Specialist Mathematics Units 1 & 2

Chapter 10 — Vectors in the plane

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Contents

Theory

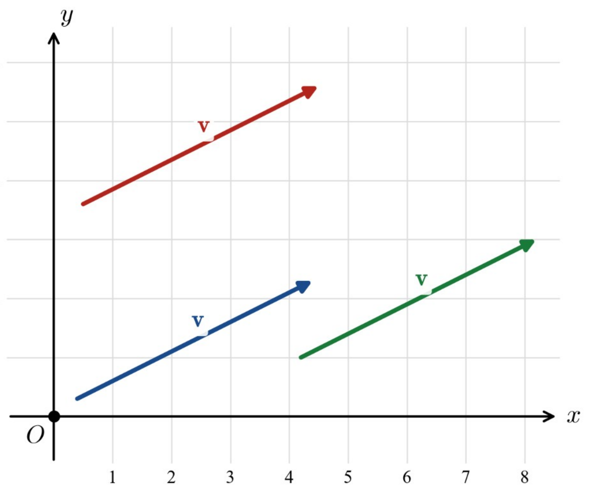
Many quantities in the plane are fixed by a single number together with a **scale** — a mass of $3\,\mathrm{kg}$, a temperature of $18\,°\mathrm{C}$, a length of $5\,\mathrm{m}$. Such a quantity is called a **scalar**. Other quantities are not settled until a **direction** is also given: a displacement of $5\,\mathrm{m}$ *north-east*, a velocity, a force. A quantity that carries both a **magnitude** (a non-negative size) and a **direction** is a **vector**. This section sets up plane (two-dimensional) vectors, measures their length and direction, and builds the unit vectors from which every direction can be read off. Everything here is in the plane; three-dimensional vectors and the products of vectors are met in later study.

Directed line segments.

A vector in the plane is pictured as a **directed line segment** — an arrow. The arrow from a point A to a point B is written $\overrightarrow{AB}$; the point A is its **tail** (initial point) and B its **tip** (terminal point). The **length** of the arrow represents the magnitude of the vector and the way it points represents the direction. Single letters in bold, such as a, u, v, also name vectors; by hand these are underlined, $\underline{a}$. The magnitude of a is written $|a|$, a non-negative real number.

Equality of vectors.

Two vectors are **equal** when they have the *same magnitude* **and** the *same direction*. Position on the page plays no part: an arrow may be slid anywhere in the plane, keeping its length and direction, without changing the vector it represents. So the three directed segments below, drawn at different places but equally long and equally directed, represent one and the same vector v.



Because a vector is free to be moved in this way, it is often called a **free vector**. Two directed segments are equal exactly when one can be slid onto the other; they need not start at the same point.

Component form.

To calculate with vectors we attach numbers to them. Let i be the vector of length 1 pointing along the positive x-axis and j the vector of length 1 pointing along the positive y-axis. Any plane vector a can then be written as

$$a = a_1 i + a_2 j,$$

where the **components** a_1 and a_2 are the horizontal and vertical steps from tail to tip. The same vector may be written as an **ordered pair** (a_1, a_2) or in **column form**

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

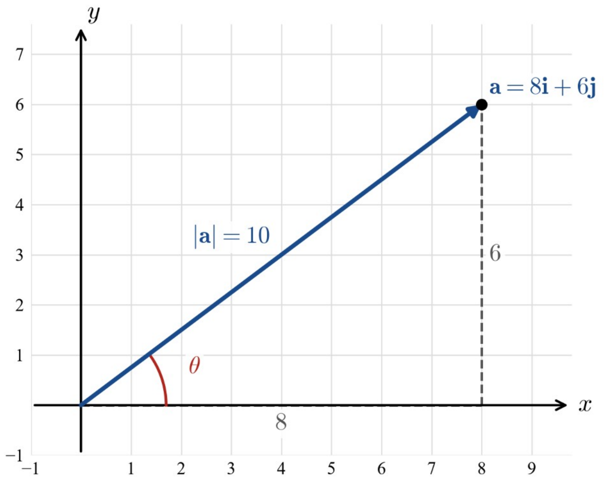
Two vectors are equal precisely when *both* pairs of components agree: $a_1 i + a_2 j = b_1 i + b_2 j$ if and only if $a_1 = b_1$ and $a_2 = b_2$. A single vector equation therefore carries **two** scalar equations, one for each component — the standard way to solve for unknowns hidden inside a vector.

Magnitude.

The **magnitude** of $a = a_1 i + a_2 j$ is the length of its arrow. The components form the two shorter sides of a right-angled triangle whose hypotenuse is the vector, so **Pythagoras' theorem** gives

$$|a| = \sqrt{a_1^2 + a_2^2}.$$

This is a non-negative real number, and $|a| = 0$ holds only for the zero vector introduced below. The figure shows $a = 8i + 6j$, whose components 8 and 6 give $|a| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$.



Direction.

The **direction** of a plane vector is described by the angle θ it makes with the positive direction of the x-axis, measured anticlockwise, with $0° \le \theta < 360°$. From the same right-angled triangle,

$$\cos\theta = \frac{a_1}{|a|}, \sin\theta = \frac{a_2}{|a|}, \tan\theta = \frac{a_2}{a_1} (a_1 \neq 0).$$

Because $\tan\theta$ takes the same value for the two opposite directions θ and $\theta + 180°$, it cannot fix the direction by itself. The safe method is to find the acute **reference angle**

$$\alpha = \tan^{-1}\left|\frac{a_2}{a_1}\right| (0° < \alpha < 90°),$$

and then read θ from the quadrant fixed by the *signs* of a_1 and a_2: in the first quadrant $\theta = \alpha$; in the second $\theta = 180° - \alpha$; in the third $\theta = 180° + \alpha$; and in the fourth $\theta = 360° - \alpha$. For example $d = -\sqrt{3} i + j$ has $|d| = \sqrt{3+1} = 2$ and, since $a_1 < 0$ while $a_2 > 0$, lies in the second quadrant; the reference angle satisfies $\tan\alpha = |1/-\sqrt{3}| = 1/\sqrt{3}$, so $\alpha = 30°$ and $\theta = 180° - 30° = 150°$.

The zero vector.

The **zero vector** $0 = 0i + 0j$ has magnitude $|0| = 0$. It is the only vector with zero length, and — having no arrow to point along — it is the one vector to which no direction is assigned. It plays the part for vectors that 0 plays for numbers.

Scalar multiplication and the negative of a vector.

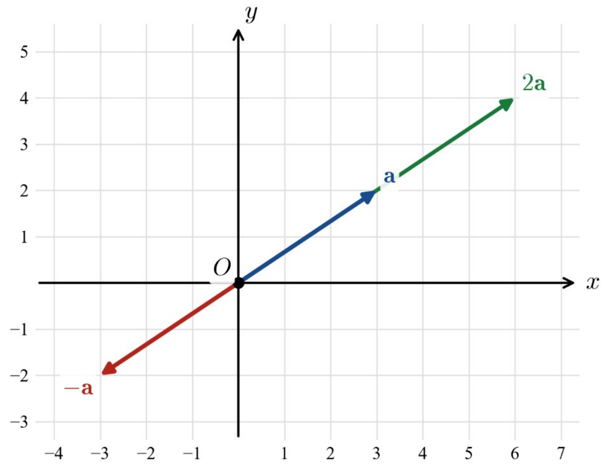
Multiplying a vector by a real number k (a **scalar**) stretches or shrinks it and may reverse it:

$$k\,a = k\,a_1\,i + k\,a_2\,j\,.$$

Its magnitude is scaled by the size of k, namely $|k\,a| = |k|\,|a|$. If $k > 0$ the result points the **same** way as a; if $k < 0$ it points the **opposite** way; and if $k = 0$ it is the zero vector. In particular, taking $k = -1$ gives the **negative** of a,

$$-a = -a_1\,i - a_2\,j\,,$$

the vector of the *same length* as a pointing in the *opposite* direction. Two non-zero vectors that are scalar multiples of one another, $b = k\,a$, point along the same line and are called **parallel**; they have the same direction when $k > 0$ and opposite directions when $k < 0$. The figure shows a together with $2\,a$ (twice as long, same direction) and $-a$ (same length, reversed).



Unit vectors.

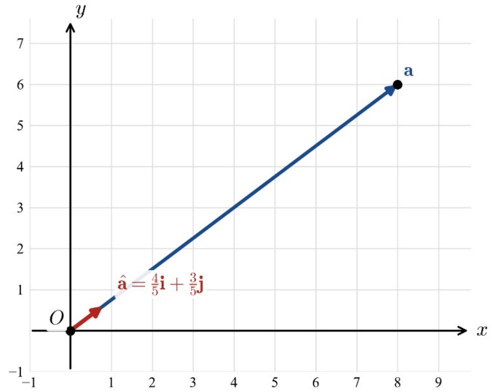
A **unit vector** is a vector of magnitude 1; the reference vectors i and j are the two standard examples. The unit vector in the direction of a non-zero vector a is written $\hat{a}$ (read "a-hat") and is obtained by dividing a by its own length:

$$\hat{a} = \frac{a}{|a|} = \frac{1}{|a|}\left(a_1\,i + a_2\,j\right).$$

Dividing by the positive scalar $|a|$ rescales the length to exactly 1 while leaving the direction unchanged, so $|\hat{a}| = 1$ and $\hat{a}$ points the same way as a. Reversing the idea, a vector of a **chosen magnitude m in the direction of a** is

$$m\,\hat{a} = \frac{m}{|a|}\,a\,,$$

and a negative value of m produces a vector of length $|m|$ pointing the opposite way. The next figure shows a vector a and its unit vector $\hat{a}$, drawn along the same ray with length shortened to 1.



Position vectors.

Fixing the origin O, each point P in the plane is located by its **position vector** $\overrightarrow{OP}$ — the directed segment from O to P. If P has coordinates (x, y) then its position vector is $\overrightarrow{OP} = x\,i + y\,j$, so the components of a position vector are simply the coordinates of the point. Given two points A and B with position vectors a and b, the vector **from** A **to** B is found by subtracting position vectors,

$$\overrightarrow{AB} = b - a = (b_1 - a_1)\,i + (b_2 - a_2)\,j,$$

because travelling out to B and back from A is the displacement tip minus tail. The **distance** between the points is the magnitude of this vector,

$$AB = |b - a| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2},$$

which is the familiar distance formula in vector dress. For instance, if $A = (2, 3)$ and $B = (9, 27)$ then $\overrightarrow{AB} = 7\,i + 24\,j$ and $AB = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$.

Worked examples

Example 1 — scaffolded

For $a = 4\,i - 2\,j$, find (a) the magnitude $|a|$, (b) the unit vector $\hat{a}$, and (c) the vector of magnitude $6\sqrt{5}$ in the direction of a.

Working	Comment
$a = 4\,i - 2\,j$, so $a_1 = 4$ and $a_2 = -2$.	Read off the components.
$\lvert a\rvert = \sqrt{4^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$.	Magnitude by Pythagoras; simplify the surd $\sqrt{20} = \sqrt{4}\sqrt{5}$.
$\hat{a} = \frac{a}{\lvert a\rvert} = \frac{1}{2\sqrt{5}}(4\,i - 2\,j) = \frac{2}{\sqrt{5}}\,i - \frac{1}{\sqrt{5}}\,j$.	Divide the vector by its magnitude.
$= \frac{2\sqrt{5}}{5}\,i - \frac{\sqrt{5}}{5}\,j$.	Rationalise each denominator by multiplying by $\frac{\sqrt{5}}{\sqrt{5}}$.
$6\sqrt{5}\,\hat{a} = \frac{6\sqrt{5}}{2\sqrt{5}}(4\,i - 2\,j) = 3(4\,i - 2\,j) = 12\,i - 6\,j$.	Scale the unit vector to the required length; the surds cancel.
Check: $\lvert 12\,i - 6\,j\rvert = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}$.	The magnitude confirms the length is $6\sqrt{5}$.

Example 2 — scaffolded

Find the magnitude of $b = \sqrt{3}\,i - j$ and the angle it makes with the positive direction of the x-axis, measured anticlockwise.

Working	Comment
$a_1 = \sqrt{3}$ and $a_2 = -1$.	Read off the components.
$\lvert b\rvert = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3 + 1} = \sqrt{4} = 2$.	Magnitude by Pythagoras.
Since $a_1 > 0$ and $a_2 < 0$, the vector lies in the fourth quadrant.	The signs of the components fix the quadrant.
$\tan\alpha = \left\lvert\frac{-1}{\sqrt{3}}\right\rvert = \frac{1}{\sqrt{3}}$, so $\alpha = 30°$.	Reference angle from the acute triangle.
Fourth quadrant: $\theta = 360° - \alpha = 360° - 30° = 330°$.	Convert the reference angle to the anticlockwise direction.

Example 3 — unscaffolded

A vector p has magnitude 12 and makes an angle of $210°$ with the positive direction of the x-axis, measured anticlockwise. Express p in component form and state the unit vector $\hat{p}$.

Resolving into horizontal and vertical parts, $p = |p|\cos\theta\, i + |p|\sin\theta\, j$ with $|p| = 12$ and $\theta = 210°$. Using the exact values $\cos 210° = -\frac{\sqrt{3}}{2}$ and $\sin 210° = -\frac{1}{2}$,

$$p = 12\left(-\frac{\sqrt{3}}{2}\right)i + 12\left(-\frac{1}{2}\right)j = -6\sqrt{3}\,i - 6\,j.$$

The unit vector is this divided by the magnitude 12, which just strips off the factor 12:

$$\hat{p} = \frac{p}{12} = -\frac{\sqrt{3}}{2}i - \frac{1}{2}j,$$

and indeed $|\hat{p}| = \sqrt{3/4 + 1/4} = 1$.

$\therefore\ p = -6\sqrt{3}\,i - 6\,j$, with $\hat{p} = -\frac{\sqrt{3}}{2}i - \frac{1}{2}j$.

Example 4 — unscaffolded

The points A and B have position vectors $a = -3i + 4j$ and $b = 17i + 25j$. Find $\overrightarrow{AB}$ in component form, the distance AB, a unit vector in the direction of $\overrightarrow{AB}$, and the vector of magnitude 87 in that direction.

The vector from A to B is tip minus tail,

$$\overrightarrow{AB} = b - a = (17 - (-3))i + (25 - 4)j = 20i + 21j.$$

Its magnitude is the distance between the points,

$$AB = |\overrightarrow{AB}| = \sqrt{20^2 + 21^2} = \sqrt{400 + 441} = \sqrt{841} = 29.$$

Dividing by this length gives the unit vector along $\overrightarrow{AB}$,

$$\widehat{\overrightarrow{AB}} = \frac{1}{29}(20i + 21j) = \frac{20}{29}i + \frac{21}{29}j.$$

Scaling this unit vector to length $87 = 3 \times 29$ multiplies the original by $\frac{87}{29} = 3$, so the required vector is $3(20i + 21j) = 60i + 63j$, whose magnitude is $\sqrt{60^2 + 63^2} = \sqrt{7569} = 87$.

