4ax$ with $4a=16$, so $a=4$; its vertex is at the origin and it opens to the right.

Q3. (a) From the equations, $\cos t=\frac{x}{3}$ and $\sin t=\frac{y}{3}$. (b) The identity is $\cos^2 t+\sin^2 t=1$. (c) Substituting, $\left(\frac{x}{3}\right)^2+\left(\frac{y}{3}\right)^2=1$. (d) Multiplying out, $\frac{x^2}{9}+\frac{y^2}{9}=1$, i.e. $x^2+y^2=9$: a circle centred at the origin of radius 3 (as expected, since $r=3$ here).

Q4. (a) Since $xy=36=6^2$, $c=6$. (b) The asymptotes of $xy=c^2$ are the coordinate axes $x=0$ and $y=0$. (c) The vertices lie on $y=x$, where $x^2=36$ gives $x=\pm 6$, so they are $(6,6)$ and $(-6,-6)$. (d) When $x=4$, $y=\frac{36}{4}=9$; and $4\times 9=36$ confirms $(4,9)$ lies on the curve.

Q5. (a) Multiplying by r, $r^2=8r\cos\theta$. (b) Using $r^2=x^2+y^2$ and $r\cos\theta=x$, $x^2+y^2=8x$. (c) Completing the square, $x^2-8x+y^2=0$ becomes $(x-4)^2-16+y^2=0$, so $(x-4)^2+y^2=16$. (d) This is a circle with centre $(4,0)$ and radius $\sqrt{16}=4$.

Q6. (a) The vertices $(\pm 5,0)$ give $a=5$. (b) The focus $(7,0)$ gives $c=7$. (c) From $b^2=c^2-a^2=49-25=24$. (d) With $a^2=25$ and $b^2=24$ the equation is $\frac{x^2}{25}-\frac{y^2}{24}=1$, and the asymptotes are $y=\pm\frac{b}{a}x=\pm\frac{\sqrt{24}}{5}x=\pm\frac{2\sqrt{6}}{5}x$ (since $\sqrt{24}=2\sqrt{6}$).

Q7. Here $a^2=9$ and $b^2=7$, so $a=3$ and $b=\sqrt{7}$. The vertices are $(\pm 3,0)$ and the asymptotes are $y=\pm\frac{\sqrt{7}}{3}x$. From $c^2=a^2+b^2=9+7=16$, $c=4$, so the foci are $(\pm 4,0)$. To sketch it, draw the guide rectangle bounded by $x=\pm 3$ and $y=\pm\sqrt{7}\approx\pm 2.6$, extend its diagonals to get the asymptotes, mark the vertices $(\pm 3,0)$ and the foci $(\pm 4,0)$, then draw the two branches opening left and right and closing on the asymptotes.

Q8. From $x=5\sec t$ and $y=4\tan t$, $\sec t=\frac{x}{5}$ and $\tan t=\frac{y}{4}$. Substituting into $\sec^2 t-\tan^2 t=1$ gives $\frac{x^2}{25}-\frac{y^2}{16}=1$, a hyperbola with $a=5$ and $b=4$. Its asymptotes are $y=\pm\frac{b}{a}x=\pm\frac{4}{5}x$. From $c^2=a^2+b^2=25+16=41$, $c=\sqrt{41}$, so the foci are $(\pm\sqrt{41},0)$. On $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ the cosine is positive and at most 1, so $\sec t\geq 1$ and $x=5\sec t\geq 5$: the parametric point never reaches the left-hand branch, and only the **right-hand branch** is traced.

Q9. From $x=7\cos t$ and $y=2\sin t$, $\cos t=\frac{x}{7}$ and $\sin t=\frac{y}{2}$; substituting into $\cos^2 t+\sin^2 t=1$ gives $\frac{x^2}{49}+\frac{y^2}{4}=1$, an ellipse. As $49>4$ the major axis is horizontal with $a^2=49$, $b^2=4$; for an ellipse $c^2=a^2-b^2=49-4=45$, so $c=\sqrt{45}=3\sqrt{5}$ and the foci are $(\pm 3\sqrt{5},0)$. (Note the ellipse uses a *difference* here, unlike the hyperbola in Q8.) For the polar form, substitute $x=r\cos\theta$ and $y=r\sin\theta$ into the Cartesian equation:

$$r^2\left(\frac{\cos^2\theta}{49}+\frac{\sin^2\theta}{4}\right)=1.$$

The sketch is an ellipse centred at the origin, cutting the axes at $(\pm 7,0)$ and $(0,\pm 2)$; at $t=0$ the point is $(7,0)$ and at $t=\frac{\pi}{2}$ it is $(0,2)$, so the curve is traced anticlockwise.

Q10. The equation $xy=25$ is a rectangular hyperbola of the form $xy=c^2$ with $c=5$; its asymptotes are the coordinate axes $x=0$ and $y=0$. Setting $y=x$ gives $x^2=25$, so $x=\pm 5$ and the line meets the curve at the vertices $(5,5)$ and $(-5,-5)$. A convenient parametric representation is $x=5t$, $y=\frac{5}{t}$ ($t\neq 0$), since then $xy=5t\cdot\frac{5}{t}=25$. The sketch has one branch in the first quadrant through $(5,5)$ and one in the third through $(-5,-5)$, each running away to both axes without meeting them.

Q11. Multiplying $r=10\sin\theta$ by r gives $r^2=10r\sin\theta$. Using $r^2=x^2+y^2$ and $r\sin\theta=y$, this becomes $x^2+y^2=10y$. Completing the square, $x^2+y^2-10y=0$ gives $x^2+(y-5)^2=25$: a circle of centre $(0,5)$ and radius 5, passing through the pole. The sketch is that circle sitting on the x-axis: it touches the axis at the origin and passes through $(\pm 5,5)$ and $(0,10)$.

Q12. (a) With $r=6$ and $\theta=\frac{5\pi}{6}$,

$$x=6\cos\frac{5\pi}{6}=6\left(-\sqrt{3}/2\right)=-3\sqrt{3},\ y=6\sin\frac{5\pi}{6}=6(1/2)=3,$$

so the point is $(-3\sqrt{3},3)$ — in the second quadrant, as $\frac{5\pi}{6}$ requires. (b) From $x=1+2t$, $t=\frac{x-1}{2}$. Substituting into $y=-2+3t$,

$$y=-2+\frac{3(x-1)}{2}\Rightarrow 2y=-4+3x-3\Rightarrow 3x-2y=7,$$

a straight line of gradient $\frac{3}{2}$ through $(1,-2)$ (the point at $t=0$). Replacing x by $r\cos\theta$ and y by $r\sin\theta$ gives $3r\cos\theta-2r\sin\theta=7$, that is $r(3\cos\theta-2\sin\theta)=7$.

Q13. The equation is a translated hyperbola with centre $(2,-1)$, and under the shift $a^2=16$, $b^2=9$, so $a=4$ and $b=3$. The vertices are $a=4$ units left and right of the centre: $(2\pm 4,-1)$, i.e. $(6,-1)$ and $(-2,-1)$. The asymptotes pass through the centre with gradients $\pm\frac{b}{a}=\pm\frac{3}{4}$: $y+1=\pm\frac{3}{4}(x-2)$. From $c^2=a^2+b^2=16+9=25$, $c=5$, so the foci are 5 units left and right of the centre: $(2\pm 5,-1)$, i.e. $(7,-1)$ and $(-3,-1)$.

Q14. The vertices lie on the y-axis, so the y-term is the positive one and the equation has the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, whose asymptotes are $y=\pm\frac{a}{b}x$. The vertices $(0,\pm 4)$ give $a=4$, and matching $\frac{a}{b}=2$ gives $b=\frac{4}{2}=2$. Hence $a^2=16$ and $b^2=4$, and the equation is $\frac{y^2}{16}-\frac{x^2}{4}=1$. From $c^2=a^2+b^2=16+4=20$, $c=\sqrt{20}=2\sqrt{5}$, and for this orientation the foci lie on the y-axis: $(0,\pm 2\sqrt{5})$.

