Specialist Mathematics Units 1 & 2

Chapter 6 — Matrices

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Contents

Theory

A great deal of mathematical information is naturally rectangular: a timetable, a price list, the scores of several teams across several rounds. A **matrix** (plural **matrices**) is a rectangular array of numbers, written inside square brackets, in which the position of each number carries meaning. Matrices give a single object to name, store and manipulate such data, and the arithmetic developed for them turns out to model an enormous range of situations — networks, transformations of the plane, systems of linear equations — studied in this chapter and beyond. This first section fixes the language (notation, dimension and the named types) and the two most basic operations, **addition** and **scalar multiplication**, together with the algebraic laws they obey. Multiplication of matrices is taken up in Section 6B, and determinants and inverses in Section 6C.

Notation and dimension.

A matrix is usually named with a capital letter. The **order** (or **dimension**, or **size**) of a matrix is the pair of counts *rows* × *columns*, always in that sequence: a matrix with m rows and n columns has order $m \times n$, read "m by n". Because the row count is quoted first, a 3×2 matrix is *not* the same shape as a 2×3 matrix. A matrix of order $m \times n$ has exactly mn **elements** (or **entries**).

The element lying in row i and column j is written a_{ij} — the lower-case letter matching the matrix name, carrying two subscripts with the **row index first** and the **column index second**. A general $m \times n$ matrix A is therefore

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

sometimes abbreviated $A = [a_{ij}]$. The subscript convention never changes: in a_{ij} the first number is the row and the second is the column, so a_{23} sits in row 2, column 3, whereas a_{32} sits in row 3, column 2 — in general a quite different entry.

Representing data.

Because each row and each column of a matrix can be given a fixed meaning, a matrix is a compact **data store**. For instance, a recycling depot records the mass, in tonnes, of glass, paper and plastic it collects in each of two months in the matrix

$$R = \begin{bmatrix} 14 & 17 \\ 23 & 19 \\ 8 & 11 \end{bmatrix},$$

where the rows are the materials (glass, paper, then plastic) and the columns are the months (March, then April). The order of R is 3×2, so R holds $3 \times 2 = 6$ figures. Reading down a column gives one month's full collection; reading across a row tracks one material over the two months. The element $r_{32} = 11$ lies in row 3, column 2, and records that 11 tonnes of plastic were collected in April. Setting data out this way lets whole tables be added, scaled or compared in one stroke, as the operations below show.

Special types of matrix.

Several shapes occur so often that they are named.

- A **row matrix** (or row vector) has a single row: order $1 \times n$, for example $\begin{bmatrix} 6 & -2 & 9 \end{bmatrix}$.
- A **column matrix** (or column vector) has a single column: order $m \times 1$, for example $\begin{bmatrix} 5 \\ 0 \\ -4 \end{bmatrix}$.

- A **square matrix** has equally many rows and columns; a square matrix of order $n \times n$ is said to be *of order* n. The entries $a_{11}, a_{22}, \dots, a_{nn}$ running from top left to bottom right form its **leading** (or **main**) **diagonal**.
- The **zero matrix** O has every element equal to 0; there is one of each order, and it plays the role of 0 in matrix addition.
- A **diagonal matrix** is a square matrix all of whose off-diagonal entries are zero; that is, $a_{ij}=0$ whenever $i \neq j$. For example $\begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix}$ is diagonal.
- The **identity matrix** I_n is the diagonal matrix of order n whose leading diagonal entries are all 1:

$$I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  It behaves as a "1" for matrix multiplication, as Section 6B shows.

Transpose.

The **transpose** of an $m \times n$ matrix A, written A^T, is the $n \times m$ matrix obtained by interchanging the rows and columns: the first row of A becomes the first column of A^T, and so on. In symbols, the entry of A^T in row i, column j is

$$(A^T)_{ij}=a_{ji}.$$

Transposing a matrix twice restores it, so $(A^T)^T=A$ always. For example,

$$A=\begin{bmatrix} 2 & 5 & -1 \\ 0 & 7 & 4 \end{bmatrix} \Rightarrow A^T=\begin{bmatrix} 2 & 0 \\ 5 & 7 \\ -1 & 4 \end{bmatrix},$$

of order 3×2.

Symmetric matrices.

A square matrix A is **symmetric** when it equals its own transpose:

$$A=A^T, \text{equivalently } a_{ij}=a_{ji} \text{ for all } i, j.$$

Every entry then matches its mirror image in the leading diagonal, so a symmetric matrix is unchanged by reflection in that diagonal. Only a square matrix can be symmetric, because A^T has order $n \times m$ while A has order $m \times n$, and the two can be equal only if $m=n$. Every diagonal matrix — and hence every identity matrix — is symmetric, since its only non-zero entries lie on the diagonal itself.

Equality.

Two matrices are **equal** exactly when *both* of the following hold: they have the **same order**, and **every pair of corresponding elements is equal**. Order alone or matching entries alone is not enough — a 2×2 matrix can never equal a 2×3 matrix, whatever the entries. When matrices containing pronumerals are declared equal, equating corresponding elements yields ordinary equations to solve, the standard route to unknowns inside a matrix statement.

Addition and subtraction.

Matrices of the **same order** are added and subtracted **element by element**; the result has that same order. If $A=[a_{ij}]$ and $B=[b_{ij}]$ are both $m \times n$,

$$(A+B)_{ij}=a_{ij}+b_{ij}, (A-B)_{ij}=a_{ij}-b_{ij}.$$

If the orders differ, the sum and difference are simply **undefined**: there is no way to pair the entries. Thus

$$\begin{bmatrix} 8 & -3 & 0 \\ 1 & 5 & -6 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 7 \\ 9 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 7 \\ 10 & 0 & -4 \end{bmatrix},$$

the sum again having order 2×3.

Scalar multiplication.

To multiply a matrix by a real number k (a **scalar**), multiply **every** element by k:

$$(kA)_{ij} = k a_{ij}.$$

The order is unchanged. Taking $k = -1$ gives the **negative** $-A$, whose entries are those of A with reversed sign; subtraction is then the same as adding a negative, $A - B = A + (-1)B$. For example,

$$5 \begin{bmatrix} -4 & 6 \\ 10 & -1 \end{bmatrix} = \begin{bmatrix} -20 & 30 \\ 50 & -5 \end{bmatrix}.$$

The algebra of matrices.

Because addition and scalar multiplication act separately on each entry, the arithmetic of matrices inherits the arithmetic of real numbers. For all matrices A, B, C of the same order, all scalars k, l, and the zero matrix O of that order:

$$\& A + B = B + A \,\&\,\& \text{(addition is commutative)}$$
$$\& (A + B) + C = A + (B + C) \,\&\,\& \text{(addition is associative)}$$
$$\& A + O = A \,\&\,\& (O \text{ is the additive identity)}$$
$$\& A + (-A) = O \,\&\,\& \text{(additive inverse)}$$
$$\& k(A + B) = kA + kB \,\&\,\& \text{(scalar over a matrix sum)}$$
$$\& (k + l)A = kA + lA \,\&\,\& \text{(scalar over a scalar sum)}$$
$$\& (kl)A = k(lA), 1A = A, 0A = O.$$

Each law is proved by checking a single typical entry. For commutativity, the entry in row i, column j of $A + B$ is $a_{ij} + b_{ij}$, while the corresponding entry of $B + A$ is $b_{ij} + a_{ij}$; these are equal because *addition of real numbers* is commutative. As this holds in every position, $A + B = B + A$. Associativity follows the same way, from $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$, and the distributive laws from $k(a_{ij} + b_{ij}) = k a_{ij} + k b_{ij}$. The key point is that matrices of equal order behave, under these two operations, exactly like the numbers inside them.

Transposition interacts neatly with both operations. For matrices of the same order and any scalar k,

$$(A + B)^T = A^T + B^T, (kA)^T = kA^T, (A^T)^T = A.$$

The first holds because interchanging rows and columns and adding entrywise can be done in either order; the others are equally immediate from the definitions. These identities make it easy to recognise, for instance, that the sum of two symmetric matrices is again symmetric.

Worked examples

Example 1 — scaffolded

A native-plant nursery records the number of seedlings of three species propagated in each of its two greenhouses in the matrix

$$N = \begin{bmatrix} 40 & 55 \\ 32 & 28 \\ 47 & 39 \end{bmatrix},$$

where the rows are the species (wattle, banksia, then grevillea) and the columns are the greenhouses (North, then South). State the order of N and its number of elements, write down n_{12} and n_{31}, explain what n_{22} represents, and state whether N is square.

Working	Comment
N has 3 rows and 2 columns, so its order is 3×2.	Order is rows × columns; the row count is written first.
Number of elements $=3\times 2=6$.	An $m\times n$ matrix has mn entries.
$n_{12}=55$.	Row 1, column 2: wattle seedlings in the South greenhouse.
$n_{31}=47$.	Row 3, column 1: grevillea seedlings in the North greenhouse.
$n_{22}=28$, the number of banksia seedlings in the South greenhouse.	In n_{ij} the subscripts name the row then the column; row 2 is banksia, column 2 is South.
Since $3\neq 2$, N is rectangular, not square.	A square matrix needs equal row and column counts.

So N is a 3×2 matrix with six elements, $n_{12}=55$, $n_{31}=47$, $n_{22}=28$ counts the banksia seedlings in the South greenhouse, and N is not square.

Example 2 — scaffolded

Let $A=\begin{bmatrix}3 & -5\\ 7 & 2\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 4\\ 6 & -8\end{bmatrix}$. Find (a) $A+B$, (b) $A-B$, (c) $4A$, and (d) $2A-3B$.

Working	Comment
A and B are both 2×2, so all four combinations are defined.	Addition and subtraction need matrices of equal order.
$A+B=\begin{bmatrix}3+(-1) & -5+4\\ 7+6 & 2+(-8)\end{bmatrix}=\begin{bmatrix}2 & -1\\ 13 & -6\end{bmatrix}$.	Add corresponding entries.
$A-B=\begin{bmatrix}3-(-1) & -5-4\\ 7-6 & 2-(-8)\end{bmatrix}=\begin{bmatrix}4 & -9\\ 1 & 10\end{bmatrix}$.	Subtract entry by entry; note $3-(-1)=4$.
$4A=\begin{bmatrix}4(3) & 4(-5)\\ 4(7) & 4(2)\end{bmatrix}=\begin{bmatrix}12 & -20\\ 28 & 8\end{bmatrix}$.	A scalar multiplies every entry.
$2A=\begin{bmatrix}6 & -10\\ 14 & 4\end{bmatrix}$, $3B=\begin{bmatrix}-3 & 12\\ 18 & -24\end{bmatrix}$.	Form each scalar multiple first.
$2A-3B=\begin{bmatrix}6-(-3) & -10-12\\ 14-18 & 4-(-24)\end{bmatrix}=\begin{bmatrix}9 & -22\\ -4 & 28\end{bmatrix}$.	Then subtract entrywise.

Example 3 — unscaffolded

Let $C=\begin{bmatrix}5 & -2 & 3\\ -2 & 8 & k\\ 3 & 7 & 1\end{bmatrix}$. Write down C^T, and find the value of k for which C is symmetric.

Interchanging the rows and columns of C,

$$C^T=\begin{bmatrix}5 & -2 & 3\\ -2 & 8 & 7\\ 3 & k & 1\end{bmatrix}.$$

The matrix C is symmetric precisely when $C=C^T$, that is when $c_{ij}=c_{ji}$ in every position. The diagonal entries match automatically, and the pairs already agree: $c_{12}=c_{21}=-2$ and $c_{13}=c_{31}=3$. The only positions still to reconcile are row 2, column 3 and row 3, column 2, where $c_{23}=k$ and $c_{32}=7$; symmetry requires $c_{23}=c_{32}$, so $k=7$.

$\therefore$ C is symmetric when $k=7$.

Example 4 — unscaffolded

The matrices $A=\begin{bmatrix}2 & -4\\ 5 & 1\end{bmatrix}$ and $B=\begin{bmatrix}11 & 8\\ -1 & 13\end{bmatrix}$ satisfy $3X+A=B$. Find the matrix X.

Treating the matrix equation exactly as its scalar analogue, subtract A from both sides and then divide by the scalar 3; every step is a legitimate matrix operation because X, A and B are all 2×2. First,

$$3X=B-A=\begin{bmatrix}11-2 & 8-(-4)\\ -1-5 & 13-1\end{bmatrix}=\begin{bmatrix}9 & 12\\ -6 & 12\end{bmatrix}.$$

Multiplying by the scalar $1/3$,

$$X=\frac{1}{3}\begin{bmatrix}9 & 12\\ -6 & 12\end{bmatrix}=\begin{bmatrix}3 & 4\\ -2 & 4\end{bmatrix}.$$

As a check, $3X+A=\begin{bmatrix}9 & 12\\ -6 & 12\end{bmatrix}+\begin{bmatrix}2 & -4\\ 5 & 1\end{bmatrix}=\begin{bmatrix}11 & 8\\ -1 & 13\end{bmatrix}=B$.

$\therefore X=\begin{bmatrix}3 & 4\\ -2 & 4\end{bmatrix}$.

Practice questions

Scaffolded

Q1. A stadium canteen records the number of pies, chips and drinks sold at each of two matches in the matrix

$$T=\begin{bmatrix}120 & 95\\ 84 & 110\\ 156 & 143\end{bmatrix},$$

where the rows are the items (pies, chips, then drinks) and the columns are the matches (Round 1, then Round 2). (a) State the order of T. (b) How many elements does T have? (c) Write down the values of t_{12} and t_{31}. (d) Explain what t_{22} represents.

Q2. Consider the matrices

$$A=\begin{bmatrix}6 & 0\\ 0 & -3\end{bmatrix}, B=\begin{bmatrix}0 & 0 & 0\end{bmatrix}, C=\begin{bmatrix}2 & -1\\ -1 & 5\end{bmatrix}, D=\begin{bmatrix}9\\ 4\end{bmatrix}, E=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(a) State the order of each matrix. (b) Classify each matrix, using every applicable name from: row, column, square, zero, diagonal, identity, symmetric.

Q3. Let $A=\begin{bmatrix}4 & -2\\ -6 & 9\end{bmatrix}$ and $B=\begin{bmatrix}7 & 3\\ 1 & -5\end{bmatrix}$. (a) Find $A+B$. (b) Find $A-B$. (c) Find $5A$. (d) Find $3A-2B$.

