Specialist Mathematics Units 1 & 2

Chapter 5 — Combinatorics

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Contents

Theory

Some of the sharpest results in combinatorics are proved not by clever construction but by a single, almost obvious, observation: if there are more objects than there are containers to put them in, then at least one container must hold more than one object. Stated so plainly it seems too simple to be useful, yet it settles a surprising range of existence questions — questions that ask us to *guarantee* that something must happen, without ever pointing to the particular objects involved. This observation is the **pigeon-hole principle**, so named from the picture of letters being sorted into a rack of pigeon-holes.

The basic principle.

> **Pigeon-hole principle.** If $n+1$ objects are placed into n boxes, then at least one box contains **two or more** objects.

The word *box* is only a label for a category, and the *objects* are whatever is being sorted into those categories; the principle is about counting, not about physical boxes. Its proof is a short argument by contradiction.

Proof. Suppose, on the contrary, that no box contained two or more objects. Then every one of the n boxes would hold at most one object, so the total number of objects would be at most

$$1+1+\cdots+1=n\,(n\,\text{boxes}).$$

But there are $n+1$ objects, and $n+1>n$, a contradiction. Hence some box must contain at least two objects, as required.

The principle guarantees *existence* only: it tells us that two objects share a box, but never *which* two, nor *which* box. That is exactly what makes it powerful — the conclusion holds for **every** possible placement, so it can be asserted with no information about the particular arrangement.

Applying it: choosing the objects and the boxes.

Using the principle is a matter of naming, for the situation at hand, what the *objects* are and what the *boxes* are, so that the number of objects exceeds the number of boxes. For instance, among any 13 people, two must share a birth **month**: take the 13 people as the objects and the 12 months as the boxes. Since $13=12+1$, two people fall in the same month-box — that is, they were born in the same month. Notice that 12 people would not force this (one could fall in each month), so the count $n+1$ is **sharp**: it is the *least* number that guarantees a repeat.

The generalised principle.

If there are far more objects than boxes, one box must be not merely shared but *heavily* loaded. The generalised principle makes this precise using the **ceiling** $\lceil x \rceil$, the least integer greater than or equal to x.

> **Generalised pigeon-hole principle.** If N objects are placed into k boxes, then at least one box contains at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Proof. Suppose, on the contrary, that every box held **fewer** than $\left\lceil \frac{N}{k} \right\rceil$ objects; that is, at most $\left\lceil \frac{N}{k} \right\rceil-1$ objects each. Because $\left\lceil \frac{N}{k} \right\rceil<\frac{N}{k}+1$, we have $\left\lceil \frac{N}{k} \right\rceil-1<\frac{N}{k}$, and so the total number of objects would be at most

$$k\left(\lceil N/k \rceil-1\right)<k\cdot\frac{N}{k}=N.$$

This says the boxes hold fewer than N objects in total — impossible, since they hold all N. Hence some box contains at least $\left\lceil \frac{N}{k} \right\rceil$ objects, as required.

Taking $N=n+1$ and $k=n$ recovers the basic principle, since $\left\lceil \frac{n+1}{n} \right\rceil = 2$. As a fuller example, among any 25 people at least $\left\lceil \frac{25}{12} \right\rceil = 3$ share a birth month, because $24 = 2 \times 12$ leaves 25 just over two full rounds of the 12 months.

A guaranteed-count threshold.

It is often cleaner to run the generalised principle the other way: *how many objects force some box to hold at least m?* If every box held at most $m-1$ objects, the total could be as large as $k(m-1)$; so as soon as

$$N \geq k(m-1)+1$$

some box is forced to contain at least m objects. Equivalently, $k(m-1)$ objects can be spread out with no box reaching m (put exactly $m-1$ in each), but one more object cannot. For the birth-month example with $k=12$, guaranteeing that **three** people share a month needs $12(3-1)+1=25$ people — matching the count found above.

Divisibility: differences and sums.

A favourite source of boxes is the set of **remainders** on division by a fixed integer n. Every integer leaves one of the n remainders $0, 1, 2, \ldots, n-1$ when divided by n, so these remainders form n boxes.

Among any $n+1$ integers, two must leave the **same** remainder on division by n; their difference is then divisible by n. (If $a=qn+r$ and $b=q'n+r$ share the remainder r, then $a-b=(q-q')n$.) For example, among any 8 integers whatsoever, two have a difference divisible by 7.

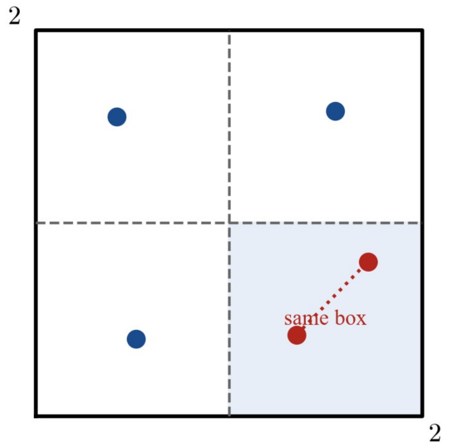
A small change of boxes handles **sums**. Pairing each remainder r with its partner $n-r$ groups the remainders into classes; two integers landing in the same class have either equal remainders (so their *difference* is divisible by n) or complementary remainders r and $n-r$ (so their *sum* is divisible by n). Choosing the boxes to fit the question is the whole art of the method.

A monotone subsequence.

Pigeon-holing can force *structure*, not just a coincidence. From a list of distinct numbers, a **subsequence** is any list obtained by deleting some terms and keeping the rest in their original order; it is **increasing** if each term exceeds the one before and **decreasing** if each is smaller. A short list can wander up and down, but a long enough list must contain a long **monotone** (increasing or decreasing) run. The precise statement is that any sequence of more than n^2 distinct numbers contains a monotone subsequence of length $n+1$; the proof, developed in the worked examples, tags each term with a *pair* of lengths and pigeon-holes the pairs.

Points in a region.

Geometry supplies boxes too: cut a region into smaller pieces, and each piece is a box. If more points than pieces are scattered in the region, two points share a piece and so lie no farther apart than that piece is wide. The figure below shows a square of side 2 divided into four unit sub-squares. Any 5 points placed in the square must, since $5>4$, put two points in a common unit sub-square; those two are then at most a sub-square diagonal — a distance of $\sqrt{2}$ — apart.



The recurring pattern in every application is the same three steps: decide what the **objects** are, decide what the **boxes** are so that objects outnumber boxes, and read off the forced conclusion. The remainder of this section practises making those two choices.

Worked examples

Example 1 — scaffolded

A café roster assigns each staff member to exactly one of the five weekdays (Monday–Friday). (a) Show that if 6 staff are rostered, two share a weekday. (b) If 23 staff are rostered, show that some weekday has at least 5 of them. (c) What is the least number of staff that guarantees some weekday has at least 6?

Working	Comment
Objects: the staff. Boxes: the 5 weekdays.	Name the objects and boxes so objects can outnumber boxes.
(a) There are 6 staff and 5 weekday-boxes, and $6=5+1$.	This is the basic principle with $n=5$.
By the pigeon-hole principle some weekday-box holds two staff.	Two staff share a weekday.
(b) With $N=23$ objects and $k=5$ boxes, $\left\lceil \frac{23}{5} \right\rceil = \lceil 4.6 \rceil = 5$.	Generalised principle: divide and round **up**.
So some weekday has at least 5 staff.	$20=4\times 5$, and 23 exceeds four full rounds.
(c) To force some box to hold at least $m=6$, need $N\geq k(m-1)+1=5(6-1)+1=26$.	Threshold form: 5 staff on each of 5 days is 25 with none reaching 6.
The least such number is 26.	The 26th staff member must lift some day to 6.

Example 2 — scaffolded

(a) Explain why, among any 8 integers, two of them have a difference divisible by 7.
(b) Find such a pair inside the list $8, 14, 17, 23, 39, 40, 55, 66$.

Working	Comment
The remainders on division by 7 are $0, 1, 2, 3, 4, 5, 6$ — that is 7 boxes.	Remainders modulo 7 are the boxes.
The 8 integers are the objects, and $8=7+1$.	Objects outnumber boxes by one.
So two of them share a remainder; if $a=7q+r$ and	Equal remainders make the difference a multiple of 7.

Working	Comment
$b=7q^{'}+r$ then $a-b=7(q-q^{'})$.	
(b) Remainders: $8 \to 1, 14 \to 0, 17 \to 3, 23 \to 2, 39 \to 4, 40 \to 5, 55 \to 6$	Compute each integer modulo 7.
17 and 66 both leave remainder 3.	The forced repeat is remainder 3.
$66-17=49=7\times 7$, divisible by 7.	A valid pair; the difference is a multiple of 7.

Example 3 — unscaffolded

Prove that from any list of six integers $a_1, a_2, \ldots, a_6$, some block of **consecutive** terms has a sum divisible by 6. Illustrate with $5, 9, 2, 7, 4, 8$.

Form the seven **partial sums**

$$S_0=0, S_1=a_1, S_2=a_1+a_2, \ldots, S_6=a_1+a_2+\cdots+a_6,$$

where each S_j ($j=1,\ldots,6$) is the total of the first j terms and $S_0=0$. These are **seven** numbers, and each leaves one of the **six** remainders $0,1,2,3,4,5$ on division by 6. Since $7=6+1$, the pigeon-hole principle forces two of the partial sums, say S_i and S_j with $i<j$, to share a remainder. Their difference is then divisible by 6:

$$S_j-S_i=a_{i+1}+a_{i+2}+\cdots+a_j,$$

which is exactly the sum of the consecutive block from term $i+1$ to term j.

For $5, 9, 2, 7, 4, 8$ the partial sums are $S_0, \ldots, S_6 = 0, 5, 14, 16, 23, 27, 35$, with remainders modulo 6 equal to $0, 5, 2, 4, 5, 3, 5$. The values S_1 and S_4 both leave remainder 5, so the block from term 2 to term 4 has sum $S_4-S_1=23-5=18=6\times 3$; indeed $9+2+7=18$.

∴ some block of consecutive terms has a sum divisible by 6, as required.

Example 4 — unscaffolded

Prove that any sequence of five distinct real numbers contains a monotone subsequence (increasing or decreasing) of length 3, and give a sequence of four distinct numbers that has no such subsequence.

Label the terms $a_1, a_2, \ldots, a_5$. To each term a_i attach the pair of counts

$$(u_i, d_i),$$

where u_i is the length of the longest **increasing** subsequence *ending* at a_i and d_i is the length of the longest **decreasing** subsequence ending at a_i. Suppose, for a contradiction, that no monotone subsequence has length 3. Then every u_i and every d_i lies in $\{1, 2\}$, so each pair (u_i, d_i) is one of only $2\times 2=4$ possibilities.

The five terms give five pairs sitting in these four possibilities, so by the pigeon-hole principle two terms a_i, a_j with $i<j$ carry the **same** pair $(u_i, d_i)=(u_j, d_j)$. But the terms are distinct, so either $a_i<a_j$ or $a_i>a_j$. If $a_i<a_j$, appending a_j to a longest increasing subsequence ending at a_i gives one longer, so $u_j\ge u_i+1>u_i$; if $a_i>a_j$, likewise $d_j\ge d_i+1>d_i$. Either way the pairs differ, contradicting $(u_i, d_i)=(u_j, d_j)$. Hence a monotone subsequence of length 3 must exist.

For four numbers this can fail: in $2, 4, 1, 3$ the longest increasing subsequences ($2, 4$ and $1, 3$ and $2, 3$) and longest decreasing subsequences ($2, 1$ and $4, 1$ and $4, 3$) all have length 2.

∴ five distinct numbers force a monotone subsequence of length 3, while four need not, as required.

Practice questions

Scaffolded

Q1. A vending machine dispenses cans in 6 different flavours. (a) What is the least number of cans that must be taken to be sure of two cans of the same flavour? (b) What is the least number that must be taken to be sure of three of the same flavour? (c) If 40 cans are taken, explain why some flavour appears at least 7 times.

Q2. Show that among any 10 integers, two have a difference divisible by 9. (a) State what the boxes are and how many there are. (b) State how many objects there are, and why a box is shared. (c) Explain why sharing a box makes a difference divisible by 9. (d) Find such a pair inside $10, 13, 14, 17, 20, 21, 24, 25, 36, 59$.

Q3. A librarian shelves 100 books across 9 shelves. (a) Identify N and k for the generalised principle. (b) Compute $\left\lceil \frac{100}{9} \right\rceil$. (c) State the conclusion about some shelf. (d) Give an arrangement of the 100 books in which no shelf holds more than your answer to (b), showing the bound cannot be improved.

Q4. Five points are placed inside a square of side 2. Show that two of them are at most $\sqrt{2}$ apart. (a) Describe how to divide the square into four boxes. (b) Explain why two of the points lie in the same box. (c) State the greatest possible distance between two points in one box. (d) Write the conclusion.

Q5. Consider the birth months of a group of people. (a) How many people guarantee that two share a birth month? (b) How many guarantee that three share a birth month? (c) How many guarantee that four share a birth month? (d) In a group of 100 people, what is the largest number that some month is guaranteed to contain?

Q6. Show that from any five integers $a_1, \ldots, a_5$, some block of consecutive terms has a sum divisible by 5. (a) Write down the partial sums $S_0, S_1, \ldots, S_5$. (b) Explain why two of them share a remainder on division by 5. (c) For $7, 3, 8, 2, 9$, list the remainders of $S_0, \ldots, S_5$ modulo 5. (d) Identify a consecutive block whose sum is divisible by 5.

Unscaffolded

Q7. In any group of $n \geq 2$ people (where friendship is mutual), prove that two people have the same number of friends **within the group**. *(Hint: the possible friend-counts are $0, 1, \ldots, n-1$, but 0 and $n-1$ cannot both occur.)*

Q8. Prove that among any 13 integers, two have a difference divisible by 12.

