Specialist Mathematics Units 1 & 2

Chapter 12 — Partial fractions and reciprocal and inverse circular functions

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Chapter 12 Partial fractions and reciprocal and inverse circular functions — 12A Partial fractions

Theory

A **rational function** is a quotient of two polynomials,

$$\frac{P(x)}{Q(x)}, Q(x) \neq 0,$$

where P and Q have real coefficients. Building such a quotient by hand is routine: to add $\frac{2}{x+1}+\frac{3}{x-3}$ we place both terms over the common denominator $(x+1)(x-3)$ and collect the top, obtaining

$$\frac{2}{x+1}+\frac{3}{x-3}=\frac{2(x-3)+3(x+1)}{(x+1)(x-3)}=\frac{5x-3}{(x+1)(x-3)}.$$

Resolving into partial fractions is the *reverse* process: starting from a single quotient such as $\frac{5x-3}{(x+1)(x-3)}$, we recover the simpler fractions $\frac{2}{x+1}$ and $\frac{3}{x-3}$ that add to give it. The simpler pieces are far easier to work with — for sketching graphs, for describing behaviour near the vertical asymptotes, and for the calculus met later — which is why the decomposition is worth carrying out systematically.

Proper and improper rational functions.

The method depends on comparing the **degrees** of the numerator and denominator. A rational function is **proper** when

$$deg\, P < deg\, Q,$$

and **improper** when $deg\, P \geq deg\, Q$. Only a *proper* rational function can be written purely as a sum of partial fractions. An **improper** rational function is first reduced by **polynomial division**: dividing P by Q produces a polynomial quotient $S(x)$ and a remainder $R(x)$ with $deg\, R < deg\, Q$, so that

$$\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)},$$

where the trailing fraction $\frac{R(x)}{Q(x)}$ is now proper and can be decomposed. Every partial-fraction problem therefore begins with the same check: *is the fraction proper?* If not, divide first.

Factorising the denominator.

The shape of the decomposition is dictated entirely by how the denominator $Q(x)$ factorises. Over the real numbers every polynomial factorises into a product of **linear factors** $x-a$ and **irreducible quadratic factors** x^2+px+q — quadratics that cannot themselves be split into real linear factors, which happens exactly when the discriminant is negative, $p^2-4q<0$. So the denominator must be fully factorised before the fractions can be set up.

The building blocks.

Each factor of the denominator contributes its own term (or terms) to the decomposition, and the numerator placed above a factor always has degree **one less** than that factor — a constant above a linear factor, a linear expression above a quadratic factor.

- A **distinct linear factor** $x-a$ contributes a single term

$$\frac{A}{x-a}.$$

- A **repeated linear factor** $(x-a)^2$ contributes one term for each power up to the repeat:

$$\frac{A}{x-a}+\frac{B}{(x-a)^2}.$$

  More generally $(x-a)^n$ contributes $ + +

- $; at this level the repeat is usually squared.
- An **irreducible quadratic factor** x^2+px+q contributes a term with a linear numerator,

$$\frac{Bx+C}{x^2+px+q}.$$

The total number of unknown constants introduced always equals the degree of $Q(x)$, which is exactly the number of conditions available to pin them down; the decomposition that results is **unique**.

Finding the constants.

Once the correct form is written, multiply both sides by the denominator $Q(x)$ to **clear the fractions**. The result is a polynomial **identity** — a statement true for *every* value of x, not an equation to be solved for a special x. Two standard techniques extract the constants, and they are freely combined.

- **Substitution of strategic values.** Because the identity holds for all x, substitute the value that makes a chosen linear factor zero. Every term still carrying that factor vanishes, isolating a single constant. Substituting each root of the denominator in turn is the quickest route to the constants sitting above distinct linear factors (this is the "cover-up" idea).
- **Equating coefficients.** Expand the right-hand side, collect like powers of x, and match the coefficient of each power on the two sides. This produces a system of linear equations for the constants. It is the reliable way to find the numerator of an irreducible quadratic term and the "middle" constant of a repeated factor, where no single substitution isolates them.

In practice one substitutes the available roots first to capture the easy constants, then equates one or two coefficients to finish off any that remain. A quick expansion of the final answer over a common denominator confirms it reproduces the original numerator.

Worked examples

Example 1 — scaffolded

Resolve $\frac{7x+3}{(x-1)(x+4)}$ into partial fractions.

Working	Comment
$deg(7x+3)=1<2=deg[(x-1)(x+4)]$, so the fraction is proper.	Check proper before decomposing; here no division is needed.
$\frac{7x+3}{(x-1)(x+4)}=\frac{A}{x-1}+\frac{B}{x+4}.$	Distinct linear factors, so one constant over each.
$7x+3=A(x+4)+B(x-1).$	Multiply both sides by $(x-1)(x+4)$ to clear fractions — an identity in x.
$x=1: 7(1)+3=A(5)+B(0)$, so $10=5A$ and $A=2$.	Substitute the root $x=1$; the B term vanishes.
$x=-4: 7(-4)+3=A(0)+B(-5)$, so $-25=-5B$	Substitute the root $x=-4$ to isolate B.

Working	Comment
and $B=5$.	
$\dfrac{7x+3}{(x-1)(x+4)}=\dfrac{2}{x-1}+\dfrac{5}{x+4}$.	State the decomposition with the constants filled in.

Example 2 — scaffolded

Resolve $\dfrac{3x^2+4x+5}{(x+2)(x-1)^2}$ into partial fractions.

Working	Comment
Degree $2<3$, so the fraction is proper.	Numerator degree below denominator degree.
$\dfrac{3x^2+4x+5}{(x+2)(x-1)^2}=\dfrac{A}{x+2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$.	A repeated factor $(x-1)^2$ needs both a $(x-1)$ and a $(x-1)^2$ term.
$3x^2+4x+5=A(x-1)^2+B(x+2)(x-1)+C(x+2)$.	Clear the denominator to obtain the identity.
$x=1: 3+4+5=C(3)$, so $12=3C$ and $C=4$.	Substitute $x=1$; only the C term survives.
$x=-2: 12-8+5=A(9)$, so $9=9A$ and $A=1$.	Substitute $x=-2$; only the A term survives.
Coefficient of x^2: $3=A+B$, so $B=3-1=2$.	No root isolates B; equate the x^2 coefficients instead.
$\dfrac{3x^2+4x+5}{(x+2)(x-1)^2}=\dfrac{1}{x+2}+\dfrac{2}{x-1}+\dfrac{4}{(x-1)^2}$.	Substitution and coefficient-matching combined.

Example 3 — unscaffolded

Resolve $\dfrac{3x^2+2x+5}{(x-1)(x^2+4)}$ into partial fractions.

The quadratic x^2+4 has discriminant $0-16<0$, so it is irreducible; the fraction is proper (degree $2<3$). The correct form places a constant over the linear factor and a linear expression over the quadratic factor:

$$\frac{3x^2+2x+5}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}.$$

Clearing the denominator gives the identity

$$3x^2+2x+5=A(x^2+4)+(Bx+C)(x-1).$$

Substituting the root $x=1$ collapses the second term:

$$3+2+5=A(1+4)\Rightarrow 10=5A\Rightarrow A=2.$$

For B and C, expand the right-hand side and equate coefficients. With $A=2$,

$$A(x^2+4)+(Bx+C)(x-1)=(A+B)x^2+(C-B)x+(4A-C).$$

Matching the coefficient of x^2 gives $A+B=3$, so $B=3-2=1$; matching the constant term gives $4A-C=5$, so $C=8-5=3$ (and the coefficient of x checks: $C-B=3-1=2$, as required).

$\therefore \dfrac{3x^2+2x+5}{(x-1)(x^2+4)}=\dfrac{2}{x-1}+\dfrac{x+3}{x^2+4}$.

Example 4 — unscaffolded

Express $\frac{2x^2+3x-4}{x^2+x-6}$ as the sum of a polynomial and partial fractions.

Here the numerator and denominator both have degree 2, so the fraction is **improper** and must be divided first. Dividing $2x^2+3x-4$ by x^2+x-6, the quotient is 2 and

$$2x^2+3x-4=2(x^2+x-6)+(x+8),$$

so the remainder is $x+8$ and

$$\frac{2x^2+3x-4}{x^2+x-6}=2+\frac{x+8}{x^2+x-6}.$$

The remaining fraction is proper. Factorising the denominator as $x^2+x-6=(x-2)(x+3)$,

$$\frac{x+8}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}\Rightarrow x+8=A(x+3)+B(x-2).$$

Substituting $x=2$ gives $10=5A$, so $A=2$; substituting $x=-3$ gives $5=-5B$, so $B=-1$.

$\therefore \frac{2x^2+3x-4}{x^2+x-6}=2+\frac{2}{x-2}-\frac{1}{x+3}.$

Practice questions

Scaffolded

Q1. Resolve $\frac{4x+5}{(x-1)(x+2)}$ into partial fractions. (a) Write the decomposition in the form $\frac{A}{x-1}+\frac{B}{x+2}$. (b) Clear the denominators to obtain an identity in x. (c) Substitute a strategic value to find A. (d) Substitute a strategic value to find B, and state the result.

Q2. Resolve $\frac{x+14}{x^2-2x-8}$ into partial fractions. (a) Factorise the denominator. (b) Write the appropriate form with unknown constants. (c) Find the constant above one factor by substitution. (d) Find the other constant and state the result.

Q3. Resolve $\frac{6x-5}{(x-2)^2}$ into partial fractions. (a) Explain why the form is $\frac{A}{x-2}+\frac{B}{(x-2)^2}$. (b) Clear the denominator to obtain an identity. (c) Substitute $x=2$ to find B. (d) Equate the coefficient of x to find A, and state the result.

Q4. Resolve $\frac{3x^2+3x+4}{(x+1)(x^2+3)}$ into partial fractions. (a) State why x^2+3 is irreducible, and write the form $\frac{A}{x+1}+\frac{Bx+C}{x^2+3}$. (b) Clear the denominators to obtain an identity. (c) Substitute $x=-1$ to find A. (d) Equate coefficients to find B and C, and state the result.

Q5. Express $\frac{x^2+3x+5}{x^2+x-2}$ as a polynomial plus partial fractions. (a) Explain why division is required. (b) Divide to write the function as $1+\frac{R(x)}{x^2+x-2}$. (c) Factorise x^2+x-2 and set up partial fractions for the remainder. (d) Find the constants and state the full result.

Q6. Resolve $\frac{6x^2-5x-5}{(x-1)(x+1)(x-2)}$ into partial fractions. (a) Write the form $\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}$. (b) Clear the denominators to obtain an identity. (c) Substitute $x=1$, $x=-1$ and $x=2$ in turn. (d) State the decomposition.

Unscaffolded

Q7. Resolve $\frac{6x-2}{(x-3)(x+1)}$ into partial fractions.

Q8. Resolve $\frac{2x-1}{x^2-x-12}$ into partial fractions.

Q9. Resolve $\frac{3x^2+11x+4}{x(x+2)^2}$ into partial fractions.

Q10. Resolve $\frac{4x^2+7x-4}{(x+3)(x^2+2)}$ into partial fractions.

Q11. Express $\frac{4x^2+11x+1}{(x-1)(x+3)}$ as a polynomial plus partial fractions.

Q12. Express $\frac{x^3+2x^2+2x-1}{x^2-1}$ as a polynomial plus partial fractions.

Q13. Resolve $\frac{4x^2+5x-6}{x(x-2)(x+3)}$ into partial fractions.

Q14. Resolve $\frac{3x^2+13x+2}{(x-1)(x+2)^2}$ into partial fractions.

Q15. Resolve $\frac{3x^2+3x-10}{(x-3)(x^2+x+1)}$ into partial fractions.

Q16. Express $\frac{2x^3+x^2+5x-3}{x^2+1}$ as a polynomial plus partial fractions.

Q17. Resolve $\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)}$ into partial fractions.

Q18. Express $\frac{2x^3+3x^2-6x+10}{(x-1)^2(x+2)}$ as a polynomial plus partial fractions.

Answers

Q1. $\frac{3}{x-1}+\frac{1}{x+2}$. **Q2.** $\frac{3}{x-4}-\frac{2}{x+2}$. **Q3.** $\frac{6}{x-2}+\frac{7}{(x-2)^2}$. **Q4.** $\frac{1}{x+1}+\frac{2x+1}{x^2+3}$. **Q5.** $1+\frac{3}{x-1}-\frac{1}{x+2}$. **Q6.** $\frac{2}{x-1}+\frac{1}{x+1}+\frac{3}{x-2}$. **Q7.** $\frac{4}{x-3}+\frac{2}{x+1}$. **Q8.** $\frac{1}{x-4}+\frac{1}{x+3}$. **Q9.** $\frac{1}{x}+\frac{2}{x+2}+\frac{3}{(x+2)^2}$. **Q10.** $\frac{1}{x+3}+\frac{3x-2}{x^2+2}$. **Q11.** $4+\frac{4}{x-1}-\frac{1}{x+3}$. **Q12.** $x+2+\frac{2}{x-1}+\frac{1}{x+1}$. **Q13.** $\frac{1}{x}+\frac{2}{x-2}+\frac{1}{x+3}$. **Q14.** $\frac{2}{x-1}+\frac{1}{x+2}+\frac{4}{(x+2)^2}$. **Q15.** $\frac{2}{x-3}+\frac{x+4}{x^2+x+1}$. **Q16.** $2x+1+\frac{3x-4}{x^2+1}$. **Q17.** $\frac{3}{x+1}+\frac{2x-1}{x^2+2x+3}$. **Q18.** $2+\frac{1}{x-1}+\frac{3}{(x-1)^2}+\frac{2}{x+2}$.