Q15. From $y=6t$, $t=\frac{y}{6}$; substituting into $x=3t^2$, $x=3\left(\frac{y}{6}\right)^2=3\cdot\frac{y^2}{36}=\frac{y^2}{12}$, so $y^2=12x$. This is a rightward parabola $y^2=4ax$ with $4a=12$, so $a=3$: vertex $(0,0)$, focus $(3,0)$ and directrix $x=-3$. For the polar form, substituting $x=r\cos\theta$ and $y=r\sin\theta$ into $y^2=12x$ gives $r^2\sin^2\theta=12r\cos\theta$; dividing by r (the vertex is the pole, where $r=0$) leaves $r\sin^2\theta=12\cos\theta$, that is $r=\frac{12\cos\theta}{\sin^2\theta}$ for $\theta\neq 0,\pi$.

Q16. Multiplying the parametric equations, $xy=7t\cdot\frac{7}{t}=49$, so the curve is the rectangular hyperbola $xy=49$ (of the form $xy=c^2$ with $c=7$). Its asymptotes are the coordinate axes $x=0$ and $y=0$; its vertices, on $y=x$, satisfy

$x^2=49$, giving $(7,7)$ and $(-7,-7)$. When $t=7$, $x=7\times7=49$ and $y=\frac{7}{7}=1$, so the point is $(49,1)$ (and $49\times1=49$ confirms it lies on the curve).

Q17. Multiplying $r=2\cos\theta+4\sin\theta$ by r gives $r^2=2r\cos\theta+4r\sin\theta$. Using $r^2=x^2+y^2$, $r\cos\theta=x$ and $r\sin\theta=y$, this is $x^2+y^2=2x+4y$. Completing the square in each variable, $(x^2-2x)+(y^2-4y)=0$ becomes $(x-1)^2-1+(y-2)^2-4=0$, so $(x-1)^2+(y-2)^2=5$: a circle of centre $(1,2)$ and radius $\sqrt{5}$.

Q18. (a) The vertical gap between $y=\frac{b}{a}x$ and $y=\frac{b}{a}\sqrt{x^2-a^2}$ is

$$\frac{b}{a}x-\frac{b}{a}\sqrt{x^2-a^2}=\frac{b}{a}\left(x-\sqrt{x^2-a^2}\right)=\frac{b}{a}\cdot\frac{\left(x-\sqrt{x^2-a^2}\right)\left(x+\sqrt{x^2-a^2}\right)}{x+\sqrt{x^2-a^2}}=\frac{b}{a}\cdot\frac{a^2}{x+\sqrt{x^2-a^2}}=\frac{ab}{x+\sqrt{x^2-a^2}},$$

using $\left(x-\sqrt{x^2-a^2}\right)\left(x+\sqrt{x^2-a^2}\right)=x^2-(x^2-a^2)=a^2$. As $x\to\infty$ the denominator increases without bound, so the gap tends to 0; the branch approaches the line as required. (b) By the symmetry of the curve the same argument applies below the axis and on the left branch, so the asymptotes are $y=\pm\frac{b}{a}x$. For $\frac{x^2}{16}-\frac{y^2}{9}=1$, $a=4$ and $b=3$, so the asymptotes are $y=\pm\frac{3}{4}x$; and $c^2=a^2+b^2=16+9=25$ gives $c=5$, so the foci are $(\pm5,0)$. (c) Substituting $x=r\cos\theta$ and $y=r\sin\theta$ into $\frac{x^2}{16}-\frac{y^2}{9}=1$ gives directly

$$r^2\left(\frac{\cos^2\theta}{16}-\frac{\sin^2\theta}{9}\right)=1.$$

Since $r^2>0$, a point exists in the direction θ only if the bracket is positive, that is if $\frac{\cos^2\theta}{16}>\frac{\sin^2\theta}{9}$. Dividing by $\cos^2\theta$ (which is non-zero in that case) gives $\tan^2\theta<\frac{9}{16}$, so the curve has a point in the direction θ exactly when $|\tan\theta|<\frac{3}{4}$, and **no** point when $|\tan\theta|\geq\frac{3}{4}$ — including $\theta=\pm\frac{\pi}{2}$, where the bracket is $-\frac{1}{9}$. Those excluded directions are exactly the directions of the asymptotes $y=\pm\frac{3}{4}x$ found in (b) and every steeper direction: the branches stay strictly inside the pair of asymptotes, which is the same fact that part (a) proved by a limiting argument.

Theory

Every real number has a **size** independent of its sign: -8 and 8 lie the same distance from 0 on the number line. The **absolute value** (or **modulus**) function records exactly this size, discarding the sign while keeping the magnitude. It is the first function met in this course whose rule changes according to a condition, and its graph — a sharp **V** rather than a smooth curve — introduces the idea of a corner where two straight pieces meet. Because "how far apart" is measured by absolute value, the function is also the natural language for distance on the number line, and hence for a whole family of equations and inequalities.

Definition.

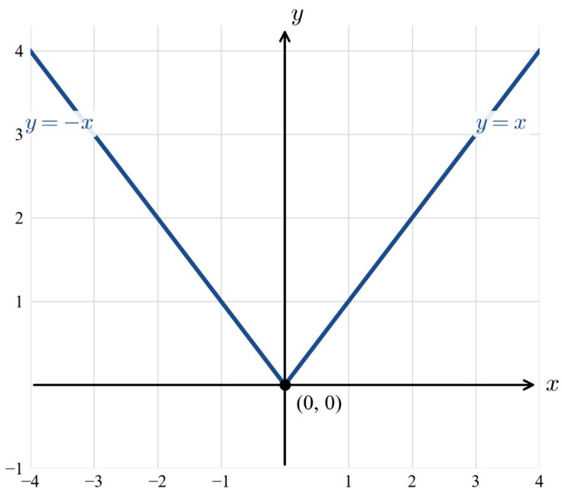
The **absolute value** of a real number x, written $|x|$, is defined by the two-part (piecewise) rule

$$|x|=\begin{cases} x & \text{if } x\geq 0, \\ -x & \text{if } x<0. \end{cases}$$

When x is already non-negative it is returned unchanged; when x is negative the rule returns $-x$, which is then positive. Thus $|8|=8$, while $|-8|=-(-8)=8$. The output is never negative. Geometrically, $|x|$ is the **distance from x to 0** on the number line, and distance carries no sign.

The graph of $y=|x|$.

Splitting the rule at $x=0$ gives two half-lines: for $x\geq 0$ the graph is $y=x$ (gradient $+1$), and for $x<0$ it is $y=-x$ (gradient -1). They meet at the origin, producing the characteristic **V-shape** with a sharp **vertex** (corner) at $(0,0)$ and arms rising symmetrically on either side.



The graph is symmetric about the y-axis, because reflecting x to $-x$ leaves $|x|$ unchanged. Its lowest point is the vertex, where $y=0$; every other value of y is positive. The single equation $y=|x|$ therefore captures both half-lines at once.

Algebraic properties.

For all real x and y the following hold, each read directly from the definition:

- **Non-negativity:** $|x|\geq 0$, and $|x|=0$ **only** when $x=0$.
- **Symmetry:** $|-x|=|x|$; the absolute value is an *even* function.
- **Square identity:** $|x|^2=x^2$, and consequently $|x|=\sqrt{x^2}$, where the radical sign denotes the *non-negative* square root. This is often the cleanest way to remove a modulus: squaring is reversible precisely because both sides are non-negative.

- **Multiplicative:** $|xy| = |x||y|$, and for $y \neq 0$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

A further consequence, the **triangle inequality**, states that

$$|x+y| \leq |x| + |y|.$$

It is proved by comparing squares: since both sides are non-negative,

$$(|x|+|y|)^2 - |x+y|^2 = (x^2 + 2|x||y| + y^2) - (x^2 + 2xy + y^2) = 2(|xy| - xy) \geq 0,$$

using $|xy| \geq xy$. Equality holds exactly when $xy = |xy|$, that is when x and y have the same sign or one of them is zero.

