Specialist Mathematics Units 1 & 2

Chapter 9 — Transformations of the plane

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Chapter 9 Transformations of the plane — 9A Points, column vectors and translations

Theory

A **transformation of the plane** is a rule that sends each point to exactly one image point, in a way that can be reversed: every point is the image of exactly one point. This chapter studies the transformations that lie behind congruence and similarity — translations, rotations, reflections and dilations. The whole machinery rests on one idea: a point can be recorded as a **column vector**, after which a transformation becomes an *operation on that vector*. This section develops the simplest transformation, the **translation**, and the vector language every later section reuses.

Points as column vectors.

Fix the usual Cartesian axes with origin O. A point P with coordinates (x, y) is recorded as the 2×1 **column vector**

$$x = \begin{bmatrix} x \\ y \end{bmatrix},$$

called the **position vector** of P: the arrow drawn from O to P. The two entries are the **components** of the vector. The point and its position vector carry exactly the same information, and we move freely between the notations $P(x, y)$ and $x = \begin{bmatrix} x \\ y \end{bmatrix}$. Writing points this way is what lets a geometric operation be carried out as arithmetic on components. Two points are the same precisely when their column vectors agree entry by entry:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2.$$

The origin itself is the **zero vector** $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Translations.

A **translation** slides every point of the plane by the same displacement. That displacement is itself a column vector,

$$t = \begin{bmatrix} a \\ b \end{bmatrix},$$

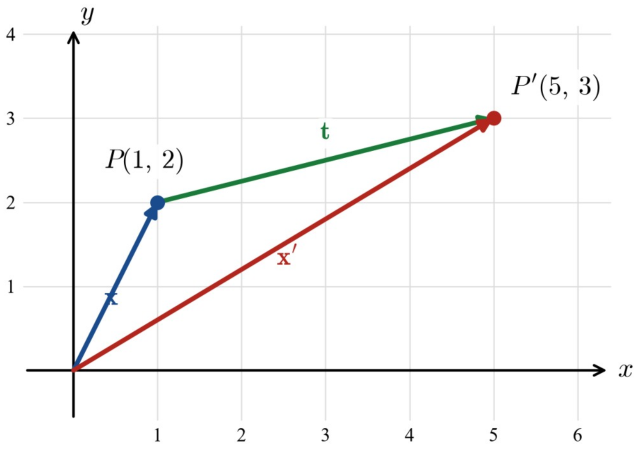
the **translation vector**: a is the shift in the x-direction and b the shift in the y-direction. Under the translation by t, the point with position vector x is sent to the point with position vector

$$x' = x + t,$$

where the sum is the ordinary component-wise addition of column vectors. The image point is written P' and its position vector x' (read "x prime"). In components,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}, \text{so } x' = x + a, y' = y + b.$$

We denote this transformation T_t, so that $T_t(x) = x + t$. The picture is a vector triangle: the position vector x of P, followed by the translation arrow t laid from P, reaches the position vector x' of P'.



Here $P(1,2)$ has position vector $x=\begin{bmatrix}1\\2\end{bmatrix}$; translating by $t=\begin{bmatrix}4\\1\end{bmatrix}$ gives $x'=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$, that is $P'(5,3)$.

Finding the pre-image.

Because the rule is simply "add t", it is undone by "subtract t". Given an image point x', the original point — its **pre-image** — is recovered by

$$x=x'-t.$$

Every point of the plane is therefore the image of exactly one point, confirming that a translation is a genuine transformation. In particular, the point that lands on a given target can always be found by subtracting the translation vector.

A translation is an isometry.

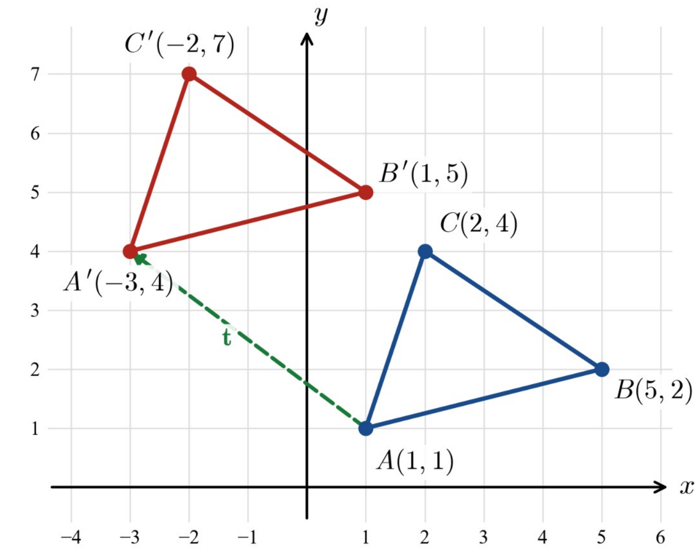
Take any two points P and Q, with position vectors p and q, and translate both by t. The displacement between the images is

$$q'-p'=(q+t)-(p+t)=q-p,$$

identical to the displacement between the originals: the translation vector cancels. The vector joining any two points is unchanged, so the distance between them is unchanged. A transformation that preserves all distances is called an **isometry**. It follows at once that a translation preserves lengths of segments, sizes of angles, areas, parallelism and orientation — the image of a figure is **congruent** to the figure, merely shifted. This is why translating a shape is done by translating its defining points and rejoining them in the same way.

The image of a shape.

To translate a polygon, translate each vertex by t and join the images in the same order. Since every point moves by the *same* arrow t, each edge maps to an edge that is equal in length and parallel to it, and the image is a congruent copy of the original.



The triangle $A(1,1)$, $B(5,2)$, $C(2,4)$ translated by $t=\begin{bmatrix}-4\\3\end{bmatrix}$ has image $A'(-3,4)$, $B'(1,5)$, $C'(-2,7)$; the dashed arrow shows the common displacement t carrying each vertex to its image.

The image of a line or a graph.

A line or a curve is translated in the same way, but it is described by an equation rather than by a list of vertices. If a set of points satisfies an equation in x and y, the equation of its image under T_t is obtained by writing the original coordinates in terms of the image ones. From $x'=x+a$ and $y'=y+b$,

$$x=x'-a,\ y=y'-b,$$

and substituting these into the original equation gives an equation satisfied by exactly the image points. Dropping the primes, the rule is: **replace x by $x-a$ and y by $y-b$.**

For the line $y=3x-2$ translated by $t=\begin{bmatrix}2\\1\end{bmatrix}$ this gives $y-1=3(x-2)-2$, that is $y=3x-7$. The gradient is unchanged: because every point moves by the same arrow, the image of a line is always a line parallel to it. Applying the same vector to the parabola $y=x^2$ gives $y-1=(x-2)^2$, that is $y=(x-2)^2+1$ — the whole curve, vertex included, has slid by t.

Composing translations.

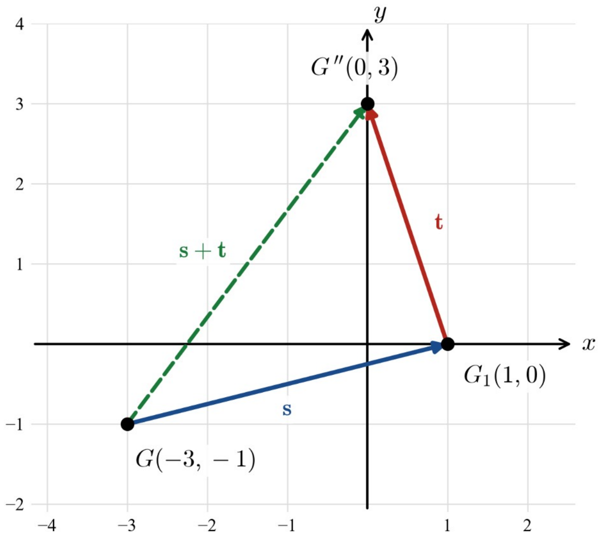
Applying one translation and then another simply adds the two displacements. If T_s is applied first and then T_t, a point x goes first to $x+s$ and then to

$$(x+s)+t=x+(s+t).$$

So the **composition** of two translations is again a translation, by the sum vector $s+t$; in operator notation

$$T_t\circ T_s=T_{s+t}.$$

Because vector addition is commutative, $s+t=t+s$: the **order of two translations does not matter**, and both reach the same final point. The **identity translation** T_0 leaves every point fixed, and each translation T_t has an **inverse** T_{-t} that undoes it, since $t+(-t)=0$.



Here the point $G(-3,-1)$ is translated by $s=\begin{bmatrix}4\\1\end{bmatrix}$ to $G_1(1,0)$, then by $t=\begin{bmatrix}-1\\3\end{bmatrix}$ to $G''(0,3)$. The single translation by $s+t=\begin{bmatrix}3\\4\end{bmatrix}$ (the dashed arrow) reaches G'' directly.

Describing the translation between two points.

If a translation maps P to P', its translation vector is the displacement from P to P':

$$t=x'-x=\overrightarrow{PP'}.$$

One point and its image therefore determine the translation completely — and, by the isometry property above, that *same* vector then carries every other point of the plane. This is the sense in which a single pair "point, image" specifies the whole transformation.

These translations are the additive part of plane transformations. In the next section the maps that fix the origin — rotations, reflections and dilations — are represented instead by a 2×1 column vector being *multiplied* by a 2×2 matrix; 9C then composes those matrix transformations, reverses them, and asks what they leave unchanged. The column-vector language set up here is what makes all of that possible.

Worked examples

Example 1 — scaffolded

A translation has vector $t=\begin{bmatrix}4\\-3\end{bmatrix}$. Find the image of (a) $A(-2,5)$ and (b) $B(6,-1)$, and (c) find the point that is mapped to the origin $O(0,0)$.

Working	Comment
Write each point as a column vector, e.g. $a=\begin{bmatrix}-2\\5\end{bmatrix}$.	A point (x,y) becomes the position vector $\begin{bmatrix}x\\y\end{bmatrix}$.
(a) $a'=\begin{bmatrix}-2\\5\end{bmatrix}+\begin{bmatrix}4\\-3\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$, so $A'(2,2)$.	Apply $x'=x+t$: add component by component.
(b) $b'=\begin{bmatrix}6\\-1\end{bmatrix}+\begin{bmatrix}4\\-3\end{bmatrix}=\begin{bmatrix}10\\-4\end{bmatrix}$, so $B'(10,-4)$.	Same rule; $-1+(-3)=-4$.
(c) Pre-image: $x=x'-t=\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}4\\-3\end{bmatrix}=\begin{bmatrix}-4\\3\end{bmatrix}$.	To reach a target, *subtract* t.

Working	Comment
So the point $(-4,3)$ maps to the origin.	Check: $\begin{bmatrix}-4\\3\end{bmatrix}+\begin{bmatrix}4\\-3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

Example 2 — scaffolded

The triangle with vertices $A(1,1)$, $B(5,2)$ and $C(2,4)$ is translated by $t=\begin{bmatrix}-4\\3\end{bmatrix}$. Find the image vertices, and verify that the side AB has the same length as its image $A'B'$.

