Specialist Mathematics Units 1 & 2

Chapter 4 — Sequences, series and recurrence relations

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Theory

Many quantities are naturally described not by a single number but by an ordered list of numbers: the balance of an account at the end of each month, the height of each bounce of a ball, the number of tiles in each row of a pattern. This section makes the idea of such a list precise, singles out the simplest and most important type — those that grow by a **fixed step** — and derives a closed formula for adding up their terms. Everything here rests on ordinary algebra; the new ingredient is a systematic notation for lists and for their sums.

Sequences.

A **sequence** is an ordered list of numbers, called its **terms**. Formally a (real) sequence is a *function* whose inputs are the positive integers: to each position $n \in \{1,2,3,\dots\}$ it assigns a term, written t_n (read "t sub n"). The first term is t_1, the second t_2, and t_n is the **general term** or **nth term**. A sequence with finitely many terms $t_1,t_2,\dots,t_k$ is **finite**; one that continues without end is **infinite**, indicated by trailing dots, as in

$$3,7,11,15,19,\dots$$

A sequence may be specified in either of two ways: by a **rule** giving t_n directly as a formula in n (for the list above, $t_n=4n-1$), or **recursively**, by stating the first term together with a rule that produces each term from the one before it. The recursive description of arithmetic and geometric sequences is developed fully in Section 4C; in this section a rule for t_n is the primary tool.

Series and partial sums.

A **series** is what results from *adding* the terms of a sequence. Given a sequence with terms $t_1,t_2,t_3,\dots$, the **nth partial sum** is the sum of its first n terms,

$$S_n=t_1+t_2+t_3+\cdots+t_n.$$

Thus $S_1=t_1$, $S_2=t_1+t_2$, and so on; S_n is itself a sequence, the **sequence of partial sums**. Because the partial sums are built by adding one further term at each stage, consecutive partial sums differ by exactly that term:

$$S_n-S_{n-1}=t_n(n\ge 2),$$

with $t_1=S_1$. This simple relation lets a term be recovered whenever the partial sums are known, and is used again when sums are studied in later chapters.

Arithmetic sequences.

A sequence is **arithmetic** if the difference between each term and the one before it is the *same* throughout. That constant, the **common difference**, is denoted d:

$$d=t_{n+1}-t_n \text{ for every } n.$$

Equivalently, each term is obtained from its predecessor by adding d, so $t_{n+1}=t_n+d$. Writing the first term as $a=t_1$, the terms read

$$a,a+d,a+2d,a+3d,\dots$$

The number of d's added is always one fewer than the position number: the second term carries one d, the third carries two, and in general the nth term carries $n-1$ of them. This gives the rule for the general term,

$$t_n=a+(n-1)d.$$

To justify it rather than merely read it off, start from the recurrence $t_{k+1}=t_k+d$ and unfold it step by step from $t_1=a$:

$$t_n = a + (d + d + \cdots + d) = a + (n-1)d,$$

each step adding one further d across the $n-1$ steps from position 1 to position n. Two features of the formula are worth noting. First, $t_n = dn + (a-d)$ is a **linear** expression in n: plotted against n the terms lie on a straight line of slope d. Second, the common difference is recovered from *any* two terms, since for positions $m < n$,

$$t_n - t_m = (a + (n-1)d) - (a + (m-1)d) = (n-m)d,$$

so $d = \frac{t_n - t_m}{n-m}$. This is the standard route to d (and then a) when two terms of a sequence are given but the first term is not.

The sum of an arithmetic series.

Adding the first n terms of an arithmetic sequence term by term would be laborious, but a symmetry discovered (in essence) by the young Gauss collapses the work to a single line. Let $\ell = t_n = a + (n-1)d$ denote the **last** term of the partial sum, and write S_n forwards and then again backwards:

$$S_n = a + (a+d) + (a+2d) + \cdots + (\ell - d) + \ell,$$

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + (a+d) + a.$$

Adding the two lines **column by column**, every one of the n columns contributes the same total $a + \ell$: the first column gives $a + \ell$, the second $(a+d) + (\ell - d) = a + \ell$, and so on down the line, each gain of d on the top matched by an equal loss on the bottom. Hence

$$2S_n = n(a + \ell), \text{so } S_n = \frac{n}{2}(a + \ell).$$

This is the compact form: the partial sum equals the number of terms times the **average of the first and last term**. Substituting $\ell = a + (n-1)d$ expresses the sum using a and d alone:

$$S_n = \frac{n}{2}(a + a + (n-1)d) = \frac{n}{2}(2a + (n-1)d).$$

The two forms

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + \ell)$$

are equivalent; use the second when the last term ℓ is known, the first when only a and d are. Because $2a + (n-1)d = dn + (2a - d)$, the sum $S_n = d/2n^2 + (a - d/2)n$ is a **quadratic** in n with no constant term — a fact that turns "how many terms are needed?" questions into quadratic equations.

Sigma notation.

A sum of many terms is written compactly using the Greek capital sigma, Σ, as a **summation sign**. The partial sum S_n of a sequence with general term t_k is written

$$S_n = \sum_{k=1}^{n} t_k = t_1 + t_2 + t_3 + \cdots + t_n.$$

The letter k is the **index of summation**; the number below the sign is where the index starts and the number above is where it stops. The index is a dummy label — any letter serves — and the expression to the right of the sign gives each term as a formula in that index. For an arithmetic sequence with $t_k = a + (k-1)d$, the partial sum in sigma notation is

$$\sum_{k=1}^{n}\left(a+(k-1)d\right)=\frac{n}{2}\left(2a+(n-1)d\right),$$

the left side merely naming the sum that the right side evaluates. A sum need not begin at $k=1$: the terms from position m to position n (with $m \le n$) are written

$$\sum_{k=m}^{n} t_k = t_m + t_{m+1} + \cdots + t_n = S_n - S_{m-1},$$

the last equality following from $S_n = S_{m-1} + (t_m + \cdots + t_n)$. Evaluating such a "middle" sum is therefore just a matter of subtracting two partial sums.

Worked examples

Example 1 — scaffolded

Consider the arithmetic sequence $-3, 5, 13, 21, \ldots$. Find (a) the common difference d, (b) a rule for the nth term t_n, (c) the value of t_{20}, and (d) which term of the sequence equals 165.

Working	Comment
$d=5-(-3)=8$ (also $13-5=8$, $21-13=8$).	The common difference is any term minus the one before it; check it is the same across the list.
$a=t_1=-3$.	The first term.
$t_n=a+(n-1)d=-3+(n-1)8=8n-11$.	Substitute $a=-3$, $d=8$ into $t_n=a+(n-1)d$ and simplify.
$t_{20}=8(20)-11=160-11=149$.	Put $n=20$ into the rule.
Set $8n-11=165$, so $8n=176$ and $n=22$.	To find *which* term equals 165, solve the rule for n; the whole-number answer $n=22$ confirms 165 is the 22nd term.

Example 2 — scaffolded

An arithmetic series has first term $a=7$ and common difference $d=3$. Find (a) the 15th term, (b) the partial sum S_{15} of the first 15 terms, and (c) express S_{15} in sigma notation.

Working	Comment
$t_{15}=7+(15-1)(3)=7+42=49$.	The last term $\ell=t_{15}$, from $t_n=a+(n-1)d$.
$S_{15}=\frac{15}{2}(a+\ell)=\frac{15}{2}(7+49)=\frac{15}{2}(56)=420$.	With ℓ known, use $S_n=n/2(a+\ell)$.
Check: $\frac{15}{2}(2(7)+14(3))=\frac{15}{2}(56)=420$.	The other form $n/2(2a+(n-1)d)$ must agree.
$t_k=7+(k-1)(3)=3k+4$, so $S_{15}=\sum_{k=1}^{15}(3k+4)$.	Write the general term as a formula in the index k, then place it under the summation sign with limits 1 to 15.

Example 3 — unscaffolded

In an arithmetic sequence the second term is $t_2=1$ and the seventh term is $t_7=46$. Find a rule for t_n and hence evaluate the partial sum S_{25}.

The two given terms are five positions apart, and $t_7-t_2=(7-2)d=5d$, so

$$5d=46-1=45 \Rightarrow d=9.$$

The first term follows from $t_2 = a + d$: $a = t_2 - d = 1 - 9 = -8$. Hence

$$t_n = a + (n-1)d = -8 + (n-1)9 = 9n - 17,$$

which correctly gives $t_2 = 1$ and $t_7 = 46$. The partial sum of the first 25 terms is

$$S_{25} = \frac{25}{2}(2a + (25-1)d) = \frac{25}{2}(2(-8) + 24(9)) = \frac{25}{2}(-16 + 216) = \frac{25}{2}(200) = 2500.$$

$\therefore t_n = 9n - 17$ and $S_{25} = 2500$.

Example 4 — unscaffolded

The arithmetic series $2 + 5 + 8 + 11 + \cdots$ has partial sum S_n. (a) Show that $S_n = n/2(3n+1)$ and write the sum in sigma notation. (b) Find how many terms are required for the partial sum to equal 155.

Here $a = 2$ and $d = 3$. Substituting into the closed form,

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(2(2) + (n-1)(3)) = \frac{n}{2}(4 + 3n - 3) = \frac{n}{2}(3n+1).$$

Since $t_k = 2 + (k-1)(3) = 3k - 1$, the same sum in sigma notation is

$$S_n = \sum_{k=1}^{n} (3k - 1).$$

Setting the partial sum equal to 155,

$$\frac{n}{2}(3n+1) = 155 \Rightarrow 3n^2 + n - 310 = 0.$$

The discriminant is $1 + 4(3)(310) = 3721 = 61^2$, so $n = \dfrac{-1 \pm 61}{6}$. Rejecting the negative root leaves $n = \dfrac{60}{6} = 10$, and indeed $S_{10} = 10/2(3(10)+1) = 5(31) = 155$.

$\therefore$ the closed form is $S_n = n/2(3n+1)$, and 10 terms are required.

Practice questions

Scaffolded

Q1. Consider the arithmetic sequence $4, 13, 22, 31, \ldots$. (a) State the first term a and the common difference d. (b) Find a rule for t_n. (c) Evaluate t_{12}. (d) Determine which term of the sequence equals 130.

Q2. An arithmetic sequence has first term $a = 3$ and common difference $d = 8$. (a) Find the tenth term t_{10}. (b) Using $S_n = n/2(2a + (n-1)d)$, find S_{10}. (c) Confirm your answer to (b) using the form $S_n = n/2(a + \ell)$.

