Specialist Mathematics Units 1 & 2

Chapter 11 — Complex numbers

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Theory

Working over the real numbers, some quadratic equations simply have no solution: $x^2+1=0$ asks for a number whose square is -1, yet the square of every real number is either positive or zero. Rather than leave such equations unsolved, we **enlarge** the number system by adjoining a single new number whose square is -1. Every algebraic rule already in use is kept; the one new fact is added to it. The result is a number system in which every quadratic — indeed every polynomial — can be solved, and which turns out to have a natural picture as a plane.

The imaginary unit.

The **imaginary unit**, written i, is defined by the single property

$$i^2=-1.$$

Thus i is a square root of -1. It is manipulated exactly like any other symbol in algebra, with the standing instruction that wherever i^2 appears it is replaced by -1. This one rule already fixes every whole-number power of i. Starting from $i^1=i$ and $i^2=-1$,

$$i^3=i^2\cdot i=-i,\quad i^4=i^2\cdot i^2=(-1)(-1)=1,$$

and because $i^4=1$ the powers then repeat in blocks of four: $i^5=i^4\cdot i=i$, $i^6=-1$, and so on. The four possible values are governed only by the remainder of the exponent on division by 4.

$n \bmod 4$	0	1	2	3
i^n	1	i	-1	$-i$

To evaluate a high power i^n, write $n=4q+r$ with remainder $r\in\{0,1,2,3\}$; then $i^n=(i^4)^q i^r=i^r$. For instance $i^{38}=i^{36}\cdot i^2=(i^4)^9 i^2=i^2=-1$, since $38=4\times 9+2$.

Complex numbers in Cartesian form.

A **complex number** is any number that can be written in the **Cartesian form** (also called **rectangular form**)

$$z=a+bi,$$

where a and b are real. The real number a is the **real part** of z and the real number b is the **imaginary part**:

$$\operatorname{Re}(z)=a,\quad \operatorname{Im}(z)=b.$$

The imaginary part is the *real coefficient* b, not bi. A number of the form bi with $b\neq 0$ — one whose real part is 0 while its imaginary part is not — is called **purely imaginary**. The set of all complex numbers is written $\mathbb{C}$:

$$\mathbb{C}=\{a+bi : a,b\in\mathbb{R}\}.$$

Each real number a can be written as $a+0i$, so every real number is also a complex number; that is, $\mathbb{R}\subset\mathbb{C}$. For example, $z=-6+2i$ has $\operatorname{Re}(z)=-6$ and $\operatorname{Im}(z)=2$, the number $z=4i$ is purely imaginary, and $z=9$ is real.

Equality.

Two complex numbers are equal exactly when they have the *same real part* **and** the *same imaginary part*. For real a,b,c,d,

$$a+bi=c+di \Leftrightarrow a=c \text{ and } b=d.$$

A single complex equation therefore contains **two** real equations, one from each part. In particular $a+bi=0$ forces both $a=0$ and $b=0$. This splitting into real and imaginary parts is the standard way of solving for unknown reals inside a complex equation.

Addition, subtraction and real multiples.

Complex numbers are added and subtracted part by part, and a real scalar k multiplies each part:

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i,$$

$$k(a+bi)=ka+kbi.$$

The answer is always returned in Cartesian form. These rules let real combinations such as $3z-2w$ be formed directly.

Multiplication.

Two complex numbers are multiplied by expanding the brackets in the ordinary way and then replacing i^2 with -1:

$$(a+bi)(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i.$$

There is no need to memorise the final line — expanding and using $i^2=-1$ reproduces it every time. Because multiplication is just bracket expansion, all the familiar algebraic identities still hold; for instance $(a+bi)(a-bi)=a^2-b^2i^2=a^2+b^2$, a fact used repeatedly below.

The complex conjugate.

The **conjugate** of $z=a+bi$ is

$$\bar{z}=a-bi,$$

formed by reversing the sign of the imaginary part alone. Combining z with its conjugate isolates the two parts:

$$z+\bar{z}=(a+bi)+(a-bi)=2a=2\,\mathrm{Re}(z),$$

$$z-\bar{z}=(a+bi)-(a-bi)=2bi=2i\,\mathrm{Im}(z).$$

So $z+\bar{z}$ is always **real**, while $z-\bar{z}$ is always **purely imaginary** unless z is real, in which case it is 0. It follows that $z=\bar{z}$ precisely when z is real, and $z=-\bar{z}$ precisely when z is purely imaginary or $z=0$. Conjugation also respects the arithmetic: for any complex z_1, z_2,

$$\overline{z_1 \pm z_2}=\bar{z}_1 \pm \bar{z}_2,\ \overline{z_1 z_2}=\bar{z}_1 \bar{z}_2,$$

and applying itself twice returns the original number, $\overline{(\bar{z})}=z$.

The modulus.

The **modulus** of $z=a+bi$ is the non-negative real number

$$|z|=\sqrt{a^2+b^2},$$

with $|z|=0$ only for $z=0$. Multiplying z by its conjugate removes the imaginary part completely and leaves exactly this squared quantity:

$$z\bar{z}=(a+bi)(a-bi)=a^2-b^2i^2=a^2+b^2=|z|^2.$$

The identity

$$z\bar{z}=|z|^2$$

is worth stating on its own: the product of a complex number with its conjugate is the **square of its modulus**, always a non-negative real number. It is the engine behind division.

Division.

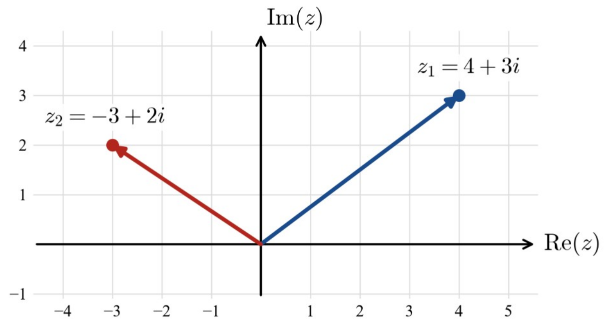
A quotient $\frac{z_1}{z_2}$ (with $z_2 \neq 0$) is put into Cartesian form by **realising the denominator** — multiplying numerator and denominator by the conjugate of the denominator. Since $z_2 \bar{z}_2 = |z_2|^2$ is real, the denominator becomes a real number and the quotient separates cleanly into real and imaginary parts:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}.$$

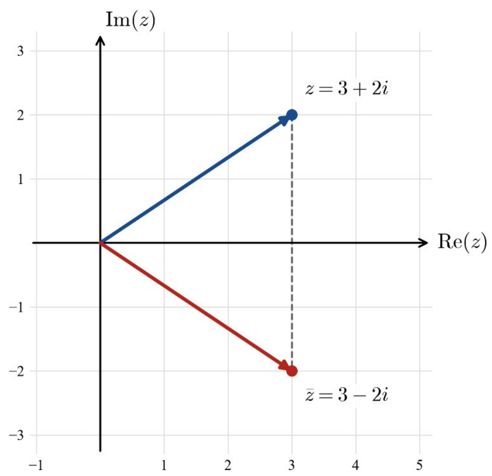
The same device gives the **reciprocal** of a non-zero z, namely $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$. Realising the denominator is the complex-number analogue of rationalising a surd denominator met in Mathematical Methods.

The Argand diagram.

Because $z = a + b i$ is completely determined by the ordered pair (a, b), it can be plotted as the point with coordinates (a, b) in a plane. Used this way the plane is called an **Argand diagram** (or the **complex plane**): the horizontal axis is the **real axis**, the vertical axis the **imaginary axis**. Equally, z may be pictured as the arrow — the **position vector** — from the origin O to the point (a, b).

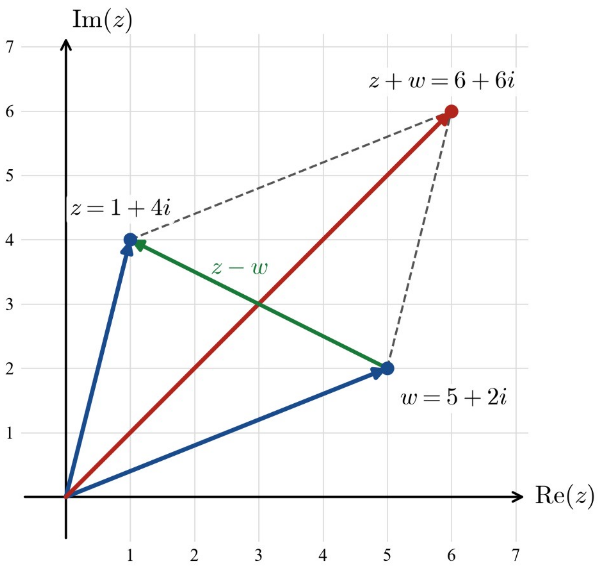


The diagram above shows $z_1 = 4 + 3i$, in the first quadrant, and $z_2 = -3 + 2i$, in the second quadrant, each drawn as a point and as a position vector from O. The conjugate $\bar{z} = a - b i$ has the same real part as z but the opposite imaginary part, so it is the **reflection of z in the real axis**.



Adding and subtracting geometrically.

Represented as position vectors, complex numbers add by the **parallelogram rule**. If z and w are drawn from O, then $z + w$ is the far corner of the parallelogram with sides $\overrightarrow{Oz}$ and $\overrightarrow{Ow}$; equivalently, it is the diagonal from O. The difference $z-w$ is the *other* diagonal: the vector drawn **from the point w to the point z**, since adding this displacement to w returns z.



Here $z=1+4i$ and $w=5+2i$ give $z+w=6+6i$ (the diagonal from O) and $z-w=-4+2i$ (the arrow from w to z). This geometric view — points and vectors in the plane — is developed much further when the modulus and direction of a complex number are studied in later sections; here it is enough to plot a complex number, read off its parts, and see addition and subtraction as vector operations.

Worked examples

Example 1 — scaffolded

Let $z=7-3i$ and $w=-2+6i$. Find (a) $z+w$, (b) $z-w$, and (c) $3z-2w$, giving each answer in the form $a+bi$.

Working	Comment
$z=7-3i$: $\mathrm{Re}(z)=7$, $\mathrm{Im}(z)=-3$; $w=-2+6i$: $\mathrm{Re}(w)=-2$, $\mathrm{Im}(w)=6$.	Identify the real and imaginary parts first.
$z+w=(7+(-2))+(-3+6)i=5+3i$.	Add the real parts and the imaginary parts separately.
$z-w=(7-(-2))+(-3-6)i=9-9i$.	Subtract part by part; note $7-(-2)=9$.
$3z=21-9i$ and $2w=-4+12i$.	A real scalar multiplies each part.
$3z-2w=(21-(-4))+(-9-12)i=25-21i$.	Combine the two results part by part.

Example 2 — scaffolded

Simplify (a) $(5+2i)(3-4i)$, (b) $(2-3i)^2$, and (c) i^{30}, giving each answer in Cartesian form.

Working	Comment
$(5+2i)(3-4i)=15-20i+6i-8i^2$.	Expand the brackets term by term.
$=15-14i-8(-1)=15-14i+8=23-14i$.	Replace i^2 with -1, then collect parts.
$(2-3i)^2=2^2-2(2)(3i)+(3i)^2=4-12i+9i^2$.	Use the perfect-square expansion.
$=4-12i-9=-5-12i$.	Since $9i^2=-9$.
$i^{30}=i^{28}\cdot i^2=(i^4)^7 i^2=1\cdot(-1)=-1$.	$30=4\times 7+2$, so $i^{30}=i^2=-1$.

Example 3 — unscaffolded

Express $\dfrac{9-7i}{2-3i}$ in Cartesian form $a+bi$.

The conjugate of the denominator $2-3i$ is $2+3i$, and multiplying it by the denominator realises it:

$$(2-3i)(2+3i)=2^2+3^2=13.$$

Multiplying numerator and denominator by this conjugate,

$$\frac{9-7i}{2-3i}=\frac{(9-7i)(2+3i)}{13}=\frac{18+27i-14i-21i^2}{13}=\frac{18+13i+21}{13}=\frac{39+13i}{13}=3+i.$$

$\therefore \dfrac{9-7i}{2-3i}=3+i.$

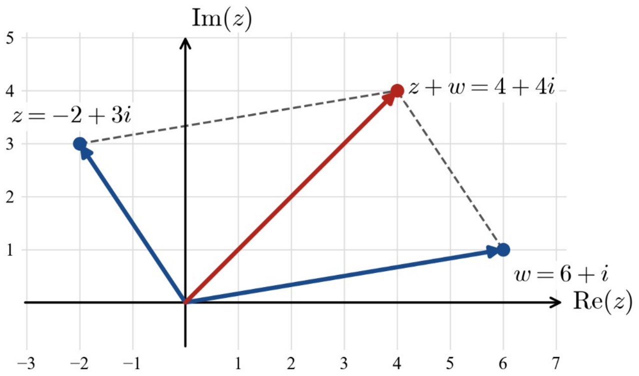
Example 4 — unscaffolded

Let $z=-2+3i$ and $w=6+i$. Find $z+w$ and $z-w$, plot z, w and $z+w$ on an Argand diagram, and interpret each result geometrically.

Adding and subtracting part by part,

$$z+w=(-2+6)+(3+1)i=4+4i, z-w=(-2-6)+(3-1)i=-8+2i.$$

Drawn as position vectors from O, the numbers z and w form two sides of a parallelogram whose far corner is $z+w=4+4i$; this point lies on the diagonal from O. The difference $z-w=-8+2i$ is the displacement from the point w to the point z, the parallelogram's other diagonal.



$\therefore z+w=4+4i$ (the diagonal from O) and $z-w=-8+2i$ (the vector from w to z).

Practice questions

Scaffolded

Q1. Let $z=6+5i$ and $w=2-7i$. (a) State $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. (b) Find $z+w$. (c) Find $z-w$. (d) Find $4z-3w$.

Q2. Simplify the following, giving each answer in Cartesian form. (a) Expand $(6+i)(2-3i)$. (b) Expand $(3-i)^2$. (c) Simplify i^{27}.

Q3. Find the real numbers x and y for which $(x+yi)(3-2i)=8-i$. (a) Expand the left-hand side and collect real and imaginary parts. (b) Equate the real parts. (c) Equate the imaginary parts. (d) Solve the two equations for x and y.

Q4. For $z=-7+24i$: (a) write down $\acute{z}$; (b) find $z+\acute{z}$ and $z-\acute{z}$; (c) find $z\acute{z}$; (d) state $|z|$ and check that $z\acute{z}=|z|^2$.

Q5. Express $\dfrac{7-i}{3+i}$ in Cartesian form. (a) State the conjugate of the denominator. (b) Multiply numerator and denominator by this conjugate. (c) Simplify the denominator to a real number. (d) Write the answer in the form $a+bi$.

Q6. Let $z=4+i$ and $w=-2+3i$. (a) Plot z and w on an Argand diagram, each as a point and as a position vector. (b) Find $z+w$. (c) Find $z-w$. (d) State which diagonal of the parallelogram on the position vectors of z and w represents $z+w$, and which vector represents $z-w$.

Unscaffolded

Q7. Given $u=9-4i$ and $v=-5+6i$, find $2u+3v$.

Q8. Expand and simplify $(6-i)(2+7i)$.

Q9. Expand and simplify $(4-5i)^2$.

Q10. Simplify $i^{53}-i^{44}$.

Q11. Expand and simplify $(1-2i)^3$.

Q12. Find the real numbers x and y for which $(x+yi)(1+4i)=-5+14i$.

Q13. For $z=-8+15i$, write down $\acute{z}$, evaluate $z\acute{z}$, and hence state $|z|$.

Q14. Express $\dfrac{7+4i}{3-2i}$ in Cartesian form.

Q15. Express the reciprocal $\dfrac{1}{2-5i}$ in Cartesian form.

Q16. Find the complex number z for which $(3-i)z=9+7i$.

Q17. Express $\dfrac{(3-i)(2-i)}{1+2i}$ in Cartesian form.