Fully-worked solutions

Q1. The fraction is proper, so write $\frac{4x+5}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$. Clearing the denominators, $4x+5=A(x+2)+B(x-1)$. Substituting $x=1$ gives $9=3A$, so $A=3$; substituting $x=-2$ gives $-3=-3B$, so $B=1$. Hence

$$\frac{4x+5}{(x-1)(x+2)}=\frac{3}{x-1}+\frac{1}{x+2}.$$

Q2. The denominator factorises as $x^2-2x-8=(x-4)(x+2)$, and the fraction is proper, so $\frac{x+14}{(x-4)(x+2)}=\frac{A}{x-4}+\frac{B}{x+2}$ and $x+14=A(x+2)+B(x-4)$. Substituting $x=4$ gives $18=6A$, so $A=3$; substituting $x=-2$ gives $12=-6B$, so $B=-2$. Hence

$$\frac{x+14}{x^2-2x-8}=\frac{3}{x-4}-\frac{2}{x+2}.$$

Q3. The denominator is the repeated linear factor $(x-2)^2$, so both a $(x-2)$ and a $(x-2)^2$ term are needed: $\frac{6x-5}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$. Clearing gives $6x-5=A(x-2)+B$. Substituting $x=2$ gives $7=B$. Equating the coefficient of x gives $A=6$. Hence

$$\frac{6x-5}{(x-2)^2}=\frac{6}{x-2}+\frac{7}{(x-2)^2}.$$

Q4. Since x^2+3 has discriminant $0-12<0$ it is irreducible, so $\frac{3x^2+3x+4}{(x+1)(x^2+3)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+3}$ and

$$3x^2+3x+4=A(x^2+3)+(Bx+C)(x+1).$$

Substituting $x=-1$ gives $3-3+4=A(1+3)$, so $4=4A$ and $A=1$. Expanding the right-hand side gives $(A+B)x^2+(B+C)x+(3A+C)$. Matching the coefficient of x^2: $A+B=3$, so $B=2$; matching the constant: $3A+C=4$, so $C=1$ (and the x coefficient checks: $B+C=3$). Hence

$$\frac{3x^2+3x+4}{(x+1)(x^2+3)}=\frac{1}{x+1}+\frac{2x+1}{x^2+3}.$$

Q5. The numerator and denominator both have degree 2, so the fraction is improper and division is required. Dividing, $x^2+3x+5=1\cdot(x^2+x-2)+(2x+7)$, so

$$\frac{x^2+3x+5}{x^2+x-2}=1+\frac{2x+7}{x^2+x-2}.$$

Factorising $x^2+x-2=(x-1)(x+2)$ and writing $\frac{2x+7}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$ gives $2x+7=A(x+2)+B(x-1)$. Substituting $x=1$: $9=3A$, so $A=3$; substituting $x=-2$: $3=-3B$, so $B=-1$. Hence

$$\frac{x^2+3x+5}{x^2+x-2}=1+\frac{3}{x-1}-\frac{1}{x+2}.$$

Q6. Three distinct linear factors give $\frac{6x^2-5x-5}{(x-1)(x+1)(x-2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}$, so

$$6x^2-5x-5=A(x+1)(x-2)+B(x-1)(x-2)+C(x-1)(x+1).$$

Substituting $x=1$: $-4=A(2)(-1)=-2A$, so $A=2$. Substituting $x=-1$: $6=B(-2)(-3)=6B$, so $B=1$. Substituting $x=2$: $9=C(1)(3)=3C$, so $C=3$. Hence

$$\frac{6x^2-5x-5}{(x-1)(x+1)(x-2)}=\frac{2}{x-1}+\frac{1}{x+1}+\frac{3}{x-2}.$$

Q7. Proper, with distinct linear factors: $\frac{6x-2}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}$, so $6x-2=A(x+1)+B(x-3)$. Substituting $x=3$: $16=4A$, so $A=4$; substituting $x=-1$: $-8=-4B$, so $B=2$. Hence

$$\frac{6x-2}{(x-3)(x+1)}=\frac{4}{x-3}+\frac{2}{x+1}.$$

Q8. Factorising, $x^2-x-12=(x-4)(x+3)$, so $\frac{2x-1}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$ and $2x-1=A(x+3)+B(x-4)$. Substituting $x=4$: $7=7A$, so $A=1$; substituting $x=-3$: $-7=-7B$, so $B=1$. Hence

$$\frac{2x-1}{x^2-x-12}=\frac{1}{x-4}+\frac{1}{x+3}.$$

Q9. A distinct factor x and a repeated factor $(x+2)^2$ give $\frac{3x^2+11x+4}{x(x+2)^2}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$, so

$$3x^2+11x+4=A(x+2)^2+Bx(x+2)+Cx.$$

Substituting $x=0$: $4=4A$, so $A=1$. Substituting $x=-2$: $12-22+4=C(-2)$, so $-6=-2C$ and $C=3$. Equating the coefficient of x^2: $3=A+B$, so $B=2$. Hence

$$\frac{3x^2+11x+4}{x(x+2)^2}=\frac{1}{x}+\frac{2}{x+2}+\frac{3}{(x+2)^2}.$$

Q10. The quadratic x^2+2 is irreducible (discriminant $-8<0$), so $\frac{4x^2+7x-4}{(x+3)(x^2+2)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+2}$ and

$$4x^2+7x-4=A(x^2+2)+(Bx+C)(x+3).$$

Substituting $x=-3$: $36-21-4=A(9+2)$, so $11=11A$ and $A=1$. Expanding, the right-hand side is $(A+B)x^2+(3B+C)x+(2A+3C)$. Matching x^2: $A+B=4$, so $B=3$; matching the constant: $2A+3C=-4$, so $3C=-6$ and $C=-2$ (the x coefficient checks: $3B+C=9-2=7$). Hence

$$\frac{4x^2+7x-4}{(x+3)(x^2+2)}=\frac{1}{x+3}+\frac{3x-2}{x^2+2}.$$

Q11. The numerator has degree 2 and $(x-1)(x+3)=x^2+2x-3$ also has degree 2, so the fraction is improper. Dividing, $4x^2+11x+1=4(x^2+2x-3)+(3x+13)$, so

$$\frac{4x^2+11x+1}{(x-1)(x+3)}=4+\frac{3x+13}{(x-1)(x+3)}.$$

Writing $\frac{3x+13}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$ gives $3x+13=A(x+3)+B(x-1)$. Substituting $x=1$: $16=4A$, so $A=4$; substituting $x=-3$: $4=-4B$, so $B=-1$. Hence

$$\frac{4x^2+11x+1}{(x-1)(x+3)}=4+\frac{4}{x-1}-\frac{1}{x+3}.$$

Q12. The numerator has degree 3 and the denominator x^2-1 has degree 2, so the fraction is improper. Dividing x^3+2x^2+2x-1 by x^2-1 gives quotient $x+2$ and remainder $3x+1$:

$$\frac{x^3+2x^2+2x-1}{x^2-1}=x+2+\frac{3x+1}{(x-1)(x+1)}.$$

Writing $\frac{3x+1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$ gives $3x+1=A(x+1)+B(x-1)$. Substituting $x=1$: $4=2A$, so $A=2$; substituting $x=-1$: $-2=-2B$, so $B=1$. Hence

$$\frac{x^3+2x^2+2x-1}{x^2-1}=x+2+\frac{2}{x-1}+\frac{1}{x+1}.$$

Q13. Three distinct linear factors give $\frac{4x^2+5x-6}{x(x-2)(x+3)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+3}$, so

$$4x^2+5x-6=A(x-2)(x+3)+Bx(x+3)+Cx(x-2).$$

Substituting $x=0$: $-6=A(-2)(3)=-6A$, so $A=1$. Substituting $x=2$: $16+10-6=B(2)(5)=10B$, so $20=10B$ and $B=2$. Substituting $x=-3$: $36-15-6=C(-3)(-5)=15C$, so $15=15C$ and $C=1$. Hence

$$\frac{4x^2+5x-6}{x(x-2)(x+3)}=\frac{1}{x}+\frac{2}{x-2}+\frac{1}{x+3}.$$

Q14. A distinct factor $x-1$ and a repeated factor $(x+2)^2$ give $\frac{3x^2+13x+2}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$, so

$$3x^2+13x+2=A(x+2)^2+B(x-1)(x+2)+C(x-1).$$

Substituting $x=1$: $18=A(9)$, so $A=2$. Substituting $x=-2$: $12-26+2=C(-3)$, so $-12=-3C$ and $C=4$. Equating the coefficient of x^2: $3=A+B$, so $B=1$. Hence

$$\frac{3x^2+13x+2}{(x-1)(x+2)^2}=\frac{2}{x-1}+\frac{1}{x+2}+\frac{4}{(x+2)^2}.$$

Q15. The quadratic x^2+x+1 has discriminant $1-4<0$ and is irreducible, so $\frac{3x^2+3x-10}{(x-3)(x^2+x+1)}=\frac{A}{x-3}+\frac{Bx+C}{x^2+x+1}$ and

$$3x^2+3x-10=A(x^2+x+1)+(Bx+C)(x-3).$$

Substituting $x=3$: $27+9-10=A(9+3+1)$, so $26=13A$ and $A=2$. Expanding, the right-hand side is $(A+B)x^2+(A-3B+C)x+(A-3C)$. Matching x^2: $A+B=3$, so $B=1$; matching the constant: $A-3C=-10$, so $-3C=-12$ and $C=4$ (the x coefficient checks: $A-3B+C=2-3+4=3$). Hence

$$\frac{3x^2+3x-10}{(x-3)(x^2+x+1)}=\frac{2}{x-3}+\frac{x+4}{x^2+x+1}.$$

Q16. The numerator has degree 3 and the irreducible denominator x^2+1 has degree 2, so the fraction is improper. Dividing $2x^3+x^2+5x-3$ by x^2+1 gives quotient $2x+1$ and remainder $3x-4$:

$$\frac{2x^3+x^2+5x-3}{x^2+1}=2x+1+\frac{3x-4}{x^2+1}.$$

Since x^2+1 is irreducible and the remainder is already linear over it, the numerator $3x-4$ is left as it stands. Hence

$$\frac{2x^3+x^2+5x-3}{x^2+1}=2x+1+\frac{3x-4}{x^2+1}.$$

Q17. The quadratic x^2+2x+3 has discriminant $4-12<0$ and is irreducible, so $\dfrac{5x^2+7x+8}{(x+1)(x^2+2x+3)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+2x+3}$ and

$$5x^2+7x+8=A(x^2+2x+3)+(Bx+C)(x+1).$$

Substituting $x=-1$: $5-7+8=A(1-2+3)$, so $6=2A$ and $A=3$. Expanding, the right-hand side is $(A+B)x^2+(2A+B+C)x+(3A+C)$. Matching x^2: $A+B=5$, so $B=2$; matching the constant: $3A+C=8$, so $C=-1$ (the x coefficient checks: $2A+B+C=6+2-1=7$). Hence

$$\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)}=\frac{3}{x+1}+\frac{2x-1}{x^2+2x+3}.$$

Q18. The numerator has degree 3 and the denominator $(x-1)^2(x+2)=x^3-3x+2$ also has degree 3, so the fraction is improper. Dividing, $2x^3+3x^2-6x+10=2(x^3-3x+2)+(3x^2+6)$, so

$$\frac{2x^3+3x^2-6x+10}{(x-1)^2(x+2)}=2+\frac{3x^2+6}{(x-1)^2(x+2)}.$$

For the proper part, write $\dfrac{3x^2+6}{(x-1)^2(x+2)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+2}$, so

$$3x^2+6=A(x-1)(x+2)+B(x+2)+C(x-1)^2.$$

Substituting $x=1$: $9=3B$, so $B=3$. Substituting $x=-2$: $18=9C$, so $C=2$. Equating the coefficient of x^2: $3=A+C$, so $A=1$. Hence

$$\frac{2x^3+3x^2-6x+10}{(x-1)^2(x+2)}=2+\frac{1}{x-1}+\frac{3}{(x-1)^2}+\frac{2}{x+2}.$$

Theory

The sine, cosine and tangent functions met in Mathematical Methods each have a **reciprocal** — the function obtained by dividing 1 by it. These three reciprocals are important enough to be given their own names and symbols, and they behave in a characteristic way: wherever the base function is zero the reciprocal shoots off to infinity, so its graph is built around a set of vertical asymptotes. This section defines the reciprocal circular functions, records their exact values, domains, ranges and graphs together with the simple transformations of those graphs, establishes two new Pythagorean identities from $\sin^2\theta+\cos^2\theta=1$, and uses all of this to simplify expressions and solve equations.

The three reciprocal circular functions.