Working	Comment
$a'=\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}-4\\3\end{bmatrix}=\begin{bmatrix}-3\\4\end{bmatrix}$, so $A'(-3,4)$.	Translate each vertex by the same t.
$b'=\begin{bmatrix}5\\2\end{bmatrix}+\begin{bmatrix}-4\\3\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$, so $B'(1,5)$.	Add component by component.
$c'=\begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}-4\\3\end{bmatrix}=\begin{bmatrix}-2\\7\end{bmatrix}$, so $C'(-2,7)$.	Join $A'B'C'$ in the same order as ABC.
Length $AB=\sqrt{(5-1)^2+(2-1)^2}=\sqrt{16+1}=\sqrt{17}$.	Length from the displacement $\begin{bmatrix}4\\1\end{bmatrix}$.
Length $A'B'=\sqrt{(1-(-3))^2+(5-4)^2}=\sqrt{16+1}=\sqrt{17}$.	Equal length: a translation is an isometry.

Example 3 — unscaffolded

A translation maps the point $D(-5,4)$ to $D'(1,-2)$. Find the translation vector t, hence find the image of $E(3,7)$ and the equation of the image of the line $y=2x+1$, and find the point whose image is the origin.

The translation vector is the displacement from D to D':

$$t=x_D'-x_D=\begin{bmatrix}1\\-2\end{bmatrix}-\begin{bmatrix}-5\\4\end{bmatrix}=\begin{bmatrix}6\\-6\end{bmatrix}.$$

This same vector carries every point. The image of $E(3,7)$ is

$$x_E'=\begin{bmatrix}3\\7\end{bmatrix}+\begin{bmatrix}6\\-6\end{bmatrix}=\begin{bmatrix}9\\1\end{bmatrix}.$$

For the line, replace x by $x-6$ and y by $y-(-6)=y+6$ in $y=2x+1$:

$$y+6=2(x-6)+1=2x-11,\text{so } y=2x-17,$$

a line of the same gradient, parallel to the original. Finally, the point mapped to the origin is found by subtracting t:

$$\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}6\\-6\end{bmatrix}=\begin{bmatrix}-6\\6\end{bmatrix}.$$

$\therefore t=\begin{bmatrix}6\\-6\end{bmatrix}$, the image of E is $E'(9,1)$, the image of $y=2x+1$ is $y=2x-17$, and $(-6,6)$ maps to the origin.

Example 4 — unscaffolded

The translation T_s with $s=\begin{bmatrix}4\\1\end{bmatrix}$ is applied to $G(-3,-1)$, and then T_t with $t=\begin{bmatrix}-1\\3\end{bmatrix}$ is applied to the result. Find the final image, show that a single translation achieves the same, and confirm that performing the two translations in the opposite order gives the same point.

Applying T_s first,

$$x_{G_1} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then applying T_t,

$$x_{G''} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

The composition is the single translation by $s + t = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and indeed

$$\begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Reversing the order, T_t first gives $\begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, and then T_s gives $\begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ — the same point.

$\therefore$ the final image is $G''(0,3)$, equal to the image under the single translation by $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and the order of the two translations does not affect the result.

Practice questions

Scaffolded

Q1. A translation has vector $t = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$. (a) Write the point $A(3, -2)$ as a column vector. (b) Find the image A'. (c) Find the image of $B(-1, -4)$. (d) Find the point that maps to the origin.

Q2. The triangle $P(-3,1)$, $Q(0,4)$, $R(2,-2)$ is translated by $t = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. (a) Find P', Q' and R'. (b) Find the length of PQ. (c) Find the length of $P'Q'$. (d) State what property of translations parts (b) and (c) illustrate.

Q3. A translation maps $M(4,7)$ to $M'(-2,3)$. (a) Find the translation vector t. (b) Find the image of $N(1,-5)$. (c) Find the point whose image is the origin. (d) Find the equation of the image of the line $y = 2x - 3$.

Q4. A point is translated by $s = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and then by $t = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$. (a) Find the single translation vector $s + t$. (b) Starting from $K(2,2)$, find the image after s, then after t. (c) Confirm this equals the image of K under the single translation from (a).

Q5. A translation maps the origin $O(0,0)$ to $O'(-7,2)$. (a) State the translation vector t. (b) Find the image of $P(5,5)$. (c) A point Q has image $Q'(3,-1)$; find Q. (d) Find the image of the origin under the inverse translation T_{-t}.

Q6. A point is translated by $a = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ and then by a translation b, after which every point has returned to its original position. (a) State the net translation vector. (b) Find b. (c) Verify your answer by translating $T(4,4)$ by a and then by b.

Unscaffolded

Q7. Find the image of $A(-6,2)$ under the translation $t=\begin{bmatrix}5\\5\end{bmatrix}$.

Q8. A translation maps $B(8,-3)$ to $B'(1,4)$. Find the translation vector.

Q9. Find the images of the triangle $D(2,5)$, $E(-3,0)$, $F(4,-2)$ under $t=\begin{bmatrix}-1\\-6\end{bmatrix}$.

Q10. Under the translation $t=\begin{bmatrix}9\\-4\end{bmatrix}$, find the point that is mapped to the origin.

Q11. A point is translated by $s=\begin{bmatrix}-3\\7\end{bmatrix}$ and then by $t=\begin{bmatrix}8\\-2\end{bmatrix}$. Find the single equivalent translation vector, and hence the image of $P(-5,1)$.

Q12. A translation $t=\begin{bmatrix}p\\q\end{bmatrix}$ maps $(-2,6)$ to $(4,-1)$. Find p and q, and hence the image of the origin.

Q13. A vertex $X(7,-5)$ of a shape has image $X'(2,-9)$ under a translation. Find the translation vector, and hence the image of the origin and the equation of the image of the parabola $y=x^2$ under the same translation.

Q14. The points $A(-4,2)$ and $B(6,8)$ are translated by $t=\begin{bmatrix}3\\-5\end{bmatrix}$. Find the midpoint of AB, the image of that midpoint, and the midpoint of $A'B'$, and comment on what you observe.

Q15. A point is translated in turn by $u=\begin{bmatrix}2\\-1\end{bmatrix}$, $v=\begin{bmatrix}-5\\3\end{bmatrix}$ and $w=\begin{bmatrix}1\\1\end{bmatrix}$. Find the single translation vector equivalent to all three, and the image of the origin.

Q16. After a translation $s=\begin{bmatrix}4\\6\end{bmatrix}$, a further translation r is applied so that the net effect is the translation $\begin{bmatrix}-1\\-2\end{bmatrix}$. Find r, and the image of $(3,3)$ under the net translation.

Q17. The square $A(-2,-2)$, $B(3,-2)$, $C(3,3)$, $D(-2,3)$ is translated by $t=\begin{bmatrix}1\\-4\end{bmatrix}$. Find the image vertices, and find the perimeter of the image.

Q18. A point $P(-3,5)$ is translated by t_1 to $P'(2,-1)$, and P' is then translated by t_2 to $P''(-4,-4)$. (a) Find t_1 and t_2. (b) Find the single translation vector that maps P directly to P''. (c) Verify that this equals t_1+t_2.

Answers

Q1. (a) $\begin{bmatrix}3\\-2\end{bmatrix}$; (b) $A'(-2,4)$; (c) $B'(-6,2)$; (d) $(5,-6)$. **Q2.** (a) $P'(-1,-4)$, $Q'(2,-1)$, $R'(4,-7)$; (b) $3\sqrt{2}$; (c) $3\sqrt{2}$; (d) a translation preserves length (it is an isometry). **Q3.** (a) $\begin{bmatrix}-6\\-4\end{bmatrix}$; (b) $N'(-5,-9)$; (c) $(6,4)$; (d) $y=2x+5$. **Q4.** (a) $\begin{bmatrix}-1\\3\end{bmatrix}$; (b) $(5,0)$ then $(1,5)$; (c) $K'(1,5)$ — the same. **Q5.** (a) $\begin{bmatrix}-7\\2\end{bmatrix}$; (b) $P'(-2,7)$; (c) $Q(10,-3)$; (d) $(7,-2)$. **Q6.** (a) $\begin{bmatrix}0\\0\end{bmatrix}$; (b) $b=\begin{bmatrix}-6\\1\end{bmatrix}$; (c) $(4,4)\rightarrow(10,3)\rightarrow(4,4)$. **Q7.** $(-1,7)$. **Q8.** $\begin{bmatrix}-7\\7\end{bmatrix}$. **Q9.** $D'(1,-1)$, $E'(-4,-6)$, $F'(3,-8)$. **Q10.** $(-9,4)$. **Q11.** $\begin{bmatrix}5\\5\end{bmatrix}$; $P''(0,6)$. **Q12.** $p=6$, $q=-7$; image of origin $(6,-7)$. **Q13.** $\begin{bmatrix}-5\\-4\end{bmatrix}$; image of origin $(-5,-4)$; image of the parabola $y=(x+5)^2-4$. **Q14.** midpoint of AB is $(1,5)$; its image is $(4,0)$; midpoint of $A'B'$ is $(4,0)$ — equal, so the translation maps midpoints to midpoints. **Q15.** $\begin{bmatrix}-2\\3\end{bmatrix}$; image of origin $(-2,3)$. **Q16.** $r=\begin{bmatrix}-5\\-8\end{bmatrix}$; image of $(3,3)$ is $(2,1)$. **Q17.** $A'(-1,-6)$, $B'(4,-6)$, $C'(4,-1)$, $D'(-1,-1)$; perimeter 20. **Q18.** (a) $t_1=\begin{bmatrix}5\\-6\end{bmatrix}$, $t_2=\begin{bmatrix}-6\\-3\end{bmatrix}$; (b) $\begin{bmatrix}-1\\-9\end{bmatrix}$; (c) $t_1+t_2=\begin{bmatrix}-1\\-9\end{bmatrix}$.

Fully-worked solutions

Q1. As a column vector $A(3,-2)=\begin{bmatrix}3\\-2\end{bmatrix}$. (b) $a'=\begin{bmatrix}3\\-2\end{bmatrix}+\begin{bmatrix}-5\\6\end{bmatrix}=\begin{bmatrix}-2\\4\end{bmatrix}$, so $A'(-2,4)$. (c) $b'=\begin{bmatrix}-1\\-4\end{bmatrix}+\begin{bmatrix}-5\\6\end{bmatrix}=\begin{bmatrix}-6\\2\end{bmatrix}$, so $B'(-6,2)$. (d) Subtracting t from the origin, $\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-5\\6\end{bmatrix}=\begin{bmatrix}5\\-6\end{bmatrix}$, so $(5,-6)$ maps to the origin.

