Specialist Mathematics Units 1 & 2

Chapter 8 — Trigonometry and identities

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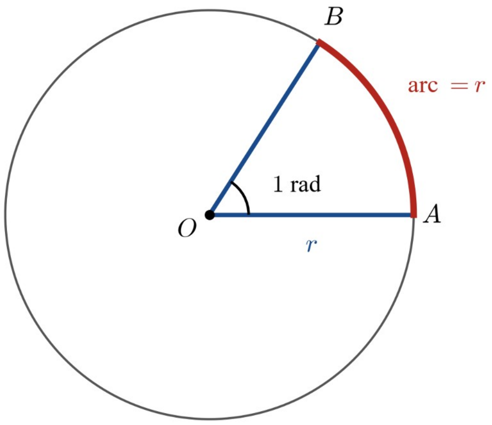
Theory

In earlier work angles were measured in **degrees**, one full revolution being divided into 360 equal parts. Degrees are convenient but arbitrary: nothing about a circle singles out the number 360. For measuring arcs and areas — and for all of the calculus of the circular functions met in Mathematical Methods — a second unit is used, one tied directly to the geometry of the circle itself. That unit is the **radian**, and it turns the length of an arc and the area of a sector into formulae with no clumsy constant attached.

The radian.

Take a circle of radius r and centre O, and mark off along the circumference an arc whose length is exactly r. The angle this arc subtends at the centre is defined to be **one radian** (written 1 rad, though usually the unit is left off entirely).



A radian is thus a *ratio of two lengths* — arc length to radius — and so carries no physical dimension. Because of this, an angle written with **no unit symbol at all is understood to be in radians**; a degree measure must always carry the ° sign.

Arc length is proportional to the angle at the centre, so doubling the angle doubles the arc. The whole circumference has length $2\pi r$, which is 2π copies of the radius r; the arc of one full revolution is therefore 2π radians. One full revolution is also 360°, so a half-revolution gives the identity from which every conversion follows:

$$\pi \text{ radians} = 180°.$$

Dividing this identity by 180 shows that $1° = \dfrac{\pi}{180}$ radians, and dividing it by π shows that 1 radian $= \dfrac{180}{\pi}$ degrees $\approx 57.30°$. In practice:

- to convert **degrees to radians**, multiply by $\dfrac{\pi}{180}$;
- to convert **radians to degrees**, multiply by $\dfrac{180}{\pi}$.

The angles met most often have exact radian measures worth committing to memory. They are the multiples of 30° and 45° around one revolution:

Degrees	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle that is a simple fraction of π is almost always left in that exact form; only when an angle is *not* a neat fraction of π — for example a measured angle of 1.2 radians — is a decimal used.

Arc length.

Let an arc subtend a central angle θ, measured in radians, in a circle of radius r. Arc length is proportional to the central angle, and a full turn of 2π radians corresponds to the whole circumference $2\pi r$. Comparing the arc with the whole,

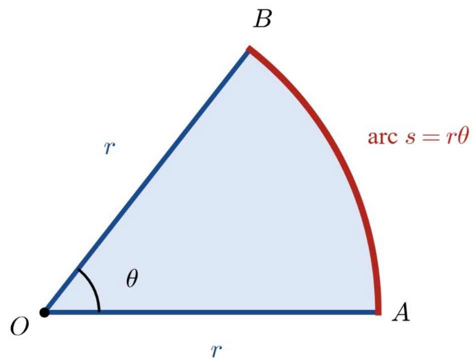
$$\frac{s}{2\pi r}=\frac{\theta}{2\pi}\Rightarrow s=r\theta\,(\theta\text{ in radians}).$$

This is the payoff of radian measure: the length of an arc is simply radius times angle, with no extra factor. (Had θ been kept in degrees the formula would read $s=\frac{\pi r\theta}{180}$, carrying the awkward constant that radians remove.)

Sectors.

A **sector** is the region enclosed by two radii and the arc between them — the shape of a slice of pie. Its area is proportional to the central angle in the same way as the arc length. A full turn of 2π radians encloses the whole disc of area πr^2, so a sector of angle θ has area

$$A=\frac{\theta}{2\pi}\times\pi r^2=1/2r^2\theta\,(\theta\text{ in radians}).$$



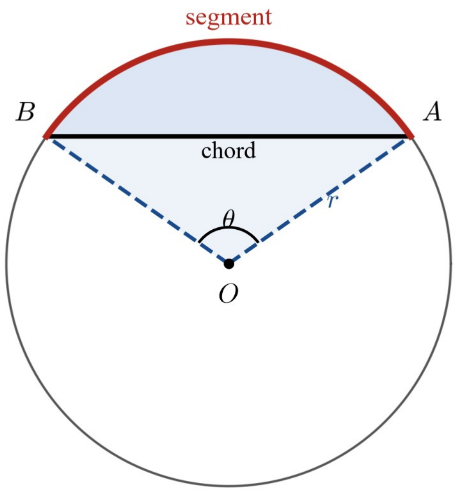
Because $s=r\theta$, the area may equally be written $A=1/2rs$, which is sometimes the quicker form when the arc length is already known. The **perimeter of a sector** is the arc plus the two bounding radii,

$$P=s+2r=r\theta+2r,$$

and must not be confused with the arc length alone.

Chords and segments.

A **chord** is a line segment joining two points A and B on a circle. It divides the disc into two regions called **segments**: the smaller **minor segment** and the larger **major segment**. The minor segment is the region between the chord and the *minor* arc, and it is exactly what is left when the triangle OAB is cut away from the sector OAB.



The triangle OAB has two sides equal to the radius r with the central angle θ between them, so by the "$1/2\,ab\sin C$" rule its area is $1/2\,r^2\sin\theta$. Subtracting this triangle from the sector gives the **area of the minor segment**:

$$A_{\text{segment}} = 1/2\,r^2\theta - 1/2\,r^2\sin\theta = 1/2\,r^2(\theta - \sin\theta)\,(\theta \text{ in radians}).$$

This formula is valid for $0<\theta<\pi$, the range in which AB genuinely cuts off a minor segment. The **major segment** on the other side of the chord is found by taking the minor segment away from the whole disc,

$$A_{\text{major}} = \pi r^2 - 1/2\,r^2(\theta - \sin\theta).$$

Two further lengths complete the picture. Dropping a perpendicular from O to the chord AB bisects both the chord and the angle θ, producing a right-angled triangle with hypotenuse r and the half-angle $\theta/2$ at O. Hence the half-chord is $r\sin\theta/2$, so the **chord length** is

$$AB = 2r\sin\theta/2,$$

and the **perimeter of the minor segment** is this chord plus the minor arc, $P = 2r\sin\theta/2 + r\theta$. Every one of these results assumes the angle is in radians; converting a degree measure to radians before substituting is the single most common safeguard against error.

Worked examples

Example 1 — scaffolded

Convert (a) 252° and (b) 40° to radians, giving exact answers; and convert (c) $\frac{11\pi}{12}$ radians and (d) 2.4 radians to degrees, giving (d) to one decimal place.

Working	Comment
To change degrees to radians, multiply by $\frac{\pi}{180}$.	The rule for this direction.
(a) 252° becomes $252\times\frac{\pi}{180} = \frac{252\pi}{180} = \frac{7\pi}{5}$.	Cancel $\frac{252}{180} = \frac{7}{5}$; leave the answer exact.
(b) 40° becomes $40\times\frac{\pi}{180} = \frac{40\pi}{180} = \frac{2\pi}{9}$.	Cancel $\frac{40}{180} = \frac{2}{9}$.
To change radians to degrees, multiply by $\frac{180}{\pi}$.	The reverse rule; the π will cancel for exact angles.

Working	Comment
(c) $\frac{11\pi}{12}\times\frac{180}{\pi}=11\times\frac{180}{12}=11\times 15=165$, i.e. 165°.	The π cancels and $\frac{180}{12}=15$.
(d) $2.4\times\frac{180}{\pi}=\frac{432}{\pi}\approx 137.5$, i.e. 137.5°.	Not a fraction of π, so give a decimal (1 dp).

Example 2 — scaffolded

A sector is cut from a circular sheet of radius 8 cm with central angle $\frac{3\pi}{8}$. Find (a) the length of its arc, (b) the perimeter of the sector, and (c) the area of the sector.

Working	Comment
$r=8$, $\theta=\frac{3\pi}{8}$, already in radians.	The formulae $s=r\theta$ and $A=1/2\,r^2\theta$ require radians.
(a) $s=r\theta=8\times\frac{3\pi}{8}=3\pi\approx 9.42$ cm.	The two 8s cancel; state the arc length in cm.
(b) $P=s+2r=3\pi+16\approx 25.42$ cm.	The sector is bounded by the arc and the two radii.
(c) $A=1/2\,r^2\theta=1/2\times 64\times\frac{3\pi}{8}=12\pi\approx 37.70$ cm².	$1/2\times 64=32$, then $32\times\frac{3\pi}{8}=12\pi$.

Example 3 — unscaffolded

A chord of a circle of radius 10 cm subtends an angle of $\frac{\pi}{3}$ at the centre. Find the exact area of the minor segment cut off by the chord, and the length of the minor arc.

The sector on this angle has area

$$1/2\,r^2\theta=1/2\times 10^2\times\frac{\pi}{3}=\frac{50\pi}{3}\,\text{cm}^2,$$

while the triangle formed by the two radii and the chord has area

$$1/2\,r^2\sin\theta=1/2\times 100\times\sin\frac{\pi}{3}=50\times\frac{\sqrt{3}}{2}=25\sqrt{3}\,\text{cm}^2.$$

The minor segment is the sector with this triangle removed:

$$A_{\text{segment}}=\frac{50\pi}{3}-25\sqrt{3}\approx 9.06\,\text{cm}^2.$$

The minor arc has length $s=r\theta=10\times\frac{\pi}{3}=\frac{10\pi}{3}\approx 10.47$ cm.

∴ the minor segment has area $\frac{50\pi}{3}-25\sqrt{3}\approx 9.06$ cm² and the minor arc has length $\frac{10\pi}{3}\approx 10.47$ cm.

Example 4 — unscaffolded

A sector of a circle has arc length 15 cm and area 60 cm². Find the radius of the circle and the central angle of the sector, giving the angle in radians and in degrees (to one decimal place).

The area of a sector can be written using its arc length: since $A=1/2\,r^2\theta$ and $s=r\theta$,

$$A=1/2\,r(r\theta)=1/2\,r s.$$

Substituting the given values,

$$60 = 1/2\,r \times 15 = 15/2\,r \Rightarrow r = \frac{120}{15} = 8\,\text{cm}.$$

The central angle then follows from $s = r\theta$:

$$\theta = \frac{s}{r} = \frac{15}{8} = 1.875\,\text{radians},$$

and in degrees $\theta = 1.875 \times \frac{180}{\pi} \approx 107.4°$.

∴ the radius is 8 cm and the central angle is $\frac{15}{8} = 1.875$ radians $\approx 107.4°$.

Practice questions

Scaffolded

Q1. Convert between degrees and radians. (a) Express 108° in radians, exactly. (b) Express 315° in radians, exactly. (c) Express $\frac{7\pi}{10}$ radians in degrees. (d) Express 1.5 radians in degrees, to one decimal place.

Q2. A sector of a circle of radius 6 cm has central angle $\frac{5\pi}{6}$. (a) Find the arc length. (b) Find the area of the sector. (c) Find the perimeter of the sector.

Q3. A sector of a circle of radius 9 m has central angle 80°. (a) Convert 80° to radians, exactly. (b) Find the length of the arc. (c) Find the area of the sector.

Q4. A chord of a circle of radius 12 cm subtends a right angle, $\frac{\pi}{2}$, at the centre. (a) Find the area of the sector. (b) Find the area of the triangle formed by the two radii and the chord. (c) Hence find the area of the minor segment.

Q5. A sector of a circle of radius 5 cm has an arc of length 8 cm. (a) Find the central angle in radians. (b) Convert this angle to degrees, to one decimal place. (c) Find the area of the sector.

Q6. A sector has central angle $\frac{2\pi}{5}$ and area 45π cm². (a) Using $A = 1/2\,r^2\theta$, find the radius. (b) Hence find the length of the arc.

Unscaffolded

Q7. Express 285° in radians, exactly.

Q8. Express $\frac{13\pi}{9}$ radians in degrees.

Q9. An arc subtends an angle of $\frac{3\pi}{7}$ at the centre of a circle of radius 14 cm. Find the length of the arc and the area of the corresponding sector.

Q10. An arc of length 22 cm subtends an angle of 2.75 radians at the centre of a circle. Find the radius.

Q11. A sector of a circle of radius 10 cm has perimeter 34 cm. Find the area of the sector.

Q12. A chord of a circle of radius 8 cm subtends an angle of $\frac{2\pi}{3}$ at the centre. Find the exact area of the minor segment, and give its value to two decimal places.

Q13. A chord of a circle of radius 20 cm subtends a right angle at the centre. Find the exact area of the **major** segment, and give its value to two decimal places.

Q14. A garden sprinkler sprays water over a sector of radius 9 m as it turns through 140°. Find the area of ground watered and the length of the curved boundary of that region.

Q15. A sector of a circle has area 96 cm² and arc length 16 cm. Find the radius of the circle and the central angle in radians.

Q16. A chord of a circle of radius 25 cm subtends an angle of 0.8 radians at the centre. (a) Find the length of the chord, to two decimal places. (b) Find the length of the minor arc. (c) Hence find the perimeter of the minor segment, to two decimal places.

Q17. A sector of a circle of radius 6 cm has area 12π cm². Find the central angle in radians and the length of the arc.