Q9. Prove that among any 6 integers, two of them have a sum **or** a difference divisible by 9.

Q10. A season of 50 matches is played over 6 days. (a) Show that some day must host at least 9 matches. (b) Exhibit a schedule in which no day hosts more than 9, confirming 9 cannot be improved.

Q11. Ten points are placed inside an equilateral triangle of side 3. Prove that two of them are at most 1 apart. *(Hint: an equilateral triangle of side 3 can be cut into nine equilateral triangles of side 1.)*

Q12. Prove that any seven distinct integers chosen from $\{1, 2, \ldots, 12\}$ include two whose sum is 13. Show also that six integers need not.

Q13. Prove that any eight distinct integers chosen from $\{1, 2, \ldots, 14\}$ include two that differ by exactly 7. Show also that seven integers need not.

Q14. Prove that from any $n+1$ integers chosen from $\{1, 2, \ldots, 2n\}$, one of them divides another. *(Hint: write each integer as $2^k \times (\text{odd})$ and box by its odd part.)*

Q15. Prove that from any $n+1$ integers chosen from $\{1, 2, \ldots, 2n\}$, two are coprime. *(Hint: box the integers into the consecutive pairs $\{1, 2\}, \{3, 4\}, \ldots$)*

Q16. Prove that any sequence of ten distinct real numbers contains a monotone subsequence of length 4. Show also that nine distinct numbers need not, by examining $3, 2, 1, 6, 5, 4, 9, 8, 7$.

Q17. Prove that every positive integer n has a positive multiple whose decimal digits are all 0s and 1s. *(Hint: consider the* $n+1$ *numbers* $1, 11, 111, \ldots$ *each made of* 1*s, and their remainders modulo* n*.)*

Q18. Seven points are placed inside a regular hexagon of side 1. Prove that two of them are at most 1 apart.

Answers

Q1. (a) 7; (b) 13; (c) $\lceil 40/6 \rceil = 7$. **Q2.** Boxes: the 9 remainders modulo 9; 10 objects force a repeat, giving a difference divisible by 9. (d) 14 and 59 (both remainder 5), $59 - 14 = 45 = 9 \times 5$. **Q3.** (a) $N = 100$, $k = 9$; (b) 12; (c) some shelf holds at least 12 books; (d) e.g. one shelf with 12 and eight shelves with 11 ($12 + 88 = 100$). **Q4.** Divide into four unit squares; two points share one; its diagonal is $\sqrt{2}$, so those two are at most $\sqrt{2}$ apart. **Q5.** (a) 13; (b) 25; (c) 37; (d) $\lceil 100/12 \rceil = 9$. **Q6.** Two of $S_0, \dots, S_5$ share a remainder modulo 5; for $7, 3, 8, 2, 9$ the remainders are $0, 2, 0, 3, 0, 4$, and the block $7 + 3 = 10$ is divisible by 5. **Q7.** The counts take only $n - 1$ possible values for n people, so two coincide. **Q8.** 12 remainders modulo 12, 13 integers; two share a remainder, difference divisible by 12. **Q9.** Five boxes $\{0\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$; six integers force two into one box, giving a sum or difference divisible by 9. **Q10.** (a) $\lceil 50/6 \rceil = 9$; (b) e.g. two days of 9 and four days of 8 ($18 + 32 = 50$). **Q11.** Cut into nine unit triangles; two of the ten points share one; its diameter is 1. **Q12.** Six pairs summing to 13; seven integers force two into one pair. $\{1, 2, 3, 4, 5, 6\}$ has no such pair. **Q13.** Seven pairs $\{i, i + 7\}$; eight integers force two into one pair. $\{1, \dots, 7\}$ has none. **Q14.** n odd parts $1, 3, \dots, 2n - 1$; $n + 1$ integers force two with the same odd part, and then one divides the other. **Q15.** n consecutive-pair boxes; $n + 1$ integers force two consecutive integers, which are coprime. **Q16.** Since $10 > 3^2$, a monotone subsequence of length 4 is forced; the nine-term list shown has none of length 4. **Q17.** Two of $1, 11, \dots$ share a remainder modulo n; their difference is a multiple of n made only of 1s and 0s. **Q18.** Six triangles from the centre; two of the seven points share one; its diameter is 1.

Fully-worked solutions

Q1. The objects are the cans and the boxes are the 6 flavours. (a) With 6 flavour-boxes, taking $6 + 1 = 7$ cans forces two into one flavour, so the answer is 7; six cans could be one of each flavour. (b) To guarantee three of one flavour, use the threshold $N \geq k(m - 1) + 1$ with $k = 6$, $m = 3$: this is $6(3 - 1) + 1 = 13$. (c) With $N = 40$ cans in $k = 6$ flavour-boxes, the generalised principle gives some flavour with at least $\lceil \frac{40}{6} \rceil = 7$ cans, since $36 = 6 \times 6$ and 40 passes six full rounds.

Q2. (a) The boxes are the remainders $0, 1, \dots, 8$ on division by 9 — nine boxes. (b) The ten integers are the objects; since $10 = 9 + 1$, two integers land in the same remainder-box. (c) If $a = 9q + r$ and $b = 9q' + r$ share remainder r, then $a - b = 9(q - q')$, a multiple of 9. (d) The remainders modulo 9 are
$10 \to 1, 13 \to 4, 14 \to 5, 17 \to 8, 20 \to 2, 21 \to 3, 24 \to 6, 25 \to 7, 36 \to 0, 59 \to 5$. The repeat is remainder 5: the integers 14 and 59, with $59 - 14 = 45 = 9 \times 5$.

Q3. (a) There are $N = 100$ objects (books) and $k = 9$ boxes (shelves). (b) $\lceil \frac{100}{9} \rceil = 12$, since $99 = 9 \times 11$ and 100 is just past eleven full rounds. (c) By the generalised principle some shelf holds at least 12 books. (d) The bound is sharp: place 12 books on one shelf and 11 on each of the other eight. This totals $12 + 8 \times 11 = 12 + 88 = 100$, and no shelf exceeds 12, so no smaller guarantee than 12 is possible.

Q4. Divide the square of side 2 into four unit sub-squares by its horizontal and vertical mid-lines; these four sub-squares are the boxes. The five points are the objects, and $5 = 4 + 1$, so by the pigeon-hole principle two of the points lie in the same unit sub-square. Within a square of side 1 the greatest distance between two points is the diagonal, of length $\sqrt{1^2 + 1^2} = \sqrt{2}$. Hence those two points are at most $\sqrt{2}$ apart.

Q5. The boxes are the 12 months. (a) $12 + 1 = 13$ people force two into one month. (b) For three in a month use $N \geq 12(3 - 1) + 1 = 25$. (c) For four in a month use $N \geq 12(4 - 1) + 1 = 37$. (d) With $N = 100$ people the generalised principle gives some month with at least $\lceil \frac{100}{12} \rceil = 9$, since $96 = 12 \times 8$ and 100 passes eight full rounds.

Q6. Form the six partial sums $S_0 = 0$ and $S_j = a_1 + \cdots + a_j$ for $j = 1, \dots, 5$. Each leaves one of the five remainders $0, 1, 2, 3, 4$ on division by 5, and there are six of them, so by the pigeon-hole principle two — say S_i and S_j with $i < j$ — share a remainder; then $S_j - S_i = a_{i+1} + \cdots + a_j$ is a consecutive block with sum divisible by 5. For $7, 3, 8, 2, 9$

the partial sums are $0,7,10,18,20,29$, with remainders $0,2,0,3,0,4$ modulo 5. Here S_0 and S_2 both leave remainder 0, so the block from term 1 to term 2 has sum $S_2 - S_0 = 10 = 5 \times 2$; indeed $7+3=10$.

Q7. Take the n people as objects. Each person's number of friends within the group is one of $0,1,\ldots,n-1$ — apparently n values. But 0 and $n-1$ cannot both occur: if some person is friends with everyone ($n-1$ friends) then no person can have 0 friends, and if some person has 0 friends then no person can be friends with all $n-1$ others. So the friend-counts actually lie in a set of only $n-1$ possible values (either $\{0,1,\ldots,n-2\}$ or $\{1,2,\ldots,n-1\}$). With n people and only $n-1$ available counts, the pigeon-hole principle forces two people to have the same number of friends, as required.

Q8. The boxes are the 12 remainders $0,1,\ldots,11$ on division by 12. The 13 integers are the objects, and $13=12+1$, so two share a remainder r. Writing them as $12q+r$ and $12q'+r$, their difference is $12(q-q')$, divisible by 12.

Q9. Group the remainders modulo 9 into the five boxes

$$\{0\},\{1,8\},\{2,7\},\{3,6\},\{4,5\},$$

each pairing a remainder r with $9-r$. The six integers are the objects, and $6=5+1$, so two of them, a and b, fall in the same box. If they have equal remainders then $a-b$ is divisible by 9; if they have complementary remainders r and $9-r$ then $a+b \equiv r+(9-r)=9\equiv 0$, so $a+b$ is divisible by 9. Either way two integers have a sum or difference divisible by 9, as required.

Q10. (a) The 50 matches are objects and the 6 days are boxes, so some day hosts at least $\left\lceil \frac{50}{6} \right\rceil = 9$ matches. (b) The value 9 cannot be lowered: since $50=6\times 8+2$, schedule two days with 9 matches and four days with 8, giving $2\times 9+4\times 8=18+32=50$ with no day above 9.

Q11. Cut the equilateral triangle of side 3 into nine equilateral triangles of side 1 (three rows of small triangles, $1+3+5=9$ in all); these are the boxes. The ten points are the objects, and $10=9+1$, so two points lie in the same small triangle. In an equilateral triangle of side 1 the greatest distance between two points is a side, of length 1, so those two points are at most 1 apart, as required.

Q12. Box the numbers $1,\ldots,12$ into the six pairs

$$\{1,12\},\{2,11\},\{3,10\},\{4,9\},\{5,8\},\{6,7\},$$

each summing to 13. Seven chosen integers are the objects, and $7=6+1$, so two fall in the same pair and sum to 13. Six integers can avoid this — for instance $\{1,2,3,4,5,6\}$, taking one number from each pair, has no two summing to 13 (the largest sum is $5+6=11$).

Q13. Box $1,\ldots,14$ into the seven pairs

$$\{1,8\},\{2,9\},\{3,10\},\{4,11\},\{5,12\},\{6,13\},\{7,14\},$$

each of the form $\{i,i+7\}$. Eight chosen integers are the objects, and $8=7+1$, so two fall in the same pair and differ by 7. Seven integers need not: $\{1,2,3,4,5,6,7\}$ has no two differing by 7.

Q14. Every integer x can be written $x=2^k\times m$ with m odd, by dividing out all factors of 2. If $x\le 2n$ then its odd part m is one of the n odd numbers $1,3,5,\ldots,2n-1$; take these as the boxes. The $n+1$ chosen integers are the objects, and $n+1>n$, so two of them, $x=2^k m$ and $y=2^\ell m$, share the same odd part m. If $k\le\ell$ then $x \,/\, y$ (since $y=2^{\ell-k}x$); otherwise $y \,/\, x$. Either way one divides the other, as required.

Q15. Box the numbers $1,\ldots,2n$ into the n consecutive pairs $\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}$. The $n+1$ chosen integers are the objects, and $n+1>n$, so two fall in the same pair; these are consecutive integers m and $m+1$. Any common divisor of m and $m+1$ divides their difference $(m+1)-m=1$, so their greatest common divisor is 1: they are coprime, as required.

Q16. As in Example 4, attach to each term a_i the pair (u_i, d_i), where u_i is the length of the longest increasing subsequence ending at a_i and d_i the longest decreasing one. Suppose no monotone subsequence has length 4; then every u_i and every d_i lies in $\{1,2,3\}$, giving only $3 \times 3 = 9$ possible pairs. As shown in Example 4, the pairs of distinct terms are all different, so the ten terms would need ten distinct pairs among only nine — impossible by the pigeon-hole principle. Hence a monotone subsequence of length 4 exists. Nine terms can avoid it: in $3,2,1,6,5,4,9,8,7$ the longest increasing subsequence has length 3 (for example $3,6,9$) and the longest decreasing one also has length 3 (for example $3,2,1$), so none reaches length 4. (Here $9 = 3^2$, one short of the guaranteeing count $3^2 + 1 = 10$.)

Q17. Consider the $n+1$ integers

$$R_1 = 1, R_2 = 11, R_3 = 111, \ldots, R_{n+1} = 11 \cdots 1(n+1 \text{ ones}),$$

each written using only the digit 1. On division by n they leave one of the n remainders $0, 1, \ldots, n-1$, so by the pigeon-hole principle two of them, R_i and R_j with $i < j$, share a remainder. Their difference

$$R_j - R_i = 1 \cdots 10 \cdots 0(j - i \text{ ones, then } i \text{ zeros})$$

is a positive multiple of n (it is $R_{j-i} \times 10^i$) whose decimal digits are all 0s and 1s. For instance with $n = 6$, $R_1 = 1$ and $R_4 = 1111$ both leave remainder 1, and $R_4 - R_1 = 1110 = 6 \times 185$. Hence every n has such a multiple, as required.

Q18. Join the centre of the regular hexagon to its six vertices; this cuts the hexagon into six equilateral triangles, each of side 1 (a regular hexagon of side 1 has each centre-to-vertex distance equal to 1). These six triangles are the boxes. The seven points are the objects, and $7 = 6 + 1$, so two points lie in the same triangle. In an equilateral triangle of side 1 the greatest distance between two points is a side, of length 1, so those two points are at most 1 apart, as required.

Theory

Counting the members of a single set is straightforward; counting the members of a **union** of overlapping sets is where mistakes begin. If 40 students study French and 36 study Japanese, it is tempting to say that $40+36=76$ study at least one of the two languages — but any student who studies **both** has been counted once in each total, and so counted twice. The remedy is to add the sizes and then **subtract the overlap**, correcting for the double-counting. This single idea, applied to two sets, three sets, or more, is the **inclusion–exclusion principle**: include everything, then exclude what has been included more than once. This section states the principle precisely, derives it, and applies it to survey and divisibility problems.