Q2. Adding $t=\begin{bmatrix}2\\-5\end{bmatrix}$ to each vertex: $P'=\begin{bmatrix}-3\\1\end{bmatrix}+\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}-1\\-4\end{bmatrix}$, $Q'=\begin{bmatrix}0\\4\end{bmatrix}+\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$, $R'=\begin{bmatrix}2\\-2\end{bmatrix}+\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}4\\-7\end{bmatrix}$. The side PQ has length $\sqrt{(0-(-3))^2+(4-1)^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$, and $P'Q'$ has length $\sqrt{(2-(-1))^2+(-1-(-4))^2}=\sqrt{9+9}=3\sqrt{2}$. The two lengths are equal because a translation is an isometry: it preserves all distances.

Q3. (a) The translation vector is the displacement from M to M': $t=\begin{bmatrix}-2\\3\end{bmatrix}-\begin{bmatrix}4\\7\end{bmatrix}=\begin{bmatrix}-6\\-4\end{bmatrix}$. (b) $N'=\begin{bmatrix}1\\-5\end{bmatrix}+\begin{bmatrix}-6\\-4\end{bmatrix}=\begin{bmatrix}-5\\-9\end{bmatrix}$, so $N'(-5,-9)$. (c) Subtracting t from the origin, $\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-6\\-4\end{bmatrix}=\begin{bmatrix}6\\4\end{bmatrix}$, so $(6,4)$ maps to the origin. (d) Here $a=-6$ and $b=-4$, so replace x by $x-(-6)=x+6$ and y by $y-(-4)=y+4$ in $y=2x-3$: $y+4=2(x+6)-3=2x+9$, that is $y=2x+5$. (Check: $(0,-3)$ lies on the original and its image $(-6,-7)$ satisfies $y=2x+5$.) The image is a parallel line, as it must be.

Q4. (a) $s+t=\begin{bmatrix}3\\-2\end{bmatrix}+\begin{bmatrix}-4\\5\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}$. (b) After s: $\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$; then after t: $\begin{bmatrix}5\\0\end{bmatrix}+\begin{bmatrix}-4\\5\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$. (c) Under the single translation, $\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}-1\\3\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$ — the same image $K'(1,5)$.

Q5. (a) $t=\begin{bmatrix}-7\\2\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}-7\\2\end{bmatrix}$. (b) $P'=\begin{bmatrix}5\\5\end{bmatrix}+\begin{bmatrix}-7\\2\end{bmatrix}=\begin{bmatrix}-2\\7\end{bmatrix}$, so $P'(-2,7)$. (c) $Q=x_Q'-t=\begin{bmatrix}3\\-1\end{bmatrix}-\begin{bmatrix}-7\\2\end{bmatrix}=\begin{bmatrix}10\\-3\end{bmatrix}$, so $Q(10,-3)$. (d) The inverse translation has vector $-t=\begin{bmatrix}7\\-2\end{bmatrix}$, so the image of the origin is $\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}7\\-2\end{bmatrix}=\begin{bmatrix}7\\-2\end{bmatrix}$, that is $(7,-2)$.

Q6. (a) Returning every point to its start means the net translation is the zero vector $\begin{bmatrix}0\\0\end{bmatrix}$ (the identity translation). (b) The composition has vector $a+b$, so $a+b=\begin{bmatrix}0\\0\end{bmatrix}$ gives $b=-a=\begin{bmatrix}-6\\1\end{bmatrix}$. (c) Translating $T(4,4)$ by a: $\begin{bmatrix}4\\4\end{bmatrix}+\begin{bmatrix}6\\-1\end{bmatrix}=\begin{bmatrix}10\\3\end{bmatrix}$; then by b: $\begin{bmatrix}10\\3\end{bmatrix}+\begin{bmatrix}-6\\1\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$ — back to T, as required.

Q7. $A'=\begin{bmatrix}-6\\2\end{bmatrix}+\begin{bmatrix}5\\5\end{bmatrix}=\begin{bmatrix}-1\\7\end{bmatrix}$, so $A'(-1,7)$.

Q8. $t=x_B'-x_B=\begin{bmatrix}1\\4\end{bmatrix}-\begin{bmatrix}8\\-3\end{bmatrix}=\begin{bmatrix}-7\\7\end{bmatrix}$.

Q9. Adding $t=\begin{bmatrix}-1\\-6\end{bmatrix}$ to each vertex: $D'=\begin{bmatrix}2\\5\end{bmatrix}+\begin{bmatrix}-1\\-6\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$, $E'=\begin{bmatrix}-3\\0\end{bmatrix}+\begin{bmatrix}-1\\-6\end{bmatrix}=\begin{bmatrix}-4\\-6\end{bmatrix}$, $F'=\begin{bmatrix}4\\-2\end{bmatrix}+\begin{bmatrix}-1\\-6\end{bmatrix}=\begin{bmatrix}3\\-8\end{bmatrix}$. So $D'(1,-1)$, $E'(-4,-6)$, $F'(3,-8)$.

Q10. Subtracting t from the origin, $\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}9\\-4\end{bmatrix}=\begin{bmatrix}-9\\4\end{bmatrix}$, so $(-9,4)$ maps to the origin.

Q11. The single vector is $s+t=\begin{bmatrix}-3\\7\end{bmatrix}+\begin{bmatrix}8\\-2\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}$. Then $P''=\begin{bmatrix}-5\\1\end{bmatrix}+\begin{bmatrix}5\\5\end{bmatrix}=\begin{bmatrix}0\\6\end{bmatrix}$, so $P''(0,6)$.

Q12. The vector is the displacement from $(-2,6)$ to $(4,-1)$: $\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}4\\-1\end{bmatrix}-\begin{bmatrix}-2\\6\end{bmatrix}=\begin{bmatrix}6\\-7\end{bmatrix}$, so $p=6$, $q=-7$. The image of the origin is $\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}6\\-7\end{bmatrix}=\begin{bmatrix}6\\-7\end{bmatrix}$, that is $(6,-7)$.

Q13. $t=\begin{bmatrix}2\\-9\end{bmatrix}-\begin{bmatrix}7\\-5\end{bmatrix}=\begin{bmatrix}-5\\-4\end{bmatrix}$. The image of the origin is $\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}-5\\-4\end{bmatrix}=\begin{bmatrix}-5\\-4\end{bmatrix}$, that is $(-5,-4)$. For the parabola, $a=-5$ and $b=-4$, so replace x by $x+5$ and y by $y+4$ in $y=x^2$: $y+4=(x+5)^2$, that is $y=(x+5)^2-4$. Its vertex $(-5,-4)$ is the image of the original vertex $(0,0)$, as expected.

Q14. The midpoint of AB is $(-4+6/2,2+8/2)=(1,5)$. Translating it by $t=\begin{bmatrix}3\\-5\end{bmatrix}$ gives $\begin{bmatrix}1\\5\end{bmatrix}+\begin{bmatrix}3\\-5\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$, i.e. $(4,0)$. The images of the endpoints are $A'=\begin{bmatrix}-4\\2\end{bmatrix}+\begin{bmatrix}3\\-5\end{bmatrix}=\begin{bmatrix}-1\\-3\end{bmatrix}$ and $B'=\begin{bmatrix}6\\8\end{bmatrix}+\begin{bmatrix}3\\-5\end{bmatrix}=\begin{bmatrix}9\\3\end{bmatrix}$, whose midpoint is $(-1+9/2,-3+3/2)=(4,0)$. The two agree: the image of the midpoint is the midpoint of the images, because a translation preserves midpoints (indeed all distances and ratios along a segment).

Q15. The combined vector is the sum $u+v+w=\begin{bmatrix}2\\-1\end{bmatrix}+\begin{bmatrix}-5\\3\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}$. The image of the origin is $\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}$, that is $(-2,3)$.

Q16. The net vector is $s+r$, so $r=\begin{bmatrix}-1\\-2\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}-5\\-8\end{bmatrix}$. Under the net translation the image of $(3,3)$ is $\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}-1\\-2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$, that is $(2,1)$.

Q17. Adding $t=\begin{bmatrix}1\\-4\end{bmatrix}$ to each vertex: $A'(-1,-6)$, $B'(4,-6)$, $C'(4,-1)$, $D'(-1,-1)$. The original square has side length 5 (for example $|AB|=3-(-2)=5$), so its perimeter is $4\times 5=20$; since a translation preserves length, the image is a congruent square and its perimeter is also 20.

Q18. (a) $t_1=x_P'-x_P=\begin{bmatrix}2\\-1\end{bmatrix}-\begin{bmatrix}-3\\5\end{bmatrix}=\begin{bmatrix}5\\-6\end{bmatrix}$, and $t_2=x_{P''}-x_P'=\begin{bmatrix}-4\\-4\end{bmatrix}-\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}-6\\-3\end{bmatrix}$. (b) The single vector from P to P'' is $\begin{bmatrix}-4\\-4\end{bmatrix}-\begin{bmatrix}-3\\5\end{bmatrix}=\begin{bmatrix}-1\\-9\end{bmatrix}$. (c) $t_1+t_2=\begin{bmatrix}5\\-6\end{bmatrix}+\begin{bmatrix}-6\\-3\end{bmatrix}=\begin{bmatrix}-1\\-9\end{bmatrix}$, which equals the single vector from (b), as required.

Chapter 9 Transformations of the plane — 9B Rotations and reflections as linear transformations

Theory

In 9A each point $P(x, y)$ of the plane was identified with its **position vector**, the column $\begin{bmatrix} x \\ y \end{bmatrix}$, and a **translation** was carried out by adding a fixed column vector. This section studies a different family of maps — the **linear transformations** — which turn out to be exactly the maps performed by *matrix multiplication*. Rotations about the origin and reflections in a line through the origin are the two most important examples, and both have simple 2×2 matrices.

Linear transformations.

A **transformation** of the plane is a rule T that assigns to each vector v an **image** vector $T(v)$. The transformation is **linear** if it respects vector addition and scalar multiplication: for all vectors u, v and every real number k,

$$T(u+v)=T(u)+T(v), T(kv)=kT(v).$$

Two consequences follow at once. Taking $k=0$ gives $T(0)=0$, so a linear transformation **fixes the origin**. Second, a straight line is the set of points $a+td$ for real t, where the direction $d \neq 0$, and the two rules give

$$T(a+td)=T(a)+tT(d),$$

which is again a straight line — through $T(a)$ in direction $T(d)$ — **provided** $T(d) \neq 0$. That proviso is genuinely needed: the linear map with matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ sends every point of the line $x=1$ to the single point $(1, 0)$, collapsing the line rather than mapping it to a line. Every transformation in this section does send non-zero vectors to non-zero vectors — a rotation and a reflection leave lengths unchanged, and a scaling $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ does so whenever $a \neq 0$ and $b \neq 0$ — so for all of them a line maps to a line, a line segment maps to a line segment, and hence a polygon maps to the polygon whose vertices are the images of the original vertices.

The matrix of a linear transformation.