Q18. A circular ornamental pond of radius 4 m is crossed by a straight footbridge that lies along a chord. The chord subtends an angle of 2 radians at the centre of the pond. Find, each to two decimal places, (a) the length of the footbridge and (b) the area of the smaller of the two regions into which the bridge divides the surface of the pond.

Answers

Q1. (a) $\frac{3\pi}{5}$; (b) $\frac{7\pi}{4}$; (c) 126°; (d) 85.9°. **Q2.** (a) $5\pi \approx 15.71$ cm; (b) $15\pi \approx 47.12$ cm²; (c) $5\pi + 12 \approx 27.71$ cm. **Q3.** (a) $\frac{4\pi}{9}$; (b) $4\pi \approx 12.57$ m; (c) $18\pi \approx 56.55$ m². **Q4.** (a) $36\pi \approx 113.10$ cm²; (b) 72 cm²; (c) $36\pi - 72 \approx 41.10$ cm². **Q5.** (a) 1.6 radians; (b) 91.7°; (c) 20 cm². **Q6.** (a) 15 cm; (b) $6\pi \approx 18.85$ cm. **Q7.** $\frac{19\pi}{12}$. **Q8.** 260°. **Q9.** arc $6\pi \approx 18.85$ cm; sector $42\pi \approx 131.95$ cm². **Q10.** 8 cm. **Q11.** 70 cm². **Q12.** $\frac{64\pi}{3} - 16\sqrt{3} \approx 39.31$ cm². **Q13.** $300\pi + 200 \approx 1142.48$ cm². **Q14.** area $\frac{63\pi}{2} \approx 98.96$ m²; boundary arc $7\pi \approx 21.99$ m. **Q15.** $r = 12$ cm; $\theta = \frac{4}{3} \approx 1.33$ radians. **Q16.** (a) $50\sin 0.4 \approx 19.47$ cm; (b) 20 cm; (c) $20 + 50\sin 0.4 \approx 39.47$ cm. **Q17.** $\theta = \frac{2\pi}{3}$ radians; arc $4\pi \approx 12.57$ cm. **Q18.** (a) $8\sin 1 \approx 6.73$ m; (b) $8(2 - \sin 2) \approx 8.73$ m².

Fully-worked solutions

Q1. (a) $108 \times \frac{\pi}{180} = \frac{108\pi}{180} = \frac{3\pi}{5}$, since $\frac{108}{180} = \frac{3}{5}$. (b) $315 \times \frac{\pi}{180} = \frac{315\pi}{180} = \frac{7\pi}{4}$, since $\frac{315}{180} = \frac{7}{4}$. (c) $\frac{7\pi}{10} \times \frac{180}{\pi} = 7 \times 18 = 126$, i.e. 126°. (d) $1.5 \times \frac{180}{\pi} = \frac{270}{\pi} \approx 85.9$, i.e. 85.9°.

Q2. With $r = 6$ and $\theta = \frac{5\pi}{6}$: (a) $s = r\theta = 6 \times \frac{5\pi}{6} = 5\pi \approx 15.71$ cm. (b) $A = 1/2\, r^2\theta = 1/2 \times 36 \times \frac{5\pi}{6} = 18 \times \frac{5\pi}{6} = 15\pi \approx 47.12$ cm². (c) The perimeter is $s + 2r = 5\pi + 12 \approx 27.71$ cm.

Q3. (a) $80 \times \frac{\pi}{180} = \frac{80\pi}{180} = \frac{4\pi}{9}$. (b) $s = r\theta = 9 \times \frac{4\pi}{9} = 4\pi \approx 12.57$ m. (c) $A = 1/2\, r^2\theta = 1/2 \times 81 \times \frac{4\pi}{9} = \frac{81}{2} \times \frac{4\pi}{9} = 18\pi \approx 56.55$ m².

Q4. With $r = 12$ and $\theta = \frac{\pi}{2}$: (a) sector area $= 1/2\, r^2\theta = 1/2 \times 144 \times \frac{\pi}{2} = 36\pi \approx 113.10$ cm². (b) triangle area $= 1/2\, r^2 \sin\theta = 1/2 \times 144 \times \sin\frac{\pi}{2} = 72 \times 1 = 72$ cm². (c) The minor segment is the sector minus the triangle, $36\pi - 72 \approx 41.10$ cm².

Q5. (a) $\theta = \frac{s}{r} = \frac{8}{5} = 1.6$ radians. (b) $1.6 \times \frac{180}{\pi} = \frac{288}{\pi} \approx 91.7$, i.e. 91.7°. (c) Using $A = 1/2\, r\, s = 1/2 \times 5 \times 8 = 20$ cm² (equivalently $1/2 \times 25 \times 1.6 = 20$).

Q6. (a) $A = 1/2\, r^2\theta$ gives $45\pi = 1/2\, r^2 \times \frac{2\pi}{5} = \frac{\pi r^2}{5}$, so $r^2 = \frac{45\pi \times 5}{\pi} = 225$ and $r = 15$ cm. (b) The arc is $s = r\theta = 15 \times \frac{2\pi}{5} = 6\pi \approx 18.85$ cm.

Q7. $285 \times \frac{\pi}{180} = \frac{285\pi}{180} = \frac{19\pi}{12}$, since $\frac{285}{180} = \frac{19}{12}$ (divide both by 15).

Q8. $\frac{13\pi}{9} \times \frac{180}{\pi} = 13 \times \frac{180}{9} = 13 \times 20 = 260$, i.e. 260°.

Q9. With $r=14$ and $\theta=\frac{3\pi}{7}$: $s=r\theta=14\times\frac{3\pi}{7}=6\pi\approx18.85$ cm, and $A=1/2r^2\theta=1/2\times196\times\frac{3\pi}{7}=98\times\frac{3\pi}{7}=42\pi\approx131.95$ cm².

Q10. From $s=r\theta$, $r=\frac{s}{\theta}=\frac{22}{2.75}=8$ cm.

Q11. The perimeter of a sector is $2r+s$, so $34=2\times10+s$ gives $s=14$ cm. The area is then $A=1/2rs=1/2\times10\times14=70$ cm². (The central angle, if wanted, is $\theta=\frac{s}{r}=1.4$ radians.)

Q12. With $r=8$ and $\theta=\frac{2\pi}{3}$, the sector area is $1/2\times64\times\frac{2\pi}{3}=\frac{64\pi}{3}$ cm² and the triangle area is $1/2\times64\times\sin\frac{2\pi}{3}=32\times\frac{\sqrt{3}}{2}=16\sqrt{3}$ cm². The minor segment is

$$\frac{64\pi}{3}-16\sqrt{3}\approx67.02-27.71=39.31\,\text{cm}^2.$$

Q13. With $r=20$ and $\theta=\frac{\pi}{2}$, the minor segment is $1/2\times400\times\left(\frac{\pi}{2}-\sin\frac{\pi}{2}\right)=200\left(\frac{\pi}{2}-1\right)=100\pi-200$ cm². The whole disc has area $\pi\times20^2=400\pi$ cm², so the major segment is

$$400\pi-(100\pi-200)=300\pi+200\approx1142.48\,\text{cm}^2.$$

Q14. The angle is $140°=140\times\frac{\pi}{180}=\frac{7\pi}{9}$ radians, and $r=9$ m. The area watered is $A=1/2r^2\theta=1/2\times81\times\frac{7\pi}{9}=\frac{9\times7\pi}{2}=\frac{63\pi}{2}\approx98.96$ m², and the curved boundary is the arc $s=r\theta=9\times\frac{7\pi}{9}=7\pi\approx21.99$ m.

Q15. Using $A=1/2rs$, $96=1/2\times r\times16=8r$, so $r=12$ cm. Then $\theta=\frac{s}{r}=\frac{16}{12}=\frac{4}{3}\approx1.33$ radians.

Q16. With $r=25$ and $\theta=0.8$ radians: (a) the chord is $2r\sin\frac{\theta}{2}=2\times25\times\sin0.4=50\sin0.4\approx19.47$ cm (the angle is already in radians, so the calculator must be set to radian mode). (b) the minor arc is $r\theta=25\times0.8=20$ cm exactly. (c) the perimeter of the minor segment is chord plus arc, $20+50\sin0.4\approx39.47$ cm.

Q17. From $A=1/2r^2\theta$, $12\pi=1/2\times36\times\theta=18\theta$, so $\theta=\frac{12\pi}{18}=\frac{2\pi}{3}$ radians. The arc is then $s=r\theta=6\times\frac{2\pi}{3}=4\pi\approx12.57$ cm.

Q18. With $r=4$ and $\theta=2$ radians: (a) the footbridge is the chord $2r\sin\frac{\theta}{2}=2\times4\times\sin1=8\sin1\approx6.73$ m. (b) the bridge cuts off the minor segment, of area $1/2r^2(\theta-\sin\theta)=1/2\times16\times(2-\sin2)=8(2-\sin2)\approx8(2-0.9093)=8.73$ m². (Since $2<\pi$, this region is indeed the smaller of the two.)

Theory

In a right-angled triangle the three basic ratios are enough to find every side and angle. Most triangles that arise in surveying, design, navigation and the geometry of solids contain **no** right angle, and two results extend trigonometry to them: the **sine rule** and the **cosine rule**. Both are theorems — each is proved below from the right-angled case — and together they make it possible to *solve* any triangle from enough given information, in the plane and in three dimensions.

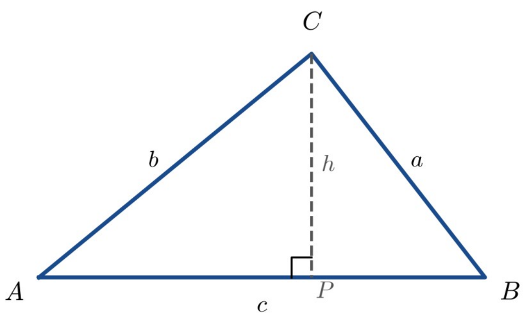
Throughout, a triangle is labelled ABC with the side opposite each vertex named by the matching lower-case letter: side $a=BC$ is opposite angle A, side $b=CA$ is opposite angle B, and side $c=AB$ is opposite angle C. The three interior angles satisfy $A+B+C=180°$, and a longer side always lies opposite a larger angle.

The sine rule.

In any triangle ABC,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Proof. Drop the perpendicular from C to the line AB, meeting it at P, and write $h=CP$ for its length.



In the right-angled triangle APC, the side h is opposite A and the hypotenuse is b, so $\sin A=\frac{h}{b}$, giving $h=b\sin A$.

In the right-angled triangle BPC, $\sin B=\frac{h}{a}$, giving $h=a\sin B$. Equating the two expressions for h,

$$b\sin A=a\sin B\Rightarrow\frac{a}{\sin A}=\frac{b}{\sin B}.$$

Dropping instead the perpendicular from A onto BC gives $\frac{b}{\sin B}=\frac{c}{\sin C}$ in the same way, and the two shared ratios join into the full statement. If one angle is obtuse the foot P falls outside the segment AB; but then the relevant right triangle uses the angle $180°-A$, and since $\sin(180°-A)=\sin A$ the identity is unchanged, as required.

The sine rule links a side to the angle **opposite** it. It is the tool to use when the given information is **two angles and any one side** (abbreviated ASA or AAS — the third angle comes from the angle sum), or **two sides and a non-included angle** (SSA), the case examined below.

The area of a triangle.

The perpendicular height from C found above is $h=b\sin A$, and taking $AB=c$ as the base gives area $=1/2\times c\times h=1/2\,bc\sin A$. The same argument from each vertex yields the symmetric statement

$$\text{Area}=1/2\,bc\sin A=1/2\,ca\sin B=1/2\,ab\sin C,$$

in words: **half the product of two sides times the sine of the angle between them**. Only two sides and their included angle are needed — no perpendicular height has to be found separately.

The cosine rule.

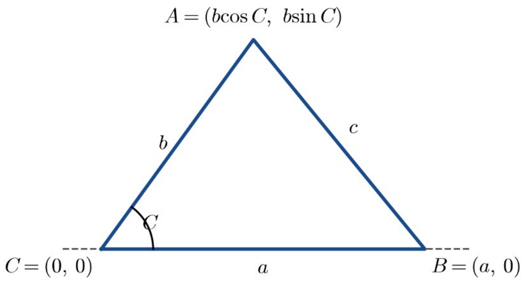
In any triangle ABC,

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

and, rearranged to give the angle,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Proof. Place C at the origin with B on the positive horizontal axis, so that $B = (a, 0)$; since the angle at C is C and $CA = b$, the third vertex is $A = (b\cos C, b\sin C)$.



The side $c = AB$ is the distance between A and B, so

$$c^2 = (a - b\cos C)^2 + (b\sin C)^2 = a^2 - 2ab\cos C + b^2\cos^2 C + b^2\sin^2 C.$$

Using $\cos^2 C + \sin^2 C = 1$ the last two terms combine to b^2, leaving $c^2 = a^2 + b^2 - 2ab\cos C$, as required.

The cosine rule contains Pythagoras' theorem as its right-angled case: when $C = 90°$, $\cos C = 0$ and the formula reduces to $c^2 = a^2 + b^2$. It is the tool to use when the given information is **two sides and the included angle** (SAS — find the third side) or **three sides** (SSS — find any angle). When an angle is found from three known sides, the cosine rule is preferred over the sine rule because it returns the angle without ambiguity: $\cos\theta$ is negative for an obtuse angle and positive for an acute one, so the sign of the result settles the matter at once.

The ambiguous case (SSA).