Q3. In an arithmetic sequence $t_3 = 20$ and $t_8 = 50$. (a) Write t_3 and t_8 in terms of a and d. (b) Subtract to find d. (c) Hence find a. (d) Find t_{20}.

Q4. Consider the sum written in sigma notation

$$\sum_{k=1}^{12} (5k + 2).$$

(a) Show that its terms form an arithmetic sequence, and state a, d and the number of terms. (b) Find the last term. (c) Evaluate the sum.

Q5. For the arithmetic series $6+10+14+\cdots$, the partial sum of the first n terms is 240. (a) Write S_n as a simplified expression in n. (b) Set your expression equal to 240. (c) Solve the resulting quadratic equation. (d) State how many terms are required.

Q6. As part of a rehabilitation program a patient walks for 8 minutes in the first week and increases the walking time by 3 minutes each week. (a) State the first term a and common difference d of the arithmetic sequence of weekly walking times. (b) Find the walking time in week 12. (c) Find the total walking time over the first 12 weeks.

Unscaffolded

Q7. Find a rule for t_n and the value of t_{30} for the arithmetic sequence $17, 12, 7, 2, \ldots$.

Q8. Determine which term of the arithmetic sequence $-4, 1, 6, 11, \ldots$ equals 91.

Q9. In an arithmetic sequence $t_5=28$ and $t_{11}=58$. Find a, d and a rule for t_n.

Q10. An arithmetic series has first term -9 and common difference 4. Find the partial sum S_{20} of its first 20 terms.

Q11. Evaluate the sum

$$\sum_{k=1}^{25}(7k-3).$$

Q12. The partial sum of the first n terms of the arithmetic series $5+9+13+\cdots$ is 495. Find n.

Q13. In an arithmetic sequence $t_7=3$ and $t_{15}=-21$. Find a rule for t_n, and hence determine the position of the first term that is negative.

Q14. An arithmetic series has first term 12, and its eighteenth term is 80. Find the partial sum S_{18} of the first 18 terms.

Q15. The expressions $2k-3$, $k+4$ and $3k-7$ are three consecutive terms of an arithmetic sequence. Find k and the three terms.

Q16. A factory produces 140 units in its first month of operation and increases output by 12 units each month. (a) Find the number of units produced in the fifteenth month. (b) Find the total number of units produced in the first 15 months.

Q17. An arithmetic sequence has nth term $t_n=6n-5$. By writing the required sum as $S_{25}-S_9$, evaluate $t_{10}+t_{11}+\cdots+t_{25}$.

Q18. For an arithmetic series the partial sum of the first 10 terms is 180 and the partial sum of the first 20 terms is 560. (a) Find the first term a and the common difference d. (b) Write a rule for t_n. (c) Show that the partial sum simplifies to $S_n=n^2+8n$.

Answers

Q1. (a) $a=4$, $d=9$; (b) $t_n=9n-5$; (c) $t_{12}=103$; (d) the 15th term. **Q2.** (a) $t_{10}=75$; (b) $S_{10}=390$; (c) $10/2(3+75)=390$. **Q3.** (a) $a+2d=20$, $a+7d=50$; (b) $d=6$; (c) $a=8$; (d) $t_{20}=122$. **Q4.** (a) $a=7$, $d=5$, 12 terms; (b) $\ell=62$; (c) 414. **Q5.** (a) $S_n=2n^2+4n$; (b) $2n^2+4n=240$; (c) $n=10$ (rejecting $n=-12$); (d) 10 terms. **Q6.** (a) $a=8$, $d=3$; (b) 41 minutes; (c) 294 minutes. **Q7.** $t_n=22-5n$; $t_{30}=-128$. **Q8.** The 20th term. **Q9.** $a=8$, $d=5$, $t_n=5n+3$. **Q10.** $S_{20}=580$. **Q11.** 2200. **Q12.** $n=15$. **Q13.** $t_n=24-3n$; the first negative term is $t_9=-3$. **Q14.** $S_{18}=828$. **Q15.** $k=6$; the terms are $9, 10, 11$. **Q16.** (a) 308 units; (b) 3360 units. **Q17.** 1600. **Q18.** (a) $a=9$, $d=2$; (b) $t_n=2n+7$; (c) $S_n=n^2+8n$.

Fully-worked solutions

Q1. The common difference is $d=13-4=9$ (and $22-13=9$), with first term $a=4$. Then $t_n=a+(n-1)d=4+(n-1)9=9n-5$. Putting $n=12$, $t_{12}=9(12)-5=103$. Solving $9n-5=130$ gives $9n=135$, so $n=15$: the 15th term equals 130.

Q2. With $a=3$ and $d=8$, the tenth term is $t_{10}=3+(10-1)(8)=3+72=75$. The partial sum is

$$S_{10}=\frac{10}{2}(2(3)+(10-1)(8))=5(6+72)=5(78)=390.$$

Since the last term is $\ell=t_{10}=75$, the other form gives $10/2(3+75)=5(78)=390$, in agreement.

Q3. Writing the two terms out, $t_3=a+2d=20$ and $t_8=a+7d=50$. Subtracting the first from the second, $5d=30$, so $d=6$. Then $a=20-2(6)=8$, giving $t_n=8+(n-1)6=6n+2$ and $t_{20}=6(20)+2=122$.

Q4. With $t_k=5k+2$, consecutive terms differ by $t_{k+1}-t_k=(5(k+1)+2)-(5k+2)=5$, a constant, so the terms are arithmetic with $d=5$; the first term is $t_1=5(1)+2=7$ and there are 12 terms. The last term is $\ell=t_{12}=5(12)+2=62$. Hence

$$\sum_{k=1}^{12}(5k+2)=\frac{12}{2}(a+\ell)=6(7+62)=6(69)=414.$$

Q5. With $a=6$ and $d=4$,

$$S_n=\frac{n}{2}(2(6)+(n-1)(4))=\frac{n}{2}(4n+8)=2n^2+4n.$$

Setting $2n^2+4n=240$ gives $n^2+2n-120=0$, that is $(n+12)(n-10)=0$. The number of terms must be positive, so $n=10$.

Q6. The weekly walking times form an arithmetic sequence with $a=8$ and $d=3$. In week 12 the time is $t_{12}=8+(12-1)(3)=8+33=41$ minutes. The total over the first 12 weeks is

$$S_{12}=\frac{12}{2}(2(8)+(12-1)(3))=6(16+33)=6(49)=294\text{ minutes.}$$

Q7. Here $a=17$ and $d=12-17=-5$, so $t_n=17+(n-1)(-5)=22-5n$. Then $t_{30}=22-5(30)=22-150=-128$.

Q8. The sequence has $a=-4$ and $d=5$, so $t_n=-4+(n-1)(5)=5n-9$. Solving $5n-9=91$ gives $5n=100$, so $n=20$: the 20th term equals 91.

Q9. Since $t_{11}-t_5=(11-5)d=6d$ and $t_{11}-t_5=58-28=30$, the common difference is $d=5$. From $t_5=a+4d=28$, $a=28-4(5)=8$, giving $t_n=8+(n-1)(5)=5n+3$ (which returns $t_5=28$ and $t_{11}=58$).

Q10. With $a=-9$, $d=4$ and $n=20$,

$$S_{20}=\frac{20}{2}(2(-9)+(20-1)(4))=10(-18+76)=10(58)=580.$$

Q11. The general term $t_k=7k-3$ is arithmetic with $t_1=4$ and $d=7$; there are 25 terms, and the last is $\ell=t_{25}=7(25)-3=172$. Therefore

$$\sum_{k=1}^{25}(7k-3)=\frac{25}{2}(a+\ell)=\frac{25}{2}(4+172)=\frac{25}{2}(176)=2200.$$

Q12. The series has $a=5$ and $d=4$, so

$$S_n=\frac{n}{2}(2(5)+(n-1)(4))=\frac{n}{2}(4n+6)=2n^2+3n.$$

Setting $2n^2+3n=495$ gives $2n^2+3n-495=0$. The discriminant is $3^2+4(2)(495)=3969=63^2$, so $n=\frac{-3\pm 63}{4}$; the positive root is $n=\frac{60}{4}=15$.

Q13. Since $t_{15}-t_7=(15-7)d=8d$ and $t_{15}-t_7=-21-3=-24$, the common difference is $d=-3$. From $t_7=a+6d=3$, $a=3-6(-3)=21$, so $t_n=21+(n-1)(-3)=24-3n$. A term is negative when $24-3n<0$, i.e. $n>8$; the first integer satisfying this is $n=9$, giving the first negative term $t_9=24-27=-3$.

Q14. The last term named is $\ell=t_{18}=80$ and the first is $a=12$, with $n=18$ terms, so the form using first and last term applies directly:

$$S_{18}=\frac{18}{2}(a+\ell)=9(12+80)=9(92)=828.$$

Q15. For three consecutive arithmetic terms the middle term minus the first equals the last minus the middle:

$$(k+4)-(2k-3)=(3k-7)-(k+4).$$

The left side is $-k+7$ and the right side is $2k-11$; setting them equal, $-k+7=2k-11$, so $3k=18$ and $k=6$. The terms are then $2(6)-3=9$, $6+4=10$ and $3(6)-7=11$, namely $9,10,11$ (common difference 1).

Q16. Monthly output is arithmetic with $a=140$ and $d=12$. In the fifteenth month the output is $t_{15}=140+(15-1)(12)=140+168=308$ units. The total over the first 15 months is

$$S_{15}=\frac{15}{2}(2(140)+(15-1)(12))=\frac{15}{2}(280+168)=\frac{15}{2}(448)=3360\text{ units.}$$

Q17. The required sum runs from the 10th to the 25th term, so it equals $S_{25}-S_9$. With $t_n=6n-5$ the terms are arithmetic with $t_1=1$, and the two partial sums are

$$S_{25}=\frac{25}{2}(1+t_{25})=\frac{25}{2}(1+145)=\frac{25}{2}(146)=1825,\ S_9=\frac{9}{2}(1+t_9)=\frac{9}{2}(1+49)=\frac{9}{2}(50)=225.$$

Hence $t_{10}+t_{11}+\cdots+t_{25}=1825-225=1600$. (As a check, the block has 16 terms with first $t_{10}=55$ and last $t_{25}=145$, giving $16/2(55+145)=8(200)=1600$.)