The **cosecant**, **secant** and **cotangent** of an angle θ are defined by

$$cosec\,\theta=\frac{1}{\sin\theta},\ sec\,\theta=\frac{1}{\cos\theta},\ \cot\theta=\frac{1}{\tan\theta}=\frac{\cos\theta}{\sin\theta}.$$

The two forms of $\cot\theta$ agree wherever both are defined, since $\frac{1}{\tan\theta}=\frac{1}{\sin\theta/\cos\theta}=\frac{\cos\theta}{\sin\theta}$; the second form $\frac{\cos\theta}{\sin\theta}$ is the more useful because it still makes sense at the angles where $\tan\theta$ itself is undefined. Note the pairing carefully: the **secant** is the reciprocal of the **cosine** and the **cosecant** is the reciprocal of the **sine** — the prefix “co-” does *not* line up, so $sec\,\theta$ goes with $\cos\theta$ while $cosec\,\theta$ goes with $\sin\theta$.

Where each is undefined.

A fraction $\frac{1}{u}$ has no value when $u=0$, so each reciprocal function is undefined exactly where its base function is zero. Since $\sin\theta=0$ at the integer multiples of π and $\cos\theta=0$ at the odd multiples of $\frac{\pi}{2}$,

$$cosec\,\theta \text{ and } \cot\theta \text{ are undefined at } \theta=n\pi,\ sec\,\theta \text{ is undefined at } \theta=\frac{\pi}{2}+n\pi,$$

for every integer n. These are precisely the positions of the vertical asymptotes on the graphs below.

Exact values.

Because each reciprocal function is 1 divided by a known circular function, its exact values come straight from the exact values of $\sin$, $\cos$ and $\tan$. The table gathers the first-quadrant values; a dash marks an angle where the function is undefined.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$cosec\,\theta$	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$sec\,\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

For example $sec\frac{\pi}{3}=\frac{1}{\cos(\pi/3)}=\frac{1}{1/2}=2$, and $cosec\frac{\pi}{6}=\frac{1}{\sin(\pi/6)}=\frac{1}{1/2}=2$. Values at angles outside the first quadrant are found the same way, keeping the sign of the base function: $sec\frac{2\pi}{3}=\frac{1}{\cos(2\pi/3)}=\frac{1}{-1/2}=-2$, and $\cot\frac{3\pi}{4}=\frac{\cos(3\pi/4)}{\sin(3\pi/4)}=\frac{-\sqrt{2}/2}{\sqrt{2}/2}=-1$.

Signs. Since $\frac{1}{u}$ always has the *same sign* as u, each reciprocal function is positive in exactly the quadrants where its base function is positive. Thus $cosec\,\theta$ is positive where $\sin\theta$ is (quadrants 1 and 2), $sec\,\theta$ where $\cos\theta$ is (quadrants 1 and 4), and $\cot\theta$ where $\tan\theta$ is (quadrants 1 and 3).

Domains, ranges and periods.

Deleting the undefined angles gives the maximal domains. The ranges follow from the fact that $\sin\theta$ and $\cos\theta$ each lie in $[-1,1]$: if $0<|\sin\theta|\leq 1$ then $|cosec\,\theta|=\frac{1}{|\sin\theta|}\geq 1$, and likewise for $sec\,\theta$. The cotangent takes every real value: $\cot\theta=\frac{1}{\tan\theta}$ runs through every non-zero real number as $\tan\theta$ does, and $\cot\theta=0$ at the angles where $\cos\theta=0$.

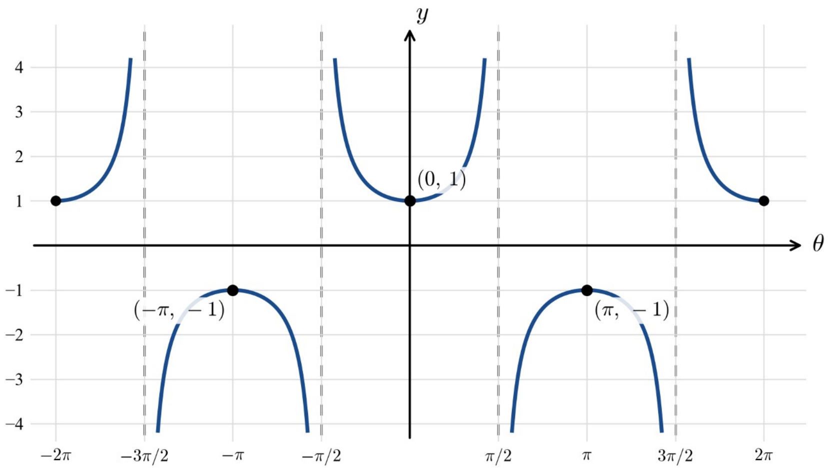
Function	Domain	Range	Period
$y=cosec\,\theta$	$\theta\neq n\pi$	$y\leq -1$ or $y\geq 1$	2π
$y=sec\,\theta$	$\theta\neq\frac{\pi}{2}+n\pi$	$y\leq -1$ or $y\geq 1$	2π
$y=\cot\theta$	$\theta\neq n\pi$	$y\in\mathbb{R}$	π

The periods are inherited from the base functions: $cosec$ and sec repeat every 2π like $\sin$ and $\cos$, while $\cot$ repeats every π like $\tan$.

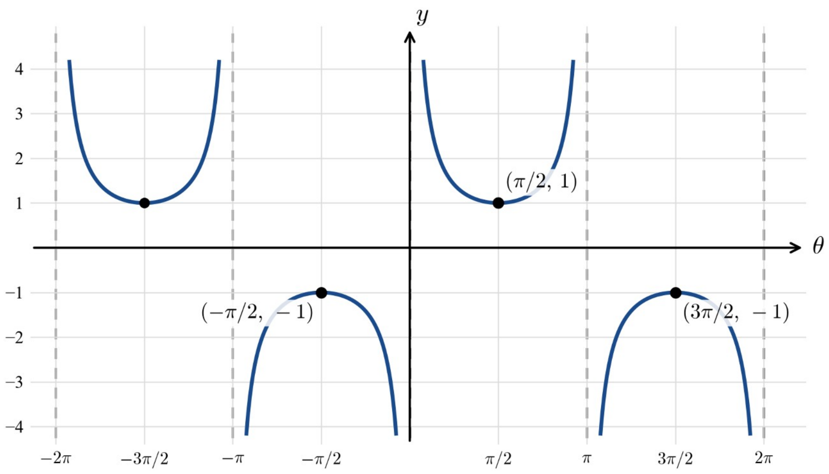
Graphs.

The graph of a reciprocal function is governed by three facts about $y=\frac{1}{u}$: it is undefined (a **vertical asymptote**) where $u=0$; it equals ± 1 where $u=\pm 1$; and as $u\to 0$ the reciprocal $\to\pm\infty$, while as $u\to\pm\infty$ the reciprocal $\to 0$. Reading these off the base curve produces each graph exactly.

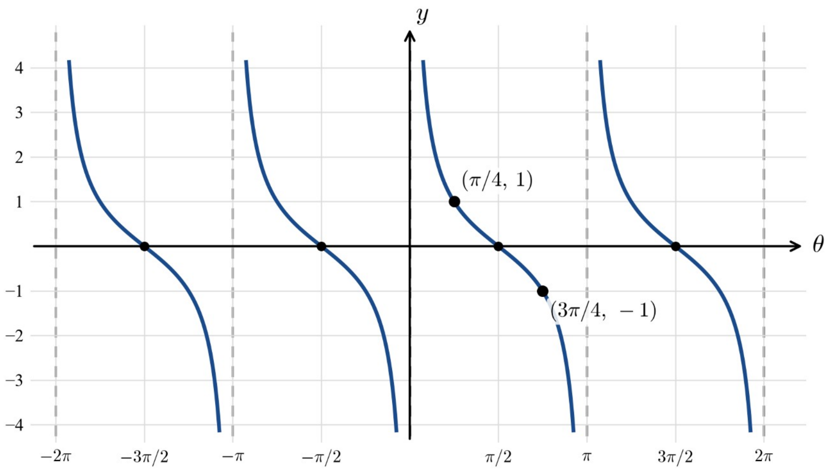
For $y=sec\,\theta$ the asymptotes stand at the zeros of $\cos\theta$, namely $\theta=\frac{\pi}{2}+n\pi$. Where $\cos\theta=1$ (at $\theta=0,\pm 2\pi,\ldots$) the secant has a local minimum of 1; where $\cos\theta=-1$ (at $\theta=\pm\pi,\ldots$) it has a local maximum of -1. The curve therefore forms a chain of $\cup$- and $\cap$-shaped branches sitting on the lines $y=1$ and $y=-1$.



For $y = cosec\,\theta$ the asymptotes stand at the zeros of $\sin\theta$, namely $\theta = n\pi$. The local minimum of 1 occurs where $\sin\theta = 1$ (at $\theta = \frac{\pi}{2} + 2n\pi$) and the local maximum of -1 where $\sin\theta = -1$. The shape is the secant graph shifted, as expected from $\sin$ being $\cos$ shifted.



For $y = \cot\theta$ the asymptotes again stand at $\theta = n\pi$, but the picture is quite different: on each interval between consecutive asymptotes the cotangent *decreases* steadily from $+\infty$ down to $-\infty$, crossing zero at the odd multiples of $\frac{\pi}{2}$ (where $\cos\theta = 0$). It passes through $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{3\pi}{4}, -1\right)$, and its period is only π.



Simple transformations of these graphs.

Each of the three graphs can be dilated, reflected and translated in the usual way. Written in the standard form

$$y = a\,sec\left(n(\theta - b)\right) + c$$

— and identically with $cosec$ or $\cot$ in place of sec — the four constants act exactly as they do on $\sin$ and $\cos$: a gives a dilation of factor $|a|$ from the θ-axis, together with a reflection in the θ-axis when $a < 0$; n gives a dilation of factor $\frac{1}{|n|}$ from the y-axis; b is a translation of b units in the positive θ-direction; and c is a translation of c units upwards. Three features are read straight off the constants.

- **Asymptotes.** The asymptotes are the images of the base asymptotes, so they are found by setting the **argument** equal to a base asymptote position and solving for θ. For the secant form solve $n(\theta - b) = \frac{\pi}{2} + k\pi$; for the cosecant and cotangent forms solve $n(\theta - b) = k\pi$. Equivalently — and this is the safer habit — the asymptotes stand at the zeros of the underlying $\cos\left(n(\theta - b)\right)$ or $\sin\left(n(\theta - b)\right)$. Note that a and c have no effect on them.

- **Period.** Only n alters the period, dividing the base period by $|n|$: the transformed secant and cosecant have period $\frac{2\pi}{|n|}$ and the transformed cotangent has period $\frac{\pi}{|n|}$.
- **Range.** The branches of sec and $cosec$ rest on the lines $y=\pm 1$; after the dilation by a and the translation by c they rest on $y=c+a$ and $y=c-a$, so the range becomes $y\le c-|a|$ or $y\ge c+|a|$. The cotangent still takes every real value, so its range stays $\mathbb{R}$ whatever a and c are.

To sketch such a graph: draw the new asymptotes first; mark the branch tips at heights $c+a$ and $c-a$ (the images of the local extrema at ± 1) directly above or below the points where the underlying $\sin$ or $\cos$ reaches ± 1; then fill in branches of the same shape as the base graph, turned upside down if $a<0$.

Sketching $y=\frac{1}{f(x)}$ in general.

The reciprocal circular functions are three instances of a single construction: given *any* function $y=f(x)$, its **reciprocal function** is $y=\frac{1}{f(x)}$, and the graph of the reciprocal can be read straight off the graph of f, with no fresh table of values. Six features transfer.

1. **Vertical asymptotes at the zeros of f.** Where $f(x)=0$ the quotient $\frac{1}{f(x)}$ is undefined, and as x approaches such a point $f(x)\to 0$, so $\left|\frac{1}{f(x)}\right|\to\infty$. Every zero of f — a point on the x-axis — becomes a **vertical asymptote** of the reciprocal.
2. **Horizontal asymptote $y=0$.** Where $|f(x)|\to\infty$ the reciprocal $\frac{1}{f(x)}\to 0$, so wherever the graph of f runs off to $\pm\infty$ the reciprocal hugs the x-axis; the line $y=0$ is a horizontal asymptote. The reciprocal never actually attains 0, since $\frac{1}{f(x)}=0$ has no solution.
3. **The two graphs meet where $f(x)=\pm 1$.** Setting $f(x)=\frac{1}{f(x)}$ gives $(f(x))^2=1$, that is $f(x)=\pm 1$. Where $f(x)=1$ the reciprocal is also 1, and where $f(x)=-1$ it is also -1; these are exactly the points at which $y=f(x)$ crosses $y=\frac{1}{f(x)}$. Plot them first — they pin the reciprocal to the lines $y=1$ and $y=-1$.
4. **Same sign as f.** Because $\frac{1}{f(x)}$ and $f(x)$ always share a sign, the reciprocal lies above the x-axis exactly where f does and below it exactly where f does. A large positive value of f gives a small positive reciprocal and a small positive value of f gives a large positive reciprocal — "big becomes small, small becomes big", with the sign preserved.
5. **Turning points swap type.** Where f has a turning point (and $f\ne 0$ there) the reciprocal has a turning point at the **same** x-value but of the **opposite** type: a local *maximum* of f becomes a local *minimum* of $\frac{1}{f}$, and a local *minimum* of f becomes a local *maximum* of $\frac{1}{f}$. The height simply inverts: if f turns at $(a, f(a))$ then the reciprocal turns at $\left(a, \frac{1}{f(a)}\right)$. (How far the new turning point sits from the x-axis therefore depends on the size of $f(a)$: a minimum value of $f(a)=4$ becomes a maximum of $\frac{1}{4}$, *nearer* the axis, whereas a minimum value of $f(a)=\frac{1}{2}$ becomes a maximum of 2, *farther* from it.)