Suppose two sides and an angle *not* between them are given — say A, a and b, with A opposite a. The sine rule gives

$$\sin B = \frac{b\sin A}{a}.$$

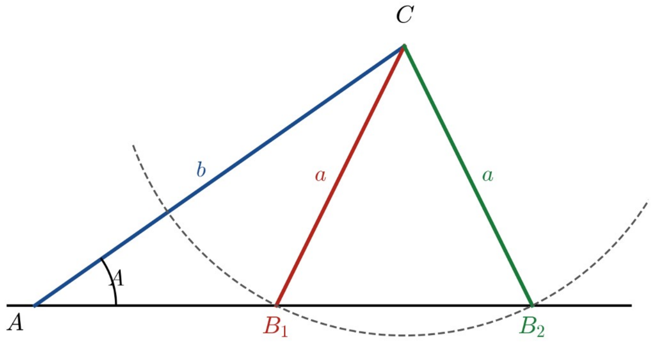
Because $\sin B = \sin(180° - B)$, a value of $\sin B$ less than 1 answers to **two** angles, $B_1 = \sin^{-1}\left(\dfrac{b\sin A}{a}\right)$ and its supplement $B_2 = 180° - B_1$. Which of these actually occurs depends on the numbers. Take first the usual case, in which **the given angle A is acute**:

- if $\dfrac{b\sin A}{a} > 1$, then $\sin B > 1$ is impossible and **no** triangle exists;
- if $\dfrac{b\sin A}{a} = 1$, then $B = 90°$ and there is exactly **one** (right) triangle;

- if $\dfrac{b\sin A}{a}<1$, the acute value B_1 always gives a triangle (both A and B_1 are acute, so $A+B_1<180°$ automatically), and the obtuse value B_2 gives a **second** triangle precisely when $A+B_2<180°$ — that is, when the given angle is opposite the shorter of the two given sides ($a<b$).

The restriction on A is essential, because the count above is not what happens when the given angle is a right angle or obtuse. If $A\geq 90°$ then a, the side opposite it, must be the longest side, so the measurements fit a triangle only when $a>b$, and then only **one**: B_1 is acute, and B_2 is rejected because no triangle can hold two angles of $90°$ or more. If $A\geq 90°$ and $a\leq b$ there is **no** triangle at all, even when $\sin B$ works out to a legitimate value — for $A=150°$, $a=6$ and $b=10$ the sine rule gives $\sin B=0.8333$ and $B_1=56.4°$, but $A+B_1=206.4°>180°$, so neither value of B survives.

The safe procedure, whatever the size of A, is: compute B_1, form $B_2=180°-B_1$, and **test each** against the angle sum $A+B<180°$; report every value of B that passes. The figure shows the geometry of the two-triangle case — with the angle A and the side $b=AC$ fixed, swinging the side of length a about the vertex C can strike the base line at two points, B_1 and B_2.



For instance, if $A=32°$, $a=9$ and $b=13$, then $\sin B=\dfrac{13\sin 32°}{9}=0.7654$, so $B=49.9°$ or $B=130.1°$. Both survive the angle-sum test (giving $C=98.1°$ or $C=17.9°$), so here two genuinely different triangles fit the data.

Choosing a rule — solving triangles.

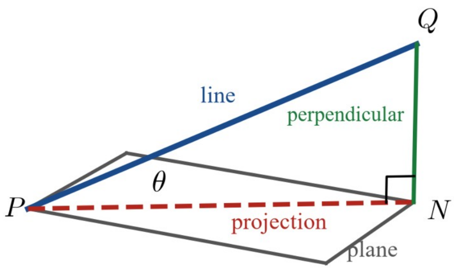
Given	Rule to start with
Two angles and a side (ASA, AAS)	sine rule
Two sides and a non-included angle (SSA)	sine rule — check the ambiguous case
Two sides and the included angle (SAS)	cosine rule (third side), then continue
Three sides (SSS)	cosine rule (an angle), then continue

After the first unknown is found, the remaining ones follow from the angle sum and whichever rule is convenient; keeping a few extra decimal places in intermediate values and rounding only at the end avoids accumulated error.

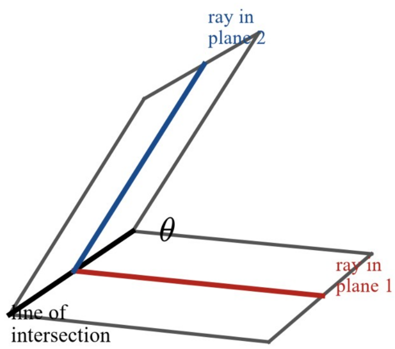
Working in three dimensions.

A three-dimensional problem is solved by finding suitable **triangles inside the solid** and applying right-angled trigonometry, the sine rule and the cosine rule to them one at a time. A large, clear diagram is essential, and it usually helps to redraw each triangle on its own as a flat picture before calculating. Three standard measurements recur.

The angle between a line and a plane. The **projection** of a point onto a plane is the foot of the perpendicular dropped from the point to the plane; the projection of a line is the line through the projections of its points. The **angle between a line and a plane** is the angle between the line and its projection on the plane — equivalently, the smallest angle the line makes with any line lying in the plane. If a slanting line meets the plane at P and Q is another point on it whose foot (projection) is N, then triangle PNQ is right-angled at N and the required angle is $\theta=\angle QPN$, with $\tan\angle QPN=\dfrac{QN}{PN}$.



The angle between two planes. Two planes that are not parallel meet in a straight line, the **line of intersection**. To measure the angle between them, take a point on that line and, in each plane, draw the ray that is perpendicular to the line of intersection at that point; the **dihedral angle** between the planes is the angle between these two rays. (Every such pair of rays gives the same angle.) Drawing that perpendicular pair correctly is the key to every “angle between two planes” problem. In the common case where one of the two planes is **horizontal**, the ray drawn in the sloping plane at right angles to the line of intersection is a **line of greatest slope** of that plane — the steepest direction in it — and the dihedral angle is the angle that ray makes with the horizontal.



The angle between two lines. When two lines meet, the angle between them is measured in the plane they share. When they do not meet, one line is imagined translated parallel to itself until it does; the angle between the directions is then measured as usual. (The vector treatment of this appears in the vectors chapter.)

Bearings. A **true bearing** is measured clockwise from north and written with three figures, from $000°$ to $360°$. A navigation problem in three dimensions typically breaks into a **horizontal** triangle — solved with the sine and cosine rules using the bearings — and a **vertical** right-angled triangle involving an angle of elevation or depression; the two are linked by a shared horizontal distance.

Worked examples

Example 1 — scaffolded

In triangle ABC, $A=52°$, $B=71°$ and $a=14.6$ cm. Find C, and then the sides b and c, giving each length to one decimal place.

Working	Comment
$C=180°-52°-71°=57°$.	Two angles are known, so the third comes from the angle sum.
$\dfrac{b}{\sin B}=\dfrac{a}{\sin A}$, so $b=\dfrac{a\sin B}{\sin A}=\dfrac{14.6\sin 71°}{\sin 52°}$.	A side and its opposite angle (a, A) are known, so use the sine rule.
$b=17.5$ cm.	Evaluate; $\sin 71°=0.9455$, $\sin 52°=0.7880$.
$c=\dfrac{a\sin C}{\sin A}=\dfrac{14.6\sin 57°}{\sin 52°}=15.5$ cm.	Same ratio with C in place of B gives the last side.

Example 2 — scaffolded

In triangle ABC, $b=8.4$ cm, $c=11.7$ cm and the included angle $A=38°$. Find the third side a, the angle B, and the area of the triangle (lengths to one decimal place, angles to the nearest tenth of a degree).

Working	Comment
$a^2=b^2+c^2-2bc\cos A=8.4^2+11.7^2-2(8.4)(11.7)$cc .	Two sides and the included angle are known (SAS): use the cosine rule.
$a^2=52.56$, so $a=7.2$ cm.	Take the positive square root.
$\cos B=\dfrac{a^2+c^2-b^2}{2ac}=\dfrac{52.56+136.89-70.56}{2(7.25)(11.7)}=0.7008$.	Find B from the rearranged cosine rule — no ambiguity, as the sign fixes acute/obtuse.
$B=\cos^{-1}(0.7008)=45.5°$.	Positive cosine, so B is acute.
Area $=1/2\,bc\sin A=1/2(8.4)(11.7)\sin 38°\approx 30.3$ cm².	Half the product of the two given sides times the sine of the included angle.

Example 3 — unscaffolded

In triangle KLM, $\angle K=29°$, $k=6.8$ m and $l=12.5$ m. Show that these measurements are consistent with two different triangles, and solve each (angles to the nearest tenth of a degree, lengths to one decimal place).

Here a side (k), its opposite angle (K) and a second side (l) are known — the ambiguous SSA case. By the sine rule,

$$\sin L=\frac{l\sin K}{k}=\frac{12.5\sin 29°}{6.8}=0.8912.$$

Since $0.8912<1$ and the known angle K is acute and opposite the shorter side ($k<l$), there are two possibilities: $L=63.0°$ or $L=180°-63.0°=117.0°$.

First triangle ($L=63.0°$): $M=180°-29°-63.0°=88.0°$, and by the sine rule

$$m=\frac{k\sin M}{\sin K}=\frac{6.8\sin 88.0°}{\sin 29°}=14.0\,\mathrm{m}.$$

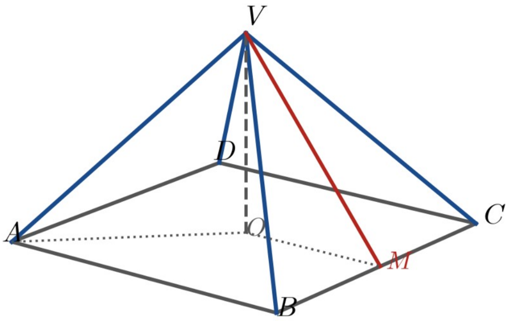
Second triangle ($L=117.0°$): the angle sum still holds, because $29°+117.0°=146.0°<180°$, giving $M=34.0°$ and

$$m=\frac{6.8\sin 34.0°}{\sin 29°}=7.8\,\mathrm{m}.$$

$\therefore$ two triangles fit the data: $(L=63.0°, M=88.0°, m=14.0\,\mathrm{m})$ and $(L=117.0°, M=34.0°, m=7.8\,\mathrm{m})$.

Example 4 — unscaffolded

A monument is a right pyramid on a horizontal square base $ABCD$ of side 24 m, with apex V standing 15 m vertically above the centre O of the base. Find, to one decimal place: (a) the length of a slant edge VA; (b) the angle a slant edge makes with the base; (c) the angle between a triangular face and the base; and (d) the apex angle $\angle AVB$ of a face.



Because the pyramid is right, O is the foot of the perpendicular from V, so $VO=15$ and VO is perpendicular to the whole base plane.

(a) The point O is the centre of the square, so OA is half a diagonal. The full diagonal of a square of side 24 is $24\sqrt{2}$, hence $OA=12\sqrt{2}\approx 16.97$ m. Triangle VOA is right-angled at O, so

$$VA=\sqrt{VO^2+OA^2}=\sqrt{15^2+(12\sqrt{2})^2}=\sqrt{225+288}=\sqrt{513}\approx 22.6\,\text{m}.$$

(b) The projection of the edge VA on the base is OA, so the angle between VA and the base is $\angle VAO$, where $\tan\angle VAO=\dfrac{VO}{OA}=\dfrac{15}{12\sqrt{2}}$, giving $\angle VAO=41.5°$.

(c) Let M be the midpoint of the base edge BC. Since O is the centre, $OM\perp BC$, and since V lies directly above O, $VM\perp BC$ as well; the two rays MO and MV are therefore both perpendicular to the line of intersection BC, so $\angle VMO$ is the dihedral angle between face VBC and the base. Here $OM=1/2(24)=12$ m and $\tan\angle VMO=\dfrac{VO}{OM}=\dfrac{15}{12}$, giving $\angle VMO=51.3°$.

(d) In the isosceles triangle AVB, $VA=VB=\sqrt{513}$ and $AB=24$. By the cosine rule,

$$\cos\angle AVB=\frac{VA^2+VB^2-AB^2}{2VA\cdot VB}=\frac{513+513-576}{2(513)}=0.4386,$$

so $\angle AVB=64.0°$.

$\therefore$ the slant edge is $VA\approx 22.6$ m, inclined at $41.5°$ to the base; each triangular face meets the base at $51.3°$; and the apex angle of a face is $\angle AVB=64.0°$.

Practice questions

Scaffolded

Q1. In triangle ABC, $B=39°$, $C=84°$ and $b=21.0$ cm. (a) Find A. (b) Find a. (c) Find c. (d) Find the area of the triangle, to the nearest square centimetre.

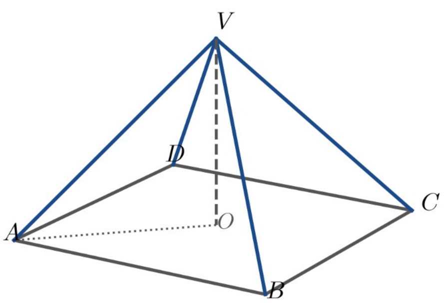
Q2. In triangle ABC, $b=9.4$ cm, $c=13.6$ cm and the included angle $A=62°$. (a) Find a. (b) Find B. (c) Find C. (d) Find the area of the triangle.

Q3. In triangle ABC, $A=35°$, $a=8.9$ cm and $b=13.4$ cm. (a) Use the sine rule to find $\sin B$. (b) Write down the two possible values of B. (c) Explain why each value gives a valid triangle. (d) Find the two corresponding values of c.

Q4. A rectangular gymnasium hall is 6.0 m long, 4.5 m wide and 3.2 m high. Let A be one bottom corner and G the top corner diagonally opposite to it. (a) Find the length of a diagonal of the floor. (b) Find the length of the space diagonal AG. (c) Find the angle AG makes with the floor. (d) Find the angle AG makes with the 6.0 m edge at A.

Q5. A survey vessel leaves port O and sails 18 km on a bearing of $072°$ to a point A, then turns and sails 25 km on a bearing of $124°$ to a point B. (a) Show that $\angle OAB=128°$. (b) Find the distance OB. (c) Find the bearing of B from O.