Q4. Let $P=\begin{bmatrix} 3 & -1 & 6 \\ 0 & 5 & -2 \end{bmatrix}$. (a) State the order of P. (b) Write down P^T and state its order. (c) Given $Q=\begin{bmatrix} 4 & -7 \\ -7 & 2 \end{bmatrix}$, show that $Q^T=Q$ and state what this tells you about Q. (d) Explain why a symmetric matrix must be square.

Q5. The matrices $\begin{bmatrix} 2x & 5 \\ y+1 & 3z \end{bmatrix}$ and $\begin{bmatrix} 14 & 5 \\ -4 & 9 \end{bmatrix}$ are equal. (a) State the two conditions for two matrices to be equal. (b) Equate corresponding elements to form equations in x, y and z. (c) Solve for x, y and z.

Q6. Let $A=\begin{bmatrix} 5 & -3 \\ 2 & 8 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 4 \\ 7 & 0 \end{bmatrix}$. (a) Find the matrix M for which $A+M=B$. (b) Evaluate $A+B$ and $B+A$, and confirm they are equal. (c) Name the property of matrix addition demonstrated in part (b).

Unscaffolded

Q7. Given $A=\begin{bmatrix} -2 & 6 \\ 4 & -9 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & -1 \\ 3 & 7 \end{bmatrix}$, find $2A+B$.

Q8. A drone-delivery service records the number of parcels delivered to three towns on each of two days in the matrix

$$D=\begin{bmatrix} 18 & 25 \\ 12 & 20 \\ 9 & 14 \end{bmatrix},$$

where the rows are the towns (Yarrawonga, Cobram, then Numurkah) and the columns are the days (Friday, then Saturday). State the order of D, explain what d_{21} represents, and find the total number of parcels delivered on Saturday.

Q9. Given $P=\begin{bmatrix} 2 & -4 & 1 \\ 0 & 6 & -3 \end{bmatrix}$ and $Q=\begin{bmatrix} 5 & 1 & -2 \\ 7 & -3 & 4 \end{bmatrix}$, find $3P-2Q$.

Q10. The matrix $A=\begin{bmatrix} 4 & p & -5 \\ 6 & 2 & 8 \\ -5 & 8 & 0 \end{bmatrix}$ is symmetric. Find the value of p.

Q11. The matrices $A=\begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 19 & 7 \\ -2 & 22 \end{bmatrix}$ satisfy $4X+A=B$. Find the matrix X.

Q12. The matrices $\begin{bmatrix} a+b & 8 \\ -3 & a-b \end{bmatrix}$ and $\begin{bmatrix} 10 & 8 \\ -3 & 4 \end{bmatrix}$ are equal. Find the values of a and b.

Q13. Write down the 3×3 diagonal matrix whose leading-diagonal entries, from top left to bottom right, are 4, -2 and 9. State, with a reason, whether this matrix is symmetric.

Q14. Let $A=\begin{bmatrix} 7 & -2 \\ -5 & 3 \end{bmatrix}$. Write down $-A$, and hence evaluate $A+(-A)$. What is the special name of the resulting matrix?

Q15. Let $A=\begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -6 \\ 0 & 7 \end{bmatrix}$. By computing both sides separately, verify that $(A+B)^T=A^T+B^T$.

Q16. A tutoring centre records the number of Year 11 and Year 12 students enrolled in Physics and Chemistry in two consecutive terms:

$$T_1=\begin{bmatrix} 14 & 9 \\ 11 & 16 \end{bmatrix}, T_2=\begin{bmatrix} 18 & 12 \\ 15 & 13 \end{bmatrix},$$

where the rows are the year levels (Year 11, then Year 12) and the columns are the subjects (Physics, then Chemistry). (a) Find T_1+T_2 and state what its entries represent. (b) Find T_2-T_1, and interpret the meaning of its negative entry.

Q17. Let A and B be symmetric matrices of the same order, so that $A^T=A$ and $B^T=B$. Prove that $A+B$ is symmetric.

Q18. Let $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. Find scalars s and t for which

$$sA+tB=\begin{bmatrix}3 & 10\\ 4 & 3\end{bmatrix}.$$

Answers

Q1. (a) 3×2; (b) 6; (c) $t_{12}=95$, $t_{31}=156$; (d) $t_{22}=110$: the number of chips sold at Round 2. **Q2.** (a) A: 2×2; B: 1×3; C: 2×2; D: 2×1; E: 3×3. (b) A square and diagonal (and symmetric); B row and zero; C square and symmetric; D column; E square, diagonal, symmetric and the identity I_3. **Q3.** (a) $\begin{bmatrix}11 & 1\\ -5 & 4\end{bmatrix}$; (b) $\begin{bmatrix}-3 & -5\\ -7 & 14\end{bmatrix}$; (c) $\begin{bmatrix}20 & -10\\ -30 & 45\end{bmatrix}$; (d) $\begin{bmatrix}-2 & -12\\ -20 & 37\end{bmatrix}$. **Q4.** (a) 2×3; (b) $P^T=\begin{bmatrix}3 & 0\\ -1 & 5\\ 6 & -2\end{bmatrix}$, order 3×2; (c) $Q^T=Q$, so Q is symmetric; (d) see solution. **Q5.** $x=7$, $y=-5$, $z=3$. **Q6.** (a) $M=\begin{bmatrix}-6 & 7\\ 5 & -8\end{bmatrix}$; (b) $A+B=B+A=\begin{bmatrix}4 & 1\\ 9 & 8\end{bmatrix}$; (c) commutativity of matrix addition. **Q7.** $\begin{bmatrix}1 & 11\\ 11 & -11\end{bmatrix}$. **Q8.** Order 3×2; $d_{21}=12$: parcels delivered to Cobram on Friday; Saturday total $=25+20+14=59$. **Q9.** $\begin{bmatrix}-4 & -14 & 7\\ -14 & 24 & -17\end{bmatrix}$. **Q10.** $p=6$. **Q11.** $X=\begin{bmatrix}4 & 2\\ -2 & 5\end{bmatrix}$. **Q12.** $a=7$, $b=3$. **Q13.** $\begin{bmatrix}4 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 9\end{bmatrix}$; symmetric, since every off-diagonal entry is 0, so it equals its transpose. **Q14.** $-A=\begin{bmatrix}-7 & 2\\ 5 & -3\end{bmatrix}$; $A+(-A)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, the zero matrix O. **Q15.** Both sides equal $\begin{bmatrix}4 & -2\\ -2 & 12\end{bmatrix}$. **Q16.** (a) $T_1+T_2=\begin{bmatrix}32 & 21\\ 26 & 29\end{bmatrix}$, the total enrolments across the two terms; (b) $T_2-T_1=\begin{bmatrix}4 & 3\\ 4 & -3\end{bmatrix}$; the -3 shows Year 12 Chemistry enrolment fell by 3. **Q17.** See solution (proof). **Q18.** $s=3$, $t=4$.

Fully-worked solutions

Q1. (a) T has 3 rows and 2 columns, so its order is 3×2. (b) An $m\times n$ matrix has mn elements, so T has $3\times2=6$ elements. (c) t_{12} is the entry in row 1, column 2: $t_{12}=95$. t_{31} is the entry in row 3, column 1: $t_{31}=156$. (d) t_{22} lies in row 2 (chips) and column 2 (Round 2), so $t_{22}=110$ is the number of chips sold at the Round 2 match.

Q2. (a) A has order 2×2; B has order 1×3; C has order 2×2; D has order 2×1; E has order 3×3. (b) A is square, and since its off-diagonal entries are both 0 it is diagonal (and therefore symmetric). B is a row matrix, and as every entry is 0 it is also a zero matrix. C is square, and because $c_{12}=c_{21}=-1$ it equals its transpose, so it is symmetric. D is a column matrix. E is square with 1s on the leading diagonal and 0s elsewhere, so it is the identity matrix I_3; being diagonal it is also symmetric.

Q3. Each combination is entrywise. (a) $A+B=\begin{bmatrix}4+7 & -2+3\\ -6+1 & 9+(-5)\end{bmatrix}=\begin{bmatrix}11 & 1\\ -5 & 4\end{bmatrix}$. (b) $A-B=\begin{bmatrix}4-7 & -2-3\\ -6-1 & 9-(-5)\end{bmatrix}=\begin{bmatrix}-3 & -5\\ -7 & 14\end{bmatrix}$. (c) $5A=\begin{bmatrix}5(4) & 5(-2)\\ 5(-6) & 5(9)\end{bmatrix}=\begin{bmatrix}20 & -10\\ -30 & 45\end{bmatrix}$. (d) $3A=\begin{bmatrix}12 & -6\\ -18 & 27\end{bmatrix}$ and $2B=\begin{bmatrix}14 & 6\\ 2 & -10\end{bmatrix}$, so $3A-2B=\begin{bmatrix}12-14 & -6-6\\ -18-2 & 27-(-10)\end{bmatrix}=\begin{bmatrix}-2 & -12\\ -20 & 37\end{bmatrix}$.

Q4. (a) P has 2 rows and 3 columns, so its order is 2×3. (b) Interchanging rows and columns, the first row $\begin{bmatrix}3 & -1 & 6\end{bmatrix}$ becomes the first column and the second row $\begin{bmatrix}0 & 5 & -2\end{bmatrix}$ the second column:

$$P^T=\begin{bmatrix}3 & 0\\ -1 & 5\\ 6 & -2\end{bmatrix},$$

of order 3×2. (c) Transposing Q interchanges the off-diagonal entries $q_{12}=-7$ and $q_{21}=-7$, which are equal, so $Q^T=\begin{bmatrix}4 & -7\\ -7 & 2\end{bmatrix}=Q$. Since $Q^T=Q$, the matrix Q is symmetric. (d) If A has order $m\times n$, then A^T has order $n\times m$. A symmetric matrix satisfies $A=A^T$, and equal matrices must have the same order, so $m\times n=n\times m$ forces $m=n$. Hence a symmetric matrix is necessarily square.

Q5. (a) Two matrices are equal when they have the same order and every pair of corresponding elements is equal. Here both matrices are 2×2, so it remains to equate corresponding entries. (b) Equating entries in each position gives $2x=14$, $5=5$ (automatically satisfied), $y+1=-4$ and $3z=9$. (c) Solving, $x=7$; from $y+1=-4$, $y=-5$; and from $3z=9$, $z=3$.

Q6. (a) From $A+M=B$, subtracting A from both sides gives $M=B-A$:

$$M=\begin{bmatrix}-1-5 & 4-(-3)\\ 7-2 & 0-8\end{bmatrix}=\begin{bmatrix}-6 & 7\\ 5 & -8\end{bmatrix}.$$

(b) $A+B=\begin{bmatrix}5+(-1) & -3+4\\ 2+7 & 8+0\end{bmatrix}=\begin{bmatrix}4 & 1\\ 9 & 8\end{bmatrix}$, and $B+A=\begin{bmatrix}-1+5 & 4+(-3)\\ 7+2 & 0+8\end{bmatrix}=\begin{bmatrix}4 & 1\\ 9 & 8\end{bmatrix}$; the two results agree. (c) The equality $A+B=B+A$ is the **commutativity** of matrix addition.

Q7. $2A=\begin{bmatrix}-4 & 12\\ 8 & -18\end{bmatrix}$, so $2A+B=\begin{bmatrix}-4+5 & 12+(-1)\\ 8+3 & -18+7\end{bmatrix}=\begin{bmatrix}1 & 11\\ 11 & -11\end{bmatrix}$.

Q8. D has 3 rows and 2 columns, so its order is 3×2. The entry d_{21} lies in row 2 (Cobram) and column 1 (Friday), so $d_{21}=12$ is the number of parcels delivered to Cobram on Friday. Saturday is column 2, with entries 25, 20 and 14, so the total number of parcels delivered on Saturday is $25+20+14=59$.

Q9. $3P=\begin{bmatrix}6 & -12 & 3\\ 0 & 18 & -9\end{bmatrix}$ and $2Q=\begin{bmatrix}10 & 2 & -4\\ 14 & -6 & 8\end{bmatrix}$, so

$$3P-2Q=\begin{bmatrix}6-10 & -12-2 & 3-(-4)\\ 0-14 & 18-(-6) & -9-8\end{bmatrix}=\begin{bmatrix}-4 & -14 & 7\\ -14 & 24 & -17\end{bmatrix}.$$

Q10. For A to be symmetric every entry must equal its mirror image, $a_{ij}=a_{ji}$. The pairs $a_{13}=a_{31}=-5$ and $a_{23}=a_{32}=8$ already agree, so the only condition left is $a_{12}=a_{21}$, that is $p=6$. With $p=6$ the matrix reads the same across the leading diagonal, confirming symmetry.

Q11. From $4X+A=B$, subtracting A gives

$$4X=B-A=\begin{bmatrix}19-3 & 7-(-1)\\ -2-6 & 22-2\end{bmatrix}=\begin{bmatrix}16 & 8\\ -8 & 20\end{bmatrix},$$

and multiplying by the scalar $1/4$,

$$X=\frac{1}{4}\begin{bmatrix}16 & 8\\ -8 & 20\end{bmatrix}=\begin{bmatrix}4 & 2\\ -2 & 5\end{bmatrix}.$$

(Check: $4X+A=\begin{bmatrix}16 & 8\\ -8 & 20\end{bmatrix}+\begin{bmatrix}3 & -1\\ 6 & 2\end{bmatrix}=\begin{bmatrix}19 & 7\\ -2 & 22\end{bmatrix}=B$.)

Q12. The matrices are both 2×2, and the entries 8 and -3 already match, so equate the remaining pairs: $a+b=10$ and $a-b=4$. Adding the two equations gives $2a=14$, so $a=7$; substituting back, $7+b=10$, so $b=3$. (Check: $a-b=7-3=4$.)

Q13. A diagonal matrix has zeros in every off-diagonal position, so

$$D=\begin{bmatrix}4 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 9\end{bmatrix}.$$

Transposing interchanges d_{ij} and d_{ji}; since every off-diagonal entry is 0, no entry changes, so $D^T=D$. Hence D is symmetric — indeed every diagonal matrix is symmetric.

Q14. The negative reverses the sign of each entry: $-A=\begin{bmatrix}-7 & 2\\ 5 & -3\end{bmatrix}$. Adding,

$$A+(-A)=\begin{bmatrix}7+(-7) & -2+2\\ -5+5 & 3+(-3)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

The result is the **zero matrix** O, illustrating the additive-inverse law $A+(-A)=O$.