6. **Opposite monotonicity on each sign interval.** On any interval where f keeps a constant sign and never vanishes, $\frac{1}{f}$ is **increasing** wherever f is **decreasing** and decreasing wherever f is increasing. Hence the reciprocal runs off towards an asymptote as f approaches a zero, and returns towards the x-axis as $|f|$ grows.

Marking the zeros (as asymptotes), the points where $f=\pm 1$ and the images of the turning points, then joining them using the sign and asymptote information above, reproduces the graph of $y=\frac{1}{f(x)}$. The three reciprocal circular functions above are just this construction applied to $f=\sin$, $f=\cos$ and $f=\tan$.

The Pythagorean identities.

The single identity $\sin^2\theta+\cos^2\theta=1$ generates two further identities once it is divided through by $\cos^2\theta$ or by $\sin^2\theta$. Dividing every term by $\cos^2\theta$ (valid wherever $\cos\theta\neq 0$),

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}\Rightarrow \tan^2\theta+1=sec^2\theta.$$

Dividing instead by $\sin^2\theta$ (valid wherever $\sin\theta\neq 0$),

$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}\Rightarrow 1+\cot^2\theta=cosec^2\theta.$$

These are usually quoted as

$$1+\tan^2\theta=sec^2\theta, 1+\cot^2\theta=cosec^2\theta.$$

Each can be rearranged as needed — for instance $sec^2\theta-\tan^2\theta=1$ and $cosec^2\theta-\cot^2\theta=1$ — and each is the reciprocal-function analogue of the original Pythagorean identity.

Simplifying and solving.

Two techniques cover most work. To **simplify** an expression, rewrite every reciprocal function in terms of $\sin\theta$ and $\cos\theta$, combine over a common denominator, and apply $\sin^2\theta+\cos^2\theta=1$; the two new identities often shorten this. To **solve** an equation, isolate a single reciprocal function and invert it back to its base function — for example $sec\,\theta=k$ becomes $\cos\theta=\frac{1}{k}$ — then solve that familiar equation over the required interval, being careful to find **every** solution there. When an equation mixes, say, $sec^2\theta$ with $\tan\theta$, use a Pythagorean identity to write it in terms of a single function first, giving a quadratic to factorise.

Worked examples

Example 1 — scaffolded

Find the exact value of (a) $sec\,\frac{\pi}{4}$, (b) $cosec\,\frac{\pi}{3}$, and (c) $\cot\frac{2\pi}{3}$.

Working	Comment
$sec\,\frac{\pi}{4}=\frac{1}{\cos(\pi/4)}=\frac{1}{\sqrt{2}/2}$.	Secant is the reciprocal of cosine; $\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$.
$=\frac{2}{\sqrt{2}}=\sqrt{2}$.	Simplify: $\frac{2}{\sqrt{2}}=\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\sqrt{2}$.
$cosec\,\frac{\pi}{3}=\frac{1}{\sin(\pi/3)}=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}$.	Cosecant is the reciprocal of sine; $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$.

Working	Comment
$=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$.	Rationalise the denominator.
$\cot\frac{2\pi}{3}=\frac{\cos(2\pi/3)}{\sin(2\pi/3)}=\frac{-1/2}{\sqrt{3}/2}$.	Use $\cot\theta=\frac{\cos\theta}{\sin\theta}$; $\frac{2\pi}{3}$ is in quadrant 2.
$=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$.	Divide, then rationalise; the result is negative, as $\cot$ is negative in quadrant 2.

Example 2 — scaffolded

Simplify (a) $\sin\theta\cot\theta$, (b) $\tan\theta\, cosec\,\theta$, and (c) $\left(1+\cot^2\theta\right)\sin^2\theta$.

Working	Comment
$\sin\theta\cot\theta=\sin\theta\cdot\frac{\cos\theta}{\sin\theta}$.	Write $\cot\theta$ as $\frac{\cos\theta}{\sin\theta}$.
$=\cos\theta$.	The factors of $\sin\theta$ cancel.
$\tan\theta\, cosec\,\theta=\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta}$.	Write each in terms of $\sin\theta$ and $\cos\theta$.
$=\frac{1}{\cos\theta}=sec\,\theta$.	The $\sin\theta$ cancels, leaving the reciprocal of $\cos\theta$.
$\left(1+\cot^2\theta\right)\sin^2\theta=cosec^2\theta\cdot\sin^2\theta$.	Apply the identity $1+\cot^2\theta=cosec^2\theta$.
$=\frac{1}{\sin^2\theta}\cdot\sin^2\theta=1$.	Since $cosec^2\theta=\frac{1}{\sin^2\theta}$.

Example 3 — unscaffolded

Solve $sec^2\theta=2\tan\theta$ for $\theta\in[0,2\pi]$.

Every term can be written using $\tan\theta$ alone. Replacing $sec^2\theta$ by $1+\tan^2\theta$,

$$1+\tan^2\theta=2\tan\theta\Rightarrow\tan^2\theta-2\tan\theta+1=0.$$

The left-hand side is a perfect square, $\left(\tan\theta-1\right)^2=0$, so $\tan\theta=1$. Over $[0,2\pi]$ the tangent equals 1 at the first- and third-quadrant angles

$$\theta=\frac{\pi}{4},\theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4}.$$

Both angles have $\cos\theta\neq0$, so $sec\,\theta$ is defined there and each is a genuine solution.

$\therefore\theta=\frac{\pi}{4}$ or $\theta=\frac{5\pi}{4}$.

Example 4 — unscaffolded

Prove that $\tan\theta+\cot\theta=sec\,\theta\, cosec\,\theta$ for all θ at which both sides are defined.

Writing each function on the left in terms of $\sin\theta$ and $\cos\theta$ and combining over the common denominator $\sin\theta\cos\theta$,

$$\tan\theta+\cot\theta=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}.$$

By the Pythagorean identity the numerator is 1, so

$$\tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta} = \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta} = sec\,\theta\, cosec\,\theta,$$

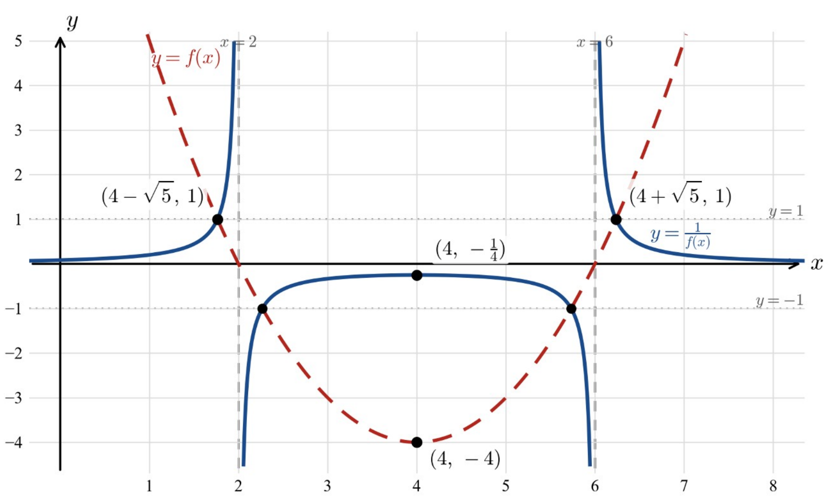
which is the right-hand side, as required.

Example 5 — scaffolded

Sketch the graph of $y = \frac{1}{f(x)}$, where $f(x) = x^2 - 8x + 12$, showing its asymptotes, its turning point, and the points at which it meets $y = f(x)$.

Working	Comment
$f(x) = x^2 - 8x + 12 = (x-2)(x-6)$, so $f(x) = 0$ at $x = 2$ and $x = 6$.	Factorise to find the zeros of f.
Vertical asymptotes $x = 2$ and $x = 6$; horizontal asymptote $y = 0$.	The reciprocal blows up at the zeros of f and tends to 0 as $f \to \pm\infty$.
f has its vertex at $x = \frac{2+6}{2} = 4$, where $f(4) = 16 - 32 + 12 = -4$.	The axis of symmetry sits midway between the zeros; this is a minimum of f.
Turning point of $\frac{1}{f}$: $\left(4, \frac{1}{-4}\right) = \left(4, -\frac{1}{4}\right)$, a local **maximum**.	Same x-value, reciprocated height; a minimum of f becomes a maximum of $\frac{1}{f}$.
Meets $y = f(x)$ where $f(x) = 1$: $x^2 - 8x + 11 = 0 \Rightarrow x = 4 \pm \sqrt{5}$.	Solve $f = 1$; the two curves cross on the line $y = 1$.
Meets $y = f(x)$ where $f(x) = -1$: $x^2 - 8x + 13 = 0 \Rightarrow x = 4 \pm \sqrt{3}$.	Solve $f = -1$; the two curves cross on the line $y = -1$.
For $x < 2$ or $x > 6$, $f > 0$ so $\frac{1}{f} > 0$; for $2 < x < 6$, $f < 0$ so $\frac{1}{f} < 0$.	The reciprocal keeps the sign of f on each interval.

Overlaying $y = f(x)$ (dashed) and $y = \frac{1}{f(x)}$ (solid) shows every feature at once: the solid curve rises to the asymptotes at $x = 2$ and $x = 6$, dips to its local maximum $\left(4, -\frac{1}{4}\right)$ on the middle branch, and crosses the dashed parabola on the lines $y = \pm 1$.



Example 6 — unscaffolded

Sketch $y=2\,sec(2\theta)-1$ for $-\pi\le\theta\le\pi$, showing the asymptotes and the branch tips, and state the period and the range.

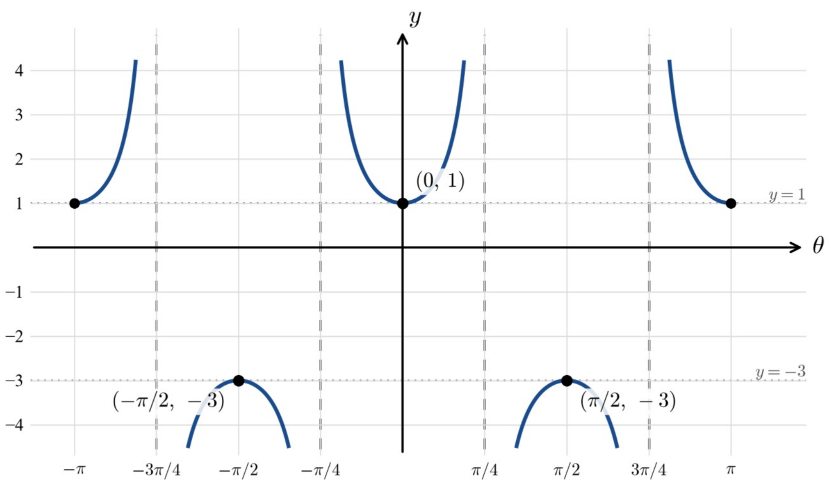
Comparing with $y=a\,sec(n(\theta-b))+c$ gives $a=2$, $n=2$, $b=0$ and $c=-1$: the secant graph is dilated by factor 2 from the θ-axis and by factor $\frac{1}{2}$ from the y-axis, then translated 1 unit down. The underlying function is $\cos 2\theta$, so the asymptotes stand where $\cos 2\theta=0$, that is

$$2\theta=\frac{\pi}{2}+k\pi\Rightarrow\theta=\frac{\pi}{4}+\frac{k\pi}{2},$$

which gives $\theta=\pm\frac{\pi}{4}$ and $\theta=\pm\frac{3\pi}{4}$ inside $[-\pi,\pi]$. Since $n=2$, the period is $\frac{2\pi}{2}=\pi$.

The branch tips lie above or below the angles at which $\cos 2\theta=\pm1$. Where $\cos 2\theta=1$ — at $2\theta=0,\pm2\pi$, i.e. $\theta=0,\pm\pi$ — the secant has its local minimum 1, so $y=2(1)-1=1$; where $\cos 2\theta=-1$ — at $2\theta=\pm\pi$, i.e. $\theta=\pm\frac{\pi}{2}$ — it has its local maximum -1, so $y=2(-1)-1=-3$. Because $a=2>0$ there is no reflection, so the $\cup$-branches still open upwards from the tips at height 1 and the $\cap$-branches open downwards from the tips at height -3. Finally $|sec\,2\theta|\ge1$, so $y\ge2(1)-1=1$ or $y\le2(-1)-1=-3$.

$\therefore$ the asymptotes are $\theta=\pm\frac{\pi}{4}$ and $\theta=\pm\frac{3\pi}{4}$, the branch tips are the minima $(0,1)$ and $(\pm\pi,1)$ and the maxima $\left(\pm\frac{\pi}{2},-3\right)$, the period is π, and the range is $y\le-3$ or $y\ge1$.