Q18. The complex numbers $z=-4+3i$ and $w=6+2i$ are plotted on an Argand diagram. (a) Find $z+w$ and $z-w$. (b) Using position vectors, explain why $z+w$ is the corner of the parallelogram with sides $\overrightarrow{Oz}$ and $\overrightarrow{Ow}$ opposite to O. (c) Describe the vector that represents $z-w$ on the same diagram.

Answers

Q1. (a) $\mathrm{Re}(z)=6$, $\mathrm{Im}(z)=5$; (b) $8-2i$; (c) $4+12i$; (d) $18+41i$. **Q2.** (a) $15-16i$; (b) $8-6i$; (c) $-i$. **Q3.** $x=2$, $y=1$. **Q4.** $\acute{z}=-7-24i$; $z+\acute{z}=-14$, $z-\acute{z}=48i$; $z\acute{z}=625$; $|z|=25$ (and $|z|^2=625$). **Q5.** $2-i$. **Q6.** (b) $2+4i$; (c) $6-2i$; (d) $z+w$ is the diagonal from O; $z-w$ is the vector from w to z. **Q7.** $3+10i$. **Q8.** $19+40i$. **Q9.** $-9-40i$. **Q10.** $-1+i$. **Q11.** $-11+2i$. **Q12.** $x=3$, $y=2$. **Q13.** $\acute{z}=-8-15i$; $z\acute{z}=289$; $|z|=17$. **Q14.** $1+2i$. **Q15.** $\frac{2}{29}+\frac{5}{29}i$. **Q16.** $z=2+3i$. **Q17.** $-1-3i$. **Q18.** (a) $z+w=2+5i$, $z-w=-10+i$; (b) see solution; (c) the vector from the point w to the point z.

Fully-worked solutions

Q1. With $z=6+5i$, $\mathrm{Re}(z)=6$ and $\mathrm{Im}(z)=5$. Adding part by part, $z+w=(6+2)+(5+(-7))i=8-2i$; subtracting, $z-w=(6-2)+(5-(-7))i=4+12i$. For the combination, $4z=24+20i$ and $3w=6-21i$, so $4z-3w=(24-6)+(20-(-21))i=18+41i$.

Q2. (a) $(6+i)(2-3i)=12-18i+2i-3i^2=12-16i+3=15-16i$. (b) $(3-i)^2=9-6i+i^2=9-6i-1=8-6i$. (c) Since $27=4\times6+3$, $i^{27}=(i^4)^6 i^3=i^3=-i$.

Q3. Expanding, $(x+yi)(3-2i)=3x-2xi+3yi-2yi^2=(3x+2y)+(-2x+3y)i$. Equating this to $8-i$ gives the real equation $3x+2y=8$ and the imaginary equation $-2x+3y=-1$. Multiplying the first by 3 and the second by 2, $9x+6y=24$ and $-4x+6y=-2$; subtracting, $13x=26$, so $x=2$, and then $2y=8-6=2$ gives $y=1$. Checking, $(2+i)(3-2i)=6-4i+3i-2i^2=8-i$.

Q4. $z=-7+24i$ has $\acute{z}=-7-24i$. Then $z+\acute{z}=-14=2\,\mathrm{Re}(z)$ and $z-\acute{z}=48i=2i\,\mathrm{Im}(z)$. The product is $z\acute{z}=(-7)^2+24^2=49+576=625$, and $|z|=\sqrt{(-7)^2+24^2}=\sqrt{625}=25$. Since $|z|^2=625=z\acute{z}$, the identity $z\acute{z}=|z|^2$ holds.

Q5. The conjugate of $3+i$ is $3-i$, and $(3+i)(3-i)=3^2+1^2=10$. The numerator is $(7-i)(3-i)=21-7i-3i+i^2=21-10i-1=20-10i$. Hence

$$\frac{7-i}{3+i}=\frac{20-10i}{10}=2-i.$$

Q6. Each $z=a+bi$ is plotted as the point (a,b). Here $z=4+i\to(4,1)$ and $w=-2+3i\to(-2,3)$, each drawn also as the position vector from O. Adding, $z+w=(4+(-2))+(1+3)i=2+4i$; subtracting, $z-w=(4-(-2))+(1-3)i=6-2i$. On the Argand diagram $z+w$ is the diagonal from O of the parallelogram with sides $\overrightarrow{Oz}$ and $\overrightarrow{Ow}$, while $z-w$ is the vector drawn from the point w to the point z.

Q7. $2u=18-8i$ and $3v=-15+18i$, so $2u+3v=(18+(-15))+(-8+18)i=3+10i$.

Q8. $(6-i)(2+7i)=12+42i-2i-7i^2=12+40i+7=19+40i$.

Q9. $(4-5i)^2=4^2-2(4)(5i)+(5i)^2=16-40i+25i^2=16-40i-25=-9-40i$.

Q10. Since $53=4\times13+1$, $i^{53}=i^1=i$; and since $44=4\times11$, $i^{44}=(i^4)^{11}=1$. Hence $i^{53}-i^{44}=i-1=-1+i$.

Q11. First $(1-2i)^2=1-4i+4i^2=1-4i-4=-3-4i$. Then $(1-2i)^3=(1-2i)^2(1-2i)=(-3-4i)(1-2i)=-3+6i-4i+8i^2=-3+2i-8=-11+2i$.

Q12. Expanding, $(x+yi)(1+4i)=x+4xi+yi+4yi^2=(x-4y)+(4x+y)i$. Equating to $-5+14i$ gives $x-4y=-5$ and $4x+y=14$. From the first, $x=4y-5$; substituting into the second, $4(4y-5)+y=14$, so $17y=34$ and $y=2$, giving $x=4(2)-5=3$. Checking, $(3+2i)(1+4i)=3+12i+2i+8i^2=-5+14i$.

Q13. $z=-8+15i$ has $\acute{z}=-8-15i$. The product is $z\acute{z}=(-8)^2+15^2=64+225=289$, so $|z|=\sqrt{289}=17$ (consistent with $z\acute{z}=|z|^2$).

Q14. The conjugate of $3-2i$ is $3+2i$, and $(3-2i)(3+2i)=3^2+2^2=13$. The numerator is $(7+4i)(3+2i)=21+14i+12i+8i^2=21+26i-8=13+26i$. Hence

$$\frac{7+4i}{3-2i}=\frac{13+26i}{13}=1+2i.$$

Q15. Using $\frac{1}{z}=\frac{\acute{z}}{|z|^2}$ with $z=2-5i$, the conjugate is $2+5i$ and $|z|^2=2^2+(-5)^2=29$, so

$$\frac{1}{2-5i}=\frac{2+5i}{29}=\frac{2}{29}+\frac{5}{29}i.$$

Q16. Dividing both sides by $3-i$, $z=\frac{9+7i}{3-i}$. The conjugate of $3-i$ is $3+i$, with $(3-i)(3+i)=3^2+1^2=10$, and the numerator is $(9+7i)(3+i)=27+9i+21i+7i^2=27+30i-7=20+30i$. Hence

$$z=\frac{20+30i}{10}=2+3i.$$

(Check: $(3-i)(2+3i)=6+9i-2i-3i^2=9+7i$.)

Q17. First expand the numerator: $(3-i)(2-i)=6-3i-2i+i^2=6-5i-1=5-5i$. Then realise the denominator with the conjugate $1-2i$, where $(1+2i)(1-2i)=1^2+2^2=5$, and $(5-5i)(1-2i)=5-10i-5i+10i^2=5-15i-10=-5-15i$. Hence

$$\frac{(3-i)(2-i)}{1+2i}=\frac{5-5i}{1+2i}=\frac{-5-15i}{5}=-1-3i.$$

Q18. (a) Adding and subtracting part by part, $z+w=(-4+6)+(3+2)i=2+5i$ and $z-w=(-4-6)+(3-2)i=-10+i$. (b) Drawn from O, the position vectors $\overrightarrow{Oz}$ and $\overrightarrow{Ow}$ are two sides of a parallelogram. Completing it by translating $\overrightarrow{Ow}$ along $\overrightarrow{Oz}$ (and vice versa) places the fourth corner at the point whose position vector is $\overrightarrow{Oz}+\overrightarrow{Ow}$; adding components, that corner is $z+w=2+5i$, opposite to O and lying on the diagonal from O. (c) $z-w=-10+i$ is represented by the vector from the point w to the point z — the parallelogram's other diagonal — since adding this displacement to w returns z.

Chapter 11 Complex numbers — 11B Quadratics over the complex field and conjugate roots

Theory

In 11A the number system was enlarged from $\mathbb{R}$ to $\mathbb{C}$ by adjoining the imaginary unit i, with $i^2=-1$, and the Cartesian arithmetic of $z=a+bi$ — its real and imaginary parts, its conjugate $\bar{z}=a-bi$ and its modulus $|z|$ — was set out. The pay-off is now collected. Over the real numbers a quadratic such as $z^2+1=0$ had **no** solution, because the quadratic formula asked for the square root of a negative number. With i available that obstruction is gone: **every** quadratic with real coefficients can be solved, and the two familiar techniques from Mathematical Methods — **completing the square** and the **quadratic formula** — carry across without change. The only new step is that a square root of a negative real is written using i.

Square roots of a negative real number.

For a positive real number k,

$$\sqrt{-k}=\sqrt{k}\,i\,,\text{because}\left(\sqrt{k}\,i\right)^2=k\,i^2=-k\,.$$

Thus $\sqrt{-4}=2i$, $\sqrt{-25}=5i$ and $\sqrt{-7}=\sqrt{7}\,i$. Since $\left(-\sqrt{k}\,i\right)^2=-k$ as well, a negative real has **two** square roots, $\pm\sqrt{k}\,i$, exactly as a positive real k has the two square roots $\pm\sqrt{k}$. When a negative number under a root has a square factor it is simplified in the usual way; for example $\sqrt{-12}=\sqrt{4\cdot 3}\,i=2\sqrt{3}\,i$.

The discriminant and the nature of the roots.

For a quadratic $az^2+bz+c=0$ with **real** coefficients a, b, c and $a\neq 0$, the **discriminant** is

$$\Delta=b^2-4ac\,.$$

Its sign classifies the roots exactly as in Methods, with the negative case now read over $\mathbb{C}$ rather than being rejected:

Discriminant	Nature of the roots
$\Delta>0$	two distinct real roots
$\Delta=0$	one repeated real root
$\Delta<0$	two distinct non-real complex roots

When $\Delta<0$ the two roots are complex numbers that are not real, and — as shown below — each is the conjugate of the other.

Solving by completing the square.

The method reshapes the quadratic so the variable appears only inside a perfect square, after which square roots are taken. For a monic quadratic $z^2+pz+q=0$,

$$z^2+pz+q=\left(z+p/2\right)^2-\frac{p^2}{4}+q\,,$$

so the equation becomes $\left(z+p/2\right)^2=\dfrac{p^2}{4}-q$. Whenever the right-hand side is negative its square roots are imaginary, and the solutions are a pair of non-real complex numbers. A non-monic quadratic is first divided through by its leading coefficient, or handled by the formula below.

The quadratic formula over $\mathbb{C}$.

Completing the square on the general quadratic $az^2+bz+c=0$ gives the same formula met in Methods,

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{\Delta}}{2a}.$$

When $\Delta<0$ we write $\sqrt{\Delta}=\sqrt{-\Delta}\,i$, with $-\Delta>0$, so that

$$z=\frac{-b\pm\sqrt{-\Delta}\,i}{2a}=-\frac{b}{2a}\pm\frac{\sqrt{-\Delta}}{2a}i.$$

The two solutions share the real part $-\frac{b}{2a}$ and have imaginary parts of equal size but opposite sign.

The conjugate root theorem.

Reading the formula above, when a,b,c are real and $\Delta<0$ the two roots are

$$\alpha=s+ti \text{ and } \acute{\alpha}=s-ti,\ s=-\frac{b}{2a},\ t=\frac{\sqrt{-\Delta}}{2a},$$

each the **conjugate** of the other. This is the **conjugate root theorem for quadratics**: *if a quadratic has real coefficients and the non-real number α is a root, then its conjugate $\acute{\alpha}$ is the other root.* Conjugation is what drives the result: it respects addition and multiplication and leaves every real number fixed, so conjugating $a\alpha^2+b\alpha+c=0$ term by term — every coefficient being real, and so unchanged — returns $a(\acute{\alpha})^2+b\acute{\alpha}+c=0$. On an Argand diagram the two roots sit symmetrically about the real axis, one directly below the other. The same conjugating argument works for a real polynomial of any degree, so its non-real zeros always pair up; that general form of the theorem, and the factorisation of cubics and quartics over $\mathbb{C}$ built on it, are **Units 3 & 4** material. Every equation in this section is a quadratic.

Sum and product of the roots.

If α and β are the roots of $az^2+bz+c=0$, then $az^2+bz+c=a(z-\alpha)(z-\beta)$. Expanding the right-hand side and comparing coefficients gives the two relations

$$\alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}.$$

Dividing through by a shows that a **monic** quadratic with roots α and β is

$$z^2-(\alpha+\beta)z+\alpha\beta=0.$$

This is the quickest route from roots to equation, and a fast check on any solution. For a conjugate pair $\alpha=s+ti$, $\acute{\alpha}=s-ti$,

$$\alpha+\acute{\alpha}=2s(\text{real}),\ \alpha\acute{\alpha}=(s+ti)(s-ti)=s^2+t^2=|\alpha|^2(\text{real}),$$

so the monic quadratic built from them,

$$z^2-2sz+(s^2+t^2)=0,$$

has **real** coefficients — consistent with the conjugate root theorem, and with $\alpha\acute{\alpha}=|\alpha|^2$ being the square of the modulus from 11A.

Factorising over the complex field.

Because a quadratic with $\Delta<0$ has the two roots α and $\acute{\alpha}$, it splits into linear factors over $\mathbb{C}$:

$$az^2+bz+c=a(z-\alpha)(z-\acute{\alpha}).$$

Multiplying the conjugate pair back together returns the real quadratic $z^2-2sz+(s^2+t^2)$, which has negative discriminant and so **cannot** be factorised any further using real numbers — it is irreducible over $\mathbb{R}$, but reducible over

$\mathbb{C}$. So **every** quadratic with real coefficients is a product of two linear factors over $\mathbb{C}$: two real factors when $\Delta \geq 0$, and a conjugate pair of non-real factors when $\Delta < 0$. In practice the equation $az^2 + bz + c = 0$ is solved first, by completing the square or by the formula, and the roots are then read straight into $a(z-\alpha)(z-\acute{\alpha})$; expanding that product is a quick check. Factorising polynomials of higher degree over $\mathbb{C}$ is taken up in Units 3 & 4.

Worked examples

Example 1 — scaffolded

Solve $z^2 - 10z + 34 = 0$ over $\mathbb{C}$ by completing the square.

Working	Comment
$z^2 - 10z = (z-5)^2 - 25$.	Halve the coefficient of z: $\dfrac{-10}{2} = -5$.
$(z-5)^2 - 25 + 34 = 0$, so $(z-5)^2 + 9 = 0$.	Substitute back and simplify the constant.
$(z-5)^2 = -9$.	Isolate the perfect square.
$z - 5 = \pm\sqrt{-9} = \pm 3i$.	Square root of a negative real: $\sqrt{-9} = 3i$.
$z = 5 + 3i$ or $z = 5 - 3i$.	A conjugate pair, since the coefficients are real.

Example 2 — scaffolded

Solve $5z^2 + 2z + 1 = 0$ over $\mathbb{C}$ using the quadratic formula.

Working	Comment
$a = 5, b = 2, c = 1$; $\Delta = 2^2 - 4(5)(1) = 4 - 20 = -16$.	Compute the discriminant; $\Delta < 0$ signals non-real roots.
$\sqrt{\Delta} = \sqrt{-16} = 4i$.	Write the root of the negative discriminant using i.
$z = \dfrac{-2 \pm 4i}{2(5)} = \dfrac{-2 \pm 4i}{10}$.	Substitute into $z = \dfrac{-b \pm \sqrt{\Delta}}{2a}$.
$z = -\dfrac{1}{5} \pm \dfrac{2}{5}i$.	Divide each part by 10 and simplify.
Check: sum $= -\dfrac{2}{5} = -\dfrac{b}{a}$, product $= \dfrac{1}{25} + \dfrac{4}{25} = \dfrac{1}{5} = \dfrac{c}{a}$.	The sum and product relations confirm the roots.