Sets, sizes and the universal set.

For a finite set A, the **cardinality** (or **size**) $|A|$ is the number of elements in A. All the sets in a counting problem are regarded as subsets of a fixed **universal set** ξ, whose size $|\xi|$ is the total number of objects under discussion. Recall from Chapter 1 the set operations used throughout: the **union** $A \cup B$ (elements in A **or** B or both), the **intersection** $A \cap B$ (elements in A **and** B), the **complement** A^c (elements of ξ **not** in A), and the **difference** $A \setminus B$ (elements in A but not in B). Two sets are **disjoint** when $A \cap B =$; for disjoint sets, and only for disjoint sets, the sizes simply add:

$$A \cap B = \quad \Rightarrow |A \cup B| = |A| + |B|.$$

Because a complement partitions ξ into A and A^c with nothing in common,

$$|A^c| = |\xi| - |A|.$$

This last identity is the key to every "how many have **none** of these properties" count.

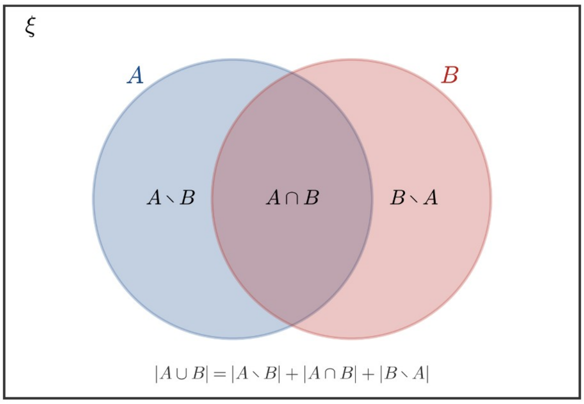
The two-set principle.

For any two finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Derivation. The union $A \cup B$ splits into **three pairwise-disjoint** pieces: the part of A outside B, namely $A \setminus B$; the shared part $A \cap B$; and the part of B outside A, namely $B \setminus A$. Since these three regions are disjoint and together make up the whole union,

$$|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A|.$$



Each original set is itself split by B (or by A) into two of these pieces:

$$|A| = |A \setminus B| + |A \cap B|, |B| = |B \setminus A| + |A \cap B|.$$

Adding these two equations,

$$|A|+|B|=|A\setminus B|+|B\setminus A|+2|A\cap B|=\big(|A\setminus B|+|A\cap B|+|B\setminus A|\big)+|A\cap B|,$$

and the bracket is exactly $|A\cup B|$. Hence $|A|+|B|=|A\cup B|+|A\cap B|$, which rearranges to the principle

$$|A\cup B|=|A|+|B|-|A\cap B|,$$

as required. In words: the overlap $A\cap B$ is counted **once** in $|A|$ and **again** in $|B|$, so subtracting it once leaves each element of the union counted exactly once.

Counting "neither".

Combining the two-set principle with the complement identity answers the common question of how many objects lie in **neither** set. The elements in neither A nor B form the complement of the union, $(A\cup B)^c$, so

$$|(A\cup B)^c|=|\xi|-|A\cup B|=|\xi|-\big(|A|+|B|-|A\cap B|\big).$$

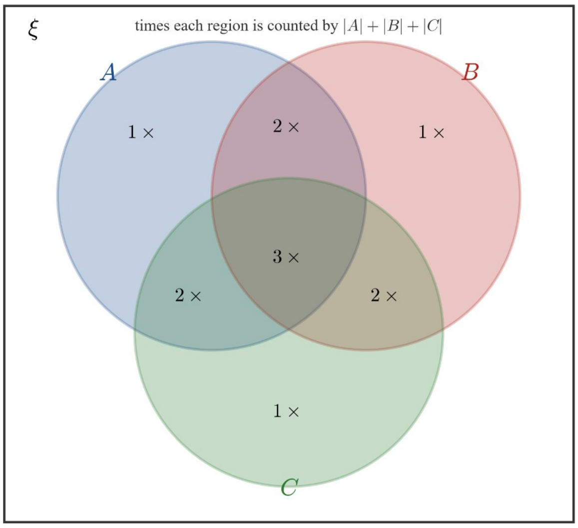
Reading a two-set problem is often easiest through its four regions — A only, B only, both, and neither — whose sizes add to $|\xi|$.

The three-set principle.

For any three finite sets A, B and C,

$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|.$$

Derivation. Three overlapping sets cut their union into **seven** disjoint regions: three "only" regions, three "exactly two" regions, and the central "all three" region. The principle is proved by checking that the right-hand side counts **every** element of the union exactly once. Track a single element according to how many of the three sets contain it.



- An element in **exactly one** set, say A alone, is counted once by $|A|$ and by nothing else on the right — net count 1.
- An element in **exactly two** sets, say A and B but not C, is counted by $|A|$ and by $|B|$ (that is $+2$), then removed once by $-|A\cap B|$ — net count $2-1=1$.
- An element in **all three** sets is counted by $|A|,|B|,|C|$ (that is $+3$), removed by each of $-|A\cap B|,-|A\cap C|,-|B\cap C|$ (that is -3), and restored once by $+|A\cap B\cap C|$ — net count $3-3+1=1$.

Every element of $A\cup B\cup C$ therefore contributes exactly 1 to the right-hand side, and elements outside the union contribute 0, so the right-hand side equals $|A\cup B\cup C|$, as required. The alternating pattern — add the singles, subtract the pairs, add back the triple — is the signature of inclusion–exclusion, and it is why the final $+|A\cap B\cap C|$ must be present. As before, the count with **none** of the three properties is $|\xi|-|A\cup B\cup C|$.

Filling a Venn diagram from the inside out.

Many three-set problems supply the individual sizes $|A|,|B|,|C|$ and the intersection sizes, from which the seven region sizes are recovered by working **inwards to outwards**. The centre is $|A \cap B \cap C|$. Each “exactly two” region is a pairwise intersection with the centre removed, for example

$$|A \cap B \cap C^c| = |A \cap B| - |A \cap B \cap C|.$$

Each “only” region is a single set with all of its overlaps removed and the centre added back once — inclusion–exclusion again — for example

$$|A \cap B^c \cap C^c| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$

Filling the diagram this way makes every subsequent count a matter of adding the relevant regions.

Application: counting multiples.

Inclusion–exclusion is the standard tool for counting the integers in a range that are divisible by given numbers. The number of integers in $\{1, 2, \ldots, N\}$ that are divisible by a positive integer d is

$$\left\lfloor \frac{N}{d} \right\rfloor,$$

the **floor** of N/d — the largest integer not exceeding N/d — since the multiples of d up to N are $d, 2d, 3d, \ldots$ and there are $\lfloor N/d \rfloor$ of them. The crucial observation is that an integer is divisible by **both** a and b exactly when it is divisible by their **lowest common multiple** $lcm(a,b)$. So if A and B are the sets of integers up to N divisible by a and by b respectively, then

$$|A| = \left\lfloor \frac{N}{a} \right\rfloor, |B| = \left\lfloor \frac{N}{b} \right\rfloor, |A \cap B| = \left\lfloor \frac{N}{lcm(a,b)} \right\rfloor,$$

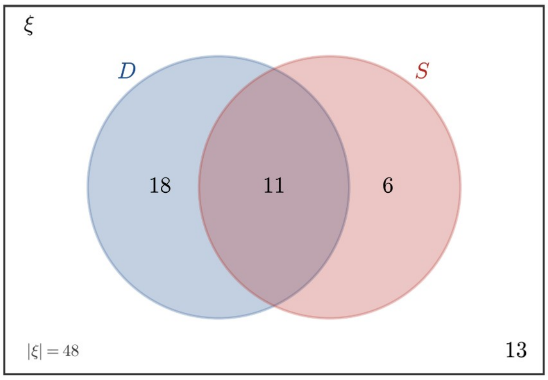
and the count divisible by a or b follows from the two-set principle. The three-set version uses lcm of each pair and of all three. This turns a divisibility question into a purely mechanical application of the principle.

Worked examples

Example 1 — scaffolded

A school robotics club has 48 members. Of these, 29 build racing drones, 17 compete in robot sumo, and 11 do both activities. Let D be the set of drone builders and S the set of sumo competitors. Find (a) $|D \cup S|$, (b) how many members do **neither** activity, and (c) how many do **exactly one** of the two.

Working	Comment
$|D|=29$, $|S|=17$ and $|D \cap S|=11$; here $|\xi|=48$.	Identify the two set sizes, their overlap, and the total.
$|D \cup S| = |D| + |S| - |D \cap S| = 29 + 17 - 11 = 35$.	Two-set principle: subtract the overlap counted twice.
Neither $= |\xi| - |D \cup S| = 48 - 35 = 13$.	The “neither” region is the complement of the union.
Drone-only $= |D| - |D \cap S| = 29 - 11 = 18$; sumo-only $= 17 - 11 = 6$.	Remove the overlap from each set to get its “only” region.
Exactly one $= 18 + 6 = 24$.	Add the two “only” regions.



Example 2 — scaffolded

Consider the integers from 1 to 500 inclusive. (a) How many are divisible by 4? (b) How many are divisible by 7? (c) How many are divisible by 4 **or** 7? (d) How many are divisible by **neither** 4 nor 7?

Working	Comment
Divisible by 4: $\lfloor 500/4 \rfloor = 125$.	Multiples of d up to N number $\lfloor N/d \rfloor$.
Divisible by 7: $\lfloor 500/7 \rfloor = 71$.	Since $7 \times 71 = 497 \le 500 < 504$.
Divisible by both means divisible by $lcm(4,7) = 28$: $\lfloor 500/28 \rfloor = 17$.	The overlap is the multiples of the lowest common multiple.
Divisible by 4 or 7 $= 125 + 71 - 17 = 179$.	Two-set principle applied to the two divisibility sets.
Divisible by neither $= 500 - 179 = 321$.	Complement of the union within the 500 integers.

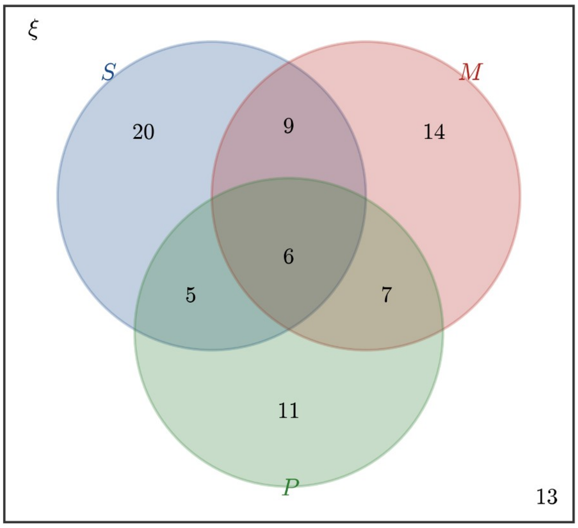
Example 3 — unscaffolded

A university hosts 85 international exchange students. During the semester, 40 visited Sydney, 36 visited Melbourne and 29 visited Perth. Of these, 15 visited both Sydney and Melbourne, 11 visited both Sydney and Perth, and 13 visited both Melbourne and Perth, while 6 visited all three cities. How many students visited at least one of the three cities, and how many visited none? Give the breakdown of all seven regions.

Write S, M, P for the sets of students who visited Sydney, Melbourne and Perth. The data are $|S| = 40$, $|M| = 36$, $|P| = 29$, $|S \cap M| = 15$, $|S \cap P| = 11$, $|M \cap P| = 13$ and $|S \cap M \cap P| = 6$. By the three-set principle,

$$|S \cup M \cup P| = 40 + 36 + 29 - 15 - 11 - 13 + 6 = 105 - 39 + 6 = 72.$$

The number who visited none is $|\xi| - |S \cup M \cup P| = 85 - 72 = 13$. Filling the Venn diagram from the inside out, the centre is 6; the "exactly two" regions are $15 - 6 = 9$ (Sydney and Melbourne only), $11 - 6 = 5$ (Sydney and Perth only) and $13 - 6 = 7$ (Melbourne and Perth only); and the "only" regions are $40 - 15 - 11 + 6 = 20$ (Sydney), $36 - 15 - 13 + 6 = 14$ (Melbourne) and $29 - 11 - 13 + 6 = 11$ (Perth).



$\therefore$ 72 students visited at least one city and 13 visited none, with region sizes $20, 14, 11$ (one city only), $9, 5, 7$ (exactly two) and 6 (all three).

Example 4 — unscaffolded

How many integers from 1 to 840 inclusive are divisible by at least one of 6, 8 and 14? Hence state how many are divisible by none of them.

Let A, B, C be the integers up to 840 divisible by 6, 8 and 14 respectively. Their sizes are

$$|A| = \lfloor 840/6 \rfloor = 140, |B| = \lfloor 840/8 \rfloor = 105, |C| = \lfloor 840/14 \rfloor = 60.$$

For the pairwise intersections, divisibility by two of the numbers means divisibility by their lowest common multiple: $lcm(6,8) = 24$, $lcm(6,14) = 42$ and $lcm(8,14) = 56$, giving

$$|A \cap B| = \lfloor 840/24 \rfloor = 35, |A \cap C| = \lfloor 840/42 \rfloor = 20, |B \cap C| = \lfloor 840/56 \rfloor = 15.$$

Divisibility by all three requires $lcm(6,8,14) = 168$, so $|A \cap B \cap C| = \lfloor 840/168 \rfloor = 5$. The three-set principle then gives

$$|A \cup B \cup C| = 140 + 105 + 60 - 35 - 20 - 15 + 5 = 305 - 70 + 5 = 240.$$

The number divisible by none is $840 - 240 = 600$.