$\therefore\ \overrightarrow{AB} = 20i + 21j$, $AB = 29$, $\widehat{\overrightarrow{AB}} = \frac{1}{29}(20i + 21j)$, and the vector of magnitude 87 is $60i + 63j$.

Practice questions

Scaffolded

Q1. For $a = 5i + 12j$: (a) find $|a|$; (b) find the unit vector $\hat{a}$; (c) find the vector of magnitude 39 in the direction of a.

Q2. For $b = -i + 3j$: (a) find $|b|$, leaving your answer as an exact surd; (b) find the unit vector $\hat{b}$, with rationalised denominators; (c) state whether $c = 3i - 9j$ is parallel to b, and if so whether it points in the same or the opposite direction.

Q3. For $u = 5i - 2j$, write in component form: (a) $2u$; (b) $-3u$; (c) $1/2u$; (d) the negative $-u$.

Q4. Find the direction of $v=-2i-2j$. (a) Find the magnitude $|v|$. (b) State the quadrant and find the reference angle α. (c) State the angle θ measured anticlockwise from the positive x-axis.

Q5. Find, in component form, the vector w of magnitude 14 that makes an angle of $240°$ with the positive x-axis. (a) Write down $\cos 240°$ and $\sin 240°$. (b) Find the i- and j-components. (c) State w in component form.

Q6. The points A and B have coordinates $A(2,-1)$ and $B(11,39)$. (a) Find $\overrightarrow{AB}$ in component form. (b) Find the distance AB. (c) Find a unit vector in the direction of $\overrightarrow{AB}$.

Unscaffolded

Q7. Find $|a|$ and $\hat{a}$ for $a=-8i+15j$.

Q8. Find the magnitude of $b=3i+5j$, giving an exact surd, and find $\hat{b}$ with rationalised denominators.

Q9. Find the vector of magnitude 74 in the direction of $d=12i+35j$.

Q10. Find the magnitude of $c=4i-4\sqrt{3}j$ and the angle it makes with the positive x-axis, measured anticlockwise.

Q11. A vector has magnitude 9 and makes an angle of $150°$ with the positive x-axis. Express it in component form.

Q12. The vectors $a=(2p-1)i+7j$ and $b=5i+(q+2)j$ are equal. Find the values of the real numbers p and q.

Q13. Find the distance between the points $A(-4,7)$ and $B(8,-2)$.

Q14. The points A and B have coordinates $A(3,-2)$ and $B(31,43)$. Find $\overrightarrow{AB}$, the distance AB, and a unit vector in the direction of $\overrightarrow{AB}$.

Q15. Find the magnitude of $v=-4i+4j$ and the angle it makes with the positive x-axis, measured anticlockwise.

Q16. Find the value of the real number t for which $a=6i+tj$ is parallel to $b=2i-3j$, and state whether a and b then point in the same or opposite direction.

Q17. For $a=3i+7j$, find the **two** unit vectors that are parallel to a.

Q18. The points A and B have coordinates $A(6,-1)$ and $B(22,62)$. Find $\overrightarrow{AB}$, the distance AB, and hence the vector of magnitude 130 in the direction of $\overrightarrow{AB}$.

Answers

Q1. (a) 13; (b) $\hat{a}=5/13i+12/13j$; (c) $15i+36j$. **Q2.** (a) $\sqrt{10}$; (b) $\hat{b}=-\sqrt{10}/10i+3\sqrt{10}/10j$; (c) parallel, since $c=-3b$, pointing in the opposite direction. **Q3.** (a) $10i-4j$; (b) $-15i+6j$; (c) $5/2i-j$; (d) $-5i+2j$. **Q4.** (a) $2\sqrt{2}$; (b) third quadrant, $\alpha=45°$; (c) $\theta=225°$. **Q5.** (a) $\cos 240°=-1/2$, $\sin 240°=-\sqrt{3}/2$; (b) -7 and $-7\sqrt{3}$; (c) $w=-7i-7\sqrt{3}j$. **Q6.** (a) $9i+40j$; (b) 41; (c) $1/41(9i+40j)$. **Q7.** $|a|=17$, $\hat{a}=1/17(-8i+15j)$. **Q8.** $|b|=\sqrt{34}$, $\hat{b}=3\sqrt{34}/34i+5\sqrt{34}/34j$. **Q9.** $24i+70j$. **Q10.** $|c|=8$, $\theta=300°$. **Q11.** $-9\sqrt{3}/2i+9/2j$. **Q12.** $p=3$, $q=5$. **Q13.** $AB=15$. **Q14.** $\overrightarrow{AB}=28i+45j$, $AB=53$, unit vector $1/53(28i+45j)$. **Q15.** $|v|=4\sqrt{2}$, $\theta=135°$. **Q16.** $t=-9$; since $a=3b$ the two point in the same direction. **Q17.** $\pm 1/\sqrt{58}(3i+7j)=\pm(3\sqrt{58}/58i+7\sqrt{58}/58j)$. **Q18.** $\overrightarrow{AB}=16i+63j$, $AB=65$, vector of magnitude 130 is $32i+126j$.

Fully-worked solutions

Q1. $a_1=5$, $a_2=12$. (a) $|a|=\sqrt{5^2+12^2}=\sqrt{25+144}=\sqrt{169}=13$. (b) $\hat{a}=\frac{a}{|a|}=\frac{1}{13}(5i+12j)=5/13i+12/13j$. (c) $39\hat{a}=\frac{39}{13}(5i+12j)=3(5i+12j)=15i+36j$; as a check $\sqrt{15^2+36^2}=\sqrt{1521}=39$.

Q2. $b=-i+3j$, so $b_1=-1$, $b_2=3$. (a) $|b|=\sqrt{(-1)^2+3^2}=\sqrt{1+9}=\sqrt{10}$. (b) $\hat{b}=\frac{1}{\sqrt{10}}(-i+3j)=-\frac{1}{\sqrt{10}}i+\frac{3}{\sqrt{10}}j=-\frac{\sqrt{10}}{10}i+\frac{3\sqrt{10}}{10}j$. (c) $c=3i-9j=-3(-i+3j)=-3b$. Since c is a scalar multiple of b it is parallel to it, and because the scalar -3 is negative, c points in the opposite direction to b.

Q3. With $u=5i-2j$, each scalar multiplies both components: (a) $2u=10i-4j$. (b) $-3u=-15i+6j$. (c) $1/2u=5/2i-j$. (d) $-u=-5i+2j$, the same length as u in the opposite direction.

Q4. $v=-2i-2j$, so $v_1=-2$, $v_2=-2$. (a) $|v|=\sqrt{(-2)^2+(-2)^2}=\sqrt{8}=2\sqrt{2}$. (b) Both components are negative, so v lies in the third quadrant. The reference angle satisfies $\tan\alpha=|-2/-2|=1$, so $\alpha=45°$. (c) In the third quadrant $\theta=180°+\alpha=180°+45°=225°$.

Q5. Resolving, $w=|w|\cos\theta i+|w|\sin\theta j$ with $|w|=14$ and $\theta=240°$. (a) $\cos 240°=-1/2$ and $\sin 240°=-\sqrt{3}/2$. (b) The i-component is $14\times(-1/2)=-7$ and the j-component is $14\times(-\sqrt{3}/2)=-7\sqrt{3}$. (c) $w=-7i-7\sqrt{3}j$.

Q6. $A(2,-1)$ and $B(11,39)$. (a) $\overrightarrow{AB}=(11-2)i+(39-(-1))j=9i+40j$. (b) $AB=\sqrt{9^2+40^2}=\sqrt{81+1600}=\sqrt{1681}=41$. (c) $\frac{\overrightarrow{AB}}{AB}=\frac{1}{41}(9i+40j)$.

Q7. $a=-8i+15j$, so $|a|=\sqrt{(-8)^2+15^2}=\sqrt{64+225}=\sqrt{289}=17$, and $\hat{a}=\frac{1}{17}(-8i+15j)$.

Q8. $b=3i+5j$, so $|b|=\sqrt{3^2+5^2}=\sqrt{34}$, which does not simplify. Then

$$\hat{b}=\frac{1}{\sqrt{34}}(3i+5j)=\frac{3}{\sqrt{34}}i+\frac{5}{\sqrt{34}}j=\frac{3\sqrt{34}}{34}i+\frac{5\sqrt{34}}{34}j.$$

Q9. First $|d|=\sqrt{12^2+35^2}=\sqrt{1369}=37$. The vector of magnitude 74 in the direction of d is

$$74\hat{d}=\frac{74}{37}(12i+35j)=2(12i+35j)=24i+70j,$$

and $\sqrt{24^2+70^2}=\sqrt{5476}=74$ confirms the length.

Q10. $c=4i-4\sqrt{3}j$, so

$$|c|=\sqrt{4^2+(4\sqrt{3})^2}=\sqrt{16+48}=\sqrt{64}=8.$$

Here $c_1=4>0$ and $c_2=-4\sqrt{3}<0$, so c lies in the fourth quadrant. The reference angle satisfies $\tan\alpha=|-4\sqrt{3}/4|=\sqrt{3}$, so $\alpha=60°$, and in the fourth quadrant $\theta=360°-60°=300°$.

Q11. Resolving with magnitude 9 and $\theta=150°$, and using $\cos 150°=-\sqrt{3}/2$, $\sin 150°=1/2$,

$$a=9\cos 150°\, i+9\sin 150°\, j=9\left(-\frac{\sqrt{3}}{2}\right)i+9\left(\frac{1}{2}\right)j=-\frac{9\sqrt{3}}{2}i+\frac{9}{2}j.$$

Q12. Equal vectors have equal components, so matching the i-parts gives $2p-1=5$ and matching the j-parts gives $7=q+2$. From the first, $2p=6$, so $p=3$; from the second, $q=5$.

Q13. The displacement is $\overrightarrow{AB}=(8-(-4))i+(-2-7)j=12i-9j$, so

$$AB=\sqrt{12^2+(-9)^2}=\sqrt{144+81}=\sqrt{225}=15.$$

Q14. $A(3,-2)$ and $B(31,43)$. $\overrightarrow{AB}=(31-3)i+(43-(-2))j=28i+45j$; $AB=\sqrt{28^2+45^2}=\sqrt{784+2025}=\sqrt{2809}=53$; and the unit vector is $\frac{1}{53}(28i+45j)$.

Q15. $v=-4i+4j$, so $|v|=\sqrt{(-4)^2+4^2}=\sqrt{32}=4\sqrt{2}$. Here $v_1<0$ while $v_2>0$, placing v in the second quadrant. The reference angle has $\tan\alpha=|4/-4|=1$, so $\alpha=45°$, and $\theta=180°-45°=135°$.

Q16. For a to be parallel to b we need $a=kb$ for some scalar k. Matching i-components, $6=2k$, so $k=3$. Matching j-components then requires $t=k\times(-3)=3\times(-3)=-9$. Since $k=3>0$, $a=3b$ points in the same direction as b.

Q17. $|a|=\sqrt{3^2+7^2}=\sqrt{58}$. The unit vector in the direction of a is $\hat{a}=\frac{1}{\sqrt{58}}(3i+7j)$, and its negative $-\hat{a}$ is the unit vector in the opposite direction; both are parallel to a. Rationalising, the two unit vectors are

$$\pm\frac{1}{\sqrt{58}}(3i+7j)=\pm\left(\frac{3\sqrt{58}}{58}i+\frac{7\sqrt{58}}{58}j\right).$$

Q18. $A(6,-1)$ and $B(22,62)$. $\overrightarrow{AB}=(22-6)i+(62-(-1))j=16i+63j$, with $AB=\sqrt{16^2+63^2}=\sqrt{4225}=65$. Scaling to length $130=2\times 65$ multiplies the vector by $\frac{130}{65}=2$, giving $2(16i+63j)=32i+126j$; as a check $\sqrt{32^2+126^2}=\sqrt{16900}=130$.

Theory

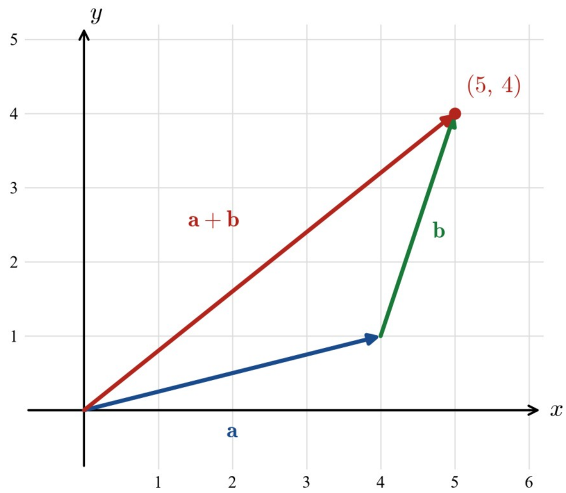
A single vector carries a magnitude and a direction, but its real power appears once vectors are **combined**. Two displacements performed one after the other produce a net displacement; two forces acting at a point produce a resultant. Each of these is a vector built from others by the operations of this section — **addition**, **subtraction** and **multiplication by a scalar**. Every such operation has a picture in the plane and an exact arithmetic in components, and the two descriptions always agree. Throughout we stay in two dimensions.

Recap: the zero vector and the negative of a vector.

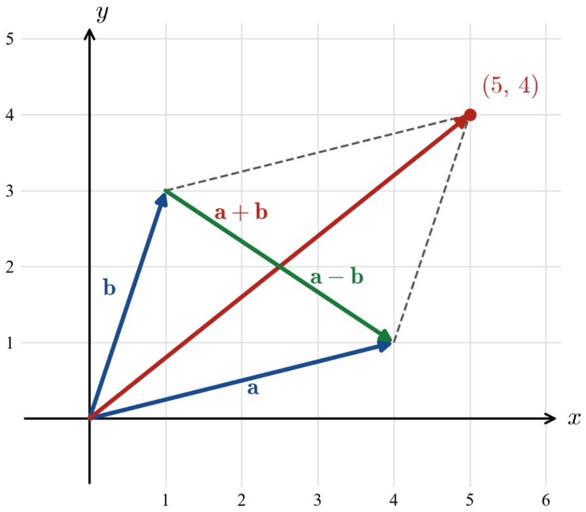
A vector is a **directed line segment**: an arrow whose length is its magnitude and whose arrowhead shows its direction. Two vectors are **equal** when they have the same magnitude and the same direction, wherever they are drawn — a vector is *free* to be slid about the plane. The **zero vector** 0 has zero magnitude and no defined direction; it begins and ends at the same point, so $\overrightarrow{AA}=0$. The **negative** of a vector a, written $-a$, has the *same magnitude* as a but the *opposite direction*. If $a=\overrightarrow{PQ}$, then $-a=\overrightarrow{QP}$: reversing the arrow reverses the vector.