Write the **standard basis vectors**

$$i=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, j=\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Every vector decomposes as $\begin{bmatrix} x \\ y \end{bmatrix}=xi+yj$. Applying a linear T and using the two rules,

$$T\begin{bmatrix} x \\ y \end{bmatrix}=T(xi+yj)=xT(i)+yT(j).$$

So T is completely determined by the two image vectors $T(i)$ and $T(j)$. Collecting these as the **columns** of a 2×2 matrix

$$A=\begin{bmatrix} T(i) & T(j) \end{bmatrix},$$

the combination $xT(i)+yT(j)$ is precisely the matrix product $A\begin{bmatrix} x \\ y \end{bmatrix}$. This proves the central fact of the section:

Every linear transformation of the plane can be written as $T(v)=Av$ for a unique 2×2 matrix A, and the columns of A are the images of the basis vectors i and j.

Reading off the images of i and j and writing them as columns is the tool used to build every matrix below.

Conversely, to find the **image of a point** under a known matrix A, simply multiply: $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$. (Note that a translation $v \mapsto v+b$ with $b \neq 0$ moves the origin, so it is *not* linear and cannot be written as Av; translations were handled separately in 9A.)

Rotations about the origin.

Let R_θ denote an **anticlockwise rotation about the origin** through angle θ (a clockwise rotation uses a negative angle). To find its matrix, rotate the basis vectors. The vector i points along the positive x-axis, so rotating it through θ carries its tip to $(\cos\theta, \sin\theta)$:

$$R_\theta(i) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

The vector j points at 90°, so rotating it through θ carries it to angle $90^\circ + \theta$; using $\cos(90^\circ + \theta) = -\sin\theta$ and $\sin(90^\circ + \theta) = \cos\theta$,

$$R_\theta(j) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Placing these as columns gives the **rotation matrix**

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The special cases used most often follow by substituting exact values:

θ	90°	180°	270°	45°	60°
R_θ	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Reflections in a line through the origin.

Take first the three simplest mirrors. Reflection in the x**-axis** keeps x and negates y, sending $i \mapsto i$ and $j \mapsto -j$, so its matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Reflection in the y**-axis** negates x: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Reflection in the line $y = x$ swaps the two coordinates, $i \mapsto j$ and $j \mapsto i$, giving $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; and reflection in $y = -x$ gives $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

All of these are special cases of one formula. Let ℓ be the line through the origin making an **angle α with the positive x-axis**, that is, $y = (\tan\alpha)x$. The key geometric fact is that reflection in ℓ sends a point at polar angle ϕ (at any distance from O) to the point at the *same distance* and polar angle $2\alpha - \phi$, because the mirror line bisects the angle between a point and its image. Applying this to the basis vectors: i is at angle $\phi = 0^\circ$, so its image is at angle 2α,

$$i \mapsto \begin{bmatrix} \cos 2\alpha \\ \sin 2\alpha \end{bmatrix};$$

and j is at angle $\phi = 90^\circ$, so its image is at angle $2\alpha - 90^\circ$, where $\cos(2\alpha - 90^\circ) = \sin 2\alpha$ and $\sin(2\alpha - 90^\circ) = -\cos 2\alpha$, giving

$$j \mapsto \begin{bmatrix} \sin 2\alpha \\ -\cos 2\alpha \end{bmatrix}.$$

The columns give the **reflection matrix**

$$M_\alpha = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

Substituting $\alpha = 0^\circ$, 90° and 45° recovers the x-axis, y-axis and $y = x$ matrices above — a useful check on the formula.

Dilations (scalings).

For completeness, an **enlargement** (or dilation) from the origin by factor k multiplies every position vector by k; since $i \mapsto k\,i$ and $j \mapsto k\,j$, its matrix is $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$. More generally a scaling by factor a parallel to the x-axis and b parallel to the y-axis has matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. These are linear too, and are combined with rotations and reflections in the next section.

Images of shapes.

Because each of the transformations above sends line segments to line segments, the image of a polygon is found by transforming each vertex and joining the images in the same order. The whole shape can be transformed in one step by writing the vertices as the columns of a matrix and multiplying on the left by A: if the vertices are $v_1, \ldots, v_n$, then $A\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ has the image vertices as its columns.

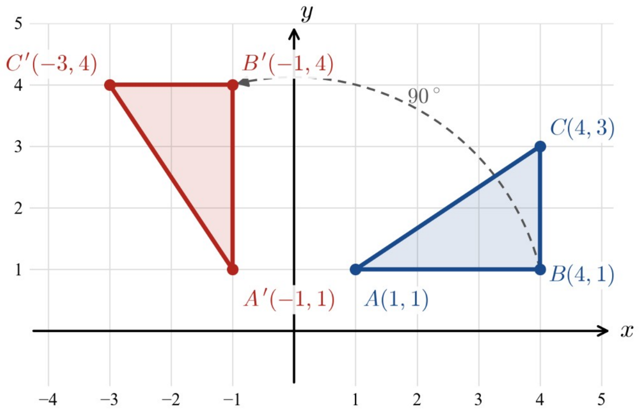
Worked examples

Example 1 — scaffolded

Consider the anticlockwise rotation of 90° about the origin. (a) Write down its matrix. (b) Find the image of the point $P(5,2)$. (c) Find the images of the vertices of the triangle $A(1,1)$, $B(4,1)$, $C(4,3)$, and describe the image.

Working	Comment
$R = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$	Substitute $\cos 90^\circ = 0$ and $\sin 90^\circ = 1$ into R_θ.
$R\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0(5) - 1(2) \\ 1(5) + 0(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$	So $P' = (-2, 5)$. A 90° turn sends (x, y) to $(-y, x)$.
$A' = R\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$	Image of $A(1,1)$.
$B' = R\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $C' = R\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$	Images of $B(4,1)$ and $C(4,3)$.

The image is the triangle $A'(-1,1)$, $B'(-1,4)$, $C'(-3,4)$ — the original turned a quarter-turn anticlockwise about O.

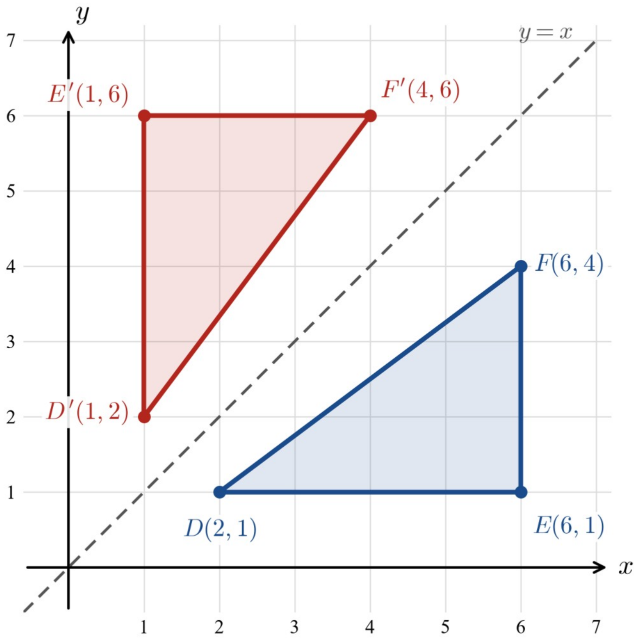


Example 2 — scaffolded

Reflect in lines through the origin. (a) Reflect $P(6,-2)$ in the x-axis. (b) Reflect $Q(-3,7)$ in the y-axis. (c) Reflect $Q(-3,7)$ in the line $y=x$. (d) Reflect the triangle $D(2,1)$, $E(6,1)$, $F(6,4)$ in the line $y=x$.

Working	Comment
$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}6\\-2\end{bmatrix}=\begin{bmatrix}6\\2\end{bmatrix}$.	x-axis reflection keeps x, negates y: $P'=(6,2)$.
$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-3\\7\end{bmatrix}=\begin{bmatrix}3\\7\end{bmatrix}$.	y-axis reflection negates x: $Q'=(3,7)$.
$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}-3\\7\end{bmatrix}=\begin{bmatrix}7\\-3\end{bmatrix}$.	Reflection in $y=x$ swaps coordinates: $Q''=(7,-3)$.
$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2 & 6 & 6\\1 & 1 & 4\end{bmatrix}=\begin{bmatrix}1 & 1 & 4\\2 & 6 & 6\end{bmatrix}$.	Transform all vertices at once: $D'=(1,2)$, $E'=(1,6)$, $F'=(4,6)$.

The reflected triangle $D'(1,2)$, $E'(1,6)$, $F'(4,6)$ is the mirror image of DEF in the line $y=x$; each point and its image are equidistant from that line.



Example 3 — unscaffolded

A linear transformation rotates the plane 120° anticlockwise about the origin. By considering the images of the basis vectors, find its matrix, and hence find the image of the point $(4,-2)$.

Rotating i, which lies at angle 0°, through 120° carries it to angle 120°, so $i \mapsto (\cos 120^\circ, \sin 120^\circ) = (-1/2, \sqrt{3}/2)$. Rotating j, at angle 90°, carries it to angle 210°, so $j \mapsto (\cos 210^\circ, \sin 210^\circ) = (-\sqrt{3}/2, -1/2)$. Writing these images as columns,

$$R_{120^\circ} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

which agrees with the general form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ at $\theta = 120^\circ$. Applying it to $(4, -2)$,

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(4) - \frac{\sqrt{3}}{2}(-2) \\ \frac{\sqrt{3}}{2}(4) - \frac{1}{2}(-2) \end{bmatrix} = \begin{bmatrix} -2+\sqrt{3} \\ 2\sqrt{3}+1 \end{bmatrix}.$$

$\therefore$ the matrix is $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ and the image of $(4, -2)$ is $(-2+\sqrt{3}, 1+2\sqrt{3})$.

Example 4 — unscaffolded

The line ℓ passes through the origin at 30° to the positive x-axis, so ℓ has equation $y = \frac{1}{\sqrt{3}}x$. Find the matrix of reflection in ℓ, use it to find the image of the point $(\sqrt{3}, 3)$, and verify the image geometrically.

Here $\alpha = 30^\circ$, so $2\alpha = 60^\circ$ with $\cos 60^\circ = 1/2$ and $\sin 60^\circ = \sqrt{3}/2$. The reflection matrix is

$$M_\alpha = \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Applying it to $(\sqrt{3}, 3)$,

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} \\ \frac{3}{2} - \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} \\ 0 \end{bmatrix}.$$

Geometrically, $(\sqrt{3}, 3)$ is at distance $\sqrt{(\sqrt{3})^2 + 3^2} = \sqrt{12} = 2\sqrt{3}$ from O and at polar angle $\arctan(3/\sqrt{3}) = \arctan\sqrt{3} = 60^\circ$. Reflection in ℓ (at 30°) sends polar angle 60° to $2(30^\circ) - 60^\circ = 0^\circ$, i.e. onto the positive x-axis at the same distance $2\sqrt{3}$ — the point $(2\sqrt{3}, 0)$, exactly as the matrix gives.

$\therefore$ the reflection matrix is $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ and the image of $(\sqrt{3}, 3)$ is $(2\sqrt{3}, 0)$.

Practice questions

Scaffolded

Q1. Consider the anticlockwise rotation of 90° about the origin. (a) Write down the rotation matrix R. (b) Find the image of the point $A(3,2)$. (c) Find the image of the point $B(-4,1)$. (d) Hence state the image of the triangle with vertices $O(0,0)$, $A(3,2)$, $B(-4,1)$.