Practice questions

Scaffolded

Q1. Find the exact value of each. (a) $sec\frac{\pi}{3}$. (b) $cosec\frac{\pi}{4}$. (c) $\cot\frac{\pi}{6}$. (d) $sec\frac{5\pi}{6}$.

Q2. Find the exact value of each, noting the quadrant. (a) $cosec\frac{2\pi}{3}$. (b) $\cot\frac{5\pi}{6}$. (c) $sec\frac{4\pi}{3}$. (d) $cosec\frac{7\pi}{6}$.

Q3. State the maximal domain, the range and the period of each function. (a) $y=cosec\,\theta$. (b) $y=sec\,\theta$. (c) $y=\cot\theta$.

Q4. An acute angle θ satisfies $\cot\theta=\frac{24}{7}$. (a) Use $1+\cot^2\theta=cosec^2\theta$ to find $cosec\,\theta$. (b) Hence find $\sin\theta$. (c) Find $\cos\theta$. (d) State $sec\,\theta$.

Q5. Simplify each expression to a single circular function or a constant. (a) $\cos\theta\, cosec\,\theta$. (b) $\frac{sec\,\theta}{cosec\,\theta}$. (c) $\cot\theta\, sec\,\theta\sin\theta$. (d) $\left(sec^2\theta-1\right)\cot^2\theta$.

Q6. Solve each equation for $\theta\in[0,2\pi]$. (a) $sec\,\theta=2$. (b) $cosec\,\theta=\sqrt{2}$. (c) $\cot\theta=1$. (d) $sec\,\theta=-\sqrt{2}$.

Unscaffolded

Q7. Find the exact value of $\cot\frac{4\pi}{3}+sec\,\pi$.

Q8. Find the exact value of $cosec\,\frac{7\pi}{4}-\cot\frac{3\pi}{4}$.

Q9. The graph of $y=cosec\,\theta$ is drawn for $0<\theta<2\pi$. (a) Give the equation of the vertical asymptote in this interval. (b) State the coordinates of the local minimum and of the local maximum. (c) State the range.

Q10. An angle θ in the second quadrant satisfies $\tan\theta=-\frac{4}{3}$. Find $sec\,\theta$ and $cosec\,\theta$.

Q11. Simplify $\frac{cosec\,\theta-\sin\theta}{\cos\theta}$.

Q12. Prove that $\frac{1+\tan^2\theta}{1+\cot^2\theta}=\tan^2\theta$.

Q13. Prove that $(sec\,\theta-1)(sec\,\theta+1)=\tan^2\theta$.

Q14. Prove that $\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}=2\,cosec^2\theta$.

Q15. Solve $cosec\,\theta=-2$ for $\theta\in[0,2\pi]$.

Q16. Solve $\cot\theta=-1$ for $\theta\in[0,2\pi]$.

Q17. Solve $\tan^2\theta=sec\,\theta+1$ for $\theta\in[0,2\pi]$.

Q18. Solve $2\cot^2\theta=3\,cosec\,\theta$ for $\theta\in[0,2\pi]$.

Q19. Let $f(x)=x-4$, and consider the reciprocal function $y=\frac{1}{f(x)}$. (a) State the equations of the vertical and horizontal asymptotes. (b) Find the coordinates of the two points at which $y=\frac{1}{f(x)}$ meets $y=f(x)$. (c) State whether $y=\frac{1}{f(x)}$ is increasing or decreasing on each side of its vertical asymptote, justifying your answer from the behaviour of f.

Q20. Let $f(x)=x^2-12x+35$. (a) State the equations of the vertical and horizontal asymptotes of $y=\frac{1}{f(x)}$. (b) Find the coordinates and the nature of the turning point of $y=\frac{1}{f(x)}$. (c) Find the y-intercept of $y=\frac{1}{f(x)}$. (d) Find the x-values at which $y=\frac{1}{f(x)}$ meets $y=f(x)$.

Q21. The graph of $y = 3\,cosec\left(\theta - \frac{\pi}{4}\right) - 2$ is drawn for $\theta \in [0, 2\pi]$. (a) State the equations of its vertical asymptotes in this interval. (b) State its period and its range. (c) Find the coordinates of the local minimum and of the local maximum in this interval. (d) Explain why the graph has no θ-intercepts.

Q22. The graph of $y = \cot(2\theta)$ is drawn for $0 < \theta < \pi$. (a) State the equation of the vertical asymptote in this interval. (b) State its period and its range. (c) Find the θ-intercepts in this interval. (d) Find the coordinates of the points in this interval at which the graph meets $y = 1$.

Answers

Q1. (a) 2; (b) $\sqrt{2}$; (c) $\sqrt{3}$; (d) $-\frac{2\sqrt{3}}{3}$. **Q2.** (a) $\frac{2\sqrt{3}}{3}$; (b) $-\sqrt{3}$; (c) -2; (d) -2. **Q3.** (a) domain $\theta \neq n\pi$, range $y \leq -1$ or $y \geq 1$, period 2π; (b) domain $\theta \neq \frac{\pi}{2}+n\pi$, range $y \leq -1$ or $y \geq 1$, period 2π; (c) domain $\theta \neq n\pi$, range $\mathbb{R}$, period π. **Q4.** (a) $cosec\,\theta=\frac{25}{7}$; (b) $\sin\theta=\frac{7}{25}$; (c) $\cos\theta=\frac{24}{25}$; (d) $sec\,\theta=\frac{25}{24}$. **Q5.** (a) $\cot\theta$; (b) $\tan\theta$; (c) 1; (d) 1. **Q6.** (a) $\theta=\frac{\pi}{3},\frac{5\pi}{3}$; (b) $\theta=\frac{\pi}{4},\frac{3\pi}{4}$; (c) $\theta=\frac{\pi}{4},\frac{5\pi}{4}$; (d) $\theta=\frac{3\pi}{4},\frac{5\pi}{4}$. **Q7.** $\frac{\sqrt{3}}{3}-1$. **Q8.** $1-\sqrt{2}$. **Q9.** (a) $\theta=\pi$; (b) local minimum $\left(\frac{\pi}{2},1\right)$, local maximum $\left(\frac{3\pi}{2},-1\right)$; (c) $y \leq -1$ or $y \geq 1$. **Q10.** $sec\,\theta=-\frac{5}{3}$, $cosec\,\theta=\frac{5}{4}$. **Q11.** $\cot\theta$. **Q12.** Proof. **Q13.** Proof. **Q14.** Proof. **Q15.** $\theta=\frac{7\pi}{6},\frac{11\pi}{6}$. **Q16.** $\theta=\frac{3\pi}{4},\frac{7\pi}{4}$. **Q17.** $\theta=\frac{\pi}{3},\pi,\frac{5\pi}{3}$. **Q18.** $\theta=\frac{\pi}{6},\frac{5\pi}{6}$. **Q19.** (a) vertical $x=4$, horizontal $y=0$; (b) $(5,1)$ and $(3,-1)$; (c) decreasing on each side of $x=4$. **Q20.** (a) vertical $x=5$ and $x=7$, horizontal $y=0$; (b) $(6,-1)$, a local maximum; (c) $\left(0,\frac{1}{35}\right)$; (d) meets where $f=1$ at $x=6\pm\sqrt{2}$, and where $f=-1$ at $x=6$. **Q21.** (a) $\theta=\frac{\pi}{4}$ and $\theta=\frac{5\pi}{4}$; (b) period 2π, range $y \leq -5$ or $y \geq 1$; (c) local minimum $\left(\frac{3\pi}{4},1\right)$, local maximum $\left(\frac{7\pi}{4},-5\right)$; (d) it would need $cosec\left(\theta-\frac{\pi}{4}\right)=\frac{2}{3}$, which is impossible since every cosecant value has modulus at least 1. **Q22.** (a) $\theta=\frac{\pi}{2}$; (b) period $\frac{\pi}{2}$, range $\mathbb{R}$; (c) $\theta=\frac{\pi}{4},\frac{3\pi}{4}$; (d) $\left(\frac{\pi}{8},1\right)$ and $\left(\frac{5\pi}{8},1\right)$.

Fully-worked solutions

Q1. Each is one over a known exact value. (a) $sec\frac{\pi}{3}=\frac{1}{\cos(\pi/3)}=\frac{1}{1/2}=2$. (b) $cosec\frac{\pi}{4}=\frac{1}{\sin(\pi/4)}=\frac{1}{\sqrt{2}/2}=\frac{2}{\sqrt{2}}=\sqrt{2}$. (c) $\cot\frac{\pi}{6}=\frac{\cos(\pi/6)}{\sin(\pi/6)}=\frac{\sqrt{3}/2}{1/2}=\sqrt{3}$. (d) $\frac{5\pi}{6}$ is in quadrant 2, where cosine is negative: $sec\frac{5\pi}{6}=\frac{1}{\cos(5\pi/6)}=\frac{1}{-\sqrt{3}/2}=-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}$.

Q2. (a) $cosec\frac{2\pi}{3}=\frac{1}{\sin(2\pi/3)}=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$. (b) $\cot\frac{5\pi}{6}=\frac{\cos(5\pi/6)}{\sin(5\pi/6)}=\frac{-\sqrt{3}/2}{1/2}=-\sqrt{3}$. (c) $sec\frac{4\pi}{3}=\frac{1}{\cos(4\pi/3)}=\frac{1}{-1/2}=-2$. (d) $cosec\frac{7\pi}{6}=\frac{1}{\sin(7\pi/6)}=\frac{1}{-1/2}=-2$.

Q3. Each reciprocal function is undefined where its base function is zero, which removes those angles from the domain; the range and period follow from the base function. (a) $y=cosec\,\theta$: undefined where $\sin\theta=0$, so domain $\theta \neq n\pi$; since $|\sin\theta| \leq 1$ the reciprocal has $|y| \geq 1$, i.e. $y \leq -1$ or $y \geq 1$; period 2π. (b) $y=sec\,\theta$: undefined where $\cos\theta=0$, so domain $\theta \neq \frac{\pi}{2}+n\pi$; range $y \leq -1$ or $y \geq 1$; period 2π. (c) $y=\cot\theta$: undefined where $\sin\theta=0$, so domain $\theta \neq n\pi$; since $\cot\theta=\frac{1}{\tan\theta}$ takes every non-zero value as $\tan\theta$ does, and $\cot\theta=0$ where $\cos\theta=0$, the range is $\mathbb{R}$; period π.

Q4. (a) Using $1+\cot^2\theta=cosec^2\theta$ with $\cot\theta=\frac{24}{7}$,

$$cosec^2\theta = 1 + \left(\frac{24}{7}\right)^2 = 1 + \frac{576}{49} = \frac{625}{49},$$

so $cosec\,\theta = \frac{25}{7}$ (positive, as θ is acute). (b) $\sin\theta = \frac{1}{cosec\,\theta} = \frac{7}{25}$. (c) From $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{24}{7}$, $\cos\theta = \frac{24}{7}\sin\theta = \frac{24}{7} \cdot \frac{7}{25} = \frac{24}{25}$. (d) $sec\,\theta = \frac{1}{\cos\theta} = \frac{25}{24}$.

Q5. Rewrite each in terms of $\sin\theta$ and $\cos\theta$. (a) $\cos\theta\, cosec\,\theta = \cos\theta \cdot \frac{1}{\sin\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta$. (b) $\frac{sec\,\theta}{cosec\,\theta} = \frac{1/\cos\theta}{1/\sin\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$. (c) $\cot\theta\, sec\,\theta \sin\theta = \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\cos\theta} \cdot \sin\theta = 1$. (d) $sec^2\theta - 1 = \tan^2\theta$, so $\left(sec^2\theta - 1\right)\cot^2\theta = \tan^2\theta \cdot \frac{1}{\tan^2\theta} = 1$.

Q6. Invert each reciprocal function, then solve over $[0, 2\pi]$. (a) $sec\,\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$. (b) $cosec\,\theta = \sqrt{2} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$. (c) $\cot\theta = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$. (d) $sec\,\theta = -\sqrt{2} \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$.

Q7. The angle $\frac{4\pi}{3}$ is in quadrant 3, where sine and cosine are both negative:

$$\cot\frac{4\pi}{3} = \frac{\cos(4\pi/3)}{\sin(4\pi/3)} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Also $sec\,\pi = \frac{1}{\cos\pi} = \frac{1}{-1} = -1$. Hence $\cot\frac{4\pi}{3} + sec\,\pi = \frac{\sqrt{3}}{3} - 1$.

Q8. The angle $\frac{7\pi}{4}$ is in quadrant 4: $cosec\frac{7\pi}{4} = \frac{1}{\sin(7\pi/4)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$. The angle $\frac{3\pi}{4}$ is in quadrant 2: $\cot\frac{3\pi}{4} = \frac{\cos(3\pi/4)}{\sin(3\pi/4)} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$. Therefore $cosec\frac{7\pi}{4} - \cot\frac{3\pi}{4} = -\sqrt{2} - (-1) = 1 - \sqrt{2}$.