Adding vectors: the triangle and parallelogram rules.

To add a and b, draw them **head to tail** — draw a, then draw b starting at the head of a. The **sum** $a+b$, also called the **resultant**, is the vector from the tail of a to the head of b, closing the triangle. This is the **triangle rule**.



Equivalently, if a and b are drawn from a *common tail*, they span a **parallelogram**, and $a+b$ is its diagonal from that common tail. This is the **parallelogram rule**; it is the same result, since the opposite side of the parallelogram is a copy of b placed head-to-tail with a. Because the parallelogram is symmetric in its two sides, $a+b=b+a$: vector addition is **commutative**.



Subtracting vectors.

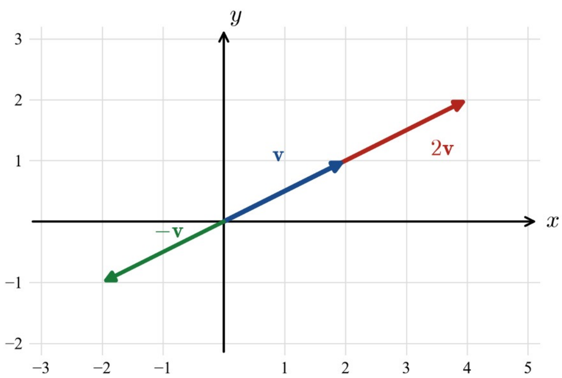
Subtraction is addition of the negative:

$$a - b = a + (-b).$$

Geometrically there is a neat reading on the same parallelogram. If a and b are drawn from a common tail O to points A and B, then $a - b$ is the vector **from the head of b to the head of a** — that is, $\overrightarrow{BA}$ — the *other* diagonal of the parallelogram. This is because $b + \overrightarrow{BA} = a$, so $\overrightarrow{BA} = a - b$. The sum is one diagonal; the difference is the other.

Scalar multiplication.

Multiplying a vector a by a real number k (a **scalar**) produces the vector $k\,a$, whose magnitude is $|k|$ times that of a and whose direction is the **same** as that of a when $k > 0$ and **opposite** when $k < 0$; and $0\,a = 0$. Thus $2\,a$ points the same way as a but is twice as long, while $-a$ is the negative met above (the case $k = -1$).



Scalar multiples of a single non-zero vector all point along one line, so they capture exactly the idea of being **parallel**. Two non-zero vectors a and b are **parallel** if and only if one is a scalar multiple of the other, that is $b = k\,a$ for some real $k \neq 0$. The vectors point the *same* way when $k > 0$ and *opposite* ways when $k < 0$.

Algebraic properties.

Vector addition and scalar multiplication obey the familiar laws of ordinary algebra. For all vectors a, b, c and all scalars h, k,

$$a + b = b + a, (a + b) + c = a + (b + c), a + 0 = a, a + (-a) = 0,$$

$$k(a + b) = k\,a + k\,b, (h + k)a = h\,a + k\,a, h(k\,a) = (hk)a.$$

These rules mean that vector expressions may be expanded, collected and rearranged just like scalar ones — the only difference is that the “numbers” being combined carry a direction.

Unit vectors and i, j components.

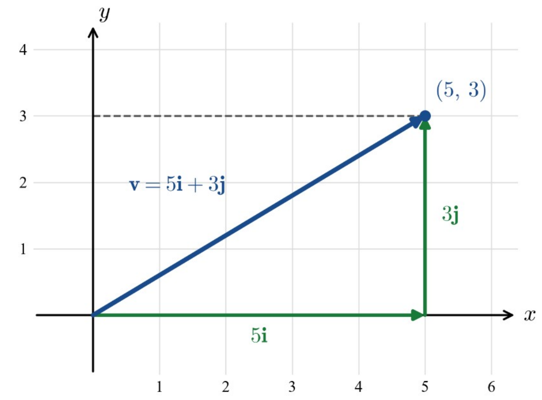
A **unit vector** is a vector of magnitude 1. Two particular unit vectors organise the whole plane: i, pointing one unit in the positive x-direction, and j, pointing one unit in the positive y-direction:

$$i=\begin{bmatrix}1\\0\end{bmatrix}, j=\begin{bmatrix}0\\1\end{bmatrix}.$$

Any plane vector a can be **resolved** into a horizontal part and a vertical part. If a moves a_1 units in the x-direction and a_2 units in the y-direction, then

$$a=a_1 i+a_2 j=\begin{bmatrix}a_1\\a_2\end{bmatrix}.$$

The numbers a_1 and a_2 are the **components** of a; the right-hand form is the **column** (or **component**) form, and the pair (a_1, a_2) is the **ordered-pair** form. The two writings $a_1 i+a_2 j$ and $\begin{bmatrix}a_1\\a_2\end{bmatrix}$ say exactly the same thing.



Because a vector is fixed by its two components, **two vectors are equal precisely when their components are equal**:

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix} \Leftrightarrow a_1=b_1 \text{ and } a_2=b_2.$$

A single vector equation therefore contains **two** scalar equations, one per component — the standard way of solving for unknowns that sit inside a vector equation.

Component form of the operations.

Resolved into components, the three operations become ordinary arithmetic carried out one row at a time:

$$a+b=\begin{bmatrix}a_1+b_1\\a_2+b_2\end{bmatrix}, a-b=\begin{bmatrix}a_1-b_1\\a_2-b_2\end{bmatrix}, k\,a=\begin{bmatrix}k\,a_1\\k\,a_2\end{bmatrix}.$$

This is the payoff of components: the geometric triangle and parallelogram rules are replaced by adding and subtracting numbers, and every scalar multiple is found by multiplying each component. The answer is always returned in component form, from which the i, j form can be read off immediately.

Position vectors.

Fixing an origin O, the **position vector** of a point A is the vector $\overrightarrow{OA}$ from O to A; it is written a, and its components are just the coordinates of A. Position vectors turn statements about points into statements about vectors. In particular, for points A and B with position vectors a and b,

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=b-a,$$

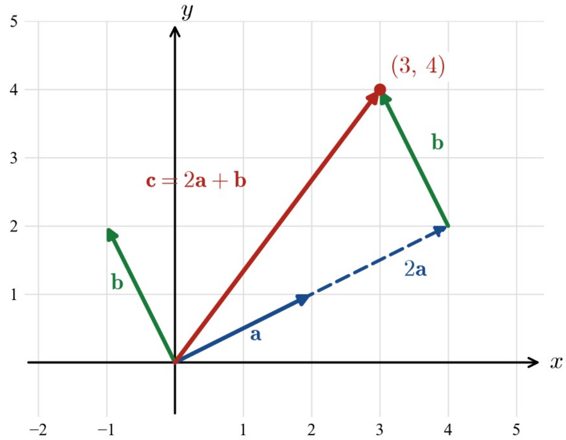
since $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$. The vector *joining* two points is the position vector of the head minus that of the tail — a fact used constantly below.

Linear combinations.

A vector built from others by scalar multiples and sums, such as

$$w=s\,a+t\,b\,(s,t\text{ real}),$$

is called a **linear combination** of a and b. Geometrically it is reached by laying the scaled vectors $s\,a$ and $t\,b$ head to tail.



If a and b are **not parallel**, then *every* vector in the plane can be written as a linear combination of them in exactly one way; the scalars s and t are found by equating components and solving the resulting pair of simultaneous equations. The vectors i and j are the simplest such pair, which is why $a=a_1 i+a_2 j$ works for every a.

A point dividing a segment in a given ratio.

Linear combinations describe points that lie on a line segment. Let P be the point on AB that **divides it in the ratio** $m:n$, meaning $AP:PB=m:n$. Then $\overrightarrow{AP}=\dfrac{m}{m+n}\overrightarrow{AB}$, and so

$$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=a+\frac{m}{m+n}(b-a)=\frac{n\,a+m\,b}{m+n}.$$

The position vector of the dividing point is a **weighted average** of a and b, the weights being the *opposite* parts of the ratio. Taking $m=n$ (the ratio $1:1$) gives the **midpoint** M of AB:

$$\overrightarrow{OM}=1/2(a+b).$$

Read the other way, a point C lies on the line through A and B exactly when $\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{AB}$; the value of that scalar fixes where on the line — and in what ratio — the point C falls. This is how vectors settle questions of **collinearity** and **ratio** without coordinate geometry's gradients.

Worked examples

Example 1 — scaffolded

Let $u=5i+3j$ and $v=2i-7j$. Find (a) $u+v$, (b) $u-v$, and (c) $3u-2v$, giving each answer in component form.

Working	Comment
$u=\begin{bmatrix}5\\3\end{bmatrix}$, $v=\begin{bmatrix}2\\-7\end{bmatrix}$.	The components are the coefficients of i and j.

Working	Comment
$u+v=\begin{bmatrix}5+2\\3+(-7)\end{bmatrix}=\begin{bmatrix}7\\-4\end{bmatrix}.$	Add corresponding components.
$u-v=\begin{bmatrix}5-2\\3-(-7)\end{bmatrix}=\begin{bmatrix}3\\10\end{bmatrix}.$	Subtract component by component; $3-(-7)=10$.
$3u=\begin{bmatrix}15\\9\end{bmatrix}, 2v=\begin{bmatrix}4\\-14\end{bmatrix}.$	A scalar multiplies each component.
$3u-2v=\begin{bmatrix}15-4\\9-(-14)\end{bmatrix}=\begin{bmatrix}11\\23\end{bmatrix}.$	Combine the two results component by component.

Example 2 — scaffolded

Let $a=3i-j$ and $b=-2i+5j$. Find the real scalars m and n for which $ma+nb=6i+11j$.

Working	Comment
$ma+nb=m\begin{bmatrix}3\\-1\end{bmatrix}+n\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}3m-2n\\-m+5n\end{bmatrix}.$	Form the linear combination in column form.
$\begin{bmatrix}3m-2n\\-m+5n\end{bmatrix}=\begin{bmatrix}6\\11\end{bmatrix}.$	Equal vectors have equal components.
$3m-2n=6$ and $-m+5n=11$.	One vector equation gives two scalar equations.
From the second, $m=5n-11$; substituting, $3(5n-11)-2n=6$, so $13n=39$ and $n=3$.	Solve the pair simultaneously.
Then $m=5(3)-11=4$.	Back-substitute for m.

So $m=4$ and $n=3$; that is, $4a+3b=6i+11j$.

Example 3 — unscaffolded

The points A and B have position vectors $a=-3i+2j$ and $b=5i+6j$ relative to an origin O. The point P divides AB in the ratio $3:1$. Find the position vector of P and the vector $\overrightarrow{AP}$.

The vector along the segment is

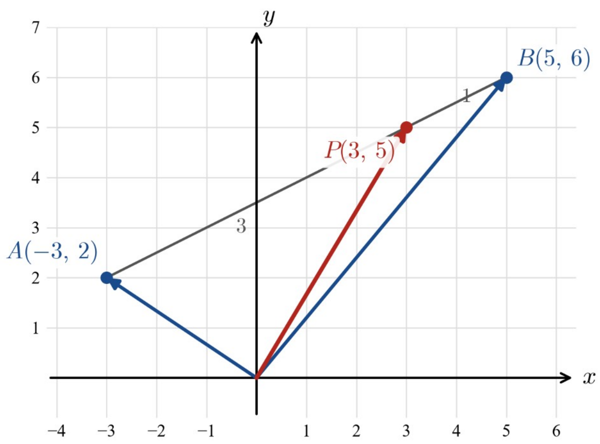
$$\overrightarrow{AB}=b-a=(5-(-3))i+(6-2)j=8i+4j.$$

Since $AP:PB=3:1$, the point P is three-quarters of the way from A to B, so

$$\overrightarrow{AP}=3/4\,\overrightarrow{AB}=3/4(8i+4j)=6i+3j.$$

Adding this to the position vector of A,

$$\overrightarrow{OP}=a+\overrightarrow{AP}=(-3+6)i+(2+3)j=3i+5j.$$



$\therefore \overrightarrow{OP}=3i+5j$ (so P is the point $(3,5)$) and $\overrightarrow{AP}=6i+3j$.

Example 4 — unscaffolded

The points A, B and C have position vectors $a=-i+4j$, $b=11i-2j$ and $c=7i$ relative to an origin O. Show that A, B and C are collinear, and find the ratio in which C divides AB.

Form the two vectors from A:

$$\overrightarrow{AB}=b-a=(11-(-1))i+(-2-4)j=12i-6j,$$

$$\overrightarrow{AC}=c-a=(7-(-1))i+(0-4)j=8i-4j.$$

Comparing them, $2/3\overrightarrow{AB}=2/3(12i-6j)=8i-4j=\overrightarrow{AC}$, so $\overrightarrow{AC}=2/3\overrightarrow{AB}$. As $\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{AB}$ and both start at A, the three points lie on one line. Then $\overrightarrow{CB}=\overrightarrow{AB}-\overrightarrow{AC}=1/3\overrightarrow{AB}$, so $AC:CB=2/3:1/3=2:1$.

$\therefore A$, B and C are collinear, and C divides AB in the ratio $2:1$.

Practice questions

Scaffolded

Q1. Let $a=6i+2j$ and $b=-i+4j$. (a) Write a and b in column form. (b) Find $a+b$. (c) Find $a-b$. (d) Find $2a+3b$.

Q2. Let $p=\begin{bmatrix}4\\-1\end{bmatrix}$ and $q=\begin{bmatrix}-2\\3\end{bmatrix}$. (a) Find $3p$. (b) Find $-2q$. (c) Find $3p-2q$. (d) Find $2p+3q$.

Q3. Let $a=\begin{bmatrix}2\\3\end{bmatrix}$ and $b=\begin{bmatrix}4\\-1\end{bmatrix}$. Find the scalars s and t for which $sa+tb=\begin{bmatrix}16\\3\end{bmatrix}$. (a) Write the linear combination $sa+tb$ in column form. (b) Equate components to obtain two equations in s and t. (c) Solve the equations. (d) State s and t.

Q4. The vector x satisfies $2x+a=b$, where $a=5i-2j$ and $b=-3i+8j$. (a) Make x the subject. (b) Substitute a and b. (c) Find x in column form. (d) State x in i, j form.