Example 3 — unscaffolded

A quadratic equation with real coefficients has $z = -7 + 2i$ as one solution. Find a monic quadratic equation that it satisfies.

Because the coefficients are real, the conjugate root theorem guarantees that the conjugate $\overline{-7+2i} = -7 - 2i$ is the second solution. The sum and product of the two roots are

$$\alpha + \beta = (-7+2i) + (-7-2i) = -14, \alpha\beta = (-7+2i)(-7-2i) = (-7)^2 + 2^2 = 49 + 4 = 53.$$

A monic quadratic with these roots is $z^2 - (\alpha+\beta)z + \alpha\beta = 0$, that is $z^2 + 14z + 53 = 0$. As a check, completing the square gives $(z+7)^2 + 4 = 0$, so $(z+7)^2 = -4$ and $z = -7 \pm 2i$, as required.

$\therefore z^2 + 14z + 53 = 0$.

Example 4 — unscaffolded

The quadratic $P(z) = 2z^2 + bz + c$ has real coefficients and $z = -1 + 6i$ as a zero. Find b and c, and write $P(z)$ as a product of linear factors over $\mathbb{C}$.

Since the coefficients are real, the conjugate root theorem makes $\overline{-1+6i}=-1-6i$ the second zero. The sum and product of the pair are

$$\alpha+\beta=(-1+6i)+(-1-6i)=-2,\alpha\beta=(-1+6i)(-1-6i)=(-1)^2+6^2=37,$$

the product being $|-1+6i|^2$, as it must be for a conjugate pair. Reading these against $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$ with $a=2$ gives $-\frac{b}{2}=-2$ and $\frac{c}{2}=37$, so $b=4$ and $c=74$. The two zeros then give the factorisation over $\mathbb{C}$,

$$P(z)=2z^2+4z+74=2\big(z-(-1+6i)\big)\big(z-(-1-6i)\big)=2(z+1-6i)(z+1+6i),$$

which expands back as $2\left((z+1)^2+36\right)=2z^2+4z+74$. The discriminant $4^2-4(2)(74)=-576$ is negative, so no factorisation into linear factors with real coefficients exists.

$\therefore b=4$, $c=74$ and $P(z)=2(z+1-6i)(z+1+6i)$.

Practice questions

Scaffolded

Q1. Solve $z^2-12z+52=0$ over $\mathbb{C}$ by completing the square. (a) Write z^2-12z in the form $(z-h)^2-h^2$. (b) Substitute back and rearrange to $(z-h)^2=k$. (c) Take square roots, expressing $\sqrt{k}$ in terms of i. (d) State the two solutions.

Q2. Solve $z^2+2z+26=0$ using the quadratic formula. (a) State a,b,c and compute the discriminant $\Delta=b^2-4ac$. (b) Since $\Delta<0$, write $\sqrt{\Delta}$ in terms of i. (c) Apply the formula $z=\frac{-b\pm\sqrt{\Delta}}{2a}$. (d) State the two solutions and confirm they are conjugates.

Q3. Consider $2z^2-6z+7=0$. (a) Compute the discriminant. (b) State the nature of the roots (real or non-real; distinct or repeated). (c) Solve the equation using the quadratic formula. (d) Write the roots as a conjugate pair $s\pm ti$.

Q4. A quadratic with real coefficients has $z=4-5i$ as one root. (a) State the other root. (b) Find the sum of the roots. (c) Find the product of the roots. (d) Write down a monic quadratic equation with these roots.

Q5. The quadratic $4z^2-8z+5=0$ has roots α and β. (a) Without solving, state $\alpha+\beta$ and $\alpha\beta$. (b) Solve the equation. (c) Verify that your solutions have the sum and product found in (a).

Q6. In the equation $z^2+mz+50=0$ the coefficient m is a real number. (a) Write the discriminant Δ in terms of m. (b) Find, in exact form, the values of m for which the equation has two distinct non-real solutions. (c) Solve the equation in the case $m=10$. (d) Hence write $z^2+10z+50$ as a product of two linear factors over $\mathbb{C}$.

Unscaffolded

Q7. Solve $z^2+49=0$ over $\mathbb{C}$.

Q8. Solve $z^2+12z+37=0$ over $\mathbb{C}$.

Q9. Solve $z^2+16z+89=0$ over $\mathbb{C}$.

Q10. Solve $z^2-4z+68=0$ over $\mathbb{C}$.

Q11. Solve $2z^2+2z+13=0$ over $\mathbb{C}$.

Q12. Solve $5z^2-6z+5=0$ over $\mathbb{C}$, giving the roots in exact form.

Q13. Find a monic quadratic equation with real coefficients that has $z = -2 + 6i$ as a root.

Q14. A quadratic with real coefficients has leading coefficient 3 and $z = 7 - 4i$ as a root. Find the quadratic.

Q15. The equation $z^2 + bz + c = 0$ has real coefficients and one solution $z = 3 - 7i$. Find b and c.

Q16. The quadratic $3z^2 - 4z + 6 = 0$ has roots α and β. State the sum and product of the roots, then solve the equation and confirm your solutions agree.

Q17. The equation $2z^2 + 10z + 21 = 0$ has roots α and β. Without solving the equation, find $\alpha^2 + \beta^2$ and $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$, and explain why both values must be real.

Q18. The equation $z^2 + bz + c = 0$ has real coefficients, and one of its solutions α satisfies $|\alpha| = 13$ and $\mathrm{Re}(\alpha) = 5$, with $\mathrm{Im}(\alpha) > 0$. Find α, and hence find b and c.

Answers

Q1. $z=6\pm4i$. **Q2.** $z=-1\pm5i$. **Q3.** $\Delta=-20$; two distinct non-real (conjugate) roots; $z=\frac{3}{2}\pm\frac{\sqrt{5}}{2}i$. **Q4.** other root $4+5i$; sum 8, product 41; $z^2-8z+41=0$. **Q5.** $\alpha+\beta=2$, $\alpha\beta=\frac{5}{4}$; $z=1\pm\frac{1}{2}i$. **Q6.** $\Delta=m^2-200$; $-10\sqrt{2}<m<10\sqrt{2}$; $z=-5\pm5i$; $(z+5-5i)(z+5+5i)$. **Q7.** $z=\pm7i$. **Q8.** $z=-6\pm i$. **Q9.** $z=-8\pm5i$. **Q10.** $z=2\pm8i$. **Q11.** $z=-\frac{1}{2}\pm\frac{5}{2}i$. **Q12.** $z=\frac{3}{5}\pm\frac{4}{5}i$. **Q13.** $z^2+4z+40=0$. **Q14.** $3z^2-42z+195=0$. **Q15.** $b=-6$, $c=58$. **Q16.** $\alpha+\beta=\frac{4}{3}$, $\alpha\beta=2$; $z=\frac{2}{3}\pm\frac{\sqrt{14}}{3}i$. **Q17.** $\alpha^2+\beta^2=4$; $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{8}{21}$. **Q18.** $\alpha=5+12i$; $b=-10$, $c=169$.

Fully-worked solutions

Q1. Completing the square, $z^2-12z=(z-6)^2-36$, so the equation becomes $(z-6)^2-36+52=0$, that is $(z-6)^2+16=0$. Hence $(z-6)^2=-16$, and taking square roots $z-6=\pm\sqrt{-16}=\pm4i$. Therefore $z=6\pm4i$. (Check: $\Delta=(-12)^2-4(1)(52)=144-208=-64<0$, so a complex conjugate pair is expected.)

Q2. Here $a=1$, $b=2$, $c=26$, so $\Delta=2^2-4(1)(26)=4-104=-100$. Since $\Delta<0$, $\sqrt{\Delta}=\sqrt{-100}=10i$. The quadratic formula gives

$$z=\frac{-2\pm10i}{2(1)}=\frac{-2\pm10i}{2}=-1\pm5i.$$

The solutions $-1+5i$ and $-1-5i$ are conjugates, consistent with the real coefficients.

Q3. With $a=2$, $b=-6$, $c=7$, the discriminant is $\Delta=(-6)^2-4(2)(7)=36-56=-20$. Because $\Delta<0$ the equation has two distinct non-real roots forming a conjugate pair. Now $\sqrt{-20}=\sqrt{4\cdot5}i=2\sqrt{5}i$, so

$$z=\frac{-(-6)\pm2\sqrt{5}i}{2(2)}=\frac{6\pm2\sqrt{5}i}{4}=\frac{3}{2}\pm\frac{\sqrt{5}}{2}i.$$

So $s=\frac{3}{2}$ and $t=\frac{\sqrt{5}}{2}$, giving the pair $z=\frac{3}{2}\pm\frac{\sqrt{5}}{2}i$.

Q4. (a) The coefficients are real, so by the conjugate root theorem the other root is the conjugate $\overline{4-5i}=4+5i$. (b) The sum of the roots is $(4-5i)+(4+5i)=8$. (c) The product is $(4-5i)(4+5i)=4^2+5^2=16+25=41$. (d) A monic quadratic with these roots is $z^2-(\text{sum})z+(\text{product})=z^2-8z+41=0$.

Q5. (a) For $4z^2-8z+5=0$, the sum and product relations give $\alpha+\beta=-\frac{b}{a}=-\frac{-8}{4}=2$ and $\alpha\beta=\frac{c}{a}=\frac{5}{4}$. (b) The discriminant is $\Delta=(-8)^2-4(4)(5)=64-80=-16$, so $\sqrt{\Delta}=4i$ and

$$z=\frac{8\pm4i}{2(4)}=\frac{8\pm4i}{8}=1\pm\frac{1}{2}i.$$

(c) The sum is $(1+1/2i)+(1-1/2i)=2$, and the product is $1^2+(1/2)^2=1+\frac{1}{4}=\frac{5}{4}$, matching part (a).

Q6. (a) With $a=1$, $b=m$ and $c=50$, the discriminant is $\Delta=m^2-4(1)(50)=m^2-200$. (b) Two distinct non-real solutions occur exactly when $\Delta<0$, that is $m^2<200$, so $|m|<\sqrt{200}=10\sqrt{2}$, giving $-10\sqrt{2}<m<10\sqrt{2}$. (c) Taking $m=10$ (which lies in that interval, since $10<10\sqrt{2}$), completing the square gives $z^2+10z+50=(z+5)^2-25+50=(z+5)^2+25$, so $(z+5)^2=-25$ and $z+5=\pm5i$. Hence $z=-5\pm5i$. (d) The two solutions supply the linear factors, so over $\mathbb{C}$

$$z^2+10z+50=(z-(-5+5i))(z-(-5-5i))=(z+5-5i)(z+5+5i),$$

which expands back as $(z+5)^2+25=z^2+10z+50$, confirming the factorisation.

Q7. $z^2+49=0$ gives $z^2=-49$, so $z=\pm\sqrt{-49}=\pm 7i$.

Q8. Completing the square, $z^2+12z+37=(z+6)^2-36+37=(z+6)^2+1$, so $(z+6)^2=-1$ and $z+6=\pm i$. Hence $z=-6\pm i$. (Here $\Delta=144-148=-4$.)

Q9. Completing the square, $z^2+16z+89=(z+8)^2-64+89=(z+8)^2+25$, so $(z+8)^2=-25$ and $z+8=\pm 5i$. Hence $z=-8\pm 5i$. (Here $\Delta=256-356=-100$.)

Q10. Completing the square, $z^2-4z+68=(z-2)^2-4+68=(z-2)^2+64$, so $(z-2)^2=-64$ and $z-2=\pm 8i$. Hence $z=2\pm 8i$. (Here $\Delta=16-272=-256$.)

Q11. With $a=2$, $b=2$, $c=13$, $\Delta=2^2-4(2)(13)=4-104=-100$, so $\sqrt{\Delta}=10i$. Then

$$z=\frac{-2\pm 10i}{2(2)}=\frac{-2\pm 10i}{4}=-\frac{1}{2}\pm\frac{5}{2}i.$$

Q12. With $a=5$, $b=-6$, $c=5$, $\Delta=(-6)^2-4(5)(5)=36-100=-64$, so $\sqrt{\Delta}=8i$. Then

$$z=\frac{6\pm 8i}{2(5)}=\frac{6\pm 8i}{10}=\frac{3}{5}\pm\frac{4}{5}i.$$

Q13. Since the coefficients are real, the conjugate $\overline{-2+6i}=-2-6i$ is the other root. The sum is $(-2+6i)+(-2-6i)=-4$ and the product is $(-2+6i)(-2-6i)=(-2)^2+6^2=4+36=40$. A monic quadratic with these roots is $z^2+4z+40=0$.

Q14. The real coefficients force the conjugate $\overline{7-4i}=7+4i$ to be the second root. A monic quadratic with roots $7\pm 4i$ has sum $(7+4i)+(7-4i)=14$ and product $(7+4i)(7-4i)=7^2+4^2=65$, giving $z^2-14z+65$. Multiplying by the required leading coefficient 3,

$$3(z^2-14z+65)=3z^2-42z+195=0.$$

Q15. The coefficients are real, so the conjugate $\overline{3-7i}=3+7i$ is the other root. Comparing $z^2+bz+c=0$ with $z^2-(\alpha+\beta)z+\alpha\beta=0$: $b=-(\alpha+\beta)=-((3-7i)+(3+7i))=-6$, and $c=\alpha\beta=(3-7i)(3+7i)=3^2+7^2=58$. Hence $b=-6$ and $c=58$.

Q16. By the sum and product relations, $\alpha+\beta=-\frac{-4}{3}=\frac{4}{3}$ and $\alpha\beta=\frac{6}{3}=2$. To solve, $\Delta=(-4)^2-4(3)(6)=16-72=-56$, so $\sqrt{-56}=\sqrt{4\cdot 14}\,i=2\sqrt{14}\,i$ and

$$z=\frac{4\pm 2\sqrt{14}\,i}{2(3)}=\frac{4\pm 2\sqrt{14}\,i}{6}=\frac{2}{3}\pm\frac{\sqrt{14}}{3}i.$$

Their sum is $\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$ and their product is $(2/3)^2+(\sqrt{14}/3)^2=\frac{4}{9}+\frac{14}{9}=\frac{18}{9}=2$, confirming the values above.

Q17. With $a=2$, $b=10$, $c=21$, the sum and product relations give $\alpha+\beta=-\frac{10}{2}=-5$ and $\alpha\beta=\frac{21}{2}$. Squaring the sum, $(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$, so

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-5)^2-2\left(\frac{21}{2}\right)=25-21=4.$$

Putting the second expression over a common denominator,

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{4}{\frac{21}{2}}=\frac{8}{21}.$$

Both values are real because each expression is symmetric in α and β, so each can be rewritten in terms of $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$ alone, and those two quantities are real whenever the coefficients are. (Here $\Delta=10^2-4(2)(21)=-68<0$, so α and β are in fact a conjugate pair, and $\alpha^2+\beta^2=\alpha^2+\overline{\alpha^2}=2\,\mathrm{Re}(\alpha^2)$ is real for that reason too.)

Q18. Write $\alpha=5+t\,i$ with t real, so that $\mathrm{Re}(\alpha)=5$. From $|\alpha|=13$,

$$|\alpha|^2=5^2+t^2=169 \quad t^2=144 \quad t=\pm 12,$$

and $\mathrm{Im}(\alpha)>0$ selects $t=12$, so $\alpha=5+12i$. The coefficients are real, so by the conjugate root theorem $\acute{\alpha}=5-12i$ is the other solution. Their sum is 10 and their product is $\alpha\acute{\alpha}=|\alpha|^2=169$, so comparing $z^2+bz+c=0$ with $z^2-(\alpha+\acute{\alpha})z+\alpha\acute{\alpha}=0$ gives $b=-10$ and $c=169$. (Check: for $z^2-10z+169=0$, $\Delta=100-676=-576$, so $\sqrt{\Delta}=24i$ and $z=\frac{10\pm 24i}{2}=5\pm 12i$.)