Q9. The graph is the reciprocal of $y = \sin\theta$. (a) A vertical asymptote occurs where $\sin\theta = 0$; within $0 < \theta < 2\pi$ this is at $\theta = \pi$. (b) The reciprocal is smallest in size where $|\sin\theta|$ is largest. Where $\sin\theta = 1$, at $\theta = \frac{\pi}{2}$, there is a local minimum $\left(\frac{\pi}{2}, 1\right)$; where $\sin\theta = -1$, at $\theta = \frac{3\pi}{2}$, there is a local maximum $\left(\frac{3\pi}{2}, -1\right)$. (c) Since $|cosec\,\theta| \geq 1$, the range is $y \leq -1$ or $y \geq 1$.

Q10. With θ in quadrant 2, cosine is negative and sine is positive. From $1 + \tan^2\theta = sec^2\theta$,

$$sec^2\theta = 1 + \left(-\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9},$$

so $sec\,\theta = \pm\frac{5}{3}$; the negative sign is taken because $\cos\theta < 0$, giving $sec\,\theta = -\frac{5}{3}$. Then $\cos\theta = -\frac{3}{5}$, and $\sin\theta = \tan\theta\cos\theta = \left(-\frac{4}{3}\right)\left(-\frac{3}{5}\right) = \frac{4}{5}$, so $cosec\,\theta = \frac{1}{\sin\theta} = \frac{5}{4}$.

Q11. Write $cosec\,\theta = \frac{1}{\sin\theta}$ and combine the numerator over $\sin\theta$:

$$\frac{cosec\,\theta-\sin\theta}{\cos\theta}=\frac{\frac{1}{\sin\theta}-\sin\theta}{\cos\theta}=\frac{\frac{1-\sin^2\theta}{\sin\theta}}{\cos\theta}=\frac{\cos^2\theta}{\sin\theta\cos\theta}=\frac{\cos\theta}{\sin\theta}=\cot\theta,$$

using $1-\sin^2\theta=\cos^2\theta$.

Q12. Apply a Pythagorean identity to the numerator and to the denominator separately, then write both in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{1+\tan^2\theta}{1+\cot^2\theta}=\frac{sec^2\theta}{cosec^2\theta}=\frac{1/\cos^2\theta}{1/\sin^2\theta}=\frac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta,$$

which is the right-hand side, as required.

Q13. Expanding the difference of two squares and applying $1+\tan^2\theta=sec^2\theta$ in the form $sec^2\theta-1=\tan^2\theta$,

$$(sec\,\theta-1)(sec\,\theta+1)=sec^2\theta-1=\tan^2\theta,$$

as required.

Q14. Adding the two fractions over the common denominator $(1-\cos\theta)(1+\cos\theta)=1-\cos^2\theta$,

$$\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}=\frac{(1+\cos\theta)+(1-\cos\theta)}{1-\cos^2\theta}=\frac{2}{1-\cos^2\theta}=\frac{2}{\sin^2\theta}=2\,cosec^2\theta,$$

using $1-\cos^2\theta=\sin^2\theta$, as required.

Q15. $cosec\,\theta=-2\Rightarrow\sin\theta=-\frac{1}{2}$. Over $[0,2\pi]$ the sine equals $-\frac{1}{2}$ in quadrants 3 and 4, at $\theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$ and $\theta=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$.

Q16. $\cot\theta=-1\Rightarrow\tan\theta=-1$. Over $[0,2\pi]$ the tangent equals -1 in quadrants 2 and 4, at $\theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$ and $\theta=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$.

Q17. Replace $\tan^2\theta$ by $sec^2\theta-1$ so the equation involves only $sec\,\theta$:

$$sec^2\theta-1=sec\,\theta+1\Rightarrow sec^2\theta-sec\,\theta-2=0.$$

Factorising, $(sec\,\theta-2)(sec\,\theta+1)=0$, so $sec\,\theta=2$ or $sec\,\theta=-1$. Then $\cos\theta=\frac{1}{2}$ gives $\theta=\frac{\pi}{3},\frac{5\pi}{3}$, and $\cos\theta=-1$ gives $\theta=\pi$. All three have $\cos\theta\neq0$, so

$$\theta=\frac{\pi}{3},\pi,\frac{5\pi}{3}.$$

Q18. Replace $\cot^2\theta$ by $cosec^2\theta-1$ so the equation involves only $cosec\,\theta$:

$$2(cosec^2\theta-1)=3\,cosec\,\theta\Rightarrow2\,cosec^2\theta-3\,cosec\,\theta-2=0.$$

Factorising, $(2\,cosec\,\theta+1)(cosec\,\theta-2)=0$, so $cosec\,\theta=-\frac{1}{2}$ or $cosec\,\theta=2$. The first is impossible, because $|cosec\,\theta|\geq1$ for every θ in the domain. The second gives $\sin\theta=\frac{1}{2}$, so over $[0,2\pi]$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6},$$

both of which have $\sin\theta \neq 0$ and so are genuine solutions.

Q19. Here $f(x) = x - 4$ has the single zero $x = 4$, so $y = \frac{1}{x-4}$. (a) The reciprocal is undefined at the zero of f, giving the vertical asymptote $x = 4$; as $x \to \pm\infty$ we have $f \to \pm\infty$, so $\frac{1}{f} \to 0$, giving the horizontal asymptote $y = 0$. (b) The two graphs meet where $f = \frac{1}{f}$, i.e. $(f(x))^2 = 1$, so $f(x) = \pm 1$. From $f(x) = 1$: $x - 4 = 1$, giving $x = 5$ and the point $(5, 1)$; from $f(x) = -1$: $x - 4 = -1$, giving $x = 3$ and the point $(3, -1)$. (c) The line $f(x) = x - 4$ is increasing everywhere (its gradient is $1 > 0$). On an interval of constant sign the reciprocal is decreasing wherever f is increasing, so $y = \frac{1}{x-4}$ is **decreasing** both on $x < 4$ and on $x > 4$.

Q20. Here $f(x) = x^2 - 12x + 35 = (x-5)(x-7)$, with zeros $x = 5$ and $x = 7$. (a) The reciprocal has vertical asymptotes at the zeros of f, namely $x = 5$ and $x = 7$, and the horizontal asymptote $y = 0$. (b) The vertex of f lies midway between the zeros, at $x = \frac{5+7}{2} = 6$, where $f(6) = 36 - 72 + 35 = -1$. This is a minimum of f, so the reciprocal has a local **maximum** at $\left(6, \frac{1}{-1}\right) = (6, -1)$. (c) At $x = 0$, $f(0) = 35$, so the y-intercept of the reciprocal is $\left(0, \frac{1}{35}\right)$. (d) The graphs meet where $f(x) = \pm 1$. From $f(x) = 1$: $x^2 - 12x + 34 = 0$, so $x = 6 \pm \sqrt{2}$. From $f(x) = -1$: $x^2 - 12x + 36 = 0$, i.e. $(x-6)^2 = 0$, so $x = 6$ — a single point, where the reciprocal's maximum just touches the line $y = -1$.

Q21. Here $a = 3$, $n = 1$, $b = \frac{\pi}{4}$ and $c = -2$, so the cosecant graph is dilated by factor 3 from the θ-axis, translated $\frac{\pi}{4}$ to the right and 2 units down. (a) The asymptotes stand at the zeros of the underlying $\sin\left(\theta - \frac{\pi}{4}\right)$, so $\theta - \frac{\pi}{4} = k\pi$, giving $\theta = \frac{\pi}{4} + k\pi$. Within $[0, 2\pi]$ these are $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$. (b) Since $n = 1$ the period is unchanged at 2π. The branches rest on $y = c + a = 1$ and $y = c - a = -5$, so the range is $y \leq -5$ or $y \geq 1$. (c) Where $\sin\left(\theta - \frac{\pi}{4}\right) = 1$, that is $\theta - \frac{\pi}{4} = \frac{\pi}{2}$ and so $\theta = \frac{3\pi}{4}$, the cosecant has its local minimum 1 and $y = 3(1) - 2 = 1$, giving $\left(\frac{3\pi}{4}, 1\right)$. Where $\sin\left(\theta - \frac{\pi}{4}\right) = -1$, that is $\theta - \frac{\pi}{4} = \frac{3\pi}{2}$ and so $\theta = \frac{7\pi}{4}$, it has its local maximum -1 and $y = 3(-1) - 2 = -5$, giving $\left(\frac{7\pi}{4}, -5\right)$. (d) A θ-intercept needs $3\,cosec\left(\theta - \frac{\pi}{4}\right) - 2 = 0$, i.e. $cosec\left(\theta - \frac{\pi}{4}\right) = \frac{2}{3}$. This is impossible, since $\left|cosec\left(\theta - \frac{\pi}{4}\right)\right| \geq 1$ wherever it is defined, while $\frac{2}{3} < 1$. (Equivalently, $y = 0$ lies in the gap $-5 < y < 1$ excluded by the range found in (b).)

Q22. Here the cotangent is dilated by factor $\frac{1}{2}$ from the y-axis and nothing else is changed, so $a = 1$, $n = 2$, $b = 0$ and $c = 0$. (a) The asymptotes stand at the zeros of $\sin 2\theta$, so $2\theta = k\pi$ and $\theta = \frac{k\pi}{2}$. The only one strictly inside $0 < \theta < \pi$ is $\theta = \frac{\pi}{2}$. (b) Since $n = 2$ the period is $\frac{\pi}{2}$, half that of $\cot\theta$. The cotangent takes every real value and neither $a = 1$ nor $c = 0$ alters that, so the range is $\mathbb{R}$. (c) $\cot 2\theta = 0$ where $\cos 2\theta = 0$, that is $2\theta = \frac{\pi}{2} + k\pi$, so $\theta = \frac{\pi}{4} + \frac{k\pi}{2}$. Within

$0<\theta<\pi$ this gives $\theta=\frac{\pi}{4}$ and $\theta=\frac{3\pi}{4}$. (d) $\cot 2\theta=1$ gives $\tan 2\theta=1$, so $2\theta=\frac{\pi}{4}+k\pi$ and $\theta=\frac{\pi}{8}+\frac{k\pi}{2}$. Within $0<\theta<\pi$ this gives $\theta=\frac{\pi}{8}$ and $\theta=\frac{5\pi}{8}$, so the points are $\left(\frac{\pi}{8},1\right)$ and $\left(\frac{5\pi}{8},1\right)$.

Theory

Each of the circular functions $\sin$, $\cos$ and $\tan$ met in Mathematical Methods is **periodic**, so it takes every value in its range infinitely often: a horizontal line $y=k$ drawn at a value k in the range meets the graph in infinitely many points. A function like this is **many-to-one**, and a many-to-one function has no inverse function — reversing it would have to send a single output back to many inputs at once. To build a genuine **inverse circular function** we must first cut the domain down to an interval on which the function is **one-to-one** while still producing *every* value in its range. This section carries out that restriction for $\sin$, $\cos$ and $\tan$, defines the three inverse functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, records their domains, their ranges (the **principal values**) and their graphs, and then uses them to find exact values, apply simple transformations and solve equations.

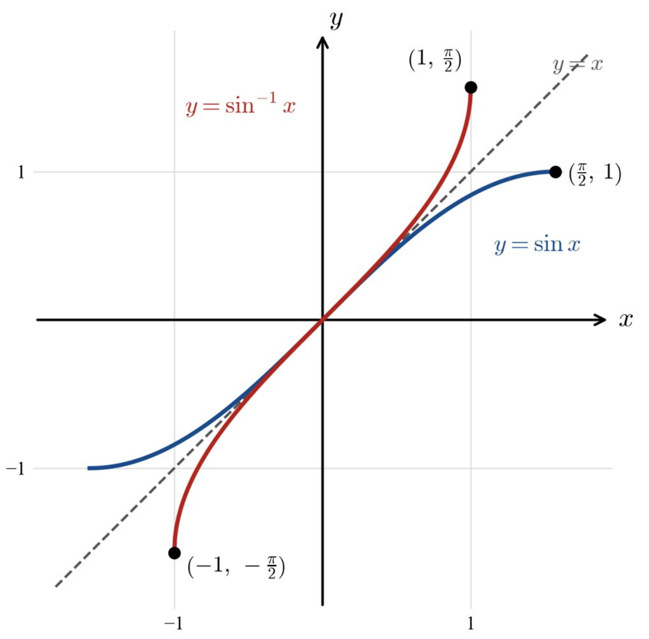
A warning about the notation. The symbol $\sin^{-1}x$ denotes the value of the **inverse function** — the angle whose sine is x — and *not* the reciprocal $\dfrac{1}{\sin x}$, which is $cosec\, x$ from the previous section. The "-1" here is the inverse-function superscript, exactly as f^{-1} denotes the inverse of a function f. The alternative names $\arcsin$, $\arccos$ and $\arctan$ carry the same meaning and avoid the clash entirely.

Restricting the sine.

On the interval $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ the sine function increases steadily from -1 up to 1, so it is one-to-one there and it already reaches every value in its full range $[-1,1]$. This interval is taken as the **principal domain** of the sine. The restricted function has an inverse: for each $x\in[-1,1]$,

$$\sin^{-1}(x)\text{ is the unique angle }\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\text{for which }\sin\theta=x.$$

Thus $\sin^{-1}$ has **domain** $[-1,1]$ and **range** $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. Because an inverse function is the reflection of the original in the line $y=x$, the graph of $y=\sin^{-1}(x)$ is the mirror image of the restricted sine curve in $y=x$; it increases through the points $\left(-1,-\dfrac{\pi}{2}\right)$, $(0,0)$ and $\left(1,\dfrac{\pi}{2}\right)$.