Q2. Reflect points in lines through the origin. (a) Write the matrix for reflection in the x-axis and use it to reflect $P(-5,3)$. (b) Write the matrix for reflection in the y-axis and use it to reflect $Q(8,-4)$. (c) Write the matrix for reflection in the line $y=x$ and use it to reflect $Q(8,-4)$.

Q3. Consider the rotation of 180° about the origin. (a) Find the image of $i=\begin{bmatrix}1\\0\end{bmatrix}$ under this rotation. (b) Find the image of $j=\begin{bmatrix}0\\1\end{bmatrix}$. (c) Using (a) and (b) as columns, write down the rotation matrix. (d) Find the image of the point $(5,-8)$.

Q4. Consider the anticlockwise rotation of 45° about the origin. (a) Write the rotation matrix, using exact values. (b) Find the image of the point $(\sqrt{2},\sqrt{2})$. (c) Find the image of the point $(3,-1)$.

Q5. The line ℓ passes through the origin at 60° to the positive x-axis, so ℓ has equation $y=\sqrt{3}x$. (a) State the value of 2α and write the matrix of reflection in ℓ. (b) Find the image of $(2,0)$. (c) Find the image of $(0,4)$. (d) Find the image of $(\sqrt{3},1)$.

Q6. The triangle $K(1,4)$, $L(5,2)$, $M(3,6)$ is reflected in the line $y=x$. (a) Write the matrix of reflection in $y=x$. (b) Find the image of K. (c) Find the image of L. (d) Find the image of M, and state the image triangle.

Unscaffolded

Q7. Find the image of $(7,-3)$ under a 90° anticlockwise rotation about the origin.

Q8. Find the image of $(-2,-6)$ under a 270° anticlockwise rotation about the origin.

Q9. Find the image of $(5,9)$ under reflection in the y-axis.

Q10. Find the image of $(8,-3)$ under reflection in the line $y=-x$.

Q11. Find the exact image of $(4,2)$ under a 60° anticlockwise rotation about the origin.

Q12. A dilation from the origin has factor 3, with matrix $\begin{bmatrix}3&0\\0&3\end{bmatrix}$. Find the image of the triangle $(1,-2)$, $(4,0)$, $(2,3)$.

Q13. By considering the images of the basis vectors, find the matrix of the 135° anticlockwise rotation about the origin, and hence find the image of $(-\sqrt{2},\sqrt{2})$.

Q14. The line ℓ through the origin makes an angle of 120° with the positive x-axis, so ℓ has equation $y=-\sqrt{3}x$. Find the matrix of reflection in ℓ and the image of the point $(0,2)$.

Q15. The square $W(2,2)$, $X(6,2)$, $Y(6,6)$, $Z(2,6)$ is rotated 90° anticlockwise about the origin. Find the image of each vertex.

Q16. Reflect the triangle $(-1,3)$, $(2,5)$, $(4,-2)$ in the x-axis.

Q17. Find the exact image of $(2,6)$ under a 30° anticlockwise rotation about the origin.

Q18. A linear transformation T maps $i \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $j \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (a) Write the matrix of T. (b) Describe T as a single rotation about the origin. (c) Find the image of the triangle $(2,5)$, $(6,5)$, $(6,8)$.

Answers

Q1. (a) $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$; (b) $(-2,3)$; (c) $(-1,-4)$; (d) $O(0,0)$, $(-2,3)$, $(-1,-4)$. **Q2.** (a) $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$, $(-5,-3)$; (b) $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$, $(-8,-4)$; (c) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, $(-4,8)$. **Q3.** (a) $(-1,0)$; (b) $(0,-1)$; (c) $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$; (d) $(-5,8)$. **Q4.** (a) $\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$; (b) $(0,2)$; (c) $(2\sqrt{2},\sqrt{2})$. **Q5.** (a) $2\alpha=120^\circ$, $\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$; (b) $(-1,\sqrt{3})$; (c) $(2\sqrt{3},2)$; (d) $(0,2)$.
Q6. (a) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$; (b) $(4,1)$; (c) $(2,5)$; (d) $(6,3)$; image triangle $(4,1)$, $(2,5)$, $(6,3)$. **Q7.** $(3,7)$. **Q8.** $(-6,2)$.

Q9. $(-5,9)$. **Q10.** $(3,-8)$. **Q11.** $(2-\sqrt{3},1+2\sqrt{3})$. **Q12.** $(3,-6)$, $(12,0)$, $(6,9)$. **Q13.** $\begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}$; image $(0,-2)$. **Q14.** $\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$; image $(-\sqrt{3},1)$. **Q15.** $(-2,2)$, $(-2,6)$, $(-6,6)$, $(-6,2)$. **Q16.** $(-1,-3)$, $(2,-5)$, $(4,2)$. **Q17.** $(\sqrt{3}-3,1+3\sqrt{3})$. **Q18.** (a) $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$; (b) a 270° anticlockwise (equivalently 90° clockwise) rotation about O; (c) $(5,-2)$, $(5,-6)$, $(8,-6)$.

Fully-worked solutions

Q1. With $\theta=90^\circ$, $R=\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\\sin 90^\circ & \cos 90^\circ\end{bmatrix}=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$, which sends (x,y) to $(-y,x)$. So $A(3,2)\mapsto(-2,3)$ and $B(-4,1)\mapsto(-1,-4)$. The origin is fixed, so the triangle OAB maps to the triangle with vertices $O(0,0)$, $(-2,3)$, $(-1,-4)$.

Q2. (a) $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-5\\3\end{bmatrix}=\begin{bmatrix}-5\\-3\end{bmatrix}$, so $P\mapsto(-5,-3)$. (b) $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}8\\-4\end{bmatrix}=\begin{bmatrix}-8\\-4\end{bmatrix}$, so $Q\mapsto(-8,-4)$. (c) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}8\\-4\end{bmatrix}=\begin{bmatrix}-4\\8\end{bmatrix}$, so $Q\mapsto(-4,8)$.

Q3. With $\theta=180^\circ$, $\cos 180^\circ=-1$ and $\sin 180^\circ=0$. (a) $i=(1,0)\mapsto(\cos 180^\circ,\sin 180^\circ)=(-1,0)$. (b) $j=(0,1)$ is at 90° and rotates to 270°, giving $(0,-1)$. (c) Writing these as columns, $R=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$. (d) $R\begin{bmatrix}5\\-8\end{bmatrix}=\begin{bmatrix}-5\\8\end{bmatrix}$, so $(5,-8)\mapsto(-5,8)$.

Q4. (a) With $\cos 45^\circ = \sin 45^\circ = \sqrt{2}/2$, $R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$. (b) $R\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}(\sqrt{2}) - \frac{\sqrt{2}}{2}(\sqrt{2}) \\ \frac{\sqrt{2}}{2}(\sqrt{2}) + \frac{\sqrt{2}}{2}(\sqrt{2}) \end{bmatrix} = \begin{bmatrix} 1-1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, so the image is $(0,2)$. (c) $R\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}(3) - \frac{\sqrt{2}}{2}(-1) \\ \frac{\sqrt{2}}{2}(3) + \frac{\sqrt{2}}{2}(-1) \end{bmatrix} = \begin{bmatrix} \frac{4\sqrt{2}}{2} \\ \frac{2\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \end{bmatrix}$, so the image is $(2\sqrt{2}, \sqrt{2})$.

Q5. Here $\alpha = 60^\circ$, so (a) $2\alpha = 120^\circ$, with $\cos 120^\circ = -1/2$ and $\sin 120^\circ = \sqrt{3}/2$, giving $M = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. (b) $M\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$, image $(-1, \sqrt{3})$. (c) $M\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} \\ 2 \end{bmatrix}$, image $(2\sqrt{3}, 2)$. (d) $M\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \\ \frac{3}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, image $(0,2)$.

Q6. Reflection in $y = x$ has matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which swaps coordinates. Hence $K(1,4) \mapsto (4,1)$, $L(5,2) \mapsto (2,5)$ and $M(3,6) \mapsto (6,3)$; the image triangle has vertices $(4,1)$, $(2,5)$, $(6,3)$.

Q7. A 90° anticlockwise rotation sends (x, y) to $(-y, x)$, so $(7, -3) \mapsto (3, 7)$.

Q8. With $\theta = 270^\circ$, $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, sending (x, y) to $(y, -x)$. Then $(-2, -6) \mapsto (-6, 2)$.

Q9. Reflection in the y-axis, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, negates x: $(5,9) \mapsto (-5,9)$.

Q10. Reflection in $y = -x$ has matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, sending (x, y) to $(-y, -x)$. Thus $(8, -3) \mapsto (3, -8)$.

Q11. With $\theta = 60^\circ$, $R = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Then

$$R\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4) - \frac{\sqrt{3}}{2}(2) \\ \frac{\sqrt{3}}{2}(4) + \frac{1}{2}(2) \end{bmatrix} = \begin{bmatrix} 2 - \sqrt{3} \\ 2\sqrt{3} + 1 \end{bmatrix},$$

so the image is $(2 - \sqrt{3}, 1 + 2\sqrt{3})$.

Q12. The dilation multiplies each vertex by 3: $(1, -2) \mapsto (3, -6)$, $(4, 0) \mapsto (12, 0)$ and $(2, 3) \mapsto (6, 9)$.

Q13. With $\theta=135^\circ$, $\cos 135^\circ=-\sqrt{2}/2$ and $\sin 135^\circ=\sqrt{2}/2$. So $i\mapsto\left(-\sqrt{2}/2,\sqrt{2}/2\right)$ and j (at 90°) rotates to 225°, giving $\left(-\sqrt{2}/2,-\sqrt{2}/2\right)$. As columns, $R=\begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}$. Then

$$R\begin{bmatrix}-\sqrt{2}\\ \sqrt{2}\end{bmatrix}=\begin{bmatrix}-\frac{\sqrt{2}}{2}\left(-\sqrt{2}\right)-\frac{\sqrt{2}}{2}\left(\sqrt{2}\right)\\ \frac{\sqrt{2}}{2}\left(-\sqrt{2}\right)-\frac{\sqrt{2}}{2}\left(\sqrt{2}\right)\end{bmatrix}=\begin{bmatrix}1-1\\ -1-1\end{bmatrix}=\begin{bmatrix}0\\ -2\end{bmatrix},$$

so the image is $\left(0,-2\right)$.

Q14. Here $\alpha=120^\circ$, so $2\alpha=240^\circ$ with $\cos 240^\circ=-1/2$ and $\sin 240^\circ=-\sqrt{3}/2$. The reflection matrix is $M=\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$, and

$$M\begin{bmatrix}0\\ 2\end{bmatrix}=\begin{bmatrix}-\frac{\sqrt{3}}{2}\left(2\right)\\ \frac{1}{2}\left(2\right)\end{bmatrix}=\begin{bmatrix}-\sqrt{3}\\ 1\end{bmatrix},$$

so the image is $\left(-\sqrt{3},1\right)$.