Q5. The points A and B have position vectors $a=\begin{bmatrix}-4\\1\end{bmatrix}$ and $b=\begin{bmatrix}6\\7\end{bmatrix}$. (a) Find $\overrightarrow{AB}$. (b) Find the position vector of the midpoint M of AB. (c) Find $\overrightarrow{AM}$. (d) Verify that $\overrightarrow{AM}=1/2\overrightarrow{AB}$.

Q6. The points A and B have position vectors $a=2i-3j$ and $b=10i+9j$. The point P divides AB in the ratio $1:3$. (a) Find $\overrightarrow{AB}$. (b) Find $\overrightarrow{AP}=1/4\overrightarrow{AB}$. (c) Find the position vector of P. (d) State the coordinates of P.

Unscaffolded

Q7. Given $a = 7i - 3j$ and $b = 2i + 5j$, find $4a - 3b$.

Q8. Given $u = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, find $2u + 5v$.

Q9. Find the scalars m and n for which $m(3i + j) + n(2i - 4j) = 13i + 9j$.

Q10. The vector r satisfies $3r - a = 2b$, where $a = 4i - j$ and $b = i + 8j$. Find r.

Q11. The points A and B have position vectors $a = i - 2j$ and $b = 9i + 6j$. The point P divides AB in the ratio $3:5$. Find the position vector of P.

Q12. Show that the vectors $a = 6i - 4j$ and $b = -3i + 2j$ are parallel, and find the scalar k for which $b = ka$.

Q13. Find the value of t for which $u = ti + 6j$ is parallel to $v = 4i + 3j$.

Q14. The points A, B and C have position vectors $a = -2i + j$, $b = 10i + 7j$ and $c = 2i + 3j$. Show that A, B and C are collinear and find the ratio in which C divides AB.

Q15. The points $A(1, -1)$, $B(5, 2)$ and $C(7, 6)$ are three vertices of a parallelogram $ABCD$, taken in order. Using position vectors, find the coordinates of the fourth vertex D.

Q16. The point $M(4, 1)$ is the midpoint of AB, where A has coordinates $(-2, 5)$. Find the coordinates of B.

Q17. Express $c = 8i - 3j$ as a linear combination of $a = 2i + j$ and $b = i - 3j$; that is, find scalars m and n with $c = ma + nb$.

Q18. $OABC$ is a parallelogram with $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = c$, so that $\overrightarrow{OB} = a + c$. Using position vectors, prove that the diagonals OB and AC bisect each other.

Answers

Q1. (a) $a=\begin{bmatrix}6\\2\end{bmatrix}$, $b=\begin{bmatrix}-1\\4\end{bmatrix}$; (b) $\begin{bmatrix}5\\6\end{bmatrix}$; (c) $\begin{bmatrix}7\\-2\end{bmatrix}$; (d) $\begin{bmatrix}9\\16\end{bmatrix}$. **Q2.** (a) $\begin{bmatrix}12\\-3\end{bmatrix}$; (b) $\begin{bmatrix}4\\-6\end{bmatrix}$; (c) $\begin{bmatrix}16\\-9\end{bmatrix}$; (d) $\begin{bmatrix}2\\7\end{bmatrix}$. **Q3.** $s=2$, $t=3$. **Q4.** $x=1/2(b-a)=\begin{bmatrix}-4\\5\end{bmatrix}=-4i+5j$. **Q5.** (a) $\overrightarrow{AB}=\begin{bmatrix}10\\6\end{bmatrix}$; (b) $\overrightarrow{OM}=\begin{bmatrix}1\\4\end{bmatrix}$; (c) $\overrightarrow{AM}=\begin{bmatrix}5\\3\end{bmatrix}$; (d) $1/2\overrightarrow{AB}=\begin{bmatrix}5\\3\end{bmatrix}=\overrightarrow{AM}$. **Q6.** (a) $\begin{bmatrix}8\\12\end{bmatrix}$; (b) $\begin{bmatrix}2\\3\end{bmatrix}$; (c) $\overrightarrow{OP}=\begin{bmatrix}4\\0\end{bmatrix}$; (d) $P=(4,0)$. **Q7.** $22i-27j$. **Q8.** $\begin{bmatrix}5\\-4\end{bmatrix}$. **Q9.** $m=5$, $n=-1$. **Q10.** $r=2i+5j$. **Q11.** $\overrightarrow{OP}=4i+j$ (the point $(4,1)$). **Q12.** $b=-1/2a$, so $k=-1/2$ (parallel, opposite directions). **Q13.** $t=8$. **Q14.** Collinear, with $\overrightarrow{AC}=1/3\overrightarrow{AB}$; C divides AB in the ratio $1:2$. **Q15.** $D=(3,3)$. **Q16.** $B=(10,-3)$. **Q17.** $m=3$, $n=2$, so $c=3a+2b$. **Q18.** Each diagonal has midpoint $1/2(a+c)$; see solution.

Fully-worked solutions

Q1. In column form $a=\begin{bmatrix}6\\2\end{bmatrix}$ and $b=\begin{bmatrix}-1\\4\end{bmatrix}$. Adding component by component, $a+b=\begin{bmatrix}6+(-1)\\2+4\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}$; subtracting, $a-b=\begin{bmatrix}6-(-1)\\2-4\end{bmatrix}=\begin{bmatrix}7\\-2\end{bmatrix}$. For the combination, $2a=\begin{bmatrix}12\\4\end{bmatrix}$ and $3b=\begin{bmatrix}-3\\12\end{bmatrix}$, so $2a+3b=\begin{bmatrix}12+(-3)\\4+12\end{bmatrix}=\begin{bmatrix}9\\16\end{bmatrix}$.

Q2. Scaling each component: $3p=\begin{bmatrix}12\\-3\end{bmatrix}$ and $-2q=\begin{bmatrix}4\\-6\end{bmatrix}$. Then $3p-2q=\begin{bmatrix}12\\-3\end{bmatrix}+\begin{bmatrix}4\\-6\end{bmatrix}=\begin{bmatrix}16\\-9\end{bmatrix}$. For the last part, $2p=\begin{bmatrix}8\\-2\end{bmatrix}$ and $3q=\begin{bmatrix}-6\\9\end{bmatrix}$, so $2p+3q=\begin{bmatrix}8+(-6)\\-2+9\end{bmatrix}=\begin{bmatrix}2\\7\end{bmatrix}$.

Q3. The linear combination is $sa+tb=\begin{bmatrix}2s+4t\\3s-t\end{bmatrix}$. Equating to $\begin{bmatrix}16\\3\end{bmatrix}$ gives $2s+4t=16$ and $3s-t=3$. From the second, $t=3s-3$; substituting, $2s+4(3s-3)=16$, so $14s=28$ and $s=2$, whence $t=3(2)-3=3$. Checking, $2\begin{bmatrix}2\\3\end{bmatrix}+3\begin{bmatrix}4\\-1\end{bmatrix}=\begin{bmatrix}16\\3\end{bmatrix}$.

Q4. Rearranging, $2x=b-a$, so $x=1/2(b-a)$. Now $b-a=\begin{bmatrix}-3-5\\8-(-2)\end{bmatrix}=\begin{bmatrix}-8\\10\end{bmatrix}$, hence $x=1/2\begin{bmatrix}-8\\10\end{bmatrix}=\begin{bmatrix}-4\\5\end{bmatrix}=-4i+5j$.

Q5. With $a=\begin{bmatrix}-4\\1\end{bmatrix}$ and $b=\begin{bmatrix}6\\7\end{bmatrix}$, $\overrightarrow{AB}=b-a=\begin{bmatrix}6-(-4)\\7-1\end{bmatrix}=\begin{bmatrix}10\\6\end{bmatrix}$. The midpoint has position vector $\overrightarrow{OM}=1/2(a+b)=1/2\begin{bmatrix}2\\8\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}$. Then $\overrightarrow{AM}=\overrightarrow{OM}-a=\begin{bmatrix}1-(-4)\\4-1\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$, which equals $1/2\overrightarrow{AB}=1/2\begin{bmatrix}10\\6\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$, as required.

Q6. Here $\overrightarrow{AB}=b-a=\begin{bmatrix}10-2\\9-(-3)\end{bmatrix}=\begin{bmatrix}8\\12\end{bmatrix}$. Since $AP:PB=1:3$, the point P is one-quarter of the way from A to B, so $\overrightarrow{AP}=1/4\overrightarrow{AB}=\begin{bmatrix}2\\3\end{bmatrix}$. Then $\overrightarrow{OP}=a+\overrightarrow{AP}=\begin{bmatrix}2+2\\-3+3\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$, so $P=(4,0)$.

Q7. $4a=\begin{bmatrix}28\\-12\end{bmatrix}$ and $3b=\begin{bmatrix}6\\15\end{bmatrix}$, so $4a-3b=\begin{bmatrix}28-6\\-12-15\end{bmatrix}=\begin{bmatrix}22\\-27\end{bmatrix}=22i-27j$.

Q8. $2u=\begin{bmatrix}-10\\16\end{bmatrix}$ and $5v=\begin{bmatrix}15\\-20\end{bmatrix}$, so $2u+5v=\begin{bmatrix}-10+15\\16+(-20)\end{bmatrix}=\begin{bmatrix}5\\-4\end{bmatrix}$.

Q9. The left side is $\begin{bmatrix}3m+2n\\m-4n\end{bmatrix}$, so $3m+2n=13$ and $m-4n=9$. From the second, $m=4n+9$; substituting, $3(4n+9)+2n=13$, so $14n=-14$ and $n=-1$, giving $m=4(-1)+9=5$. Checking, $5(3i+j)-(2i-4j)=13i+9j$.

Q10. Rearranging, $3r=a+2b$. Now $a+2b=\begin{bmatrix}4\\-1\end{bmatrix}+\begin{bmatrix}2\\16\end{bmatrix}=\begin{bmatrix}6\\15\end{bmatrix}$, so $r=1/3\begin{bmatrix}6\\15\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}=2i+5j$.

Q11. First $\overrightarrow{AB}=b-a=\begin{bmatrix}9-1\\6-(-2)\end{bmatrix}=\begin{bmatrix}8\\8\end{bmatrix}$. With $AP:PB=3:5$, the point P is $3/8$ of the way along, so $\overrightarrow{AP}=3/8\begin{bmatrix}8\\8\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$. Hence $\overrightarrow{OP}=a+\overrightarrow{AP}=\begin{bmatrix}1+3\\-2+3\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}=4i+j$, so $P=(4,1)$.

Q12. Testing whether b is a scalar multiple of a: $-1/2a=-1/2\begin{bmatrix}6\\-4\end{bmatrix}=\begin{bmatrix}-3\\2\end{bmatrix}=b$. So $b=ka$ with $k=-1/2$; the vectors are parallel, pointing in opposite directions because $k<0$.

Q13. For u parallel to v there must be a scalar k with $\begin{bmatrix}t\\6\end{bmatrix}=k\begin{bmatrix}4\\3\end{bmatrix}$. The second component gives $6=3k$, so $k=2$; then the first component gives $t=4k=8$.

Q14. From A, $\overrightarrow{AB}=b-a=\begin{bmatrix}10-(-2)\\7-1\end{bmatrix}=\begin{bmatrix}12\\6\end{bmatrix}$ and $\overrightarrow{AC}=c-a=\begin{bmatrix}2-(-2)\\3-1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$. Since $1/3\overrightarrow{AB}=\begin{bmatrix}4\\2\end{bmatrix}=\overrightarrow{AC}$, the vector $\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{AB}$ and both begin at A, so A, B, C are collinear. Then $\overrightarrow{CB}=\overrightarrow{AB}-\overrightarrow{AC}=2/3\overrightarrow{AB}$, so $AC:CB=1/3:2/3=1:2$.

Q15. Write position vectors $a=\begin{bmatrix}1\\-1\end{bmatrix}$, $b=\begin{bmatrix}5\\2\end{bmatrix}$, $c=\begin{bmatrix}7\\6\end{bmatrix}$. In a parallelogram $ABCD$ the diagonals AC and BD share a midpoint, so $1/2(a+c)=1/2(b+d)$, giving $d=a+c-b=\begin{bmatrix}1+7-5\\-1+6-2\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$. Hence $D=(3,3)$. (Equivalently $\overrightarrow{AD}=\overrightarrow{BC}$ gives the same point.)

Q16. If M is the midpoint of AB then $\overrightarrow{OM}=1/2(a+b)$, so $b=2\overrightarrow{OM}-a=2\begin{bmatrix}4\\1\end{bmatrix}-\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}8+2\\2-5\end{bmatrix}=\begin{bmatrix}10\\-3\end{bmatrix}$. Hence $B=(10,-3)$.

Q17. Seek m,n with $m\begin{bmatrix}2\\1\end{bmatrix}+n\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}8\\-3\end{bmatrix}$, i.e. $\begin{bmatrix}2m+n\\m-3n\end{bmatrix}=\begin{bmatrix}8\\-3\end{bmatrix}$. Then $2m+n=8$ and $m-3n=-3$. From the first, $n=8-2m$; substituting, $m-3(8-2m)=-3$, so $7m=21$ and $m=3$, giving $n=8-6=2$. Thus $c=3a+2b$ (check: $3(2i+j)+2(i-3j)=8i-3j$).

Q18. Take O as origin, so A, B, C have position vectors a, $a+c$ and c respectively. The midpoint of diagonal OB has position vector

$$1/2(\overrightarrow{OO}+\overrightarrow{OB})=1/2(a+c).$$

The midpoint of diagonal AC has position vector

$$1/2(\overrightarrow{OA}+\overrightarrow{OC})=1/2(a+c).$$

The two midpoints have the *same* position vector, so they are the same point. Hence the diagonals OB and AC cut each other at their common midpoint, that is, they bisect each other, as required.

Theory

The vector operations met so far — addition, subtraction and multiplication by a scalar — each combine vectors to produce **another vector**. This section introduces a completely different way of combining two vectors, one that produces a **number** (a scalar): the **scalar product**, almost always called the **dot product** after the raised dot written between the two vectors, $a \cdot b$. The dot product measures how much two vectors point in the same direction. It is the single tool behind three of the most useful geometric questions about plane vectors: what is the **angle** between them, are they **perpendicular**, and how much of one vector lies **along** another (its **projection**). Throughout this section every vector lies in the plane; a two-dimensional vector is written in component form as $a = a_1 i + a_2 j$, where i and j are the perpendicular unit vectors along the axes.

The component definition.

For two plane vectors $a = a_1 i + a_2 j$ and $b = b_1 i + b_2 j$, the **dot product** is

$$a \cdot b = a_1 b_1 + a_2 b_2.$$

In words: multiply the matching components and add the two results. The answer is a single real number, not a vector — a point worth stressing, because writing an arrow or a vector on the right-hand side is the commonest early error. For example, with $a = 3i + 5j$ and $b = 8i + 2j$,

$$a \cdot b = (3)(8) + (5)(2) = 24 + 10 = 34.$$

The geometric definition.