Q6. A right pyramid has a square base of side 18 cm and a vertical height of 12 cm.



(a) Find the length of a diagonal of the base.
(b) Find the length of a slant edge.
(c) Find the angle a slant edge makes with the base.
(d) Find the angle between a triangular face and the base.

Unscaffolded

Q7. In triangle DEF, $D=48°$, $E=59°$ and $d=12.5$ cm. Find f.

Q8. In triangle ABC, $a=17$ m, $b=23$ m and $C=109°$. Find c.

Q9. A triangle has sides of length 26 cm, 19 cm and 14 cm. Find the size of its largest angle.

Q10. A triangular vineyard plot has two straight edges of length 34 m and 41 m meeting at an angle of $76°$. Find the area of the plot and its perimeter.

Q11. In triangle ABC, $A=41°$, $a=9.2$ cm and $b=12.0$ cm. Determine how many triangles are consistent with these measurements, and solve each one completely.

Q12. In triangle ABC, $B=58°$, $b=10.5$ cm and $c=13.9$ cm. Show that no such triangle exists.

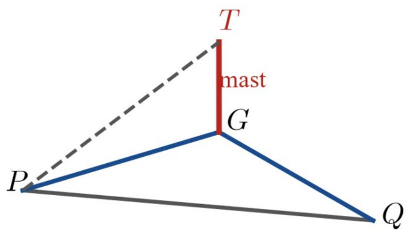
Q13. A light aircraft flies from airfield A on a bearing of $118°$ for 46 km to a turning point B, then flies on a bearing of $203°$ for a further 38 km to C. (a) Find the distance AC. (b) Find the bearing of A from C.

Q14. A shipping container is a rectangular box 12 m long, 2.5 m wide and 2.9 m high. (a) Find the length of the longest straight rod that fits inside it. (b) Find the angle this rod makes with the base of the container.

Q15. A symmetric roof is built on a rectangular building 10 m wide; the ridge line runs 3.5 m above the level of the eaves. (a) Find the angle each sloping roof face makes with the horizontal. (b) Find the angle between the two roof faces (the dihedral angle at the ridge).

Q16. A right pyramid has a square base of side 30 cm and a vertical height of 20 cm. (a) Find the length of a slant edge. (b) Find the apex angle of a triangular face. (c) Find the angle between two opposite triangular faces.

Q17. Two survey pegs P and Q lie 140 m apart on level ground. A vertical mast stands at a point G on the same ground, with $\angle GPQ = 63°$ and $\angle GQP = 48°$. From P the angle of elevation of the top T of the mast is $19°$.



(a) Find the distance PG.
(b) Find the height of the mast.

Q18. A rectangular box has dimensions 8 cm, 6 cm and 4 cm. (a) Find the length of a space diagonal. (b) Two of the space diagonals lie in the rectangular cross-section that contains a pair of diagonally opposite 8 cm edges, and cross at the centre of the box. Find the acute angle between these two diagonals.

Answers

Q1. (a) $A = 57°$; (b) $a = 28.0$ cm; (c) $c = 33.2$ cm; (d) 292 cm². **Q2.** (a) $a = 12.4$ cm; (b) $B = 42.1°$; (c) $C = 75.9°$; (d) area $= 56.4$ cm². **Q3.** (a) $\sin B = 0.8636$; (b) $B = 59.7°$ or $120.3°$; (c) each passes the angle-sum test, since $35° + 59.7° < 180°$ and $35° + 120.3° < 180°$; (d) $c = 15.5$ cm or 6.5 cm. **Q4.** (a) 7.5 m; (b) 8.2 m; (c) $23.1°$; (d) $42.6°$. **Q5.** (b) $OB = 38.8$ km; (c) bearing $102.5°$. **Q6.** (a) 25.5 cm; (b) 17.5 cm; (c) $43.3°$; (d) $53.1°$. **Q7.** $f = 16.1$ cm. **Q8.** $c = 32.8$ m. **Q9.** $102.9°$ (opposite the 26 cm side). **Q10.** Area $= 676.3$ m²; perimeter $= 121.5$ m. **Q11.** Two triangles: $(B = 58.8°, C = 80.2°, c = 13.8 \text{ cm})$ and $(B = 121.2°, C = 17.8°, c = 4.3 \text{ cm})$. **Q12.** No triangle: the sine rule requires $\sin C = 1.1227 > 1$, which is impossible. **Q13.** (a) $AC = 62.2$ km; (b) bearing of A from C is $335.5°$. **Q14.** (a) 12.6 m; (b) $13.3°$. **Q15.** (a) $35.0°$; (b) $110.0°$. **Q16.** (a) 29.2 cm; (b) $61.9°$; (c) $73.7°$. **Q17.** (a) $PG = 111.4$ m; (b) height $= 38.4$ m. **Q18.** (a) 10.8 cm; (b) $84.1°$.

Fully-worked solutions

Q1. The angle sum gives $A = 180° - 39° - 84° = 57°$. By the sine rule with b and B known,

$$a = \frac{b \sin A}{\sin B} = \frac{21.0 \sin 57°}{\sin 39°} = 28.0 \,\text{cm}, c = \frac{b \sin C}{\sin B} = \frac{21.0 \sin 84°}{\sin 39°} = 33.2 \,\text{cm}.$$

The area, using the two sides a and b and their included angle C, is $1/2\, a b \sin C = 1/2 (28.0)(21.0) \sin 84° \approx 292$ cm² (to the nearest square centimetre).

Q2. Two sides and the included angle are known, so the cosine rule gives

$$a^2 = 9.4^2 + 13.6^2 - 2(9.4)(13.6)\cos 62° = 153.29, a = 12.4 \,\text{cm}.$$

Then $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{153.29 + 184.96 - 88.36}{2(12.38)(13.6)} = 0.7420$, so $B = 42.1°$, and $C = 180° - 62° - 42.1° = 75.9°$. The area is $1/2\, b c \sin A = 1/2 (9.4)(13.6) \sin 62° = 56.4$ cm².

Q3. By the sine rule, $\sin B = \dfrac{b \sin A}{a} = \dfrac{13.4 \sin 35°}{8.9} = 0.8636$. Hence $B = \sin^{-1}(0.8636) = 59.7°$ or $B = 180° - 59.7° = 120.3°$. Both are possible: $35° + 59.7° = 94.7° < 180°$ and $35° + 120.3° = 155.3° < 180°$, so each leaves a positive third angle (the ambiguous case, since A is acute and $a < b$). For $B = 59.7°$, $C = 85.3°$ and $c = \dfrac{a \sin C}{\sin A} = \dfrac{8.9 \sin 85.3°}{\sin 35°} = 15.5$ cm; for $B = 120.3°$, $C = 24.7°$ and $c = \dfrac{8.9 \sin 24.7°}{\sin 35°} = 6.5$ cm.

Q4. The floor is a 6.0 by 4.5 rectangle, so its diagonal is $\sqrt{6.0^2 + 4.5^2} = \sqrt{56.25} = 7.5$ m. The space diagonal stands on this floor diagonal and the vertical edge, so

$$AG = \sqrt{6.0^2 + 4.5^2 + 3.2^2} = \sqrt{66.49} = 8.2 \,\text{m}.$$

Its projection on the floor is the floor diagonal (7.5 m), so the angle with the floor is $\tan^{-1} \dfrac{3.2}{7.5} = 23.1°$. Along the 6.0 m edge AB, the segment BG is perpendicular to AB, so in right-angled triangle ABG the angle at A has $\cos(\angle GAB) = \dfrac{AB}{AG} = \dfrac{6.0}{8.2}$, giving $\angle GAB = 42.6°$.

Q5. (a) The leg $O \to A$ has bearing $072°$, so the reverse direction $A \to O$ has bearing $072° + 180° = 252°$. The next leg $A \to B$ has bearing $124°$, so the angle turned through inside the triangle at A is $\angle OAB = 252° - 124° = 128°$. (b) By the cosine rule,

$$OB^2 = 18^2 + 25^2 - 2(18)(25)\cos 128° = 1503.1, OB = 38.8 \,\text{km}.$$

(c) By the sine rule, $\sin(\angle AOB) = \frac{AB\sin 128°}{OB} = \frac{25\sin 128°}{38.8} = 0.5081$, so $\angle AOB = 30.5°$. Since B lies clockwise of the first leg, the bearing of B from O is $072° + 30.5° = 102.5°$.

Q6. The base is an 18 cm square, so a diagonal is $18\sqrt{2} = 25.5$ cm and the centre O lies $1/2(18\sqrt{2}) = 9\sqrt{2} \approx 12.73$ cm from each corner. With vertical height 12 cm, a slant edge is

$$\sqrt{12^2 + (9\sqrt{2})^2} = \sqrt{144 + 162} = \sqrt{306} = 17.5\,\text{cm},$$

inclined to the base at $\tan^{-1}\frac{12}{9\sqrt{2}} = 43.3°$. For a face, the midpoint of a base edge is $1/2(18) = 9$ cm from O, so the dihedral angle between the face and the base is $\tan^{-1}\frac{12}{9} = 53.1°$.

Q7. $F = 180° - 48° - 59° = 73°$, so by the sine rule $f = \frac{d\sin F}{\sin D} = \frac{12.5\sin 73°}{\sin 48°} = 16.1$ cm.

Q8. By the cosine rule, $c^2 = 17^2 + 23^2 - 2(17)(23)\cos 109° = 1072.6$, so $c = 32.8$ m.

Q9. The largest angle lies opposite the longest side (26 cm). By the cosine rule,

$$\cos\theta = \frac{19^2 + 14^2 - 26^2}{2(19)(14)} = \frac{-119}{532} = -0.2237,$$

so $\theta = 102.9°$ (obtuse, as the negative cosine indicates).

Q10. The area is $1/2(34)(41)\sin 76° = 676.3$ m². The third side is $\sqrt{34^2 + 41^2 - 2(34)(41)\cos 76°} = \sqrt{2162.5} = 46.5$ m, so the perimeter is $34 + 41 + 46.5 = 121.5$ m.

Q11. By the sine rule, $\sin B = \frac{b\sin A}{a} = \frac{12.0\sin 41°}{9.2} = 0.8557$, so $B = 58.8°$ or $B = 121.2°$. As A is acute and $a < b$, both are admissible ($41° + 121.2° = 162.2° < 180°$), giving **two** triangles. For $B = 58.8°$: $C = 80.2°$ and $c = \frac{9.2\sin 80.2°}{\sin 41°} = 13.8$ cm. For $B = 121.2°$: $C = 17.8°$ and $c = \frac{9.2\sin 17.8°}{\sin 41°} = 4.3$ cm.

Q12. If such a triangle existed, the sine rule would give $\sin C = \frac{c\sin B}{b} = \frac{13.9\sin 58°}{10.5} = 1.1227$. But $\sin C$ can never exceed 1, so no triangle has these measurements. (Geometrically the side $b = 10.5$ is too short to reach across from the $58°$ angle to the far side.)

Q13. At B the incoming leg $A \to B$ has bearing $118°$, so $B \to A$ has bearing $298°$; the outgoing leg $B \to C$ has bearing $203°$, so $\angle ABC = 298° - 203° = 95°$. By the cosine rule,

$$AC^2 = 46^2 + 38^2 - 2(46)(38)\cos 95° = 3864.7,\ AC = 62.2\,\text{km}.$$

For the return bearing, the sine rule gives $\sin(\angle BCA) = \frac{AB\sin 95°}{AC} = \frac{46\sin 95°}{62.2} = 0.7371$, so $\angle BCA = 47.5°$. Taking north–south and east–west components of the two legs places C south-east of A; the direction $C \to A$ then works out to a bearing of $335.5°$.

Q14. The longest rod is the space diagonal, $\sqrt{12^2 + 2.5^2 + 2.9^2} = \sqrt{158.66} = 12.6$ m. Its projection on the base is the floor diagonal $\sqrt{12^2 + 2.5^2} = 12.26$ m, so the angle with the base is $\tan^{-1}\frac{2.9}{12.26} = 13.3°$.

Q15. A cross-section perpendicular to the ridge is an isosceles triangle of base 10 m and height 3.5 m. Each half-base is 5 m, so each roof face makes an angle $\tan^{-1}\frac{3.5}{5}=35.0°$ with the horizontal. This cross-section is perpendicular to the ridge line, so the angle at its apex is the dihedral angle between the faces: $180° - 2(35.0°)=110.0°$.

Q16. The base diagonal is $30\sqrt{2}$, so the centre is $15\sqrt{2}$ from each corner and a slant edge is $\sqrt{20^2+(15\sqrt{2})^2}=\sqrt{850}=29.2$ cm. For a face, the triangle has two edges $\sqrt{850}$ and base 30, so the apex angle satisfies $\cos\theta=\frac{850+850-30^2}{2(850)}=0.4706$, giving $\theta=61.9°$. Opposite faces meet only at the apex; the cross-section through the apex and the midpoints of two opposite base edges is isosceles with equal sides equal to the slant height $\sqrt{20^2+15^2}=25$ cm and base 30 cm, so the angle between the faces is $\cos^{-1}\frac{25^2+25^2-30^2}{2(25)^2}=\cos^{-1}(0.28)=73.7°$.

Q17. In the horizontal triangle PGQ, $\angle PGQ=180°-63°-48°=69°$, so by the sine rule

$$PG=\frac{PQ\sin(\angle GQP)}{\sin(\angle PGQ)}=\frac{140\sin 48°}{\sin 69°}=111.4\,\text{m}.$$

The mast GT is vertical, so triangle PGT is right-angled at G and $GT=PG\tan 19°=111.4\tan 19°=38.4$ m.