Q18. Applying $S_n=n/2(2a+(n-1)d)$ at $n=10$ and $n=20$,

$$S_{10}=5(2a+9d)=180 \Rightarrow 2a+9d=36, S_{20}=10(2a+19d)=560 \Rightarrow 2a+19d=56.$$

Subtracting the first equation from the second, $10d=20$, so $d=2$; then $2a=36-9(2)=18$, giving $a=9$. The general term is $t_n=9+(n-1)(2)=2n+7$, and the partial sum simplifies to

$$S_n=\frac{n}{2}(2(9)+(n-1)(2))=\frac{n}{2}(2n+16)=n(n+8)=n^2+8n,$$

which returns $S_{10}=180$ and $S_{20}=560$ as required.

Theory

The previous section built sequences by *adding* a fixed amount at each step. This section studies the sequences built by *multiplying* by a fixed amount. They grow (or decay) far faster than arithmetic sequences, and — remarkably — an unending geometric sequence can have a **finite** total. Making that idea precise leads to the sum to infinity and to a clean proof that every recurring decimal is a fraction.

Geometric sequences.

A **geometric sequence** is one in which each term after the first is obtained by multiplying the previous term by a fixed non-zero constant. That constant is the **common ratio**, written r; the first term is non-zero as well, so no term of the sequence is ever 0 and r is recovered by dividing any term by the one before it:

$$r=\frac{t_2}{t_1}=\frac{t_3}{t_2}=\frac{t_4}{t_3}=\cdots.$$

Writing the first term as $a=t_1$, the terms are $a, ar, ar^2, ar^3, \ldots$; reaching the nth term from the first takes $n-1$ multiplications by r, so the **explicit rule** is

$$t_n=ar^{n-1}.$$

Throughout this section sequences are numbered from t_1, so the power of r is always $n-1$. A list of numbers is geometric exactly when *every* ratio of consecutive terms is the same; if the ratios disagree, the sequence is not geometric. Two terms that lie k places apart differ by a factor of r^k: since $t_{m+k}=t_m r^k$, any two given terms determine r^k, and once r is known the first term a follows by stepping back to t_1. The root must be taken with care. When k is **odd** the value of r^k fixes r uniquely; when k is **even** it fixes only $|r|$, so *two* sequences fit — for instance $t_1=6$ and $t_3=54$ give $r^2=9$ and hence $r=3$ or $r=-3$, that is $6, 18, 54, \ldots$ or $6, -18, 54, \ldots$ — while an even gap whose two terms have opposite signs, such as $t_1=6$ and $t_3=-54$, has no real common ratio at all.

Geometric series and their partial sums.

Adding the terms of a geometric sequence produces a **geometric series**. The sum of the first n terms is called the nth **partial sum**, written S_n:

$$S_n=a+ar+ar^2+\cdots+ar^{n-1}.$$

There is a short and important trick for evaluating this without adding term by term. Multiply every term of that line by the common ratio r:

$$rS_n=ar+ar^2+ar^3+\cdots+ar^n.$$

The two lines share all of their terms except two: the first has a leading a that the second lacks, and the second has a trailing ar^n that the first lacks. Subtracting the second line from the first, every middle term cancels in pairs and only those two survive:

$$S_n-rS_n=a-ar^n, \text{so } S_n(1-r)=a(1-r^n).$$

Provided $r\neq 1$ we may divide by $1-r$, giving the **partial-sum formula**

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{a(r^n-1)}{r-1}.$$

The two expressions are identical — multiplying the numerator and denominator of either by -1 produces the other — so either may be quoted. In practice the right-hand form $\frac{a(r^n-1)}{r-1}$ is tidier when $r>1$ (both parts positive) and the left-hand form $\frac{a(1-r^n)}{1-r}$ is tidier when $r<1$. The one case the formula excludes is $r=1$; but then every term equals a, so the partial sum is simply $S_n=na$.

Limiting behaviour of the terms.

Whether a geometric sequence settles down or runs away is decided entirely by the common ratio, through the behaviour of the powers r^{n-1} as n increases without bound. There are four cases.

Common ratio r	Behaviour of $t_n=ar^{n-1}$ as n grows
$\lvert r\rvert<1$	the terms shrink towards 0
$r=1$	the terms are constant, every one equal to a
$r=-1$	the terms oscillate $a,-a,a,-a,\ldots$ and never settle
$\lvert r\rvert>1$	the sizes of the terms grow without bound

The decisive case for what follows is $|r|<1$: repeatedly multiplying by a number smaller than 1 in size drives the terms towards zero, written

$$r^n\to 0\,\text{as}\,n\to\infty\,(|r|<1).$$

For example, with $r=1/2$ the powers $1/2,1/4,1/8,1/16,\ldots$ close in on 0; with $r=3/2$ the powers $3/2,9/4,27/8,\ldots$ grow beyond every bound.

The sum to infinity.

When $|r|<1$ the terms being added grow ever smaller, and the partial sums S_n approach a fixed limit. To see the value, look at where n appears in the partial-sum formula: only inside r^n. As $n\to\infty$ that power tends to 0, so

$$S_n=\frac{a(1-r^n)}{1-r}\to\frac{a(1-0)}{1-r}=\frac{a}{1-r}(|r|<1).$$

This limiting value is the **sum to infinity**, written S_∞:

$$S_\infty=\frac{a}{1-r},|r|<1.$$

It is the total of the whole unending series $a+ar+ar^2+\cdots$. The condition $|r|<1$ is essential: if $|r|\geq 1$ the terms do **not** shrink to zero — they stay the same size, oscillate, or grow — the partial sums never settle, and no finite sum to infinity exists. A geometric series therefore has a sum to infinity precisely when its common ratio satisfies $|r|<1$.

Recurring decimals as geometric series.

An infinite geometric series with $|r|<1$ is exactly what a recurring decimal is. Consider $0.\overline{7}=0.7777\ldots$; by place value this is the sum

$$0.7+0.07+0.007+\cdots,$$

a geometric series with first term $a=7/10$ and common ratio $r=1/10$. Since $|r|<1$ the sum to infinity applies:

$$0.\overline{7}=\frac{7/10}{1-1/10}=\frac{7/10}{9/10}=\frac{7}{9}.$$

The same method turns any recurring decimal into a fraction, which is why every recurring decimal is a rational number. When only part of the decimal recurs, split off the non-recurring part first and apply the sum to infinity to the repeating tail alone, then add the two pieces.

Worked examples

Example 1 — scaffolded

A geometric sequence has first term $t_1 = 3$ and common ratio $r = 4$. (a) Write the explicit rule and find t_5. (b) Find S_6, the sum of the first six terms. (c) State which form of the partial-sum formula is more convenient here, and why.

Working	Comment
(a) $a = 3$, $r = 4$, so $t_n = 3 \times 4^{n-1}$.	Reaching t_n from t_1 takes $n-1$ multiplications by r.
$t_5 = 3 \times 4^4 = 3 \times 256 = 768$.	Substitute $n = 5$ into the explicit rule.
(b) $S_6 = \dfrac{a\left(r^6 - 1\right)}{r-1} = \dfrac{3\left(4^6 - 1\right)}{4-1}$.	Partial-sum formula with $a = 3$, $r = 4$, $n = 6$.
$= \dfrac{3\left(4096 - 1\right)}{3} = \dfrac{3 \times 4095}{3} = 4095$.	$4^6 = 4096$, then the factor of 3 cancels.
(c) The form $\dfrac{a\left(r^n - 1\right)}{r-1}$ is neater.	With $r = 4 > 1$ both $r^n - 1$ and $r - 1$ are positive, so no signs to juggle.

Example 2 — scaffolded

Released from rest, a pendulum sweeps through an arc of 81 cm on its first swing, and each following swing is $2/3$ as long as the one before. (a) Explain why the swing lengths are geometric, and write the explicit rule. (b) Find the length of the fifth swing. (c) Assuming the motion continues indefinitely, find the total distance the pendulum bob travels. (d) Find the distance travelled in the first four swings, and comment on the terms.

Working	Comment
(a) Each swing is $2/3$ of the previous, so $r = 2/3$; with $a = 81$, $t_n = 81\left(2/3\right)^{n-1}$.	A constant ratio between consecutive swings makes the lengths geometric.
(b) $t_5 = 81\left(2/3\right)^4 = 81 \times \dfrac{16}{81} = 16$.	The fifth swing sweeps an arc of 16 cm.
(c) $S_\infty = \dfrac{a}{1-r} = \dfrac{81}{1-\frac{2}{3}} = \dfrac{81}{\frac{1}{3}} = 243$.	Valid since $\lvert r \rvert = 2/3 < 1$; the total distance is 243 cm.
(d) $S_4 = 81 + 54 + 36 + 24 = 195$.	The first four swings already cover 195 cm of the 243 cm total.
Because $\lvert r \rvert < 1$, the terms $81\left(2/3\right)^{n-1} \to 0$.	Each swing is shorter than the last, so the running total closes in on S_∞.

Example 3 — unscaffolded

Express the recurring decimal $0.\overline{351} = 0.351351351\ldots$ as a fraction in simplest form.

By place value the digits repeat in blocks of three, so the decimal is the infinite geometric series

$$0.\overline{351} = \frac{351}{1000} + \frac{351}{1000^2} + \frac{351}{1000^3} + \cdots,$$

with first term $a = \dfrac{351}{1000}$ and common ratio $r = \dfrac{1}{1000}$. Since $\lvert r \rvert < 1$, the sum to infinity gives

$$0.\overline{351}=\frac{a}{1-r}=\frac{351/1000}{1-1/1000}=\frac{351/1000}{999/1000}=\frac{351}{999}.$$

Cancelling the common factor 27 (as $351=27\times 13$ and $999=27\times 37$),

$$\frac{351}{999}=\frac{13}{37}.$$

$\therefore 0.\overline{351}=\frac{13}{37}.$

Example 4 — unscaffolded

A ball is dropped from a height of 12 m onto a hard floor. After each bounce it rises to $3/5$ of the height it fell from. Find the total distance the ball travels before coming to rest.

The ball first falls 12 m. After that it makes a matched pair of movements for each bounce — up to a rebound height and back down through the same height — so every rebound height is counted twice. The rebound heights themselves form a geometric sequence: the first is $12\times 3/5=7.2$ m, and each later rebound is $3/5$ of the one before, so $a=7.2$ and $r=3/5$. Because $|r|<1$, the total of all the rebound heights is

$$\frac{a}{1-r}=\frac{7.2}{1-\frac{3}{5}}=\frac{7.2}{\frac{2}{5}}=18\,\text{m}.$$

Each of these 18 m is travelled once going up and once coming down, and to them we add the single initial fall of 12 m:

$$\text{total distance}=12+2\times 18=48\,\text{m}.$$

$\therefore$ the ball travels a total distance of 48 m.