The dot product also has a purely geometric description. Place the two non-zero vectors a and b tail to tail and let θ be the angle between them, measured so that $0° \le \theta \le 180°$. Then

$$a \cdot b = |a||b| \cos\theta.$$

Because $|a|$ and $|b|$ are positive lengths, the **sign** of the dot product is exactly the sign of $\cos\theta$, a fact used repeatedly below.

The two definitions agree.

These two formulae describe the *same number*. Take non-zero $a = a_1 i + a_2 j$ and $b = b_1 i + b_2 j$ drawn tail to tail with angle θ between them. The third side of the triangle they form is the vector $a - b$, joining the tip of b to the tip of a, so the **cosine rule** applied to that triangle gives

$$|a - b|^2 = |a|^2 + |b|^2 - 2|a||b| \cos\theta.$$

Computing the same length from components instead,

$$|a - b|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2(a_1 b_1 + a_2 b_2) = |a|^2 + |b|^2 - 2(a_1 b_1 + a_2 b_2).$$

Comparing the two expressions for $|a - b|^2$, the terms $|a|^2$ and $|b|^2$ cancel and what is left is

$$a_1 b_1 + a_2 b_2 = |a||b| \cos\theta,$$

so the component sum and the geometric product are indeed one and the same number. The component form is the quicker to evaluate; the geometric form is the one that carries the meaning.

The square of a vector.

Setting $b = a$ in the component definition, or reading the geometric form with $\theta = 0°$ and $\cos 0° = 1$, gives the important identity

$$a \cdot a = a_1^2 + a_2^2 = |a|^2.$$

A vector dotted with itself is the square of its length, so $|a| = \sqrt{a \cdot a}$. This is the bridge between the dot product and magnitude, and it turns questions about lengths into dot-product algebra.

Properties.

For all plane vectors a, b, c and every scalar k, the dot product obeys the following rules, each of which follows in one line from the component definition:

- **Commutative:** $a \cdot b = b \cdot a$, since $a_1 b_1 + a_2 b_2 = b_1 a_1 + b_2 a_2$.
- **Distributive over addition:** $a \cdot (b + c) = a \cdot b + a \cdot c$, because $a_1(b_1 + c_1) + a_2(b_2 + c_2)$ expands and regroups into the two dot products.
- **Scalar factors:** $(ka) \cdot b = k(a \cdot b) = a \cdot (kb)$.
- **Square of a vector:** $a \cdot a = |a|^2 \geq 0$, with equality only for $a = 0$.

These rules mean a dot product of sums may be expanded exactly like ordinary algebra. In particular, replacing every product by a dot product,

$$(a + b) \cdot (a + b) = a \cdot a + 2a \cdot b + b \cdot b = |a|^2 + 2a \cdot b + |b|^2,$$

which is the identity behind finding the length of a sum or difference of two vectors from their magnitudes and the angle between them.

The angle between two vectors.

Rearranging the geometric definition isolates the cosine of the angle, giving it directly from components:

$$\cos\theta = \frac{a \cdot b}{|a||b|}, 0° \leq \theta \leq 180°.$$

Evaluate $a \cdot b$, $|a|$ and $|b|$ from the components, substitute, and take the inverse cosine; because θ is restricted to $[0°, 180°]$ the answer is unique. The sign of the dot product classifies the angle at a glance, without any inverse cosine at all:

$a \cdot b$	angle θ
positive	$0° \leq \theta < 90°$: acute, or $\theta = 0°$ (parallel, same direction)
zero	$\theta = 90°$: a right angle
negative	$90° < \theta \leq 180°$: obtuse, or $\theta = 180°$ (parallel, opposite directions)

The two end values $\theta = 0°$ and $\theta = 180°$ are the parallel cases treated below; for any two vectors that are not parallel, a positive dot product means the angle is acute and a negative one means it is obtuse.

Perpendicular vectors.

Two non-zero vectors are **perpendicular** (also called *orthogonal*) exactly when the angle between them is $90°$. Since $\cos 90° = 0$, the geometric definition then makes the dot product zero; conversely, if the dot product of two non-zero vectors is zero then $\cos\theta = 0$, forcing $\theta = 90°$. Hence the **perpendicularity test**

$$a \perp b \Leftrightarrow a \cdot b = 0.$$

This is the most-used consequence of the dot product: to decide whether two vectors are at right angles, simply check whether their dot product is zero. It also solves for an unknown — setting $a \cdot b = 0$ gives a single equation whose solution makes the vectors perpendicular.

Parallel vectors.

Two non-zero vectors are **parallel** when one is a scalar multiple of the other, $a = k\,b$ for some scalar $k \neq 0$: they point the same way when $k > 0$ (so $\theta = 0°$) and opposite ways when $k < 0$ (so $\theta = 180°$). In either case $\cos\theta = \pm 1$, so the dot product reaches its extreme value

$$a \cdot b = \pm |a||b|,$$

positive for like directions and negative for opposite directions. In practice the quickest parallel test is to check whether the components are in a common ratio, $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, which is precisely the condition that one vector is a scalar multiple of the other.

Scalar projection.

The dot product answers the question: *how far does a reach in the direction of b?* Only the direction of b matters, so work with the unit vector

$$\hat{b} = \frac{b}{|b|},$$

which points the same way as b but has length 1. The **scalar projection** of a on b (also called the **scalar resolute**) is the signed length of the shadow a casts on the line of b:

$$a \cdot \hat{b} = \frac{a \cdot b}{|b|} = |a| \cos\theta.$$

It is a **number**, and its sign records the angle exactly as in the table above: positive when $\theta < 90°$ (the shadow points the same way as b), zero when the vectors are perpendicular, and negative when $\theta > 90°$ (the shadow falls on the opposite side).

Vector projection.

Attaching the direction $\hat{b}$ to that signed length turns the shadow back into a vector — the **vector projection** of a on b, written $proj_b a$ or $a_{/\!/}$. It is the part of a that lies **parallel** to b:

$$a_{/\!/} = (a \cdot \hat{b})\hat{b} = \frac{a \cdot b}{|b|^2} b = \frac{a \cdot b}{b \cdot b} b.$$

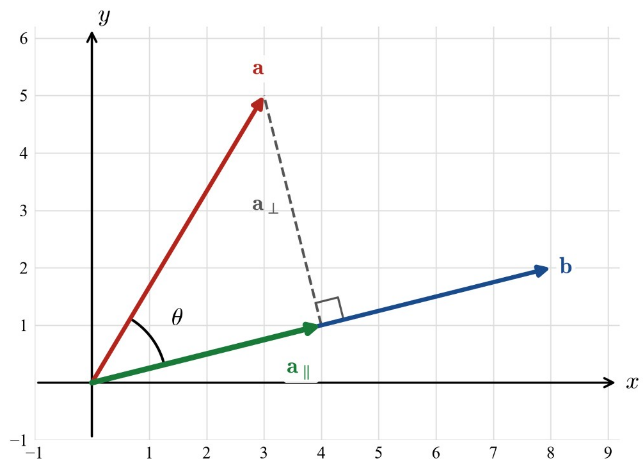
The last two forms are equal because $\hat{b} = \frac{b}{|b|}$ and $|b|^2 = b \cdot b$; the final form avoids surds and is usually the quicker to compute. Whatever is left of a after removing this parallel part is the **perpendicular component**

$$a_{\perp} = a - a_{/\!/}, \text{so that } a = a_{/\!/} + a_{\perp}.$$

The perpendicular component genuinely is perpendicular to b, since

$$a_{\perp} \cdot b = \left(a - \frac{a \cdot b}{b \cdot b} b\right) \cdot b = a \cdot b - \frac{a \cdot b}{b \cdot b}(b \cdot b) = 0.$$

Evaluating $a_{\perp} \cdot b$ is the standard arithmetic check on a resolution: if it does not come to 0, a slip has been made. The diagram shows the construction for $a = 3i + 5j$ and $b = 8i + 2j$: the vector projection $a_{/\!/} = 4i + j$ lies along the line of b, and $a_{\perp}$ drops from its tip to the tip of a, meeting that line at a right angle. (Here the angle is $\theta = 45°$ and the scalar projection is $a \cdot \hat{b} = \sqrt{17}$.)



Worked examples

Example 1 — scaffolded

Let $a=4i+7j$ and $b=5i-2j$. Find (a) $a\cdot b$, (b) $a\cdot a$ and hence $|a|$, and (c) state whether the angle between a and b is acute, right or obtuse.

Working	Comment
$a\cdot b=(4)(5)+(7)(-2)=20-14=6$.	Multiply matching components and add; the result is a scalar.
$a\cdot a=(4)(4)+(7)(7)=16+49=65$.	The same rule with b replaced by a.
$\lvert a\rvert=\sqrt{a\cdot a}=\sqrt{65}$.	Uses $a\cdot a=\lvert a\rvert^2$.
$a\cdot b=6>0$, so the angle is acute.	The sign of the dot product fixes the type of angle.

Example 2 — scaffolded

Find the angle θ between $a=2i+j$ and $b=-3i+j$.

Working	Comment
$a\cdot b=(2)(-3)+(1)(1)=-6+1=-5$.	Dot product from components.
$\lvert a\rvert=\sqrt{2^2+1^2}=\sqrt{5}$, $\lvert b\rvert=\sqrt{(-3)^2+1^2}=\sqrt{10}$.	Magnitude of each vector.
$\cos\theta=\dfrac{a\cdot b}{\lvert a\rvert\lvert b\rvert}=\dfrac{-5}{\sqrt{5}\sqrt{10}}=\dfrac{-5}{\sqrt{50}}$.	Substitute into the angle formula.
$=\dfrac{-5}{5\sqrt{2}}=-\dfrac{1}{\sqrt{2}}$.	Since $\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}$.
$\theta=\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)=135°$.	The unique angle with $0°\le\theta\le 180°$; the dot product is negative, so θ had to be obtuse.

Example 3 — unscaffolded

The triangle ABC has vertices $A(1,4)$, $B(2,1)$ and $C(5,2)$. Show that the triangle is right-angled, and state at which vertex the right angle occurs.

The angle at a vertex is the angle between the two side-vectors drawn from that vertex, and two such vectors meet at a right angle exactly when their dot product is zero. Testing vertex B, the vectors along the sides BA and BC are

$$\overrightarrow{BA}=A-B=-i+3j,\overrightarrow{BC}=C-B=3i+j,$$

with dot product $\overrightarrow{BA} \cdot \overrightarrow{BC}=(-1)(3)+(3)(1)=-3+3=0$. A zero dot product between two non-zero vectors means they are perpendicular, so the angle at B is a right angle. (The other vertices are not right angles: at A, $\overrightarrow{AB}=i-3j$ and $\overrightarrow{AC}=4i-2j$ give $\overrightarrow{AB} \cdot \overrightarrow{AC}=4+6=10\neq 0$.)

∴ triangle ABC is right-angled at B.

Example 4 — unscaffolded

Resolve $a=10i+5j$ into components parallel and perpendicular to $b=9i+12j$, and state the scalar projection of a on b.

Work with the unit vector in the direction of b. Since $|b|=\sqrt{9^2+12^2}=\sqrt{225}=15$,

$$\hat{b}=\frac{1}{15}(9i+12j)=3/5i+4/5j.$$

The dot product is $a \cdot b=(10)(9)+(5)(12)=90+60=150$, so the scalar projection of a on b is

$$a \cdot \hat{b}=\frac{a \cdot b}{|b|}=\frac{150}{15}=10.$$

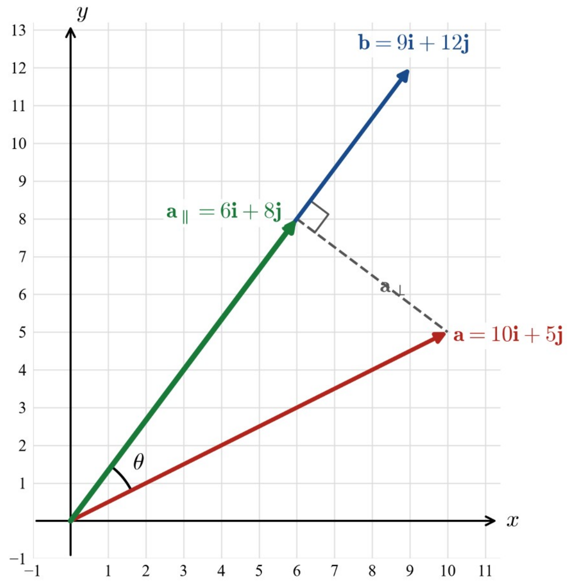
The vector projection (the part of a parallel to b) is

$$a_{\parallel}=\frac{a \cdot b}{|b|^2}b=\frac{150}{225}(9i+12j)=2/3(9i+12j)=6i+8j,$$

and the perpendicular part is what remains,

$$a_{\perp}=a-a_{\parallel}=(10-6)i+(5-8)j=4i-3j.$$

As a check, $a_{\perp} \cdot b=(4)(9)+(-3)(12)=36-36=0$, confirming $a_{\perp}$ is perpendicular to b.



∴ $a=(6i+8j)+(4i-3j)$, with scalar projection 10 and vector projection $6i+8j$.

Practice questions

Scaffolded

Q1. Let $a=3i+8j$ and $b=6i-5j$. (a) Find $a \cdot b$. (b) Find $a \cdot a$ and hence $|a|$. (c) Find $b \cdot b$ and hence $|b|$. (d) State whether the angle between a and b is acute, right or obtuse.

Q2. Find the angle θ between $a=6i+2j$ and $b=i+3j$, to the nearest degree. (a) Find $a \cdot b$. (b) Find $|a|$ and $|b|$. (c) Find $\cos\theta$. (d) State θ.

Q3. Find the value of t for which $a=ti+3j$ and $b=6i+4j$ are perpendicular. (a) Write $a \cdot b$ in terms of t. (b) Set the dot product to zero and solve for t.

Q4. Let $a=22i+19j$ and $b=5i+12j$. (a) Find $|b|$ and the unit vector $\hat{b}$. (b) Find $a \cdot b$. (c) Find the scalar projection of a on b. (d) Find the vector projection $a_{/\!/}$ of a on b. (e) Find the perpendicular component $a_{\perp}$ and verify that $a_{\perp} \cdot b=0$.

Q5. Determine whether $a=3i-6j$ and $b=-2i+4j$ are parallel, perpendicular or neither. (a) Is b a scalar multiple of a? (b) Confirm your answer by comparing $a \cdot b$ with $|a||b|$.