Q18. A space diagonal is $\sqrt{8^2+6^2+4^2}=\sqrt{116}=10.8$ cm. The cross-section through a pair of diagonally opposite 8 cm edges is a rectangle measuring 8 cm by $\sqrt{6^2+4^2}=\sqrt{52}$ cm, and its two diagonals are space diagonals of the box; they bisect each other at the centre O, so each is split into halves of length $1/2\sqrt{116}$. In the triangle formed by O and the two ends A, B of one of those 8 cm edges, $OA=OB=1/2\sqrt{116}$ and $AB=8$, so by the cosine rule

$$\cos(\angle AOB)=\frac{OA^2+OB^2-AB^2}{2OA\cdot OB}=\frac{29+29-64}{2(29)}=-0.1034,$$

giving $\angle AOB=95.9°$. The diagonals therefore cross at $95.9°$ and $180°-95.9°=84.1°$; the acute angle between them is $84.1°$. (The cross-section has to be named, because a different one pairs different diagonals: the $6\times\sqrt{80}$ section gives $67.7°$ and the 4×10 section gives $43.6°$.)

Theory

The circular functions met in Mathematical Methods are evaluated exactly only at a short list of special angles. What is missing is a way of relating the sine and cosine of a **sum** or **difference** of angles to the sines and cosines of the separate angles. It is tempting — and completely wrong — to suppose that $\sin(A+B)$ equals $\sin A+\sin B$; a single numerical check, such as $\sin\frac{\pi}{2}=1$ against $\sin\frac{\pi}{4}+\sin\frac{\pi}{4}=\sqrt{2}$, disposes of that idea. The correct relationships are the **compound-angle formulas**, and from them follow the **double-angle formulas** and a compact way of rewriting $a\cos\theta+b\sin\theta$. These identities let us find new exact values, simplify expressions, prove further identities, and solve equations that mix an angle with its double.

The compound-angle formula for $\cos(A-B)$.

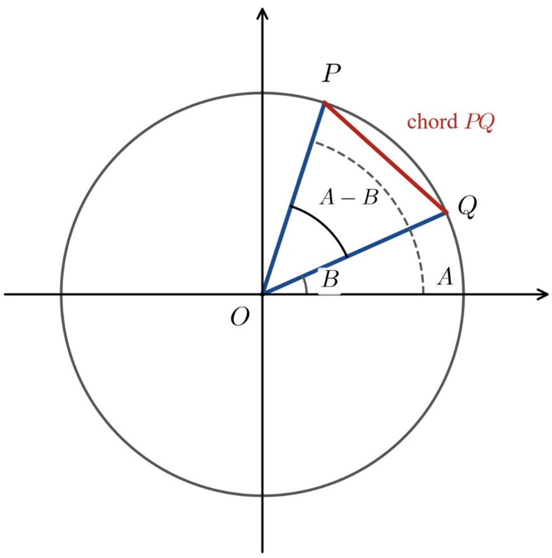
Everything in this section rests on one result, which we prove from first principles:

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

Consider the unit circle centred at the origin O. Let P be the point at angle A and Q the point at angle B, measured anticlockwise from the positive horizontal axis, so that

$$P=(\cos A,\sin A),Q=(\cos B,\sin B).$$

Both lie at distance 1 from O, and the angle $\angle POQ$ between the radii OP and OQ is $A-B$. We compute the squared length of the chord PQ in two different ways and equate the results.



First, by the distance formula,

$$PQ^2=(\cos A-\cos B)^2+(\sin A-\sin B)^2.$$

Expanding and grouping the squared terms,

$$PQ^2=(\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)-2(\cos A\cos B+\sin A\sin B).$$

Each bracketed pair equals 1 by the Pythagorean identity, so

$$PQ^2=2-2(\cos A\cos B+\sin A\sin B).$$

Second, apply the **cosine rule** to triangle POQ, in which $OP=OQ=1$ and the included angle is $A-B$:

$$PQ^2 = 1^2 + 1^2 - 2(1)(1)\cos(A-B) = 2 - 2\cos(A-B).$$

The two expressions for PQ^2 must agree. Cancelling the common 2 and dividing by -2 gives $\cos(A-B) = \cos A\cos B + \sin A\sin B$, as claimed.

The figure above is drawn with A and B acute, so that POQ is an ordinary triangle and the cosine rule applies to it directly. The identity nevertheless holds for **all** real A and B: the distance-formula calculation never used the sizes of A and B at all, and however the two radii happen to lie, the angle actually turned through from OQ to OP differs from $A-B$ only in sign or by a whole number of revolutions — and cosine, being even and of period 2π, is blind to both.

The remaining compound-angle formulas.

The other three formulas now follow with no further pictures. Replacing B by $-B$ and using the fact that cosine is even and sine is odd, $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, converts the difference formula into the **sum** formula:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B.$$

For sine, use the complementary-angle relations $\sin\theta = \cos(\pi/2 - \theta)$ and $\cos\theta = \sin(\pi/2 - \theta)$. Writing $\sin(A+B) = \cos(\pi/2 - A - B) = \cos((\pi/2 - A) - B)$ and expanding with the difference formula,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$

and replacing B by $-B$ once more gives the sine difference formula. Collected together,

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

Read the signs vertically: the sine formula keeps the sign, while the cosine formula **reverses** it. Dividing the sine formula by the cosine formula and then dividing every term by $\cos A\cos B$ produces the tangent formula,

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B},$$

valid whenever the tangents involved are defined and the denominator is non-zero.

The double-angle formulas.

Setting $B = A$ in the sum formulas collapses each compound angle into a **double angle**. From $\sin(A+A)$,

$$\sin 2A = 2\sin A\cos A.$$

From $\cos(A+A)$,

$$\cos 2A = \cos^2 A - \sin^2 A.$$

This cosine result has **three equivalent forms**, obtained by substituting the Pythagorean identity $\sin^2 A + \cos^2 A = 1$. Replacing $\sin^2 A$ by $1 - \cos^2 A$ gives a form in cosine alone; replacing $\cos^2 A$ by $1 - \sin^2 A$ gives a form in sine alone:

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

Which form to reach for depends on the problem — the sine-only form is ideal when $\sin A$ is known, the cosine-only form when $\cos A$ is known. Setting $B = A$ in the tangent formula gives

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

Rearranging the last two cosine forms isolates the squares, giving the **power-reduction** identities used to lower an even power of a circular function by one:

$$\cos^2 A = \frac{1+\cos 2A}{2}, \sin^2 A = \frac{1-\cos 2A}{2}.$$

Exact values and equations.

Because $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ and $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$, the compound-angle formulas extend the table of exact values to every multiple of $\frac{\pi}{12}$ (that is, every multiple of $15\,°$). The double-angle formulas run the other way: given one circular ratio of an angle, together with the quadrant that fixes the signs, they deliver the ratios of the doubled angle directly. When an equation contains both an angle and its double — for instance $\cos 2x + \sin x = 0$ — the standard tactic is to use a double-angle formula to write everything in terms of a **single** circular function of the single angle, producing a polynomial equation that can be factorised.

Expressing $a\cos\theta + b\sin\theta$ in the form $R\cos(\theta - \alpha)$.

A sum of a cosine and a sine wave of the same period is itself a single sinusoid. For constants a and b not both zero, expand $R\cos(\theta-\alpha)$ with the compound formula:

$$R\cos(\theta-\alpha) = R\cos\alpha\cos\theta + R\sin\alpha\sin\theta.$$

Matching this against $a\cos\theta + b\sin\theta$ requires

$$R\cos\alpha = a, R\sin\alpha = b.$$

Squaring and adding removes α (since $\cos^2\alpha + \sin^2\alpha = 1$), giving $R^2 = a^2 + b^2$, while dividing the two equations removes R:

$$R = \sqrt{a^2+b^2}\,(R>0), \tan\alpha = \frac{b}{a}.$$

When a and b are both positive the angle α lies in the first quadrant, and it is fixed uniquely by the two requirements $\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$. The same expression may equally be written as $R\sin(\theta+\beta)$; the choice is one of convenience. Because $\cos$ ranges over $[-1, 1]$, the rewritten form shows at a glance that $a\cos\theta + b\sin\theta$ oscillates between $-R$ and R: its **maximum** is R, attained when $\theta - \alpha = 0$, and its **minimum** is $-R$. The single-sinusoid form also turns an equation such as $a\cos\theta + b\sin\theta = c$ into the elementary $\cos(\theta-\alpha) = \frac{c}{R}$.

Worked examples

Example 1 — scaffolded

Using $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$, find the exact value of (a) $\cos\frac{\pi}{12}$ and (b) $\tan\frac{\pi}{12}$.

Working	Comment
$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}.$	Apply $\cos(A-B) = \cos A\cos B + \sin A\sin B$.
$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$	Substitute the known exact ratios and combine over 4.
$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}}.$	Apply $\tan(A-B)$ with $\tan\frac{\pi}{3} = \sqrt{3}$, $\tan\frac{\pi}{4} = 1$.

Working	Comment
$=\dfrac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}=\dfrac{4-2\sqrt{3}}{2}=2-\sqrt{3}.$	Rationalise by multiplying top and bottom by $\sqrt{3}-1$.

Example 2 — scaffolded

The angle θ is acute with $\tan\theta=\dfrac{2}{3}$. Find the exact values of (a) $\sin\theta$ and $\cos\theta$, (b) $\sin 2\theta$ and $\cos 2\theta$, and (c) $\tan 2\theta$.

Working	Comment
A right triangle with opposite 2 and adjacent 3 has hypotenuse $\sqrt{2^2+3^2}=\sqrt{13}$.	$\tan\theta=\dfrac{2}{3}$, and θ is acute so both ratios are positive.
$\sin\theta=\dfrac{2}{\sqrt{13}},\cos\theta=\dfrac{3}{\sqrt{13}}.$	Read the ratios from the triangle.
$\sin 2\theta=2\sin\theta\cos\theta=2\cdot\dfrac{2}{\sqrt{13}}\cdot\dfrac{3}{\sqrt{13}}=\dfrac{12}{13}.$	Double-angle sine; the $\sqrt{13}$ factors give 13.
$\cos 2\theta=\cos^2\theta-\sin^2\theta=\dfrac{9}{13}-\dfrac{4}{13}=\dfrac{5}{13}.$	Double-angle cosine in the form $\cos^2\theta-\sin^2\theta$.
$\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2\cdot\frac{2}{3}}{1-\frac{4}{9}}=\dfrac{\frac{4}{3}}{\frac{5}{9}}=\dfrac{12}{5}.$	Matches $\dfrac{\sin 2\theta}{\cos 2\theta}=\dfrac{12/13}{5/13}=\dfrac{12}{5}$.

Example 3 — unscaffolded

Solve $\cos 2x+\sin x=0$ for $0\le x\le 2\pi$.

To reduce the equation to a single circular function of x, use the sine-only form of the double-angle cosine, $\cos 2x=1-2\sin^2 x$:

$$1-2\sin^2 x+\sin x=0.$$

Multiplying by -1 and writing the quadratic in $\sin x$ in standard order,

$$2\sin^2 x-\sin x-1=0\Rightarrow(2\sin x+1)(\sin x-1)=0,$$

so $\sin x=-\dfrac{1}{2}$ or $\sin x=1$. Over $0\le x\le 2\pi$, the value $\sin x=1$ gives $x=\dfrac{\pi}{2}$, while $\sin x=-\dfrac{1}{2}$ gives the two third- and fourth-quadrant solutions $x=\pi+\dfrac{\pi}{6}=\dfrac{7\pi}{6}$ and $x=2\pi-\dfrac{\pi}{6}=\dfrac{11\pi}{6}$.

$\therefore x=\dfrac{\pi}{2},\dfrac{7\pi}{6},\dfrac{11\pi}{6}.$

Example 4 — unscaffolded

Express $\cos\theta+\sqrt{3}\sin\theta$ in the form $R\cos(\theta-\alpha)$ with $R>0$ and $0<\alpha<\dfrac{\pi}{2}$. Hence state its maximum value and solve $\cos\theta+\sqrt{3}\sin\theta=\sqrt{3}$ for $0\le\theta\le 2\pi$.

Expanding the target form, $R\cos(\theta-\alpha)=R\cos\alpha\cos\theta+R\sin\alpha\sin\theta$, and matching coefficients against $\cos\theta+\sqrt{3}\sin\theta$ gives $R\cos\alpha=1$ and $R\sin\alpha=\sqrt{3}$. Then $R^2=1^2+(\sqrt{3})^2=4$, so $R=2$, and $\tan\alpha=\frac{\sqrt{3}}{1}=\sqrt{3}$ with α in the first quadrant gives $\alpha=\frac{\pi}{3}$. Hence

$$\cos\theta+\sqrt{3}\sin\theta=2\cos\left(\theta-\frac{\pi}{3}\right).$$

Since cos never exceeds 1, the maximum value is 2, attained when $\theta-\frac{\pi}{3}=0$. Solving the equation, $2\cos\left(\theta-\frac{\pi}{3}\right)=\sqrt{3}$ becomes $\cos\left(\theta-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$. As θ runs over $[0,2\pi]$, the shifted angle $\theta-\frac{\pi}{3}$ runs over $\left[-\frac{\pi}{3},\frac{5\pi}{3}\right]$, on which $\cos=\frac{\sqrt{3}}{2}$ at $\theta-\frac{\pi}{3}=-\frac{\pi}{6}$ and $\theta-\frac{\pi}{3}=\frac{\pi}{6}$, giving $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{2}$.

$\therefore\cos\theta+\sqrt{3}\sin\theta=2\cos\left(\theta-\frac{\pi}{3}\right)$, with maximum value 2, and the equation has solutions $\theta=\frac{\pi}{6},\frac{\pi}{2}$.