Practice questions

Scaffolded

Q1. A geometric sequence has first term $t_1=2$ and common ratio $r=5$. (a) Write down the explicit rule for t_n. (b) Find t_5. (c) Find S_5, the sum of the first five terms. (d) State which form of the partial-sum formula is more convenient here, and why.

Q2. Consider the geometric series $40+20+10+5+\cdots$. (a) State the common ratio and the explicit rule for t_n. (b) Find the fifth term t_5. (c) Find the sum to infinity. (d) Explain, using the value of r, why this series has a finite sum to infinity, and describe what happens to the terms and to the partial sums.

Q3. Express the recurring decimal $0.\overline{6}=0.6666\ldots$ as a fraction, using an infinite geometric series. (a) Write the decimal as the series $0.6+0.06+0.006+\cdots$ and state a and r. (b) State the condition on r needed to use the sum to infinity, and check it holds. (c) Evaluate the sum to infinity. (d) Give the fraction in simplest form.

Q4. A geometric sequence is numbered from t_1, and $t_2=20$ and $t_5=1280$. (a) Find the common ratio r. (b) Find the first term t_1. (c) Write down the explicit rule. (d) Find S_6.

Q5. A wind-up toy travels 5 m in the first second after release, and in each following second travels $4/5$ of the distance it travelled in the previous second. (a) Explain why the distances travelled in successive seconds are geometric, and state r. (b) Find the distance travelled during the third second. (c) Find the total distance the toy would travel if it ran indefinitely. (d) State why the total is finite.

Q6. A chain email is opened by 7 people on the first day, and the number of new people who open it doubles each day. (a) Find the number of new people who open it on the fourth day. (b) Write down the explicit rule for the number of new openers t_n on day n. (c) Find the total number of people who have opened it over the first six days. (d) State which form of the partial-sum formula you used and why.

Unscaffolded

Q7. Find the sum of the first nine terms of the geometric series $11+22+44+\cdots$.

Q8. Find the sum to infinity of the geometric series $20+12+7.2+\cdots$.

Q9. Express the recurring decimal $0.4\overline{6}=0.4666\ldots$ as a fraction in simplest form, by splitting off the non-recurring part first.

Q10. For each geometric series below, state whether it has a finite sum to infinity; if it does, find it. In every case the first term is $a=8$. (a) $r=3/4$ (b) $r=5/4$ (c) $r=-1/2$ (d) $r=-1$

Q11. A geometric series has first term a and common ratio r, with $r\neq 1$, and $S_n=a+ar+ar^2+\cdots+ar^{n-1}$. By forming $r\,S_n$ and subtracting it from S_n, prove that

$$S_n=\frac{a\left(r^n-1\right)}{r-1}.$$

Q12. A geometric sequence has first term 9 and common ratio 2. Find the smallest value of n for which the sum of the first n terms exceeds 5000.

Q13. A geometric series has first term 18 and a sum to infinity of 30. Find its common ratio.

Q14. A geometric series has a sum to infinity equal to five times its first term. Find its common ratio.

Q15. An equilateral triangle has side length 27 cm. A second triangle is drawn by joining the midpoints of its sides, a third by joining the midpoints of the second, and so on without end. Find the sum of the perimeters of all the triangles.

Q16. A ribbon is cut into infinitely many pieces whose lengths form a geometric sequence with common ratio $3/4$, and whose total length (the sum to infinity) is 96 cm. (a) Find the length of the first piece. (b) Find the length of the third piece.

Q17. Find the sum of the first six terms of the geometric series $4-8+16-32+\cdots$.

Q18. A geometric sequence is numbered from t_1, and $t_3=20$ and $t_6=2.5$. (a) Find the common ratio and the first term. (b) Find the sum to infinity. (c) Find the sum of the first five terms.

Answers

Q1. (a) $t_n = 2 \times 5^{n-1}$; (b) $t_5 = 1250$; (c) $S_5 = 1562$; (d) the form $\dfrac{a(r^n - 1)}{r-1}$, since $r = 5 > 1$ keeps both parts positive.

Q2. (a) $r = 1/2$, $t_n = 40(1/2)^{n-1}$; (b) $t_5 = 2.5$; (c) $S_\infty = 80$; (d) $|r| < 1$, so the terms tend to 0 and the partial sums approach 80. **Q3.** (a) $a = 6/10$, $r = 1/10$; (b) $|r| < 1$ (holds, as $1/10 < 1$); (c) $6/9$; (d) $2/3$. **Q4.** (a) $r = 4$; (b) $t_1 = 5$; (c) $t_n = 5 \times 4^{n-1}$; (d) $S_6 = 6825$. **Q5.** (a) each second's distance is $4/5$ of the previous, so $r = 4/5$; (b) 3.2 m; (c) 25 m; (d) $|r| < 1$. **Q6.** (a) 56; (b) $t_n = 7 \times 2^{n-1}$; (c) 441; (d) the form $\dfrac{a(r^n - 1)}{r-1}$, since $r = 2 > 1$. **Q7.** 5621. **Q8.** 50. **Q9.** $\dfrac{7}{15}$. **Q10.** (a) finite, 32; (b) no finite sum ($|r| > 1$); (c) finite, $16/3$; (d) no finite sum ($r = -1$). **Q11.** Proof — see solutions. **Q12.** $n = 10$. **Q13.** $r = 2/5$. **Q14.** $r = 4/5$. **Q15.** 162 cm. **Q16.** (a) 24 cm; (b) 13.5 cm. **Q17.** -84. **Q18.** (a) $r = 1/2$, $t_1 = 80$; (b) $S_\infty = 160$; (c) $S_5 = 155$.

Fully-worked solutions

Q1. With $a = 2$ and $r = 5$, the explicit rule is $t_n = 2 \times 5^{n-1}$, so $t_5 = 2 \times 5^4 = 2 \times 625 = 1250$. The partial sum is

$$S_5 = \frac{a(r^5 - 1)}{r-1} = \frac{2(5^5 - 1)}{5-1} = \frac{2(3125-1)}{4} = \frac{2 \times 3124}{4} = 1562.$$

Because $r = 5 > 1$, the form $\dfrac{a(r^n - 1)}{r-1}$ is more convenient: both $r^n - 1$ and $r - 1$ are positive, so no negative signs appear.

Q2. Dividing consecutive terms, $\dfrac{20}{40} = \dfrac{10}{20} = \dfrac{1}{2}$, so $r = 1/2$ and, with $a = 40$, $t_n = 40(1/2)^{n-1}$. Then $t_5 = 40(1/2)^4 = \dfrac{40}{16} = 2.5$. Since $|r| = 1/2 < 1$,

$$S_\infty = \frac{a}{1-r} = \frac{40}{1 - \frac{1}{2}} = \frac{40}{\frac{1}{2}} = 80.$$

Because $|r| < 1$, the terms shrink towards 0, and each new term adds less than the one before, so the partial sums climb towards — but never pass — the limit 80.

Q3. Writing the decimal out, $0.\overline{6} = 0.6 + 0.06 + 0.006 + \cdots$, a geometric series with $a = 6/10$ and $r = 1/10$. The sum to infinity may be used because $|r| = 1/10 < 1$. Then

$$0.\overline{6} = \frac{a}{1-r} = \frac{6/10}{1 - 1/10} = \frac{6/10}{9/10} = \frac{6}{9} = \frac{2}{3}.$$

Q4. From t_2 to t_5 is three steps, so $t_5 = t_2 r^3$, giving $1280 = 20r^3$, hence $r^3 = \dfrac{1280}{20} = 64$ and $r = 4$. Stepping back one place, $t_1 = \dfrac{t_2}{r} = \dfrac{20}{4} = 5$, so $t_n = 5 \times 4^{n-1}$. The partial sum is

$$S_6 = \frac{5(4^6 - 1)}{4-1} = \frac{5(4096-1)}{3} = \frac{5 \times 4095}{3} = 6825.$$

Q5. Each second's distance is $4/5$ of the previous second's, a constant ratio, so the distances are geometric with $r = 4/5$ and $a = 5$. The third second's distance is $t_3 = 5(4/5)^2 = 5 \times \dfrac{16}{25} = \dfrac{16}{5} = 3.2$ m. Since $|r| = 4/5 < 1$, the total distance is

$$S_\infty = \frac{a}{1-r} = \frac{5}{1-\frac{4}{5}} = \frac{5}{\frac{1}{5}} = 25\,\mathrm{m}.$$

The total is finite precisely because $|r|<1$: the per-second distances tend to 0 fast enough for the running total to settle at 25 m.

Q6. The number of new openers doubles each day, so $r=2$ with $a=7$, giving $t_n = 7\times 2^{n-1}$. On the fourth day the number of new openers is $t_4 = 7\times 2^3 = 7\times 8 = 56$. The total over the first six days is the partial sum

$$S_6 = \frac{a\left(r^6-1\right)}{r-1} = \frac{7\left(2^6-1\right)}{2-1} = 7\times 63 = 441.$$

The form $\dfrac{a\left(r^n-1\right)}{r-1}$ was used because $r=2>1$, keeping both parts positive.

Q7. Here $a=11$ and $r=\dfrac{22}{11}=2$, so

$$S_9 = \frac{a\left(r^9-1\right)}{r-1} = \frac{11\left(2^9-1\right)}{2-1} = 11\left(512-1\right) = 11\times 511 = 5621.$$

Q8. The ratio is $r=\dfrac{12}{20}=\dfrac{3}{5}$, and $a=20$. Since $|r|=3/5<1$,

$$S_\infty = \frac{a}{1-r} = \frac{20}{1-\frac{3}{5}} = \frac{20}{\frac{2}{5}} = 50.$$

Q9. Split off the non-recurring part: $0.4\overline{6} = 0.4 + 0.0\overline{6}$. The first piece is $0.4 = 2/5$. The recurring tail is $0.0\overline{6} = 0.06 + 0.006 + 0.0006 + \cdots$, a geometric series with $a = 6/100$ and $r=1/10$, so

$$0.0\overline{6} = \frac{6/100}{1-1/10} = \frac{6/100}{9/10} = \frac{6}{90} = \frac{1}{15}.$$

Adding the two pieces,

$$0.4\overline{6} = \frac{2}{5} + \frac{1}{15} = \frac{6}{15} + \frac{1}{15} = \frac{7}{15}.$$

Q10. A finite sum to infinity exists exactly when $|r|<1$, and then it equals $\dfrac{a}{1-r}$ with $a=8$. (a) $|r|=3/4<1$: $S_\infty = \dfrac{8}{1-\frac{3}{4}} = \dfrac{8}{\frac{1}{4}} = 32$. (b) $|r|=5/4>1$: the terms grow, so there is no finite sum. (c) $|r|=1/2<1$: $S_\infty = \dfrac{8}{1-\left(-\frac{1}{2}\right)} = \dfrac{8}{\frac{3}{2}} = \dfrac{16}{3}$. (d) $r=-1$, so $|r|=1$: the partial sums oscillate $8, 0, 8, 0, \ldots$ and never settle, so there is no finite sum.