Distance on the number line.

Because $|x|$ is the distance from x to 0, replacing x by $x-a$ measures distance from a shifted centre: for any real a and b,

$$|x-a| \text{ is the distance between the points } x \text{ and } a.$$

In particular $|a-b| = |b-a|$ is the distance between a and b, so the order of the two numbers does not matter. This single interpretation drives every absolute-value equation and inequality below: "$|x-a| = b$" says x is a distance b from a, and "$|x-a| < b$" says x is *within* b of a.

Transformations: $y = a|x-h|+k$.

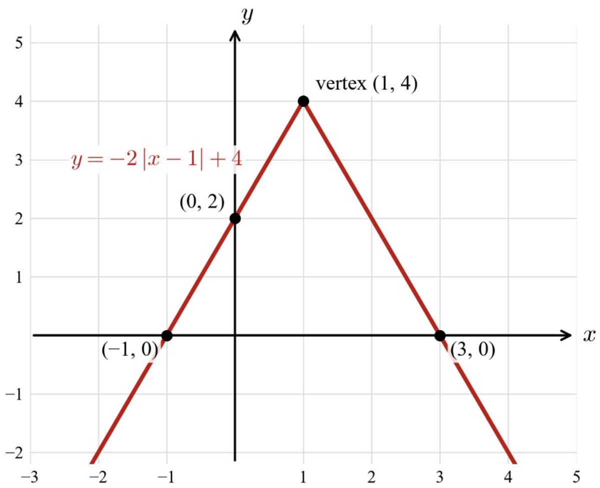
The base graph $y = |x|$ is transformed exactly as any function is. Writing

$$y = a|x-h|+k,$$

the constants act independently:

- h translates the graph **horizontally** by h (right if $h > 0$), and k translates it **vertically** by k, so the **vertex moves to** (h, k);
- a **dilates** the arms: their gradients become $\pm a$, so $|a| > 1$ makes the V narrower and $0 < |a| < 1$ makes it wider;
- the **sign of** a sets the opening — $a > 0$ opens **upward** (vertex a minimum), $a < 0$ opens **downward** (vertex a maximum), the downward case being a reflection in the horizontal line through the vertex.

To sketch $y = a|x-h|+k$: plot the vertex (h, k), decide the direction and steepness from a, then find the intercepts. The diagram below shows $y = -2|x-1|+4$: here $a = -2$ (opening downward, arms of gradient ± 2) and $(h, k) = (1, 4)$, giving vertex $(1, 4)$, x-intercepts $(-1, 0)$ and $(3, 0)$, and y-intercept $(0, 2)$.



The graph of $y = |f(x)|$.

For any function f, the graph of $y=|f(x)|$ is obtained from the graph of $y=f(x)$ by a single rule: **wherever $f(x)$ is negative, reflect that part up in the x-axis**; wherever $f(x)\geq 0$, leave it unchanged. This is immediate from the definition, since $|f(x)|=f(x)$ when $f(x)\geq 0$ and $|f(x)|=-f(x)$ when $f(x)<0$. A **corner** appears at each point where the graph of f *crosses* the x-axis with non-zero gradient, because there a falling piece is folded into a rising one; a zero at which the graph merely *touches* the axis without changing sign — such as $x=0$ for $f(x)=x^2$ — leaves the graph smooth, since nothing on either side of it is reflected. No part of $y=|f(x)|$ ever lies below the x-axis. For a *linear* f whose graph crosses the axis this reflection simply reproduces a V; for a curved f such as $f(x)=x^2-4$ the dipping portion is flipped up into a hump, with corners at the crossings $x=-2$ and $x=2$.

Solving equations involving absolute values.

The key fact is that an absolute value is never negative, so $|X|=c$ can be solved by inspection of c:

$$|X|=c \Rightarrow \begin{cases} X=c \text{ or } X=-c & \text{if } c>0, \\ X=0 & \text{if } c=0, \\ \text{no solution} & \text{if } c<0. \end{cases}$$

Applied to $|x-a|=b$ (with $b>0$) this gives $x=a\pm b$ at once — the two points a distance b from a; when $b=0$ these coalesce into the single point $x=a$. An equation $|f(x)|=g(x)$ is solved by **cases** on the sign of $f(x)$: solve $f(x)=g(x)$ and $f(x)=-g(x)$, then **discard any solution for which the right-hand side $g(x)$ is negative** (an absolute value cannot equal a negative number). Every candidate should be checked in the original equation. Equivalently, the solutions are the x-coordinates of the intersections of $y=|f(x)|$ and $y=g(x)$ — the **graphical method** — which also reveals *how many* solutions to expect.

Solving inequalities involving absolute values.

The distance picture reads off the two standard forms directly. For $c>0$,

$$|X|<c \Leftrightarrow -c<X<c, \quad |X|>c \Leftrightarrow X<-c \text{ or } X>c,$$

with ≤ and ≥ giving closed endpoints in the obvious way. A "<" (or "≤") statement collapses to a **single interval** centred on the solution of the inside; a ">" (or "≥") statement opens out to **two rays**. Thus $|x-a|<b$ gives the interval $a-b<x<a+b$, and $|x-a|\geq b$ gives $x\leq a-b$ or $x\geq a+b$. Two edge cases follow from non-negativity: if $c<0$ then $|X|<c$ has *no* solution while $|X|>c$ holds for *all* real x; and $|X|\geq 0$ is always true.

Worked examples

Example 1 — scaffolded

Evaluate, simplifying inside the bars first: (a) $|-11|$ and $|7|-|-4|$; (b) $|3-10|$ and $|(-2)(9)|$, and confirm $|xy|=|x||y|$; (c) $\sqrt{(-6)^2}$.

Working	Comment
(a) $|-11|=11$, and $|7|-|-4|=7-4=3$.	Absolute value strips the sign of each number; then combine.
(b) $|3-10|=|-7|=7$.	Always simplify the expression *inside* the bars first.
(b) $|(-2)(9)|=|-18|=18$, and $|-2||9|=2\times 9=18$.	The two agree, illustrating $|xy|=|x||y|$.
(c) $\sqrt{(-6)^2}=\sqrt{36}=6=|-6|$.	The identity $|x|=\sqrt{x^2}$: the root is the non-negative value.

Example 2 — scaffolded

Sketch $y=3|x+2|-6$, stating the vertex, the direction of opening, and all axis intercepts.

Working	Comment
Write in standard form: $y=a\|x-h\|+k$ with $a=3$, $h=-2$, $k=-6$.	Note $x+2=x-(-2)$, so $h=-2$.
Vertex at $(h,k)=(-2,-6)$.	The vertex sits where the inside, $x+2$, is zero.
$a=3>0$: opens **upward**, arms of gradient ± 3 (steeper than $y=\|x\|$).	Sign of a gives direction; $\|a\|=3>1$ narrows the V.
x-intercepts: $3\|x+2\|-6=0 \Rightarrow \|x+2\|=2 \Rightarrow x+2=\pm 2$, so $x=0$ or $x=-4$.	Set $y=0$; solve the resulting absolute-value equation.
y-intercept: $x=0$ gives $y=3\|2\|-6=0$, the point $(0,0)$.	Substitute $x=0$. The graph is an upward V with vertex $(-2,-6)$ through $(-4,0)$ and $(0,0)$.

Example 3 — unscaffolded

Solve $|2x-4|=x+1$ algebraically, and confirm the result graphically.

Since the left-hand side is an absolute value, split on the sign of the inside, $2x-4$.