$\therefore$ 240 of the integers are divisible by at least one of 6, 8, 14, and 600 are divisible by none.

Practice questions

Scaffolded

Q1. In a group of 60 tourists, 34 have visited Tasmania (T) and 25 have visited Kakadu (K); 12 have visited both. (a) Find $|T \cup K|$. (b) How many have visited neither? (c) How many have visited Tasmania only? (d) How many have visited exactly one of the two places?

Q2. Consider the integers from 1 to 400 inclusive. (a) How many are divisible by 5? (b) How many are divisible by 8 ? (c) How many are divisible by both 5 and 8? (d) How many are divisible by 5 or 8?

Q3. In a class of 30 students, 18 play a string instrument (S), 14 play a wind instrument (W), and 5 play neither. (a) How many play at least one of the two kinds of instrument? (b) Use the inclusion–exclusion principle to find how many play both. (c) How many play a string instrument only? (d) How many play a wind instrument only?

Q4. Fifty bushwalkers are surveyed on three long-distance trails they have walked: the Overland (O), the Larapinta (L) and the Bibbulmun (B). The figures are $|O| = 24$, $|L| = 20$, $|B| = 18$, $|O \cap L| = 8$, $|O \cap B| = 6$, $|L \cap B| = 7$ and

$|O \cap L \cap B| = 3$. (a) How many have walked the Overland only? (b) Find the size of each of the seven regions inside the three circles of the Venn diagram. (c) Use the inclusion–exclusion principle to find $|O \cup L \cup B|$. (d) How many have walked none of the three trails?

Q5. Consider the integers from 1 to 360 inclusive. (a) How many are divisible by 4, by 5, and by 6 (answer each separately)? (b) How many are divisible by 4 and 5, by 4 and 6, and by 5 and 6 (each separately)? (c) How many are divisible by 4, 5 and 6 all together? (d) How many are divisible by at least one of 4, 5, 6?

Q6. A gym with 80 members offers two classes. A Venn diagram of attendance shows 22 attend yoga only, 15 attend both yoga and pilates, and 19 attend pilates only. (a) Find $|Y|$, the number who attend yoga. (b) Find $|P|$, the number who attend pilates. (c) Find $|Y \cup P|$. (d) How many members attend neither class?

Unscaffolded

Q7. Of 118 shoppers at a farmers' market, 74 bought vegetables, 46 bought flowers and 21 bought both. How many bought neither?

Q8. How many integers from 1 to 750 inclusive are divisible by 6 or 9?

Q9. How many integers from 1 to 600 inclusive are divisible by neither 8 nor 10?

Q10. In a cohort of 200 university students, 130 study mathematics and 95 study physics; every student studies at least one of the two. How many study both?

Q11. Of 150 travellers, 78 have visited Japan, 64 have visited Korea and 58 have visited Vietnam. Also 30 have visited both Japan and Korea, 25 both Japan and Vietnam, and 22 both Korea and Vietnam, while 12 have visited all three. How many have visited at least one of the three countries?

Q12. A council surveys 90 households about three home upgrades: solar panels (S), a home battery (B) and an electric-vehicle charger (E). The results are $|S| = 50$, $|B| = 34$, $|E| = 28$, $|S \cap B| = 18$, $|S \cap E| = 14$, $|B \cap E| = 11$ and $|S \cap B \cap E| = 7$. How many households have none of the three upgrades?

Q13. How many integers from 1 to 1260 inclusive are divisible by at least one of 4, 9 and 14?

Q14. In a year level of 60 students, 30 study French, 28 study German and 24 study Japanese. Also 14 study both French and German, 12 both French and Japanese, and 11 both German and Japanese, while 5 students study none of the three languages. How many study all three languages?

Q15. How many integers from 1 to 500 inclusive are divisible by exactly one of 5 and 6?

Q16. A survey of 200 sports fans records which of three sports they follow: cricket (C), netball (N) and Australian football (A). The figures are $|C| = 112$, $|N| = 74$, $|A| = 96$, $|C \cap N| = 40$, $|C \cap A| = 55$, $|N \cap A| = 30$ and $|C \cap N \cap A| = 18$. How many fans follow exactly one of the three sports?

Q17. (a) By partitioning $A \cup B$ into the three disjoint regions $A \setminus B$, $A \cap B$ and $B \setminus A$, derive the two-set inclusion–exclusion principle $|A \cup B| = |A| + |B| - |A \cap B|$. (b) In a library survey of 250 members, 160 borrow fiction and 95 borrow non-fiction, while 40 borrow neither. Use the principle to find how many borrow both.

Q18. At a music festival, 300 attendees each visited at least one of three stages: the Main stage (M), the Grove (G) and the Barn (B). Records show $|M| = 180$, $|G| = 140$, $|B| = 120$, $|M \cap G| = 60$, $|M \cap B| = 50$ and $|G \cap B| = 40$. (a) How many attendees visited all three stages? (b) How many visited exactly two of the three stages?

Answers

Q1. (a) 47; (b) 13; (c) 22; (d) 35. **Q2.** (a) 80; (b) 50; (c) 10; (d) 120. **Q3.** (a) 25; (b) 7; (c) 11; (d) 7. **Q4.** (a) 13; (b) Overland only 13, Larapinta only 8, Bibbulmun only 8, Overland–Larapinta only 5, Overland–Bibbulmun only 3, Larapinta–Bibbulmun only 4, all three 3; (c) 44; (d) 6. **Q5.** (a) 90, 72, 60; (b) 18, 30, 12; (c) 6; (d) 168. **Q6.** (a) 37; (b) 34; (c) 56; (d) 24. **Q7.** 19. **Q8.** 167. **Q9.** 480. **Q10.** 25. **Q11.** 135. **Q12.** 14. **Q13.** 460. **Q14.** 10. **Q15.** 151. **Q16.** 86. **Q17.** (a) see solution; (b) 45. **Q18.** (a) 10; (b) 120.

Fully-worked solutions

Q1. With $|T|=34$, $|K|=25$ and $|T\cap K|=12$: (a) $|T\cup K|=34+25-12=47$. (b) Neither $=60-47=13$. (c) Tasmania only $=|T|-|T\cap K|=34-12=22$. (d) Exactly one = (Tasmania only) + (Kakadu only) $=22+(25-12)=22+13=35$.

Q2. For $N=400$: (a) $\lfloor 400/5\rfloor=80$. (b) $\lfloor 400/8\rfloor=50$. (c) Divisible by both means divisible by $lcm(5,8)=40$, so $\lfloor 400/40\rfloor=10$. (d) By inclusion–exclusion, $80+50-10=120$.

Q3. Here $|S|=18$, $|W|=14$, and 5 of the 30 play neither. (a) At least one $=30-5=25$. (b) The inclusion–exclusion principle gives $|S\cup W|=|S|+|W|-|S\cap W|$, so $25=18+14-|S\cap W|$, giving $|S\cap W|=32-25=7$. (c) String only $=18-7=11$. (d) Wind only $=14-7=7$.

Q4. Work inwards to outwards from the given data. (a) Overland only $=|O|-|O\cap L|-|O\cap B|+|O\cap L\cap B|=24-8-6+3=13$. (b) The centre is 3. The "exactly two" regions are $8-3=5$ (Overland–Larapinta), $6-3=3$ (Overland–Bibbulmun) and $7-3=4$ (Larapinta–Bibbulmun). The "only" regions are 13 (Overland, from part (a)), $20-8-7+3=8$ (Larapinta) and $18-6-7+3=8$ (Bibbulmun). (c) $|O\cup L\cup B|=24+20+18-8-6-7+3=62-21+3=44$. (As a check, the seven regions sum to $13+8+8+5+3+4+3=44$.) (d) None $=50-44=6$.

Q5. For $N=360$: (a) $\lfloor 360/4\rfloor=90$, $\lfloor 360/5\rfloor=72$, $\lfloor 360/6\rfloor=60$. (b) $lcm(4,5)=20$ gives $\lfloor 360/20\rfloor=18$; $lcm(4,6)=12$ gives $\lfloor 360/12\rfloor=30$; $lcm(5,6)=30$ gives $\lfloor 360/30\rfloor=12$. (c) $lcm(4,5,6)=60$ gives $\lfloor 360/60\rfloor=6$. (d) By the three-set principle, $90+72+60-18-30-12+6=222-60+6=168$.

Q6. The four regions of the diagram are yoga only $=22$, both $=15$, pilates only $=19$, and neither. (a) $|Y|=22+15=37$. (b) $|P|=15+19=34$. (c) $|Y\cup P|=22+15+19=56$ (equivalently $37+34-15=56$). (d) Neither $=80-56=24$.

Q7. With $|V|=74$, $|F|=46$ and $|V\cap F|=21$, the number who bought at least one is $|V\cup F|=74+46-21=99$. Hence neither $=118-99=19$.

Q8. $\lfloor 750/6\rfloor=125$ and $\lfloor 750/9\rfloor=83$. Divisible by both means divisible by $lcm(6,9)=18$, giving $\lfloor 750/18\rfloor=41$. By inclusion–exclusion the count divisible by 6 or 9 is $125+83-41=167$.

Q9. $\lfloor 600/8\rfloor=75$ and $\lfloor 600/10\rfloor=60$. Since $lcm(8,10)=40$, the count divisible by both is $\lfloor 600/40\rfloor=15$, so the count divisible by 8 or 10 is $75+60-15=120$. The count divisible by neither is the complement, $600-120=480$.

Q10. Every student studies at least one subject, so $|M\cup P|=200$. The inclusion–exclusion principle gives $200=130+95-|M\cap P|$, hence $|M\cap P|=225-200=25$ students study both.

Q11. By the three-set principle,

$$|J\cup K\cup V|=78+64+58-30-25-22+12=200-77+12=135.$$

So 135 travellers have visited at least one of the three countries.

Q12. By the three-set principle,

$$|S \cup B \cup E| = 50 + 34 + 28 - 18 - 14 - 11 + 7 = 112 - 43 + 7 = 76.$$

The number with none of the three upgrades is $90 - 76 = 14$.

Q13. With $N = 1260$: $\lfloor 1260/4 \rfloor = 315$, $\lfloor 1260/9 \rfloor = 140$, $\lfloor 1260/14 \rfloor = 90$. The pairwise lowest common multiples are $lcm(4,9) = 36$, $lcm(4,14) = 28$ and $lcm(9,14) = 126$, giving $\lfloor 1260/36 \rfloor = 35$, $\lfloor 1260/28 \rfloor = 45$ and $\lfloor 1260/126 \rfloor = 10$. Finally $lcm(4,9,14) = 252$ gives $\lfloor 1260/252 \rfloor = 5$. By the three-set principle the count is

$$315 + 140 + 90 - 35 - 45 - 10 + 5 = 545 - 90 + 5 = 460.$$

Q14. Since 5 of the 60 study none, $|F \cup G \cup J| = 60 - 5 = 55$. Writing t for $|F \cap G \cap J|$, the three-set principle gives

$$55 = 30 + 28 + 24 - 14 - 12 - 11 + t = 82 - 37 + t = 45 + t,$$

so $t = 55 - 45 = 10$ students study all three languages.

Q15. Let A be the multiples of 5 and B the multiples of 6 up to 500. Then $|A| = \lfloor 500/5 \rfloor = 100$, $|B| = \lfloor 500/6 \rfloor = 83$, and, since $lcm(5,6) = 30$, $|A \cap B| = \lfloor 500/30 \rfloor = 16$. "Exactly one" excludes the overlap from **each** set, so the count is

$$(|A| - |A \cap B|) + (|B| - |A \cap B|) = |A| + |B| - 2|A \cap B| = 100 + 83 - 32 = 151.$$

Q16. First find the seven regions. The centre is 18. The "exactly two" regions are $40 - 18 = 22$, $55 - 18 = 37$ and $30 - 18 = 12$. The "only" regions are $112 - 40 - 55 + 18 = 35$ (cricket), $74 - 40 - 30 + 18 = 22$ (netball) and $96 - 55 - 30 + 18 = 29$ (Australian football). Fans who follow exactly one sport number $35 + 22 + 29 = 86$.

Q17. (a) The union $A \cup B$ is the disjoint union of $A \setminus B$, $A \cap B$ and $B \setminus A$, so $|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A|$. Also $|A| = |A \setminus B| + |A \cap B|$ and $|B| = |B \setminus A| + |A \cap B|$. Adding the last two, $|A| + |B| = |A \setminus B| + |B \setminus A| + 2|A \cap B| = |A \cup B| + |A \cap B|$, which rearranges to $|A \cup B| = |A| + |B| - |A \cap B|$, as required. (b) The members who borrow at least one kind number $250 - 40 = 210$, so $|F \cup N| = 210$. By the principle, $210 = 160 + 95 - |F \cap N|$, giving $|F \cap N| = 255 - 210 = 45$ members who borrow both.

Q18. Every attendee visited at least one stage, so $|M \cup G \cup B| = 300$. (a) Writing $t = |M \cap G \cap B|$, the three-set principle gives

$$300 = 180 + 140 + 120 - 60 - 50 - 40 + t = 440 - 150 + t = 290 + t,$$

so $t = 10$ attendees visited all three stages. (b) Each "exactly two" region is a pairwise intersection with the centre removed: $60 - 10 = 50$, $50 - 10 = 40$ and $40 - 10 = 30$. Attendees visiting exactly two stages number $50 + 40 + 30 = 120$.

Theory

Combinatorics is the art of counting the outcomes of a procedure **without listing them one by one**. The questions look innocent — how many codes, how many teams, how many seatings — but for even modest sizes the outcomes run to millions, so a systematic method is essential. Every technique in this section is built from a single idea, the multiplication principle, refined step by step: first to ordered arrangements, then to arrangements when repetition is allowed or some objects are identical, then to arrangements in a ring, and finally to unordered selections, both without repetition and with it. Throughout, the two questions to ask of any counting problem are: **does order matter?** and **may an object be used more than once?**

The multiplication principle.