Q11. Write the partial sum and its multiple by r:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}, \quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Every term of rS_n also appears in S_n except the final ar^n, and every term of S_n appears in rS_n except the leading a. Subtracting therefore cancels all the shared terms:

$$S_n - r S_n = a - a r^n.$$

Factorising each side, $S_n(1-r) = a(1-r^n)$. Since $r \neq 1$, the factor $1-r$ is non-zero and division is allowed. Dividing, and then multiplying numerator and denominator by -1,

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},$$

as required.

Q12. With $a=9$ and $r=2$, the partial sum is $S_n = \dfrac{9(2^n-1)}{2-1} = 9(2^n-1)$. Requiring $S_n > 5000$ gives $2^n - 1 > \dfrac{5000}{9} \approx 555.6$, so $2^n > 556.6$. Now $2^9 = 512$ is too small and $2^{10} = 1024$ is large enough, so $n=10$ is the first value that works. Checking: $S_9 = 9(512-1) = 4599$ (not yet over 5000) while $S_{10} = 9(1024-1) = 9207$ (over). So $n=10$.

Q13. Using $S_\infty = \dfrac{a}{1-r}$ with $a=18$ and $S_\infty = 30$,

$$\frac{18}{1-r} = 30 \Rightarrow 1-r = \frac{18}{30} = \frac{3}{5} \Rightarrow r = \frac{2}{5}.$$

Since $|r| = 2/5 < 1$, a sum to infinity indeed exists, so $r = 2/5$.

Q14. Let the first term be a (with $a \neq 0$). The condition $S_\infty = 5a$ gives

$$\frac{a}{1-r} = 5a \Rightarrow \frac{1}{1-r} = 5 \Rightarrow 1-r = \frac{1}{5} \Rightarrow r = \frac{4}{5}.$$

As $|r| = 4/5 < 1$, the series does have a sum to infinity, so $r = 4/5$.

Q15. The first triangle has perimeter $3 \times 27 = 81$ cm. Joining the midpoints of a triangle's sides produces a triangle with each side half as long — a well-known midpoint property — so each successive perimeter is $1/2$ of the previous one. The perimeters therefore form a geometric series with $a=81$ and $r=1/2$, and since $|r|<1$,

$$S_\infty = \frac{a}{1-r} = \frac{81}{1-\frac{1}{2}} = \frac{81}{\frac{1}{2}} = 162\,\text{cm}.$$

Q16. Let the first piece have length a. The total length is the sum to infinity, so with $r = 3/4$,

$$\frac{a}{1-\frac{3}{4}} = 96 \Rightarrow \frac{a}{\frac{1}{4}} = 96 \Rightarrow 4a = 96 \Rightarrow a = 24\,\text{cm}.$$

The third piece is $t_3 = ar^2 = 24(3/4)^2 = 24 \times \dfrac{9}{16} = 13.5$ cm.

Q17. Here $a=4$ and $r = \dfrac{-8}{4} = -2$, so

$$S_6 = \frac{a(r^6-1)}{r-1} = \frac{4((-2)^6-1)}{-2-1} = \frac{4(64-1)}{-3} = \frac{4 \times 63}{-3} = -84.$$

(Check by direct addition: $4 - 8 + 16 - 32 + 64 - 128 = -84$.)

Q18. From t_3 to t_6 is three steps, so $t_6 = t_3 r^3$, giving $2.5 = 20r^3$, hence $r^3 = \frac{2.5}{20} = \frac{1}{8}$ and $r = 1/2$. Stepping back from $t_3 = ar^2$, $20 = a(1/2)^2 = \frac{a}{4}$, so $a = 80$. Since $|r| = 1/2 < 1$,

$$S_\infty = \frac{a}{1-r} = \frac{80}{1-\frac{1}{2}} = 160.$$

The sum of the first five terms is

$$S_5 = \frac{a(1-r^5)}{1-r} = \frac{80\left(1-\left(\frac{1}{2}\right)^5\right)}{1-\frac{1}{2}} = 160\left(1-\frac{1}{32}\right) = 160 \times \frac{31}{32} = 155.$$

Theory

The last two sections built sequences from a *rule for the nth term* — a formula into which the position n is substituted directly. This section takes the opposite, and often more natural, point of view: each term is described only in relation to the one before it. Such a **step-by-step** description is called a recurrence relation. It models any process that repeats the same operation over and over — a bank balance, a managed population, the concentration of a drug — and the central problem of the section is to turn that step-by-step description back into a single formula for t_n. For the important family of *first-order linear* recurrences this can always be done, and the method that does it — locating a fixed value that the rule leaves unchanged — is the main idea of the section.

Defining a sequence by recursion.

A sequence is defined **recursively** when two things are given: a **starting value**, and a **rule** that produces each new term from the term immediately before it. Together these form a **recurrence relation**. It is convenient here to number the terms from t_0, the starting value, so that the subscript n counts the number of steps taken; the term after t_n is t_{n+1}. For example,

$$t_0=2,t_{n+1}=3t_n-4$$

says “start at 2, and at each step multiply by 3 and subtract 4”. Applying the rule repeatedly **generates** the terms one at a time:

$$t_1=3(2)-4=2,t_2=3(2)-4=2,\ldots$$

so this particular sequence is constant. Both parts of the definition are essential: the same rule started at $t_0=5$ instead gives $5,11,29,83,\ldots$, a completely different sequence. A rule without a starting value does not determine a sequence.

First-order linear recurrence relations.

The recurrences of this course all have the form

$$t_{n+1}=at_n+b,$$

where a and b are **constants**. Such a relation is called **first-order** because each term is computed from just *one* previous term (a single step back), and **linear** because the previous term is only multiplied by a constant and has a constant added. Two familiar sequences are special cases. When $a=1$ the rule adds the fixed amount b at every step, so the sequence is **arithmetic** with common difference b; when $b=0$ the rule multiplies by the fixed factor a at every step, so the sequence is **geometric** with common ratio a. The general first-order linear recurrence does both at once: at each step it multiplies by a and then adds b.

Equilibrium values and the closed-form rule.

Generating terms one at a time answers “what is t_3?” but not “what is t_{100}?” without a hundred steps. What is wanted is a **closed-form** (or **explicit**) rule giving t_n directly as a function of n. The key is to look for a value that the rule does not change. A number L is an **equilibrium** (or **fixed point**) of the recurrence if starting there keeps the sequence there forever — that is, if substituting $t_n=L$ returns $t_{n+1}=L$:

$$L=aL+b.$$

Provided $a\neq 1$ this has exactly one solution,

$$L=\frac{b}{1-a}.$$

The equilibrium is the pivot of the whole analysis. Measure each term by its **gap from the equilibrium**, writing $u_n = t_n - L$. Subtracting the equilibrium equation $L = aL + b$ from the recurrence $t_{n+1} = a t_n + b$ makes the constant b cancel:

$$t_{n+1} - L = (a t_n + b) - (aL + b) = a(t_n - L),$$

that is, $u_{n+1} = a u_n$. So the gaps form a **geometric** sequence with ratio a and first gap $u_0 = t_0 - L$; hence $u_n = (t_0 - L)a^n$. Adding L back gives the closed-form rule

$$t_n = L + (t_0 - L)a^n, L = \frac{b}{1-a}, (a \neq 1).$$

The same formula can be obtained by unrolling the recurrence directly. Applying the rule n times to t_0,

$$t_n = a^n t_0 + b\left(a^{n-1} + a^{n-2} + \cdots + a + 1\right).$$

The bracket is a geometric series of n terms with first term 1 and ratio a, which by the partial-sum formula of Section 4B equals $\frac{a^n - 1}{a - 1} = \frac{1 - a^n}{1 - a}$. Therefore

$$t_n = a^n t_0 + \frac{b}{1-a}(1 - a^n) = a^n t_0 + L(1 - a^n) = L + (t_0 - L)a^n,$$

in exact agreement with the fixed-point derivation. The single excluded case $a = 1$ is the arithmetic one, where the equilibrium equation collapses to $L = L + b$: no value of L satisfies it when $b \neq 0$, and *every* value satisfies it when $b = 0$, so in neither case is there a single pivot for the gap method to work from. The closed form is instead simply $t_n = t_0 + nb$.

Long-term behaviour.

Because n enters the closed form only through the power a^n, the entire long-term behaviour of a first-order linear recurrence is governed by the multiplier a, exactly as the behaviour of a geometric sequence was in Section 4B. The gap from the equilibrium, $(t_0 - L)a^n$, either shrinks, holds, or grows.

Multiplier a	Behaviour of $t_n = L + (t_0 - L)a^n$ as n grows
$0 < a < 1$	gap shrinks with the same sign; terms approach L steadily from one side
$-1 < a < 0$	gap shrinks but changes sign each step; terms approach L, alternating above and below
$a = 1$	arithmetic: $t_n = t_0 + nb$ increases or decreases without bound (unless $b = 0$)
$a = -1$	terms alternate between t_0 and $b - t_0$ forever, never settling (unless $t_0 = L = b/2$, when the two values coincide and the sequence is constant)
$\lvert a \rvert > 1$	gap grows without bound; terms run away from L (unless $t_0 = L$ exactly)

The decisive case is $\lvert a \rvert < 1$: then $a^n \to 0$ as $n \to \infty$, so $t_n \to L$ whatever the starting value. The equilibrium is then a genuine **limiting value** to which every solution is drawn, and the starting value affects only how the terms approach it, not where they end up. When $\lvert a \rvert > 1$ the equilibrium still exists algebraically but repels: unless the sequence begins exactly at L, its terms move ever further away.