Theory

In 11A a single complex number $z = x + yi$ was plotted as the point (x, y) on an **Argand diagram**, whose horizontal axis is the real axis and whose vertical axis is the imaginary axis. This section studies whole **sets** of complex numbers. A **complex relation** is an equation or inequality that z must satisfy, and the set of every point that satisfies it is a subset of the Argand plane. An **equation** (using =) usually describes a *curve* — a line or a circle — while an **inequality** (using $<, \le, >, \ge$) usually describes a *region* with area. A curve traced out by a moving point subject to a condition is called a **locus** (plural *loci*).

The method never changes: write $z = x + yi$, translate the relation into an ordinary **Cartesian** condition on the real variables x and y, then sketch the resulting curve or region. Two drawing conventions are used throughout. A boundary that **is** part of the set (from =, ≤ or ≥) is drawn as a **solid** line or curve; a boundary that is **excluded** (from < or >, or an undefined point) is drawn **dashed**. The region belonging to the set is **shaded**. Two readings of the modulus drive almost everything below, and both follow from what 11A already gives. Since $|z| = \sqrt{x^2 + y^2}$, Pythagoras in the right-angled triangle with legs x and y makes $|z|$ the **distance of z from the origin**. More generally $z_1 - z_2$ is the displacement drawn **from the point z_2 to the point z_1**, so its modulus $|z_1 - z_2|$ is the **distance between the points z_1** and z_2.

Circles and discs.

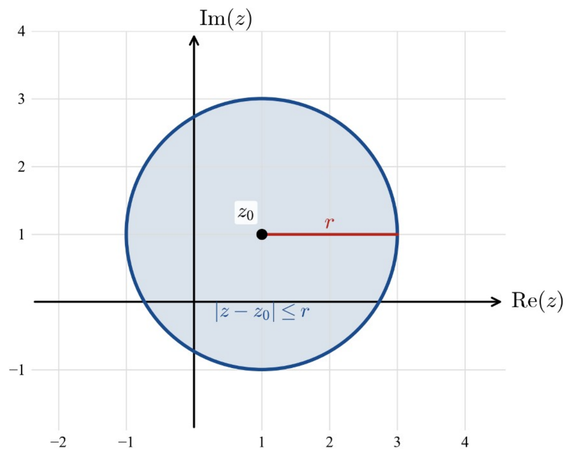
For a fixed complex number z_0 and a positive real number r, the relation

$$|z - z_0| = r$$

collects every point whose distance from z_0 is exactly r — a **circle** of **centre** z_0 and **radius** r. Writing $z = x + yi$ and $z_0 = a + bi$, the displacement is $z - z_0 = (x - a) + (y - b)i$, whose modulus squared is $(x - a)^2 + (y - b)^2$. Squaring $|z - z_0| = r$ therefore gives the Cartesian equation of the circle,

$$(x - a)^2 + (y - b)^2 = r^2.$$

The associated inequalities describe **regions**. The relation $|z - z_0| \le r$ is the **closed disc** — the interior together with its boundary circle; $|z - z_0| < r$ is the **open disc**, with the boundary dashed and excluded; and $|z - z_0| \ge r$ or $> r$ is the **exterior** of the circle.

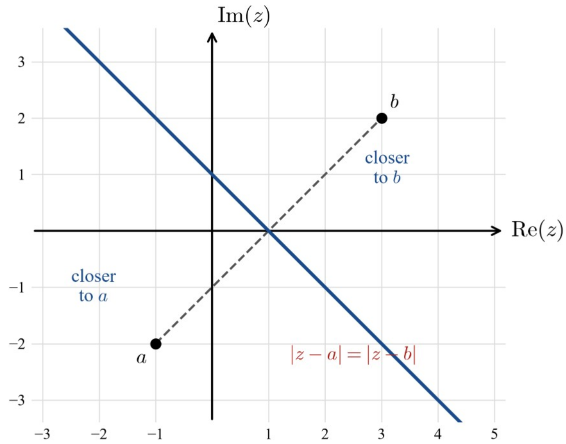


Perpendicular bisectors and half-planes.

For two fixed distinct points a and b, the relation

$$|z - a| = |z - b|$$

says that z is **equidistant** from a and b. Every such point lies on the **perpendicular bisector** of the line segment joining a and b. To find its equation, put $z = x + yi$ and square both sides. Because $|z-a|^2$ and $|z-b|^2$ each contribute the *same* x^2 and y^2 terms, those terms cancel and a **straight line** remains. The strict inequalities give **half-planes**: $|z-a| < |z-b|$ is the side of the bisector **closer to** a, and $|z-a| > |z-b|$ is the other side, closer to b.



Ellipses.

The bisector relation $|z - z_1| = |z - z_2|$ fixes the *difference* of the two distances from z to the fixed points z_1 and z_2 at zero. Fixing instead their **sum** produces a new locus. For two fixed points z_1 and z_2 — now called the **foci** — and a positive constant k, the relation

$$|z - z_1| + |z - z_2| = k$$

collects every point the sum of whose distances to the two foci equals k. Going from z_1 to z_2 by way of z can never beat the straight segment joining them, so a point satisfying the relation can exist only when $k \geq |z_1 - z_2|$: if $k = |z_1 - z_2|$ the locus collapses to the **line segment** joining the foci, and if $k < |z_1 - z_2|$ it is **empty**. Whenever

$$k > |z_1 - z_2|,$$

the locus is a genuine **ellipse** enclosing both foci.

To find its Cartesian equation, take the symmetric case with the foci on the real axis at $z_1 = -c$ and $z_2 = c$, so that $|z_1 - z_2| = 2c$, and write the constant sum as $k = 2a$ with $a > c > 0$. Putting $z = x + yi$,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Isolating one root and squaring removes it; the squared terms x^2, y^2 and c^2 then cancel from the two sides, leaving $a\sqrt{(x-c)^2 + y^2} = a^2 - cx$, and squaring a second time and gathering terms gives

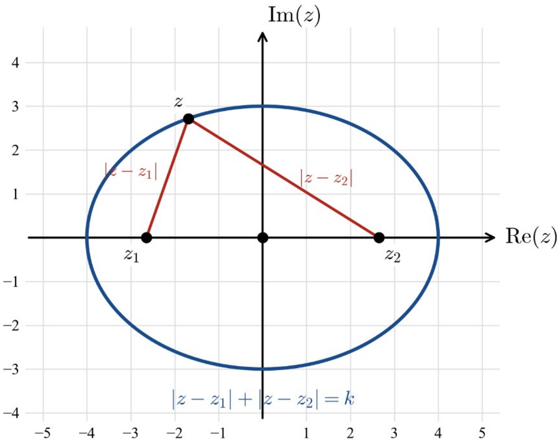
$$(a^2 - c^2)x^2 + a^2 y^2 = a^2(a^2 - c^2).$$

Writing $b^2 = a^2 - c^2$ — positive precisely because $a > c$ — and dividing through by $a^2 b^2$ yields the **standard ellipse**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b^2 = a^2 - c^2.$$

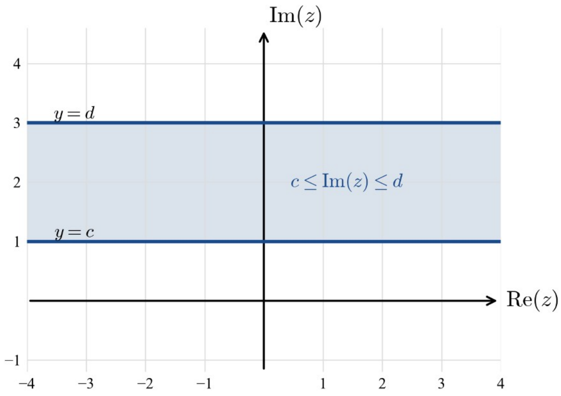
Here a is the **semi-major axis** and b the **semi-minor axis**: the curve meets the real axis at the **vertices** $(\pm a, 0)$ and the imaginary axis at $(0, \pm b)$, and the foci sit on the major axis a distance $c = \sqrt{a^2 - b^2}$ either side of the centre.

For a general pair of foci the same three numbers do all the work. The **centre** of the ellipse is the midpoint $\frac{1}{2}(z_1 + z_2)$ of the foci; the semi-major axis is $a = \frac{1}{2}k$; the focal distance is $c = \frac{1}{2}|z_1 - z_2|$; and $b = \sqrt{a^2 - c^2}$. The major axis lies along the line through the two foci, so when the foci share the same real part that axis is **vertical** and the roles of a and b beneath x and y are exchanged. Note the contrast with the earlier locus: equal distances, $|z - z_1| = |z - z_2|$, give a **line** (the perpendicular bisector), whereas a constant **sum** of distances gives an **ellipse** — and that ellipse's centre always lies on the very bisector, being equidistant from the two foci.



Conditions on the real and imaginary parts.

Since $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$, a condition on a part of z is simply a condition on a coordinate. Thus $\mathrm{Re}(z) = k$ is the **vertical line** $x = k$, while $\mathrm{Im}(z) = k$ is the **horizontal line** $y = k$. The matching inequalities are half-planes: $\mathrm{Re}(z) \geq k$ is everything on or to the right of $x = k$, and $\mathrm{Im}(z) < k$ is everything strictly below $y = k$. Pairing two such conditions produces a **band** (an infinite strip): for instance $c \leq \mathrm{Im}(z) \leq d$ is the horizontal band between the lines $y = c$ and $y = d$. Combining a real-part band with an imaginary-part band traps the point inside a **rectangle**.



Rays and sectors.

A point of the plane is fixed by its distance from the origin together with its **direction**, and the second of these needs a name. The **argument** of a non-zero complex number z, written $arg(z)$, is the angle that the position vector from O to z makes with the **positive real axis**, measured **anticlockwise as positive**; adding a whole turn returns to the same direction, so $arg(z)$ is fixed only to within multiples of 2π, and $arg(0)$ is undefined. The point at distance $r = |z|$ in the direction $\theta = arg(z)$ has coordinates $(r\cos\theta, r\sin\theta)$, so

$$z = r(\cos\theta + i\sin\theta) = r\,cis\,\theta,$$

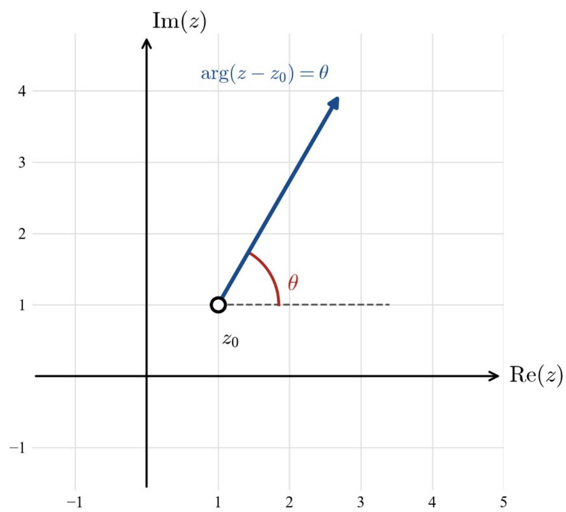
writing $cis\,\theta$ for the shorthand $\cos\theta + i\sin\theta$. This is the **polar (modulus–argument) form**, developed properly in 11D; here it is needed only to name directions. More generally $arg(z - z_0)$ is the angle that the displacement **from** z_0 **to** z makes with the positive real direction. Fixing this angle,

$$arg(z - z_0) = \theta,$$

forces z to lie in a single direction out of z_0, so the set is a **ray** — a half-line that starts at z_0 and points in the direction θ. Every point of the ray has the form

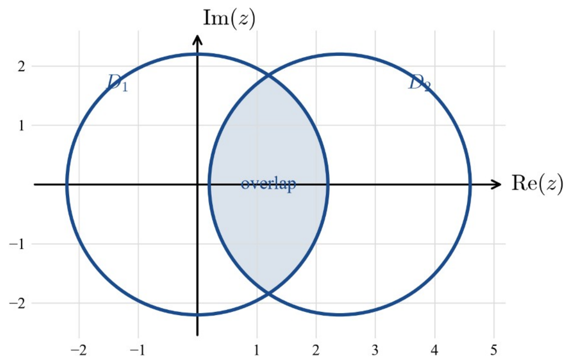
$$z = z_0 + t\,cis\,\theta, t > 0,$$

so the endpoint z_0 itself (where $t = 0$) is **excluded**: $arg(0)$ is undefined, and the endpoint is drawn as an open dot. In Cartesian terms, with $z_0 = a + bi$, the ray lies on the line through (a, b) of gradient $\tan\theta$, but only the **half** of that line on the side matching the direction θ is taken — the sign of $\cos\theta$ decides whether $x > a$ or $x < a$. (When $\theta = \pm\frac{\pi}{2}$ there is no gradient: the ray is the vertical half-line $x = a$ with $y > b$ or $y < b$ respectively.) A pair of such conditions, $\alpha \le arg(z - z_0) \le \beta$, describes a **sector**: the infinite wedge between the rays at angles α and β.



Intersections of regions.

The word **and** — equivalently the set intersection ∩ — requires a point to satisfy **every** relation at once. To sketch such a compound set, draw each region on its own and keep only the **overlap**. Richer shapes are assembled this way: a **half-disc** (a disc met by a line through its centre), a **circular segment** (a disc cut by a line that misses the centre), an **annulus** or ring ($r_1 \le |z| \le r_2$, the region between two concentric circles), a **rectangle**, or a **lens** (the overlap of two discs). The **corner points** of a compound region are the points where its boundary curves cross, found by solving their equations simultaneously.



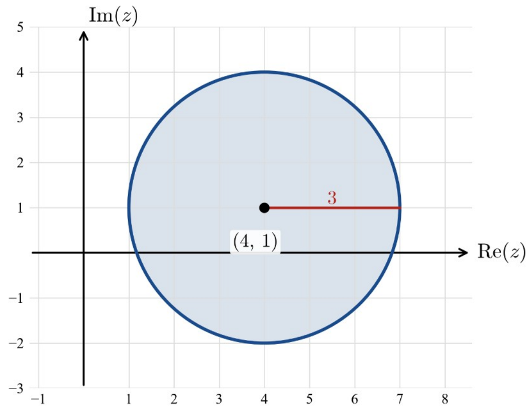
Worked examples

Example 1 — scaffolded

Sketch the set $\{z : |z - (4 + i)| = 3\}$, give its Cartesian equation, and describe the region $\{z : |z - (4 + i)| \le 3\}$.

Working	Comment
Here $z_0 = 4 + i$, so the centre is $(4, 1)$, and $r = 3$.	$\lvert z - z_0\rvert = r$ is the circle of centre z_0 and radius r.
Let $z = x + yi$. Then $z - z_0 = (x - 4) + (y - 1)i$, so $\lvert(x - 4) + (y - 1)i\rvert = 3$.	Substitute and group the real and imaginary parts of $z - z_0$.

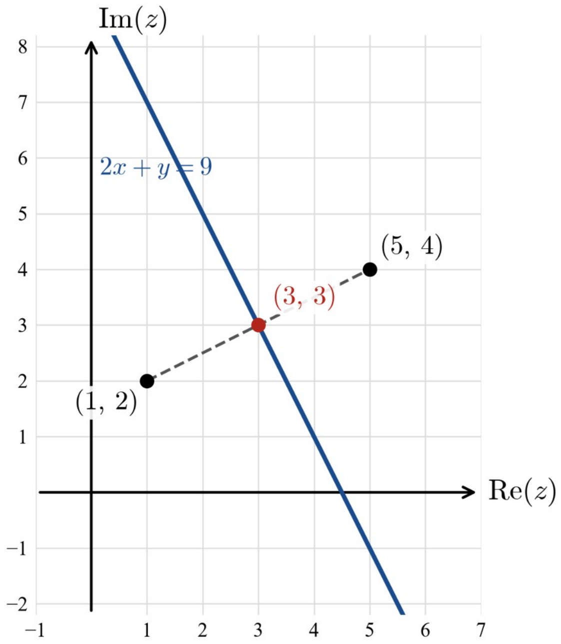
Working	Comment
Square both sides: $(x-4)^2+(y-1)^2=9$.	$\text{distance}^2=r^2$ clears the modulus.
So the set is the circle of centre $(4,1)$ and radius 3, drawn solid.	Every point of an = relation is included, so the curve is solid.
The region $\lvert z-(4+i)\rvert\le 3$ is the **closed disc**: this circle together with its interior, shaded.	≤ keeps the boundary (solid) and every interior point.