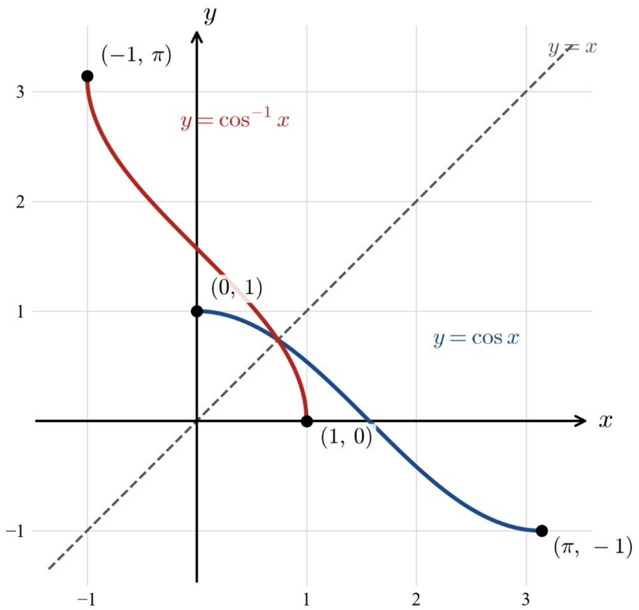


Restricting the cosine.

The cosine cannot be inverted on any interval containing a turning point in its **interior**, because the curve rises on one side of such a point and falls on the other, so the same value is taken twice. The symmetric interval used for sine will not do, since $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ has the maximum $(0,1)$ strictly inside it. Instead the standard choice is $[0,\pi]$, on which the cosine **decreases** steadily from 1 down to -1; the turning points $(0,1)$ and $(\pi,-1)$ sit at the two **endpoints**, so the function is one-to-one there and again attains every value in $[-1,1]$. This is the **principal domain** of the cosine, and it gives the inverse: for each $x\in[-1,1]$,

$$\cos^{-1}(x)\text{ is the unique angle }\theta\in[0,\pi]\text{ for which }\cos\theta=x.$$

So $\cos^{-1}$ has **domain** $[-1,1]$ and **range** $[0,\pi]$. Its graph is the reflection of the restricted cosine in $y=x$; it decreases through $(-1,\pi)$, $\left(0,\frac{\pi}{2}\right)$ and $(1,0)$.

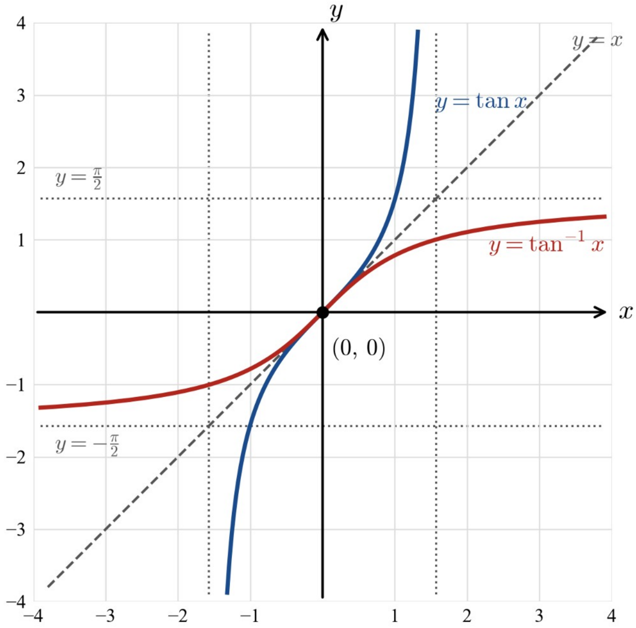


Restricting the tangent.

On the open interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ the tangent increases from $-\infty$ to $+\infty$, taking every real value exactly once; the interval is open because tan is undefined at $\pm\frac{\pi}{2}$. This is the **principal domain** of the tangent, and its inverse is defined for **every** real number: for each $x\in\mathbb{R}$,

$$\tan^{-1}(x)\text{ is the unique angle }\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\text{ for which }\tan\theta=x.$$

Hence $\tan^{-1}$ has **domain** $\mathbb{R}$ and **range** $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Its graph is the reflection of the restricted tangent in $y=x$; it increases through $(0,0)$, and the two vertical asymptotes $x=\pm\frac{\pi}{2}$ of the tangent reflect into the two **horizontal asymptotes** $y=\pm\frac{\pi}{2}$ that the inverse curve approaches but never reaches.



Domains and ranges together.

The three principal ranges are worth committing to memory, as every exact value and every transformation depends on them.

Function	Domain	Range (principal values)
$y=\sin^{-1}(x)$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\cos^{-1}(x)$	$[-1,1]$	$[0,\pi]$
$y=\tan^{-1}(x)$	$\mathbb{R}$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Exact values.

To evaluate an inverse circular function at an exact argument, find the angle **inside the principal range** whose sine, cosine or tangent equals that argument. The principal range is what makes the answer unique. For instance $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$, because $\frac{\pi}{6}$ lies in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and has sine $\frac{1}{2}$; and $\cos^{-1}(0)=\frac{\pi}{2}$, since $\frac{\pi}{2}$ lies in $[0,\pi]$ with cosine 0. A negative argument is handled the same way, but the sign of the answer depends on the range. Since sine and tangent are **odd** and their principal ranges are symmetric about 0,

$$\sin^{-1}(-x)=-\sin^{-1}(x),\tan^{-1}(-x)=-\tan^{-1}(x).$$

The cosine's principal range $[0,\pi]$ is **not** symmetric about 0, so cosine behaves differently: a negative argument gives an angle greater than $\frac{\pi}{2}$, and

$$\cos^{-1}(-x)=\pi-\cos^{-1}(x).$$

For example $\cos^{-1}\left(-\frac{1}{2}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

Compositions.

Applying a function and then its inverse returns the starting value, provided that value is legal for the first function applied. Reading in one order,

$$\sin\left(\sin^{-1}x\right)=x \text{ for all } x\in[-1,1],\ \tan\left(\tan^{-1}x\right)=x \text{ for all } x\in\mathbb{R},$$

and likewise $\cos\left(\cos^{-1}x\right)=x$ for $x\in[-1,1]$. Reading in the *other* order needs care, because the answer must land in the principal range:

$$\sin^{-1}(\sin\theta)=\theta \text{ only when } \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

If θ lies outside that interval, first replace $\sin\theta$ by its exact value and then take the inverse, which returns the *principal* angle with that sine rather than θ itself. A composition of a **different** pair, such as $\cos\left(\sin^{-1}x\right)$, is found with the Pythagorean identity: put $\theta=\sin^{-1}x$, so $\sin\theta=x$; then $\cos\theta=\sqrt{1-x^2}$, the positive root being correct because $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ keeps $\cos\theta\geq 0$.

Transformations.

A transformed inverse function such as $y=a\sin^{-1}(bx)+c$ is read exactly like any other transformation. The factor b multiplies the input, so for $\sin^{-1}$ and $\cos^{-1}$ the requirement that the input stay in $[-1,1]$ becomes $-1\leq bx\leq 1$, giving the **domain** $-\frac{1}{|b|}\leq x\leq\frac{1}{|b|}$; for $\tan^{-1}$ the domain remains all of $\mathbb{R}$. The outer constant a **scales** the principal range and c **shifts** it, so the two ends of the new range are $a\times(\text{lower end})+c$ and $a\times(\text{upper end})+c$. When $a>0$ these are the lower and upper ends respectively, but when $a<0$ they swap over, because multiplying by a negative number reverses order: always identify which of the two is the smaller before writing the range down. For example $y=2\tan^{-1}(x)$ has range $(-\pi,\pi)$, while $y=\cos^{-1}(3x)$ has domain $\left[-\frac{1}{3},\frac{1}{3}\right]$ and range $[0,\pi]$.

Solving basic equations.

To solve an equation such as $\sin^{-1}(x)=k$, apply the sine to both sides: $x=\sin k$ — but this has a solution only when k actually lies in the range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, since the left-hand side can equal nothing outside it. When the inverse function is transformed, first undo the outer operations (subtract c, divide by a) to isolate the inverse function, then apply the matching circular function to both sides, and finally check that any value found keeps the original input inside the correct domain.

Worked examples

Example 1 — scaffolded

Find the exact value of (a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, (b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, and (c) $\tan^{-1}(\sqrt{3})$.

Working	Comment
(a) Seek $\theta\in[0,\pi]$ with $\cos\theta=\frac{\sqrt{3}}{2}$.	The principal range of $\cos^{-1}$ is $[0,\pi]$.
$\theta=\frac{\pi}{6}$, so $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$.	$\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ and $\frac{\pi}{6}\in[0,\pi]$.
(b) Seek $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ with $\sin\theta=-\frac{\sqrt{2}}{2}$.	The principal range of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
$\theta=-\frac{\pi}{4}$, so $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$.	A negative argument gives a negative angle, since $\sin^{-1}$ is odd.

Working	Comment
(c) Seek $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with $\tan\theta=\sqrt{3}$.	The principal range of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.
$\theta=\frac{\pi}{3}$, so $\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$.	$\tan\frac{\pi}{3}=\sqrt{3}$ and $\frac{\pi}{3}$ lies in the range.

Example 2 — scaffolded

State the maximal domain and the range of (a) $y=3\sin^{-1}(x)$, (b) $y=\cos^{-1}(2x)$, and (c) $y=\tan^{-1}(x)+\frac{\pi}{2}$.

Working	Comment
(a) Domain: need $-1\le x\le 1$, so the domain is $[-1,1]$.	Only the input to $\sin^{-1}$ is restricted; the outer factor 3 does not affect it.
Range: $\sin^{-1}(x)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $3\sin^{-1}(x)\in\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]$.	Multiply each end of the principal range by 3.
(b) Domain: need $-1\le 2x\le 1$, so $-\frac{1}{2}\le x\le\frac{1}{2}$; the domain is $\left[-\frac{1}{2},\frac{1}{2}\right]$.	The input to $\cos^{-1}$ is $2x$; solve $-1\le 2x\le 1$.
Range: $\cos^{-1}(2x)\in[0,\pi]$, unchanged.	There is no outer scaling or shift.
(c) Domain: $\tan^{-1}$ accepts every real number, so the domain is $\mathbb{R}$.	The tangent's inverse is defined for all x.
Range: $\tan^{-1}(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, so adding $\frac{\pi}{2}$ gives $(0,\pi)$.	Shift each end up by $\frac{\pi}{2}$; the interval stays open.

Example 3 — unscaffolded

Evaluate exactly (a) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$ and (b) $\cos\left(\sin^{-1}\frac{3}{5}\right)$.

(a) The angle $\frac{5\pi}{6}$ does **not** lie in the range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ of $\sin^{-1}$, so the answer is not $\frac{5\pi}{6}$. Evaluating the inner sine first, $\sin\frac{5\pi}{6}=\frac{1}{2}$, and then $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$, the angle in the principal range whose sine is $\frac{1}{2}$.

(b) Put $\theta=\sin^{-1}\frac{3}{5}$, so $\sin\theta=\frac{3}{5}$ with $\theta\in\left[0,\frac{\pi}{2}\right]$ (positive, since $\frac{3}{5}>0$). On this interval $\cos\theta\ge 0$, so

$$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}.$$

$\therefore \sin^{-1}\left(\sin\frac{5\pi}{6}\right)=\frac{\pi}{6}$ and $\cos\left(\sin^{-1}\frac{3}{5}\right)=\frac{4}{5}$.

Example 4 — unscaffolded

Solve $2\cos^{-1}(x)-\pi=\frac{\pi}{3}$ for x, giving an exact value, and state where the graph of $y=2\cos^{-1}(x)-\pi$ crosses the x-axis.

Isolating the inverse function by adding π and dividing by 2,

$$2\cos^{-1}(x)=\frac{\pi}{3}+\pi=\frac{4\pi}{3}\Rightarrow\cos^{-1}(x)=\frac{2\pi}{3}.$$

Because $\frac{2\pi}{3}$ lies in the range $[0,\pi]$, it is a value the inverse cosine genuinely attains, so applying the cosine gives

$$x=\cos\frac{2\pi}{3}=-\frac{1}{2}.$$

The curve meets the x-axis where $y=0$, that is $2\cos^{-1}(x)=\pi$, so $\cos^{-1}(x)=\frac{\pi}{2}$ and $x=\cos\frac{\pi}{2}=0$.

$\therefore x=-\frac{1}{2}$, and the graph crosses the x-axis at the origin $(0,0)$.

Practice questions

Scaffolded

Q1. Find the exact value of each. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. (b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. (c) $\sin^{-1}(1)$. (d) $\sin^{-1}(0)$.

Q2. Find the exact value of each. (a) $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$. (b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. (c) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$. (d) $\tan^{-1}(-1)$.

Q3. State the maximal domain and the range of each function. (a) $y=\sin^{-1}(x)$. (b) $y=2\cos^{-1}(x)$. (c) $y=\tan^{-1}(3x)$. (d) $y=\tan^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{2}$.