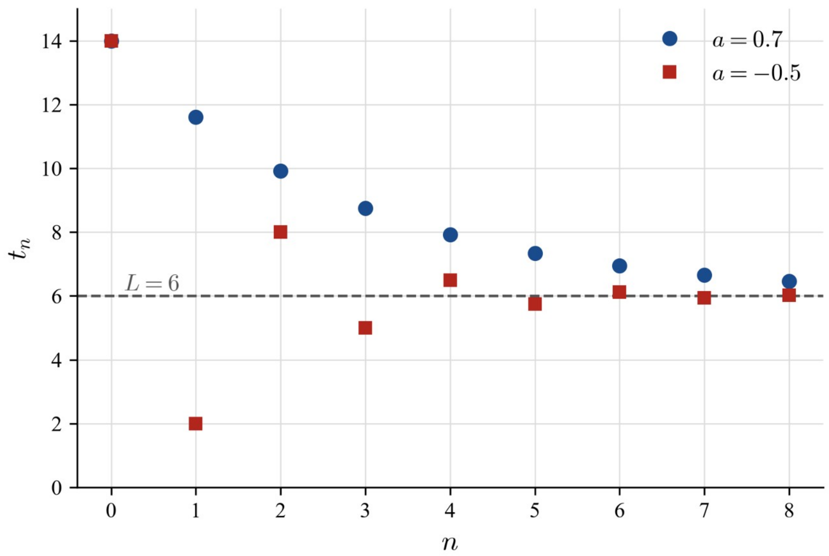
Applications: financial modelling and populations.

First-order linear recurrences describe a wide range of real situations in which a quantity is repeatedly scaled and then adjusted by a fixed amount. To set one up, take the initial quantity as t_0 and translate one step of the process into the rule: a percentage growth or decay becomes the multiplier a, and a fixed addition or subtraction becomes the constant b; always state what t_n measures and what one step of n represents.

In **financial modelling**, interest compounding at $r\,\%$ per period multiplies a balance by the growth multiplier $R=1+\frac{r}{100}$ each period. A pure compound-interest investment, or a reducing-balance depreciation with $R=1-\frac{r}{100}$, is the case $b=0$: purely geometric. Adding a constant b each period covers the richer situations that need the full theory — a savings plan with a regular deposit d obeys $V_{n+1}=R\,V_n+d$, and a loan of balance B_n charged interest and then reduced by a regular repayment P obeys $B_{n+1}=R\,B_n-P$. For a loan the equilibrium $L=\frac{P}{R-1}$ is the balance at which the interest each period exactly equals the repayment. A repayment larger than the interest on the opening balance — equivalently, an opening balance below L — drives the balance down to zero; a repayment smaller than that interest never covers it, and the debt grows without bound.

In **population modelling**, a managed population in which a fixed proportion survives each period and a fixed number is added (by stocking, birth programs or immigration) obeys $P_{n+1}=a\,P_n+b$ with $0<a<1$. Here the equilibrium $L=\frac{b}{1-a}$ is a **sustainable level**: because $|a|<1$, the population approaches L in the long run regardless of its starting size, rising towards it if it begins below and falling towards it if it begins above.

The two sequences below share the equilibrium $L=6$: one has $a=0.7$ (approaching steadily from above) and the other $a=-0.5$ (approaching while alternating from side to side). The terms are plotted as separate points — a sequence has a value only at whole numbers n, so the points are never joined.



Worked examples

Example 1 — scaffolded

A sequence is defined by the recurrence relation $t_0=1$, $t_{n+1}=2t_n+5$. Generate the first five terms t_0 to t_4, state the constants a and b, and describe the behaviour of the sequence.

Working	Comment
Starting value $t_0=1$.	The recurrence states the first term directly.
$t_1=2(1)+5=7$.	Apply the rule with $n=0$: multiply the current term by 2 , then add 5.

Working	Comment
$t_2=2(7)+5=19$, $t_3=2(19)+5=43$, $t_4=2(43)+5=91$.	Each new term is computed from the term just found.
The constants are $a=2$ and $b=5$.	Comparing $t_{n+1}=2t_n+5$ with $t_{n+1}=at_n+b$.
$1, 7, 19, 43, 91, \ldots$ is increasing, and grows without bound.	Each term exceeds the one before; since $a=2>1$ the terms run away, roughly doubling each step.

Example 2 — scaffolded

A sequence is defined by $t_0=4$, $t_{n+1}=0.2t_n+8$. Find (a) the equilibrium value L, (b) the closed-form rule for t_n, (c) the exact value of t_5, and (d) the limiting value of the sequence.

Working	Comment
(a) $L=0.2L+8 \Rightarrow 0.8L=8 \Rightarrow L=10$.	The equilibrium satisfies $L=aL+b$; solve for L.
(b) $t_n=L+(t_0-L)a^n=10+(4-10)(0.2)^n=10-6(0.2)^n$.	Substitute $L=10$, $t_0=4$, $a=0.2$ into $t_n=L+(t_0-L)a^n$.
Check: $t_1=10-6(0.2)=8.8$, matching $0.2(4)+8=8.8$.	The closed form must agree with one step of the recurrence.
(c) $t_5=10-6(0.2)^5=10-6(0.00032)=9.99808$.	Substitute $n=5$ into the closed form.
(d) Since $\lvert a\rvert=0.2<1$, $(0.2)^n \to 0$, so $t_n \to 10$.	The gap $-6(0.2)^n$ shrinks to zero; the limiting value is the equilibrium $L=10$.

Example 3 — unscaffolded

A commercial apiary manages 5000 beehives. Each year 70% of the hives come through the winter still productive, and each spring the beekeepers split strong colonies to establish a further 900 hives. Let P_n be the number of hives after n years. Set up a recurrence relation, find the equilibrium number of hives, obtain a closed-form rule, find the number of hives after 3 years, and describe the long-term behaviour.

Each year the surviving hives number $0.7P_n$ and then 900 are added, so with $P_0=5000$,

$$P_{n+1}=0.7P_n+900.$$

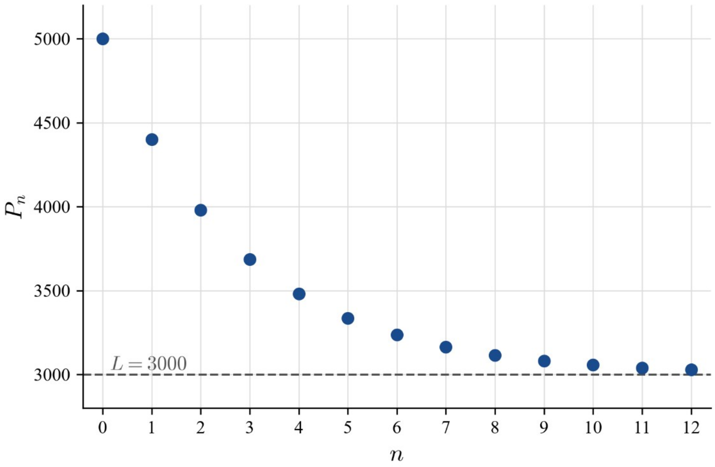
The equilibrium satisfies $L=0.7L+900$, giving $0.3L=900$ and $L=3000$. Measuring from the equilibrium, the closed-form rule is

$$P_n=L+(P_0-L)(0.7)^n=3000+(5000-3000)(0.7)^n=3000+2000(0.7)^n.$$

After three years,

$$P_3=3000+2000(0.7)^3=3000+2000(0.343)=3686.$$

Because $\lvert 0.7\rvert<1$, the term $2000(0.7)^n$ shrinks to zero, so the number of hives falls steadily towards the equilibrium and never drops below it. Plotting the first twelve years shows the terms closing on the sustainable level $L=3000$ from above.



$\therefore P_{n+1} = 0.7P_n + 900$ with closed form $P_n = 3000 + 2000(0.7)^n$; there are 3686 hives after 3 years, and in the long run the apiary approaches a sustainable level of 3000 hives.

Example 4 — unscaffolded

Rowan opens an investment account with \$2000. The account pays 5% per annum compounding annually, and at the end of each year Rowan deposits a further \$1000. Let V_n be the balance after n years. Write a recurrence relation, find a closed-form rule, and find the balance after 10 years, correct to the nearest cent.

Each year the balance is multiplied by the growth multiplier $R = 1 + \frac{5}{100} = 1.05$ and then \$1000 is added, so with $V_0 = 2000$,

$$V_{n+1} = 1.05V_n + 1000.$$

Solving $L = 1.05L + 1000$ gives $-0.05L = 1000$, so $L = -20000$. Since $|1.05| > 1$ this equilibrium is not a level the balance approaches — it is negative and merely a pivot for the algebra — but the closed-form rule still holds:

$$V_n = L + (V_0 - L)(1.05)^n = -20000 + (2000 - (-20000))(1.05)^n = -20000 + 22000(1.05)^n.$$

After ten years,

$$V_{10} = -20000 + 22000(1.05)^{10} = -20000 + 22000(1.628894\ldots) = 15835.681\ldots,$$

which is \$15835.68 to the nearest cent. Rowan has contributed $2000 + 10 \times 1000 = \$12000$ in total, so the interest earned is $15835.68 - 12000 = \$3835.68$.

$\therefore V_{n+1} = 1.05V_n + 1000$ with closed form $V_n = -20000 + 22000(1.05)^n$; the balance after 10 years is \$15835.68, of which \$3835.68 is interest.

Practice questions

Scaffolded

Q1. A sequence is defined by $t_0 = 2$, $t_{n+1} = 3t_n - 2$. (a) Generate the terms t_1 to t_4. (b) State the constants a and b. (c) Find the equilibrium value L. (d) Verify that the closed-form rule $t_n = 1 + 3^n$ gives the correct value of t_4.

Q2. A sequence is defined by $t_0 = 1$, $t_{n+1} = 0.25t_n + 6$. (a) Generate the terms t_1, t_2 and t_3. (b) Find the equilibrium value L. (c) Write down the closed-form rule for t_n. (d) State the limiting value of the sequence, and explain how the value of a determines it.

Q3. A wildlife reserve holds 800 native birds. Each year 80% of the birds remain in the reserve, and 120 more are introduced at the end of the year. Let P_n be the number of birds after n years. (a) Write a recurrence relation for P_n. (b) Generate the terms P_1, P_2 and P_3. (c) Find the equilibrium population L. (d) Describe the long-term behaviour of the sequence.

Q4. Imogen invests \$3000 in an account paying 4% per annum compounding annually, and deposits a further \$500 at the end of each year. Let V_n be the balance after n years. (a) Write a recurrence relation for V_n. (b) Find V_1 and V_2, correct to the nearest cent. (c) Show that the closed-form rule is $V_n = -12500 + 15500(1.04)^n$. (d) Find the balance after 5 years, correct to the nearest cent.