Suppose a task is carried out in a sequence of independent stages. If the first stage can be completed in n_1 ways, and *whatever* the outcome of the first stage the second can be completed in n_2 ways, and so on to the k-th stage in n_k ways, then the whole task can be completed in

$$n_1 \times n_2 \times \cdots \times n_k$$

ways. The word "independent" here means only that the *number* of choices at each stage does not depend on the earlier outcomes; the choices themselves may well differ. For instance, a field tag made from one of 5 colours followed by one of 3 shapes can be produced in $5 \times 3 = 15$ ways, and a three-course set menu offering 4 entrées, 6 mains and 2 desserts yields $4 \times 6 \times 2 = 48$ different meals. This principle underlies everything that follows.

Factorials and arranging distinct objects.

Recall the **factorial** $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, the product of the whole numbers from n down to 1, together with the convention $0! = 1$. Arranging n **distinct** objects in a row is a task in n stages: the first position may be filled in n ways, the next in $n-1$ ways (one object now used), and so on down to 1 way for the last. By the multiplication principle the number of arrangements is

$$n(n-1)(n-2)\cdots 2 \cdot 1 = n!.$$

So six different books can be ordered on a shelf in $6! = 720$ ways.

Permutations: arrangements without repetition.

More generally we may arrange only **some** of the objects. An ordered arrangement of r objects selected from n distinct objects, with no object used more than once, is called a **permutation** of n objects taken r at a time. Filling r ordered positions, the first can be filled in n ways, the second in $n-1$ ways, and so on until the r-th in $n-r+1$ ways. By the multiplication principle the number of such permutations is

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}\,(r\text{ factors}),$$

a quantity written ${}^{n}\mathrm{P}_{r}$ and read "*n* pick *r*". The compact factorial form follows because dividing $n!$ by $(n-r)!$ cancels exactly the tail $(n-r)(n-r-1)\cdots 1$, leaving the first r factors. Taking $r = n$ arranges every object and gives ${}^{n}\mathrm{P}_{n} = \frac{n!}{0!} = n!$, recovering the count above. For example, the number of ways to award gold, silver and bronze among 10 finalists is ${}^{10}\mathrm{P}_{3} = 10 \times 9 \times 8 = 720$; here order matters, since a different assignment of the same three athletes to the medals is a different outcome.

Arrangements with repetition.

If instead each position may be filled from the **full** set of n objects every time — that is, repetition is allowed — then each of the r positions independently has n choices, and the number of arrangements is

$$n \times n \times \cdots \times n = n^r \,(r \text{ factors}).$$

This is the model for codes and passwords drawn from a fixed alphabet. A 4-symbol code in which each symbol is any one of 5 available symbols can be formed in $5^4 = 625$ ways. Note the contrast: n^r counts arrangements *with* repetition, whereas ${}^n\mathrm{P}_r$ counts them *without*.

Arrangements with repeated (identical) objects.

A different kind of repetition arises when the objects being arranged are not all distinguishable. Suppose n objects are arranged in a row, but among them n_1 are identical of one kind, n_2 identical of a second kind, and so on, with $n_1 + n_2 + \cdots + n_k = n$. Were the objects all distinct there would be $n!$ arrangements; but permuting the n_1 identical objects of the first kind among themselves (in $n_1!$ ways) produces no new arrangement, and likewise for each other kind. Each distinguishable arrangement is therefore counted $n_1!\,n_2!\cdots n_k!$ times over, so the number of **distinguishable arrangements** is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

For instance, the letters of the word **TATTOO** (with three T's and two O's) can be arranged in $\dfrac{6!}{3!\,2!} = \dfrac{720}{12} = 60$ distinguishable ways.

Circular arrangements.

When objects are arranged around a circle, what usually matters is their order *relative to one another*, not their absolute positions; rotating everyone one seat to the left gives the **same** arrangement. To count arrangements of n distinct objects around a circle, **fix one object** in a reference position to remove this rotational duplication, then arrange the remaining $n-1$ objects around it in the $n-1$ places. This gives

$$(n-1)!$$

distinct circular arrangements. Thus 5 people can be seated around a round table in $(5-1)! = 24$ ways. If, in addition, the arrangement may be **turned over** — as with the beads of a bracelet or a necklace, where a ring and its mirror image are indistinguishable — then each arrangement is paired with its reflection and the count is halved to $\dfrac{(n-1)!}{2}$ (for $n \geq 3$).

Combinations: selections without regard to order.

Often the *order* of a selection is irrelevant: a committee of three is the same committee however its members are listed. An unordered selection of r objects from n distinct objects is called a **combination**, and their number is written ${}^n\mathrm{C}_r$ or $\dbinom{n}{r}$ (read "n choose r"). To count combinations, compare them with permutations. Every permutation of r objects can be obtained in two stages: first choose *which* r objects (a combination), then arrange those r objects in order (in $r!$ ways). By the multiplication principle,

$$\text{number of permutations} = \binom{n}{r} \times r!,$$

that is, ${}^n\mathrm{P}_r = \dbinom{n}{r} \times r!$. Dividing by $r!$ and replacing ${}^n\mathrm{P}_r$ by $\dfrac{n!}{(n-r)!}$ gives

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

The values at the ends are $\binom{n}{0}=\binom{n}{n}=1$ (there is exactly one way to choose nothing, and one way to choose everything). Choosing which r objects to **include** is the same as choosing which $n-r$ to **leave out**, which gives the useful symmetry

$$\binom{n}{r}=\binom{n}{n-r},$$

handy for keeping the arithmetic small — for example $\binom{20}{18}=\binom{20}{2}=190$. As a first illustration, the number of ways to choose 2 subjects from 6 is $\binom{6}{2}=\frac{6!}{2!4!}=15$.

Selections with repetition.

In many problems the objects being selected come in **types**, with an unlimited supply of each, so that a type may be taken more than once. Order is still irrelevant, and such a selection is settled by *how many* objects of each type are taken. To count the selections of r objects from n types, record a selection as a row of r symbols x — one for each object chosen — divided into n blocks by $n-1$ bars, the k-th block holding the symbols for the objects of type k; a block may be empty. With $n=3$ types and $r=5$ objects, for instance, the row

$$x\,x\,/\,/\,x\,x\,x$$

records two objects of type 1, none of type 2 and three of type 3. Every selection gives exactly one such row and every row exactly one selection, so it is enough to count the rows. A row has $r+(n-1)$ places in all, and it is fixed as soon as we decide **which r of those places hold an x** (the rest being bars) — a plain combination. Hence the number of selections of r objects from n types, with repetition allowed and order disregarded, is

$$\binom{n+r-1}{r}.$$

So a box of 5 pastries chosen from 3 kinds can be made up in $\binom{3+5-1}{5}=\binom{7}{5}=21$ ways. Contrast the two counts in which repetition is allowed: n^r when the order of the choices matters, $\binom{n+r-1}{r}$ when it does not.

Selections and arrangements with restrictions.

Most problems attach a condition, and the count is adjusted by one of a few standard manoeuvres.

- **Forced inclusion or exclusion.** If a particular object *must* be included in a selection of r from n, place it first and choose the remaining $r-1$ from the other $n-1$: $\binom{n-1}{r-1}$ ways. If it must be *excluded*, simply choose all r from the other $n-1$: $\binom{n-1}{r}$ ways.
- **Grouped selections.** When a selection must contain set numbers from separate groups (say, so many from one category and so many from another), count each group with its own combination and **multiply**, by the multiplication principle.
- **Every type present.** In a selection of r objects from n types with repetition allowed, if each type must appear at least once, set one object of every type aside first; the remaining $r-n$ objects are then selected freely from the n types, giving $\binom{n+(r-n)-1}{r-n}=\binom{r-1}{r-n}$ ways.
- **"At least" and "at most".** These are handled either by adding the counts of the allowed cases, or — often more quickly — by the **complement**: count everything and subtract the unwanted outcomes, using (favourable) = (total) − (unwanted).

- **Objects together (the block method).** To arrange objects in a row with a particular group required to stay **adjacent**, treat that group as a single block, arrange the block among the other items, then arrange the members *within* the block. A block of m items among t other items contributes $(t+1)!\times m!$ arrangements.
- **Objects apart.** The number of arrangements in which the group is **not** together is found by the complement: (all arrangements) − (arrangements with the group together).

These devices, combined with the core formulae — ${}^{n}\mathrm{P}_{r}$ for an ordered selection without repetition, n^r for an ordered selection with repetition, $\dfrac{n!}{n_1!\cdots n_k!}$ for a row of partly identical objects, $(n-1)!$ for a ring, $\dbinom{n}{r}$ for an unordered selection without repetition and $\dbinom{n+r-1}{r}$ for an unordered selection with repetition — cover the counting problems of this course. The two habits that matter most are to decide, before counting, **whether order matters** (a permutation or arrangement) or **not** (a combination or selection), and **whether an object may be used more than once**.

Worked examples

Example 1 — scaffolded

At a surf-lifesaving carnival a club fields a relay in which four different swimmers each swim one leg, so the order of the swimmers matters. The club has 6 swimmers available. (a) In how many ways can an ordered team of 4 be chosen from the 6 swimmers? (b) Evaluate this count as a permutation ${}^{6}\mathrm{P}_{4}$ using the factorial formula, and confirm it agrees with part (a).

Working	Comment
Leg 1 can be filled by any of 6 swimmers, leg 2 by any of the remaining 5, leg 3 by 4, leg 4 by 3.	Order matters and no swimmer repeats, so this is a permutation.
Number of teams $=6\times5\times4\times3=360$.	Multiplication principle over the four legs.
${}^{6}\mathrm{P}_{4}=\dfrac{6!}{(6-4)!}=\dfrac{6!}{2!}=\dfrac{720}{2}$.	Apply ${}^{n}\mathrm{P}_{r}=\dfrac{n!}{(n-r)!}$ with $n=6$, $r=4$.
$=360$, which matches the count in part (a).	The two methods agree.

Example 2 — scaffolded

(a) A wildlife survey tags each animal with a sequence of 4 symbols, where each symbol is one of 6 colours and colours may repeat. How many different tags are possible?
(b) How many distinguishable arrangements are there of all the letters of the word **KANGAROO**?

Working	Comment
(a) Each of the 4 positions can independently be any of the 6 colours.	Repetition is allowed, so this is an arrangement with repetition.
Number of tags $=6^4=1296$.	Use n^r with $n=6$, $r=4$.
(b) KANGAROO has 8 letters, with the letter A appearing twice and O appearing twice.	Identify the repeated (identical) letters first.
Distinguishable arrangements $=\dfrac{8!}{2!2!}=\dfrac{40320}{4}$.	Divide $n!$ by a factorial for each repeated kind.
$=10080$.	This is the final count.

Example 3 — unscaffolded

A wildlife-rescue group has 7 volunteers trained in reptile handling and 5 trained in bird handling. (a) A response team of 5 volunteers is to be formed containing exactly 3 reptile handlers and 2 bird handlers. In how many ways can the team be chosen? (b) The group also orders 4 first-aid kits, each of which may be any one of the 6 stocked types; kits of the same type are interchangeable, so only the number ordered of each type matters. How many different orders are possible?

For (a), order does not matter within the team, so each part is a combination. The 3 reptile handlers are chosen from 7 in $\binom{7}{3}$ ways, and independently the 2 bird handlers are chosen from 5 in $\binom{5}{2}$ ways. Evaluating,

$$\binom{7}{3}=\frac{7!}{3!4!}=35,\binom{5}{2}=\frac{5!}{2!3!}=10.$$

By the multiplication principle the two independent choices combine, so the number of possible teams is $\binom{7}{3}\times\binom{5}{2}=35\times 10=350$.

For (b), the kits are chosen from $n=6$ types, repetition is allowed and the order of the choices is irrelevant, so this is a selection with repetition with $r=4$:

$$\binom{n+r-1}{r}=\binom{6+4-1}{4}=\binom{9}{4}=\frac{9!}{4!5!}=126.$$

∴ (a) the team can be formed in 350 ways, and (b) there are 126 possible orders of kits.

Example 4 — unscaffolded

Eight members of a book club sit around a circular table, where only positions relative to one another are distinguished. (a) How many distinct seating arrangements are there? (b) In how many of these do the host and the guest speaker sit next to each other? (c) In how many do the host and the guest speaker **not** sit next to each other?

For (a), with $n=8$ people around a circle we fix one member to remove rotational duplication and arrange the other seven, giving $(8-1)!=7!=5040$ arrangements.

For (b), regard the host and guest speaker as a single **block**, so that 7 items (the block and the other 6 members) are arranged in a circle: $(7-1)!=6!=720$ ways. Within the block the two people may sit in either order, contributing a further $2!=2$, so the number of arrangements with the pair together is $720\times 2=1440$.

For (c), the arrangements in which they are not adjacent are the remainder, $5040-1440=3600$.

∴ there are (a) 5040 arrangements in total, (b) 1440 with the pair together, and (c) 3600 with the pair apart.

Practice questions

Scaffolded

Q1. A boutique makes personalised phone cases. (a) A case is made by choosing one of 8 base colours, then one of 5 patterns, then one of 4 finishes. How many different cases are possible? (b) The boutique arranges 5 different cases in a row in its window. How many arrangements are there? (c) From 9 designs, an ordered trio (first, second and third featured) is chosen for a banner. Evaluate this as a permutation $^9P_3$.

Q2. (a) A festival wristband shows a row of 5 symbols, each one of 4 shapes, with repetition allowed. How many different wristbands are possible? (b) How many distinguishable arrangements are there of all the letters of the town name **BALLARAT**? (c) A native-plant nursery grows 5 species of seedling. A tray holds 4 seedlings, a species may be repeated and only the number of each species in the tray matters. How many different trays are possible?