If $2x-4\geq 0$ (that is, $x\geq 2$), the bars come off unchanged:

$$2x-4=x+1 \Rightarrow x=5.$$

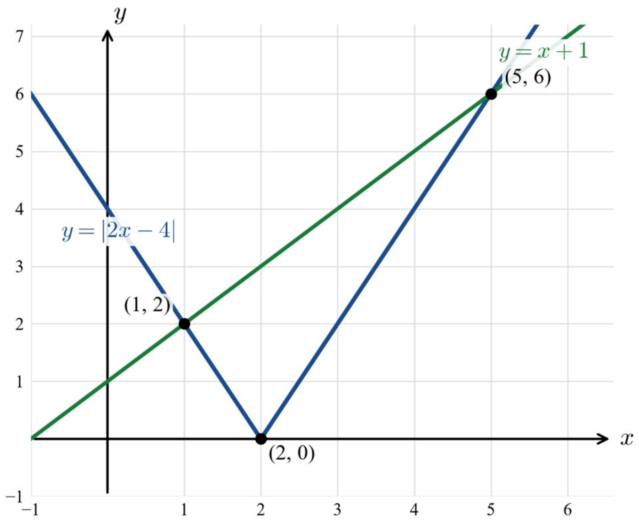
Here $x=5$ satisfies $x\geq 2$, and the right-hand side $x+1=6\geq 0$, so it is valid.

If $2x-4<0$ (that is, $x<2$), the bars introduce a sign change:

$$-(2x-4)=x+1 \Rightarrow -2x+4=x+1 \Rightarrow -3x=-3 \Rightarrow x=1.$$

Here $x=1$ satisfies $x<2$, and $x+1=2\geq 0$, so it too is valid.

Graphically, the solutions are where $y=|2x-4|$ (a V with vertex $(2,0)$) meets the line $y=x+1$; the two graphs cross at $(1,2)$ and $(5,6)$, confirming both values.



$\therefore x=1$ or $x=5$.

Example 4 — unscaffolded

Solve the inequality $|2x+3|\geq 5$, and describe the solution set on a number line.

The statement $|2x+3|\geq 5$ says the quantity $2x+3$ is at least 5 units from 0, so it must be ≥ 5 or ≤ -5. These two cases give

$$2x+3\geq 5 \Rightarrow 2x\geq 2 \Rightarrow x\geq 1,$$

$$2x+3 \le -5 \Rightarrow 2x \le -8 \Rightarrow x \le -4.$$

Because the original is a "≥" statement, the solution opens out into two rays rather than a single interval. On the number line this is everything from -4 leftward together with everything from 1 rightward, with both endpoints filled (closed).

$\therefore x \le -4$ or $x \ge 1$; that is, $x \in (-\infty, -4] \cup [1, \infty)$.

Practice questions

Scaffolded

Q1. Evaluate each, simplifying inside the bars first. (a) $|-13|$. (b) $|5-12|$. (c) $|-4||-7|$, and compare with $|(-4)(-7)|$. (d) $\sqrt{(-9)^2}$.

Q2. Consider $y=|x-3|$. (a) Write the rule as a piecewise (two-part) definition. (b) State the coordinates of the vertex. (c) Find the x- and y-intercepts. (d) Describe the shape and the gradients of the two arms.

Q3. Consider $y=-|x+1|+5$. (a) State the vertex. (b) State the direction of opening and the effect of the coefficient $a=-1$. (c) Find the x-intercepts. (d) Find the y-intercept.

Q4. Solve for x. (a) $|x-4|=6$. (b) $|2x+1|=9$. (c) $|3x-5|=0$. (d) $|x+2|=-3$.

Q5. Solve each inequality, giving the answer as an interval or union of intervals. (a) $|x-2|<5$. (b) $|x+3|\le 1$. (c) $|2x-1|>7$. (d) $|x-6|\ge 0$.

Q6. Use the distance interpretation of absolute value. (a) Write "the distance from x to 4 is 3" as an equation and solve it. (b) Write "the distance from x to -2 is less than 5" as an inequality and solve it. (c) Interpret $|x-7|=|x-1|$ geometrically, and hence solve it.

Unscaffolded

Q7. Evaluate $|-3|+|-8|-|4-11|$.

Q8. Solve $|x-7|=2$.

Q9. Solve $|3x+4|=10$.

Q10. Solve $|5-2x|=3$.

Q11. For $y=2|x-3|-4$, state the vertex and find all axis intercepts.

Q12. For $y=-3|x+1|+6$, state the vertex, the direction of opening, and all axis intercepts.

Q13. Solve $|x+5|<4$.

Q14. Solve $|2x-7|\ge 3$.

Q15. Solve $|6-2x|>4$.

Q16. Solve $|x-1|=3x-5$, taking care to reject any extraneous solution.

Q17. Solve $|2x+1|=|x-4|$.

Q18. Consider $y=|x^2-4|$. (a) State the x-intercepts. (b) Find the coordinates of the highest point of the graph on the interval $-2\le x\le 2$. (c) State the range, and describe how the graph is obtained from that of $y=x^2-4$.

Answers

Q1. (a) 13; (b) 7; (c) 28 and $|(-4)(-7)|=28$ (equal); (d) 9. **Q2.** (a) $y=x-3$ for $x\geq 3$, $y=3-x$ for $x<3$; (b) $(3,0)$; (c) x-intercept $(3,0)$, y-intercept $(0,3)$; (d) an upward V with vertex $(3,0)$, arms of gradient ± 1. **Q3.** (a) $(-1,5)$; (b) opens downward (vertex a maximum), $a=-1$ reflects the base V in the horizontal through the vertex; (c) $(-6,0)$ and $(4,0)$; (d) $(0,4)$. **Q4.** (a) $x=10$ or $x=-2$; (b) $x=4$ or $x=-5$; (c) $x=5/3$; (d) no solution. **Q5.** (a) $-3<x<7$; (b) $-4\leq x\leq -2$; (c) $x<-3$ or $x>4$; (d) all real x. **Q6.** (a) $|x-4|=3$, so $x=1$ or $x=7$; (b) $|x+2|<5$, so $-7<x<3$; (c) x is equidistant from 7 and 1, so $x=4$ (the midpoint). **Q7.** 4. **Q8.** $x=9$ or $x=5$. **Q9.** $x=2$ or $x=-14/3$. **Q10.** $x=1$ or $x=4$. **Q11.** Vertex $(3,-4)$; x-intercepts $(1,0)$ and $(5,0)$; y-intercept $(0,2)$. **Q12.** Vertex $(-1,6)$; opens downward; x-intercepts $(-3,0)$ and $(1,0)$; y-intercept $(0,3)$. **Q13.** $-9<x<-1$. **Q14.** $x\leq 2$ or $x\geq 5$. **Q15.** $x<1$ or $x>5$. **Q16.** $x=2$ only (the value $x=3/2$ is extraneous). **Q17.** $x=-5$ or $x=1$. **Q18.** (a) $(-2,0)$ and $(2,0)$; (b) $(0,4)$; (c) range $[0,\infty)$; the dip of $y=x^2-4$ between $x=-2$ and $x=2$ is reflected up in the x-axis, the rest unchanged.

Fully-worked solutions

Q1. (a) $|-13|=13$. (b) $|5-12|=|-7|=7$. (c) $|-4||-7|=4\times 7=28$, while $|(-4)(-7)|=|28|=28$; the two agree, as $|x\,y|=|x||y|$ requires. (d) Using $|x|=\sqrt{x^2}$, $\sqrt{(-9)^2}=\sqrt{81}=9$.

Q2. (a) By the definition applied to $x-3$,

$$y=|x-3|=\begin{cases} x-3 & \text{if } x\geq 3, \\ 3-x & \text{if } x<3. \end{cases}$$

(b) The inside is zero at $x=3$, giving the vertex $(3,0)$. (c) Setting $y=0$ gives $|x-3|=0$, so $x=3$: the x-intercept is $(3,0)$ (the vertex itself). At $x=0$, $y=|-3|=3$, so the y-intercept is $(0,3)$. (d) The graph is an upward V with vertex $(3,0)$ and arms of gradient -1 (left) and $+1$ (right).