Q15. First the left-hand side: $A+B=\begin{bmatrix}1+3 & 4+(-6)\\ -2+0 & 5+7\end{bmatrix}=\begin{bmatrix}4 & -2\\ -2 & 12\end{bmatrix}$, and transposing interchanges the two off-diagonal entries (both -2), giving $(A+B)^T=\begin{bmatrix}4 & -2\\ -2 & 12\end{bmatrix}$. Now the right-hand side: $A^T=\begin{bmatrix}1 & -2\\ 4 & 5\end{bmatrix}$ and $B^T=\begin{bmatrix}3 & 0\\ -6 & 7\end{bmatrix}$, so $A^T+B^T=\begin{bmatrix}1+3 & -2+0\\ 4+(-6) & 5+7\end{bmatrix}=\begin{bmatrix}4 & -2\\ -2 & 12\end{bmatrix}$. The two sides agree, so $(A+B)^T=A^T+B^T$.

Q16. (a) Adding entrywise, $T_1+T_2=\begin{bmatrix}14+18 & 9+12\\ 11+15 & 16+13\end{bmatrix}=\begin{bmatrix}32 & 21\\ 26 & 29\end{bmatrix}$; each entry is the combined enrolment in that year level and subject across the two terms (for instance 32 Year 11 Physics enrolments in all). (b) Subtracting, $T_2-T_1=\begin{bmatrix}18-14 & 12-9\\ 15-11 & 13-16\end{bmatrix}=\begin{bmatrix}4 & 3\\ 4 & -3\end{bmatrix}$. The negative entry -3 is in row 2, column 2 (Year 12 Chemistry): the enrolment there *fell* by 3 from Term 1 to Term 2, whereas the positive entries record increases.

Q17. Since A and B have the same order, $A+B$ is defined and has that order too. Using the transpose-of-a-sum identity, and then the hypotheses $A^T=A$ and $B^T=B$,

$$(A+B)^T=A^T+B^T=A+B.$$

A square matrix equal to its own transpose is symmetric, so $A+B$ is symmetric, as required.

Q18. Forming the scalar multiples and adding,

$$sA+tB=\begin{bmatrix}s & 2s\\ 0 & s\end{bmatrix}+\begin{bmatrix}0 & t\\ t & 0\end{bmatrix}=\begin{bmatrix}s & 2s+t\\ t & s\end{bmatrix}.$$

Equating this to $\begin{bmatrix}3 & 10\\ 4 & 3\end{bmatrix}$, the entry in row 1, column 1 gives $s=3$, and the entry in row 2, column 1 gives $t=4$. These also satisfy the remaining entries, since $2s+t=2(3)+4=10$ and $s=3$ match. Hence $s=3$ and $t=4$.

Theory

Matrices of the same order are added, subtracted and scaled entry by entry, as in the previous section. **Matrix multiplication** is different: it is *not* carried out entry by entry. Its rule is deliberately chosen so that a single product performs exactly the pairing-and-adding that arises when costs are totalled, when the effect of one linear map is followed by another, or when the paths through a network are counted. The rule takes a little care to state, but once fixed it obeys most — though not all — of the laws of ordinary algebra, and those laws can be proved directly from the definition.

Throughout, a matrix with m rows and n columns has **order** $m \times n$, and a_{ij} denotes the entry of A in **row i, column j**.

The row-by-column rule. The single building block of every product is a **row** multiplied by a **column** of the same length: multiply entries in matching positions and add. For a row of length n and a column of length n,

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{bmatrix},$$

a single number (a 1×1 matrix). For instance $\begin{bmatrix} 4 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 4(3)+(-1)(5)+2(-2) \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}$. The two lengths **must** agree, or the pairing breaks down — and this single requirement is what controls when a whole product exists.

When the product exists, and its order. For the product AB to be defined, each row of A must have the same length as each column of B; that is, the number of **columns of** A must equal the number of **rows of** B. Writing the two orders side by side,

$$(m \times n)(n \times p) \to m \times p,$$

the **inner** numbers must match for AB to exist, and the **outer** numbers give the order of the result. If A is $m \times n$ and B is $n \times p$ then AB is $m \times p$; if the inner numbers differ, AB is **undefined**.

The definition. When AB exists, the entry of AB in row i, column j is **row i of A** multiplied by **column j of B**: work *across* the row and *down* the column. If A is $m \times n$ and B is $n \times p$, then for $1 \le i \le m$ and $1 \le j \le p$,

$$(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

This one formula is the source of every property below. As a 2×2 illustration,

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2(1)+(-1)(5) & 2(4)+(-1)(-2) \\ 0(1)+3(5) & 0(4)+3(-2) \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ 15 & -6 \end{bmatrix}.$$

Order matters: $AB \neq BA$ in general. For real numbers $xy = yx$, but matrix multiplication is **not commutative**. Several things can go wrong when the order is reversed: AB may exist while BA does not; both may exist but have different orders; and even when both exist and share an order, their entries are usually different. A single example settles the point. With $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

so $AB \neq BA$. Because order matters, we speak of multiplying B **on the left** by A (forming AB) or **on the right** by A (forming BA); the two are different operations and must never be interchanged.

The laws that do hold. Reversing the order is not the only familiar law to fail. The **null factor law** — that a product can be zero only when one of its factors is zero — also breaks down: for $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ the row-by-column rule gives $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$, and yet $N \neq O$. **Cancellation** goes with it: from $AB = AC$ one may *not* conclude $B = C$, since NN and NO are both O and yet $N \neq O$. Neither failure follows from non-commutativity, so each must be guarded against in its own right. Beyond these, though, the familiar laws do carry over. Provided the orders are such that every product below is defined, matrix multiplication is **associative** and **distributive** over addition:

$$(AB)C = A(BC), A(B+C) = AB + AC, (A+B)C = AC + BC,$$

and it commutes with scalars, $k(AB) = (kA)B = A(kB)$. Each follows from the definition: associativity because both $(AB)C$ and $A(BC)$ collect the same triple sums $a_{ik} b_{k\ell} c_{\ell j}$, and distributivity because the row-by-column sum splits over a sum in B and C (this is proved in full in Example 4). Associativity is what makes a product such as ABC unambiguous: it may be built as $(AB)C$ or as $A(BC)$ with the same result.

The identity matrix. The **identity matrix** I_n is the $n \times n$ matrix with 1s on the leading diagonal and 0s elsewhere,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It plays the role of the number 1: for any $m \times n$ matrix A,

$$I_m A = A \text{ and } A I_n = A,$$

and in particular, for a **square** matrix A, $AI = IA = A$. This is one case in which the order does not matter. It holds because a row of I has a single 1 (in the diagonal position) and 0s elsewhere, so the row-by-column sum picks out exactly one entry of A and leaves it unchanged. The subscript n is usually dropped when the size is clear, and I is written for the identity of whatever order the context requires.

Powers of a square matrix. Only a **square** matrix can be multiplied by itself, since $A^2 = AA$ requires the product $(m \times n)(m \times n)$, which exists only when $n = m$. For a square matrix the **powers** are defined by

$$A^1 = A, A^{k+1} = A^k A, \text{and by convention } A^0 = I.$$

Thus $A^2 = AA$, $A^3 = A^2 A = A A^2$, and so on; associativity guarantees that a power may be built up in any grouping without changing the result. The ordinary **index laws** for a *single* matrix follow at once,

$$A^m A^n = A^{m+n}, (A^m)^n = A^{mn}.$$

However, because multiplication is not commutative, the schoolbook rule $(AB)^2 = A^2 B^2$ **fails** in general: expanding gives $(AB)^2 = ABAB$, and reaching $A^2 B^2 = AABB$ from it means replacing the middle BA by AB — a step that is justified *whenever* $BA = AB$, but not otherwise. So $AB = BA$ is enough to guarantee $(AB)^2 = A^2 B^2$; without it the two products need not agree, and usually do not.

Application 1 — costing and state products. If each row of a **quantities matrix** Q lists amounts of several items (one row per buyer, per branch or per day) and a **price column** P lists the unit cost of each item in the same order, then $C = QP$ is a column of totals: the row-by-column rule multiplies each amount by its price and adds. The rows of the product keep the meaning of the rows of Q, a useful check that the factors were written in the correct order. The same matrix–column product describes one step of a **process**: a state column s multiplied on the left by a matrix T that encodes the rules of change gives the next state Ts, and repeated steps are then powers $T^2 s, T^3 s, \ldots$ — a use of powers developed further in later work.

Application 2 — connection matrices. A network of direct links can be recorded in a square **connection matrix** (or **adjacency matrix**) M: the entry in row i, column j is 1 when there is a direct link from i to j and 0 when there is not. The power M^2 then counts **two-step links**: by the definition its row i, column j entry is the sum of the products $m_{ik} m_{kj}$ over all k, and each such product is 1 exactly when both the link $i \to k$ and the link $k \to j$ are present — so the entry counts the intermediate members k giving a two-step route from i to j. For an undirected network M is symmetric, and a diagonal entry of M^2 then records the number of direct links a member has (each out-and-back trip $i \to k \to i$).

Application 3 — a matrix as a linear map. A 2×2 matrix M acts on a point (x, y), written as a position vector, by multiplication on the left:

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix},$$

sending the point to its **image** (x', y'). For example $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ sends (x, y) to $(x, -y)$, which is a reflection in the x-axis. Applying one map and then a second corresponds to multiplying their matrices, and the non-commutativity of matrix products mirrors the plain fact that performing two transformations in the opposite order generally gives a different result. This geometric reading — matrices *are* linear maps — is the deeper reason the row-by-column rule is defined as it is.

Worked examples

Example 1 — scaffolded

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 5 & 0 \\ -2 & 3 \end{bmatrix}$. Explain why AB exists, state its order, and find AB. Then state whether BA exists and give its order.

Working	Comment
A is 2×3 and B is 3×2: $(2 \times 3)(3 \times 2)$, and the inner numbers agree ($3 = 3$), so AB exists.	Columns of A must equal rows of B.
The outer numbers give the order of AB: 2×2.	Outer numbers give the order of the product.
Row 1 of A against the columns of B: $2(1) + (-1)(5) + 3(-2) = -9$ and $2(2) + (-1)(0) + 3(3) = 13$.	Work across row 1, down each column of B.
Row 2 of A against the columns of B: $0(1) + 4(5) + 1(-2) = 18$ and $0(2) + 4(0) + 1(3) = 3$.	The entry in row i, column j uses row i of A and column j of B.
$AB = \begin{bmatrix} -9 & 13 \\ 18 & 3 \end{bmatrix}$.	Assemble the four entries.
For BA: $(3 \times 2)(2 \times 3)$ — inner numbers agree ($2 = 2$), so BA exists, with order 3×3.	The same check, in the reversed order.

Here AB is 2×2 but BA is 3×3: the two products do not even share an order, so certainly $AB \neq BA$.

Example 2 — scaffolded

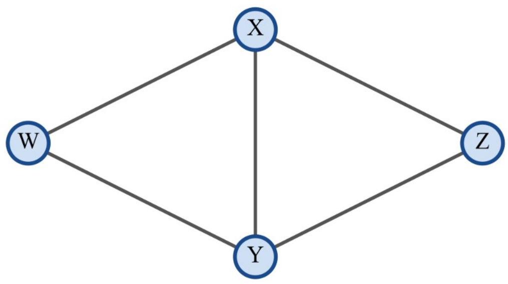
Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$. Find (a) AB, (b) BA and hence confirm $AB \neq BA$; (c) A^2; and (d) verify that $AI = IA = A$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Working	Comment
$AB=\begin{bmatrix}3(2)+(-2)(-1) & 3(0)+(-2)(5)\\ 1(2)+4(-1) & 1(0)+4(5)\end{bmatrix}=\begin{bmatrix}8 & -\\ -2 & 2\end{bmatrix}$.	Rows of A times columns of B.
$BA=\begin{bmatrix}2(3)+0(1) & 2(-2)+0(4)\\ -1(3)+5(1) & -1(-2)+5(4)\end{bmatrix}=\begin{bmatrix}6 & -4\\ 2 & 22\end{bmatrix}$.	Now rows of B times columns of A.
Same order but $8\neq 6$ already, so $AB\neq BA$.	Reversing the factors changes the entries.
$A^2=AA=\begin{bmatrix}3(3)+(-2)(1) & 3(-2)+(-2)(4)\\ 1(3)+4(1) & 1(-2)+4(4)\end{bmatrix}=\begin{bmatrix}7\\ 7\end{bmatrix}$.	A power multiplies the matrix by itself.
$AI=\begin{bmatrix}3(1)+(-2)(0) & 3(0)+(-2)(1)\\ 1(1)+4(0) & 1(0)+4(1)\end{bmatrix}=\begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}=$, and likewise $IA=A$.	Each row of I selects one entry of A unchanged.

So $AB=\begin{bmatrix}8 & -10\\ -2 & 20\end{bmatrix}$, $BA=\begin{bmatrix}6 & -4\\ 2 & 22\end{bmatrix}$ (unequal), $A^2=\begin{bmatrix}7 & -14\\ 7 & 14\end{bmatrix}$, and I leaves A unchanged from either side.

Example 3 — unscaffolded

Four data centres W, X, Y, Z are joined by direct fibre-optic links: W–X, W–Y, X–Y, X–Z and Y–Z, as shown.



Write down the connection matrix M (rows and columns in the order W, X, Y, Z), find M^2, and interpret the entry in row W, column Z, the entry in row W, column X, and the diagonal entry in row X.

Each entry of M is 1 for a directly linked pair and 0 otherwise, with 0s on the diagonal:

$$M=\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 0 & 1 & 1 & 0\end{bmatrix}.$$

Each entry of M^2 is a row of M times a column of M. For instance the entry in row W, column Z is $0(0)+1(1)+1(1)+0(0)=2$, and the diagonal entry in row X is $1(1)+0(0)+1(1)+1(1)=3$. Computing all sixteen entries the same way,

$$M^2=\begin{bmatrix}2 & 1 & 1 & 2\\ 1 & 3 & 2 & 1\\ 1 & 2 & 3 & 1\\ 2 & 1 & 1 & 2\end{bmatrix}.$$

The entry 2 in row W, column Z counts **two two-step routes** from W to Z: W–X–Z and W–Y–Z. The entry 1 in row W, column X counts the single two-step route W–Y–X (the link W–X itself is direct, a one-step route, and is not counted by M^2). The diagonal entry 3 in row X counts the out-and-back trips $X–W–X$, $X–Y–X$ and $X–Z–X$, one through each of X's three direct links — so it equals the number of links at X.