Q6. The vectors a and b have $|a|=5$, $|b|=6$ and the angle between them is $60°$. (a) Find $a \cdot b$. (b) Expand $(a+b)\cdot(a+b)$. (c) Hence find $|a+b|$.

Unscaffolded

Q7. Given $a=9i-4j$ and $b=2i+5j$, find $a \cdot b$ and state whether the angle between a and b is acute, right or obtuse.

Q8. Find the angle between $a=2i+3j$ and $b=5i-j$, to the nearest degree.

Q9. Find the values of k for which $a=ki+2j$ and $b=ki-8j$ are perpendicular.

Q10. Show that $a=8i-6j$ and $b=3i+4j$ are perpendicular.

Q11. The points $A(1,1)$, $B(7,5)$ and $C(3,4)$ are given. Find the size of angle BAC, to the nearest degree.

Q12. Find the scalar projection of $a=6i+2j$ in the direction of $b=4i+3j$.

Q13. Find the vector projection of $a=4i+3j$ on $b=i+2j$.

Q14. Find the value of t for which $a=2ti+3j$ is perpendicular to $b=i-4j$.

Q15. The vectors a and b satisfy $|a|=8$, $|b|=3$ and the angle between them is $120°$. Find $|a-b|$.

Q16. Resolve $a=11i+3j$ into components parallel and perpendicular to $b=3i-j$.

Q17. Find the value of s for which $a=12i+sj$ is parallel to $b=8i-6j$.

Q18. The triangle ABC has vertices $A(5,2)$, $B(7,6)$ and $C(3,3)$. (a) Show that the triangle is right-angled, and state at which vertex. (b) Find the area of the triangle. (c) Find the size of angle BCA, to the nearest degree.

Answers

Q1. (a) $a \cdot b = -22$; (b) $a \cdot a = 73$, $|a| = \sqrt{73}$; (c) $b \cdot b = 61$, $|b| = \sqrt{61}$; (d) obtuse. **Q2.** $a \cdot b = 12$, $|a| = 2\sqrt{10}$, $|b| = \sqrt{10}$, $\cos\theta = \frac{3}{5}$; $\theta \approx 53°$. **Q3.** $t = -2$. **Q4.** (a) $|b| = 13$, $\hat{b} = 5/13\, i + 12/13\, j$; (b) 338; (c) 26; (d) $a_{/\!/} = 10i + 24j$; (e) $a_\perp = 12i - 5j$. **Q5.** $b = -2/3\, a$, so parallel (opposite directions); $a \cdot b = -30 = -|a||b|$. **Q6.** (a) 15; (b) $|a+b|^2 = 91$; (c) $|a+b| = \sqrt{91}$. **Q7.** $a \cdot b = -2$; obtuse. **Q8.** $\theta \approx 68°$. **Q9.** $k = 4$ or $k = -4$. **Q10.** $a \cdot b = 0$, so perpendicular. **Q11.** $\angle BAC \approx 23°$. **Q12.** 6. **Q13.** $2i + 4j$. **Q14.** $t = 6$. **Q15.** $|a - b| = \sqrt{97}$. **Q16.** $a_{/\!/} = 9i - 3j$, $a_\perp = 2i + 6j$. **Q17.** $s = -9$. **Q18.** (a) right-angled at A; (b) area $= 5$; (c) $\angle BCA \approx 63°$.

Fully-worked solutions

Q1. From components, $a \cdot b = (3)(6) + (8)(-5) = 18 - 40 = -22$. Also $a \cdot a = (3)(3) + (8)(8) = 9 + 64 = 73$, so $|a| = \sqrt{73}$, and $b \cdot b = (6)(6) + (-5)(-5) = 36 + 25 = 61$, so $|b| = \sqrt{61}$. Since $a \cdot b = -22 < 0$, the angle between the vectors is obtuse.

Q2. The dot product is $a \cdot b = (6)(1) + (2)(3) = 12$, and the magnitudes are $|a| = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$ and $|b| = \sqrt{1^2 + 3^2} = \sqrt{10}$. Then

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{12}{2\sqrt{10}\sqrt{10}} = \frac{12}{20} = \frac{3}{5},$$

so $\theta = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.13°$, which is $53°$ to the nearest degree.

Q3. In terms of t, $a \cdot b = (t)(6) + (3)(4) = 6t + 12$. The vectors are perpendicular when $a \cdot b = 0$, that is $6t + 12 = 0$, giving $t = -2$.

Q4. The magnitude is $|b| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$, so $\hat{b} = 1/13(5i + 12j) = 5/13\, i + 12/13\, j$. The dot product is $a \cdot b = (22)(5) + (19)(12) = 110 + 228 = 338$. The scalar projection is

$$a \cdot \hat{b} = \frac{a \cdot b}{|b|} = \frac{338}{13} = 26.$$

The vector projection is

$$a_{/\!/} = \frac{a \cdot b}{|b|^2} b = \frac{338}{169}(5i + 12j) = 2(5i + 12j) = 10i + 24j,$$

and the perpendicular component is $a_\perp = a - a_{/\!/} = (22 - 10)i + (19 - 24)j = 12i - 5j$. The check gives $a_\perp \cdot b = (12)(5) + (-5)(12) = 60 - 60 = 0$, as required.

Q5. Comparing components, $b = -2i + 4j = -2/3(3i - 6j) = -2/3\, a$, so b is a scalar multiple of a and the vectors are **parallel**, pointing in opposite directions because the scalar is negative. As a check, $a \cdot b = (3)(-2) + (-6)(4) = -6 - 24 = -30$, while $|a| = \sqrt{45} = 3\sqrt{5}$ and $|b| = \sqrt{20} = 2\sqrt{5}$, so $|a||b| = 6 \times 5 = 30$. Thus $a \cdot b = -|a||b|$, which corresponds to $\theta = 180°$, confirming the vectors are parallel and opposite.

Q6. Using the geometric definition with $\cos 60° = 1/2$, $a \cdot b = |a||b|\cos 60° = 5 \times 6 \times 1/2 = 15$. Expanding with the distributive rule,

$$(a + b) \cdot (a + b) = |a|^2 + 2a \cdot b + |b|^2 = 25 + 2(15) + 36 = 91.$$

Since $(a + b) \cdot (a + b) = |a + b|^2$, it follows that $|a + b| = \sqrt{91}$.

Q7. The dot product is $a \cdot b=(9)(2)+(-4)(5)=18-20=-2$. Because it is negative, $\cos\theta<0$, so the angle between the vectors is obtuse.

Q8. The dot product is $a \cdot b=(2)(5)+(3)(-1)=10-3=7$, and the magnitudes are $|a|=\sqrt{2^2+3^2}=\sqrt{13}$ and $|b|=\sqrt{5^2+(-1)^2}=\sqrt{26}$. Then

$$\cos\theta=\frac{7}{\sqrt{13}\sqrt{26}}=\frac{7}{\sqrt{338}}=\frac{7}{13\sqrt{2}}\approx 0.3808,$$

so $\theta=\cos^{-1}(0.3808)\approx 67.6°$, which is $68°$ to the nearest degree.

Q9. The dot product is $a \cdot b=(k)(k)+(2)(-8)=k^2-16$. The vectors are perpendicular when $k^2-16=0$, that is $k^2=16$, so $k=4$ or $k=-4$.

Q10. The dot product is $a \cdot b=(8)(3)+(-6)(4)=24-24=0$. A zero dot product between two non-zero vectors means the angle between them is $90°$, so a and b are perpendicular.

Q11. The angle BAC is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ drawn from A. Now $\overrightarrow{AB}=B-A=6i+4j$ and $\overrightarrow{AC}=C-A=2i+3j$, with $\overrightarrow{AB} \cdot \overrightarrow{AC}=(6)(2)+(4)(3)=12+12=24$. The magnitudes are $|\overrightarrow{AB}|=\sqrt{52}=2\sqrt{13}$ and $|\overrightarrow{AC}|=\sqrt{13}$, so

$$\cos(\angle BAC)=\frac{24}{2\sqrt{13}\cdot\sqrt{13}}=\frac{24}{26}=\frac{12}{13},$$

giving $\angle BAC=\cos^{-1}\left(\frac{12}{13}\right)\approx 22.6°$, which is $23°$ to the nearest degree.

Q12. The magnitude is $|b|=\sqrt{4^2+3^2}=5$, and $a \cdot b=(6)(4)+(2)(3)=24+6=30$. The scalar projection is

$$a \cdot \hat{b}=\frac{a \cdot b}{|b|}=\frac{30}{5}=6.$$

Q13. Here $b \cdot b=1^2+2^2=5$ and $a \cdot b=(4)(1)+(3)(2)=4+6=10$. The vector projection is

$$a_{/\!/}=\frac{a \cdot b}{b \cdot b}b=\frac{10}{5}(i+2j)=2(i+2j)=2i+4j.$$

Q14. In terms of t, $a \cdot b=(2t)(1)+(3)(-4)=2t-12$. The vectors are perpendicular when $2t-12=0$, so $t=6$.

Q15. Using $\cos 120°=-1/2$, $a \cdot b=|a||b|\cos 120°=8\times 3\times(-1/2)=-12$. For the magnitude of the difference,

$$|a-b|^2=|a|^2-2a \cdot b+|b|^2=64-2(-12)+9=64+24+9=97,$$

so $|a-b|=\sqrt{97}$.

Q16. Here $|b|$ is a surd, so use the form that avoids it: $b \cdot b=3^2+(-1)^2=10$, and $a \cdot b=(11)(3)+(3)(-1)=33-3=30$. The parallel (vector projection) component is

$$a_{/\!/}=\frac{a \cdot b}{b \cdot b}b=\frac{30}{10}(3i-j)=3(3i-j)=9i-3j,$$

and the perpendicular component is $a_{\perp}=a-a_{/\!/}=(11-9)i+(3-(-3))j=2i+6j$. As a check, $a_{\perp} \cdot b=(2)(3)+(6)(-1)=6-6=0$.

Q17. For a to be parallel to b, one vector must be a scalar multiple of the other, so the components are in a common ratio: $\frac{12}{8}=\frac{s}{-6}$. Hence $\frac{3}{2}=\frac{s}{-6}$, giving $s=-6\times\frac{3}{2}=-9$. Then $a=12i-9j=3/2(8i-6j)=3/2b$, confirming the vectors are parallel.

Q18. (a) Test each vertex for a right angle using the dot product of the two side-vectors from that vertex. At A, $\overrightarrow{AB}=B-A=2i+4j$ and $\overrightarrow{AC}=C-A=-2i+j$, with $\overrightarrow{AB}\cdot\overrightarrow{AC}=(2)(-2)+(4)(1)=-4+4=0$. Since this dot product is zero, $\overrightarrow{AB}\perp\overrightarrow{AC}$, so the triangle is right-angled at A. (The angle at B is not a right angle: $\overrightarrow{BA}=-2i-4j$ and $\overrightarrow{BC}=-4i-3j$ give $\overrightarrow{BA}\cdot\overrightarrow{BC}=8+12=20\neq0$.) (b) With the right angle at A, the two perpendicular sides are $|\overrightarrow{AB}|=\sqrt{2^2+4^2}=\sqrt{20}=2\sqrt{5}$ and $|\overrightarrow{AC}|=\sqrt{(-2)^2+1^2}=\sqrt{5}$, so the area is $1/2|\overrightarrow{AB}||\overrightarrow{AC}|=1/2\times2\sqrt{5}\times\sqrt{5}=1/2\times10=5$. (c) At C, $\overrightarrow{CB}=B-C=4i+3j$ and $\overrightarrow{CA}=A-C=2i-j$, with $\overrightarrow{CB}\cdot\overrightarrow{CA}=(4)(2)+(3)(-1)=8-3=5$. The magnitudes are $|\overrightarrow{CB}|=\sqrt{25}=5$ and $|\overrightarrow{CA}|=\sqrt{5}$, so

$$\cos(\angle BCA)=\frac{5}{5\sqrt{5}}=\frac{1}{\sqrt{5}},$$

giving $\angle BCA=\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\approx63.4°$, which is $63°$ to the nearest degree.

Theory

The earlier sections of this chapter built the machinery of plane vectors: addition and subtraction, multiplication by a scalar, components relative to the perpendicular unit vectors i and j, the magnitude $|v|$, and the **scalar (dot) product**. This section puts that machinery to work in two directions. First, a purely geometric fact — that two lines are perpendicular, that a point bisects a segment, that three lines pass through one point — can be *proved* by turning the figure into vectors and calculating. Second, because a vector carries both a size and a direction, it is exactly the right tool for **displacement**, **velocity** and **force**, so the same algebra settles practical questions about motion and equilibrium.

Position vectors and displacement.

Fix a point O as **origin**. Every point A is then named by its **position vector** $a=\overrightarrow{OA}$, the vector from O to A; points get capital letters and their position vectors the matching bold lower-case letters. The vector joining A to B is the difference of the position vectors,

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=b-a,$$

since $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$. Keeping the position vectors **general** — mere symbols $a, b, \ldots$ subject only to the conditions of the figure — makes any conclusion hold for *every* figure of that type, not just for one drawn case. This is what turns a vector calculation into a proof.

Midpoints and dividing a segment.

The midpoint M of AB has position vector the average of the ends,

$$\overrightarrow{OM}=1/2(a+b).$$

More generally the point dividing AB in the ratio $t:(1-t)$ has position vector $(1-t)a+tb=a+t(b-a)$, and $t=1/2$ recovers the midpoint. Two non-zero vectors are **parallel** when one is a scalar multiple of the other, and three points A, B, C are **collinear** when $\overrightarrow{AC}=\lambda\overrightarrow{AB}$ for some scalar λ.

The scalar product as a proof tool.

The scalar product satisfies $a\cdot b=|a||b|\cos\theta$, where θ is the angle between the vectors. Three facts make it the key to geometric proofs.

- **Squared length:** $a\cdot a=|a|^2$, so a length squared is a dot product and can be expanded algebraically.
- **Perpendicularity test:** for non-zero vectors, $a\cdot b=0$ if and only if a and b are perpendicular.
- **Algebra:** the dot product is commutative, distributes over addition, and lets scalars pull out, so a product of vector brackets multiplies out exactly like an ordinary product.

For example, the difference of two squares survives intact:

$$(a+b)\cdot(a-b)=a\cdot a-a\cdot b+b\cdot a-b\cdot b=|a|^2-|b|^2,$$

because the two middle terms cancel. Several proofs below are a single expansion of just this kind.

A strategy for vector proofs.

1. **Choose an origin** at a convenient point of the figure — often a vertex or a centre — so that as many position vectors as possible are simple.
2. **Assign position vectors** to the remaining points and record each imposed condition as a vector equation: equal lengths as $|a|=|b|$, a right angle as $a\cdot b=0$, a point on a circle of radius r as $|a|=r$.