Q3. (a) In the form $y=a|x-h|+k$, $h=-1$ and $k=5$, so the vertex is $(-1,5)$. (b) Since $a=-1<0$ the graph opens downward, the vertex being a maximum; the negative sign reflects the base V in the horizontal line through the vertex, and $|a|=1$ leaves the arm gradients at ± 1. (c) Setting $y=0$: $-|x+1|+5=0$, so $|x+1|=5$, giving $x+1=\pm 5$ and $x=4$ or $x=-6$; the x-intercepts are $(4,0)$ and $(-6,0)$. (d) At $x=0$, $y=-|1|+5=4$, so the y-intercept is $(0,4)$.

Q4. (a) $|x-4|=6\Rightarrow x-4=\pm 6\Rightarrow x=10$ or $x=-2$. (b) $|2x+1|=9\Rightarrow 2x+1=\pm 9$; from $2x+1=9$, $x=4$, and from $2x+1=-9$, $x=-5$. (c) $|3x-5|=0\Rightarrow 3x-5=0\Rightarrow x=5/3$. (d) $|x+2|=-3$ has no solution, because an absolute value is never negative.

Q5. (a) $|x-2|<5\Rightarrow -5<x-2<5\Rightarrow -3<x<7$. (b) $|x+3|\leq 1\Rightarrow -1\leq x+3\leq 1\Rightarrow -4\leq x\leq -2$. (c) $|2x-1|>7\Rightarrow 2x-1>7$ or $2x-1<-7$, giving $x>4$ or $x<-3$. (d) $|x-6|\geq 0$ is true for every real x, since an absolute value is always non-negative; the solution is all real x.

Q6. (a) The distance from x to 4 is $|x-4|$, so the statement is $|x-4|=3$; then $x-4=\pm 3$ gives $x=1$ or $x=7$. (b) The distance from x to -2 is $|x-(-2)|=|x+2|$, so the statement is $|x+2|<5$; then $-5<x+2<5$ gives $-7<x<3$. (c) $|x-7|=|x-1|$ says x is the same distance from 7 as from 1, so x is their midpoint, $x=7+1/2=4$. (Algebraically, $x-7=-(x-1)$ gives $2x=8$, $x=4$; the case $x-7=x-1$ is impossible.)

Q7. $|-3|+|-8|-|4-11|=3+8-|-7|=3+8-7=4$.

Q8. $|x-7|=2\Rightarrow x-7=\pm 2$, so $x=9$ or $x=5$.

Q9. $|3x+4|=10\Rightarrow 3x+4=\pm 10$. From $3x+4=10$, $3x=6$, $x=2$; from $3x+4=-10$, $3x=-14$, $x=-14/3$.

Q10. $|5-2x|=3\Rightarrow 5-2x=\pm 3$. From $5-2x=3$, $-2x=-2$, $x=1$; from $5-2x=-3$, $-2x=-8$, $x=4$.

Q11. Here $a=2$, $h=3$, $k=-4$, so the vertex is $(3,-4)$ and the graph opens upward. Setting $y=0$: $2|x-3|=4$, so $|x-3|=2$ and $x-3=\pm 2$, giving $x=1$ or $x=5$; the x-intercepts are $(1,0)$ and $(5,0)$. At $x=0$, $y=2|-3|-4=6-4=2$, so the y-intercept is $(0,2)$.

Q12. Here $a=-3$, $h=-1$, $k=6$, so the vertex is $(-1,6)$ and, since $a<0$, the graph opens downward. Setting $y=0$: $-3|x+1|+6=0$, so $|x+1|=2$ and $x+1=\pm 2$, giving $x=1$ or $x=-3$; the x-intercepts are $(-3,0)$ and $(1,0)$. At $x=0$, $y=-3|1|+6=3$, so the y-intercept is $(0,3)$.

Q13. $|x+5|<4 \Rightarrow -4<x+5<4$; subtracting 5 throughout gives $-9<x<-1$.

Q14. $|2x-7|\geq 3 \Rightarrow 2x-7\geq 3$ or $2x-7\leq -3$. The first gives $2x\geq 10$, $x\geq 5$; the second gives $2x\leq 4$, $x\leq 2$. Hence $x\leq 2$ or $x\geq 5$.

Q15. $|6-2x|>4 \Rightarrow 6-2x>4$ or $6-2x<-4$. The first gives $-2x>-2$, so $x<1$ (the inequality reverses on dividing by -2); the second gives $-2x<-10$, so $x>5$. Hence $x<1$ or $x>5$.

Q16. Split on the sign of $x-1$. If $x\geq 1$: $x-1=3x-5 \Rightarrow -2x=-4 \Rightarrow x=2$; this satisfies $x\geq 1$ and the right-hand side $3(2)-5=1\geq 0$, so it is valid. If $x<1$: $-(x-1)=3x-5 \Rightarrow -x+1=3x-5 \Rightarrow -4x=-6 \Rightarrow x=3/2$; but $3/2$ does not satisfy $x<1$ (and the right-hand side there would be negative), so it is extraneous and rejected. Hence $x=2$ is the only solution. (Check: $|2-1|=1$ and $3(2)-5=1$.)

Q17. Both sides are absolute values, so $|2x+1|=|x-4|$ requires $2x+1=x-4$ or $2x+1=-(x-4)$. The first gives $x=-5$; the second gives $2x+1=-x+4$, so $3x=3$ and $x=1$. Both check: at $x=-5$, $|-9|=9=|-9|$; at $x=1$, $|3|=3=|-3|$. Hence $x=-5$ or $x=1$.

Q18. (a) The x-intercepts occur where $|x^2-4|=0$, i.e. $x^2-4=0$, so $x=\pm 2$: the points $(-2,0)$ and $(2,0)$. (b) On $-2\leq x\leq 2$ the inside x^2-4 is negative (or zero), so $|x^2-4|=4-x^2$, a downward parabola with maximum at $x=0$; there $y=4-0=4$, giving the highest point $(0,4)$. (c) An absolute value is never negative, and on the outer arms x^2-4 takes every non-negative value, so the range is $[0,\infty)$. The graph of $y=x^2-4$ dips below the x-axis between $x=-2$ and $x=2$; reflecting that dip up in the x-axis (and leaving the two outer arms, where $x^2-4\geq 0$, unchanged) turns the trough into a hump peaking at $(0,4)$.

Topic 2 Test A — Technology-active

One bound reference and one approved technology (CAS calculator or software) are permitted. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

A rooftop-solar maintenance crew inspects one building each visit. Let X be the number of panels flagged for cleaning on a visit, with the probability distribution below.

x	1	2	3	4	5
$p(x)$	0.24	0.31	0.22	0.15	0.08

a. Find $E(X)$ and $Var(X)$. 2 marks

b. The crew's time on site is modelled as $T = 8X + 15$ minutes. Find $E(T)$, and find $sd(T)$ correct to two decimal places. 2 marks

c. On an independent contract a second crew flags Y panels per visit, where Y has mean 3.4 and standard deviation 1.8. The combined number flagged by the two crews on a day is $W = X + Y$. Find $E(W)$, and find $sd(W)$ correct to two decimal places. 2 marks

d. The first crew visits n buildings, and $\acute{X}$ is the mean number of panels flagged per building. Find the smallest value of n for which $sd(\acute{X}) < 0.35$. 1 mark

Question 2 (7 marks)

Three survey pegs A, B and C are set out on level parkland, with $AB = 88$ m, $BC = 57$ m and the included angle $\angle ABC = 79°$.

a. Find the distance AC, correct to two decimal places. 2 marks

b. Find the angle $\angle BAC$, correct to the nearest degree. 2 marks

c. Find the area of triangle ABC, correct to the nearest square metre. 1 mark

d. A vertical lighting mast of height 15 m stands at C. From A, find the angle of elevation of the top of the mast, correct to the nearest degree. 2 marks

Question 3 (7 marks)