Q15. A 90° anticlockwise rotation sends $\left(x,y\right)$ to $\left(-y,x\right)$. Applying this to each vertex: $W\left(2,2\right)\mapsto\left(-2,2\right)$, $X\left(6,2\right)\mapsto\left(-2,6\right)$, $Y\left(6,6\right)\mapsto\left(-6,6\right)$ and $Z\left(2,6\right)\mapsto\left(-6,2\right)$.

Q16. Reflection in the x-axis negates y: $\left(-1,3\right)\mapsto\left(-1,-3\right)$, $\left(2,5\right)\mapsto\left(2,-5\right)$ and $\left(4,-2\right)\mapsto\left(4,2\right)$.

Q17. With $\theta=30^\circ$, $R=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$. Then

$$R\begin{bmatrix}2\\ 6\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2}\left(2\right)-\frac{1}{2}\left(6\right)\\ \frac{1}{2}\left(2\right)+\frac{\sqrt{3}}{2}\left(6\right)\end{bmatrix}=\begin{bmatrix}\sqrt{3}-3\\ 1+3\sqrt{3}\end{bmatrix},$$

so the image is $\left(\sqrt{3}-3,1+3\sqrt{3}\right)$.

Q18. (a) The images of i and j are the columns, so $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$. (b) Comparing with $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ gives $\cos\theta=0$ and $\sin\theta=-1$, so $\theta=270^\circ$: a 270° anticlockwise (equivalently 90° clockwise) rotation about O, sending $\left(x,y\right)$ to $\left(y,-x\right)$. (c) $\left(2,5\right)\mapsto\left(5,-2\right)$, $\left(6,5\right)\mapsto\left(5,-6\right)$ and $\left(6,8\right)\mapsto\left(8,-6\right)$.

Theory

A **linear transformation** of the plane is a rule $x \mapsto M\,x$ in which each point $x = \begin{bmatrix} x \\ y \end{bmatrix}$ is sent to the point $M\,x$ for a fixed 2×2 **transformation matrix** M. As established in the previous two sections, such a map always fixes the origin, and provided $det\,M \neq 0$ (the determinant is defined below) it sends straight lines to straight lines; when $det\,M = 0$ some lines are crushed to a single point, a case treated in its own paragraph below. The reflections, rotations and dilations met there each have their own matrix. This section combines those transformations (**composition**), reverses them (**inverses**), measures how they change area (the **determinant**), and asks which points and lines are left unmoved (**invariance**). Throughout, the standard building blocks are

$$\text{reflect in the } x\text{-axis} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{reflect in } y = x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{rotate } \theta \text{ about } O \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

together with the **dilation** $\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$ of factor k_x parallel to the x-axis and k_y parallel to the y-axis, and the **identity** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which leaves every point where it is.

Composition of transformations.

To apply one transformation and then another is to **compose** them. Suppose a point x is first transformed by the matrix A, giving $A\,x$, and this image is then transformed by B. The final image is

$$B(A\,x) = (B\,A)\,x,$$

using the associativity of matrix multiplication. So the single matrix of the composite "B **after** A" is the **product** $B\,A$. The order of writing is the reverse of the order of doing: the matrix applied *first* sits on the *right*, next to x, and the matrix applied *last* sits on the *left*. Composition of linear transformations is therefore exactly matrix multiplication.

Order matters.

Matrix multiplication is **not commutative**: in general $B\,A \neq A\,B$. Because composing transformations *is* multiplying matrices, the order in which two transformations are carried out generally changes the result. Reflecting and then rotating need not give the same map as rotating and then reflecting. This is a genuinely geometric fact, not an accident of notation, and every composite below must be built with the first-applied matrix on the right.

The inverse transformation.

A transformation "undoes" another if performing one after the other returns every point to its start. Since the identity I is the transformation that fixes every point, the **inverse** of M is the matrix M^{-1} with

$$M^{-1} M = M\,M^{-1} = I.$$

For a 2×2 matrix the inverse is

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow M^{-1} = \frac{1}{a\,d - b\,c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided the number $a\,d - b\,c$ is not zero. This number is the **determinant** of M, written $det\,M = a\,d - b\,c$. A transformation can be undone **exactly when** $det\,M \neq 0$; the inverse transformation then has matrix M^{-1}, and to recover the original point (the **pre-image**) of an image y we solve $M\,x = y$ as $x = M^{-1}\,y$.

Undoing a composite.

To reverse the composite "B after A", the last-done transformation must be undone first, so the inverse is taken **in the reverse order**:

$$(BA)^{-1} = A^{-1}B^{-1}.$$

This mirrors everyday experience — to undo putting on socks then shoes, one removes shoes first, then socks.

Composing with a translation.

A translation by $t \neq 0$ moves the origin, so it is *not* linear and cannot be written as a 2×2 matrix; but it can still be composed with one. Applying M first and then translating by t gives the rule

$$x \mapsto Mx + t,$$

while translating first and then applying M gives $x \mapsto M(x+t) = Mx + Mt$ — a different rule unless $Mt = t$, so order matters here too. When $det\, M \neq 0$ the first composite is undone by reversing both steps in the opposite order: subtract t, then apply M^{-1}, so that the point x carried to a given image x' is

$$x = M^{-1}(x' - t).$$

The image of a curve or graph.

A curve is simply a set of points, so a transformation carries it to another curve, the **image** of the original. Its equation is found by working *backwards*, because the equation of a curve is a test applied to the coordinates of a point: writing $x' = Mx$ for the image of a general point and assuming $det\, M \neq 0$,

$$x = M^{-1}x'$$

expresses x and y in terms of x' and y'. The point (x', y') lies on the image exactly when its pre-image (x, y) lies on the original curve, so the method is to **substitute these expressions for x and y into the equation of the original curve**, then rename x' as x and y' as y. For a composite with a translation the same substitution is used, with $x = M^{-1}(x' - t)$.

For instance, under $M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ the inverse is $M^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}$, so $x = 1/4\, x'$ and $y = y'$. The parabola $y = x^2$ therefore has image $y' = (1/4\, x')^2$, that is $y = 1/16\, x^2$. Because x and y are replaced by *linear* expressions in x' and y', the image of a straight line is again a straight line, and the image of a curve of a given polynomial degree has the same degree — which is why an invertible linear transformation cannot bend a line or turn a parabola into anything but another parabola.

The determinant as an area scale factor.

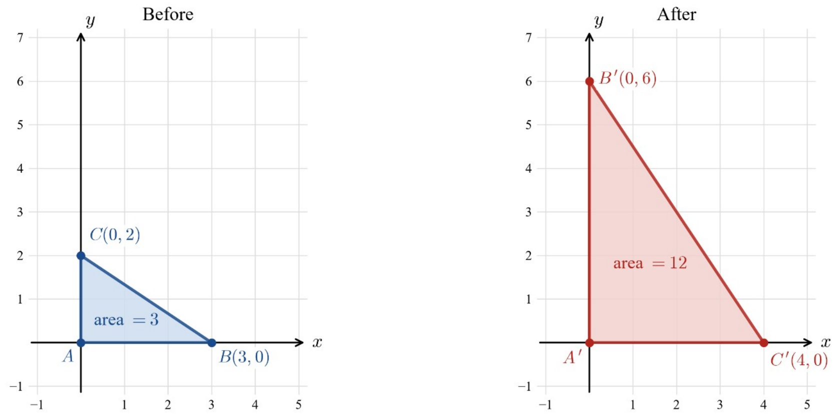
The two columns of M are the images of the basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The unit square built on e_1 and e_2 (area 1) is therefore mapped to the parallelogram built on the columns of M, whose area works out to be exactly $|det\, M|$. Since any region can be tiled by many small squares, and each is scaled the same way, a linear transformation multiplies **every** area by the same factor:

$$\text{area of image} = |det\, M| \times \text{area of original}.$$

Because the area factors of successive transformations multiply, and composition is multiplication of matrices, the determinant is **multiplicative**:

$$det(BA) = det(B)\, det(A).$$

Composite $M=[0\ 2;\ 2\ 0]$ (reflect in $y=x$, then enlarge by 2): $\det M=-4$, so $|\det M|=4$ — area $\times 4$, orientation reversed



For example, reflecting in the line $y=x$ and then enlarging by a factor of 2 from the origin gives the composite

$$M=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=\begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}, det\,M=-4.$$

The figure shows a triangle of area 3 carried to an image of area 12: the area is multiplied by $|det\,M|=4$, exactly as claimed.

Sign of the determinant and orientation.

The *sign* of the determinant records what the transformation does to **orientation**. If the vertices of the unit square are listed anticlockwise, their images run anticlockwise when $det\,M>0$ and **clockwise** when $det\,M<0$. A positive determinant therefore **preserves orientation**, while a negative determinant **reverses** it — the plane has been flipped over, so the transformation contains a reflection. In the figure above $det\,M=-4<0$, and indeed the vertex order of the image triangle is reversed.

A zero determinant: collapse to a line.

When $det\,M=0$ the two columns of M are parallel, so the image parallelogram is squashed perfectly flat and has area 0. Every point of the plane is then sent onto a single **line through the origin** (or, if M is the zero matrix, onto the origin alone). Such a transformation destroys area and **cannot be undone**: no inverse exists, consistent with division by $det\,M=0$ being impossible in the inverse formula.

Invariant points.

A point x is an **invariant point** (or **fixed point**) of M if it is left exactly where it is:

$$M\,x=x\,,\text{equivalently}\,(M-I)\,x=0\,.$$

The origin always satisfies this, so a linear transformation has the origin as an invariant point no matter what. Whether there are others is decided by $det\,(M-I)$: if $det\,(M-I)\neq 0$ the origin is the **only** invariant point, while if $det\,(M-I)=0$ there are infinitely many others. In that second case there are two possibilities. If $M=I$ then every point of the plane is invariant, since the identity moves nothing. For every other M with $det\,(M-I)=0$ the invariant points form a whole **line of invariant points** through the origin — for instance the mirror line of a reflection, every point of which stays put.

Invariant lines.

A line ℓ is an **invariant line** of M if the transformation maps it **into itself** as a whole set: the image of every point of ℓ again lies on ℓ, even though individual points may slide along it. This is weaker than a *line of invariant points*, where each point is fixed one by one. For a line through the origin with direction d, the line is invariant precisely when $M\,d$ points along d, that is

$$M\,d=\lambda\,d$$

for some scalar λ: the points of the line are stretched by the factor λ but never leave it. When $\lambda = 1$ the whole line is fixed pointwise; when $\lambda \neq 1$ every point of the line except the origin genuinely moves, and in the extreme case $\lambda = 0$ the whole line is collapsed onto the origin. To test the lines $y = m x$, require $M\begin{bmatrix}1\\m\end{bmatrix}$ to be parallel to $\begin{bmatrix}1\\m\end{bmatrix}$, and check the vertical line $x = 0$ separately.