Q3. A community garden has 10 volunteers. (a) In how many ways can a working group of 4 be chosen? (b) In how many ways if a particular volunteer, the coordinator, must be in the group? (c) In how many ways if two particular volunteers refuse to serve together (so the group cannot contain both)?

Q4. A trivia team of 5 is chosen from 6 science students and 4 arts students. (a) How many teams are possible with no restriction? (b) How many contain exactly 3 science and 2 arts students? (c) How many contain at least 3 science students?

Q5. Eight distinct trophies are arranged in a row on a shelf. (a) How many arrangements are there in total? (b) How many arrangements have the two oldest trophies adjacent? (c) How many arrangements have the two oldest trophies **not** adjacent? (d) How many arrangements have the three engraved trophies all adjacent?

Q6. Six friends sit around a circular café table, with only relative position distinguished. (a) How many distinct seating arrangements are there? (b) How many have two particular friends sitting next to each other? (c) How many have those two friends **not** next to each other?

Unscaffolded

Q7. A build-your-own smoothie is made by choosing one of 5 bases, one of 7 fruits and one of 4 boosters. How many different smoothies are possible?

Q8. At a regional running final, 12 athletes compete and there are no ties. In how many ways can the gold, silver and bronze medals be awarded?

Q9. A locker code consists of 3 characters, each of which may be any of the 10 digits or any of the 26 capital letters (36 options in all), with repetition allowed. How many different codes are possible?

Q10. How many distinguishable arrangements are there of all the letters of the word **CROCODILE**?

Q11. A hardware shop stocks bolts in 4 sizes, with plenty of each size. A builder buys a packet of 12 bolts, and only the number of each size in the packet matters. (a) How many different packets are possible? (b) How many are possible if the packet must contain at least one bolt of each size?

Q12. A review panel of 4 is selected from 5 teachers and 6 parents, and must include at least one teacher. How many different panels are possible?

Q13. Ten beads, all of different colours, are threaded onto a bracelet. (a) How many distinct arrangements are there if the bracelet may be rotated but not turned over? (b) How many distinct arrangements are there if the bracelet may also be turned over?

Q14. Four different mathematics books and three different physics books are arranged on a shelf so that all the books of each subject are kept together. In how many ways can this be done?

Q15. A committee of 5 is chosen from a group of 9 people that includes one married couple. In how many ways can the committee be chosen if the couple must either both serve or both not serve?

Q16. Using the digits $0, 1, 2, 3, 4, 5, 6$ with no digit repeated: (a) how many 4-digit numbers can be formed (a 4-digit number cannot begin with 0)? (b) how many of these 4-digit numbers are even?

Q17. A quiz team of 4 is chosen from 8 junior and 5 senior members and must contain at least 2 seniors. The four chosen members are then seated in a row for a photograph. How many different photographs are possible?

Q18. At a formal dinner, four delegates from Australia and four delegates from New Zealand are seated around a circular table. In how many distinct arrangements do the delegates alternate by nationality around the table?

Answers

Q1. (a) 160; (b) 120; (c) 504. **Q2.** (a) 1024; (b) 3360; (c) 70. **Q3.** (a) 210; (b) 84; (c) 182. **Q4.** (a) 252; (b) 120; (c) 186. **Q5.** (a) 40320; (b) 10080; (c) 30240; (d) 4320. **Q6.** (a) 120; (b) 48; (c) 72. **Q7.** 140. **Q8.** 1320. **Q9.** 46656. **Q10.** 90720. **Q11.** (a) 455; (b) 165. **Q12.** 315. **Q13.** (a) 362880; (b) 181440. **Q14.** 288. **Q15.** 56. **Q16.** (a) 720; (b) 420. **Q17.** 8760. **Q18.** 144.

Fully-worked solutions

Q1. (a) The three stages are independent, so by the multiplication principle the number of cases is $8\times5\times4=160$. (b) Arranging 5 distinct cases in a row gives $5!=120$ arrangements. (c) Order matters for the ranked trio, so ${}^{9}P_{3}$ $=\frac{9!}{6!}=9\times8\times7=504$.

Q2. (a) Each of the 5 positions independently takes any of the 4 shapes, and repetition is allowed, so the number of wristbands is $4^5=1024$. (b) BALLARAT has 8 letters, with A appearing three times and L appearing twice. The number of distinguishable arrangements is $\frac{8!}{3!2!}=\frac{40320}{12}=3360$. (c) The order of the seedlings in the tray is irrelevant and a species may repeat, so this is a selection with repetition with $n=5$ types and $r=4$ objects: $\binom{n+r-1}{r}=\binom{8}{4}=\frac{8!}{4!4!}=70$ trays.

Q3. (a) Order does not matter in a working group, so the count is $\binom{10}{4}=\frac{10!}{4!6!}=210$. (b) With the coordinator fixed in the group, the remaining 3 members are chosen from the other 9: $\binom{9}{3}=84$. (c) From the total, subtract the groups that contain **both** awkward volunteers; fixing both leaves 2 more to choose from the other 8, namely $\binom{8}{2}=28$. Hence $210-28=182$.

Q4. (a) $\binom{10}{5}=\frac{10!}{5!5!}=252$. (b) Choose 3 of 6 science and 2 of 4 arts students and multiply: $\binom{6}{3}\binom{4}{2}=20\times6=120$. (c) "At least 3 science" means exactly 3, 4 or 5 science students:

$$\binom{6}{3}\binom{4}{2}+\binom{6}{4}\binom{4}{1}+\binom{6}{5}\binom{4}{0}=120+(15)(4)+(6)(1)=120+60+6=186.$$

Q5. (a) Eight distinct trophies in a row: $8!=40320$. (b) Treat the two oldest as a single block, giving 7 items to arrange in a row ($7!=5040$), and multiply by the $2!$ internal orders of the block: $5040\times2=10080$. (c) The "not adjacent" count is the complement of part (b): $40320-10080=30240$. (d) Treat the three engraved trophies as a single block, giving 6 items to arrange in a row ($6!=720$), and multiply by the $3!$ internal orders of the block: $720\times6=4320$.

Q6. (a) Around a circle, fix one friend and arrange the other five: $(6-1)!=5!=120$. (b) Treating the two particular friends as a block gives 5 items in a circle, $(5-1)!=4!=24$, times the $2!$ internal orders: $24\times2=48$. (c) The complement of part (b): $120-48=72$.

Q7. The three independent choices multiply: $5\times7\times4=140$ smoothies.

Q8. The medals are distinct and order matters, so the count is a permutation: ${}^{12}P_{3}=12\times11\times10=1320$.

Q9. Each of the 3 positions may independently be any of the 36 characters, with repetition allowed, so the number of codes is $36^3=46656$.

Q10. CROCODILE has 9 letters, with C appearing twice and O appearing twice. The number of distinguishable arrangements is $\dfrac{9!}{2!2!}=\dfrac{362880}{4}=90720$.

Q11. (a) Only the number of bolts of each size matters, so the packet is a selection with repetition of $r=12$ objects from $n=4$ types:

$$\binom{n+r-1}{r}=\binom{15}{12}=\binom{15}{3}=\frac{15\times14\times13}{3!}=455.$$

(b) Set one bolt of each of the 4 sizes aside first. The remaining $12-4=8$ bolts are then chosen freely from the 4 sizes, in $\binom{4+8-1}{8}=\binom{11}{8}=\binom{11}{3}=165$ ways, so 165 packets contain every size.

Q12. Count all panels and subtract those with no teacher (all parents). Total panels of 4 from 11 people: $\binom{11}{4}=330$.

All-parent panels: $\binom{6}{4}=15$. Hence panels with at least one teacher number $330-15=315$.

Q13. (a) The beads are distinct and only relative position matters, so a rotatable bracelet has $(10-1)!=9!=362880$ arrangements. (b) If the bracelet may also be turned over, each arrangement is identified with its mirror image, halving the count: $\dfrac{9!}{2}=181440$.

Q14. Keep each subject together by forming two blocks, one of the 4 mathematics books and one of the 3 physics books. The two blocks are arranged in a row in $2!$ ways; within the blocks the books are arranged in $4!$ and $3!$ ways respectively. By the multiplication principle the total is $2!\times4!\times3!=2\times24\times6=288$.

Q15. Split into the two allowed cases. If the couple both serve, the remaining 3 members come from the other 7 people: $\binom{7}{3}=35$. If neither serves, all 5 members come from the other 7: $\binom{7}{5}=21$. These cases are exclusive, so the total is $35+21=56$.

Q16. (a) The first digit must be non-zero (6 choices from 1–6); the remaining three places are an ordered selection of 3 from the other 6 digits (which now include 0), namely ${}^6P_3=6\times5\times4=120$. Hence $6\times120=720$. (b) A number is even when its last digit is even, so split on the last digit. *Last digit* 0*:* the first place is any of $\{1,2,3,4,5,6\}$ (6 choices) and the two middle places are an ordered selection of 2 from the remaining 5 digits, ${}^5P_2=20$, giving $6\times20=120$. *Last digit* 2*,* 4 *or* 6 (3 choices): the first place must be non-zero and different from the last digit, leaving 5 choices, and the two middle places are ${}^5P_2=20$ from the remaining digits, giving $3\times5\times20=300$. The total number of even values is $120+300=420$.

Q17. First select the team, then seat it. With 5 seniors and 8 juniors, a team of 4 with at least 2 seniors splits into exactly 2, 3 or 4 seniors:

$$\binom{5}{2}\binom{8}{2}+\binom{5}{3}\binom{8}{1}+\binom{5}{4}\binom{8}{0}=(10)(28)+(10)(8)+(5)(1)=280+80+5=365.$$

Each chosen team of 4 is then seated in a row in $4!=24$ ways, so the number of possible photographs is $365\times24=8760$.

Q18. Seat the four Australian delegates around the circle first. Fixing one Australian to remove rotational duplication, the other three Australians take the alternate seats of the same class in $(4-1)!=3!=6$ ways. The four alternate seats between them are then filled by the four New Zealand delegates in $4!=24$ ways. By the multiplication principle the number of alternating arrangements is $6\times24=144$.

Theory

The binomial coefficient $\binom{n}{r}$ was introduced as a **count**: for integers $0 \le r \le n$ it is the number of ways of choosing an unordered selection of r objects from n distinct objects, and it is given by the factorial formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \binom{n}{0} = \binom{n}{n} = 1.$$

These numbers are far from independent of one another: they obey a web of exact relationships called **combinatorial identities**. Every identity in this section can be established in two complementary ways. An **algebraic proof** manipulates the factorial formula directly. A **combinatorial proof** — also called a **double-counting proof** — counts one carefully chosen collection of objects in two different ways and equates the two answers, since a single collection has only one size. The combinatorial argument usually shows *why* an identity must hold, while the algebra confirms it mechanically. Specialist Mathematics expects both, so each identity below is argued both ways.

The symmetry rule.

For integers $0 \le r \le n$,

$$\binom{n}{r} = \binom{n}{n-r}.$$

Combinatorial argument. Choosing which r of the n objects to **take** is the very same decision as choosing which $n-r$ objects to **leave behind**. Each selection of r objects determines exactly one selection of the remaining $n-r$ objects (its complement), and vice versa, so the selections of size r pair off one-to-one with the selections of size $n-r$. Two collections in one-to-one correspondence have equal size, hence the two counts are equal.

Algebraic argument. Replacing r by $n-r$ in the factorial formula and using $n-(n-r)=r$,

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

The rule lets awkward coefficients be replaced by easier ones: for instance $\binom{100}{98} = \binom{100}{2} = 4950$.

Pascal's rule.

For integers $1 \le r \le n-1$,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Combinatorial argument. Among the n distinct objects, single out one particular object and call it $\star$. Every selection of r objects either **contains** $\star$ or **does not**, and these two cases never overlap. A selection that contains $\star$ is completed by choosing the other $r-1$ members from the remaining $n-1$ objects, which can be done in $\binom{n-1}{r-1}$ ways. A selection that omits $\star$ must take all r members from those same $n-1$ objects, in $\binom{n-1}{r}$ ways. As the two cases are exclusive and together account for every r-selection, the total $\binom{n}{r}$ is their sum.

Algebraic argument. Placing the two right-hand terms over the common denominator $r!(n-r)!$,

$$\binom{n-1}{r-1}+\binom{n-1}{r}=\frac{(n-1)!}{(r-1)!(n-r)!}+\frac{(n-1)!}{r!(n-1-r)!}=\frac{(n-1)!r+(n-1)!(n-r)}{r!(n-r)!}.$$

Factoring $(n-1)!$ from the numerator gives $(n-1)!(r+(n-r))=(n-1)!n=n!$, so

$$\binom{n-1}{r-1}+\binom{n-1}{r}=\frac{n!}{r!(n-r)!}=\binom{n}{r}.$$

Pascal's triangle.

Pascal's rule says that each binomial coefficient is the **sum of the two coefficients directly above it** in the previous row. Starting from the boundary values $\binom{n}{0}=\binom{n}{n}=1$, this lets every coefficient be built by simple addition, with no factorials at all. Arranging $\binom{n}{r}$ in row n and column r produces **Pascal's triangle**; the columns below are indexed by r, and a blank cell means $r>n$ (no such selection exists).

n	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Each interior entry is the sum of the pair above it, exactly as Pascal's rule requires; for example the $\binom{8}{3}=56$ in the last row is $\binom{7}{2}+\binom{7}{3}=21+35$. The symmetry rule shows as the **left–right mirror symmetry** of each row: the entries read the same forwards and backwards.

The row sum.

Adding a complete row of Pascal's triangle gives

$$\sum_{r=0}^{n}\binom{n}{r}=2^n.$$

Combinatorial argument. The left-hand side counts **all** subsets of a set of n objects, sorted by size: $\binom{n}{r}$ of them have size r, and r runs from 0 (the empty set) to n (the whole set). But a subset is built by deciding, independently for each of the n objects, whether to include it — two choices each, giving 2^n subsets in all. Counting the same subsets both ways gives 2^n.