Practice questions

Scaffolded

Q1. Use $\frac{5\pi}{12}=\frac{\pi}{6}+\frac{\pi}{4}$. (a) Find the exact value of $\sin\frac{5\pi}{12}$. (b) Find the exact value of $\cos\frac{5\pi}{12}$. (c) Hence find the exact value of $\tan\frac{5\pi}{12}$, giving the answer in simplest surd form.

Q2. The angle θ is acute with $\tan\theta=\frac{2}{5}$. (a) Write down $\sin\theta$ and $\cos\theta$ in surd form. (b) Find $\sin 2\theta$ and $\cos 2\theta$. (c) Find $\tan 2\theta$ and verify that it equals $\frac{\sin 2\theta}{\cos 2\theta}$.

Q3. Prove the identity $\cos^4\theta-\sin^4\theta=\cos 2\theta$. (a) Factorise the left-hand side as a difference of two squares. (b) Simplify one factor using the Pythagorean identity. (c) Identify the remaining factor as a double-angle form.

Q4. Solve $\sin 2x=\cos x$ for $0\le x\le 2\pi$. (a) Replace $\sin 2x$ by $2\sin x\cos x$. (b) Move all terms to one side and factorise out $\cos x$. (c) Solve each factor equal to zero over the given domain.

Q5. Express $f(\theta)=3\sqrt{3}\cos\theta+3\sin\theta$ in the form $R\cos(\theta-\alpha)$ with $R>0$ and $0<\alpha<\frac{\pi}{2}$. (a) Expand $R\cos(\theta-\alpha)$ and equate coefficients of $\cos\theta$ and $\sin\theta$. (b) Find R. (c) Find α. (d) State the maximum value of f and the value of θ in $[0,2\pi]$ at which it occurs.

Q6. The angle θ is acute and $\cos 2\theta=\frac{7}{25}$. (a) Use $\cos 2\theta=1-2\sin^2\theta$ to find $\sin\theta$. (b) Use $\cos 2\theta=2\cos^2\theta-1$ to find $\cos\theta$. (c) Hence find $\sin 2\theta$.

Unscaffolded

Q7. Find the exact value of $\cos\frac{7\pi}{12}$.

Q8. Simplify $\cos\left(\theta+\frac{\pi}{3}\right)+\cos\left(\theta-\frac{\pi}{3}\right)$.

Q9. The angle A lies in the third quadrant with $\sin A = -\frac{8}{17}$. Find the exact values of $\sin 2A$ and $\cos 2A$.

Q10. Solve $\cos 2x = \sin x$ for $0 \le x \le 2\pi$.

Q11. Prove the identity $\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$.

Q12. Solve $\sin 2x = \sin x$ for $0 \le x \le 2\pi$.

Q13. Express $\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta - \alpha)$ with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, and hence state its maximum value and the value of θ in $[0, 2\pi]$ at which the maximum occurs.

Q14. Solve $\cos x - \sin x = 1$ for $0 \le x \le 2\pi$.

Q15. Given that $A + B = \frac{\pi}{4}$, prove that $(1+\tan A)(1+\tan B) = 2$.

Q16. Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ by writing $3\theta = 2\theta + \theta$.

Q17. Solve $\cos 2x + 3\cos x + 2 = 0$ for $0 \le x \le 2\pi$.

Q18. Let $f(\theta) = 4\sin\theta + 3\cos\theta$. (a) Express $f(\theta)$ in the form $R\sin(\theta + \alpha)$ with $R > 0$ and α the first-quadrant angle satisfying $\tan\alpha = \frac{3}{4}$. (b) State the maximum and minimum values of $f(\theta)$. (c) Hence state the maximum value of $\frac{1}{4\sin\theta + 3\cos\theta + 7}$.

Answers

Q1. (a) $\frac{\sqrt{6}+\sqrt{2}}{4}$; (b) $\frac{\sqrt{6}-\sqrt{2}}{4}$; (c) $2+\sqrt{3}$. **Q2.** (a) $\sin\theta=\frac{2}{\sqrt{29}}$, $\cos\theta=\frac{5}{\sqrt{29}}$; (b) $\sin 2\theta=\frac{20}{29}$, $\cos 2\theta=\frac{21}{29}$; (c) $\tan 2\theta=\frac{20}{21}$. **Q3.** Proof. **Q4.** $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{3\pi}{2}$. **Q5.** $f(\theta)=6\cos\left(\theta-\frac{\pi}{6}\right)$; maximum 6 at $\theta=\frac{\pi}{6}$. **Q6.** (a) $\sin\theta=\frac{3}{5}$; (b) $\cos\theta=\frac{4}{5}$; (c) $\sin 2\theta=\frac{24}{25}$. **Q7.** $\frac{\sqrt{2}-\sqrt{6}}{4}$. **Q8.** $\cos\theta$. **Q9.** $\sin 2A=\frac{240}{289}$, $\cos 2A=\frac{161}{289}$. **Q10.** $x=\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2}$. **Q11.** Proof. **Q12.** $x=0,\frac{\pi}{3},\pi,\frac{5\pi}{3},2\pi$. **Q13.** $2\sin\left(\theta-\frac{\pi}{3}\right)$; maximum 2 at $\theta=\frac{5\pi}{6}$. **Q14.** $x=0,\frac{3\pi}{2},2\pi$. **Q15.** Proof. **Q16.** Proof. **Q17.** $x=\frac{2\pi}{3},\pi,\frac{4\pi}{3}$. **Q18.** (a) $f(\theta)=5\sin(\theta+\alpha)$, $\tan\alpha=\frac{3}{4}$; (b) maximum 5, minimum -5; (c) $\frac{1}{2}$.

Fully-worked solutions

Q1. With $\frac{5\pi}{12}=\frac{\pi}{6}+\frac{\pi}{4}$: (a) $\sin\frac{5\pi}{12}=\sin\frac{\pi}{6}\cos\frac{\pi}{4}+\cos\frac{\pi}{6}\sin\frac{\pi}{4}=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}=\frac{\sqrt{6}+\sqrt{2}}{4}$. (b) $\cos\frac{5\pi}{12}=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$. (c) $\tan\frac{5\pi}{12}=\frac{\sin(5\pi/12)}{\cos(5\pi/12)}=\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$. Multiplying top and bottom by $\sqrt{6}+\sqrt{2}$,

$$\tan\frac{5\pi}{12}=\frac{(\sqrt{6}+\sqrt{2})^2}{(\sqrt{6})^2-(\sqrt{2})^2}=\frac{8+2\sqrt{12}}{4}=\frac{8+4\sqrt{3}}{4}=2+\sqrt{3}.$$

Q2. Since $\tan\theta=\frac{2}{5}$ with θ acute, a right triangle with opposite 2 and adjacent 5 has hypotenuse $\sqrt{29}$, so (a) $\sin\theta=\frac{2}{\sqrt{29}}$ and $\cos\theta=\frac{5}{\sqrt{29}}$. (b) Then $\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{2}{\sqrt{29}}\cdot\frac{5}{\sqrt{29}}=\frac{20}{29}$ and $\cos 2\theta=\cos^2\theta-\sin^2\theta=\frac{25}{29}-\frac{4}{29}=\frac{21}{29}$. (c) $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\cdot\frac{2}{5}}{1-\frac{4}{25}}=\frac{4/5}{21/25}=\frac{20}{21}$, which equals $\frac{\sin 2\theta}{\cos 2\theta}=\frac{20/29}{21/29}=\frac{20}{21}$.

Q3. Factorising the left-hand side as a difference of two squares and using $\cos^2\theta+\sin^2\theta=1$,

$$\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)=(\cos^2\theta-\sin^2\theta)(1)=\cos^2\theta-\sin^2\theta=\cos 2\theta,$$

as required.

Q4. Writing $\sin 2x=2\sin x\cos x$, the equation $2\sin x\cos x=\cos x$ becomes $2\sin x\cos x-\cos x=0$, that is $\cos x(2\sin x-1)=0$. Hence $\cos x=0$ or $\sin x=\frac{1}{2}$. Over $0\le x\le 2\pi$, $\cos x=0$ gives $x=\frac{\pi}{2},\frac{3\pi}{2}$, and $\sin x=\frac{1}{2}$ gives $x=\frac{\pi}{6},\frac{5\pi}{6}$. Collecting, $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{3\pi}{2}$.

Q5. Expanding, $R\cos(\theta-\alpha)=R\cos\alpha\cos\theta+R\sin\alpha\sin\theta$; equating coefficients gives $R\cos\alpha=3\sqrt{3}$ and $R\sin\alpha=3$. (b) Squaring and adding, $R^2=(3\sqrt{3})^2+3^2=27+9=36$, so $R=6$. (c) Dividing, $\tan\alpha=\frac{3}{3\sqrt{3}}=\frac{1}{\sqrt{3}}$, and as α is in the first quadrant, $\alpha=\frac{\pi}{6}$. Thus $f(\theta)=6\cos\left(\theta-\frac{\pi}{6}\right)$. (d) The maximum value is 6, attained when $\theta-\frac{\pi}{6}=0$, i.e. $\theta=\frac{\pi}{6}$.

Q6. (a) From $\cos 2\theta=1-2\sin^2\theta=\frac{7}{25}$, $2\sin^2\theta=1-\frac{7}{25}=\frac{18}{25}$, so $\sin^2\theta=\frac{9}{25}$ and, as θ is acute, $\sin\theta=\frac{3}{5}$. (b) From $\cos 2\theta=2\cos^2\theta-1=\frac{7}{25}$, $2\cos^2\theta=\frac{32}{25}$, so $\cos^2\theta=\frac{16}{25}$ and $\cos\theta=\frac{4}{5}$. (c) Then $\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$.

Q7. Using $\frac{7\pi}{12}=\frac{\pi}{3}+\frac{\pi}{4}$,

$$\cos\frac{7\pi}{12}=\cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}-\sqrt{6}}{4}.$$

Q8. Expanding each term with the compound-angle cosine formula,

$$\cos\left(\theta+\frac{\pi}{3}\right)+\cos\left(\theta-\frac{\pi}{3}\right)=(\cos\theta\cos\pi/3-\sin\theta\sin\pi/3)+(\cos\theta\cos\pi/3+\sin\theta\sin\pi/3).$$

The sine terms cancel, leaving $2\cos\theta\cos\frac{\pi}{3}=2\cos\theta\cdot\frac{1}{2}=\cos\theta$.

Q9. In the third quadrant both sine and cosine are negative. With $\sin A=-\frac{8}{17}$, the identity $\cos^2 A=1-\sin^2 A=1-\frac{64}{289}=\frac{225}{289}$ gives $\cos A=-\frac{15}{17}$. Then $\sin 2A=2\sin A\cos A=2\cdot\left(-\frac{8}{17}\right)\cdot\left(-\frac{15}{17}\right)=\frac{240}{289}$, and $\cos 2A=1-2\sin^2 A=1-2\cdot\frac{64}{289}=1-\frac{128}{289}=\frac{161}{289}$.

Q10. Using $\cos 2x=1-2\sin^2 x$, the equation $1-2\sin^2 x=\sin x$ rearranges to $2\sin^2 x+\sin x-1=0$, that is $(2\sin x-1)(\sin x+1)=0$. Hence $\sin x=\frac{1}{2}$ or $\sin x=-1$. Over $0\le x\le 2\pi$, $\sin x=\frac{1}{2}$ gives $x=\frac{\pi}{6},\frac{5\pi}{6}$, and $\sin x=-1$ gives $x=\frac{3\pi}{2}$. Thus $x=\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2}$.

Q11. Using the double-angle forms $1-\cos 2\theta=2\sin^2\theta$ (from $\cos 2\theta=1-2\sin^2\theta$) and $\sin 2\theta=2\sin\theta\cos\theta$,

$$\frac{1-\cos 2\theta}{\sin 2\theta}=\frac{2\sin^2\theta}{2\sin\theta\cos\theta}=\frac{\sin\theta}{\cos\theta}=\tan\theta,$$

as required.

Q12. Writing $\sin 2x=2\sin x\cos x$, the equation $2\sin x\cos x=\sin x$ gives $\sin x(2\cos x-1)=0$, so $\sin x=0$ or $\cos x=\frac{1}{2}$. Over $0\le x\le 2\pi$, $\sin x=0$ gives $x=0,\pi,2\pi$, and $\cos x=\frac{1}{2}$ gives $x=\frac{\pi}{3},\frac{5\pi}{3}$. Thus $x=0,\frac{\pi}{3},\pi,\frac{5\pi}{3},2\pi$.

Q13. Expanding $R\sin(\theta-\alpha)=R\sin\theta\cos\alpha-R\cos\theta\sin\alpha$ and matching against $\sin\theta-\sqrt{3}\cos\theta$ gives $R\cos\alpha=1$ and $R\sin\alpha=\sqrt{3}$. Then $R^2=1+3=4$, so $R=2$, and $\tan\alpha=\sqrt{3}$ with α in the first quadrant gives $\alpha=\frac{\pi}{3}$. Hence $\sin\theta-\sqrt{3}\cos\theta=2\sin\left(\theta-\frac{\pi}{3}\right)$. The maximum value is 2, attained when $\theta-\frac{\pi}{3}=\frac{\pi}{2}$, i.e. $\theta=\frac{5\pi}{6}$.

Q14. Writing the left side in single-sinusoid form, $\cos x - \sin x = \sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$ (here $R=\sqrt{1^2+1^2}=\sqrt{2}$, with $\cos\alpha = 1/\sqrt{2}$, $\sin\alpha = -1/\sqrt{2}$ giving the shift $+\frac{\pi}{4}$). The equation becomes $\cos\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$. As x runs over $[0,2\pi]$, the angle $x+\frac{\pi}{4}$ runs over $\left[\frac{\pi}{4},\frac{9\pi}{4}\right]$, on which $\cos=\frac{1}{\sqrt{2}}$ at $\frac{\pi}{4},\frac{7\pi}{4}$ and $\frac{9\pi}{4}$. These give $x=0,\frac{3\pi}{2}$ and 2π respectively. Thus $x=0,\frac{3\pi}{2},2\pi$.