$\therefore M^2$ counts two-step routes: W reaches Z two ways, W reaches X in one two-step route besides the direct link, and X's diagonal entry 3 equals its number of direct links.

Example 4 — unscaffolded

Prove, directly from the definition of matrix multiplication, that $A(B+C)=AB+AC$ whenever A is $m\times n$ and B, C are both $n\times p$.

Both sides are defined and of order $m\times p$: $B+C$ is $n\times p$, so $A(B+C)$ is $m\times p$, and AB, AC are each $m\times p$, so their sum is too. It therefore suffices to show that the two sides agree in every entry. Fix i with $1\le i\le m$ and j with $1\le j\le p$. The entry of $B+C$ in row k, column j is $b_{kj}+c_{kj}$, so by the definition of the product,

$$(A(B+C))_{ij}=\sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj})=\sum_{k=1}^{n}(a_{ik}b_{kj}+a_{ik}c_{kj}).$$

Each term is a sum of two real numbers, and a finite sum splits over addition, so

$$(A(B+C))_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}+\sum_{k=1}^{n} a_{ik}c_{kj}=(AB)_{ij}+(AC)_{ij}=(AB+AC)_{ij}.$$

Since the entries in row i, column j agree for every i and j, the matrices are equal, $A(B+C)=AB+AC$, as required.

Practice questions

Scaffolded

Q1. Matrix P has order 3×4, matrix Q has order 4×2, matrix R has order 2×3 and matrix S has order 3×3. (a) Explain why PQ exists and state its order. (b) Does QP exist? Explain. (c) State the order of QR. (d) State the order of RP. (e) Exactly one of P, Q, R, S can be squared. Which one, and why?

Q2. Let $R=\begin{bmatrix}-3 & 6 & 2\end{bmatrix}$ and $C=\begin{bmatrix}5\\1\\-4\end{bmatrix}$. (a) Find RC and state its order. (b) State the order of CR without evaluating it. (c) Find CR. (d) Is $RC=CR$? What does this show?

Q3. Let $A=\begin{bmatrix}2 & 1\\-3 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & -2\\0 & 5\end{bmatrix}$ and $C=\begin{bmatrix}3 & 1\\2 & -1\end{bmatrix}$. (a) Find AB. (b) Find AC. (c) Find $B+C$ and hence $A(B+C)$. (d) Verify that $A(B+C)=AB+AC$.

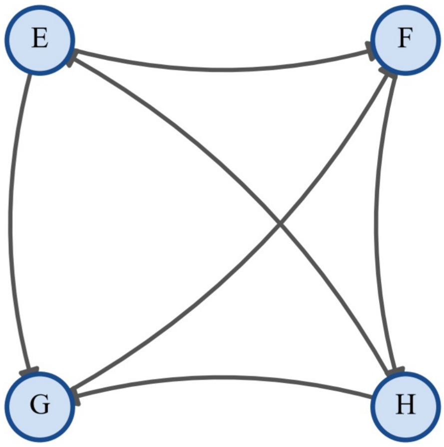
Q4. A print shop sells flyers at \$2 each, posters at \$8 each and banners at \$25 each. Its Northgate branch sold 120 flyers, 45 posters and 10 banners in a week; its Riverside branch sold 90 flyers, 60 posters and 15 banners. Let

$$Q=\begin{bmatrix}120 & 45 & 10\\90 & 60 & 15\end{bmatrix},\ P=\begin{bmatrix}2\\8\\25\end{bmatrix},$$

where row 1 of Q is Northgate and row 2 is Riverside. (a) Write the orders side by side and hence state the order of QP. (b) Find QP. (c) Interpret each entry of QP in context.

Q5. Let $A=\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$. (a) Find A^2. (b) Use $A^3=A^2A$ to find A^3. (c) Check that AA^2 gives the same result as part (b), illustrating $A^2A=AA^2$. (d) Explain why a 2×3 matrix has no powers.

Q6. In a project team, tasks are passed one way only. Member E can pass a task directly to F and to G; F can pass to H; G can pass to F; and H can pass to E and to G. The arrows below point from sender to receiver.



The communication matrix M has a 1 in row i, column j when the member in row i can pass a task directly to the member in column j. (a) Write down M, with rows and columns in the order E, F, G, H. (b) Find M^2. (c) Interpret the entry in row H, column F of M^2. (d) Explain what the 0 in row E, column E of M^2 tells you.

Unscaffolded

Q7. Matrix A has order 2×5, matrix B has order 5×5, matrix C has order 5×1 and matrix D has order 1×2. State whether each of AB, BC, CD, DA, BA and ABC exists, and give the order of each product that does.

Q8. For $A=\begin{bmatrix}5 & -2\\ 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}0 & 4\\ -1 & 2\end{bmatrix}$, find AB and BA, and hence show that $AB\neq BA$.

Q9. For $A=\begin{bmatrix}1 & -2 & 0\\ 3 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 0 & -1\\ 4 & 3\end{bmatrix}$, find AB, and state (without evaluating it) the order of BA.

Q10. A stationery supplier delivers to two schools. Notebooks cost \$5, pens \$2 and folders \$4. Ashfield College ordered 40 notebooks, 150 pens and 25 folders; Bexley Grammar ordered 60 notebooks, 90 pens and 35 folders. Write a quantities matrix Q (rows Ashfield then Bexley) and a price column P, and use QP to find each school's total bill.

Q11. The matrix $R=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ represents an anticlockwise rotation of 90° about the origin, acting on a point by $R\begin{bmatrix}x\\ y\end{bmatrix}$. Let $p=\begin{bmatrix}3\\ 1\end{bmatrix}$. (a) Find Rp and state the image of $(3,1)$. (b) Find R^2 and describe the transformation it represents. (c) Verify that $R^2p=-p$, and explain why this is geometrically expected.

Q12. Let $B=\begin{bmatrix}1 & -1\\ 2 & 0\end{bmatrix}$. Find B^2 and B^3.

Q13. Let $A=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}$, $B=\begin{bmatrix}3 & 1\\ -2 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0\\ 1 & 5\end{bmatrix}$. Find $(AB)C$ and $A(BC)$ separately, and confirm that they agree, illustrating the associative law.

Q14. Let $A=\begin{bmatrix}2 & -1\\ 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 4\\ -2 & 1\end{bmatrix}$ and $C=\begin{bmatrix}0 & 2\\ 3 & -1\end{bmatrix}$. By computing both sides, verify the distributive law $(A+B)C=AC+BC$.

Q15. Let $A=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}$. Find $(AB)^2$ and A^2B^2, show that they are unequal, and explain in terms of commutativity why $(AB)^2=A^2B^2$ need not hold.

Q16. Let $T=\begin{bmatrix}-1 & 2\\ 1 & 0\end{bmatrix}$. Find T^2 and T^3.

Q17. A 2×2 matrix is **upper triangular** if its bottom-left entry is 0, that is, of the form $\begin{bmatrix}a & b\\ 0 & c\end{bmatrix}$. Prove that the product of two upper-triangular 2×2 matrices is again upper triangular.

Q18. Four monitoring sensors around a reservoir link by short-range radio. Sensor 4 is a central base station on the dam wall; sensors 1, 2 and 3 are spaced around the shoreline, each within range of the base station but too far apart to reach one another directly (all links are two-way). Thus sensors 1, 2 and 3 each reach only sensor 4, while sensor 4 reaches sensors 1, 2 and 3. The connection matrix, with rows and columns in the order $1,2,3,4$, is

$$M=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}.$$

Find M^2 and $M+M^2$. Explain what the entries of $M+M^2$ count, and hence explain why every sensor can reach every other sensor either directly or through a single relay.

Answers

Q1. (a) Columns of P = rows of $Q=4$; order 3×2. (b) No — Q has 2 columns but P has 3 rows. (c) 4×3. (d) 2×4. (e) S (it is square, 3×3); the others are not square, so cannot be squared. **Q2.** (a) $RC=\begin{bmatrix}-17\end{bmatrix}$, order 1×1. (b) 3×3. (c) $CR=\begin{bmatrix}-15 & 30 & 10\\ -3 & 6 & 2\\ 12 & -24 & -8\end{bmatrix}$. (d) No — they differ in order, showing multiplication is not commutative. **Q3.** (a) $\begin{bmatrix}2 & 1\\ -3 & 26\end{bmatrix}$. (b) $\begin{bmatrix}8 & 1\\ -1 & -7\end{bmatrix}$. (c) $B+C=\begin{bmatrix}4 & -1\\ 2 & 4\end{bmatrix}$, $A(B+C)=\begin{bmatrix}10 & 2\\ -4 & 19\end{bmatrix}$. (d) $AB+AC=\begin{bmatrix}10 & 2\\ -4 & 19\end{bmatrix}$, equal. **Q4.** (a) $(2\times3)(3\times1)\to2\times1$. (b) $QP=\begin{bmatrix}850\\ 1035\end{bmatrix}$. (c) Northgate's takings are \$850 and Riverside's \$1035. **Q5.** (a) $A^2=\begin{bmatrix}4 & 1\\ 0 & 1\end{bmatrix}$. (b) $A^3=\begin{bmatrix}8 & 3\\ 0 & -1\end{bmatrix}$. (c) $AA^2=\begin{bmatrix}8 & 3\\ 0 & -1\end{bmatrix}$, the same. (d) $(2\times3)(2\times3)$ has inner numbers $3\neq2$, so A^2 is undefined. **Q6.** (a) $M=\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\end{bmatrix}$. (b) $M^2=\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 2 & 1 & 0\end{bmatrix}$. (c) Two two-step routes from H to F ($H\to E\to F$ and $H\to G\to F$). (d) No task from E returns to E in exactly two steps. **Q7.** AB: 2×5. BC: 5×1. CD: 5×2. DA: 1×5. BA: does not exist ($5\neq2$). ABC: 2×1. **Q8.** $AB=\begin{bmatrix}2 & 16\\ -3 & 10\end{bmatrix}$, $BA=\begin{bmatrix}4 & 12\\ -3 & 8\end{bmatrix}$; entries differ, so $AB\neq BA$. **Q9.** $AB=\begin{bmatrix}2 & 3\\ 14 & 8\end{bmatrix}$; BA has order 3×3. **Q10.** $P=\begin{bmatrix}5\\ 2\\ 4\end{bmatrix}$, $QP=\begin{bmatrix}600\\ 620\end{bmatrix}$: Ashfield \$600, Bexley \$620. **Q11.** (a) $Rp=\begin{bmatrix}-1\\ 3\end{bmatrix}$; image $(-1,3)$. (b) $R^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$, a rotation of 180° about the origin. (c) $R^2p=\begin{bmatrix}-3\\ -1\end{bmatrix}=-p$; two 90° rotations make a half-turn, which sends $(3,1)$ to $(-3,-1)$. **Q12.** $B^2=\begin{bmatrix}-1 & -1\\ 2 & -2\end{bmatrix}$, $B^3=\begin{bmatrix}-3 & 1\\ -2 & -2\end{bmatrix}$. **Q13.** $(AB)C=A(BC)=\begin{bmatrix}7 & 45\\ 0 & -20\end{bmatrix}$. **Q14.** $(A+B)C=AC+BC=\begin{bmatrix}9 & 3\\ 3 & 1\end{bmatrix}$. **Q15.** $(AB)^2=\begin{bmatrix}2 & -14\\ 7 & 23\end{bmatrix}$, $A^2B^2=\begin{bmatrix}4 & -12\\ 5 & 21\end{bmatrix}$; unequal because $BA\neq AB$. **Q16.** $T^2=\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}$, $T^3=\begin{bmatrix}-5 & 6\\ 3 & -2\end{bmatrix}$. **Q17.** The product is $\begin{bmatrix}ad & ae+bf\\ 0 & cf\end{bmatrix}$, whose bottom-left entry is 0; hence it is upper triangular. **Q18.** $M^2=\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$, $M+M^2=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 3\end{bmatrix}$; its entries count routes of one or two steps, and every off-diagonal entry is positive, so every sensor reaches every other directly or through one relay.

Fully-worked solutions

Q1. Write the orders side by side each time. (a) $(3\times4)(4\times2)$: inner numbers both 4, so PQ exists; the outer numbers give order 3×2. (b) $(4\times2)(3\times4)$: the inner numbers are 2 and 3, which differ, so QP does not exist. (c) $(4\times2)(2\times3)$: inner numbers both 2, order 4×3. (d) $(2\times3)(3\times4)$: inner numbers both 3, order 2×4. (e) A matrix can be squared only when it is square. Of the four, only S is square (3×3), so only S^2 exists; P, Q, R are not square.

Q2. (a) $(1\times3)(3\times1)\to1\times1$: $RC=[-3(5)+6(1)+2(-4)]=[-15+6-8]=[-17]$. (b) $(3\times1)(1\times3)\to3\times3$. (c) Each entry is one entry of C times one entry of R:

$$CR=\begin{bmatrix}5(-3)&5(6)&5(2)\\1(-3)&1(6)&1(2)\\-4(-3)&-4(6)&-4(2)\end{bmatrix}=\begin{bmatrix}-15&30&10\\-3&6&2\\12&-24&-8\end{bmatrix}.$$

(d) No: RC is 1×1 while CR is 3×3. Reversing the order changes even the order of the answer, so matrix multiplication is not commutative.

Q3. (a) $AB=\begin{bmatrix}2(1)+1(0)&2(-2)+1(5)\\-3(1)+4(0)&-3(-2)+4(5)\end{bmatrix}=\begin{bmatrix}2&1\\-3&26\end{bmatrix}$. (b) $AC=\begin{bmatrix}2(3)+1(2)&2(1)+1(-1)\\-3(3)+4(2)&-3(1)+4(-1)\end{bmatrix}=\begin{bmatrix}8&1\\-1&-7\end{bmatrix}$. (c) $B+C=\begin{bmatrix}1+3&-2+1\\0+2&5-1\end{bmatrix}=\begin{bmatrix}4&-1\\2&4\end{bmatrix}$, so $A(B+C)=\begin{bmatrix}2(4)+1(2)&2(-1)+1(4)\\-3(4)+4(2)&-3(-1)+4(4)\end{bmatrix}=\begin{bmatrix}10&2\\-4&19\end{bmatrix}$. (d) Adding the answers to (a) and (b), $AB+AC=\begin{bmatrix}2+8&1+1\\-3-1&26-7\end{bmatrix}=\begin{bmatrix}10&2\\-4&19\end{bmatrix}$, which equals $A(B+C)$, confirming the distributive law here.