Argument from Pascal's rule. Write S_n for the sum of the entries of row n. In building row n from row $n-1$, each entry of row $n-1$ is added into **exactly two** entries of the new row — the one below and to its left, and the one below and to its right. Every entry of row $n-1$ therefore contributes twice to S_n, so $S_n=2S_{n-1}$ for $n\geq 1$. Starting from $S_0=\binom{0}{0}=1$ and doubling n times gives $S_n=2^n$.

The alternating sum.

Adding a row with alternating signs instead gives

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}=0\,(n\geq 1).$$

Separating the sum into its positive and negative parts, the statement is that a set of $n\geq 1$ objects has **as many subsets of even size as subsets of odd size**.

Combinatorial argument. Single out one object and call it $\star$. Pair each subset S with the subset obtained from it by **toggling** $\star$: take $\star$ out of S if it is there, and put $\star$ into S if it is not. Toggling changes the size of a subset by exactly 1, so it always pairs an even-sized subset with an odd-sized one; and toggling a second time restores the original subset, so the pairing is one-to-one and no subset is left unpaired. The even-sized and the odd-sized subsets therefore occur in equal numbers, which is precisely what the alternating sum asserts. As the two kinds together make up all 2^n subsets, there are exactly 2^{n-1} subsets of each parity. This is a quick route to counts such as "how many even-sized selections are there", used in the worked examples.

Argument from Pascal's rule. Replace each coefficient using $\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$, reading $\binom{n}{0}=\binom{n-1}{0}$ and $\binom{n}{n}=\binom{n-1}{n-1}$ at the two ends:

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots=\binom{n-1}{0}-\left[\binom{n-1}{0}+\binom{n-1}{1}\right]+\left[\binom{n-1}{1}+\binom{n-1}{2}\right]-\cdots.$$

Every entry of row $n-1$ now appears exactly twice, once with a plus sign and once with a minus sign, so the whole expression cancels to 0.

Double counting: the committee–captain identity.

Combinatorial proofs are not limited to the boundary identities above. Consider counting the number of ways to choose, from n people, a team of r members and to appoint one of those members as captain. Counting the team first and the captain second gives $\binom{n}{r}\cdot r$; counting the captain first (any of the n people) and then the remaining $r-1$ team members from the other $n-1$ gives $n\cdot\binom{n-1}{r-1}$. The two counts describe the same arrangements, so

$$r\binom{n}{r}=n\binom{n-1}{r-1}.$$

The factorial formula confirms it: both sides equal $\dfrac{n!}{(r-1)!(n-r)!}$.

The hockey-stick identity.

Reading down a diagonal of Pascal's triangle gives another double-counting result. For integers $0\leq r\leq n$,

$$\sum_{i=r}^{n}\binom{i}{r}=\binom{r}{r}+\binom{r+1}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1}.$$

Combinatorial argument. Count the selections of $r+1$ objects from $\{1,2,\ldots,n+1\}$; there are $\binom{n+1}{r+1}$ of them. Now classify each selection by its **largest** element. If the largest element is $i+1$ (where i ranges over $r,r+1,\ldots,n$), the remaining r elements are chosen from $\{1,2,\ldots,i\}$, in $\binom{i}{r}$ ways. Adding these disjoint cases over all possible largest elements recovers the total, giving the identity. The name comes from the picture: the summed entries lie along a diagonal of the triangle and the answer sits just off its end, so the highlighted cells form the shape of a hockey stick.

Argument from Pascal's rule. Peel the right-hand side apart one row at a time. Pascal's rule gives $\binom{n+1}{r+1}=\binom{n}{r}+\binom{n}{r+1}$; applying it again to the second term gives $\binom{n}{r+1}=\binom{n-1}{r}+\binom{n-1}{r+1}$, and so on. Each step releases one further term $\binom{i}{r}$ of the sum and leaves behind a coefficient of the same shape one row further up the triangle, until the process ends with $\binom{r+1}{r+1}=\binom{r}{r}$. The terms released, together with that last one, are exactly $\binom{n}{r},\binom{n-1}{r},\ldots,\binom{r}{r}$, so their total is $\binom{n+1}{r+1}$.

Extension note — where the name "binomial coefficient" comes from.

The entries of row n are also the coefficients that appear when a sum of two terms is raised to the power n; using row 4, for instance,

$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4.$$

In general the coefficient of $x^{n-r}y^r$ is $\binom{n}{r}$, because such a term arises exactly when y is taken from r of the n brackets in $(x+y)(x+y)\cdots(x+y)$ and x from the other $n-r$. That is why $\binom{n}{r}$ is called a **binomial coefficient**, and the expansion itself is known as the **binomial theorem**. That result lies **outside** the Specialist Mathematics Units 1 & 2 course; it is recorded here only to account for the name, and no example, question, answer or solution below uses it. Every identity in this section is established by counting arguments, the factorial formula and Pascal's rule alone.

Worked examples

Example 1 — scaffolded

(a) Use the symmetry rule to rewrite $\binom{17}{14}$ with a smaller lower entry and hence evaluate it. (b) Verify Pascal's rule in the form $\binom{9}{4}=\binom{8}{3}+\binom{8}{4}$.

Working	Comment
$\binom{17}{14}=\binom{17}{17-14}=\binom{17}{3}$.	Symmetry: $\binom{n}{r}=\binom{n}{n-r}$ with $n=17$, $r=14$.
$\binom{17}{3}=\frac{17\times16\times15}{3\times2\times1}=\frac{4080}{6}=680$.	Evaluate the easier coefficient.
$\binom{9}{4}=\frac{9\times8\times7\times6}{4!}=\frac{3024}{24}=126$.	Left-hand side of Pascal's rule.
$\binom{8}{3}=56$ and $\binom{8}{4}=70$.	Read from row 8 of Pascal's triangle.
$56+70=126=\binom{9}{4}$.	The two sides agree, as Pascal's rule requires.

Example 2 — scaffolded

(a) Apply Pascal's rule twice to derive the identity $\binom{n}{r}=\binom{n-2}{r-2}+2\binom{n-2}{r-1}+\binom{n-2}{r}$, valid for integers $2\le r\le n-2$. (b) Verify it when $n=9$ and $r=3$.

Working	Comment
$\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$.	Pascal's rule applied once to $\binom{n}{r}$.
$\binom{n-1}{r-1}=\binom{n-2}{r-2}+\binom{n-2}{r-1}$.	Pascal's rule again, on the first term.
$\binom{n-1}{r}=\binom{n-2}{r-1}+\binom{n-2}{r}$.	Pascal's rule again, on the second term.
Adding, $\binom{n}{r}=\binom{n-2}{r-2}+2\binom{n-2}{r-1}+\binom{n-2}{r}$.	$\binom{n-2}{r-1}$ came out of **both** applications, hence the factor 2.
(b) $\binom{9}{3}=\frac{9\times8\times7}{3\times2\times1}=\frac{504}{6}=84$.	Left-hand side with $n=9$, $r=3$.
$\binom{7}{1}+2\binom{7}{2}+\binom{7}{3}=7+2(21)+35=84$.	Right-hand side; read 7, 21, 35 from row 7 of the triangle.
The two sides agree, as the derivation requires.	The identity holds at $n=9$, $r=3$.

Example 3 — unscaffolded

A tasting menu offers 10 distinct small dishes, and a diner may choose any subset of them — from none at all to all ten. How many of the possible selections contain an **even** number of dishes (counting the empty selection, which has zero dishes, as even)?

Each dish is either taken or not, so the total number of selections is the number of subsets of a 10-element set,

$$\sum_{r=0}^{10}\binom{10}{r}=2^{10}=1024\,.$$

Now pair each selection with the selection obtained by toggling one fixed dish — the oyster, say — in or out. This pairing is one-to-one and always matches an even-sized selection with an odd-sized one, so the alternating sum vanishes,

$$\sum_{r=0}^{10}(-1)^r\binom{10}{r}=0\,,$$

and the even-sized selections and the odd-sized selections occur in **equal** numbers. As the two kinds together total 1024, each kind numbers $1/2(1024)=512=2^9$.

∴ there are 512 selections containing an even number of dishes.

Example 4 — unscaffolded

Prove the **subset-of-a-subset** identity: for integers $0\le k\le r\le n$,

$$\binom{n}{r}\binom{r}{k}=\binom{n}{k}\binom{n-k}{r-k}.$$

Give both a combinatorial and an algebraic argument.

Combinatorial argument. Count the number of ways to choose, from n people, a committee of r members and then a subcommittee of k of those members. Choosing the committee first ($\binom{n}{r}$ ways) and the subcommittee from within it ($\binom{r}{k}$ ways) gives the left-hand side. Alternatively, choose the k subcommittee members directly from all n people ($\binom{n}{k}$ ways), then fill the remaining $r-k$ ordinary committee places from the $n-k$ people not on the subcommittee ($\binom{n-k}{r-k}$ ways); this gives the right-hand side. Both procedures count exactly the same committee-with-subcommittee arrangements, so the two expressions are equal.

Algebraic argument. Using the factorial formula on each side,

$$\binom{n}{r}\binom{r}{k}=\frac{n!}{r!(n-r)!}\cdot\frac{r!}{k!(r-k)!}=\frac{n!}{(n-r)!k!(r-k)!},$$

$$\binom{n}{k}\binom{n-k}{r-k}=\frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(r-k)!(n-r)!}=\frac{n!}{k!(r-k)!(n-r)!}.$$

The two final fractions are identical, so the identity holds for all $0\le k\le r\le n$, as required.

Practice questions

Scaffolded

Q1. Work with the symmetry and Pascal rules. (a) Use symmetry to rewrite $\binom{20}{17}$ with a lower entry of 3, then evaluate it. (b) Verify Pascal's rule $\binom{11}{5}=\binom{10}{4}+\binom{10}{5}$ by evaluating all three coefficients. (c) Given $\binom{11}{4}=330$ and $\binom{11}{5}=462$, use Pascal's rule to write down $\binom{12}{5}$. (d) Use symmetry to evaluate $\binom{14}{11}$.

Q2. Relate coefficients two rows apart, as in Example 2. (a) Write down row 5 of Pascal's triangle. (b) Apply Pascal's rule twice to write $\binom{7}{2}$ in terms of entries of row 5 only. (c) Evaluate both sides of the result in part (b) and check that they agree.

Q3. Consider the sum $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}$. (a) Evaluate each term and hence the sum. (b) Write down the single binomial coefficient that the hockey-stick identity predicts for this sum, and check that it agrees with part (a). (c) Use the identity to state the value of $\binom{2}{2}+\binom{3}{2}+\cdots+\binom{12}{2}$.

Q4. Let $n=7$. (a) Evaluate $\sum_{r=0}^{7}\binom{7}{r}$. (b) Explain the value in part (a) by counting the subsets of a set of 7 objects. (c) State the value of the alternating sum $\sum_{r=0}^{7}(-1)^r\binom{7}{r}$. (d) Hence evaluate $\binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6}$.

Q5. Build part of Pascal's triangle. (a) Write out rows $n=0$ to $n=4$. (b) Use Pascal's rule to extend it to rows 5 and 6. (c) Add the entries of each of the rows 0 to 6 and state the identity that the resulting sequence of totals illustrates.

Q6. A class of 10 students must send r of its members to a science excursion. (a) In how many ways can the r students who go be chosen, in terms of a binomial coefficient? (b) Equivalently, one could choose the $10-r$ students who stay behind. In how many ways? (c) Explain why the two counts must be equal, and state the identity this proves. (d) Verify the identity when $r=7$ by evaluating both $\binom{10}{7}$ and $\binom{10}{3}$.

Unscaffolded

Q7. Use the symmetry rule to evaluate $\binom{19}{16}$.

Q8. Verify Pascal's rule $\binom{13}{6}=\binom{12}{5}+\binom{12}{6}$ by evaluating all three coefficients.

Q9. Use Pascal's rule twice to show that, for integers $2\le r\le n-2$,

$$\binom{n}{r}=\binom{n-2}{r-2}+2\binom{n-2}{r-1}+\binom{n-2}{r},$$

and verify the result when $n=10$ and $r=4$.

Q10. Show that $\binom{n}{2}+\binom{n+1}{2}=n^2$ for every integer $n\ge 1$, and verify the result when $n=7$.

Q11. A band shortlists 6 of its 15 recorded tracks for a set list, and then chooses 2 of those 6 tracks to release as singles. Count the possible outcomes in two ways — choosing the set list first, and choosing the singles first — and state the identity your two equal answers illustrate.

Q12. Use the symmetry rule together with the row sum to show that, for a positive integer n,

$$\binom{2n}{0}+\binom{2n}{1}+\cdots+\binom{2n}{n-1}=1/2\left(2^{2n}-\binom{2n}{n}\right),$$

and evaluate this sum when $n=4$.

Q13. By counting the subsets of a set of 9 objects in two different ways, show that $\binom{9}{0}+\binom{9}{1}+\cdots+\binom{9}{9}=512$.

Q14. By pairing each subset of a 6-element set with the subset obtained by toggling one fixed element, show that $\sum_{r=0}^{6}(-1)^r\binom{6}{r}=0$, and hence evaluate $\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6}$.

Q15. A laboratory prepares a sample by optionally adding any selection of 8 distinct reagents. (a) How many different mixtures are possible, counting the mixture with no reagents added? (b) How many mixtures contain at least one reagent? (c) How many mixtures contain an even number of reagents (with the empty mixture counted as even)?

Q16. Evaluate $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\binom{6}{3}+\binom{7}{3}$ and show that it equals $\binom{8}{4}$, illustrating the hockey-stick identity.

Q17. A working party of 5 is to be chosen from 12 people, and one of the five is to be named as leader. (a) Count the arrangements by choosing the working party first and then its leader. (b) Count them again by choosing the leader first and then the rest of the party. (c) State the identity your two equal answers illustrate.