The transformation A is the dilation of factor 9 parallel to the x-axis, and B is the reflection in the line $y = -x$. The transformation T is "A followed by B", with matrix $M = BA$.

a. Write down the matrices A and B. 1 mark

b. Show that $M = \begin{bmatrix} 0 & -1 \\ -9 & 0 \end{bmatrix}$. 1 mark

c. Find $det\, M$, hence the area of the image under T of a triangle of area 5, and state whether T preserves or reverses orientation. 2 marks

d. Show that the origin is the only invariant point of T, and find the invariant lines of T through the origin. 3 marks

Question 4 (6 marks)

Let $a = 7i + 4j$ and $b = -2i + 6j$.

a. Find $a \cdot b$, and hence state whether the angle between a and b is acute, right or obtuse. 2 marks

b. Find the angle between a and b, correct to the nearest degree. 2 marks

c. Find the exact scalar projection of a in the direction of b. 1 mark

d. Find the value of t for which a is perpendicular to $c=3i+tj$. 1 mark

Question 5 (7 marks)

Let $z=-7-7\sqrt{3}i$ and $w=7\,cis\,\frac{5\pi}{6}$.

a. Express z in polar form $r\,cis\,\theta$, using the principal argument. 2 marks

b. Find zw and $\frac{z}{w}$ in polar form, using the principal argument. 2 marks

c. Find z^3, and hence state the smallest positive integer n for which z^n is a positive real number. 1 mark

d. On an Argand diagram, sketch the region $S=\{z\in\mathbb{C}:1\le|z|\le 2 \text{ and } 0\le arg(z)\le 2\pi/3\}$, and find its exact area. 2 marks

Question 6 (6 marks)

a. For the ellipse $\frac{x^2}{52}+\frac{y^2}{16}=1$, state the coordinates of the vertices (the endpoints of the major axis) and of the endpoints of the minor axis. 2 marks

b. Find the coordinates of the foci of this ellipse. 1 mark

Let $g(x)=|3x+6|-9$.

c. State the coordinates of the vertex of the graph of $y=g(x)$, and find all of its axis intercepts. 2 marks

d. Solve the equation $g(x)=x$. 1 mark

End of Topic 2 Test A

Test A — answers and solutions

Question 1.

a. The five probabilities sum to 1, so the distribution is valid. The mean is

$$E(X)=1(0.24)+2(0.31)+3(0.22)+4(0.15)+5(0.08)=0.24+0.62+0.66+0.60+0.40=2.52.$$

For the variance,
$E(X^2)=1(0.24)+4(0.31)+9(0.22)+16(0.15)+25(0.08)=0.24+1.24+1.98+2.40+2.00=7.86$, so

$$Var(X)=E(X^2)-\mu^2=7.86-2.52^2=7.86-6.3504=1.5096.$$

b. With $T=8X+15$ (so $a=8$, $b=15$), $E(T)=8(2.52)+15=35.16$ minutes. The shift $+15$ does not affect the spread, so $sd(T)=|8|sd(X)=8\sqrt{1.5096}=9.8293\ldots\approx 9.83$ minutes.

c. The means add, $E(W)=E(X)+E(Y)=2.52+3.4=5.92$. Since the two crews are independent the variances add,

$$Var(W)=Var(X)+Var(Y)=1.5096+1.8^2=1.5096+3.24=4.7496,$$

so $sd(W)=\sqrt{4.7496}=2.1794\ldots\approx 2.18$ panels.

d. The sample mean has $sd(\acute{X})=\dfrac{sd(X)}{\sqrt{n}}=\dfrac{\sqrt{1.5096}}{\sqrt{n}}$. Requiring this to be less than 0.35,

$$\frac{1.22866}{\sqrt{n}}<0.35 \Rightarrow \sqrt{n}>\frac{1.22866}{0.35}=3.5105 \Rightarrow n>12.32.$$

The smallest integer is $n=13$ (giving $sd(\acute{X})\approx 0.341<0.35$, while $n=12$ gives ≈ 0.355).

Question 2.

a. Two sides and the included angle are known, so the cosine rule gives

$$AC^2=88^2+57^2-2(88)(57)\cos 79°=7744+3249-10032\cos 79°=9078.80,$$

so $AC=\sqrt{9078.80}=95.28$ m.

b. By the sine rule, $\dfrac{\sin(\angle BAC)}{BC}=\dfrac{\sin(\angle ABC)}{AC}$, so

$$\sin(\angle BAC)=\frac{57\sin 79°}{95.28}=0.58725, \angle BAC=\sin^{-1}(0.58725)=36.0°\approx 36°.$$

(The angle is acute, since $BC<AC$.)

c. Using the two given sides and their included angle,

$$\text{Area}=1/2(AB)(BC)\sin 79°=1/2(88)(57)\sin 79°=2461.9\approx 2462\,\text{m}^2.$$

d. The mast is vertical at C, so its top, the base of A and C form a right-angled triangle with the horizontal leg $AC=95.28$ m. The angle of elevation ϕ satisfies

$$\tan\phi=\frac{15}{95.28}=0.15743, \phi=\tan^{-1}(0.15743)=8.9°\approx 9°.$$

Question 3.

a. The dilation of factor 9 parallel to the x-axis and the reflection in $y=-x$ have matrices

$$A=\begin{bmatrix}9 & 0\\0 & 1\end{bmatrix}, B=\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}.$$

b. "A followed by B" places B on the left:

$$M=BA=\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}9 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}0 & -1\\-9 & 0\end{bmatrix}.$$

c. $det\, M=(0)(0)-(-1)(-9)=-9$. Areas are multiplied by $|det\, M|=9$, so the image of a triangle of area 5 has area $9\times 5=45$. Since $det\, M<0$, T **reverses** orientation.

d. An invariant point satisfies $Mx=x$, that is $-y=x$ and $-9x=y$. Substituting $y=-x$ into $-9x=y$ gives $-9x=-x$, so $x=0$ and then $y=0$: the origin is the only invariant point. For an invariant line $y=mx$, the direction $\begin{bmatrix}1\\m\end{bmatrix}$ maps to $M\begin{bmatrix}1\\m\end{bmatrix}=\begin{bmatrix}-m\\-9\end{bmatrix}$, which is parallel to $\begin{bmatrix}1\\m\end{bmatrix}$ when $(1)(-9)-(m)(-m)=0$, i.e. $m^2-9=0$, so $m=\pm 3$. The vertical line $x=0$ maps $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}-1\\0\end{bmatrix}$, which is not parallel, so it is not invariant. The invariant lines through the origin are $y=3x$ and $y=-3x$. The two behave differently: on $y=-3x$ each point is stretched by 3 along the same ray, since $\begin{bmatrix}1\\-3\end{bmatrix}\mapsto\begin{bmatrix}3\\-9\end{bmatrix}=3\begin{bmatrix}1\\-3\end{bmatrix}$, whereas on $y=3x$ each point is carried to the opposite ray, three times as far from the origin, since $\begin{bmatrix}1\\3\end{bmatrix}\mapsto\begin{bmatrix}-3\\-9\end{bmatrix}=-3\begin{bmatrix}1\\3\end{bmatrix}$.

Question 4.

a. $a\cdot b=(7)(-2)+(4)(6)=-14+24=10$. Since $a\cdot b=10>0$, the angle between them is **acute**.

b. $|a|=\sqrt{7^2+4^2}=\sqrt{65}$ and $|b|=\sqrt{(-2)^2+6^2}=\sqrt{40}=2\sqrt{10}$, so

$$\cos\theta=\frac{a\cdot b}{|a||b|}=\frac{10}{\sqrt{65}\cdot 2\sqrt{10}}=\frac{10}{2\sqrt{650}}=0.19612,$$

so $\theta=\cos^{-1}(0.19612)=78.7°\approx 79°$.

c. The scalar projection of a on b is

$$a\cdot\hat{b}=\frac{a\cdot b}{|b|}=\frac{10}{2\sqrt{10}}=\frac{5}{\sqrt{10}}=\frac{\sqrt{10}}{2}.$$

d. Perpendicularity requires $a\cdot c=0$: $(7)(3)+(4)(t)=21+4t=0$, so $t=-\frac{21}{4}$.