Q4. (a) $(2\times3)(3\times1)$: inner numbers match ($3=3$), so QP exists with order 2×1. (b) $QP=\begin{bmatrix}120(2)+45(8)+10(25)\\90(2)+60(8)+15(25)\end{bmatrix}=\begin{bmatrix}240+360+250\\180+480+375\end{bmatrix}=\begin{bmatrix}850\\1035\end{bmatrix}$. (c) Each row of the product keeps the meaning of the matching row of Q: the Northgate branch took \$850 and the Riverside branch took \$1035.

Q5. (a) $A^2=\begin{bmatrix}2(2)+1(0)&2(1)+1(-1)\\0(2)+(-1)(0)&0(1)+(-1)(-1)\end{bmatrix}=\begin{bmatrix}4&1\\0&1\end{bmatrix}$. (b) $A^3=A^2A=\begin{bmatrix}4(2)+1(0)&4(1)+1(-1)\\0(2)+1(0)&0(1)+1(-1)\end{bmatrix}=\begin{bmatrix}8&3\\0&-1\end{bmatrix}$. (c) $AA^2=\begin{bmatrix}2(4)+1(0)&2(1)+1(1)\\0(4)+(-1)(0)&0(1)+(-1)(1)\end{bmatrix}=\begin{bmatrix}8&3\\0&-1\end{bmatrix}$, the same as part (b); associativity guarantees $A^2A=AA^2$.
(d) Squaring a 2×3 matrix needs $(2\times3)(2\times3)$, whose inner numbers are 3 and 2; they differ, so the product is undefined. Only square matrices have powers.

Q6. (a) Reading the arrows (sender in the row, receiver in the column): $M=\begin{bmatrix}0&1&1&0\\0&0&0&1\\0&1&0&0\\1&0&1&0\end{bmatrix}$. (b) Each entry of M^2 is a row of M times a column of M; for example the entry in row H, column F is $1(1)+0(0)+1(1)+0(0)=2$. Completing all sixteen entries,

$$M^2=\begin{bmatrix}0&1&0&1\\1&0&1&0\\0&0&0&1\\0&2&1&0\end{bmatrix}.$$

(c) The 2 in row H, column F means there are two two-step routes by which a task from H reaches F: $H\to E\to F$ and $H\to G\to F$. (d) The 0 in row E, column E means no task sent by E returns to E in exactly two steps — there is no member that E can reach who can pass a task back to E.

Q7. Writing orders side by side: AB: $(2\times5)(5\times5)$ — inner match, order 2×5. BC: $(5\times5)(5\times1)$ — inner match, order 5×1. CD: $(5\times1)(1\times2)$ — inner match, order 5×2. DA: $(1\times2)(2\times5)$ — inner match, order 1×5

. BA: $(5\times5)(2\times5)$ — inner numbers $5\neq2$, does not exist. ABC: AB is 2×5, and $(2\times5)(5\times1)$ gives order 2×1.

Q8. $AB=\begin{bmatrix}5(0)+(-2)(-1) & 5(4)+(-2)(2)\\ 1(0)+3(-1) & 1(4)+3(2)\end{bmatrix}=\begin{bmatrix}2 & 16\\ -3 & 10\end{bmatrix}$ and

$BA=\begin{bmatrix}0(5)+4(1) & 0(-2)+4(3)\\ -1(5)+2(1) & -1(-2)+2(3)\end{bmatrix}=\begin{bmatrix}4 & 12\\ -3 & 8\end{bmatrix}$. The two products have the same order but different entries (for example $2\neq4$ in row 1, column 1), so $AB\neq BA$.

Q9. $(2\times3)(3\times2)\rightarrow2\times2$:

$$AB=\begin{bmatrix}1(2)+(-2)(0)+0(4) & 1(1)+(-2)(-1)+0(3)\\ 3(2)+1(0)+2(4) & 3(1)+1(-1)+2(3)\end{bmatrix}=\begin{bmatrix}2 & 3\\ 14 & 8\end{bmatrix}.$$

For BA: $(3\times2)(2\times3)$ — the inner numbers match, and the outer numbers give order 3×3.

Q10. With rows Ashfield then Bexley and columns notebooks, pens, folders, $Q=\begin{bmatrix}40 & 150 & 25\\ 60 & 90 & 35\end{bmatrix}$ and $P=\begin{bmatrix}5\\ 2\\ 4\end{bmatrix}$. Then

$$QP=\begin{bmatrix}40(5)+150(2)+25(4)\\ 60(5)+90(2)+35(4)\end{bmatrix}=\begin{bmatrix}200+300+100\\ 300+180+140\end{bmatrix}=\begin{bmatrix}600\\ 620\end{bmatrix},$$

so Ashfield College’s bill is \$600 and Bexley Grammar’s is \$620.

Q11. (a) $Rp=\begin{bmatrix}0(3)+(-1)(1)\\ 1(3)+0(1)\end{bmatrix}=\begin{bmatrix}-1\\ 3\end{bmatrix}$, so $(3,1)$ maps to $(-1,3)$. (b) $R^2=\begin{bmatrix}0(0)+(-1)(1) & 0(-1)+(-1)(0)\\ 1(0)+0(1) & 1(-1)+0(0)\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$, which sends (x,y) to $(-x,-y)$: a rotation of 180° about the origin. (c) $R^2p=\begin{bmatrix}-1(3)+0(1)\\ 0(3)+(-1)(1)\end{bmatrix}=\begin{bmatrix}-3\\ -1\end{bmatrix}=-p$. This is expected: two successive 90° rotations amount to a half-turn, which sends every point to its opposite.

Q12. $B^2=\begin{bmatrix}1(1)+(-1)(2) & 1(-1)+(-1)(0)\\ 2(1)+0(2) & 2(-1)+0(0)\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 2 & -2\end{bmatrix}$. Then $B^3=B^2B=\begin{bmatrix}-1(1)+(-1)(2) & -1(-1)+(-1)(0)\\ 2(1)+(-2)(2) & 2(-1)+(-2)(0)\end{bmatrix}=\begin{bmatrix}-3 & 1\\ -2 & -2\end{bmatrix}$.

Q13. First $AB=\begin{bmatrix}1(3)+2(-2) & 1(1)+2(4)\\ 0(3)+(-1)(-2) & 0(1)+(-1)(4)\end{bmatrix}=\begin{bmatrix}-1 & 9\\ 2 & -4\end{bmatrix}$, so

$$(AB)C=\begin{bmatrix}-1(2)+9(1) & -1(0)+9(5)\\ 2(2)+(-4)(1) & 2(0)+(-4)(5)\end{bmatrix}=\begin{bmatrix}7 & 45\\ 0 & -20\end{bmatrix}.$$

Next $BC=\begin{bmatrix}3(2)+1(1) & 3(0)+1(5)\\ -2(2)+4(1) & -2(0)+4(5)\end{bmatrix}=\begin{bmatrix}7 & 5\\ 0 & 20\end{bmatrix}$, so

$$A(BC)=\begin{bmatrix}1(7)+2(0) & 1(5)+2(20)\\ 0(7)+(-1)(0) & 0(5)+(-1)(20)\end{bmatrix}=\begin{bmatrix}7 & 45\\ 0 & -20\end{bmatrix}.$$

The two agree, illustrating $(AB)C=A(BC)$.

Q14. First $A+B=\begin{bmatrix} 2+1 & -1+4 \\ 3-2 & 0+1 \end{bmatrix}=\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$, so

$$(A+B)C=\begin{bmatrix} 3(0)+3(3) & 3(2)+3(-1) \\ 1(0)+1(3) & 1(2)+1(-1) \end{bmatrix}=\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}.$$

Separately, $AC=\begin{bmatrix} 2(0)+(-1)(3) & 2(2)+(-1)(-1) \\ 3(0)+0(3) & 3(2)+0(-1) \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ 0 & 6 \end{bmatrix}$ and $BC=\begin{bmatrix} 1(0)+4(3) & 1(2)+4(-1) \\ -2(0)+1(3) & -2(2)+1(-1) \end{bmatrix}=\begin{bmatrix} 12 & -2 \\ 3 & -5 \end{bmatrix}$, so $AC+BC=\begin{bmatrix} -3+12 & 5-2 \\ 0+3 & 6-5 \end{bmatrix}=\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$, equal to $(A+B)C$.

Q15. First $AB=\begin{bmatrix} 2(1)+0(0) & 2(-1)+0(2) \\ 1(1)+3(0) & 1(-1)+3(2) \end{bmatrix}=\begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$, so $(AB)^2=\begin{bmatrix} 2(2)+(-2)(1) & 2(-2)+(-2)(5) \\ 1(2)+5(1) & 1(-2)+5(5) \end{bmatrix}=\begin{bmatrix} 2 & -14 \\ 7 & 23 \end{bmatrix}$. Next $A^2=\begin{bmatrix} 4 & 0 \\ 5 & 9 \end{bmatrix}$ and $B^2=\begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$, so $A^2B^2=\begin{bmatrix} 4(1)+0(0) & 4(-3)+0(4) \\ 5(1)+9(0) & 5(-3)+9(4) \end{bmatrix}=\begin{bmatrix} 4 & -12 \\ 5 & 21 \end{bmatrix}$. Since $\begin{bmatrix} 2 & -14 \\ 7 & 23 \end{bmatrix}\neq\begin{bmatrix} 4 & -12 \\ 5 & 21 \end{bmatrix}$, $(AB)^2\neq A^2B^2$. The reason is that $(AB)^2=ABAB$ while $A^2B^2=AABB$: passing from the first to the second means replacing the middle BA by AB, which is justified whenever $BA=AB$. Here $BA=\begin{bmatrix} 1 & -3 \\ 2 & 6 \end{bmatrix}$ while $AB=\begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$, so that step is not available, and the two products do indeed differ.

Q16. $T^2=\begin{bmatrix} -1(-1)+2(1) & -1(2)+2(0) \\ 1(-1)+0(1) & 1(2)+0(0) \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$. Then $T^3=T^2T=\begin{bmatrix} 3(-1)+(-2)(1) & 3(2)+(-2)(0) \\ -1(-1)+2(1) & -1(2)+2(0) \end{bmatrix}=\begin{bmatrix} -5 & 6 \\ 3 & -2 \end{bmatrix}$.

Q17. Take two upper-triangular matrices $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $B=\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$. Multiplying,

$$AB=\begin{bmatrix} a(d)+b(0) & a(e)+b(f) \\ 0(d)+c(0) & 0(e)+c(f) \end{bmatrix}=\begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}.$$

The bottom-left entry is $0(d)+c(0)=0$, so AB has the form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ and is upper triangular, as required.

Q18. Squaring M, each entry being a row of M times a column of M (for example the entry in row 1, column 2 is $0(0)+0(0)+0(0)+1(1)=1$, and the diagonal entry in row 4 is $1(1)+1(1)+1(1)+0(0)=3$):

$$M^2=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Adding entry by entry,

$$M+M^2=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

An entry of M counts the direct (one-step) links between two sensors and the matching entry of M^2 counts the two-step links, so each entry of $M + M^2$ counts the routes between two sensors using **at most two steps** — at most one relay through a middle sensor. Every off-diagonal entry of $M + M^2$ is positive: the base station 4 links directly to each of sensors 1, 2 and 3, and although those three sensors are out of range of one another, any two of them are joined by a single two-step route through the base station — for instance $1–4–2$, counted by the entry 1 in row 1, column 2. Since no off-diagonal entry is 0, every sensor can reach every other sensor either directly or through a single relay.

Theory

Earlier sections of this chapter built up the algebra of matrices: matrices of matching dimensions are added entry by entry, and matrices whose inner dimensions agree are multiplied row-into-column. This section asks the reverse question. Multiplication by a non-zero number is undone by multiplication by its reciprocal, because $5\times 1/5=1$; can matrix multiplication be undone in the same way? The answer leads to the **determinant** — a single number that decides the question — and to the **inverse matrix**, which together give a compact and rigorous method for solving systems of linear equations.

The identity matrix.

The 2×2 **identity matrix** is

$$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

carrying 1s on the leading diagonal and 0s elsewhere. It plays the role that the number 1 plays for real numbers: for every 2×2 matrix A,

$$AI=IA=A.$$

There is an identity matrix of each square size; the 3×3 identity has three 1s down its leading diagonal. Multiplying any square matrix by the identity of the right size leaves it unchanged, and I is the only matrix with this property.

The determinant of a 2×2 matrix.

For

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix},$$

the **determinant** is the real number

$$det(A)=ad-bc,$$

also written $|A|$: the product of the leading-diagonal entries minus the product of the other two. The determinant is a *number*, not a matrix, and — as the next result shows — it decides completely whether A can be inverted. (In Chapter 9 the determinant reappears geometrically: a linear transformation scales every area by the factor $|det(A)|$, while the *sign* of $det(A)$ records whether orientation is preserved or reversed. Here it is a computational tool.)

The inverse of a 2×2 matrix.

The **inverse** of a square matrix A is a matrix A^{-1} of the same size for which

$$AA^{-1}=A^{-1}A=I.$$

To find it for a 2×2 matrix, pair A with the related matrix $\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$ (the leading-diagonal entries swapped and the other two negated) and multiply:

$$A\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\begin{bmatrix}ad-bc & -ab+ba\\ cd-dc & -cb+da\end{bmatrix}=\begin{bmatrix}ad-bc & 0\\ 0 & ad-bc\end{bmatrix}=(ad-bc)I.$$

The same product taken in the other order also gives $(ad-bc)I$. Provided the scalar $ad-bc$ is not zero, dividing through by it isolates the identity, and we read off

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

So the inverse **exists exactly when** $det(A) \neq 0$. A matrix with $det(A) = 0$ is called **singular** and has *no* inverse; a matrix with $det(A) \neq 0$ is **non-singular** (or **invertible**). The reasoning above also proves the singular case rigorously: if $det(A) = 0$ then $A\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the zero matrix, and were an inverse A^{-1} to exist, multiplying on the left by it would force $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ to be the zero matrix, hence $a = b = c = d = 0$ and A itself the zero matrix — which certainly has no inverse. A singular matrix therefore cannot be inverted.