Example 2 — scaffolded

Describe and sketch the set $\{z : \lvert z-(1+2i)\rvert = \lvert z-(5+4i)\rvert\}$, and give its Cartesian equation.

Working	Comment
z is equidistant from $a=1+2i$, the point $(1,2)$, and $b=5+4i$, the point $(5,4)$.	$\lvert z-a\rvert=\lvert z-b\rvert$ is the perpendicular bisector of segment ab.
Put $z=x+yi$ and square: $(x-1)^2+(y-2)^2=(x-5)^2+(y-4)^2$.	Equal distances ⇔ equal squared distances.
Expand: $x^2-2x+1+y^2-4y+4=x^2-10x+25+y^2-8y+16$.	Expand each bracket.
The x^2 and y^2 cancel: $-2x-4y+5=-10x-8y+41$, so $8x+4y=36$.	Collect the linear terms.
Hence $2x+y=9$, that is $y=9-2x$: a line of gradient -2.	The bisector; its midpoint $(3,3)$ checks, since $9-6=3$.



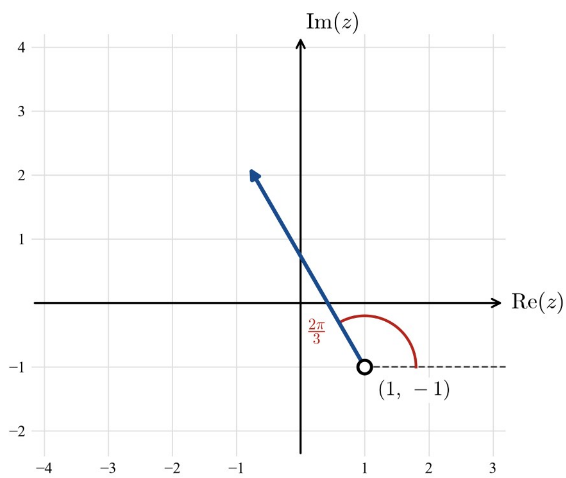
Example 3 — unscaffolded

Sketch the set $\{z : arg(z-(1-i))=\frac{2\pi}{3}\}$ and describe it in Cartesian form.

Write $z_0=1-i$, the point $(1,-1)$. The relation fixes the argument of the displacement from z_0 to z, so the set is a **ray** starting at $(1,-1)$ and pointing in the direction $\frac{2\pi}{3}$; every point of it is $z=(1-i)+t\,cis\frac{2\pi}{3}$ for some real $t>0$. Because $\tan\frac{2\pi}{3}=-\sqrt{3}$, the ray lies on the line through $(1,-1)$ of gradient $-\sqrt{3}$,

$$y-(-1)=-\sqrt{3}(x-1), \text{that is } y=-\sqrt{3}(x-1)-1.$$

The direction $\frac{2\pi}{3}$ points up and to the left, since $\cos\frac{2\pi}{3}=-\frac{1}{2}<0$ and $\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}>0$; so only the part of the line with $x<1$ (equivalently $y>-1$) belongs to the set. The endpoint $(1,-1)$ is excluded because $arg(0)$ is undefined, and is drawn as an open dot.



∴ the set is the open ray $y=-\sqrt{3}(x-1)-1$, $x<1$, a half-line from — but not including — the point $(1,-1)$.

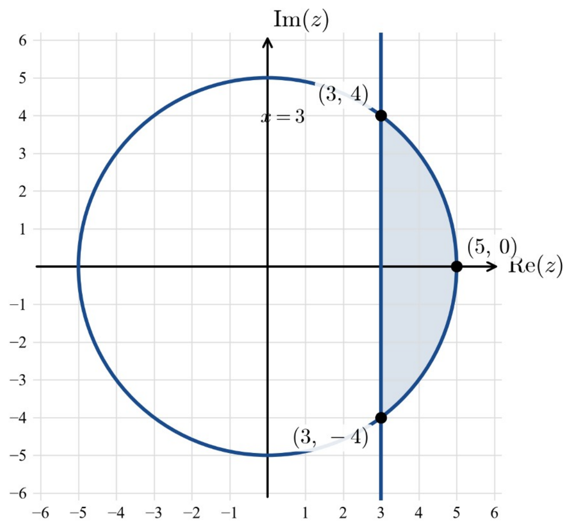
Example 4 — unscaffolded

Sketch and describe the region $\{z : |z|\le 5 \text{ and } \mathrm{Re}(z)\ge 3\}$, stating its corner points.

The first relation is the **closed disc** of centre O and radius 5, that is $x^2+y^2\le 25$. The second relation, $\operatorname{Re}(z)\ge 3$, is the closed half-plane on and to the right of the vertical line $x=3$. The set is their **intersection**: the slice of the disc lying on or to the right of $x=3$. The line $x=3$ meets the boundary circle where

$$3^2+y^2=25 \Rightarrow y^2=16 \Rightarrow y=\pm 4,$$

so the chord runs from $(3,-4)$ to $(3,4)$, and the disc reaches as far right as $(5,0)$. The overlap is therefore a **circular segment**, bounded on the left by the vertical chord along $x=3$ and on the right by the arc of the circle; both boundaries are solid and included.



∴ the region is the circular segment of the disc $x^2+y^2\le 25$ with $x\ge 3$, having corner points $(3,4)$ and $(3,-4)$.

Example 5 — unscaffolded

Sketch the locus $\{z:|z-2|+|z+2|=8\}$ and find its Cartesian equation.

The foci are $z_1=2$, the point $(2,0)$, and $z_2=-2$, the point $(-2,0)$, a distance $|z_1-z_2|=4$ apart; since the constant sum $k=8$ exceeds 4, the locus is an ellipse with these two foci. Writing $z=x+yi$, the relation $|z-2|+|z+2|=8$ becomes

$$\sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=8.$$

Move the second root to the right and square both sides:

$$(x-2)^2+y^2=64-16\sqrt{(x+2)^2+y^2}+(x+2)^2+y^2.$$

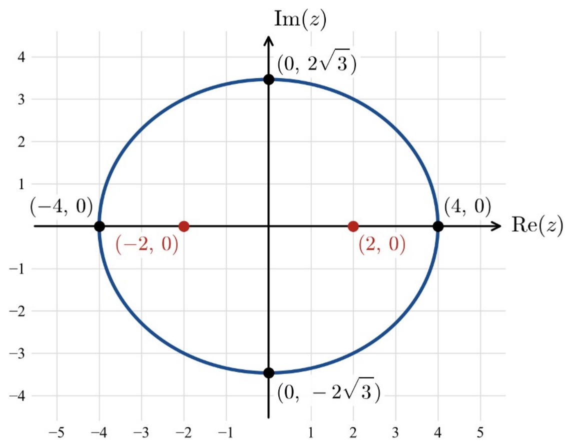
The two squared brackets differ by $(x-2)^2-(x+2)^2=-8x$, so $-8x=64-16\sqrt{(x+2)^2+y^2}$, which rearranges to $\sqrt{(x+2)^2+y^2}=4+\frac{x}{2}$. Squaring a second time,

$$(x+2)^2+y^2=16+4x+\frac{x^2}{4}, \text{so } x^2+4x+4+y^2=16+4x+\frac{x^2}{4}.$$

The $4x$ terms cancel and $x^2-\frac{x^2}{4}=\frac{3x^2}{4}$, leaving $\frac{3x^2}{4}+y^2=12$; dividing by 12,

$$\frac{x^2}{16}+\frac{y^2}{12}=1.$$

This is an ellipse centred at the origin, with semi-major axis $a=4$ along the real axis and semi-minor axis $b=2\sqrt{3}$ along the imaginary axis; the check $b^2=a^2-c^2=16-4=12$ confirms the foci $(\pm 2,0)$ lie inside it. The whole curve is drawn solid.



$\therefore$ the locus is the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$, with centre O, vertices $(\pm 4, 0)$, co-vertices $(0, \pm 2\sqrt{3})$ and foci $(\pm 2, 0)$.

Practice questions

Scaffolded

Q1. Consider the relation $|z-(-2+5i)|=4$. (a) State the centre and radius of this circle. (b) Taking $z=x+yi$, show its Cartesian equation is $(x+2)^2+(y-5)^2=16$. (c) Show that the circle passes through the point $2+5i$. (d) Sketch the circle, marking the centre.

Q2. Consider the relation $|z-(3-i)|<2$. (a) Write it in the form $|z-z_0|<r$ and state z_0 and r. (b) Is the boundary circle part of the set? What does this mean for the sketch? (c) Give the Cartesian description of the region. (d) Sketch the region, shading it.

Q3. Consider the relation $|z-2|=|z-8|$. (a) State the two points a and b that z is equidistant from. (b) Find the midpoint of the segment joining them. (c) Show that the locus is the vertical line $x=5$. (d) Sketch the line.

Q4. Consider the region $\{z: 1 \le \text{Im}(z) \le 4\}$. (a) What line is $\text{Im}(z)=1$? What line is $\text{Im}(z)=4$? (b) Are these boundary lines included in the set? (c) Describe the region in words. (d) Sketch the region, shading it.

Q5. Consider the relation $arg(z-1)=\frac{\pi}{4}$. (a) From which point does the ray start, and is that point included? (b) State the gradient of the line on which the ray lies. (c) Give the Cartesian description, including the restriction on x. (d) Sketch the ray.

Q6. Consider the region $\{z: |z| \le 3 \text{ and } \text{Im}(z) \ge 0\}$. (a) Describe each of the two relations separately. (b) The intersection keeps only points in both. Which half of the disc is it? (c) State the two pieces of its boundary. (d) Sketch the region, shading it.

Unscaffolded

Q7. State the centre, radius and Cartesian equation of $|z-7i|=3$, and sketch it.

Q8. Sketch the region $\{z: |z+2| \le 4\}$, describe it, and state where its boundary meets the axes.

Q9. Find the centre and radius of the circle $|z-3+4i|=2$, and give its Cartesian equation.

Q10. Describe and sketch the set $\{z: |z-(3+i)|=|z+(3+i)|\}$.

Q11. Show that $\{z: |z-4i|=|z-6|\}$ is a straight line, and find its equation.

Q12. Sketch the region $\{z: \text{Re}(z)>2\}$ and describe it.

Q13. Sketch the region $\{z: -2 \le \text{Re}(z) \le 1 \text{ and } 0 \le \text{Im}(z) \le 5\}$ and describe it, stating its corner points.

Q14. Sketch the set $\{z : arg(z+2) = \frac{\pi}{3}\}$ and give its Cartesian description.

Q15. Sketch the set $\{z : arg(z-4i) = -\frac{\pi}{3}\}$ and give its Cartesian description.

Q16. Sketch the region $\{z : \frac{\pi}{4} \le arg(z) \le \frac{\pi}{2}\}$ and describe it.

Q17. Sketch the region $\{z : 2 \le |z| \le 4\}$ and describe it.

Q18. Sketch the region $\{z : |z| \le 3 \text{ and } |z-4| \le 3\}$, find the two points where the boundary circles meet, and describe the region.

Q19. Sketch the locus $\{z : |z-4i| + |z+4i| = 10\}$, give its Cartesian equation, and state the coordinates of its vertices and foci.

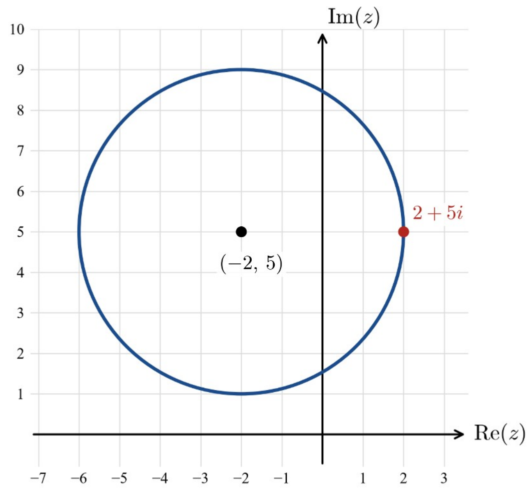
Q20. The points $z_1 = 2+i$ and $z_2 = 2+7i$ are the foci of an ellipse on which a point z moves so that $|z-z_1| + |z-z_2| = 10$. Find the centre, the semi-major and semi-minor axes, and the Cartesian equation of the ellipse.

Answers

Q1. Centre $(-2,5)$, radius 4; $(x+2)^2+(y-5)^2=16$; passes through $2+5i$ since $|(2+5i)-(-2+5i)|=4$. **Q2.** $z_0=3-i$, $r=2$; open disc (boundary dashed, excluded), $(x-3)^2+(y+1)^2<4$. **Q3.** Equidistant from $(2,0)$ and $(8,0)$; midpoint $(5,0)$; perpendicular bisector is the vertical line $x=5$. **Q4.** Closed horizontal band between $y=1$ and $y=4$, both lines included. **Q5.** Ray from $(1,0)$ (excluded) along $y=x-1$ with $x>1$; gradient 1. **Q6.** Upper half of the closed disc of radius 3: $x^2+y^2\le 9$ with $y\ge 0$. **Q7.** Centre $(0,7)$, radius 3; $x^2+(y-7)^2=9$. **Q8.** Closed disc centre $(-2,0)$, radius 4: $(x+2)^2+y^2\le 16$; boundary meets the x-axis at $(2,0)$ and $(-6,0)$ and the y-axis at $(0,2\sqrt{3})$ and $(0,-2\sqrt{3})$. **Q9.** Centre $(3,-4)$, radius 2; $(x-3)^2+(y+4)^2=4$. **Q10.** Perpendicular bisector of $(3,1)$ and $(-3,-1)$: the line $y=-3x$. **Q11.** Straight line $y=\frac{3x-5}{2}$ (perpendicular bisector of $(0,4)$ and $(6,0)$). **Q12.** Open half-plane to the right of the line $x=2$ (boundary dashed, excluded). **Q13.** Closed rectangle $-2\le x\le 1$, $0\le y\le 5$; corners $(-2,0)$, $(1,0)$, $(1,5)$, $(-2,5)$. **Q14.** Ray from $(-2,0)$ (excluded) on $y=\sqrt{3}(x+2)$ with $x>-2$. **Q15.** Ray from $(0,4)$ (excluded) on $y=4-\sqrt{3}x$ with $x>0$. **Q16.** Sector from O (excluded) between the rays $arg(z)=\frac{\pi}{4}$ and $arg(z)=\frac{\pi}{2}$: the wedge between $y=x$ ($x>0$) and the positive imaginary axis. **Q17.** Closed annulus $2\le|z|\le 4$: the ring between the circles of radius 2 and 4, both included. **Q18.** Circles meet at $(2,\sqrt{5})$ and $(2,-\sqrt{5})$; the region is the lens-shaped overlap of the two closed discs. **Q19.** Ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$ (major axis vertical); vertices $(0,\pm 5)$, co-vertices $(\pm 3,0)$, foci $(0,\pm 4)$. **Q20.** Centre $(2,4)$; semi-major axis $a=5$, semi-minor axis $b=4$; $\frac{(x-2)^2}{16}+\frac{(y-4)^2}{25}=1$.