Q5. A car loan of \$8000 is charged interest at 1.5% per month, and \$300 is repaid at the end of each month. Let B_n be the balance owing after n months. (a) Write a recurrence relation for B_n. (b) Find B_1 and B_2. (c) Show that the closed-form rule is $B_n = 20000 - 12000(1.015)^n$. (d) The equilibrium of the recurrence is $L = 20000$. Explain what a balance of \$20000 would represent for this loan, and why the actual balance falls.

Q6. For each recurrence relation below, state a and b, find the equilibrium value L, and describe the long-term behaviour of the sequence. (a) $t_0 = 2$, $t_{n+1} = 0.9t_n + 2$. (b) $t_0 = 0$, $t_{n+1} = -0.75t_n + 14$. (c) $t_0 = 30$, $t_{n+1} = 1.2t_n - 4$. (d) $t_0 = 3$, $t_{n+1} = -t_n + 12$.

Unscaffolded

Q7. A sequence is defined by $t_0 = 6$, $t_{n+1} = 1.5t_n - 4$. Find the equilibrium value, generate the terms t_1 to t_4, obtain the closed-form rule, and describe the behaviour of the sequence.

Q8. A sequence is defined by $t_0 = 100$, $t_{n+1} = 0.6t_n + 20$. Find the closed-form rule, evaluate t_4, and state the limiting value.

Q9. A koala colony of 500 animals is managed so that each year 85% of the colony survives and 60 koalas are introduced. Let P_n be the number after n years. Write a recurrence relation, find the equilibrium population and a closed-form rule, find the population after 3 years, and state the long-term population.

Q10. The first four terms of a sequence generated by a recurrence of the form $t_{n+1} = at_n + b$ are $5, 17, 53, 161$. Find a and b, find the next term, and obtain the closed-form rule.

Q11. Marika invests \$5000 in a superannuation-style account paying 6% per annum compounding annually, and adds \$1500 at the end of each year. Let V_n be the balance after n years. Write a recurrence relation, find the closed-form rule, and find the balance after 8 years and the total interest earned, each correct to the nearest cent.

Q12. A loan of \$15000 is charged interest at 1% per month, and \$400 is repaid at the end of each month. Let B_n be the balance owing after n months. Find the closed-form rule and determine after how many months the loan is first fully repaid (that is, the first month in which the balance is zero or less).

Q13. A sequence is defined by $t_{n+1} = 4t_n - 9$. (a) Find the value of t_0 for which the sequence is constant. (b) For $t_0 = -2$, find t_1, t_2 and t_3, obtain the closed-form rule, and describe the behaviour of the sequence.

Q14. A sequence is defined by $t_0 = 20$, $t_{n+1} = -0.4t_n + 7$. Find the equilibrium value and the closed-form rule, evaluate t_3, and describe the long-term behaviour of the sequence.

Q15. A machine is bought for \$24000 and depreciates by 12% of its value each year on a reducing-balance basis. Let V_n be its value after n years. (a) Write a recurrence relation for V_n, and explain why it is the special case $b = 0$ of a first-order linear recurrence, stating its equilibrium value. (b) Write down the closed-form rule. (c) Determine after how many years the value first falls below \$10000.

Q16. Each year a lake loses 40% of a dissolved pollutant through natural outflow, while a nearby works discharges a further 90 kg. The lake starts with 600 kg of the pollutant. Let P_n be the mass, in kilograms, after n years. Write a recurrence relation, find the equilibrium mass and a closed-form rule, and determine after how many years the mass first falls below 250 kg.

Q17. Thanh invests \$4000 at 5% per annum compounding annually and adds \$600 at the end of each year. Let V_n be the balance after n years. Find the closed-form rule and determine after how many years the balance first reaches \$20000.

Q18. A car-share company manages a fleet in which, each year, 75% of the cars remain roadworthy and a fixed number F of new cars is added. Let P_n be the number of cars after n years. (a) The fleet size settles towards 8000 cars in the long run. Find F. (b) With F as in part (a) and $P_0 = 5000$, write the closed-form rule and find P_4. (c) Explain why the long-term fleet size is 8000 whatever the starting number of cars.

Answers

Q1. (a) $t_1=4$, $t_2=10$, $t_3=28$, $t_4=82$; (b) $a=3$, $b=-2$; (c) $L=1$; (d) $1+3^4=82$, which matches t_4. **Q2.** (a) $t_1=6.25$, $t_2=7.5625$, $t_3=7.890625$; (b) $L=8$; (c) $t_n=8-7(0.25)^n$; (d) limiting value 8, since $|a|=0.25<1$. **Q3.** (a) $P_0=800$, $P_{n+1}=0.8P_n+120$; (b) $P_1=760$, $P_2=728$, $P_3=702.4$; (c) $L=600$; (d) decreasing towards the limiting value 600. **Q4.** (a) $V_0=3000$, $V_{n+1}=1.04V_n+500$; (b) $V_1=\$3620$, $V_2=\$4264.80$; (c) shown; (d) $V_5=\$6358.12$. **Q5.** (a) $B_0=8000$, $B_{n+1}=1.015B_n-300$; (b) $B_1=\$7820$, $B_2=\$7637.30$; (c) shown; (d) at \$20000 the monthly interest (1.5% of \$20000 $=\$300$) equals the repayment, so the balance would hold; below it the repayment exceeds the interest, so the balance falls. **Q6.** (a) $a=0.9$, $b=2$, $L=20$; increasing towards 20. (b) $a=-0.75$, $b=14$, $L=8$; oscillating towards 8. (c) $a=1.2$, $b=-4$, $L=20$; since $t_0=30>20$ and $|a|>1$, increasing without bound. (d) $a=-1$, $b=12$, $L=6$; alternates $3,9,3,9,\ldots$, never settling. **Q7.** $L=8$; $t_1=5$, $t_2=3.5$, $t_3=1.25$, $t_4=-2.125$; $t_n=8-2(1.5)^n$; decreasing without bound (diverges from $L=8$). **Q8.** $t_n=50+50(0.6)^n$; $t_4=56.48$; limiting value 50. **Q9.** $P_0=500$, $P_{n+1}=0.85P_n+60$; $L=400$; $P_n=400+100(0.85)^n$; $P_3=461.4125$ (about 461 koalas); long-term population 400. **Q10.** $a=3$, $b=2$; next term 485; $t_n=-1+6(3)^n$. **Q11.** $V_0=5000$, $V_{n+1}=1.06V_n+1500$; $V_n=-25000+30000(1.06)^n$; $V_8=\$22815.44$; total interest $\$5815.44$. **Q12.** $B_n=40000-25000(1.01)^n$; the loan is first repaid in month 48 ($B_{47}\approx\$93.41>0$, $B_{48}\approx-\$305.65$). **Q13.** (a) $t_0=3$; (b) $t_1=-17$, $t_2=-77$, $t_3=-317$; $t_n=3-5(4)^n$; decreasing without bound. **Q14.** $L=5$; $t_n=5+15(-0.4)^n$; $t_3=4.04$; oscillating towards the limiting value 5. **Q15.** (a) $V_0=24000$, $V_{n+1}=0.88V_n$ ($b=0$, purely geometric; equilibrium $L=0$); (b) $V_n=24000(0.88)^n$; (c) after 7 years ($V_6\approx\$11145.70$, $V_7\approx\$9808.21$). **Q16.** $P_0=600$, $P_{n+1}=0.6P_n+90$; $L=225$; $P_n=225+375(0.6)^n$; first below 250 kg after 6 years ($P_5\approx254.16$, $P_6\approx242.50$). **Q17.** $V_n=-12000+16000(1.05)^n$; the balance first reaches \$20000 after 15 years ($V_{14}\approx\$19678.91$, $V_{15}\approx\$21262.85$). **Q18.** (a) $F=2000$; (b) $P_n=8000-3000(0.75)^n$, $P_4=7050.78125$ (about 7051 cars); (c) since $|0.75|<1$, the gap $-3000(0.75)^n$ shrinks to zero for any P_0, so $P_n\to 8000$.

Fully-worked solutions

Q1. (a) Iterating $t_{n+1}=3t_n-2$ from $t_0=2$: $t_1=3(2)-2=4$, $t_2=3(4)-2=10$, $t_3=3(10)-2=28$, $t_4=3(28)-2=82$. (b) Comparing with $t_{n+1}=at_n+b$ gives $a=3$, $b=-2$. (c) The equilibrium satisfies $L=3L-2$, so $-2L=-2$ and $L=1$. (d) $t_4=1+3^4=1+81=82$, which matches the value generated in part (a). (In general $t_n=L+(t_0-L)a^n=1+(2-1)3^n=1+3^n$.)

Q2. (a) $t_1=0.25(1)+6=6.25$; $t_2=0.25(6.25)+6=7.5625$; $t_3=0.25(7.5625)+6=7.890625$. (b) The equilibrium satisfies $L=0.25L+6$, so $0.75L=6$ and $L=8$. (c) $t_n=L+(t_0-L)a^n=8+(1-8)(0.25)^n=8-7(0.25)^n$. (d) Since $|a|=0.25<1$, the power $(0.25)^n\to 0$ as $n\to\infty$, so the gap $-7(0.25)^n$ shrinks to zero and $t_n\to 8$: the limiting value is the equilibrium $L=8$.

Q3. (a) Each year 80% of the birds remain (multiply by 0.8) and 120 are added, so $P_0=800$, $P_{n+1}=0.8P_n+120$. (b) $P_1=0.8(800)+120=760$; $P_2=0.8(760)+120=728$; $P_3=0.8(728)+120=702.4$. (c) The equilibrium satisfies $L=0.8L+120$, so $0.2L=120$ and $L=600$. (d) The closed form is $P_n=600+(800-600)(0.8)^n=600+200(0.8)^n$; since $|0.8|<1$ the gap shrinks to zero, so the number of birds decreases steadily towards the limiting value 600.

Q4. (a) The growth multiplier is $R=1+\dfrac{4}{100}=1.04$, and \$500 is added each year, so $V_0=3000$, $V_{n+1}=1.04V_n+500$. (b) $V_1=1.04(3000)+500=3620$; $V_2=1.04(3620)+500=4264.80$. (c) The equilibrium satisfies $L=1.04L+500$, so $-0.04L=500$ and $L=-12500$. Then $V_n=L+(V_0-L)R^n=-12500+\left(3000-(-12500)\right)(1.04)^n=-12500+15500(1.04)^n$. (d)

$V_5 = -12500 + 15500(1.04)^5 = -12500 + 15500(1.2166529\ldots) = 6358.119\ldots$, which is \$6358.12 to the nearest cent.