The inverse is **unique**: if both B and C satisfied $AB = BA = I$ and $AC = CA = I$, then $B = BI = B(AC) = (BA)C = IC = C$. After finding an inverse by hand it is always worth forming AA^{-1} and confirming that it equals I — this single check catches nearly every arithmetic slip.

Properties of determinants and inverses.

The determinant of the identity is $det(I) = 1$. The single most useful property is that the determinant is **multiplicative**: for any two 2×2 matrices A and B,

$$det(AB) = det(A)\,det(B).$$

This is verified by direct expansion. Writing $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, the product is $AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$, so

$$det(AB) = (ap + br)(cq + ds) - (aq + bs)(cp + dr) = ad(ps - qr) - bc(ps - qr) = (ad - bc)(ps - qr),$$

after the terms $apcq$ and $aqcp$ cancel and likewise $brds$ and $bsdr$. The right-hand side is exactly $det(A)\,det(B)$. Two consequences follow at once. First, taking determinants of $AA^{-1} = I$ gives $det(A)\,det(A^{-1}) = 1$, so a non-singular matrix has

$$det(A^{-1}) = \frac{1}{det(A)}.$$

Second, if A and B are both invertible then so is their product, with $(AB)^{-1} = B^{-1}A^{-1}$ (the order reverses), since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$. Note that the determinant is **not** additive: $det(A + B)$ is generally not $det(A) + det(B)$.

Matrix equations.

Because matrices can be inverted, equations *between* matrices can be solved much as numerical equations are. Suppose A is non-singular and X is unknown. To solve

$$AX = B,$$

multiply **on the left** by A^{-1}: then $A^{-1}(AX) = A^{-1}B$, so $(A^{-1}A)X = A^{-1}B$, that is $IX = A^{-1}B$, giving

$$X = A^{-1}B.$$

To solve $XA = B$ instead, multiply **on the right** by A^{-1}, giving $X = BA^{-1}$. Matrix multiplication is not commutative, so the side on which A^{-1} is applied matters: $A^{-1}B$ and BA^{-1} are usually different matrices. If A is singular the method fails, because A^{-1} does not exist.

Simultaneous linear equations.

A pair of linear equations in two unknowns, written in standard form $a\,x+b\,y=e$ and $c\,x+d\,y=f$, is captured by a single matrix equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} e \\ f \end{bmatrix}, \text{that is } A\,x=b,$$

where A is the **coefficient matrix**, x the column of unknowns and b the column of constants. Multiplying out the left-hand side reproduces $a\,x+b\,y$ and $c\,x+d\,y$, so the one matrix equation says exactly what the two equations say. When $det(A)\neq 0$ the coefficient matrix has an inverse, and the equation has the **unique** solution

$$x=A^{-1}b.$$

When $det(A)=0$ there is no inverse and no unique solution: the two lines the equations describe are then either **parallel and distinct** (no solution) or **coincident** (the same line, hence infinitely many solutions). Which of these occurs is settled by comparing the constant terms once the coefficients are seen to be proportional.

Larger systems and the use of technology.

The same structure $A\,x=b$, solved by $x=A^{-1}b$, governs systems of three or more equations, with A the corresponding square coefficient matrix. A 3×3 matrix also has a determinant, and it too is non-zero exactly when the matrix is invertible and the system has a unique solution; by hand you are expected to invert only 2×2 matrices, and the determinant and inverse of a 3×3 matrix are obtained with technology. Entering A and b into a calculator's matrix facility returns $det(A)$, A^{-1} and the product $A^{-1}b$ directly; a reported error such as "singular matrix" is the machine stating that $det(A)=0$, so no unique solution exists. Knowing the by-hand method for the 2×2 case is what lets you interpret and check what the technology reports.

Worked examples

Example 1 — scaffolded

For $A=\begin{bmatrix} 3 & 7 \\ 2 & 6 \end{bmatrix}$, find $det(A)$, find A^{-1}, and confirm the inverse by computing $A\,A^{-1}$.

Working	Comment
$det(A)=a\,d-b\,c=3\times 6-7\times 2=18-14=4$.	Leading-diagonal product minus the product of the other two entries.
Since $det(A)=4\neq 0$, the inverse exists.	A non-zero determinant guarantees an inverse.
$A^{-1}=\frac{1}{4}\begin{bmatrix} 6 & -7 \\ -2 & 3 \end{bmatrix}$.	Swap the 3 and the 6, negate the 7 and the 2, and divide by $det(A)$.
$A\,A^{-1}=\frac{1}{4}\begin{bmatrix} 3 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 3 \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 18-14 & -21+21 \\ 12-12 & -14+18 \end{bmatrix}$.	Each entry is a row of A into a column of A^{-1}; the product is I, confirming the inverse.

So $det(A)=4$ and $A^{-1}=\frac{1}{4}\begin{bmatrix} 6 & -7 \\ -2 & 3 \end{bmatrix}$.

Example 2 — scaffolded

Use an inverse matrix to solve the simultaneous equations $3\,x+5\,y=17$ and $2\,x+4\,y=12$.

Working	Comment
$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 17 \\ 12 \end{bmatrix}$, that is $A\,x=b$.	The coefficients fill A; the constants form the right-hand column.
$det(A)=3\times 4-5\times 2=12-10=2\neq 0$.	A unique solution exists because the determinant is non-

Working	Comment
	zero.
$A^{-1}=\frac{1}{2}\begin{bmatrix}4 & -5\\ -2 & 3\end{bmatrix}$.	Swap, negate, divide by $det(A)=2$.
$x=A^{-1}b=\frac{1}{2}\begin{bmatrix}4 & -5\\ -2 & 3\end{bmatrix}\begin{bmatrix}17\\ 12\end{bmatrix}=\frac{1}{2}\begin{bmatrix}68-60\\ -34+36\end{bmatrix}=\frac{1}{2}\begin{bmatrix}8\\ 2\end{bmatrix}=\big[$.	Multiply the constant column by A^{-1}, with A^{-1} in front.
As a check, $3\times 4+5\times 1=17$ and $2\times 4+4\times 1=12$.	Substituting into both original equations recovers the constants.

So $x=4$ and $y=1$.

Example 3 — unscaffolded

Find the 2×2 matrix X satisfying $\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}X=\begin{bmatrix}4 & 1\\ 9 & 2\end{bmatrix}$.

This is a matrix equation of the form $AX=B$ with $A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$ and $B=\begin{bmatrix}4 & 1\\ 9 & 2\end{bmatrix}$. Because A multiplies X on the left, the equation is solved by multiplying on the left by A^{-1}, giving $X=A^{-1}B$. The determinant is $det(A)=2\times 3-1\times 5=1$, so

$$A^{-1}=\frac{1}{1}\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}.$$

Therefore

$$X=A^{-1}B=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}\begin{bmatrix}4 & 1\\ 9 & 2\end{bmatrix}=\begin{bmatrix}12-9 & 3-2\\ -20+18 & -5+4\end{bmatrix}=\begin{bmatrix}3 & 1\\ -2 & -1\end{bmatrix}.$$

As a check, $AX=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ -2 & -1\end{bmatrix}=\begin{bmatrix}6-2 & 2-1\\ 15-6 & 5-3\end{bmatrix}=\begin{bmatrix}4 & 1\\ 9 & 2\end{bmatrix}=B$.

$\therefore X=\begin{bmatrix}3 & 1\\ -2 & -1\end{bmatrix}$.

Example 4 — unscaffolded

Prove that if A is a 2×2 matrix with $det(A)\neq 0$, then its inverse satisfies $det(A^{-1})=\frac{1}{det(A)}$.

Since $det(A)\neq 0$, the inverse A^{-1} exists and satisfies $AA^{-1}=I$ by definition. Taking the determinant of both sides of this matrix identity,

$$det(AA^{-1})=det(I).$$

The determinant is multiplicative, so $det(AA^{-1})=det(A)\,det(A^{-1})$; and the identity matrix has $det(I)=1$. Hence

$$det(A)\,det(A^{-1})=1.$$

Because $det(A)\neq 0$, both sides may be divided by it, giving $det(A^{-1})=\frac{1}{det(A)}$, as required.

Practice questions

Scaffolded

Q1. Let $A=\begin{bmatrix} 6 & 5 \\ 2 & 3 \end{bmatrix}$. (a) Write down the 2×2 identity matrix I. (b) Find $det(A)$. (c) Find A^{-1}. (d) Confirm your inverse by computing $A\,A^{-1}$.

Q2. Find the determinant of each matrix. (a) $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} -3 & 5 \\ 2 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ (e) Exactly one of these matrices is singular. Which one, and how do you know?

Q3. Let $A=\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$. (a) Find $det(A)$ and $det(B)$. (b) Compute the product $A\,B$ and hence $det(A\,B)$. (c) Verify that $det(A\,B)=det(A)\,det(B)$.

Q4. Consider the simultaneous equations $5x+2y=12$ and $3x+4y=10$. (a) Write the pair as a matrix equation $A\,x=b$. (b) Find $det(A)$ and A^{-1}. (c) Solve the system using $x=A^{-1}b$. (d) Check your solution in both original equations.

Q5. Let $A=\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$. A matrix X satisfies $X\,A=B$. (a) Explain why $X=B\,A^{-1}$, and why this differs from $A^{-1}B$. (b) Find A^{-1}. (c) Compute $X=B\,A^{-1}$. (d) Check your answer by computing $X\,A$.

Q6. Let $M=\begin{bmatrix} k & 4 \\ 3 & 6 \end{bmatrix}$. (a) Find $det(M)$ in terms of k. (b) For what value of k is M singular? (c) When $k=5$, find M^{-1}.

Unscaffolded

Q7. Find the determinant and the inverse of $\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$.

Q8. Find the determinant and the inverse of $\begin{bmatrix} 8 & 6 \\ 2 & 3 \end{bmatrix}$.

Q9. Show that $B=\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is the inverse of $A=\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ by computing both $A\,B$ and $B\,A$.

Q10. Two of the matrices $P=\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$, $Q=\begin{bmatrix} 8 & 9 \\ 3 & 4 \end{bmatrix}$ and $R=\begin{bmatrix} 10 & 4 \\ 15 & 6 \end{bmatrix}$ are singular. Identify them, and find the inverse of the matrix that is not singular.

Q11. Use an inverse matrix to solve $4x+5y=7$ and $2x+3y=5$.

Q12. Use an inverse matrix to solve $7x+4y=1$ and $5x+3y=2$.

Q13. A workshop assembles two products, P and Q. Each unit of P requires 5 machine-hours and 3 labour-hours, and each unit of Q requires 6 machine-hours and 4 labour-hours. In one week the workshop logs 38 machine-hours and 24 labour-hours, all used on these two products. Letting x and y be the numbers of units of P and Q made, write the situation as a matrix equation and use an inverse matrix to find x and y.

Q14. An investor divides a sum between two accounts. Account A pays 4% simple interest per year and account B pays 7% per year. The total invested is \$9000 and the interest earned in one year is \$540. Letting x and y be the amounts (in dollars) in accounts A and B, form a matrix equation and solve it with an inverse matrix to find each amount.

Q15. Consider the system $2x+6y=10$ and $3x+9y=15$. (a) Show that the coefficient matrix is singular. (b) Determine whether the system has no solution or infinitely many, justifying your answer. (c) State what changes if the second equation is replaced by $3x+9y=18$.

Q16. Find all values of k for which $\begin{bmatrix} k & 8 \\ 2 & k \end{bmatrix}$ is singular.

Q17. Use technology to solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 11 \\ 14 \end{bmatrix},$$

stating the determinant of the coefficient matrix and verifying that a unique solution exists.

Q18. Find the 2×2 matrix X satisfying $\begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix}X=\begin{bmatrix} 10 & 2 \\ 5 & 3 \end{bmatrix}$.

Answers

Q1. (a) $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$; (b) $det(A)=8$; (c) $A^{-1}=\frac{1}{8}\begin{bmatrix}3 & -5\\ -2 & 6\end{bmatrix}$; (d) $AA^{-1}=I$. **Q2.** (a) 2; (b) 0; (c) 2; (d) 1; (e) matrix (b), because its determinant is 0. **Q3.** (a) $det(A)=7$, $det(B)=-5$; (b) $AB=\begin{bmatrix}9 & 8\\ 19 & 13\end{bmatrix}$, $det(AB)=-35$; (c) $-35=7\times(-5)$. **Q4.** (a) $\begin{bmatrix}5 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}12\\ 10\end{bmatrix}$; (b) $det(A)=14$, $A^{-1}=\frac{1}{14}\begin{bmatrix}4 & -2\\ -3 & 5\end{bmatrix}$; (c) $x=2$, $y=1$; (d) both equations hold. **Q5.** (b) $A^{-1}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$; (c) $X=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$. **Q6.** (a) $det(M)=6k-12$; (b) $k=2$; (c) $M^{-1}=\frac{1}{18}\begin{bmatrix}6 & -4\\ -3 & 5\end{bmatrix}$. **Q7.** $det=-1$; inverse $=\begin{bmatrix}-2 & 3\\ 5 & -7\end{bmatrix}$. **Q8.** $det=12$; inverse $=\frac{1}{12}\begin{bmatrix}3 & -6\\ -2 & 8\end{bmatrix}$. **Q9.** $AB=BA=I$, so $B=A^{-1}$. **Q10.** P and R are singular (each has determinant 0); $det(Q)=5$ and $Q^{-1}=\frac{1}{5}\begin{bmatrix}4 & -9\\ -3 & 8\end{bmatrix}$. **Q11.** $x=-2$, $y=3$. **Q12.** $x=-5$, $y=9$. **Q13.** $x=4$, $y=3$: 4 units of P and 3 units of Q. **Q14.** $x=3000$, $y=6000$: \$3000 in account A and \$6000 in account B. **Q15.** (a) $det=0$; (b) infinitely many solutions (both equations reduce to $x+3y=5$); (c) then no solution — the lines are parallel and distinct. **Q16.** $k=4$ or $k=-4$. **Q17.** $det=3\neq 0$; $x=1$, $y=2$, $z=3$. **Q18.** $X=\begin{bmatrix}-5 & -7\\ 5 & 5\end{bmatrix}$.