Fully-worked solutions

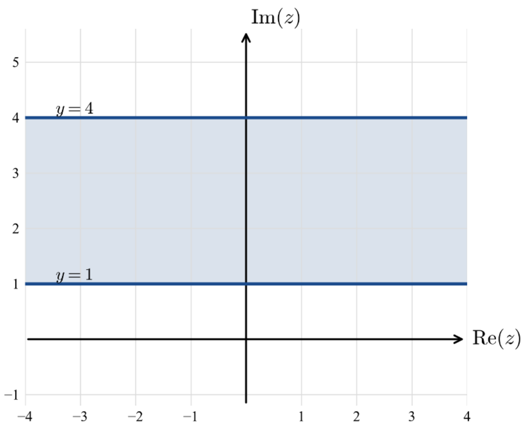
Q1. The relation $|z-(-2+5i)|=4$ has $z_0=-2+5i$ and $r=4$, so the centre is $(-2,5)$ and the radius is 4. With $z=x+yi$, $z-z_0=(x+2)+(y-5)i$; squaring $|z-z_0|=4$ gives $(x+2)^2+(y-5)^2=16$. The point $2+5i$ lies on the circle because its distance from the centre is $|(2+5i)-(-2+5i)|=|4|=4=r$.



Q2. Rewrite $|z-(3-i)|<2$, so $z_0=3-i$ (the point $(3,-1)$) and $r=2$. The inequality is **strict** (<), so the boundary circle is **not** part of the set and is drawn dashed; the set is the open disc of interior points only. With $z=x+yi$ the Cartesian description is $(x-3)^2+(y+1)^2<4$: every point less than 2 units from $(3,-1)$.

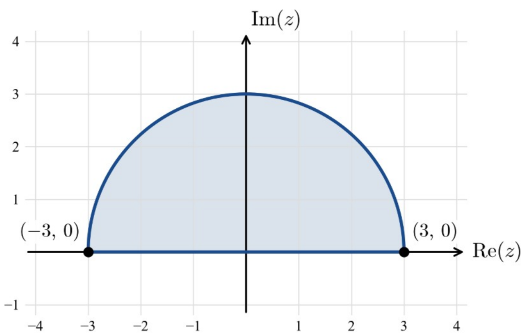
Q3. Here z is equidistant from $a=2$, the point $(2,0)$, and $b=8$, the point $(8,0)$. Their midpoint is $\left(\frac{2+8}{2},0\right)=(5,0)$. Putting $z=x+yi$ and squaring, $(x-2)^2+y^2=(x-8)^2+y^2$, so $x^2-4x+4=x^2-16x+64$, giving $12x=60$ and $x=5$. As a and b both lie on the real axis, the perpendicular bisector is the vertical line $x=5$ through their midpoint.

Q4. The line $\mathrm{Im}(z)=1$ is the horizontal line $y=1$, and $\mathrm{Im}(z)=4$ is the horizontal line $y=4$. Both use $\leq$, so both boundary lines are **included** (solid). The region $1\leq\mathrm{Im}(z)\leq 4$ places no restriction on $\mathrm{Re}(z)$, so it is the closed horizontal **band** consisting of every point whose imaginary part lies between 1 and 4 inclusive — an infinite strip between the lines $y=1$ and $y=4$.



Q5. The relation $arg(z-1)=\frac{\pi}{4}$ is a ray starting at $z_0=1$, the point $(1,0)$; the endpoint itself is **excluded** because $arg(0)$ is undefined. The ray lies on the line through $(1,0)$ of gradient $\tan\frac{\pi}{4}=1$, namely $y-0=1(x-1)$, so $y=x-1$. Since $\frac{\pi}{4}$ points into the first quadrant ($\cos\frac{\pi}{4}>0$), only the part with $x>1$ (and hence $y>0$) belongs to the set. So the Cartesian description is $y=x-1$, $x>1$, with an open endpoint at $(1,0)$.

Q6. The relation $|z|\leq 3$ is the closed disc of centre O and radius 3, that is $x^2+y^2\leq 9$; the relation $\mathrm{Im}(z)\geq 0$ is the closed half-plane on and above the real axis, $y\geq 0$. Their intersection keeps points satisfying both, so it is the **upper half** of the disc. Its boundary has two pieces: the upper semicircle $x^2+y^2=9$ (with $y\geq 0$) and the segment of the real axis from $(-3,0)$ to $(3,0)$; both are solid.



Q7. The relation $|z-7i|=3$ has $z_0=7i$ (the point $(0,7)$) and $r=3$. With $z=x+yi$, $z-7i=x+(y-7)i$, and squaring $|z-7i|=3$ gives $x^2+(y-7)^2=9$: the circle of centre $(0,7)$ and radius 3, drawn solid.

Q8. The relation $|z+2|\leq 4$ is $|z-(-2)|\leq 4$, the closed disc of centre $(-2,0)$ and radius 4, that is $(x+2)^2+y^2\leq 16$ — every point on or inside that circle. The boundary meets the x-axis where $y=0$: $(x+2)^2=16$, so $x=2$ or $x=-6$, giving $(2,0)$ and $(-6,0)$. It meets the y-axis where $x=0$: $4+y^2=16$, so $y^2=12$ and $y=\pm 2\sqrt{3}$, giving $(0,2\sqrt{3})$ and $(0,-2\sqrt{3})$.

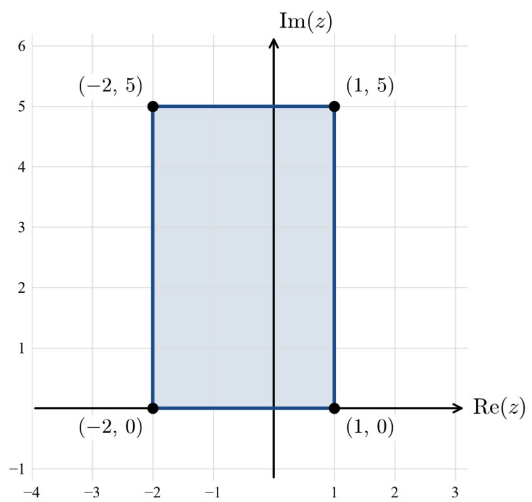
Q9. Rewrite $|z-3+4i|=|z-(3-4i)|$, so $z_0=3-4i$ (the point $(3,-4)$) and $r=2$. With $z=x+yi$, $z-z_0=(x-3)+(y+4)i$, and squaring the relation gives $(x-3)^2+(y+4)^2=4$: the circle of centre $(3,-4)$ and radius 2.

Q10. Rewrite the relation as $|z-(3+i)|=|z-(-3-i)|$, so z is equidistant from $a=3+i$, the point $(3,1)$, and $b=-3-i$, the point $(-3,-1)$. Put $z=x+yi$ and square: $(x-3)^2+(y-1)^2=(x+3)^2+(y+1)^2$. Expanding, $-6x-2y+10=6x+2y+10$, so $-12x-4y=0$, giving $y=-3x$. The segment from $(3,1)$ to $(-3,-1)$ has gradient $\frac{1}{3}$ and midpoint O, and $y=-3x$ passes through O with the perpendicular gradient -3, confirming the perpendicular bisector.

Q11. Here z is equidistant from $a=4i$, the point $(0,4)$, and $b=6$, the point $(6,0)$. Put $z=x+yi$ and square: $x^2+(y-4)^2=(x-6)^2+y^2$. Expanding, $x^2+y^2-8y+16=x^2-12x+36+y^2$, so $-8y+16=-12x+36$, giving $12x-8y=20$ and hence $y=\frac{3x-5}{2}$. This is a straight line (gradient $\frac{3}{2}$, through $\left(0,-\frac{5}{2}\right)$), the perpendicular bisector of the segment from $(0,4)$ to $(6,0)$ — its midpoint $(3,2)$ satisfies $2=\frac{3(3)-5}{2}$.

Q12. Since $\operatorname{Re}(z)=x$, the relation $\operatorname{Re}(z)>2$ is $x>2$: the open half-plane of all points strictly to the right of the vertical line $x=2$. The boundary line $x=2$ is dashed and excluded, and the region to its right (with no restriction on y) is shaded.

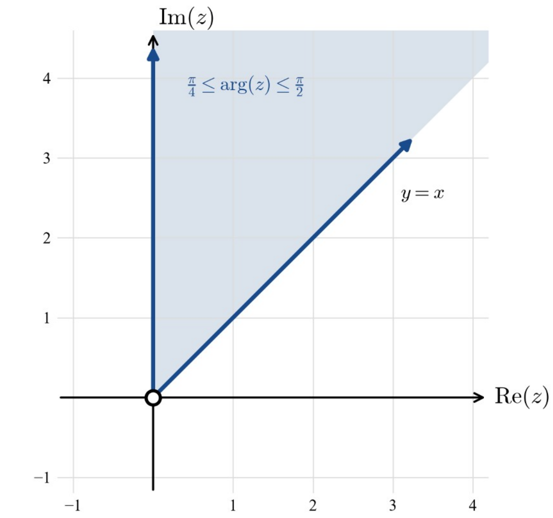
Q13. The pair of conditions $-2\le\operatorname{Re}(z)\le1$ and $0\le\operatorname{Im}(z)\le5$ becomes $-2\le x\le1$ and $0\le y\le5$. The first is the vertical band between $x=-2$ and $x=1$; the second is the horizontal band between $y=0$ and $y=5$. Their intersection is the closed **rectangle** with corners $(-2,0)$, $(1,0)$, $(1,5)$ and $(-2,5)$; all four edges are solid.



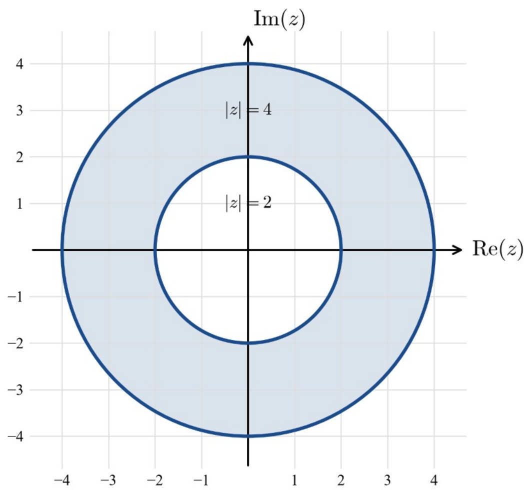
Q14. The set $arg(z+2)=\frac{\pi}{3}$ is a ray from $z_0=-2$, the point $(-2,0)$, which is excluded. It lies on the line through $(-2,0)$ of gradient $\tan\frac{\pi}{3}=\sqrt{3}$: $y-0=\sqrt{3}(x+2)$, so $y=\sqrt{3}(x+2)$. The direction $\frac{\pi}{3}$ points up and to the right ($\cos\frac{\pi}{3}>0$), so only the part with $x>-2$ (and $y>0$) belongs. Cartesian description: $y=\sqrt{3}(x+2)$, $x>-2$, open at $(-2,0)$.

Q15. The set $arg(z-4i)=-\frac{\pi}{3}$ is a ray from $z_0=4i$, the point $(0,4)$, which is excluded. It lies on the line through $(0,4)$ of gradient $\tan\left(-\frac{\pi}{3}\right)=-\sqrt{3}$: $y-4=-\sqrt{3}(x-0)$, so $y=4-\sqrt{3}x$. The direction $-\frac{\pi}{3}$ has $\cos\left(-\frac{\pi}{3}\right)>0$, so only the part with $x>0$ (and $y<4$) belongs. Cartesian description: $y=4-\sqrt{3}x$, $x>0$, open at $(0,4)$.

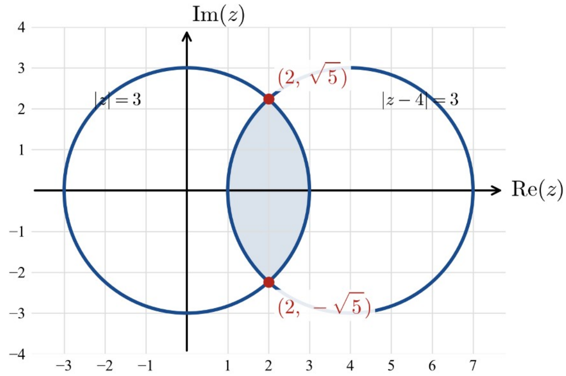
Q16. The relation $\frac{\pi}{4} \le arg(z) \le \frac{\pi}{2}$ is a **sector** of rays from the origin whose argument lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Its lower bounding ray is $arg(z) = \frac{\pi}{4}$, lying on $y = x$ ($\tan\frac{\pi}{4} = 1$) with $x > 0$; its upper bounding ray is $arg(z) = \frac{\pi}{2}$, the positive imaginary axis $x = 0$, $y > 0$. The set is the first-quadrant wedge between these two rays (both included), with the origin excluded.



Q17. Since $|z|$ is the distance of z from O, the relation $2 \le |z| \le 4$ keeps points whose distance from the origin is between 2 and 4 inclusive. This is the closed **annulus** (ring) bounded by the circle $x^2 + y^2 = 4$ on the inside and the circle $x^2 + y^2 = 16$ on the outside; both circles are solid, and the ring between them is shaded.



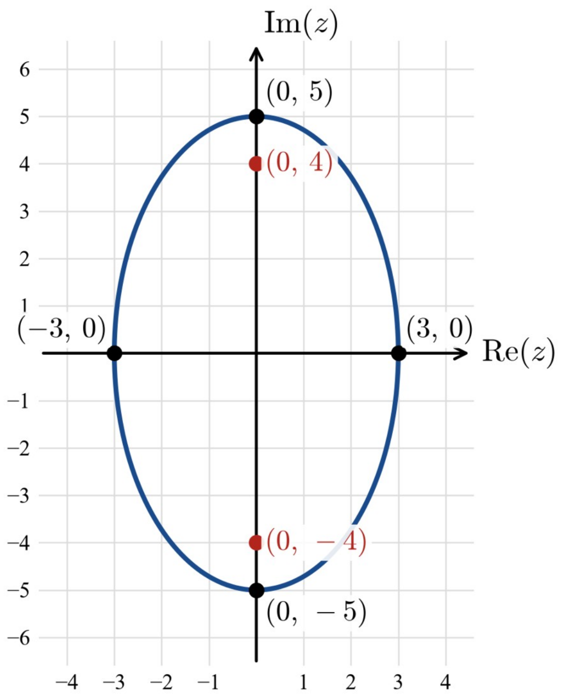
Q18. The relation $|z| \le 3$ is the closed disc of centre $(0,0)$, radius 3, and $|z-4| \le 3$ is the closed disc of centre $(4,0)$, radius 3. The two boundary circles $x^2 + y^2 = 9$ and $(x-4)^2 + y^2 = 9$ meet where the left-hand sides are equal: $x^2 = (x-4)^2$, so $x = 2$; then $y^2 = 9 - 4 = 5$, giving $y = \pm\sqrt{5}$. Hence the circles cross at $(2, \sqrt{5})$ and $(2, -\sqrt{5})$. The intersection of the two discs is the **lens**-shaped region between the two arcs joining these points — every z within 3 units of both $(0,0)$ and $(4,0)$.



Q19. The relation $|z-4i|+|z+4i|=10$ fixes the sum of the distances from z to the foci $z_1=4i$, the point $(0,4)$, and $z_2=-4i$, the point $(0,-4)$, at $k=10$. Since $|z_1-z_2|=8<10$, the locus is an ellipse; the foci lie on the imaginary axis, so the major axis is vertical. With $z=x+yi$,

$$\sqrt{x^2+(y-4)^2}+\sqrt{x^2+(y+4)^2}=10.$$

Isolating and squaring the first root gives $x^2+(y-4)^2=100-20\sqrt{x^2+(y+4)^2}+x^2+(y+4)^2$; the brackets differ by $(y-4)^2-(y+4)^2=-16y$, so $-16y=100-20\sqrt{x^2+(y+4)^2}$ and hence $\sqrt{x^2+(y+4)^2}=5+\frac{4y}{5}$. Squaring again, $x^2+y^2+8y+16=25+8y+\frac{16y^2}{25}$, so $x^2+y^2-\frac{16y^2}{25}=9$, that is $x^2+\frac{9y^2}{25}=9$. Dividing by 9 gives $\frac{x^2}{9}+\frac{y^2}{25}=1$. Thus $a=5$, $b=3$ and $c=\sqrt{25-9}=4$: an ellipse with vertices $(0,\pm 5)$, co-vertices $(\pm 3,0)$ and foci $(0,\pm 4)$.