Q18. Let n be a positive integer. (a) Give a combinatorial proof that $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$ by counting subsets of a set of n objects. (b) Deduce the value of $\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$, the number of **non-empty** subsets, and evaluate it for $n=6$.

Answers

Q1. (a) $\binom{20}{17}=\binom{20}{3}=1140$; (b) $\binom{11}{5}=462$, $\binom{10}{4}=210$, $\binom{10}{5}=252$, and $210+252=462$; (c) $\binom{12}{5}=792$; (d) $\binom{14}{11}=\binom{14}{3}=364$. **Q2.** (a) $1,5,10,10,5,1$; (b) $\binom{7}{2}=\binom{5}{0}+2\binom{5}{1}+\binom{5}{2}$; (c) both sides equal 21 (since $1+2(5)+10=21$). **Q3.** (a) $1+3+6+10+15=35$; (b) $\binom{7}{3}=35$, which agrees; (c) $\binom{13}{3}=286$. **Q4.** (a) 128; (b) each of 7 objects is in or out, giving $2^7=128$ subsets; (c) 0; (d) 64. **Q5.** (a)–(b) rows 0–6 are 1; $1,1$; $1,2,1$; $1,3,3,1$; $1,4,6,4,1$; $1,5,10,10,5,1$; $1,6,15,20,15,6,1$; (c) the totals are $1,2,4,8,16,32,64$, each double the one before, illustrating the row sum $\sum_{r=0}^{n}\binom{n}{r}=2^n$. **Q6.** (a) $\binom{10}{r}$; (b) $\binom{10}{10-r}$; (c) both count the same split of the class, so $\binom{10}{r}=\binom{10}{10-r}$ (the symmetry rule); (d) $\binom{10}{7}=\binom{10}{3}=120$. **Q7.** $\binom{19}{16}=\binom{19}{3}=969$. **Q8.** $\binom{13}{6}=1716$, $\binom{12}{5}=792$, $\binom{12}{6}=924$, and $792+924=1716$. **Q9.** See solution; at $n=10$, $r=4$ both sides equal 210 (since $28+2(56)+70=210$). **Q10.** See solution; at $n=7$ both sides equal 49 (since $21+28=49$). **Q11.** $\binom{15}{6}\binom{6}{2}=\binom{15}{2}\binom{13}{4}=75075$; the identity is the subset-of-a-subset rule $\binom{n}{r}\binom{r}{k}=\binom{n}{k}\binom{n-k}{r-k}$ with $n=15$, $r=6$, $k=2$. **Q12.** See solution; when $n=4$ the sum is $1/2(256-70)=93$. **Q13.** Counting subsets by size and by in-or-out decisions gives $2^9=512$. **Q14.** The toggle pairing makes the even-sized and odd-sized subsets equally numerous, so the alternating sum is 0; hence the even-index sum is $1/2(2^6)=32$. **Q15.** (a) $2^8=256$; (b) 255; (c) 128. **Q16.** $1+4+10+20+35=70=\binom{8}{4}$. **Q17.** (a) $\binom{12}{5}\times 5=3960$; (b) $12\times\binom{11}{4}=3960$; (c) $r\binom{n}{r}=n\binom{n-1}{r-1}$ with $n=12$, $r=5$. **Q18.** (a) see solution; (b) 2^n-1, which is 63 when $n=6$.

Fully-worked solutions

Q1. (a) By symmetry $\binom{20}{17}=\binom{20}{20-17}=\binom{20}{3}=\frac{20\times19\times18}{3!}=\frac{6840}{6}=1140$. (b) Directly, $\binom{11}{5}=\frac{11!}{5!6!}=462$, while $\binom{10}{4}=210$ and $\binom{10}{5}=252$; since $210+252=462$, Pascal's rule is confirmed. (c) Pascal's rule with $n=12$, $r=5$ gives $\binom{12}{5}=\binom{11}{4}+\binom{11}{5}=330+462=792$. (d) By symmetry $\binom{14}{11}=\binom{14}{3}=\frac{14\times13\times12}{6}=364$.

Q2. (a) Row 5 of Pascal's triangle is $1,5,10,10,5,1$. (b) Pascal's rule with $n=7$, $r=2$ gives $\binom{7}{2}=\binom{6}{1}+\binom{6}{2}$, and a second application to each term gives $\binom{6}{1}=\binom{5}{0}+\binom{5}{1}$ and $\binom{6}{2}=\binom{5}{1}+\binom{5}{2}$. Adding, $\binom{5}{1}$ appears twice, so

$$\binom{7}{2}=\binom{5}{0}+2\binom{5}{1}+\binom{5}{2}.$$

(c) The left-hand side is $\binom{7}{2}=\frac{7\times6}{2}=21$; the right-hand side is $1+2(5)+10=21$. The two sides agree.

Q3. (a) Each term is a column-2 entry of the triangle: $\binom{2}{2}=1$, $\binom{3}{2}=3$, $\binom{4}{2}=6$, $\binom{5}{2}=10$ and $\binom{6}{2}=15$, so the sum is $1+3+6+10+15=35$. (b) The hockey-stick identity $\sum_{i=r}^{n}\binom{i}{r}=\binom{n+1}{r+1}$ with $r=2$ and $n=6$ predicts $\binom{7}{3}=\frac{7\times6\times5}{6}=35$, which agrees with part (a). (c) The same identity with $r=2$ and $n=12$ gives

$$\binom{2}{2}+\binom{3}{2}+\cdots+\binom{12}{2}=\binom{13}{3}=\frac{13\times12\times11}{6}=286.$$

Q4. (a) $\sum_{r=0}^{7}\binom{7}{r}=2^7=128$. (b) A subset of a 7-element set is formed by an independent in-or-out decision for each of the 7 objects, giving $2^7=128$ subsets; grouping them by size gives the sum of the binomial coefficients, so the two counts agree. (c) Toggling one fixed object in or out pairs each subset with a subset whose size differs by 1, so the even-sized and odd-sized subsets are equally numerous and $\sum_{r=0}^{7}(-1)^r\binom{7}{r}=0$. (d) The vanishing alternating sum makes the even-index and odd-index parts equal, so each is half the total: $\binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6}=1/2(128)=64$. (Directly, $1+21+35+7=64$.)

Q5. (a)–(b) Starting each row and ending each row with 1 and adding neighbouring pairs:

$$\begin{gathered}n=0:1\\ n=1:1\,1\\ n=2:1\,2\,1\\ n=3:1\,3\,3\,1\\ n=4:1\,4\,6\,4\,1\\ n=5:1\,5\,10\,10\,5\,1\\ n=6:1\,6\,15\,20\,15\,6\,1\end{gathered}$$

For instance the 15 in row 6 is $5+10$ from row 5. (c) The row totals are

$$1,2,4,8,16,32,64,$$

each twice the one before, because every entry of a row is used exactly twice in forming the next row. These are the powers $2^0,2^1,\ldots,2^6$, illustrating the row-sum identity $\sum_{r=0}^{n}\binom{n}{r}=2^n$.

Q6. (a) Choosing which r students go is $\binom{10}{r}$. (b) Choosing instead which $10-r$ students stay is $\binom{10}{10-r}$. (c) The two descriptions specify exactly the same division of the class into a group that goes and a group that stays, so they count the same thing: $\binom{10}{r}=\binom{10}{10-r}$. This is the symmetry rule. (d) With $r=7$, $\binom{10}{7}=\binom{10}{3}=\frac{10\times9\times8}{6}=120$, and indeed both sides equal 120.

Q7. By symmetry $\binom{19}{16}=\binom{19}{3}=\frac{19\times18\times17}{6}=\frac{5814}{6}=969$.

Q8. Evaluating each coefficient, $\binom{13}{6}=1716$, $\binom{12}{5}=792$ and $\binom{12}{6}=924$. Since $792+924=1716$, Pascal's rule $\binom{13}{6}=\binom{12}{5}+\binom{12}{6}$ holds.

Q9. Apply Pascal's rule to $\binom{n}{r}$, then to each of the two coefficients it produces:

$$\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r},\binom{n-1}{r-1}=\binom{n-2}{r-2}+\binom{n-2}{r-1},\binom{n-1}{r}=\binom{n-2}{r-1}+\binom{n-2}{r}.$$

Substituting the last two into the first, the coefficient $\binom{n-2}{r-1}$ arises from both, so

$$\binom{n}{r}=\binom{n-2}{r-2}+2\binom{n-2}{r-1}+\binom{n-2}{r}.$$

With $n=10$ and $r=4$ the left-hand side is $\binom{10}{4}=\frac{10\times9\times8\times7}{24}=210$, while the right-hand side is $\binom{8}{2}+2\binom{8}{3}+\binom{8}{4}=28+2(56)+70=210$. The two agree.

Q10. Using the factorial formula on each term,

$$\binom{n}{2}+\binom{n+1}{2}=\frac{n(n-1)}{2}+\frac{(n+1)n}{2}=\frac{n((n-1)+(n+1))}{2}=\frac{n(2n)}{2}=n^2,$$

as required. (Combinatorially: n^2 counts the ordered pairs (a,b) drawn from $\{1,2,\dots,n\}$, of which $\binom{n}{2}$ have $a<b$, $\binom{n}{2}$ have $a>b$ and n have $a=b$; since Pascal's rule gives $\binom{n+1}{2}=\binom{n}{1}+\binom{n}{2}=n+\binom{n}{2}$, the two counts match.) With $n=7$, $\binom{7}{2}+\binom{8}{2}=21+28=49=7^2$.

Q11. Choosing the set list first, the 6 tracks can be chosen in $\binom{15}{6}$ ways and the 2 singles from within them in $\binom{6}{2}$ ways, giving

$$\binom{15}{6}\binom{6}{2}=5005\times15=75075.$$

Choosing the singles first, the 2 singles come from all 15 tracks in $\binom{15}{2}$ ways, and the remaining 4 set-list tracks from the 13 tracks not released as singles in $\binom{13}{4}$ ways, giving

$$\binom{15}{2}\binom{13}{4}=105\times715=75075.$$

Both procedures count the same (set list, singles) outcomes, so the counts agree. This is the subset-of-a-subset identity $\binom{n}{r}\binom{r}{k}=\binom{n}{k}\binom{n-k}{r-k}$ with $n=15$, $r=6$, $k=2$.

Q12. Row $2n$ has the $2n+1$ entries $\binom{2n}{0},\binom{2n}{1},\dots,\binom{2n}{2n}$, which total 2^{2n} by the row sum. Remove the single middle entry $\binom{2n}{n}$; the remaining $2n$ entries total $2^{2n}-\binom{2n}{n}$. By the symmetry rule $\binom{2n}{r}=\binom{2n}{2n-r}$, those remaining entries fall into matching pairs, one from each side of the middle, so the n entries to the left of the middle total exactly half:

$$\binom{2n}{0}+\binom{2n}{1}+\cdots+\binom{2n}{n-1}=1/2\left(2^{2n}-\binom{2n}{n}\right).$$

With $n=4$ this is $1/2\left(2^8-\binom{8}{4}\right)=1/2(256-70)=93$. (Directly, $1+8+28+56=93$.)

Q13. Count the subsets of a set of 9 objects in two ways. Sorting them by size, there are $\binom{9}{r}$ subsets of size r, so the total is $\binom{9}{0}+\binom{9}{1}+\cdots+\binom{9}{9}$. Alternatively, a subset is fixed by an independent in-or-out decision for each of the 9 objects, giving 2^9 subsets. The same collection has been counted both times, so

$$\binom{9}{0}+\binom{9}{1}+\cdots+\binom{9}{9}=2^9=512.$$

Q14. Fix one element $\star$ of the 6-element set and pair each subset with the subset obtained by toggling $\star$ — removing it if present, inserting it if absent. Toggling changes the size by exactly 1, so each pair consists of one even-sized and one odd-sized subset, and toggling twice restores the original subset, so every subset lies in exactly one such pair. Hence the even-sized subsets and the odd-sized subsets are equally numerous, which says precisely that $\sum_{r=0}^{6}(-1)^r\binom{6}{r}=0$. As the combined total is $\sum_{r=0}^{6}\binom{6}{r}=2^6=64$, the even-index sum is half of this:

$$\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6}=1/2(64)=32.$$

(Directly, $1+15+15+1=32$.)

Q15. Each of the 8 reagents is independently added or not. (a) The number of mixtures is the number of subsets of the 8 reagents, $2^8=256$. (b) Excluding the single empty mixture leaves $256-1=255$ mixtures with at least one reagent. (c) Toggling one fixed reagent in or out pairs every mixture with a mixture whose reagent count differs by 1, so the alternating sum $\sum_{r=0}^{8}(-1)^r\binom{8}{r}=0$ and the even-sized and odd-sized mixtures are equal in number; each kind therefore numbers $1/2(256)=128$.

Q16. Adding the diagonal entries,

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\binom{6}{3}+\binom{7}{3}=1+4+10+20+35=70.$$

The hockey-stick identity with $r=3$ and $n=7$ predicts $\binom{7+1}{3+1}=\binom{8}{4}=70$, which matches.

Q17. (a) Choosing the working party of 5 and then its leader gives $\binom{12}{5}\times 5=792\times 5=3960$. (b) Choosing the leader from all 12 people and then the other 4 party members from the remaining 11 gives $12\times\binom{11}{4}=12\times 330=3960$. (c) The two equal counts illustrate the committee–captain identity $r\binom{n}{r}=n\binom{n-1}{r-1}$ with $n=12$, $r=5$.

Q18. (a) A subset of a set of n objects is determined by deciding, independently for each object, whether to include it — two possibilities per object, hence 2^n subsets altogether. Sorting the same subsets by their size r gives $\binom{n}{r}$ of size r, so $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}$ counts the identical collection; the two counts must agree, giving $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$.

(b) Removing the single empty subset (the $\binom{n}{0}=1$ term) leaves the non-empty subsets, so $\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n-1$. For $n=6$ this is $2^6-1=63$.