3. **Translate the claim** into vectors: *perpendicular* means a dot product is 0; *bisect* means two midpoints coincide; *parallel and half the length* means one displacement is a scalar multiple of another; *concurrent* means one point lies on several lines.
4. **Compute** with vector algebra and the dot product until the required statement appears, then **state the conclusion in words**.

Everything here stays in the plane and uses only position vectors and the scalar product; the vector equations of lines and the vector (cross) product belong to later study.

Displacement, velocity and speed.

A moving object has a **velocity** v, a vector whose magnitude $|v|$ is the **speed** and whose direction is the direction of motion. Throughout this section i points east and j points north, so a velocity $v=p\,i+q\,j$ has speed $\sqrt{p^2+q^2}$ and a direction usually reported as a **true bearing** (the angle, measured clockwise from north, given by $\tan\theta=p/q$ when both components are positive). If the velocity is **constant**, then in time t the object undergoes a **displacement** $t\,v$, and a body starting at position vector r_0 reaches

$$r(t)=r_0+t\,v.$$

Displacement is itself a vector; the straight-line distance from the start is its magnitude, which need not equal the total length of the path travelled.

Relative velocity.

Motion is always measured against some observer. If two objects A and B have velocities v_A and v_B relative to the ground, then the velocity of A **relative to** B — the velocity A appears to have to an observer moving with B — is the difference

$$v_{A\,\mathrm{rel}\,B}=v_A-v_B.$$

Its magnitude $|v_A-v_B|$ is the **relative speed**: the speed at which A appears to move when B is treated as fixed. Reversing the roles simply reverses the vector, $v_{B\,\mathrm{rel}\,A}=-v_{A\,\mathrm{rel}\,B}$. The same subtraction handles a boat in a current or an aircraft in a wind: the velocity over the ground is the object's velocity through the water or air **plus** the velocity of that water or air.

Force, resultant and equilibrium.

A **force** is a vector. When several forces act at a point their combined effect is their vector sum, the **resultant**

$$R=F_1+F_2+\ldots+F_n.$$

Two forces can be added by the parallelogram rule, or by adding components. A particle is in **equilibrium** when it has no tendency to accelerate, which happens exactly when the resultant of all the forces on it is the zero vector,

$$F_1+F_2+\ldots+F_n=0.$$

If a set of forces has resultant R, the single extra force that would produce equilibrium is the **equilibrant** $-R$: equal in magnitude to the resultant but opposite in direction. Forces are measured in newtons (N).

Motion under a constant force.

Because a force is a vector, Newton's second law also reads as a vector equation: a constant net force F (in newtons) on a particle of mass m (in kilograms) produces a **constant acceleration**

$$a=\frac{F}{m},$$

measured in metres per second squared ($\mathrm{m/s^2}$) and pointing in the same direction as F. While the acceleration is constant the velocity and position obey the same **constant-acceleration relations** as in one dimension, now read component by component as vector equations,

$$v = u + at, s = ut + 1/2at^2,$$

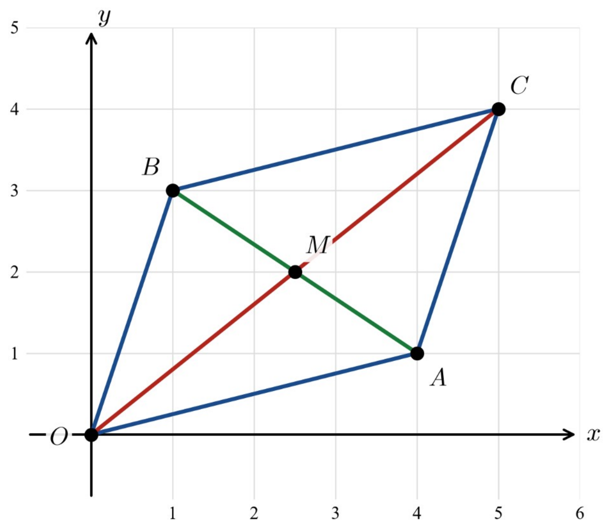
where u is the **initial velocity**, v the velocity after time t, and s the **displacement** from the starting point. These relations are stated here and applied to plane motion; obtaining them from the acceleration by integration belongs to the calculus of later study.

Worked examples

Example 1 — scaffolded

Prove that the diagonals of a parallelogram bisect each other.

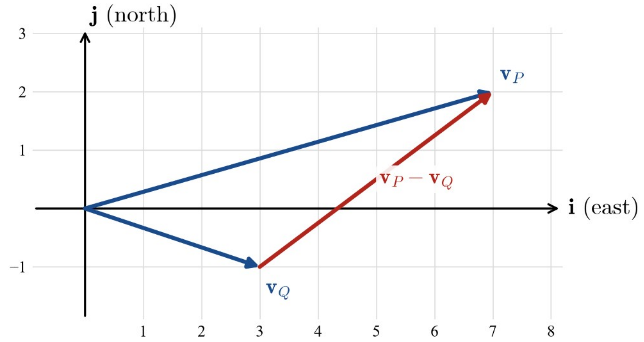
Take the parallelogram $OACB$ with O at the origin, $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$; its diagonals are OC and AB.



Working	Comment
Put O at the origin with $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$.	Name each vertex by a position vector; a vertex origin keeps the algebra short.
The fourth vertex has $\overrightarrow{OC} = a + b$.	In parallelogram $OACB$, $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$.
The midpoint of diagonal OC is $1/2(0 + (a + b)) = 1/2(a + b)$.	A midpoint is the average of the two ends.
The midpoint of diagonal AB is $1/2(a + b)$.	Average of a and b.
The two midpoints are the same point $M = 1/2(a + b)$.	One point lies on both diagonals and halves each of them.
Hence the diagonals of a parallelogram bisect each other, as required.	Concluding sentence.

Example 2 — scaffolded

At noon a yacht P has velocity $v_P = (7i + 2j)$ km/h and a ferry Q has velocity $v_Q = (3i - j)$ km/h, where i points east and j north. Find the velocity of P relative to Q, and give its speed and bearing.



Working	Comment
The velocity of P relative to Q is $v_P - v_Q$.	Relative velocity subtracts the observer's velocity.
$v_P - v_Q = (7-3)i + (2-(-1))j = 4i + 3j$.	Subtract component by component.
Speed $= \lvert 4i+3j \rvert = \sqrt{4^2+3^2} = \sqrt{25} = 5$.	Magnitude gives the relative speed, in km/h.
The vector is 4 east and 3 north, so $\tan\theta = 4/3$ measured clockwise from north, giving $\theta \approx 53.13^\circ$.	East-component over north-component gives the bearing angle.
So P appears from Q to move at 5 km/h on a bearing of about 053°.	Concluding statement.

Example 3 — unscaffolded

Prove that the diagonals of a rhombus are perpendicular.

Let the rhombus be $OACB$ with O at the origin, $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. All sides of a rhombus are equal, so $|a| = |b|$; and as in any parallelogram $\overrightarrow{OC} = a + b$, so the diagonals are $\overrightarrow{OC} = a + b$ and $\overrightarrow{AB} = b - a$. Testing them with the scalar product and expanding as an ordinary product,

$$\overrightarrow{OC} \cdot \overrightarrow{AB} = (a+b) \cdot (b-a) = a \cdot b - a \cdot a + b \cdot b - b \cdot a = |b|^2 - |a|^2.$$

The cross terms cancel, and $|a| = |b|$ makes the result 0. A zero scalar product of two non-zero vectors means the vectors are perpendicular.

$\therefore \overrightarrow{OC} \cdot \overrightarrow{AB} = 0$, so the diagonals of a rhombus are perpendicular, as required.

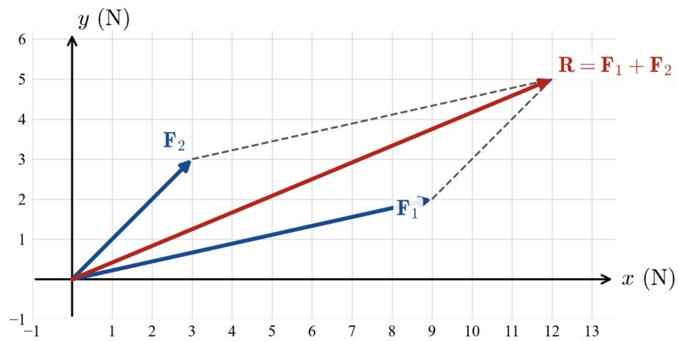
Example 4 — unscaffolded

Two forces $F_1 = (9i + 2j)\,\mathrm{N}$ and $F_2 = (3i + 3j)\,\mathrm{N}$ act at a point. Find the resultant force and its magnitude, and the additional force needed to hold the point in equilibrium.

The resultant is the vector sum

$$R = F_1 + F_2 = (9+3)i + (2+3)j = 12i + 5j,$$

with magnitude $|R| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13\,\mathrm{N}$. It makes an angle θ with the positive x-direction (east) where $\tan\theta = 5/12$, so $\theta \approx 22.62^\circ$.



For equilibrium the three forces must sum to 0, so the added force is the equilibrant

$$F_3 = -R = -12i - 5j,$$

equal in magnitude to R, that is $13\,\mathrm{N}$, but opposite in direction.

∴ the resultant is $12i + 5j$ N, of magnitude 13 N, and equilibrium requires an added force of $-12i - 5j$ N.

Example 5 — scaffolded

A particle of mass $3\,\mathrm{kg}$ moves on a smooth horizontal plane under a single constant force $F = (21i - 18j)\,\mathrm{N}$, where i points east and j north. At the instant it passes the origin its velocity is $u = (5i + 20j)\,\mathrm{m/s}$. Find its acceleration, and its velocity and position 2 seconds later.

Working	Comment
The net force is constant, so by Newton's second law $a = \frac{F}{m} = 1/3(21i - 18j) = 7i - 6j$.	Divide each component by the mass; the acceleration is constant, in $\mathrm{m/s^2}$.
With $u = 5i + 20j$ and $t = 2$, use $v = u + at = (5i + 20j) + 2(7i - 6j)$.	Substitute the constant a into the velocity relation.
$v = (5 + 14)i + (20 - 12)j = 19i + 8j$.	Add component by component; the velocity is in $\mathrm{m/s}$.
Its speed is $\lvert v \rvert = \sqrt{19^2 + 8^2} = \sqrt{425} = 5\sqrt{17}$.	Magnitude of the velocity gives the speed, about $20.6\,\mathrm{m/s}$.
The displacement is $s = ut + 1/2\,at^2 = 2(5i + 20j) + 1/2(7i - 6j)(4)$.	Position from the origin, using $t^2 = 4$.
$s = (10i + 40j) + 2(7i - 6j) = 24i + 28j$.	So the particle is at $24i + 28j$, in metres, after 2 s.

Practice questions

Scaffolded

Q1. In triangle ABC the points M and N are the midpoints of sides AB and AC. The vertices have position vectors a, b, c relative to an origin O. Prove that MN is parallel to BC and half its length (the midpoint theorem). (a) Write down the position vectors of M and N. (b) Find $\overrightarrow{MN}$. (c) Find $\overrightarrow{BC}$. (d) Compare $\overrightarrow{MN}$ with $\overrightarrow{BC}$ and state the conclusion.

Q2. A delivery drone leaves a depot and makes two straight flights: first a displacement $d_1 = (12i + 3j)\,\mathrm{km}$, then $d_2 = (-4i + 12j)\,\mathrm{km}$, with i east and j north. (a) Write d_1 and d_2 as column vectors. (b) Find the drone's resultant displacement from the depot. (c) Find its distance from the depot. (d) Find the bearing of the drone from the depot.

Q3. Two ships move with constant velocities $v_A = (20i - 15j)\,\mathrm{km/h}$ and $v_B = (8i + 20j)\,\mathrm{km/h}$ (i east, j north). (a) Write down v_A and v_B. (b) Find the velocity of A relative to B. (c) Find the speed of A relative to B. (d) Find the bearing on which A appears to move as seen from B.

Q4. AB is a diameter of a circle with centre O, and X is any other point on the circle. Take O as origin with $\overrightarrow{OA} = a$ (so $\overrightarrow{OB} = -a$) and $\overrightarrow{OX} = x$, where $\lvert a \rvert = \lvert x \rvert = r$. Prove that $\angle AXB$ is a right angle. (a) Write down $\overrightarrow{XA}$ and $\overrightarrow{XB}$. (b) Expand the scalar product $\overrightarrow{XA} \cdot \overrightarrow{XB}$. (c) Use $\lvert a \rvert = \lvert x \rvert = r$ to evaluate it. (d) State the conclusion.

Q5. A particle stays in equilibrium under three forces. Two of them are $F_1 = (10i + 15j)\,\mathrm{N}$ and $F_2 = (-17i + 9j)\,\mathrm{N}$. (a) Find $F_1 + F_2$. (b) State the condition on the third force F_3 for equilibrium. (c) Find F_3. (d) Find the magnitude of F_3.

Q6. Triangle ABC has vertices with position vectors a, b, c, and G is the point with position vector $g = 1/3(a + b + c)$. Let M_A be the midpoint of BC. (a) Write down the position vector of M_A. (b) Write down the position vector of the point dividing the median AM_A from A in the ratio $2:1$. (c) Simplify it. (d) Show it equals g, and explain why the same point lies on the medians from B and C.

Unscaffolded

Q7. A yacht sails $20\,\mathrm{km}$ due east and then $21\,\mathrm{km}$ due north. Find its distance and bearing from the starting point.

Q8. Prove that the three medians of a triangle are concurrent, meeting at the centroid G with position vector $1/3(a+b+c)$, where a, b, c are the position vectors of the vertices.

Q9. A police car travels at $90\,\mathrm{km/h}$ due north and a suspect's car at $56\,\mathrm{km/h}$ due east. Find the speed and bearing of the suspect's car relative to the police car.

Q10. Prove that the diagonals of a square are equal in length and perpendicular to each other.

Q11. A boat is steered so that it points directly across a river, moving at $8\,\mathrm{km/h}$ relative to the water, while the river flows at $6\,\mathrm{km/h}$. Find the boat's resultant speed over the ground and the angle its actual path makes with the direction straight across.

Q12. A and B are points on a circle with centre O, and M is the midpoint of the chord AB. Prove that OM is perpendicular to AB (the perpendicular from the centre bisects the chord).

Q13. Two forces $F_1=(28i-5j)\,\mathrm{N}$ and $F_2=(-4i+15j)\,\mathrm{N}$ act on a body. Find the single force F_3 that must also act for the body to be in equilibrium, and state its magnitude.

Q14. In triangle OAB the angle at O is a right angle. Prove that the midpoint M of the hypotenuse AB is equidistant from all three vertices O, A and B.