Q20. Because the foci $z_1=2+i$, the point $(2,1)$, and $z_2=2+7i$, the point $(2,7)$, share the real part $x=2$, they lie on the vertical line $x=2$; hence the major axis is vertical and the centre is their midpoint $\left(2,\frac{1+7}{2}\right)=(2,4)$. The focal distance is $c=\frac{1}{2}|z_1-z_2|=\frac{1}{2}(6)=3$, and the constant sum gives the semi-major axis $a=\frac{1}{2}k=\frac{1}{2}(10)=5$; then the semi-minor axis is $b=\sqrt{a^2-c^2}=\sqrt{25-9}=4$. Measuring from the centre $(2,4)$, with the major axis vertical the larger denominator $a^2=25$ sits under the y-term:

$$\frac{(x-2)^2}{16}+\frac{(y-4)^2}{25}=1.$$

Theory

In 11A a complex number was written in **Cartesian form** $z=a+bi$, and that form is ideal for adding and subtracting: real parts and imaginary parts combine separately. For **multiplying, dividing and raising to a power**, however, Cartesian form quickly becomes heavy. This section develops a second description — **polar form** — in which each of these three operations becomes a single, simple rule, and in which multiplication acquires a vivid geometric meaning: a rotation combined with a scaling.

Modulus and argument.

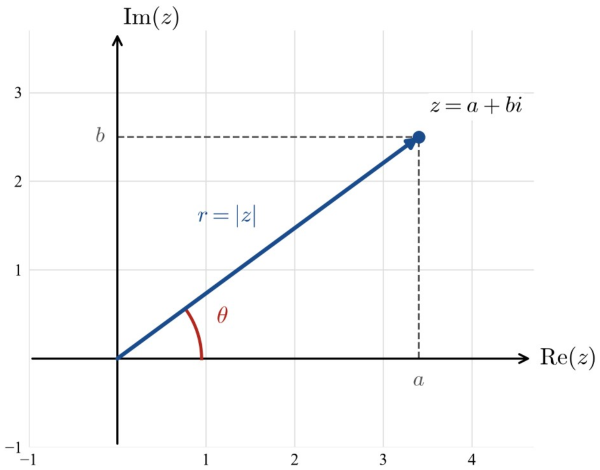
Plot $z=a+bi$ as the point (a,b) on the Argand diagram. Two numbers locate that point just as well as the pair (a,b) does: *how far* it is from the origin, and *in which direction*. The distance from the origin O is the **modulus**, already met in 11A,

$$r=|z|=\sqrt{a^2+b^2}\geq 0,$$

and the direction is the **argument** $arg(z)$: the angle the position vector from O to (a,b) makes with the **positive real axis**, measured **anticlockwise as positive**. From the right-angled triangle formed by the point and its foot on the real axis,

$$a=r\cos\theta, b=r\sin\theta, \tan\theta=\frac{b}{a}(a\neq 0).$$

Adding a whole turn 2π returns to the same direction, so the argument is only fixed up to multiples of 2π: if θ is an argument of z, so is $\theta+2\pi k$ for every integer k. The argument of $z=0$ is undefined.



Principal argument.

To make the direction a single number we select the argument lying in one revolution. The **principal argument**, written $Arg(z)$, is the one value with

$$-\pi<Arg(z)\leq\pi.$$

Points above the real axis take a positive principal argument (up to π on the negative real axis); points below it take a negative principal argument (down to, but not including, $-\pi$).

Polar form.

Substituting $a=r\cos\theta$ and $b=r\sin\theta$ into $z=a+bi$ gives $z=r\cos\theta+ir\sin\theta=r(\cos\theta+i\sin\theta)$. Writing $cis\theta$ as shorthand for $\cos\theta+i\sin\theta$, the **polar form** (or **modulus–argument form**) of a complex number is

$$z=r\,cis\theta=r(\cos\theta+i\sin\theta), r=|z|\geq 0,$$

where θ is any argument of z; the principal argument is normally quoted. Two non-zero numbers in polar form are **equal** exactly when their moduli agree and their arguments differ by a whole number of turns: $r_1 cis\,\theta_1 = r_2 cis\,\theta_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$ for some integer k.

Changing between Cartesian and polar form.

To pass from **Cartesian to polar**, compute $r = \sqrt{a^2 + b^2}$, find the acute **reference angle**

$$\alpha = \tan^{-1}\left|\frac{b}{a}\right| (0 < \alpha < \pi/2),$$

and then read the principal argument from the quadrant in which z lies. The equation $\tan\theta = \frac{b}{a}$ alone cannot fix the quadrant, since directions π apart share a tangent, so a quick sketch of the sign pattern is essential.

Quadrant of z	signs (a, b)	$Arg(z)$
first	$a > 0, b > 0$	α
second	$a < 0, b > 0$	$\pi - \alpha$
third	$a < 0, b < 0$	$-(\pi - \alpha)$
fourth	$a > 0, b < 0$	$-\alpha$

A number on an axis is read directly: a positive real has $Arg(z) = 0$, a negative real has $Arg(z) = \pi$, a positive imaginary has $Arg(z) = \frac{\pi}{2}$, and a negative imaginary has $Arg(z) = -\frac{\pi}{2}$. To pass from **polar to Cartesian**, simply evaluate $a = r\cos\theta$ and $b = r\sin\theta$. Standard angles keep the answers exact.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Multiplication in polar form.

Let $z_1 = r_1 cis\,\theta_1$ and $z_2 = r_2 cis\,\theta_2$ be any two complex numbers. Multiplying out the two brackets and using $i^2 = -1$,

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The two inner brackets are precisely the **addition formulas** for cosine and sine,

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2,$$

so the product collapses to

$$z_1 z_2 = r_1 r_2\, cis(\theta_1 + \theta_2).$$

In words: **to multiply, multiply the moduli and add the arguments.** A useful special case is $|z_1 z_2| = |z_1||z_2|$ and $arg(z_1 z_2) = arg(z_1) + arg(z_2)$.

Division in polar form.

First find the **reciprocal** of $z_2 = r_2 \, cis\, \theta_2$ (with $z_2 \neq 0$). The number $\frac{1}{r_2} cis(-\theta_2)$ works, because multiplying it by z_2 gives $r_2 \cdot \frac{1}{r_2} cis(\theta_2 - \theta_2) = 1\, cis\, 0 = 1$. Multiplying z_1 by this reciprocal,

$$\frac{z_1}{z_2} = r_1\, cis\, \theta_1 \times \frac{1}{r_2} cis(-\theta_2) = \frac{r_1}{r_2} cis(\theta_1 - \theta_2).$$

Put in words: **dividing divides the moduli and subtracts the arguments.**

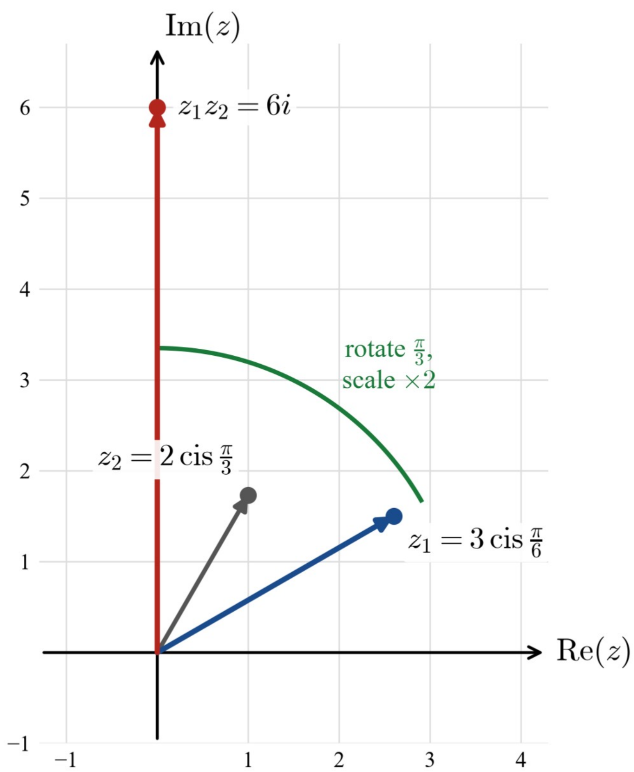
Bringing an out-of-range argument back.

Adding or subtracting two arguments can land outside $(-\pi, \pi]$. Since a whole turn changes nothing, add or subtract 2π as needed to bring the result back into the principal range before quoting the answer. For instance an argument of $\frac{4\pi}{3}$ exceeds π, so subtract one turn: $\frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$.

A geometric picture: rotation and enlargement.

The multiplication rule gives multiplication a striking geometric meaning. Multiplying a point z by $w = r\, cis\, \phi$ produces a new point whose modulus is r times that of z and whose argument is ϕ more than that of z. So **multiplying by $w = r\, cis\, \phi$ rotates the point z anticlockwise about the origin through the angle ϕ and scales its distance from O by the factor r.** Two cases stand out:

- if $|w| = 1$ (that is $w = cis\, \phi$) the effect is a **pure rotation** through ϕ, with no change of modulus;
- since $i = cis \frac{\pi}{2}$, multiplying by i rotates a point a quarter-turn anticlockwise about the origin.



The diagram shows the product of $z_1 = 3\, cis \frac{\pi}{6}$ and $z_2 = 2\, cis \frac{\pi}{3}$. Multiplying z_1 by z_2 turns it anticlockwise through $arg(z_2) = \frac{\pi}{3}$ and stretches its length by $|z_2| = 2$; the moduli give $3 \times 2 = 6$ and the arguments give $\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$, so $z_1 z_2 = 6\, cis \frac{\pi}{2} = 6i$.

Integer powers.

Applying the multiplication rule to $z = r\,cis\,\theta$ taken against itself,

$$z^2 = z \cdot z = r \cdot r\,cis(\theta+\theta) = r^2\,cis(2\theta),$$

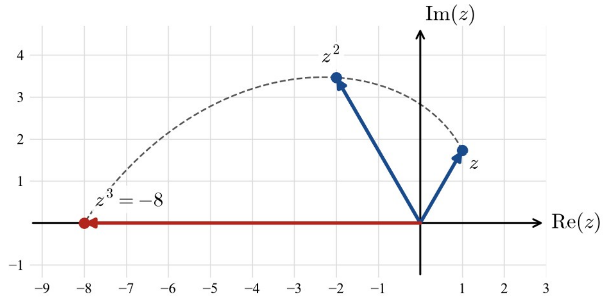
and multiplying by z once more,

$$z^3 = z^2 \cdot z = r^2 \cdot r\,cis(2\theta+\theta) = r^3\,cis(3\theta).$$

Each additional factor of z multiplies the modulus by r and adds θ to the argument, so for every **positive integer** n,

$$(r\,cis\,\theta)^n = r^n\,cis(n\theta).$$

This is the natural extension of repeated multiplication: the modulus is raised to the power n and the argument is multiplied by n. Geometrically, each factor repeats the same rotation-and-scale, so the points $z, z^2, z^3, \ldots$ spiral outward (if $r > 1$) or inward (if $r < 1$), turning by the constant angle θ at each step. As always, reduce the final argument to the principal range. The general statement — proved for all integers n by mathematical induction, and named **De Moivre's theorem** — is met in Units 3 & 4; here the positive-integer case obtained by repeated multiplication is all that is needed.



The diagram shows the powers of $z = 2\,cis\,\frac{\pi}{3}$. Each step turns a further $\frac{\pi}{3}$ anticlockwise and doubles the modulus, so $z^2 = 4\,cis\,\frac{2\pi}{3}$ and $z^3 = 2^3\,cis\left(3 \times \frac{\pi}{3}\right) = 8\,cis\,\pi = -8$.

Worked examples

Example 1 — scaffolded

(a) Express $z = -3+3i$ in polar form, using the principal argument.

(b) Express $w = 6\,cis\,\frac{2\pi}{3}$ in Cartesian form.

Working	Comment
(a) $a=-3, b=3$: $a<0, b>0$, so z is in the second quadrant.	Fix the quadrant from the signs before finding the angle.
$r = \sqrt{(-3)^2+3^2} = \sqrt{18} = 3\sqrt{2}$.	Modulus $r = \sqrt{a^2+b^2}$; simplify the surd.
$\tan\alpha = \left\lvert\frac{3}{-3}\right\rvert = 1$, so $\alpha = \frac{\pi}{4}$.	Reference angle from the acute triangle.
Second quadrant: $Arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.	Read the principal argument from the quadrant table.
$z = 3\sqrt{2}\,cis\,\frac{3\pi}{4}$.	Assemble $r\,cis\,\theta$.

Working	Comment
(b) $\cos\frac{2\pi}{3}=-\frac{1}{2}$, $\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$.	Read the exact values for the standard angle.
$w=6\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-3+3\sqrt{3}i$.	$a=r\cos\theta$, $b=r\sin\theta$.

Example 2 — scaffolded

Given $z_1=8\,cis\frac{5\pi}{6}$ and $z_2=2\,cis\frac{\pi}{4}$, find (a) z_1z_2 and (b) $\frac{z_1}{z_2}$, each in polar form using the principal argument.

Working	Comment
(a) $r_1r_2=8\times2=16$.	To multiply, multiply the moduli.
$\theta_1+\theta_2=\frac{5\pi}{6}+\frac{\pi}{4}=\frac{10\pi}{12}+\frac{3\pi}{12}=\frac{13\pi}{12}$.	Add the arguments over a common denominator.
$\frac{13\pi}{12}>\pi$, so subtract a turn: $\frac{13\pi}{12}-2\pi=-\frac{11\pi}{12}$.	Bring the argument into $(-\pi,\pi]$.
$z_1z_2=16\,cis\left(-\frac{11\pi}{12}\right)$.	Combine the new modulus and principal argument.
(b) $\frac{r_1}{r_2}=\frac{8}{2}=4$.	To divide, divide the moduli.
$\theta_1-\theta_2=\frac{5\pi}{6}-\frac{\pi}{4}=\frac{10\pi}{12}-\frac{3\pi}{12}=\frac{7\pi}{12}$.	Subtract the arguments; already in $(-\pi,\pi]$.
$\frac{z_1}{z_2}=4\,cis\frac{7\pi}{12}$.	Combine the new modulus and argument.

Example 3 — unscaffolded

Let $z=-2+2i$. Convert z to polar form, and hence find z^3, giving the answer in both polar and Cartesian form.

The real part $a=-2<0$ and the imaginary part $b=2>0$, so z lies in the second quadrant. Its modulus is $r=\sqrt{(-2)^2+2^2}=\sqrt{8}=2\sqrt{2}$, and the reference angle satisfies $\tan\alpha=\left|\frac{2}{-2}\right|=1$, so $\alpha=\frac{\pi}{4}$ and $Arg(z)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$.

Thus $z=2\sqrt{2}\,cis\frac{3\pi}{4}$. Raising to the third power multiplies the modulus to the power 3 and the argument by 3:

$$z^3=(2\sqrt{2})^3\,cis\left(3\times\frac{3\pi}{4}\right)=16\sqrt{2}\,cis\frac{9\pi}{4}.$$

Since $\frac{9\pi}{4}>\pi$, subtract a turn: $\frac{9\pi}{4}-2\pi=\frac{\pi}{4}$, so $z^3=16\sqrt{2}\,cis\frac{\pi}{4}$. Converting with $\cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$,

$$z^3=16\sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=16+16i.$$

$\therefore z^3=16\sqrt{2}\,cis\frac{\pi}{4}=16+16i$.

Example 4 — unscaffolded

The point P on the Argand diagram represents $z=3+4i$. Find the complex number obtained by rotating P a quarter-turn anticlockwise about O and doubling its distance from O, and verify the result by direct multiplication.