Fully-worked solutions

Q1. (a) $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. (b) $det(A)=6\times 3-5\times 2=18-10=8$. (c) $A^{-1}=\frac{1}{8}\begin{bmatrix}3 & -5\\ -2 & 6\end{bmatrix}$ (swap the 6 and the 3, negate the 5 and the 2, divide by 8). (d) $AA^{-1}=\frac{1}{8}\begin{bmatrix}6 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & -5\\ -2 & 6\end{bmatrix}=\frac{1}{8}\begin{bmatrix}18-10 & -30+30\\ 6-6 & -10+18\end{bmatrix}=\frac{1}{8}\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}=I$, confirming the inverse.

Q2. (a) $7\times 2-4\times 3=14-12=2$. (b) $6\times 3-9\times 2=18-18=0$. (c) $(-3)\times(-4)-5\times 2=12-10=2$. (d) $8\times 2-3\times 5=16-15=1$. (e) The matrix in (b) is singular, since its determinant is 0; a matrix is invertible exactly when its determinant is non-zero, and the other three have non-zero determinants.

Q3. (a) $det(A)=3\times 5-2\times 4=15-8=7$ and $det(B)=1\times 1-2\times 3=1-6=-5$. (b) $AB=\begin{bmatrix}3 & 2\\ 4 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}3+6 & 6+2\\ 4+15 & 8+5\end{bmatrix}=\begin{bmatrix}9 & 8\\ 19 & 13\end{bmatrix}$, so $det(AB)=9\times 13-8\times 19=117-152=-35$. (c) $det(A)det(B)=7\times(-5)=-35=det(AB)$, confirming the multiplicative property.

Q4. (a) $\begin{bmatrix}5 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}12\\ 10\end{bmatrix}$. (b) $det(A)=5\times 4-2\times 3=20-6=14$, so $A^{-1}=\frac{1}{14}\begin{bmatrix}4 & -2\\ -3 & 5\end{bmatrix}$. (c) $x=A^{-1}b=\frac{1}{14}\begin{bmatrix}4 & -2\\ -3 & 5\end{bmatrix}\begin{bmatrix}12\\ 10\end{bmatrix}=\frac{1}{14}\begin{bmatrix}48-20\\ -36+50\end{bmatrix}=\frac{1}{14}\begin{bmatrix}28\\ 14\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}$, so $x=2$ and $y=1$. (d) $5\times 2+2\times 1=12$ and $3\times 2+4\times 1=10$: both equations hold.

Q5. (a) In $XA=B$ the unknown X is multiplied by A **on its right**, so to undo it we multiply the whole equation on the right by A^{-1}: $XAA^{-1}=BA^{-1}$, giving $XI=BA^{-1}$ and hence $X=BA^{-1}$. This is generally different from $A^{-1}B$ because matrix multiplication is not commutative. (b) $det(A)=2\times 2-3\times 1=1$, so $A^{-1}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$. (c) $X=BA^{-1}=\begin{bmatrix}5 & 8\\ 3 & 5\end{bmatrix}\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}10-8 & -15+16\\ 6-5 & -9+10\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$. (d) $XA=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}4+1 & 6+2\\ 2+1 & 3+2\end{bmatrix}=\begin{bmatrix}5 & 8\\ 3 & 5\end{bmatrix}=B$, confirming X.

Q6. (a) $det(M) = k \times 6 - 4 \times 3 = 6k - 12$. (b) M is singular when $det(M) = 0$: $6k - 12 = 0$ gives $k = 2$. (c) When $k = 5$, $det(M) = 30 - 12 = 18$, so $M^{-1} = \frac{1}{18}\begin{bmatrix} 6 & -4 \\ -3 & 5 \end{bmatrix}$.

Q7. $det = 7 \times 2 - 3 \times 5 = 14 - 15 = -1$, so $\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -3 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix}$. (A negative determinant is non-zero, so the inverse still exists.)

Q8. $det = 8 \times 3 - 6 \times 2 = 24 - 12 = 12$, so the inverse is $\frac{1}{12}\begin{bmatrix} 3 & -6 \\ -2 & 8 \end{bmatrix}$.

Q9. $AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, and $BA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Since the product is the identity both ways, $B = A^{-1}$.

Q10. $det(P) = 4 \times 9 - 6 \times 6 = 36 - 36 = 0$ and $det(R) = 10 \times 6 - 4 \times 15 = 60 - 60 = 0$, so P and R are singular. The remaining matrix has $det(Q) = 8 \times 4 - 9 \times 3 = 32 - 27 = 5 \neq 0$, so $Q^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -9 \\ -3 & 8 \end{bmatrix}$.

Q11. As a matrix equation, $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$. The determinant is $4 \times 3 - 5 \times 2 = 2$, so $A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$ and $x = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 21-25 \\ -14+20 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. So $x = -2$ and $y = 3$; checking, $4 \times (-2) + 5 \times 3 = 7$ and $2 \times (-2) + 3 \times 3 = 5$.

Q12. As a matrix equation, $\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The determinant is $7 \times 3 - 4 \times 5 = 1$, so $A^{-1} = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$ and $x = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-8 \\ -5+14 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$. So $x = -5$ and $y = 9$; checking, $7 \times (-5) + 4 \times 9 = 1$ and $5 \times (-5) + 3 \times 9 = 2$.

Q13. The machine-hours and labour-hours give $5x + 6y = 38$ and $3x + 4y = 24$, that is $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 38 \\ 24 \end{bmatrix}$. The determinant is $5 \times 4 - 6 \times 3 = 20 - 18 = 2$, so $A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -3 & 5 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 38 \\ 24 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 152-144 \\ -114+120 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. So the workshop makes 4 units of P and 3 units of Q; checking, $5 \times 4 + 6 \times 3 = 38$ and $3 \times 4 + 4 \times 3 = 24$.

Q14. The total invested gives $x + y = 9000$, and the interest gives $0.04x + 0.07y = 540$; clearing decimals in the second equation ($\times 100$) yields $4x + 7y = 54000$. Hence $\begin{bmatrix} 1 & 1 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9000 \\ 54000 \end{bmatrix}$. The determinant is $1 \times 7 - 1 \times 4 = 3$, so $A^{-1} = \frac{1}{3}\begin{bmatrix} 7 & -1 \\ -4 & 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 9000 \\ 54000 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 63000-54000 \\ -36000+54000 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9000 \\ 18000 \end{bmatrix} = \begin{bmatrix} 3000 \\ 6000 \end{bmatrix}$. So \$3000 is in account A and \$6000 in account B; checking, the amounts total \$9000 and the interest is $0.04 \times 3000 + 0.07 \times 6000 = 120 + 420 = 540$ dollars.

Q15. (a) The coefficient matrix is $\begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$, with determinant $2 \times 9 - 6 \times 3 = 18 - 18 = 0$, so it is singular and has no inverse. (b) Dividing the first equation by 2 gives $x + 3y = 5$, and dividing the second by 3 gives $x + 3y = 5$ as well: the two equations describe the *same* line, so every point on it is a solution and the system has **infinitely many** solutions. (c) Replacing the second equation by $3x + 9y = 18$ makes it $x + 3y = 6$, which contradicts $x + 3y = 5$: the lines are now parallel and distinct, so the system has **no solution**. In both cases the singular coefficient matrix signals the absence of a unique solution; the constants decide which case occurs.

Q16. The matrix is singular when its determinant is zero: $det = k \times k - 8 \times 2 = k^2 - 16$, and $k^2 - 16 = 0$ gives $k^2 = 16$, so $k = 4$ or $k = -4$.

Q17. Entering the coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ and the constant column $b = \begin{bmatrix} 6 \\ 11 \\ 14 \end{bmatrix}$ into technology gives $det(A) = 3$. Since $det(A) \neq 0$, the matrix is invertible and the system has a unique solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

So $x = 1,\ y = 2,\ z = 3$. As a check, $1 + 2 + 3 = 6$, $2 \times 1 + 3 \times 2 + 3 = 11$ and $1 + 2 \times 2 + 3 \times 3 = 14$, matching all three equations.

Q18. This has the form $AX = B$ with $A = \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & 2 \\ 5 & 3 \end{bmatrix}$, so $X = A^{-1} B$. The determinant is $4 \times 2 - 6 \times 1 = 2$, so $A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -6 \\ -1 & 4 \end{bmatrix}$ and

$$X = \frac{1}{2} \begin{bmatrix} 2 & -6 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 10 & 2 \\ 5 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 20 - 30 & 4 - 18 \\ -10 + 20 & -2 + 12 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -10 & -14 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} -5 & -7 \\ 5 & 5 \end{bmatrix}.$$

As a check, $AX = \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -5 & -7 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} -20 + 30 & -28 + 30 \\ -5 + 10 & -7 + 10 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 5 & 3 \end{bmatrix} = B$.

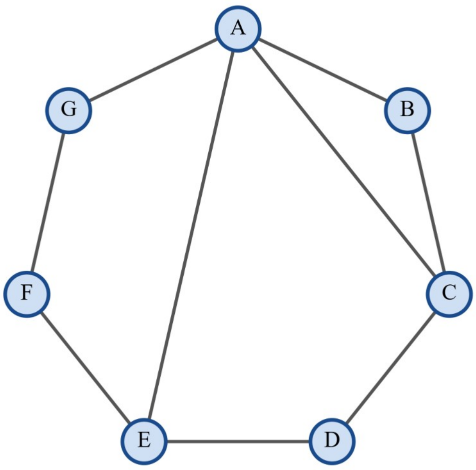
Topic 1 Test A — Technology-active

One bound reference and one approved technology (CAS calculator or software) are permitted. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (8 marks)

The connected graph H has vertices A, B, C, D, E, F, G and the nine edges

$$AB, BC, CD, DE, EF, FG, GA, AC, AE.$$



a. Write down the degree of each of the seven vertices, and hence verify the handshaking lemma for H. 2 marks

b. Explain why H is not a tree. Given that H is planar, use Euler's formula to determine the number of faces in a planar drawing of H. 3 marks

c. Determine, with reference to the vertex degrees, whether H has an Eulerian trail and whether it has an Eulerian circuit. Write down a Hamiltonian cycle in H. 3 marks

Question 2 (6 marks)

An online discussion forum tracks the number of *active* threads at the end of each week. During each week, half of the currently active threads become inactive, and 110 new threads are started. Let T_n be the number of active threads at the end of week n, so that

$$T_0 = 60, T_{n+1} = 0.5T_n + 110.$$

a. Find the number of active threads at the end of week 1 and at the end of week 2. 2 marks

b. Find the steady-state value L for which $T_{n+1} = T_n$, and interpret it in the context of the forum. 2 marks

c. Explain, with reference to the recurrence relation, why the number of active threads increases each week yet never reaches the value L. 2 marks

Question 3 (7 marks)

A school debating club has 12 members, of whom 7 are girls and 5 are boys.

a. In how many ways can a captain, a vice-captain and a scorer be chosen from the 12 members, assuming the three roles are held by different people? 1 mark

b. In how many ways can a team of 5 members be selected from the 12, if the order of selection does not matter? 1 mark

c. In how many ways can a team of 5 be selected that contains exactly 3 girls and 2 boys? 2 marks

d. Of the 12 members, 8 have competed in interschool debating and 6 have competed in public speaking, while 3 have competed in both. Using the inclusion–exclusion principle, find how many members have competed in at least one of these two activities, and how many have competed in neither. 3 marks

Question 4 (7 marks)

a. Construct a truth table for the compound proposition $(p \wedge q) \vee (\neg p \wedge r)$, showing a column for each of $p \wedge q$ and $\neg p \wedge r$. State the rows (as ordered triples of truth values) in which the proposition is true. 3 marks

b. Use the laws of Boolean algebra to simplify $x\,\acute{y}+x\,y+\acute{x}\,y$ to a single sum, and name the one logic gate that produces this output. 2 marks

c. The algorithm below reads a positive integer $n \geq 2$. Complete a trace for $n=6$ and state the output, then name the well-known sequence the algorithm generates. 2 marks

```
input n
a ← 1
b ← 1
for i from 3 to n do
    c ← a + b
    a ← b
    b ← c
end for
output b
```

Question 5 (6 marks)

Let $A=\begin{bmatrix}4 & 3\\1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1\\3 & 5\end{bmatrix}$.

a. Evaluate the matrix product AB. 2 marks

b. Find $det(A)$ and hence write down A^{-1}. 2 marks

c. Hence use A^{-1} to solve the simultaneous equations $4x+3y=14$ and $x+2y=1$. 2 marks

Question 6 (6 marks)

a. Express 4620 as a product of prime powers, and hence write down the number of positive divisors of 4620.2 marks

b. Prove by mathematical induction that

$$\sum_{r=1}^{n} r(r+1)=\frac{n(n+1)(n+2)}{3}$$

for all integers $n \geq 1$. 4 marks

End of Topic 1 Test A

Test A — answers and solutions

Question 1.

a. Listing the neighbours of each vertex,

$$deg(A)=4, deg(B)=2, deg(C)=3, deg(D)=2, deg(E)=3, deg(F)=2, deg(G)=2.$$

The degree sum is $4+2+3+2+3+2+2=18$, which is twice the number of edges: $2\times 9=18$. Hence the handshaking lemma $\sum_v deg(v)=2e$ is satisfied.

b. A tree on 7 vertices has exactly $7-1=6$ edges and contains no cycle. Since H has 9 edges (more than 6) it must contain cycles — for instance A-B-C-A — so H is not a tree. For a connected planar graph Euler's formula gives $v-e+f=2$, so

$$7-9+f=2 \Rightarrow f=4.$$

There are 4 faces (three bounded faces together with the outer face).

c. Exactly two vertices, C and E, have odd degree; the other five have even degree. A connected graph has an Eulerian trail exactly when it has 0 or 2 vertices of odd degree, so H **has an Eulerian trail** (one beginning at C and ending at E). It has an Eulerian circuit only when *every* vertex has even degree, so H **has no Eulerian circuit**. A Hamiltonian cycle — one passing through every vertex exactly once — is

$$A\rightarrow B\rightarrow C\rightarrow D\rightarrow E\rightarrow F\rightarrow G\rightarrow A.$$

Question 2.

a. $T_1=0.5(60)+110=30+110=140$ threads, and $T_2=0.5(140)+110=70+110=180$ threads.

b. At the steady state $L=0.5L+110$, so $0.5L=110$ and $L=220$. In the long term the number of active threads settles at 220: the number becoming inactive each week (half of 220, i.e. 110) exactly balances the 110 started.

c. Whenever $T_n<220$, the change over the week is

$$T_{n+1}-T_n=110-0.5T_n>110-0.5(220)=0,$$

so the count increases. At the same time $T_{n+1}=0.5T_n+110<0.5(220)+110=220$, so the count stays below 220. Starting from $T_0=60<220$, the number of active threads therefore rises each week but never reaches 220.