Worked examples

Example 1 — scaffolded

Let $A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ be reflection in the line $y = x$ and $B = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$ the dilation of factor 3 parallel to the x-axis. Find the matrix of (a) "B after A" and (b) "A after B"; (c) find the image of $(-2, 5)$ under each; and (d) comment on the determinants.

Working	Comment
"B after A" has matrix $BA = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 3\\1 & 0\end{bmatrix}$.	First-applied matrix A on the right, last-applied B on the left.
"A after B" has matrix $AB = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\3 & 0\end{bmatrix}$.	Reversing the order swaps the product; $AB \neq BA$.
$BA\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}0(-2)+3(5)\\1(-2)+0(5)\end{bmatrix} = \begin{bmatrix}15\\-2\end{bmatrix}$.	Apply the composite as a matrix–vector product.
$AB\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}0(-2)+1(5)\\3(-2)+0(5)\end{bmatrix} = \begin{bmatrix}5\\-6\end{bmatrix}$.	A different image: the order of the two transformations matters.
$det\,A = -1$, $det\,B = 3$, so $det(BA) = det(AB) = (3)(-1) = -3$.	Each composite scales area by 3 and reverses orientation.

Example 2 — scaffolded

A linear transformation has matrix $M = \begin{bmatrix}4 & 2\\1 & 3\end{bmatrix}$. (a) Find $det\,M$ and the factor by which areas are scaled. (b) Find M^{-1}. (c) A triangle has area 5; find the area of its image. (d) A point maps to $(10, 5)$; find its pre-image. (e) Find the equation of the image of the line $x + y = 4$.

Working	Comment
$det\,M = (4)(3) - (2)(1) = 10$.	$det\begin{bmatrix}a & b\\c & d\end{bmatrix} = ad - bc$.
Areas are scaled by $\lvert det\,M\rvert = 10$; since $det\,M > 0$, orientation is preserved.	The area factor is the modulus of the determinant.
$M^{-1} = \frac{1}{10}\begin{bmatrix}3 & -2\\-1 & 4\end{bmatrix}$.	Swap a, d, negate b, c, divide by $det\,M$.
Image area $= 10 \times 5 = 50$.	Multiply the original area by the area factor.
Pre-image $= M^{-1}\begin{bmatrix}10\\5\end{bmatrix} = \frac{1}{10}\begin{bmatrix}3(10)-2(5)\\-1(10)+4(5)\end{bmatrix} = \frac{1}{10}\begin{bmatrix}20\\10\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$.	Solve $Mx = y$ as $x = M^{-1}y$. Check: $M\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix}$.
From $x = M^{-1}x'$, $x = 1/10(3x' - 2y')$ and $y = 1/10(-x' + 4y')$.	Work backwards: express the pre-image coordinates in terms of the image coordinates.
Substituting into $x + y = 4$:	(x', y') lies on the image exactly when its pre-image

Working	Comment
$1/10(3x'-2y')+1/10(-x'+4y')=4$, that is $1/10(2x'+2y')=4$, so $2x'+2y'=40$ and the image is $x+y=20$.	lies on $x+y=4$; finally rename x' as x and y' as y. Check: $(4,0)$ maps to $(16,4)$, and $16+4=20$.

Example 3 — unscaffolded

Find all invariant points of the transformation $M=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$, and find its invariant lines through the origin.

An invariant point satisfies $Mx=x$. Writing this out,

$$\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}\Rightarrow\begin{cases}2x=x\\ x+y=y\end{cases}$$

Both equations reduce to $x=0$, with y free. Every point of the y-axis is therefore fixed; the invariant points form a whole line, consistent with $det(M-I)=det\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}=0$.

For an invariant line $y=mx$ through the origin, the image of the direction $\begin{bmatrix}1\\ m\end{bmatrix}$ must be parallel to it. Here $M\begin{bmatrix}1\\ m\end{bmatrix}=\begin{bmatrix}2\\ 1+m\end{bmatrix}$, and parallelism requires $\frac{1+m}{2}=m$, giving $1+m=2m$, so $m=1$: the line $y=x$ is invariant (its points are stretched by a factor of 2 but stay on it). The vertical line $x=0$ must be checked separately: $M\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$ is parallel to $\begin{bmatrix}0\\ 1\end{bmatrix}$, so the y-axis is invariant too (indeed pointwise fixed, as found above).

∴ the invariant points are exactly the points of the y-axis, and the invariant lines through the origin are $x=0$ and $y=x$.

Example 4 — unscaffolded

A point is rotated 90° anticlockwise about the origin and then reflected in the line $y=x$. Find the single matrix of this composite, identify it as a familiar transformation, state what its determinant says about area and orientation, and find its invariant points and lines. Comment on what happens if the two transformations are done in the opposite order.

Write $R=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ for the rotation and $S=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ for the reflection. “Reflect after rotate” places the reflection on the left:

$$SR=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

which is reflection in the x-axis. Its determinant is $det(SR)=(1)(-1)-0=-1$: since $|det|=1$ area is preserved, and since the determinant is negative orientation is reversed — as expected for a reflection.

For invariant points, $SR\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ -y\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}$ forces $-y=y$, so $y=0$: the x-axis is a line of invariant points. Testing a line $y=mx$, the direction $\begin{bmatrix}1\\ m\end{bmatrix}$ maps to $\begin{bmatrix}1\\ -m\end{bmatrix}$, parallel to the original only when $m=0$ (the x-axis); the vertical line $x=0$ maps $\begin{bmatrix}0\\ 1\end{bmatrix}$ to $\begin{bmatrix}0\\ -1\end{bmatrix}$, still on the y-axis, so the y-axis is invariant as well (its points are sent to their negatives).

Doing the transformations in the opposite order gives $RS=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$, reflection in the y-axis — a different result, confirming that order matters.

$\therefore$ rotating 90° anticlockwise and then reflecting in $y=x$ is reflection in the x-axis, with $det=-1$ (area unchanged, orientation reversed); its invariant points form the x-axis and its invariant lines are the x-axis and the y-axis, while the opposite order instead yields reflection in the y-axis.

Practice questions

Scaffolded

Q1. Let $A=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$ (dilation of factor 2 parallel to the x-axis) and $B=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ (rotation of 90° anticlockwise). (a) Find the matrix BA of "B after A". (b) Find the matrix AB of "A after B". (c) State whether $BA=AB$. (d) Find the image of $(3,4)$ under "B after A".

Q2. Let $M=\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}$. (a) Find $det\,M$. (b) Find M^{-1}. (c) Verify that $M^{-1}M=I$. (d) Use M^{-1} to solve $Mx=\begin{bmatrix}17\\ 6\end{bmatrix}$. (e) Find the equation of the image of the line $x+2y=3$ under M.

Q3. A transformation has matrix $M=\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}$. (a) Find $det\,M$. (b) State the factor by which areas are scaled. (c) State whether orientation is preserved or reversed. (d) A rectangle has area 6; find the area of its image.

Q4. Let $M=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$. (a) Write the two equations obtained from $Mx=x$. (b) Solve them. (c) Describe the set of invariant points. (d) State whether the x-axis is a line of invariant points.

Q5. Let $F=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ (reflection in $y=x$) and $G=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$ (reflection in $y=-x$). (a) Find the matrix of "reflect in $y=x$, then reflect in $y=-x$". (b) Identify the resulting single transformation. (c) Find its determinant. (d) State whether orientation is preserved or reversed.

Q6. Let $A=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, and let $C=BA$. (a) Find C. (b) Find $det\,C$. (c) Find C^{-1}. (d) Verify that $C^{-1}=A^{-1}B^{-1}$.

Unscaffolded

Q7. Let $P=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ (reflection in the x-axis) and $Q=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ (reflection in $y=x$). Find PQ and QP, and state the single transformation each represents.

Q8. With $A=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}$, find the image of $(1,-2)$ under "A after B".

Q9. For $M=\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}$, find M^{-1} and hence the point whose image under M is $(1,1)$.

Q10. Let $M=\begin{bmatrix}2 & 6\\ 1 & 3\end{bmatrix}$. Find $det\,M$ and describe what M does to the plane, including where a general point (x,y) is sent.

Q11. Find all invariant points of $M=\begin{bmatrix}3 & 1\\0 & 2\end{bmatrix}$.

Q12. Find all invariant points of $M=\begin{bmatrix}3 & -1\\4 & -1\end{bmatrix}$ and describe them geometrically.

Q13. For the dilation $M=\begin{bmatrix}5 & 0\\0 & 2\end{bmatrix}$, find all invariant lines through the origin, and state whether either is a line of invariant points.

Q14. The parabola $y=x^2$ is transformed by $M=\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$. Find the equation of the image curve, and verify that the image of the point $(2,4)$ lies on it.

Q15. Let $A=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ (enlargement of factor 2) and $B=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ (rotation of 90° anticlockwise), and let $T=BA$. Find the matrix that undoes T, and verify that it equals $A^{-1}B^{-1}$.

Q16. A triangle has vertices $O(0,0)$, $A(3,1)$ and $B(1,4)$, and is transformed by $M=\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}$. Find the area of the original triangle and of its image, and confirm that their ratio equals $|det\,M|$.

Q17. For $M=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (reflection in $y=x$), find its invariant points and all its invariant lines through the origin.

Q18. Let $R=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ be rotation of 90° anticlockwise and $S=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ be reflection in the x-axis. (a) Find the matrix of "reflect in the x-axis after rotating 90° anticlockwise" and identify it as a single reflection. (b) Find the matrix of the reverse-order composite and identify it. (c) State the determinant of each and interpret it (area and orientation). (d) For the composite in (a), find its invariant points and its invariant lines through the origin.

Answers

Q1. (a) $BA=\begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}$; (b) $AB=\begin{bmatrix}0 & -2\\ 1 & 0\end{bmatrix}$; (c) no, $BA\neq AB$; (d) $(-4,6)$. **Q2.** (a) $\det M=1$; (b) $M^{-1}=\begin{bmatrix}2 & -5\\ -1 & 3\end{bmatrix}$; (c) product is I; (d) $x=(4,1)$; (e) $y=3$. **Q3.** (a) 7; (b) factor 7; (c) preserved; (d) 42. **Q4.** (a) $x+2y=x$ and $y=y$; (b) $y=0$; (c) the x-axis; (d) yes. **Q5.** (a) $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$; (b) rotation of 180° about the origin; (c) 1; (d) preserved. **Q6.** (a) $C=\begin{bmatrix}0 & 3\\ -1 & 0\end{bmatrix}$; (b) $\det C=3$; (c) $C^{-1}=\begin{bmatrix}0 & -1\\ 1/3 & 0\end{bmatrix}$; (d) equal, as shown. **Q7.** $PQ=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ (rotation 90° clockwise); $QP=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ (rotation 90° anticlockwise). **Q8.** $AB=\begin{bmatrix}5 & 1\\ 3 & 1\end{bmatrix}$; image $(3,1)$. **Q9.** $M^{-1}=\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}$; pre-image $(-1,2)$. **Q10.** $\det M=0$; M collapses the plane onto the line $y=1/2x$, sending (x,y) to $(x+3y)\begin{bmatrix}2\\ 1\end{bmatrix}$; no inverse exists. **Q11.** Only the origin $(0,0)$. **Q12.** $y=2x$: a line of invariant points through the origin. **Q13.** The x-axis (points scaled by 5) and the y-axis (points scaled by 2); neither is a line of invariant points, since only the origin is fixed. **Q14.** $y=x^2+2x$; the image of $(2,4)$ is $(2,8)$, and $2^2+2(2)=8$. **Q15.** $T=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$, $T^{-1}=\begin{bmatrix}0 & 1/2\\ -1/2 & 0\end{bmatrix}=A^{-1}B^{-1}$. **Q16.** Original area $11/2$; $\det M=5$; image area $55/2$; ratio $5=|\det M|$. **Q17.** Invariant points: the line $y=x$; invariant lines: $y=x$ and $y=-x$. **Q18.** (a) $\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$, reflection in $y=-x$; (b) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, reflection in $y=x$; (c) each has determinant -1 (area preserved, orientation reversed); (d) invariant points: the line $y=-x$; invariant lines: $y=-x$ and $y=x$.