Q5. (a) Interest at 1.5% per month multiplies the balance by $R = 1.015$, and \$300 is repaid, so $B_0 = 8000$, $B_{n+1} = 1.015B_n - 300$. (b) $B_1 = 1.015(8000) - 300 = 8120 - 300 = 7820$; $B_2 = 1.015(7820) - 300 = 7937.30 - 300 = 7637.30$. (c) The equilibrium satisfies $L = 1.015L - 300$, so $-0.015L = -300$ and $L = 20000$. Then $B_n = L + (B_0 - L)R^n = 20000 + (8000 - 20000)(1.015)^n = 20000 - 12000(1.015)^n$. (d) A balance of \$20000 is the equilibrium: the interest for a month, 1.5% of \$20000, is exactly \$300, the same as the repayment, so a loan sitting at \$20000 would neither grow nor shrink. Because the actual balance \$8000 is below \$20000, its monthly interest is less than \$300, so each repayment more than covers the interest and the balance falls.

Q6. In each case the equilibrium solves $L = aL + b$, i.e. $L = \dfrac{b}{1-a}$ when $a \neq 1$, and the behaviour follows from the size and sign of a. (a) $a = 0.9$, $b = 2$, $L = \dfrac{2}{0.1} = 20$. Since $0 < a < 1$ and $t_0 = 2 < 20$, the terms increase steadily towards 20. (b) $a = -0.75$, $b = 14$, $L = \dfrac{14}{1.75} = 8$. Since $-1 < a < 0$, the terms approach 8 while alternating above and below it. (c) $a = 1.2$, $b = -4$, $L = \dfrac{-4}{-0.2} = 20$. Since $|a| > 1$ and $t_0 = 30 > 20$, the gap $(30 - 20)(1.2)^n$ grows, so the terms increase without bound. (d) $a = -1$, $b = 12$, $L = 6$. With $a = -1$ the rule gives $t_1 = -3 + 12 = 9$, $t_2 = -9 + 12 = 3$, and the terms alternate $3, 9, 3, 9, \ldots$ forever, never settling to $L = 6$.

Q7. The equilibrium satisfies $L = 1.5L - 4$, so $-0.5L = -4$ and $L = 8$. Iterating from $t_0 = 6$: $t_1 = 1.5(6) - 4 = 5$, $t_2 = 1.5(5) - 4 = 3.5$, $t_3 = 1.5(3.5) - 4 = 1.25$, $t_4 = 1.5(1.25) - 4 = -2.125$. The closed form is $t_n = 8 + (6 - 8)(1.5)^n = 8 - 2(1.5)^n$. Since $|1.5| > 1$ and $t_0 = 6 < 8$, the gap $-2(1.5)^n$ grows more negative, so the terms decrease without bound, moving away from the equilibrium 8.

Q8. The equilibrium satisfies $L = 0.6L + 20$, so $0.4L = 20$ and $L = 50$. Then $t_n = 50 + (100 - 50)(0.6)^n = 50 + 50(0.6)^n$, and $t_4 = 50 + 50(0.6)^4 = 50 + 50(0.1296) = 56.48$. Since $|0.6| < 1$, the gap shrinks to zero and the limiting value is $L = 50$.

Q9. Each year 85% survive and 60 are introduced, so $P_0 = 500$, $P_{n+1} = 0.85P_n + 60$. The equilibrium satisfies $L = 0.85L + 60$, so $0.15L = 60$ and $L = 400$. The closed form is $P_n = 400 + (500 - 400)(0.85)^n = 400 + 100(0.85)^n$, giving $P_3 = 400 + 100(0.85)^3 = 400 + 100(0.614125) = 461.4125$, that is, about 461 koalas. Because $|0.85| < 1$ the gap shrinks to zero, so the colony approaches a long-term population of 400.

Q10. Writing the rule for the first two steps, $t_1 = at_0 + b$ and $t_2 = at_1 + b$ give $5a + b = 17$ and $17a + b = 53$. Subtracting the first from the second, $12a = 36$, so $a = 3$, and then $b = 17 - 5(3) = 2$. (Check: $3(53) + 2 = 161$, as given.) The next term is $t_4 = 3(161) + 2 = 485$. The equilibrium is $L = \dfrac{2}{1-3} = -1$, so the closed form is $t_n = -1 + (5 - (-1))3^n = -1 + 6(3)^n$.

Q11. The growth multiplier is $R = 1.06$ and \$1500 is added each year, so $V_0 = 5000$, $V_{n+1} = 1.06V_n + 1500$. The equilibrium satisfies $L = 1.06L + 1500$, so $-0.06L = 1500$ and $L = -25000$; hence $V_n = -25000 + (5000 - (-25000))(1.06)^n = -25000 + 30000(1.06)^n$. After eight years, $V_8 = -25000 + 30000(1.06)^8 = -25000 + 30000(1.5938481\ldots) = 22815.442\ldots$, i.e. \$22815.44. Marika has contributed $5000 + 8 \times 1500 = \$17000$, so the interest earned is $22815.44 - 17000 = \$5815.44$.

Q12. Interest at 1% per month gives $R = 1.01$, and \$400 is repaid, so $B_{n+1} = 1.01B_n - 400$ with $B_0 = 15000$. The equilibrium satisfies $L = 1.01L - 400$, so $-0.01L = -400$ and $L = 40000$; hence $B_n = 40000 + (15000 - 40000)(1.01)^n = 40000 - 25000(1.01)^n$. The balance reaches zero when

$25000(1.01)^n=40000$, i.e. $(1.01)^n=1.6$, giving $n=\dfrac{\log_{10}1.6}{\log_{10}1.01}=\dfrac{0.20412\ldots}{0.00432\ldots}=47.2\ldots$. Since n must be a whole number of months, check either side: $B_{47}=40000-25000(1.01)^{47}\approx\$93.41>0$, while $B_{48}=40000-25000(1.01)^{48}\approx-\$305.65<0$. So the loan is first fully repaid in month 48 (the final repayment clears the small remaining balance).

Q13. (a) The sequence is constant exactly when it starts at the equilibrium. Solving $L=4L-9$ gives $-3L=-9$, so $L=3$; thus $t_0=3$ produces the constant sequence $3,3,3,\ldots$ (b) With $t_0=-2$: $t_1=4(-2)-9=-17$, $t_2=4(-17)-9=-77$, $t_3=4(-77)-9=-317$. The closed form is $t_n=3+(-2-3)(4)^n=3-5(4)^n$. Since $|4|>1$ and $t_0=-2<3$, the gap $-5(4)^n$ grows more negative, so the terms decrease without bound.

Q14. The equilibrium satisfies $L=-0.4L+7$, so $1.4L=7$ and $L=5$; hence $t_n=5+(20-5)(-0.4)^n=5+15(-0.4)^n$. Then $t_3=5+15(-0.4)^3=5+15(-0.064)=5-0.96=4.04$. Because $-1<-0.4<0$, the gap $15(-0.4)^n$ shrinks in size while changing sign each step, so the terms approach the limiting value 5, alternating above and below it.

Q15. (a) Reducing-balance depreciation keeps $100\%-12\%=88\%$ of the value each year, so $V_0=24000$, $V_{n+1}=0.88V_n$. Comparing with $t_{n+1}=at_n+b$ this is the case $a=0.88$, $b=0$: nothing is added, so the sequence is purely geometric, and its equilibrium is $L=\dfrac{0}{1-0.88}=0$. (b) With $b=0$ the closed form is simply $V_n=24000(0.88)^n$. (c) Setting $24000(0.88)^n<10000$ gives $(0.88)^n<0.41666\ldots$, so $n>\dfrac{\log_{10}0.41666\ldots}{\log_{10}0.88}=6.84\ldots$. Checking whole years, $V_6=24000(0.88)^6\approx\$11145.70$ (still above \$10000) and $V_7=24000(0.88)^7\approx\$9808.21$ (below \$10000), so the value first falls below \$10000 after 7 years.

Q16. Losing 40% each year keeps 60%, and 90 kg is added, so $P_0=600$, $P_{n+1}=0.6P_n+90$. The equilibrium satisfies $L=0.6L+90$, so $0.4L=90$ and $L=225$; hence $P_n=225+(600-225)(0.6)^n=225+375(0.6)^n$. Setting $225+375(0.6)^n<250$ gives $375(0.6)^n<25$, i.e. $(0.6)^n<0.06666\ldots$, so $n>\dfrac{\log_{10}0.06666\ldots}{\log_{10}0.6}=5.30\ldots$. Checking whole years, $P_5=225+375(0.6)^5\approx254.16$ kg (above 250) and $P_6=225+375(0.6)^6\approx242.50$ kg (below 250), so the mass first falls below 250 kg after 6 years.

Q17. The growth multiplier is $R=1.05$ and \$600 is added each year, so $V_{n+1}=1.05V_n+600$ with $V_0=4000$. The equilibrium satisfies $L=1.05L+600$, so $-0.05L=600$ and $L=-12000$; hence $V_n=-12000+(4000-(-12000))(1.05)^n=-12000+16000(1.05)^n$. The balance reaches \$20000 when $16000(1.05)^n=32000$, i.e. $(1.05)^n=2$, giving $n=\dfrac{\log_{10}2}{\log_{10}1.05}=14.2\ldots$. Checking whole years, $V_{14}\approx\$19678.91$ (below \$20000) and $V_{15}\approx\$21262.85$ (above \$20000), so the balance first reaches \$20000 after 15 years.

Q18. (a) The fleet obeys $P_{n+1}=0.75P_n+F$. Its long-term size is the equilibrium $L=\dfrac{F}{1-0.75}=\dfrac{F}{0.25}=4F$. Setting $4F=8000$ gives $F=2000$. (b) With $F=2000$ the equilibrium is $L=8000$, so $P_n=8000+(5000-8000)(0.75)^n=8000-3000(0.75)^n$, giving $P_4=8000-3000(0.75)^4=8000-3000(0.31640625)=7050.78125$, that is, about 7051 cars. (c) The starting value affects only the gap $(P_0-8000)(0.75)^n$. Since $|0.75|<1$, this gap shrinks to zero as n grows no matter what P_0 is, so every solution approaches the same long-term fleet size of 8000 cars.