Q15. An aircraft is pointed due north and flies at an airspeed of $240\,\mathrm{km/h}$. A wind blows from the west at $70\,\mathrm{km/h}$. Find the aircraft's ground speed and its actual bearing.

Q16. $ABCD$ is any quadrilateral, and P, Q, R, S are the midpoints of the sides AB, BC, CD, DA respectively. Prove that $PQRS$ is a parallelogram (Varignon's theorem).

Q17. Two forces of magnitude $5\,\mathrm{N}$ and $3\,\mathrm{N}$ act at a point with an angle of 60° between them. Find the magnitude of their resultant and the angle it makes with the $5\,\mathrm{N}$ force.

Q18. A ferry can travel at $41\,\mathrm{km/h}$ in still water and must make good a course due north across a current flowing due east at $9\,\mathrm{km/h}$. Find the direction in which the ferry must be steered and its resulting speed due north.

Q19. A $4\,\mathrm{kg}$ puck slides on a frictionless plane, acted on by a single constant net force $F=(8i+24j)\,\mathrm{N}$, with i east and j north. As it passes the origin its velocity is $u=(10i-15j)\,\mathrm{m/s}$. Find the puck's acceleration, and its velocity and position 3 seconds later.

Answers

Q1. $\overrightarrow{MN}=1/2(c-b)=1/2\overrightarrow{BC}$, so $MN \parallel BC$ and $|MN|=1/2|BC|$. **Q2.** (a) $\begin{bmatrix}12\\3\end{bmatrix}$, $\begin{bmatrix}-4\\12\end{bmatrix}$; (b) $8i+15j$; (c) 17 km; (d) about 028°. **Q3.** (b) $12i-35j$; (c) 37 km/h; (d) about 161°. **Q4.** $\overrightarrow{XA}\cdot\overrightarrow{XB}=|x|^2-|a|^2=r^2-r^2=0$, so $\angle AXB=90°$. **Q5.** (a) $-7i+24j$; (c) $F_3=7i-24j$; (d) 25 N. **Q6.** (a) $1/2(b+c)$; (c) $1/3(a+b+c)$; (d) $=g$, symmetric in a,b,c. **Q7.** 29 km on a bearing of about 044°. **Q8.** Each median passes through $1/3(a+b+c)$, dividing it $2:1$ from the vertex. **Q9.** 106 km/h on a bearing of about 148°. **Q10.** Diagonals $a+b$ and $b-a$: equal length and dot product 0. **Q11.** 10 km/h, at about 36.87° to the straight-across direction (downstream). **Q12.** $\overrightarrow{OM}\cdot\overrightarrow{AB}=1/2(|b|^2-|a|^2)=0$; perpendicular. **Q13.** $F_3=-24i-10j$, magnitude 26 N. **Q14.** $|MO|^2=|MA|^2=|MB|^2=1/4(|a|^2+|b|^2)$. **Q15.** Ground speed 250 km/h on a bearing of about 016°. **Q16.** $\overrightarrow{PQ}=1/2(c-a)=\overrightarrow{SR}$; a parallelogram. **Q17.** 7 N, at about 21.79° to the 5 N force. **Q18.** Steer about 12.68° west of north (bearing 347°); resulting speed 40 km/h due north. **Q19.** $a=2i+6j$ m/s^2; $v=16i+3j$ m/s; $s=39i-18j$ m.

Fully-worked solutions

Q1. The midpoints are $M=1/2(a+b)$ and $N=1/2(a+c)$. Subtracting,

$$\overrightarrow{MN}=N-M=1/2(a+c)-1/2(a+b)=1/2(c-b).$$

The third side is $\overrightarrow{BC}=c-b$, so $\overrightarrow{MN}=1/2\overrightarrow{BC}$. Being a scalar multiple of $\overrightarrow{BC}$, the vector $\overrightarrow{MN}$ is parallel to BC, and taking magnitudes gives $|MN|=1/2|BC|$. Hence the segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length, as required.

Q2. (a) In column form $d_1=\begin{bmatrix}12\\3\end{bmatrix}$ and $d_2=\begin{bmatrix}-4\\12\end{bmatrix}$. (b) The resultant displacement is

$$s=d_1+d_2=(12-4)i+(3+12)j=8i+15j.$$

(c) Its magnitude is $|s|=\sqrt{8^2+15^2}=\sqrt{289}=17$, so the drone is 17 km from the depot. (d) The displacement is 8 east and 15 north, so the bearing θ satisfies $\tan\theta=8/15$, giving $\theta\approx 28.07^{\circ}$ — a bearing of about 028°.

Q3. (a) $v_A=20i-15j$ and $v_B=8i+20j$. (b) The velocity of A relative to B is

$$v_A-v_B=(20-8)i+(-15-20)j=12i-35j.$$

(c) Its speed is $\sqrt{12^2+(-35)^2}=\sqrt{1369}=37$ km/h. (d) The vector is 12 east and 35 south, so it points into the south-east. Its bearing is $180°-\arctan 12/35=180°-18.92°=161.08°$, a bearing of about 161°.

Q4. (a) With $\overrightarrow{OA}=a$, $\overrightarrow{OB}=-a$ and $\overrightarrow{OX}=x$,

$$\overrightarrow{XA}=a-x,\overrightarrow{XB}=-a-x.$$

(b) Expanding the scalar product,

$$\overrightarrow{XA}\cdot\overrightarrow{XB}=(a-x)\cdot(-a-x)=-a\cdot a-a\cdot x+x\cdot a+x\cdot x=|x|^2-|a|^2.$$

(c) Both A and X lie on the circle of radius r, so $|a|=|x|=r$ and the product equals $r^2-r^2=0$. (d) A zero scalar product of the two non-zero chords $\overrightarrow{XA}$ and $\overrightarrow{XB}$ means they are perpendicular, so $\angle AXB$ is a right angle: the angle in a semicircle is a right angle, as required.

Q5. (a) $F_1+F_2=(10-17)i+(15+9)j=-7i+24j$. (b) Equilibrium requires the three forces to sum to 0, so $F_3=-(F_1+F_2)$. (c) Hence $F_3=-(-7i+24j)=7i-24j$. (d) Its magnitude is $|F_3|=\sqrt{7^2+(-24)^2}=\sqrt{625}=25$ N.

Q6. (a) The midpoint of BC is $M_A = 1/2(b+c)$. (b) The point dividing AM_A from A in the ratio $2:1$ is $a + 2/3(M_A - a)$. (c) Simplifying,

$$a + 2/3\left(1/2(b+c) - a\right) = a - 2/3a + 1/3(b+c) = 1/3(a+b+c).$$

(d) This is exactly g, so G lies on the median from A. Because $1/3(a+b+c)$ is symmetric in the three vertices, the identical calculation on the medians from B and C gives the same point, so G lies on all three.

Q7. Taking i east and j north, the successive displacements give a resultant $20i + 21j$. The distance from the start is

$$|20i + 21j| = \sqrt{20^2 + 21^2} = \sqrt{841} = 29\,\text{km}.$$

The bearing θ satisfies $\tan\theta = 20/21$ (east over north), so $\theta \approx 43.60^\circ$, a bearing of about 044°.

Q8. Let the vertices A, B, C have position vectors a, b, c. A median joins a vertex to the midpoint of the opposite side, so the feet of the three medians are

$$m_A = 1/2(b+c), m_B = 1/2(a+c), m_C = 1/2(a+b).$$

Consider the point G with $g = 1/3(a+b+c)$. The point two-thirds of the way from A to m_A is

$$a + 2/3(m_A - a) = a + 2/3\left(1/2(b+c) - a\right) = 1/3(a+b+c) = g,$$

so G lies on the median from A. The expression is symmetric in a, b, c, so the same working gives $b + 2/3(m_B - b) = g$ and $c + 2/3(m_C - c) = g$. All three medians therefore pass through the single point $G = 1/3(a+b+c)$, dividing each in the ratio $2:1$ from the vertex. Hence the medians of a triangle are concurrent, as required.

Q9. With i east and j north, the suspect's velocity is $v_S = 56i$ and the police velocity is $v_P = 90j$. The velocity of the suspect relative to the police car is

$$v_S - v_P = 56i - 90j,$$

with speed $\sqrt{56^2 + 90^2} = \sqrt{11236} = 106$ km/h. The vector is 56 east and 90 south, so its bearing is $180° - \arctan 56/90 = 180° - 31.89° = 148.11°$, about 148°.

Q10. Take the square $OACB$ with $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. A square has equal sides and a right angle, so $|a| = |b|$ and $a \cdot b = 0$. Its diagonals are $\overrightarrow{OC} = a + b$ and $\overrightarrow{AB} = b - a$. For their lengths,

$$|a+b|^2 = |a|^2 + 2a \cdot b + |b|^2 = |a|^2 + |b|^2,$$

$$|b-a|^2 = |a|^2 - 2a \cdot b + |b|^2 = |a|^2 + |b|^2,$$

so $|\overrightarrow{OC}| = |\overrightarrow{AB}|$: the diagonals are equal. For their directions,

$$(a+b) \cdot (b-a) = |b|^2 - |a|^2 = 0,$$

since $|a| = |b|$, so the diagonals are perpendicular. Hence the diagonals of a square are equal in length and perpendicular, as required.

Q11. Resolve along the boat's own axes: let j' point straight across the river and i' point downstream. The boat's velocity relative to the water is $8j'$ and the current is $6i'$, so the resultant velocity over the ground is

$$v = 6i' + 8j', |v| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10\,\text{km/h}.$$

The angle α between the actual path and the straight-across direction satisfies $\tan\alpha = 6/8$, so $\alpha \approx 36.87^\circ$ towards downstream. Hence the boat moves at 10 km/h along a path about 36.87° from straight across.

Q12. Put the centre O at the origin, so A and B have position vectors a and b with $|a|=|b|=r$. The midpoint of the chord is $M=1/2(a+b)$, so $\overrightarrow{OM}=1/2(a+b)$ and $\overrightarrow{AB}=b-a$. Then

$$\overrightarrow{OM}\cdot\overrightarrow{AB}=1/2(a+b)\cdot(b-a)=1/2(|b|^2-|a|^2)=1/2(r^2-r^2)=0.$$

Hence OM is perpendicular to AB: the perpendicular from the centre bisects the chord, as required.

Q13. For equilibrium the three forces must sum to 0, so $F_3=-(F_1+F_2)$. Adding the two given forces,

$$F_1+F_2=(28-4)i+(-5+15)j=24i+10j,$$

so $F_3=-24i-10j$, with magnitude $\sqrt{(-24)^2+(-10)^2}=\sqrt{676}=26$ N.

Q14. Put the origin at O, with $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$; the right angle at O gives $a\cdot b=0$. The midpoint of the hypotenuse is $M=1/2(a+b)$, and its distances to the vertices satisfy

$$|MO|^2=|1/2(a+b)|^2=1/4(|a|^2+2a\cdot b+|b|^2)=1/4(|a|^2+|b|^2),$$

$$|MA|^2=|a-1/2(a+b)|^2=|1/2(a-b)|^2=1/4(|a|^2+|b|^2),$$

$$|MB|^2=|b-1/2(a+b)|^2=|1/2(b-a)|^2=1/4(|a|^2+|b|^2),$$

each using $a\cdot b=0$. All three are equal, so $|MO|=|MA|=|MB|$: the midpoint of the hypotenuse is equidistant from the three vertices, as required.

Q15. With i east and j north, the aircraft's velocity through the air is $240j$. A wind *from* the west blows towards the east, adding $70i$, so the ground velocity is

$$v=70i+240j,|v|=\sqrt{70^2+240^2}=\sqrt{62500}=250\,\text{km/h}.$$

The bearing θ satisfies $\tan\theta=70/240$ (east over north), so $\theta\approx16.26^\circ$: a ground speed of 250 km/h on a bearing of about 016°.

Q16. Let A, B, C, D have position vectors a, b, c, d. The midpoints of the sides are

$$P=1/2(a+b),Q=1/2(b+c),R=1/2(c+d),S=1/2(d+a).$$

Then

$$\overrightarrow{PQ}=Q-P=1/2(c-a),\overrightarrow{SR}=R-S=1/2(c-a).$$

Since $\overrightarrow{PQ}=\overrightarrow{SR}$, the sides PQ and SR are equal in length and parallel, which makes $PQRS$ a parallelogram. (Both equal $1/2\overrightarrow{AC}$, so each is parallel to the diagonal AC and half its length.) Hence the midpoints of the sides of any quadrilateral form a parallelogram, as required.

Q17. Placing the two forces tip-to-tail to build the resultant, the triangle of forces has sides 5 and 3 with the angle between those sides equal to $180°-60°=120°$. By the cosine rule the resultant R satisfies

$$R^2=5^2+3^2+2(5)(3)\cos60^\circ=25+9+15=49,$$

so $R=\sqrt{49}=7$ N. (Equivalently, resolving with the 5 N force along an axis gives components $(5+3\cos60^\circ,3\sin60^\circ)=(6.5,2.598)$, whose magnitude is 7.) The angle β the resultant makes with the 5 N force satisfies, by the sine rule,

$$\frac{\sin\beta}{3}=\frac{\sin120^\circ}{7}\Rightarrow\sin\beta=\frac{3\sin120^\circ}{7}\approx0.3712,$$

so $\beta\approx21.79^\circ$. The resultant is 7 N at about 21.79° to the 5 N force.

Q18. Let i point east and j north. The ferry's velocity relative to the water has magnitude 41; to cancel the 9 km/h eastward current the ferry must be steered so that this velocity has a westward component of 9, that is a component $-9i$. Its northward component is then $\sqrt{41^2-9^2}=\sqrt{1600}=40$. Adding the current $9i$, the resultant velocity over the ground is

$$(-9i+40j)+9i=40j,$$

which is due north at 40 km/h. The steering direction makes an angle $\arctan 9/40\approx 12.68^\circ$ west of north (a bearing of about 347°). Hence the ferry is steered about 12.68° west of north and makes good 40 km/h due north.

Q19. The force is constant, so the acceleration is constant and, by Newton's second law in vector form,

$$a=\frac{F}{m}=1/4(8i+24j)=2i+6j,$$

in m/s^2. The constant-acceleration relations then give the velocity after $t=3$ s,

$$v=u+at=(10i-15j)+3(2i+6j)=16i+3j,$$

in m/s, and the displacement from the origin,

$$s=ut+1/2at^2=3(10i-15j)+1/2(2i+6j)(9)=(30i-45j)+(9i+27j)=39i-18j.$$

So after 3 seconds the puck has acceleration $2i+6j$ m/s^2, velocity $16i+3j$ m/s, and position $39i-18j$ m relative to its starting point.