Question 5.

a. Here $a=-7<0$ and $b=-7\sqrt{3}<0$, so z lies in the third quadrant. Its modulus is $r=\sqrt{(-7)^2+(-7\sqrt{3})^2}=\sqrt{49+147}=\sqrt{196}=14$, and the reference angle satisfies $\tan\alpha=\left|\frac{-7\sqrt{3}}{-7}\right|=\sqrt{3}$, so $\alpha=\frac{\pi}{3}$. The principal argument in the third quadrant is negative, $Arg(z)=-\left(\pi-\frac{\pi}{3}\right)=-\frac{2\pi}{3}$, giving $z=14\,cis\left(-\frac{2\pi}{3}\right)$.

b. To multiply, multiply the moduli and add the arguments:

$$zw=(14)(7)\,cis\left(-\frac{2\pi}{3}+\frac{5\pi}{6}\right)=98\,cis\frac{\pi}{6},$$

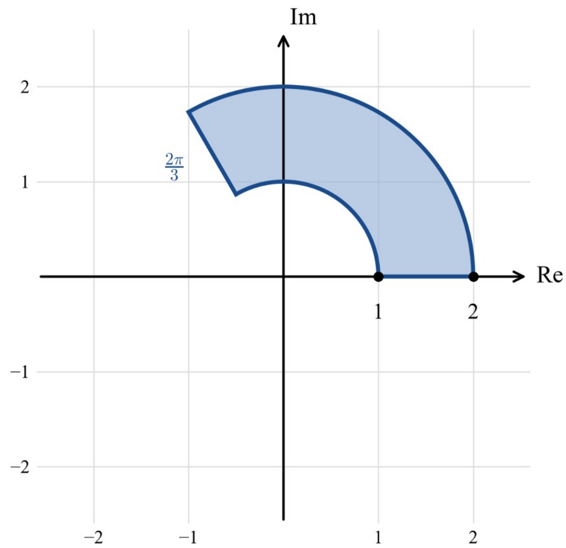
and $\frac{\pi}{6}$ is already principal. To divide, divide the moduli and subtract the arguments:

$$\frac{z}{w}=\frac{14}{7}cis\left(-\frac{2\pi}{3}-\frac{5\pi}{6}\right)=2\,cis\left(-\frac{3\pi}{2}\right)=2\,cis\frac{\pi}{2},$$

the last step adding 2π to bring the argument back into $(-\pi,\pi]$.

c. Raising to the third power, $z^3=14^3cis\left(3\times\left(-\frac{2\pi}{3}\right)\right)=2744\,cis(-2\pi)=2744$, a positive real number. In general $z^n=14^n cis\left(-\frac{2n\pi}{3}\right)$ is a positive real exactly when $\frac{2n\pi}{3}$ is a whole multiple of 2π, i.e. when $\frac{n}{3}$ is an integer; the smallest positive value is $n=3$.

d. The condition $1\le|z|\le2$ is the closed annulus between the circles of radius 1 and 2 centred at O; the condition $0\le arg(z)\le\frac{2\pi}{3}$ restricts to the wedge of directions from the positive real axis anticlockwise to $\frac{2\pi}{3}$. Their intersection is the annular sector shown.



Its area is the fraction $\frac{2\pi/3}{2\pi}=\frac{1}{3}$ of the full annulus of area $\pi(2^2-1^2)=3\pi$, that is $1/2(2^2-1^2)\left(\frac{2\pi}{3}\right)=1/2(3)\left(\frac{2\pi}{3}\right)=\pi$. The exact area is π.

Question 6.

a. Since $52>16$, the larger denominator lies under x^2, so the major axis is horizontal with $a^2=52$ and $b^2=16$; thus $a=\sqrt{52}=2\sqrt{13}$ and $b=4$. The vertices are $(\pm2\sqrt{13},0)$ and the endpoints of the minor axis are $(0,\pm4)$.

b. Using $c^2=a^2-b^2=52-16=36$, $c=6$; as the major axis is horizontal the foci are $(\pm6,0)$.

c. Writing $g(x)=3|x+2|-9$ in the form $a|x-h|+k$ gives $a=3$, $h=-2$, $k=-9$, so the vertex is $(-2,-9)$. Setting $y=0$: $3|x+2|=9$, so $|x+2|=3$ and $x+2=\pm3$, giving x-intercepts $(1,0)$ and $(-5,0)$. At $x=0$, $g(0)=|6|-9=-3$, so the y-intercept is $(0,-3)$.

d. The equation $|3x+6|-9=x$ becomes $|3x+6|=x+9$. If $3x+6\ge0$ (that is $x\ge-2$): $3x+6=x+9$, so $2x=3$ and $x=\frac{3}{2}$; this satisfies $x\ge-2$ and the right side $x+9=\frac{21}{2}>0$, so it is valid. If $3x+6<0$ (that is $x<-2$):

$-(3x+6)=x+9$, so $-3x-6=x+9$, giving $-4x=15$ and $x=-\frac{15}{4}$; this satisfies $x<-2$ and $x+9=\frac{21}{4}>0$, so it is also valid. Hence $x=\frac{3}{2}$ or $x=-\frac{15}{4}$.

Topic 2 Test B — Technology-free

Technology-free. No calculator. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given. Where more than one mark is available, appropriate working must be shown.

Question 1 (6 marks)

A prize wheel awards X tickets per spin, with the probability distribution below.

x	2	5	8
$p(x)$	1/2	1/3	1/6

a. Find $E(X)$. 2 marks

b. Show that $Var(X)=5$. 2 marks

c. The wheel is spun n times independently, and $\acute{X}$ is the mean number of tickets per spin. Find the value of n for which $sd(\acute{X})=1$. 2 marks

Question 2 (8 marks)

a. A sector of a circle of radius 14 cm has central angle $\frac{3\pi}{4}$. Find the exact length of its arc, the exact area of the sector, and the exact area of the minor segment it cuts off. 3 marks

b. The angle θ is acute with $\sin\theta=\frac{12}{13}$. Find the exact values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$. 3 marks

c. Using a product-to-sum identity, find the exact value of $\sin 75° \cos 15°$. 2 marks

Question 3 (7 marks)

Let $u=8+3i$ and $v=1-4i$.

a. Find $u+2v$ and uv, giving each answer in Cartesian form. 2 marks

b. Express $\frac{u}{v}$ in the form $a+bi$. 2 marks

c. Solve $z^2-12z+45=0$ over $\mathbb{C}$. 2 marks

d. The equation $z^2+bz+c=0$ has real coefficients and $z=-5+4i$ is one solution. Find b and c. 1 mark

Question 4 (7 marks)

a. Resolve $\frac{4x^2+x+1}{(x+2)(x^2+1)}$ into partial fractions. 3 marks

b. Find the exact value of $sec\frac{7\pi}{6}$, $cosec\frac{5\pi}{4}$ and $\cot\frac{11\pi}{6}$. 2 marks

c. Find the exact value of $\tan^{-1}(-\sqrt{3})$ and $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. 2 marks

Question 5 (6 marks)

Let $a=15i+8j$ and $b=-6i+7j$.

a. Find $|a|$ and the unit vector $\hat{a}$. 2 marks

b. Find $a \cdot b$, and hence state whether the angle between a and b is acute, right or obtuse. 2 marks

The points P, Q and R have position vectors $p = i + 2j$, $q = 5i + 4j$ and $r = 3i + 8j$.

c. Show that triangle PQR is right-angled, and state at which vertex the right angle occurs. 2 marks