Q15. Since $A+B=\frac{\pi}{4}$ and $\tan\frac{\pi}{4}=1$, the tangent sum formula gives

$$1=\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B},$$

so $\tan A+\tan B=1-\tan A\tan B$, that is $\tan A+\tan B+\tan A\tan B=1$. Therefore

$$(1+\tan A)(1+\tan B)=1+\tan A+\tan B+\tan A\tan B=1+1=2,$$

as required.

Q16. Writing $3\theta=2\theta+\theta$ and applying the sine sum formula,

$$\sin 3\theta=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta.$$

Substituting $\sin 2\theta=2\sin\theta\cos\theta$ and $\cos 2\theta=1-2\sin^2\theta$,

$$\sin 3\theta=2\sin\theta\cos^2\theta+(1-2\sin^2\theta)\sin\theta.$$

Replacing $\cos^2\theta$ by $1-\sin^2\theta$,

$$\sin 3\theta=2\sin\theta(1-\sin^2\theta)+\sin\theta-2\sin^3\theta=2\sin\theta-2\sin^3\theta+\sin\theta-2\sin^3\theta=3\sin\theta-4\sin^3\theta,$$

as required.

Q17. Using $\cos 2x=2\cos^2 x-1$, the equation $2\cos^2 x-1+3\cos x+2=0$ becomes $2\cos^2 x+3\cos x+1=0$, that is $(2\cos x+1)(\cos x+1)=0$. Hence $\cos x=-\frac{1}{2}$ or $\cos x=-1$. Over $0\le x\le 2\pi$, $\cos x=-\frac{1}{2}$ gives $x=\frac{2\pi}{3},\frac{4\pi}{3}$, and $\cos x=-1$ gives $x=\pi$. Thus $x=\frac{2\pi}{3},\pi,\frac{4\pi}{3}$.

Q18. (a) Expanding $R\sin(\theta+\alpha)=R\cos\alpha\sin\theta+R\sin\alpha\cos\theta$ and matching against $4\sin\theta+3\cos\theta$ gives $R\cos\alpha=4$ and $R\sin\alpha=3$, so $R^2=4^2+3^2=25$, $R=5$, and $\tan\alpha=\frac{3}{4}$ with α in the first quadrant. Hence $f(\theta)=5\sin(\theta+\alpha)$. (b) Because $\sin(\theta+\alpha)$ ranges over $[-1,1]$, the maximum value of f is 5 and the minimum is -5. (c) The denominator $4\sin\theta+3\cos\theta+7=f(\theta)+7$ ranges over $[-5+7,5+7]=[2,12]$, and is always positive, so the fraction is largest when the denominator is smallest, namely 2. The maximum value is therefore $\frac{1}{2}$.

Chapter 8 Trigonometry and identities — 8D Product-to-sum and sum-to-product identities

Theory

The compound-angle formulas of the previous section expand $\sin(A \pm B)$ and $\cos(A \pm B)$ as combinations of the sines and cosines of A and B separately. Read in the other direction they do something equally useful: they let a **product** of two sines or cosines be rewritten as a **sum or difference**, and — by a simple change of variable — let a sum or difference of two sines or cosines be rewritten as a **product**. These two families of results are the **product-to-sum** and **sum-to-product** identities. They are the standard tool for finding exact values of awkward combinations, for simplifying trigonometric expressions to a single term, and for solving equations in which a sum of sines or cosines is set to zero; the same identities reappear later as the key to integrating products of trigonometric functions. Every result below is *derived* from the compound-angle formulas, so nothing new need be memorised beyond the method.

The compound-angle formulas.

The whole section rests on the four compound-angle formulas established in $8\mathrm{C}$. For all angles A and B,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,\ \sin(A-B)=\sin A\cos B-\cos A\sin B,$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B,\ \cos(A-B)=\cos A\cos B+\sin A\sin B.$$

Only these are assumed; the eight identities of this section follow from them by addition, subtraction, and one substitution.

Products to sums.

Add and subtract the two sine formulas. Adding removes the $\cos A\sin B$ term and doubles the $\sin A\cos B$ term; subtracting does the reverse:

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B,$$

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B.$$

Doing the same to the two cosine formulas — where now *adding* keeps the $\cos A\cos B$ term and *subtracting* keeps the $\sin A\sin B$ term — gives

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B,$$

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B.$$

Note the order in the last line: the difference is taken as $\cos(A-B)-\cos(A+B)$ so that the result is $+2\sin A\sin B$ rather than its negative. Reading each line from right to left states the four **product-to-sum identities**:

$$2\sin A\cos B=\sin(A+B)+\sin(A-B),$$

$$2\cos A\sin B=\sin(A+B)-\sin(A-B),$$

$$2\cos A\cos B=\cos(A+B)+\cos(A-B),$$

$$2\sin A\sin B=\cos(A-B)-\cos(A+B).$$

Each converts a product of two circular functions into a sum or difference of two others, the new angles being the **sum** $A+B$ and the **difference** $A-B$ of the originals. The left-hand coefficient is always 2; to expand a bare product such as $\sin A\sin B$, apply the identity and then halve, for example

$$\sin A\sin B=1/2\left[\cos(A-B)-\cos(A+B)\right].$$

Sums to products.

The reverse direction is obtained by naming the angles that appear on the right. In the product-to-sum identities the two angles are the sum $A+B$ and the difference $A-B$; so set

$$P=A+B,Q=A-B.$$

Adding and subtracting these recovers A and B as

$$A=\frac{P+Q}{2},B=\frac{P-Q}{2}.$$

Substituting into the four product-to-sum identities and reading them from right to left turns every sum or difference on the left into a single product on the right. This gives the four **sum-to-product identities**:

$$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2},$$

$$\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2},$$

$$\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2},$$

$$\cos P-\cos Q=-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

In words: a sum or difference of two sines, or of two cosines, equals **twice** a product of a function of the **half-sum** $P+Q/2$ and a function of the **half-difference** $P-Q/2$. The last identity carries a leading minus sign — a difference of cosines is the *negative* of a product of two sines — and this sign is the commonest slip, so it is worth committing to memory separately.

Using the identities.

Three kinds of task recur.

- **Exact values.** A sum such as $\cos 105^\circ+\cos 15^\circ$ has half-sum 60° and half-difference 45°, both exact angles, so sum-to-product turns it into $2\cos 60^\circ\cos 45^\circ$, which evaluates exactly. Product-to-sum does the same job for a product such as $2\cos 75^\circ\cos 15^\circ$.
- **Simplification.** When a ratio has a sum of sines or cosines on top and another underneath, converting both to products usually lets a common factor cancel, collapsing the expression to a single tangent, or to a simple quotient of one circular function by another — for instance
$$\frac{\sin P+\sin Q}{\cos P+\cos Q}=\frac{2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}}{2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}}=\tan\frac{P+Q}{2}.$$
- **Solving equations.** An equation of the form (sum of two sines) $=0$, or (difference of two cosines) $=0$, becomes a *product* $=0$ after applying sum-to-product; the equation then splits into two simpler equations, one for each factor, exactly as a factorised polynomial does.

Throughout, angles are measured in radians when equations are solved over an interval, and the exact special angles $\pi/6,\pi/4,\pi/3$ (equivalently $30^\circ,45^\circ,60^\circ$) supply the exact values.

Worked examples

Example 1 — scaffolded

Write each product as a sum or difference of two circular functions: (a) $2\sin 5\theta\cos 2\theta$, (b) $2\cos 7\theta\cos 3\theta$, (c) $\sin 4\theta\sin\theta$.

Working	Comment
(a) Use $2\sin A\cos B=\sin(A+B)+\sin(A-B)$ with $A=5\theta$, $B=2\theta$.	The product $\sin\times\cos$ matches the first identity.
$2\sin 5\theta\cos 2\theta=\sin 7\theta+\sin 3\theta$.	Sum of angles 7θ, difference 3θ.
(b) Use $2\cos A\cos B=\cos(A+B)+\cos(A-B)$ with $A=7\theta$, $B=3\theta$.	A $\cos\times\cos$ product gives two cosines.
$2\cos 7\theta\cos 3\theta=\cos 10\theta+\cos 4\theta$.	Angles 10θ and 4θ.
(c) The coefficient here is 1, so apply $2\sin A\sin B=\cos(A-B)-\cos(A+B)$ and halve.	Insert the factor 2, then divide by 2 at the end.
$\sin 4\theta\sin\theta=1/2[\cos 3\theta-\cos 5\theta]$.	$A-B=3\theta$, $A+B=5\theta$.

Example 2 — scaffolded

Write each sum or difference as a product: (a) $\sin 8\theta+\sin 2\theta$, (b) $\cos 9\theta-\cos 5\theta$, (c) $\sin 6\theta-\sin 2\theta$.

Working	Comment
(a) Use $\sin P+\sin Q=2\sin\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$ with $P=8\theta$, $Q=2\theta$.	Half-sum 5θ, half-difference 3θ.
$\sin 8\theta+\sin 2\theta=2\sin 5\theta\cos 3\theta$.	Twice $\sin(\text{half-sum})\cos(\text{half-difference})$.
(b) Use $\cos P-\cos Q=-2\sin\dfrac{P+Q}{2}\sin\dfrac{P-Q}{2}$ with $P=9\theta$, $Q=5\theta$.	A difference of cosines — mind the leading minus.
$\cos 9\theta-\cos 5\theta=-2\sin 7\theta\sin 2\theta$.	Half-sum 7θ, half-difference 2θ.
(c) Use $\sin P-\sin Q=2\cos\dfrac{P+Q}{2}\sin\dfrac{P-Q}{2}$ with $P=6\theta$, $Q=2\theta$.	A difference of sines uses $\cos$ then $\sin$.
$\sin 6\theta-\sin 2\theta=2\cos 4\theta\sin 2\theta$.	Half-sum 4θ, half-difference 2θ.

Example 3 — unscaffolded

Find the exact value of $\cos 105^\circ+\cos 15^\circ$.

The two angles have half-sum $1/2(105^\circ+15^\circ)=60^\circ$ and half-difference $1/2(105^\circ-15^\circ)=45^\circ$, both exact angles, so the sum-to-product identity for two cosines applies:

$$\cos 105^\circ+\cos 15^\circ=2\cos\frac{105^\circ+15^\circ}{2}\cos\frac{105^\circ-15^\circ}{2}=2\cos 60^\circ\cos 45^\circ.$$

Substituting the exact values $\cos 60^\circ=1/2$ and $\cos 45^\circ=\sqrt{2}/2$,

$$2\cos 60^\circ\cos 45^\circ=2\times 1/2\times\sqrt{2}/2=\frac{\sqrt{2}}{2}.$$

$\therefore \cos 105^\circ+\cos 15^\circ=\dfrac{\sqrt{2}}{2}$.

Example 4 — unscaffolded

Solve $\sin 3x+\sin x=\sin 2x$ for $x\in[0,2\pi]$.

The left-hand side is a sum of two sines, with half-sum $1/2(3x+x)=2x$ and half-difference $1/2(3x-x)=x$, so sum-to-product gives

$$\sin 3x + \sin x = 2\sin 2x\cos x.$$

The equation becomes $2\sin 2x\cos x = \sin 2x$; bringing every term to one side and taking out the common factor $\sin 2x$,

$$2\sin 2x\cos x - \sin 2x = 0 \Rightarrow \sin 2x(2\cos x - 1) = 0.$$

A product is zero when one of its factors is zero, so $\sin 2x = 0$ or $\cos x = 1/2$. For $\sin 2x = 0$ with $x \in [0, 2\pi]$, the angle $2x$ runs through $[0, 4\pi]$, giving $2x = 0, \pi, 2\pi, 3\pi, 4\pi$ and hence $x = 0, \pi/2, \pi, 3\pi/2, 2\pi$. For $\cos x = 1/2$ with $x \in [0, 2\pi]$, $x = \pi/3$ or $x = 5\pi/3$. Collecting and ordering all solutions,

$\therefore x = 0, \pi/3, \pi/2, \pi, 3\pi/2, 5\pi/3, 2\pi.$

Practice questions

Scaffolded

Q1. Write each product as a sum or difference of two circular functions. (a) $2\sin 6\theta\cos\theta$ (b) $2\cos 5\theta\cos 2\theta$ (c) $2\sin 6\theta\sin 3\theta$ (d) $2\cos 4\theta\sin\theta$

Q2. Write each sum or difference as a product. (a) $\sin 7\theta + \sin 3\theta$ (b) $\cos 8\theta + \cos 2\theta$ (c) $\sin 9\theta - \sin 5\theta$ (d) $\cos 4\theta - \cos 10\theta$

Q3. Find the exact value of $\sin 75^\circ + \sin 15^\circ$. (a) State the half-sum and half-difference of the two angles. (b) Apply the sum-to-product identity for $\sin P + \sin Q$. (c) Substitute the exact values of the special angles. (d) Simplify to a single exact value.

Q4. Find the exact value of $2\cos 75^\circ\cos 15^\circ$. (a) Identify A and B and apply $2\cos A\cos B = \cos(A+B) + \cos(A-B)$. (b) Evaluate the two cosines that result. (c) State the exact value.

Q5. Simplify $\dfrac{\sin 5\theta + \sin 3\theta}{\cos 5\theta + \cos 3\theta}$. (a) Write the numerator as a product using sum-to-product. (b) Write the denominator as a product using sum-to-product. (c) Cancel the common factor and simplify to a single trigonometric function.