Question 3.

a. The three roles are distinct, so this is an ordered selection: $12\times 11\times 10=1320$ ways.

b. Order does not matter, so the number of teams is $\binom{12}{5}=792$.

c. Choose 3 of the 7 girls and 2 of the 5 boys independently:

$$\binom{7}{3}\binom{5}{2}=35\times 10=350 \text{ ways.}$$

d. Let D be the set who have debated and S the set who have done public speaking, with $|D|=8$, $|S|=6$ and $|D\cap S|=3$. By inclusion–exclusion,

$$|D\cup S|=|D|+|S|-|D\cap S|=8+6-3=11,$$

so 11 members have competed in at least one activity. The number who have done neither is $12-11=1$.

Question 4.

a.

p	q	r	$p \wedge q$	$\neg p \wedge r$	$(p \wedge q) \vee (\neg p \wedge r)$
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	F	F

The proposition is true in exactly the rows $(p,q,r)=(\mathrm{T},\mathrm{T},\mathrm{T})$, $(\mathrm{T},\mathrm{T},\mathrm{F})$, $(\mathrm{F},\mathrm{T},\mathrm{T})$ and $(\mathrm{F},\mathrm{F},\mathrm{T})$.

b. Grouping the first two terms and using $y'+y=1$,

$$x y' + x y + x' y = x(y'+y) + x' y = x + x' y.$$

By the absorption identity $x + x' y = x + y$. The simplified output $x+y$ is produced by a single **OR gate**.

c. With $a=1$ and $b=1$ initially, each pass computes $c=a+b$ and shifts the pair:

i	$c=a+b$	a	b
3	2	1	2
4	3	2	3
5	5	3	5
6	8	5	8

After the loop $b=8$, so the output is 8. The values produced, $1,1,2,3,5,8,\ldots$, are the **Fibonacci sequence**, and the output is the n-th Fibonacci number.

Question 5.

a. Multiplying row by column,

$$AB=\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 5\end{bmatrix}=\begin{bmatrix}4(2)+3(3) & 4(-1)+3(5)\\ 1(2)+2(3) & 1(-1)+2(5)\end{bmatrix}=\begin{bmatrix}17 & 11\\ 8 & 9\end{bmatrix}.$$

b. $det(A)=4\times 2-3\times 1=8-3=5$. Since $det(A)\neq 0$,

$$A^{-1}=\frac{1}{5}\begin{bmatrix}2 & -3\\ -1 & 4\end{bmatrix}.$$

c. The system is $A\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}14\\ 1\end{bmatrix}$, so

$$\begin{bmatrix}x\\ y\end{bmatrix}=A^{-1}\begin{bmatrix}14\\ 1\end{bmatrix}=\frac{1}{5}\begin{bmatrix}2 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}14\\ 1\end{bmatrix}=\frac{1}{5}\begin{bmatrix}28-3\\ -14+4\end{bmatrix}=\frac{1}{5}\begin{bmatrix}25\\ -10\end{bmatrix}=\begin{bmatrix}5\\ -2\end{bmatrix}.$$

Hence $x=5$ and $y=-2$. (Check: $4(5)+3(-2)=14$ and $5+2(-2)=1$.)

Question 6.

a. Factoring, $4620=2^2\times 3\times 5\times 7\times 11$. The number of positive divisors is the product of one more than each exponent:

$$(2+1)(1+1)(1+1)(1+1)(1+1)=3\times2\times2\times2\times2=48.$$

b. Let $P(n)$ be the statement $\sum_{r=1}^{n} r(r+1)=\frac{n(n+1)(n+2)}{3}$.

Base case. For $n=1$ the left side is $1\times2=2$ and the right side is $\frac{1\times2\times3}{3}=2$, so $P(1)$ is true.

Inductive step. Assume $P(k)$ holds for some integer $k\geq1$, that is $\sum_{r=1}^{k} r(r+1)=\frac{k(k+1)(k+2)}{3}$. Then

$$\sum_{r=1}^{k+1} r(r+1)=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)=(k+1)(k+2)\left(\frac{k}{3}+1\right)=\frac{(k+1)(k+2)(k+3)}{3}.$$

This is exactly the formula with n replaced by $k+1$, so $P(k)\Rightarrow P(k+1)$.

Conclusion. Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for all $k\geq1$, by the principle of mathematical induction $P(n)$ is true for all integers $n\geq1$.

Topic 1 Test B — Technology-free

Technology-free. No calculator. Reading time: 3 minutes. Writing time: 45 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

Let $A=\{n \in \mathbb{Z}: 1 \le n \le 15\}$.

a. Let $P=\{n \in A: n \text{ is prime}\}$ and $S=\{n \in A: n \text{ is a perfect square}\}$. List the elements of P and of S, and hence write down $P \cap S$ and $P \cup S$. 2 marks

b. Prove, using the contrapositive, that for every integer n, if $5n-3$ is even then n is odd. 2 marks

c. Prove, by contradiction, that $\sqrt{3}$ is irrational. You may assume that for an integer p, if 3 divides p^2 then 3 divides p. 3 marks

Question 2 (6 marks)

a. Using the handshaking lemma, prove that in any graph the number of vertices of odd degree is even. Hence explain why there is no graph with exactly 5 vertices, each of degree 3. 3 marks

b. A tree has 18 vertices. State the number of edges it has, and explain why adding a single new edge between two of its existing vertices must create a cycle. 2 marks

c. A connected planar graph has 10 vertices and 15 edges. Using Euler's formula, find the number of faces in a planar drawing of the graph. 1 mark

Question 3 (7 marks)

a. By constructing truth tables, show that $\neg(p \Rightarrow q)$ is logically equivalent to $p \wedge \neg q$. 3 marks

b. Use the laws of Boolean algebra to simplify $(a+b)(a+b')$. 2 marks

c. Evaluate $10011_2 + 1110_2$, giving your answer as a binary numeral. 2 marks

Question 4 (7 marks)

a. An arithmetic sequence has first term 8 and common difference 5. Find the 15th term and the sum of the first 15 terms. 2 marks

b. A geometric sequence has first term 5 and common ratio 3. Find the 6th term and the sum of the first 6 terms. 2 marks

c. Find the sum to infinity of the geometric series with first term 15 and common ratio $\frac{2}{5}$. 1 mark

d. A sequence is defined by $t_1=4$ and $t_{n+1}=3t_n-5$. Find t_2, t_3 and t_4. 2 marks

Question 5 (7 marks)

a. A bag holds a large number of marbles, each one of 5 different colours. Explain, using the pigeon-hole principle, why drawing 6 marbles guarantees that at least two of them are the same colour. 1 mark

b. For the same bag, find the least number of marbles that must be drawn to guarantee that at least three of them are the same colour. 2 marks

c. Using the inclusion–exclusion principle, find how many of the integers from 1 to 90 inclusive are divisible by 3 or by 5, and hence how many are divisible by neither. 2 marks

d. By evaluating both sides, show that $\binom{12}{9}=\binom{12}{3}$, and explain this equality by a combinatorial argument. 2 marks

Question 6 (6 marks)

Let $A=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}$ and $C=\begin{bmatrix}5 & -2\\ 1 & 6\end{bmatrix}$.

a. Evaluate $2A-C$. 2 marks

b. Find $det(A)$ and hence write down A^{-1}. 2 marks

c. Hence solve the simultaneous equations $3x+4y=10$ and $2x+3y=6$. 2 marks

End of Topic 1 Test B

Test B — answers and solutions

Question 1.

a. The primes in A are $P=\{2,3,5,7,11,13\}$ and the perfect squares in A are $S=\{1,4,9\}$. These sets have no element in common, so $P \cap S =$ (they are disjoint), while

$$P \cup S=\{1,2,3,4,5,7,9,11,13\}.$$

b. The contrapositive of "if $5n-3$ is even then n is odd" is "if n is even then $5n-3$ is odd." Suppose n is even, so $n=2k$ for some integer k. Then

$$5n-3=10k-3=2(5k-2)+1,$$

which is odd, since $5k-2$ is an integer. Thus the contrapositive is true, and because a statement and its contrapositive are logically equivalent, the original statement is true.

c. Suppose, for contradiction, that $\sqrt{3}$ is rational. Then $\sqrt{3}=\frac{p}{q}$ for integers p,q with $q\neq 0$, and we may take the fraction in lowest terms, so p and q have no common factor greater than 1. Squaring gives $3=\frac{p^2}{q^2}$, hence

$$p^2=3q^2.$$

So 3 divides p^2, and by the given assumption 3 divides p; write $p=3m$. Substituting, $9m^2=3q^2$, so $q^2=3m^2$, meaning 3 divides q^2 and hence 3 divides q. But then p and q share the factor 3, contradicting the assumption that the fraction was in lowest terms. This contradiction shows $\sqrt{3}$ is irrational.

Question 2.

a. By the handshaking lemma, $\sum_{v} deg(v)=2e$, which is even. Split the vertices into those of even degree and those of odd degree. The even-degree contributions sum to an even number, so the odd-degree contributions must also sum to an even number. A sum of odd numbers is even only when there is an even count of them, so the number of odd-degree vertices is even. A graph with 5 vertices each of degree 3 would have degree sum $5\times 3=15$, which is odd and so cannot equal $2e$; equivalently it would have 5 (an odd number of) odd-degree vertices. Hence no such graph exists.

b. A tree with n vertices has $n-1$ edges, so a tree with 18 vertices has 17 edges. Any two vertices of a tree are joined by exactly one path. Adding a new edge between two existing vertices u and v creates a second route between them (the new edge together with the existing u–v path), and these two distinct routes together form a cycle.

c. Euler's formula for a connected planar graph is $v-e+f=2$. With $v=10$ and $e=15$,

$$10-15+f=2 \Rightarrow f=7.$$

There are 7 faces.

Question 3.

a.

p	q	$p\Rightarrow q$	$\neg(p\Rightarrow q)$	$\neg q$	$p\wedge\neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

The columns for $\neg(p\Rightarrow q)$ and $p\wedge\neg q$ are identical in every row, so the two propositions are logically equivalent.

b. Using the distributive law $(a+X)(a+Y)=a+XY$ with $X=b$, $Y=b'$,

$$(a+b)(a+b')=a+bb'=a+0=a,$$

since $bb'=0$. The expression simplifies to a.

c. In base ten $10011_2=16+2+1=19$ and $1110_2=8+4+2=14$, so the sum is 33. Converting 33 back to binary, $33=32+1=100001_2$. Adding directly column by column confirms

$$10011_2+1110_2=100001_2.$$

Question 4.

a. With $a=8$ and $d=5$, the 15th term is $t_{15}=a+14d=8+14(5)=78$. The sum of the first 15 terms is

$$S_{15}=\frac{15}{2}(a+t_{15})=\frac{15}{2}(8+78)=\frac{15}{2}\times 86=645.$$

b. With $a=5$ and $r=3$, the 6th term is $t_6=ar^5=5\times 3^5=5\times 243=1215$. The sum of the first 6 terms is

$$S_6=\frac{a(r^6-1)}{r-1}=\frac{5(729-1)}{3-1}=\frac{5\times 728}{2}=1820.$$

c. Since $|r|=\frac{2}{5}<1$, the sum to infinity is

$$S_\infty=\frac{a}{1-r}=\frac{15}{1-\frac{2}{5}}=\frac{15}{\frac{3}{5}}=25.$$

d. Applying $t_{n+1}=3t_n-5$ from $t_1=4$:

$$t_2=3(4)-5=7,\, t_3=3(7)-5=16,\, t_4=3(16)-5=43.$$

Question 5.

a. Treat the 5 colours as boxes and the drawn marbles as objects. Drawing 6 marbles places 6 objects into 5 boxes, and since $6>5$ at least one box holds two or more marbles. That box is a single colour, so at least two of the marbles share a colour.

b. To have at most two of each colour, one could draw 2 marbles of each of the 5 colours, a total of 10 without any colour reaching three. The next marble drawn must repeat a colour already chosen twice, giving three of that colour. Hence the least number that guarantees three of one colour is $2\times 5+1=11$.

c. Among 1 to 90: the number divisible by 3 is $\lfloor 90/3\rfloor=30$; the number divisible by 5 is $\lfloor 90/5\rfloor=18$; and the number divisible by both (that is, by 15) is $\lfloor 90/15\rfloor=6$. By inclusion–exclusion the number divisible by 3 or 5 is

$$30+18-6=42.$$

The number divisible by neither is $90-42=48$.

d. Evaluating each side,

$$\binom{12}{9}=\frac{12!}{9!3!}=220,\binom{12}{3}=\frac{12!}{3!9!}=220,$$

so the two are equal. Combinatorially, choosing which 9 of the 12 objects to **include** is the same act as choosing which 3 to **leave out**; each selection of 9 corresponds to exactly one selection of 3, so the two counts must agree. This is the identity $\binom{n}{r}=\binom{n}{n-r}$ with $n=12$, $r=9$.

Question 6.

a. $2A=\begin{bmatrix}6 & 8\\ 4 & 6\end{bmatrix}$, so

$$2A-C=\begin{bmatrix}6 & 8\\ 4 & 6\end{bmatrix}-\begin{bmatrix}5 & -2\\ 1 & 6\end{bmatrix}=\begin{bmatrix}1 & 10\\ 3 & 0\end{bmatrix}.$$

b. $det(A)=3\times3-4\times2=9-8=1$. Since $det(A)=1$,

$$A^{-1}=\frac{1}{1}\begin{bmatrix}3 & -4\\ -2 & 3\end{bmatrix}=\begin{bmatrix}3 & -4\\ -2 & 3\end{bmatrix}.$$

c. The system is $A\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}10\\ 6\end{bmatrix}$, so

$$\begin{bmatrix}x\\ y\end{bmatrix}=A^{-1}\begin{bmatrix}10\\ 6\end{bmatrix}=\begin{bmatrix}3 & -4\\ -2 & 3\end{bmatrix}\begin{bmatrix}10\\ 6\end{bmatrix}=\begin{bmatrix}30-24\\ -20+18\end{bmatrix}=\begin{bmatrix}6\\ -2\end{bmatrix}.$$

Hence $x=6$ and $y=-2$. (Check: $3(6)+4(-2)=10$ and $2(6)+3(-2)=6$.)