Question 6 (6 marks)

a. Express $x^2 + y^2 + 6x - 4y - 12 = 0$ in the form $(x - h)^2 + (y - k)^2 = r^2$, and state the centre and radius. 2 marks

b. Using the focus–directrix definition, find the equation of the parabola with focus $(4, 0)$ and directrix $x = -4$, and state its vertex. 2 marks

c. Solve the equation $|3x - 2| = |x + 6|$. 2 marks

End of Topic 2 Test B

Test B — answers and solutions

Question 1.

a. Weighting each value by its probability,

$$E(X)=2\cdot 1/2+5\cdot 1/3+8\cdot 1/6=1+5/3+4/3=1+9/3=1+3=4.$$

b. The second moment is

$$E(X^2)=4\cdot 1/2+25\cdot 1/3+64\cdot 1/6=2+25/3+32/3=2+57/3=2+19=21,$$

so $Var(X)=E(X^2)-\mu^2=21-4^2=21-16=5$, as required.

c. For a sample of size n, $sd(\acute{X})=\frac{\sigma}{\sqrt{n}}=\frac{\sqrt{5}}{\sqrt{n}}$. Setting this equal to 1,

$$\frac{\sqrt{5}}{\sqrt{n}}=1\Rightarrow\sqrt{n}=\sqrt{5}\Rightarrow n=5.$$

Question 2.

a. With $r=14$ and $\theta=\frac{3\pi}{4}$ (in radians), the arc length is $s=r\theta=14\times\frac{3\pi}{4}=\frac{21\pi}{2}$ cm. The sector area is $1/2r^2\theta=1/2(196)\frac{3\pi}{4}=\frac{147\pi}{2}$ cm². The triangle formed by the two radii and the chord has area $1/2r^2\sin\theta=1/2(196)\sin\frac{3\pi}{4}=98\cdot\frac{\sqrt{2}}{2}=49\sqrt{2}$ cm², so the minor segment has area

$$1/2r^2(\theta-\sin\theta)=\frac{147\pi}{2}-49\sqrt{2}\,\mathrm{cm}^2.$$

b. Since θ is acute and $\sin\theta=\frac{12}{13}$, $\cos\theta=\sqrt{1-\frac{144}{169}}=\frac{5}{13}$. Then

$$\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{12}{13}\cdot\frac{5}{13}=\frac{120}{169},$$

$$\cos 2\theta=1-2\sin^2\theta=1-2\cdot\frac{144}{169}=1-\frac{288}{169}=-\frac{119}{169},$$

and $\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{120/169}{-119/169}=-\frac{120}{119}$. (Here 2θ is obtuse, so $\cos 2\theta<0$.)

c. Using $2\sin A\cos B=\sin(A+B)+\sin(A-B)$ with $A=75°$ and $B=15°$ and then halving,

$$\sin 75°\cos 15°=1/2[\sin 90°+\sin 60°]=1/2\left[1+\frac{\sqrt{3}}{2}\right]=\frac{2+\sqrt{3}}{4}.$$

Question 3.

a. $u+2v=(8+3i)+2(1-4i)=(8+2)+(3-8)i=10-5i$. Expanding the product,

$$uv=(8+3i)(1-4i)=8-32i+3i-12i^2=8-29i+12=20-29i.$$

b. Realising the denominator with the conjugate $1+4i$, where $(1-4i)(1+4i)=1+16=17$,

$$\frac{u}{v}=\frac{(8+3i)(1+4i)}{17}=\frac{8+32i+3i+12i^2}{17}=\frac{8+35i-12}{17}=\frac{-4+35i}{17}=-\frac{4}{17}+\frac{35}{17}i.$$

c. Completing the square, $z^2-12z+45=(z-6)^2-36+45=(z-6)^2+9=0$, so $(z-6)^2=-9$ and $z-6=\pm 3i$. Hence $z=6+3i$ or $z=6-3i$.

d. With real coefficients the conjugate $\overline{-5+4i}=-5-4i$ is the other root. Their sum is $(-5+4i)+(-5-4i)=-10=-b$, so $b=10$; their product is $(-5+4i)(-5-4i)=(-5)^2+4^2=25+16=41=c$. Hence $b=10$ and $c=41$.

Question 4.

a. The quadratic x^2+1 is irreducible, so the fraction is proper with the form

$$\frac{4x^2+x+1}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}, 4x^2+x+1=A(x^2+1)+(Bx+C)(x+2).$$

Substituting $x=-2$: $16-2+1=A(4+1)$, so $15=5A$ and $A=3$. Expanding the right-hand side gives $(A+B)x^2+(2B+C)x+(A+2C)$. Matching the coefficient of x^2: $A+B=4$, so $B=1$; matching the constant: $A+2C=1$, so $2C=-2$ and $C=-1$ (the x coefficient checks: $2B+C=2-1=1$). Hence

$$\frac{4x^2+x+1}{(x+2)(x^2+1)}=\frac{3}{x+2}+\frac{x-1}{x^2+1}.$$

b. Each angle has reference angle $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$ respectively; the quadrant fixes the sign of the base function, and the reciprocal carries that sign.

$$sec\frac{7\pi}{6}=\frac{1}{\cos(7\pi/6)}=\frac{1}{-\sqrt{3}/2}=-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}, cosec\frac{5\pi}{4}=\frac{1}{\sin(5\pi/4)}=\frac{1}{-\sqrt{2}/2}=-\frac{2}{\sqrt{2}}=-\sqrt{2},$$

$$\cot\frac{11\pi}{6}=\frac{\cos(11\pi/6)}{\sin(11\pi/6)}=\frac{\sqrt{3}/2}{-1/2}=-\sqrt{3}.$$

c. For $\tan^{-1}(-\sqrt{3})$, seek the angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with tangent $-\sqrt{3}$: since $\tan^{-1}$ is odd and $\tan\frac{\pi}{3}=\sqrt{3}$, $\tan^{-1}(-\sqrt{3})=-\frac{\pi}{3}$. For $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, seek the angle in $[0,\pi]$ with cosine $-\frac{\sqrt{2}}{2}$: this is $\pi-\frac{\pi}{4}=\frac{3\pi}{4}$.

Question 5.

a. $|a|=\sqrt{15^2+8^2}=\sqrt{289}=17$, so $\hat{a}=\frac{1}{17}(15i+8j)=\frac{15}{17}i+\frac{8}{17}j$.

b. $a\cdot b=(15)(-6)+(8)(7)=-90+56=-34$. Since $a\cdot b=-34<0$, the angle between a and b is **obtuse**.

c. The side vectors from Q are $\overrightarrow{QP}=p-q=-4i-2j$ and $\overrightarrow{QR}=r-q=-2i+4j$. Their dot product is

$$\overrightarrow{QP}\cdot\overrightarrow{QR}=(-4)(-2)+(-2)(4)=8-8=0,$$

so $\overrightarrow{QP}\perp\overrightarrow{QR}$ and triangle PQR is right-angled at Q.

Question 6.

a. Grouping and completing the square, $(x^2+6x)+(y^2-4y)=12$ becomes $(x+3)^2-9+(y-2)^2-4=12$, so

$$(x+3)^2+(y-2)^2=12+9+4=25.$$

The centre is $(-3,2)$ and the radius is $\sqrt{25}=5$.

b. Let $P=(x,y)$. Equidistance from the focus $(4,0)$ and the directrix $x=-4$ gives $\sqrt{(x-4)^2+y^2}=|x+4|$; squaring, $(x-4)^2+y^2=(x+4)^2$, so

$$y^2=(x+4)^2-(x-4)^2=16x.$$

This is the parabola $y^2=16x$, opening to the right, with vertex $(0,0)$.

c. Two numbers have equal modulus exactly when their squares are equal, so squaring loses nothing here: $(3x-2)^2=(x+6)^2$ gives $9x^2-12x+4=x^2+12x+36$, that is $8x^2-24x-32=0$, or $x^2-3x-4=0$ after dividing by 8. Factorising, $(x-4)(x+1)=0$, so $x=4$ or $x=-1$. Substituting back confirms both: at $x=4$, $|10|=10=|10|$; at $x=-1$, $|-5|=5=|5|$.