Q6. Solve $\cos 3x + \cos x = 0$ for $x \in [0, 2\pi]$. (a) Use sum-to-product to write the left-hand side as a product. (b) Set each factor to zero. (c) Solve each equation over the given interval. (d) List all solutions.

Unscaffolded

Q7. Write $2\sin 8\theta\cos 3\theta$ as a sum of two circular functions.

Q8. Write $\cos 6\theta\sin 2\theta$ as a difference of two circular functions.

Q9. Write $\cos 2\theta - \cos 6\theta$ as a product.

Q10. Find the exact value of $\cos 15^\circ - \cos 75^\circ$.

Q11. Find the exact value of $\sin 105^\circ\sin 45^\circ$.

Q12. Find the exact value of $2\sin 195^\circ\cos 75^\circ$.

Q13. Simplify $\dfrac{\sin 7\theta - \sin\theta}{\cos 7\theta + \cos\theta}$.

Q14. Show that $\dfrac{\sin 4\theta + \sin 2\theta}{\cos 4\theta - \cos 2\theta} = -\dfrac{\cos\theta}{\sin\theta}$.

Q15. Solve $\sin 5x - \sin 3x = 0$ for $x \in [0, \pi]$.

Q16. Solve $\cos 7x - \cos 3x = 0$ for $x \in [0, \pi]$.

Q17. Solve $\cos 5x + \cos x = \cos 2x$ for $x \in [0, \pi]$.

Q18. Prove that $\sin\theta + \sin 3\theta + \sin 5\theta = \sin 3\theta(1 + 2\cos 2\theta)$ for all θ.

Answers

Q1. (a) $\sin 7\theta + \sin 5\theta$; (b) $\cos 7\theta + \cos 3\theta$; (c) $\cos 3\theta - \cos 9\theta$; (d) $\sin 5\theta - \sin 3\theta$. **Q2.** (a) $2\sin 5\theta \cos 2\theta$; (b) $2\cos 5\theta \cos 3\theta$; (c) $2\cos 7\theta \sin 2\theta$; (d) $2\sin 7\theta \sin 3\theta$. **Q3.** $\frac{\sqrt{6}}{2}$. **Q4.** $\frac{1}{2}$. **Q5.** $\tan 4\theta$. **Q6.** $x = \pi/4, \pi/2, 3\pi/4, 5\pi/4, 3\pi/2, 7\pi/4$. **Q7.** $\sin 11\theta + \sin 5\theta$. **Q8.** $1/2(\sin 8\theta - \sin 4\theta)$. **Q9.** $2\sin 4\theta \sin 2\theta$. **Q10.** $\frac{\sqrt{2}}{2}$. **Q11.** $\frac{1+\sqrt{3}}{4}$. **Q12.** $\frac{\sqrt{3}-2}{2}$. **Q13.** $\tan 3\theta$. **Q14.** Proof; see solution. **Q15.** $x = 0, \pi/8, 3\pi/8, 5\pi/8, 7\pi/8, \pi$. **Q16.** $x = 0, \pi/5, 2\pi/5, \pi/2, 3\pi/5, 4\pi/5, \pi$. **Q17.** $x = \pi/9, \pi/4, 5\pi/9, 3\pi/4, 7\pi/9$. **Q18.** Proof; see solution.

Fully-worked solutions

Q1. (a) With $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ and $A = 6\theta$, $B = \theta$, $2\sin 6\theta \cos\theta = \sin 7\theta + \sin 5\theta$. (b) With $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ and $A = 5\theta$, $B = 2\theta$, $2\cos 5\theta \cos 2\theta = \cos 7\theta + \cos 3\theta$. (c) With $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ and $A = 6\theta$, $B = 3\theta$, $2\sin 6\theta \sin 3\theta = \cos 3\theta - \cos 9\theta$. (d) With $2\cos A \sin B = \sin(A+B) - \sin(A-B)$ and $A = 4\theta$, $B = \theta$, $2\cos 4\theta \sin\theta = \sin 5\theta - \sin 3\theta$.

Q2. (a) Half-sum 5θ, half-difference 2θ: $\sin 7\theta + \sin 3\theta = 2\sin 5\theta \cos 2\theta$. (b) Half-sum 5θ, half-difference 3θ: $\cos 8\theta + \cos 2\theta = 2\cos 5\theta \cos 3\theta$. (c) $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$ with $P = 9\theta$, $Q = 5\theta$ gives $\sin 9\theta - \sin 5\theta = 2\cos 7\theta \sin 2\theta$. (d) $\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$ with $P = 4\theta$, $Q = 10\theta$ gives half-sum 7θ and half-difference -3θ, so $\cos 4\theta - \cos 10\theta = -2\sin 7\theta \sin(-3\theta) = 2\sin 7\theta \sin 3\theta$, using $\sin(-3\theta) = -\sin 3\theta$.

Q3. (a) Half-sum $1/2(75^\circ + 15^\circ) = 45^\circ$ and half-difference $1/2(75^\circ - 15^\circ) = 30^\circ$. (b) By $\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$,

$$\sin 75^\circ + \sin 15^\circ = 2\sin 45^\circ \cos 30^\circ.$$

(c),(d) With $\sin 45^\circ = \sqrt{2}/2$ and $\cos 30^\circ = \sqrt{3}/2$,

$$2\sin 45^\circ \cos 30^\circ = 2 \times \sqrt{2}/2 \times \sqrt{3}/2 = \frac{\sqrt{6}}{2}.$$

Q4. (a) With $A = 75^\circ$, $B = 15^\circ$, $2\cos 75^\circ \cos 15^\circ = \cos(75^\circ + 15^\circ) + \cos(75^\circ - 15^\circ) = \cos 90^\circ + \cos 60^\circ$. (b) Here $\cos 90^\circ = 0$ and $\cos 60^\circ = 1/2$. (c) Hence $2\cos 75^\circ \cos 15^\circ = 0 + 1/2 = 1/2$.

Q5. The numerator is a sum of two sines with half-sum 4θ and half-difference θ: $\sin 5\theta + \sin 3\theta = 2\sin 4\theta \cos\theta$. The denominator is a sum of two cosines with the same half-sum and half-difference: $\cos 5\theta + \cos 3\theta = 2\cos 4\theta \cos\theta$. Dividing and cancelling the common factor $2\cos\theta$,

$$\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \frac{2\sin 4\theta \cos\theta}{2\cos 4\theta \cos\theta} = \frac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta.$$

Q6. By sum-to-product with half-sum $2x$ and half-difference x, $\cos 3x + \cos x = 2\cos 2x \cos x$, so the equation is $2\cos 2x \cos x = 0$, giving $\cos 2x = 0$ or $\cos x = 0$. For $\cos 2x = 0$ with $x \in [0, 2\pi]$, the angle $2x$ runs through $[0, 4\pi]$, so $2x = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$ and $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. For $\cos x = 0$, $x = \pi/2, 3\pi/2$. Ordering all solutions, $x = \pi/4, \pi/2, 3\pi/4, 5\pi/4, 3\pi/2, 7\pi/4$.

Q7. With $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ and $A = 8\theta$, $B = 3\theta$, $2\sin 8\theta \cos 3\theta = \sin 11\theta + \sin 5\theta$.

Q8. The coefficient is 1, so use $2\cos A\sin B=\sin(A+B)-\sin(A-B)$ and halve. With $A=6\theta$, $B=2\theta$, $2\cos 6\theta\sin 2\theta=\sin 8\theta-\sin 4\theta$, hence $\cos 6\theta\sin 2\theta=1/2(\sin 8\theta-\sin 4\theta)$.

Q9. Using $\cos P-\cos Q=-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$ with $P=2\theta$, $Q=6\theta$, the half-sum is 4θ and the half-difference is -2θ, so $\cos 2\theta-\cos 6\theta=-2\sin 4\theta\sin(-2\theta)=2\sin 4\theta\sin 2\theta$, since $\sin(-2\theta)=-\sin 2\theta$.

Q10. By $\cos P-\cos Q=-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$ with $P=15^\circ$, $Q=75^\circ$, the half-sum is 45° and the half-difference is -30°, so

$$\cos 15^\circ-\cos 75^\circ=-2\sin 45^\circ\sin(-30^\circ)=2\sin 45^\circ\sin 30^\circ=2\times\sqrt{2}/2\times 1/2=\frac{\sqrt{2}}{2}.$$

Q11. The coefficient is 1, so use $2\sin A\sin B=\cos(A-B)-\cos(A+B)$ and halve, with $A=105^\circ$, $B=45^\circ$:

$$\sin 105^\circ\sin 45^\circ=1/2[\cos 60^\circ-\cos 150^\circ]=1/2[1/2-(-\sqrt{3}/2)]=1/2\cdot\frac{1+\sqrt{3}}{2}=\frac{1+\sqrt{3}}{4},$$

using $\cos 150^\circ=-\sqrt{3}/2$.

Q12. By $2\sin A\cos B=\sin(A+B)+\sin(A-B)$ with $A=195^\circ$, $B=75^\circ$, the sum of the angles is 270° and the difference is 120°, so

$$2\sin 195^\circ\cos 75^\circ=\sin 270^\circ+\sin 120^\circ=-1+\sqrt{3}/2=\frac{\sqrt{3}-2}{2},$$

using $\sin 270^\circ=-1$ and $\sin 120^\circ=\sqrt{3}/2$.

Q13. The numerator is a difference of two sines with half-sum 4θ and half-difference 3θ: $\sin 7\theta-\sin\theta=2\cos 4\theta\sin 3\theta$. The denominator is a sum of two cosines with the same half-sum and half-difference: $\cos 7\theta+\cos\theta=2\cos 4\theta\cos 3\theta$. Cancelling the common factor $2\cos 4\theta$,

$$\frac{\sin 7\theta-\sin\theta}{\cos 7\theta+\cos\theta}=\frac{2\cos 4\theta\sin 3\theta}{2\cos 4\theta\cos 3\theta}=\frac{\sin 3\theta}{\cos 3\theta}=\tan 3\theta.$$

Q14. The numerator is a sum of two sines with half-sum 3θ and half-difference θ: $\sin 4\theta+\sin 2\theta=2\sin 3\theta\cos\theta$. The denominator is a difference of two cosines with the same half-sum and half-difference: $\cos 4\theta-\cos 2\theta=-2\sin 3\theta\sin\theta$. Dividing and cancelling the common factor $2\sin 3\theta$,

$$\frac{\sin 4\theta+\sin 2\theta}{\cos 4\theta-\cos 2\theta}=\frac{2\sin 3\theta\cos\theta}{-2\sin 3\theta\sin\theta}=-\frac{\cos\theta}{\sin\theta},$$

as required.

Q15. By $\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$ with $P=5x$, $Q=3x$, the half-sum is $4x$ and the half-difference is x, so $\sin 5x-\sin 3x=2\cos 4x\sin x$ and the equation is $2\cos 4x\sin x=0$. Hence $\cos 4x=0$ or $\sin x=0$. For $\cos 4x=0$ with $x\in[0,\pi]$, the angle $4x$ runs through $[0,4\pi]$, so $4x=\pi/2,3\pi/2,5\pi/2,7\pi/2$ and $x=\pi/8,3\pi/8,5\pi/8,7\pi/8$. For $\sin x=0$, $x=0,\pi$. Ordering all solutions, $x=0,\pi/8,3\pi/8,5\pi/8,7\pi/8,\pi$.

Q16. By $\cos P-\cos Q=-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$ with $P=7x$, $Q=3x$, the half-sum is $5x$ and the half-difference is $2x$, so $\cos 7x-\cos 3x=-2\sin 5x\sin 2x$ and the equation is $-2\sin 5x\sin 2x=0$. Hence $\sin 5x=0$ or $\sin 2x=0$. For $\sin 5x=0$ with $x\in[0,\pi]$, the angle $5x$ runs through $[0,5\pi]$, so $5x=0,\pi,2\pi,3\pi,4\pi,5\pi$ and $x=0,\pi/5,2\pi/5,3\pi/5,4\pi/5,\pi$. For $\sin 2x=0$, $2x=0,\pi,2\pi$ and $x=0,\pi/2,\pi$. Ordering all solutions, $x=0,\pi/5,2\pi/5,\pi/2,3\pi/5,4\pi/5,\pi$.

Q17. The left-hand side is a sum of two cosines with half-sum $3x$ and half-difference $2x$, so $\cos 5x+\cos x=2\cos 3x\cos 2x$ and the equation becomes $2\cos 3x\cos 2x=\cos 2x$. Taking out the common factor, $\cos 2x(2\cos 3x-1)=0$, so $\cos 2x=0$ or $\cos 3x=1/2$. For $\cos 2x=0$ with $x\in[0,\pi]$, $2x\in[0,2\pi]$ gives $2x=\pi/2, 3\pi/2$ and $x=\pi/4, 3\pi/4$. For $\cos 3x=1/2$ with $x\in[0,\pi]$, $3x\in[0,3\pi]$ gives $3x=\pi/3, 5\pi/3, 7\pi/3$ and $x=\pi/9, 5\pi/9, 7\pi/9$. Ordering all solutions, $x=\pi/9, \pi/4, 5\pi/9, 3\pi/4, 7\pi/9$.

Q18. Pair the first and last terms and apply sum-to-product, with half-sum $1/2(\theta+5\theta)=3\theta$ and half-difference $1/2(5\theta-\theta)=2\theta$:

$$\sin\theta+\sin 5\theta=2\sin 3\theta\cos 2\theta.$$

Adding the remaining middle term $\sin 3\theta$ to both sides,

$$\sin\theta+\sin 3\theta+\sin 5\theta=2\sin 3\theta\cos 2\theta+\sin 3\theta=\sin 3\theta(2\cos 2\theta+1)=\sin 3\theta(1+2\cos 2\theta),$$

as required.