Fully-worked solutions

Q1. "B after A" multiplies with B on the left: $BA=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}$, while $AB=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & -2\\ 1 & 0\end{bmatrix}$. These differ, so $BA\neq AB$. The image of $(3,4)$ under BA is $\begin{bmatrix}0(3)-1(4)\\ 2(3)+0(4)\end{bmatrix}=\begin{bmatrix}-4\\ 6\end{bmatrix}$, that is $(-4,6)$.

Q2. $\det M=(3)(2)-(5)(1)=1$. Since the determinant is 1, $M^{-1}=\begin{bmatrix}2 & -5\\ -1 & 3\end{bmatrix}$. Checking, $M^{-1}M=\begin{bmatrix}2 & -5\\ -1 & 3\end{bmatrix}\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}=\begin{bmatrix}6-5 & 10-10\\ -3+3 & -5+6\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$. Then $x=M^{-1}\begin{bmatrix}17\\ 6\end{bmatrix}=\begin{bmatrix}2(17)-5(6)\\ -1(17)+3(6)\end{bmatrix}=\begin{bmatrix}4\\ 1\end{bmatrix}$, so $x=(4,1)$ (check: $M\begin{bmatrix}4\\ 1\end{bmatrix}=\begin{bmatrix}17\\ 6\end{bmatrix}$). For the image of the line, $x=M^{-1}x'$ gives $x=2x'-5y'$ and $y=-x'+3y'$. Substituting into $x+2y=3$,

$$(2x'-5y')+2(-x'+3y')=3\Rightarrow y'=3,$$

so the image is the horizontal line $y=3$. (Check: $(3,0)$ lies on $x+2y=3$ and maps to $(9,3)$, which lies on $y=3$.)

Q3. $\det M=(4)(2)-(1)(1)=7$, so areas are scaled by the factor $|\det M|=7$. As $\det M>0$, orientation is preserved. A rectangle of area 6 maps to a region of area $7\times 6=42$.

Q4. From $Mx=x$, $\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$ gives $x+2y=x$ and $y=y$. The first reduces to $2y=0$, so $y=0$ with x free. The invariant points are exactly the points $(x,0)$, that is the whole x-axis, so the x-axis is a line of invariant points.

Q5. The second-applied reflection goes on the left: $GF=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$, which sends every point (x,y) to $(-x,-y)$: the rotation of 180° about the origin. Its determinant is $(-1)(-1)-(0)(0)=1$, so area is unchanged and orientation is preserved — as it must be, since each of the two reflections reverses orientation and the two reversals cancel.

Q6. $C=BA=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}=\begin{bmatrix}0&3\\-1&0\end{bmatrix}$, with $\det C=(0)(0)-(3)(-1)=3$. Hence $C^{-1}=\frac{1}{3}\begin{bmatrix}0&-3\\1&0\end{bmatrix}=\begin{bmatrix}0&-1\\1/3&0\end{bmatrix}$. To undo a composite the pieces are reversed: $A^{-1}=\begin{bmatrix}1&0\\0&1/3\end{bmatrix}$ and $B^{-1}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, so $A^{-1}B^{-1}=\begin{bmatrix}1&0\\0&1/3\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}=\begin{bmatrix}0&-1\\1/3&0\end{bmatrix}=C^{-1}$, as required.

Q7. $PQ=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, which matches $\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$ with $\theta=-90^\circ$: rotation of 90° clockwise. Reversing, $QP=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, rotation of 90° anticlockwise. The two orders give opposite rotations.

Q8. "A after B" has matrix $AB=\begin{bmatrix}2&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}=\begin{bmatrix}5&1\\3&1\end{bmatrix}$. Then the image of $(1,-2)$ is $\begin{bmatrix}5(1)+1(-2)\\3(1)+1(-2)\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$, that is $(3,1)$. (Applying B first sends $(1,-2)$ to $(1,1)$, and then A sends $(1,1)$ to $(3,1)$.)

Q9. $\det M=(5)(2)-(3)(3)=1$, so $M^{-1}=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}$. The point whose image is $(1,1)$ is $M^{-1}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2-3\\-3+5\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$, that is $(-1,2)$. (Check: $M(-1,2)=(-5+6,-3+4)=(1,1)$.)

Q10. $\det M=(2)(3)-(6)(1)=0$, so M has no inverse and destroys area. A general point maps to $\begin{bmatrix}2x+6y\\x+3y\end{bmatrix}=(x+3y)\begin{bmatrix}2\\1\end{bmatrix}$, which is always a multiple of $\begin{bmatrix}2\\1\end{bmatrix}$. Every point of the plane is therefore sent onto the single line through the origin with direction $(2,1)$, namely $y=1/2x$.

Q11. $Mx=x$ gives $3x+y=x$ and $2y=y$. The second forces $y=0$, and then $3x+0=x$ gives $2x=0$, so $x=0$. The only invariant point is the origin (consistent with $\det(M-I)=\det\begin{bmatrix}2&1\\0&1\end{bmatrix}=2\neq0$).

Q12. $Mx=x$ gives $3x-y=x$ and $4x-y=y$. The first is $2x=y$ and the second is $4x=2y$, that is $y=2x$ again: the two equations carry the same information, so $y=2x$ with x free. The invariant points are the points $(x,2x)$ — every point of the line $y=2x$ — so that line is a line of invariant points through the origin, and every point off it is moved. (This is consistent with $\det(M-I)=\det\begin{bmatrix}2&-1\\4&-2\end{bmatrix}=-4+4=0$, and $M\neq I$, so the invariant points form a line rather than the whole plane. Check: $M\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$.)

Q13. A line $y=mx$ is invariant when $M\begin{bmatrix}1\\m\end{bmatrix}=\begin{bmatrix}5\\2m\end{bmatrix}$ is parallel to $\begin{bmatrix}1\\m\end{bmatrix}$, requiring $\frac{2m}{5}=m$, so $3m=0$ and $m=0$: the x-axis, whose points are scaled by 5. The vertical line $x=0$ is also invariant, since $M\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$ stays on the y-axis, whose points are scaled by 2. Neither is a line of invariant points, because the scale factors 5 and 2 differ from 1 (only the origin is fixed).

Q14. Here $\det M=(1)(1)-(0)(2)=1\neq 0$, so $M^{-1}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}$. Writing $x'=Mx$ for the image of a general point and working backwards, $x=M^{-1}x'$ gives $x=x'$ and $y=-2x'+y'$. The point (x',y') lies on the image exactly when (x,y) lies on $y=x^2$, that is when

$$-2x'+y'=(x')^2\Rightarrow y'=(x')^2+2x'.$$

Renaming, the image is the parabola $y=x^2+2x$ — still a parabola, as it must be, since x and y have been replaced by linear expressions. Checking the given point: $(2,4)$ lies on $y=x^2$, and its image is $M\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}2\\2(2)+4\end{bmatrix}=\begin{bmatrix}2\\8\end{bmatrix}$; since $2^2+2(2)=8$, the point $(2,8)$ does lie on $y=x^2+2x$.

Q15. $T=BA=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}=\begin{bmatrix}0 & -2\\2 & 0\end{bmatrix}$, with $\det T=(0)(0)-(-2)(2)=4$, so $T^{-1}=\frac{1}{4}\begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}=\begin{bmatrix}0 & 1/2\\-1/2 & 0\end{bmatrix}$. Undoing in reverse order, $A^{-1}=\begin{bmatrix}1/2 & 0\\0 & 1/2\end{bmatrix}$ and $B^{-1}=\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$, so $A^{-1}B^{-1}=\begin{bmatrix}1/2 & 0\\0 & 1/2\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1/2\\-1/2 & 0\end{bmatrix}=T^{-1}$, as required.

Q16. With O at the origin, the original triangle has area $1/2|x_A y_B-x_B y_A|=1/2|(3)(4)-(1)(1)|=11/2$. Since $\det M=(2)(2)-(-1)(1)=5$, the image area is $|\det M|\times 11/2=5\times 11/2=55/2$. Directly, the images are $A'=M(3,1)=(5,5)$ and $B'=M(1,4)=(-2,9)$, giving image area $1/2|(5)(9)-(-2)(5)|=1/2(55)=55/2$. The ratio of image area to original area is $55/2/11/2=5=|\det M|$.

Q17. For invariant points, $Mx=x$ gives $y=x$ and $x=y$, the single condition $y=x$: every point of the line $y=x$ is fixed, a line of invariant points. For invariant lines $y=mx$, the direction $\begin{bmatrix}1\\m\end{bmatrix}$ maps to $\begin{bmatrix}m\\1\end{bmatrix}$, parallel to the original when $(1)(1)-(m)(m)=1-m^2=0$, so $m=\pm 1$. The vertical line $x=0$ maps $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}1\\0\end{bmatrix}$, which is not parallel, so it is not invariant. The invariant lines are $y=x$ (fixed pointwise) and $y=-x$ (points sent to their negatives).

Q18. (a) "Reflect after rotate" gives $SR=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}=\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$, which is reflection in the line $y=-x$. (b) The reverse order gives $RS=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, reflection in $y=x$ — a different result. (c) $\det(SR)=(0)(0)-(-1)(-1)=-1$ and $\det(RS)=(0)(0)-(1)(1)=-1$; each preserves area and reverses orientation, as befits a reflection. (d) For $SR=\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$, invariance $Mx=x$ gives $-y=x$ and $-x=y$, both the condition $y=-x$: the line $y=-x$ is fixed pointwise. Testing $y=mx$, the direction $\begin{bmatrix}1\\m\end{bmatrix}$ maps to $\begin{bmatrix}-m\\-1\end{bmatrix}$, parallel when $(1)(-1)-(m)(-m)=m^2-1=0$, so $m=\pm 1$; the vertical line $x=0$ is not invariant. The invariant lines are $y=-x$ (fixed pointwise) and $y=x$ (points sent to their negatives).