A quarter-turn anticlockwise with a scaling by 2 is exactly multiplication by $w=2\,cis\frac{\pi}{2}=2i$, since $|w|=2$ and $Arg(w)=\frac{\pi}{2}$. The image is therefore wz. By direct multiplication,

$$wz=2i(3+4i)=6i+8i^2=-8+6i.$$

As a check on the geometry, $|z|=\sqrt{3^2+4^2}=5$ while $|wz|=\sqrt{(-8)^2+6^2}=\sqrt{100}=10=2|z|$, confirming the distance from O has doubled; and the direction of $(3,4)$ turned a quarter-turn anticlockwise becomes $(-4,3)$, which scaled by 2 is indeed $(-8,6)$.

∴ the required number is $wz=-8+6i$, with $|wz|=10=2|z|$.

Practice questions

Scaffolded

Q1. Express $z=-4-4i$ in polar form, using the principal argument. (a) State the signs of a and b and the quadrant of z. (b) Find the modulus r. (c) Find the reference angle α, and hence the principal argument. (d) State z as $r\,cis\,\theta$.

Q2. Express $z=10\,cis\frac{2\pi}{3}$ in Cartesian form. (a) Write down $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$. (b) Find $a=r\cos\theta$ and $b=r\sin\theta$. (c) Hence write z as $a+bi$.

Q3. Given $z_1=3\,cis\frac{\pi}{4}$ and $z_2=6\,cis\frac{\pi}{3}$, find $z_1 z_2$ in polar form. (a) Multiply the moduli. (b) Add the arguments using a common denominator. (c) Confirm the sum lies in $(-\pi,\pi]$. (d) State $z_1 z_2$ as $r\,cis\,\theta$.

Q4. Given $z_1=20\,cis\frac{3\pi}{4}$ and $z_2=4\,cis\frac{\pi}{3}$, find $\frac{z_1}{z_2}$ in polar form. (a) Divide the moduli. (b) Subtract the second argument from the first. (c) Confirm the argument is principal. (d) State $\frac{z_1}{z_2}$ in polar form.

Q5. Let $z=3\,cis\frac{\pi}{6}$. Find z^3 by repeated multiplication. (a) Find $z^2=z\cdot z$ (square the modulus, double the argument). (b) Find $z^3=z^2\cdot z$. (c) State z^3 in polar form and in Cartesian form.

Q6. The point representing $z=1+2i$ is multiplied by $w=3i$. (a) Express w in polar form. (b) Describe the geometric effect of multiplying by w. (c) Compute wz in Cartesian form. (d) Verify that $|wz|=3|z|$.

Unscaffolded

Q7. Express $z=\sqrt{3}-i$ in polar form, using the principal argument.

Q8. Express $z=12\,cis\left(-\frac{\pi}{6}\right)$ in Cartesian form.

Q9. Given $z_1=6\,cis\frac{5\pi}{6}$ and $z_2=4\,cis\frac{2\pi}{3}$, find $z_1 z_2$ in polar form, using the principal argument.

Q10. Given $z_1=9\,cis\left(-\frac{3\pi}{4}\right)$ and $z_2=3\,cis\frac{5\pi}{6}$, find $\frac{z_1}{z_2}$ in polar form, using the principal argument.

Q11. Let $z=1-i$. Using polar form, find z^4.

Q12. Let $z = -\sqrt{3} + i$. Find the smallest positive integer n for which z^n is a real number, and state the value of z^n for that n.

Q13. The point representing $z = 2 - 3i$ is rotated a quarter-turn anticlockwise about the origin. Find the resulting complex number, and state the multiplication that achieves it.

Q14. Given $z_1 = 10\,cis\,\frac{3\pi}{4}$, $z_2 = 2\,cis\,\frac{\pi}{2}$ and $z_3 = 4\,cis\,\frac{\pi}{6}$, find $\frac{z_1 z_2}{z_3}$ in polar form.

Q15. The number $z = 3\,cis\,\frac{\pi}{4}$ is multiplied by $w = 2\,cis\,\frac{\pi}{3}$. Describe the geometric effect on z, and state $w\,z$ in polar form.

Q16. Express $(\sqrt{3} + i)^5$ in Cartesian form.

Q17. Let $z = -2 + 2\sqrt{3}\,i$. Express z in polar form, and hence show that z^3 is real and find its value.

Q18. Let $w = cis\,\frac{\pi}{6}$. (a) Find w^6 in Cartesian form. (b) Find w^{12}. (c) State the smallest positive integer n for which $w^n = 1$, and explain the answer geometrically.

Answers

Q1. $r=4\sqrt{2}$, $Arg(z)=-\frac{3\pi}{4}$; $z=4\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right)$. **Q2.** $z=-5+5\sqrt{3}i$. **Q3.** $z_1z_2=18\,cis\frac{7\pi}{12}$. **Q4.** $\frac{z_1}{z_2}=5\,cis\frac{5\pi}{12}$. **Q5.** $z^3=27\,cis\frac{\pi}{2}=27i$. **Q6.** (a) $w=3\,cis\frac{\pi}{2}$; (b) a quarter-turn anticlockwise about O together with a scaling by 3; (c) $wz=-6+3i$; (d) $|wz|=3\sqrt{5}=3|z|$. **Q7.** $z=2\,cis\left(-\frac{\pi}{6}\right)$. **Q8.** $z=6\sqrt{3}-6i$. **Q9.** $z_1z_2=24\,cis\left(-\frac{\pi}{2}\right)=-24i$. **Q10.** $\frac{z_1}{z_2}=3\,cis\frac{5\pi}{12}$. **Q11.** $z^4=-4$. **Q12.** $n=6$; $z^6=-64$. **Q13.** $3+2i$, achieved by multiplying by i. **Q14.** $\frac{z_1z_2}{z_3}=5\,cis\left(-\frac{11\pi}{12}\right)$. **Q15.** A rotation of $\frac{\pi}{3}$ anticlockwise about O together with a scaling by 2; $wz=6\,cis\frac{7\pi}{12}$. **Q16.** $-16\sqrt{3}+16i$. **Q17.** $z=4\,cis\frac{2\pi}{3}$; $z^3=64$. **Q18.** (a) $w^6=-1$; (b) $w^{12}=1$; (c) $n=12$, since each multiplication by w turns the point $\frac{\pi}{6}$ anticlockwise and 12 such turns complete one full revolution of 2π.

Fully-worked solutions

Q1. $a=-4$ and $b=-4$ are both negative, so z is in the third quadrant. $r=\sqrt{(-4)^2+(-4)^2}=\sqrt{32}=4\sqrt{2}$. The reference angle satisfies $\tan\alpha=\left|\frac{-4}{-4}\right|=1$, so $\alpha=\frac{\pi}{4}$; in the third quadrant the principal argument is $Arg(z)=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{4}\right)=-\frac{3\pi}{4}$. Hence $z=4\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right)$.

Q2. With $r=10$ and $\theta=\frac{2\pi}{3}$, use $\cos\frac{2\pi}{3}=-\frac{1}{2}$ and $\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$. Then $a=10\times\left(-\frac{1}{2}\right)=-5$ and $b=10\times\frac{\sqrt{3}}{2}=5\sqrt{3}$, so $z=-5+5\sqrt{3}i$.

Q3. Multiply the moduli: $r_1r_2=3\times 6=18$. Add the arguments: $\frac{\pi}{4}+\frac{\pi}{3}=\frac{3\pi}{12}+\frac{4\pi}{12}=\frac{7\pi}{12}$, which satisfies $-\pi<\frac{7\pi}{12}\le\pi$. Hence $z_1z_2=18\,cis\frac{7\pi}{12}$.

Q4. Divide the moduli: $\frac{r_1}{r_2}=\frac{20}{4}=5$. Subtract the arguments: $\frac{3\pi}{4}-\frac{\pi}{3}=\frac{9\pi}{12}-\frac{4\pi}{12}=\frac{5\pi}{12}$, already principal. Hence $\frac{z_1}{z_2}=5\,cis\frac{5\pi}{12}$.

Q5. Squaring, $z^2=3^2\,cis\left(2\times\frac{\pi}{6}\right)=9\,cis\frac{\pi}{3}$. Multiplying by z again, $z^3=z^2\cdot z=9\times 3\,cis\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=27\,cis\frac{\pi}{2}$, and $\frac{\pi}{2}$ is principal. Converting with $\cos\frac{\pi}{2}=0$, $\sin\frac{\pi}{2}=1$, $z^3=27(0+i)=27i$. Hence $z^3=27\,cis\frac{\pi}{2}=27i$.

Q6. (a) $w=3i$ has $|w|=3$ and $Arg(w)=\frac{\pi}{2}$, so $w=3\,cis\frac{\pi}{2}$. (b) Multiplying by w adds $\frac{\pi}{2}$ to the argument and multiplies the modulus by 3: a quarter-turn anticlockwise about O together with a scaling by 3. (c) $wz=3i(1+2i)=3i+6i^2=-6+3i$. (d) $|z|=\sqrt{1^2+2^2}=\sqrt{5}$ and $|wz|=\sqrt{(-6)^2+3^2}=\sqrt{45}=3\sqrt{5}=3|z|$, as expected.

Q7. $a=\sqrt{3}>0$ and $b=-1<0$, so z is in the fourth quadrant. $r=\sqrt{(\sqrt{3})^2+(-1)^2}=\sqrt{3+1}=2$. The reference angle satisfies $\tan\alpha=\left|\frac{-1}{\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$, so $\alpha=\frac{\pi}{6}$; in the fourth quadrant $Arg(z)=-\alpha=-\frac{\pi}{6}$. Hence $z=2\,cis\left(-\frac{\pi}{6}\right)$.

Q8. With $r=12$ and $\theta=-\frac{\pi}{6}$, use $\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$. Then $a=12\times\frac{\sqrt{3}}{2}=6\sqrt{3}$ and $b=12\times\left(-\frac{1}{2}\right)=-6$, so $z=6\sqrt{3}-6i$.

Q9. Multiply the moduli: $6\times4=24$. Add the arguments: $\frac{5\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6}+\frac{4\pi}{6}=\frac{9\pi}{6}=\frac{3\pi}{2}$. Since $\frac{3\pi}{2}>\pi$, subtract a turn: $\frac{3\pi}{2}-2\pi=-\frac{\pi}{2}$. Hence $z_1z_2=24\,cis\left(-\frac{\pi}{2}\right)=-24i$.

Q10. Divide the moduli: $\frac{9}{3}=3$. Subtract the arguments: $-\frac{3\pi}{4}-\frac{5\pi}{6}=-\frac{9\pi}{12}-\frac{10\pi}{12}=-\frac{19\pi}{12}$. Since $-\frac{19\pi}{12}\le-\pi$, add a turn: $-\frac{19\pi}{12}+2\pi=-\frac{19\pi}{12}+\frac{24\pi}{12}=\frac{5\pi}{12}$. Hence $\frac{z_1}{z_2}=3\,cis\frac{5\pi}{12}$.

Q11. First write $z=1-i$ in polar form: $r=\sqrt{1^2+(-1)^2}=\sqrt{2}$, and with $a>0$, $b<0$ (fourth quadrant) the reference angle is $\frac{\pi}{4}$, so $Arg(z)=-\frac{\pi}{4}$ and $z=\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$. Then $z^4=(\sqrt{2})^4\,cis\left(4\times\left(-\frac{\pi}{4}\right)\right)=4\,cis(-\pi)$. Since $-\pi$ represents the same direction as π (the negative real axis), $z^4=4\,cis\,\pi=4(\cos\pi+i\sin\pi)=-4$.

Q12. In polar form $z=-\sqrt{3}+i$ has $r=\sqrt{(-\sqrt{3})^2+1^2}=2$; it is in the second quadrant with reference angle $\tan\alpha=\frac{1}{\sqrt{3}}$, $\alpha=\frac{\pi}{6}$, so $Arg(z)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$ and $z=2\,cis\frac{5\pi}{6}$. Then $z^n=2^n\,cis\frac{5n\pi}{6}$, which is real exactly when its argument $\frac{5n\pi}{6}$ is a whole multiple of π, that is when $\frac{5n}{6}$ is an integer. Since 5 and 6 share no factor, this first happens at $n=6$. Then $z^6=2^6\,cis\left(6\times\frac{5\pi}{6}\right)=64\,cis(5\pi)$; subtracting two turns, $5\pi-4\pi=\pi$, so $z^6=64\,cis\,\pi=-64$.

Q13. A quarter-turn anticlockwise about the origin is multiplication by $i=cis\frac{\pi}{2}$. Hence the image of $z=2-3i$ is $iz=i(2-3i)=2i-3i^2=3+2i$. So the resulting number is $3+2i$, obtained by multiplying z by i.

Q14. The moduli combine as $\frac{r_1r_2}{r_3}=\frac{10\times2}{4}=5$. The arguments combine as $\frac{3\pi}{4}+\frac{\pi}{2}-\frac{\pi}{6}=\frac{9\pi}{12}+\frac{6\pi}{12}-\frac{2\pi}{12}=\frac{13\pi}{12}$. Since $\frac{13\pi}{12}>\pi$, subtract a turn: $\frac{13\pi}{12}-2\pi=-\frac{11\pi}{12}$. Hence $\frac{z_1z_2}{z_3}=5\,cis\left(-\frac{11\pi}{12}\right)$.

Q15. Since $|w|=2$ and $Arg(w)=\frac{\pi}{3}$, multiplying by w turns z anticlockwise about O through $\frac{\pi}{3}$ and scales its distance from O by 2. The result has modulus $3\times2=6$ and argument $\frac{\pi}{4}+\frac{\pi}{3}=\frac{3\pi}{12}+\frac{4\pi}{12}=\frac{7\pi}{12}$, which is principal. Hence $wz=6\,cis\frac{7\pi}{12}$.

Q16. Write $\sqrt{3}+i$ in polar form: $r=\sqrt{(\sqrt{3})^2+1^2}=2$, and in the first quadrant $\tan\alpha=\frac{1}{\sqrt{3}}$ gives $\alpha=\frac{\pi}{6}$, so $\sqrt{3}+i=2\,cis\,\frac{\pi}{6}$. Raising to the fifth power, $(\sqrt{3}+i)^5=2^5\,cis\left(5\times\frac{\pi}{6}\right)=32\,cis\,\frac{5\pi}{6}$, and $\frac{5\pi}{6}$ is principal. Converting with $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6}=\frac{1}{2}$,

$$(\sqrt{3}+i)^5=32\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=-16\sqrt{3}+16i.$$

Q17. For $z=-2+2\sqrt{3}i$, $r=\sqrt{(-2)^2+(2\sqrt{3})^2}=\sqrt{4+12}=\sqrt{16}=4$. With $a<0$, $b>0$ (second quadrant) and $\tan\alpha=\left|\frac{2\sqrt{3}}{-2}\right|=\sqrt{3}$, the reference angle is $\alpha=\frac{\pi}{3}$, so $Arg(z)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ and $z=4\,cis\,\frac{2\pi}{3}$. Cubing, $z^3=4^3\,cis\left(3\times\frac{2\pi}{3}\right)=64\,cis(2\pi)$. Since 2π is a whole turn, $cis(2\pi)=cis\,0=1$, so $z^3=64\,cis\,0=64$, a (positive) real number.

Q18. (a) $w^6=cis\left(6\times\frac{\pi}{6}\right)=cis\,\pi=\cos\pi+i\sin\pi=-1$. (b) $w^{12}=cis\left(12\times\frac{\pi}{6}\right)=cis(2\pi)=cis\,0=1$. (c) $w^n=cis\,\frac{n\pi}{6}$ equals 1 exactly when $\frac{n\pi}{6}$ is a whole multiple of 2π, that is when $\frac{n}{12}$ is an integer, so the smallest positive value is $n=12$. Geometrically, $|w|=1$, so each multiplication by w is a pure rotation of $\frac{\pi}{6}$ anticlockwise about O; twelve such rotations amount to $12\times\frac{\pi}{6}=2\pi$, one complete revolution, returning the point to 1.