Q4. Evaluate each, taking care with the principal range. (a) $\sin^{-1}\left(\sin\frac{\pi}{5}\right)$. (b) $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$. (c) $\cos^{-1}\left(\cos\frac{6\pi}{5}\right)$. (d) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Q5. Find the exact value of each, using a right-angled triangle or the Pythagorean identity. (a) $\cos\left(\sin^{-1}\frac{7}{25}\right)$. (b) $\sin\left(\cos^{-1}\frac{5}{13}\right)$. (c) $\tan\left(\sin^{-1}\frac{8}{17}\right)$. (d) $\cos\left(\tan^{-1}\frac{4}{3}\right)$.

Q6. Solve each equation for x, giving an exact value. (a) $\sin^{-1}(x)=\frac{\pi}{4}$. (b) $\cos^{-1}(x)=\frac{5\pi}{6}$. (c) $\tan^{-1}(x)=-\frac{\pi}{3}$. (d) $2\sin^{-1}(x)=\frac{\pi}{3}$.

Unscaffolded

Q7. Find the exact value of $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$.

Q8. Find the exact value of $\tan^{-1}(\sqrt{3})-\tan^{-1}(-1)$.

Q9. Evaluate $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$, explaining why the answer is not $\frac{7\pi}{6}$.

Q10. Evaluate $\sin^{-1}\left(\sin\frac{11\pi}{6}\right)$.

Q11. Find the exact value of $\sin\left(\cos^{-1}\frac{2}{3}\right)$.

Q12. Find the exact value of $\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$.

Q13. State the maximal domain and the range of $y=3\tan^{-1}(2x)$.

Q14. State the maximal domain and the range of $y=\cos^{-1}\left(\frac{x}{2}\right)+\frac{\pi}{2}$.

Q15. Solve $\sin^{-1}(2x)=\frac{\pi}{3}$ for x, giving an exact value, and confirm the solution is admissible.

Q16. Solve $2\cos^{-1}(x)+\frac{\pi}{3}=\pi$ for x.

Q17. Solve $\tan^{-1}(3x)=\frac{\pi}{6}$ for x, giving an exact value.

Q18. A function $y=a\sin^{-1}(bx)+c$, with $a>0$ and $b>0$, has domain $\left[-\frac{1}{2},\frac{1}{2}\right]$ and range $[0,2\pi]$. Find a, b and c.

Answers

Q1. (a) $\frac{\pi}{3}$; (b) $-\frac{\pi}{3}$; (c) $\frac{\pi}{2}$; (d) 0. **Q2.** (a) $\frac{\pi}{4}$; (b) $\frac{5\pi}{6}$; (c) $\frac{\pi}{6}$; (d) $-\frac{\pi}{4}$. **Q3.** (a) domain $[-1,1]$, range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; (b) domain $[-1,1]$, range $[0,2\pi]$; (c) domain $\mathbb{R}$, range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$; (d) domain $\mathbb{R}$, range $(-\pi,0)$. **Q4.** (a) $\frac{\pi}{5}$; (b) $\frac{\pi}{5}$; (c) $\frac{4\pi}{5}$; (d) $-\frac{\pi}{4}$. **Q5.** (a) $\frac{24}{25}$; (b) $\frac{12}{13}$; (c) $\frac{8}{15}$; (d) $\frac{3}{5}$. **Q6.** (a) $\frac{\sqrt{2}}{2}$; (b) $-\frac{\sqrt{3}}{2}$; (c) $-\sqrt{3}$; (d) $\frac{1}{2}$. **Q7.** $\frac{\pi}{2}$. **Q8.** $\frac{7\pi}{12}$. **Q9.** $\frac{5\pi}{6}$. **Q10.** $-\frac{\pi}{6}$. **Q11.** $\frac{\sqrt{5}}{3}$. **Q12.** $-\frac{4}{3}$. **Q13.** domain $\mathbb{R}$, range $\left(-\frac{3\pi}{2},\frac{3\pi}{2}\right)$. **Q14.** domain $[-2,2]$, range $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$. **Q15.** $x=\frac{\sqrt{3}}{4}$. **Q16.** $x=\frac{1}{2}$. **Q17.** $x=\frac{\sqrt{3}}{9}$. **Q18.** $a=2$, $b=2$, $c=\pi$.

Fully-worked solutions

Q1. Each is the angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ with the given sine. (a) $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, so $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$. (b) Since $\sin^{-1}$ is odd, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$ by part (a). (c) $\sin\frac{\pi}{2}=1$, so $\sin^{-1}(1)=\frac{\pi}{2}$ (the upper end of the range). (d) $\sin 0=0$, so $\sin^{-1}(0)=0$.

Q2. (a) $\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ with $\frac{\pi}{4}\in[0,\pi]$, so $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$. (b) Using $\cos^{-1}(-x)=\pi-\cos^{-1}(x)$, $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$. (c) $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$, so $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)=\frac{\pi}{6}$. (d) Since $\tan^{-1}$ is odd and $\tan\frac{\pi}{4}=1$, $\tan^{-1}(-1)=-\frac{\pi}{4}$.

Q3. The domain is set by the input restriction and the range by scaling or shifting the principal range. (a) $y=\sin^{-1}(x)$: domain $[-1,1]$, range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. (b) $y=2\cos^{-1}(x)$: the input x still needs $-1\le x\le 1$, so domain $[-1,1]$; doubling the range $[0,\pi]$ gives $[0,2\pi]$. (c) $y=\tan^{-1}(3x)$: the tangent's inverse accepts all reals, so domain $\mathbb{R}$; there is no outer change, so range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. (d) $y=\tan^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{2}$: the inverse tangent accepts every real number and the inner factor $\frac{1}{2}$ does not change that, so domain $\mathbb{R}$; subtracting $\frac{\pi}{2}$ from each end of $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ gives range $(-\pi,0)$.

Q4. Each answer must lie in the principal range of the outer inverse function. (a) $\frac{\pi}{5}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\frac{\pi}{5}\right)=\frac{\pi}{5}$. (b) $\sin\frac{4\pi}{5}=\sin\left(\pi-\frac{\pi}{5}\right)=\sin\frac{\pi}{5}$, and $\frac{\pi}{5}$ is in the range, so the value is $\frac{\pi}{5}$. (c) $\cos\frac{6\pi}{5}=\cos\left(2\pi-\frac{6\pi}{5}\right)=\cos\frac{4\pi}{5}$, and $\frac{4\pi}{5}\in[0,\pi]$, so $\cos^{-1}\left(\cos\frac{6\pi}{5}\right)=\frac{4\pi}{5}$. (d) $\tan\frac{3\pi}{4}=-1$, and $\tan^{-1}(-1)=-\frac{\pi}{4}$, which lies in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Q5. In each case set θ equal to the inner inverse function and use $\sin^2\theta+\cos^2\theta=1$; every θ here is a first-quadrant angle, so both $\sin\theta$ and $\cos\theta$ are non-negative. (a) $\theta=\sin^{-1}\frac{7}{25}$: $\sin\theta=\frac{7}{25}$, so $\cos\theta=\sqrt{1-\frac{49}{625}}=\sqrt{\frac{576}{625}}=\frac{24}{25}$. (b) $\theta=\cos^{-1}\frac{5}{13}$: $\cos\theta=\frac{5}{13}$, so $\sin\theta=\sqrt{1-\frac{25}{169}}=\frac{12}{13}$. (c) $\theta=\sin^{-1}\frac{8}{17}$: $\sin\theta=\frac{8}{17}$ gives $\cos\theta=\frac{15}{17}$, so $\tan\theta=\frac{8/17}{15/17}=\frac{8}{15}$. (d) $\theta=\tan^{-1}\frac{4}{3}$: a right triangle with opposite 4 and adjacent 3 has hypotenuse 5, so $\cos\theta=\frac{3}{5}$.

Q6. Apply the matching circular function to both sides, checking the target angle is in range. (a) $\sin^{-1}(x)=\frac{\pi}{4}$, so $x=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$. (b) $\cos^{-1}(x)=\frac{5\pi}{6}$, so $x=\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$. (c) $\tan^{-1}(x)=-\frac{\pi}{3}$, so $x=\tan\left(-\frac{\pi}{3}\right)=-\sqrt{3}$. (d) $2\sin^{-1}(x)=\frac{\pi}{3}$ gives $\sin^{-1}(x)=\frac{\pi}{6}$, so $x=\sin\frac{\pi}{6}=\frac{1}{2}$.

Q7. From the odd rule $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$, and from $\cos^{-1}(-x)=\pi-\cos^{-1}(x)$, $\cos^{-1}\left(-\frac{1}{2}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$. Adding,

$$-\frac{\pi}{6}+\frac{2\pi}{3}=-\frac{\pi}{6}+\frac{4\pi}{6}=\frac{3\pi}{6}=\frac{\pi}{2}.$$

Q8. $\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$ and, since $\tan^{-1}$ is odd, $\tan^{-1}(-1)=-\frac{\pi}{4}$. Hence

$$\tan^{-1}(\sqrt{3})-\tan^{-1}(-1)=\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)=\frac{4\pi}{12}+\frac{3\pi}{12}=\frac{7\pi}{12}.$$

Q9. The angle $\frac{7\pi}{6}$ is outside the range $[0,\pi]$ of $\cos^{-1}$, so it cannot be the answer. Reducing the inner cosine, $\cos\frac{7\pi}{6}=-\frac{\sqrt{3}}{2}$, and the angle in $[0,\pi]$ with this cosine is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$. (Equivalently, for $\theta\in[\pi,2\pi]$ the reduction is $\cos^{-1}(\cos\theta)=2\pi-\theta=2\pi-\frac{7\pi}{6}=\frac{5\pi}{6}$.)

Q10. The angle $\frac{11\pi}{6}$ is outside $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Its sine is $\sin\frac{11\pi}{6}=-\frac{1}{2}$, so $\sin^{-1}\left(\sin\frac{11\pi}{6}\right)=\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$, which lies in the principal range. (Equivalently $\frac{11\pi}{6}-2\pi=-\frac{\pi}{6}$.)

Q11. Let $\theta=\cos^{-1}\frac{2}{3}$, so $\cos\theta=\frac{2}{3}$ with $\theta\in\left[0,\frac{\pi}{2}\right]$, where $\sin\theta\geq 0$. Then

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{4}{9}}=\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{3}.$$

Q12. Let $\theta=\cos^{-1}\left(-\frac{3}{5}\right)$, so $\cos\theta=-\frac{3}{5}$ with $\theta\in\left(\frac{\pi}{2},\pi\right)$ — the second quadrant, where sine is positive. Then $\sin\theta=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$, and

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{4/5}{-3/5}=-\frac{4}{3}.$$

Q13. For $y=3\tan^{-1}(2x)$ the inverse tangent accepts all reals, so the domain is $\mathbb{R}$. Multiplying the range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ by 3 gives the range $\left(-\frac{3\pi}{2},\frac{3\pi}{2}\right)$.

Q14. For $y=\cos^{-1}\left(\frac{x}{2}\right)+\frac{\pi}{2}$ the input must satisfy $-1\leq\frac{x}{2}\leq1$, that is $-2\leq x\leq2$, so the domain is $[-2,2]$. Adding $\frac{\pi}{2}$ to each end of the range $[0,\pi]$ gives $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$.

Q15. Applying the sine, $2x=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, so $x=\frac{\sqrt{3}}{4}$. The solution is admissible because the input then satisfies $2x=\frac{\sqrt{3}}{2}$, and $\left|\frac{\sqrt{3}}{2}\right|=\frac{\sqrt{3}}{2}<1$ lies in $[-1,1]$; equivalently $\frac{\pi}{3}$ lies in the range of $\sin^{-1}$.

Q16. Isolating the inverse function, $2\cos^{-1}(x)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$, so $\cos^{-1}(x)=\frac{\pi}{3}$. Since $\frac{\pi}{3}\in[0,\pi]$, applying the cosine gives $x=\cos\frac{\pi}{3}=\frac{1}{2}$.

Q17. Applying the tangent, $3x=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$, so $x=\frac{\sqrt{3}}{9}$. The target angle $\frac{\pi}{6}$ lies in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, so the solution is valid.

Q18. The domain of $\sin^{-1}(bx)$ is where $-1\leq bx\leq1$, i.e. $-\frac{1}{b}\leq x\leq\frac{1}{b}$ (as $b>0$). Matching this to $\left[-\frac{1}{2},\frac{1}{2}\right]$ gives $\frac{1}{b}=\frac{1}{2}$, so $b=2$. Because $a>0$ is given, the function increases, so its least value occurs at $x=-\frac{1}{2}$ and its greatest at $x=\frac{1}{2}$:

$$a\sin^{-1}(-1)+c=-\frac{a\pi}{2}+c=0,\quad a\sin^{-1}(1)+c=\frac{a\pi}{2}+c=2\pi.$$

Adding the two equations gives $2c=2\pi$, so $c=\pi$; subtracting gives $a\pi=2\pi$, so $a=2$. Thus $a=2$, $b=2$ and $c=\pi$, and indeed $y=2\sin^{-1}(2x)+\pi$ has domain $\left[-\frac{1}{2},\frac{1}{2}\right]$ and range $[0,2\pi]$.