# Specialist Mathematics Units 1 & 2

## Chapter 2 — Graph theory



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### Contents

## Theory

Much of mathematics is about *quantity* — how large, how many, how fast. Graph theory is about something more basic: **which things are connected to which**. Strip away every measurement from a road map, a molecule or a friendship group and keep only the pattern of connections, and what remains is a **graph**. The subject was born in 1736 when Euler settled a puzzle about the bridges of Königsberg using nothing but this pattern; it is now the language of networks throughout science. This chapter builds the theory carefully, and — this being Specialist Mathematics — proves the results rather than merely asserting them.

**Graphs, vertices and edges.**

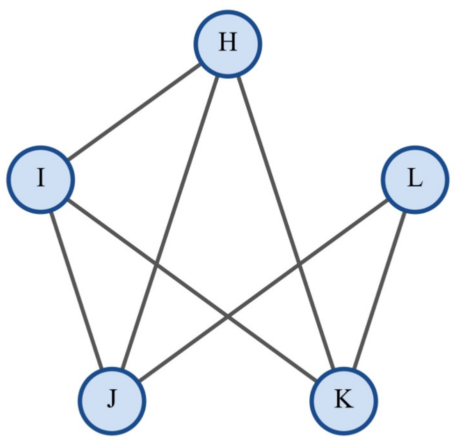
A **graph** $G$ consists of a finite set of **vertices** (singular **vertex**) together with a set of **edges**, each edge joining a pair of vertices. We write

$$G=(V,E),$$

where $V$ is the set of vertices and $E$ the set of edges. The number of vertices, $n=|V|$, is the **order** of the graph, and the number of edges is written $e$. Drawn on the page, each vertex is a dot and each edge is a line joining two dots. In this chapter every edge is **undirected**: an edge joining $u$ and $v$ may be travelled in either direction, so the pair of vertices it joins is *unordered* — $\{u,v\}$ and $\{v,u\}$ name the same edge. Two vertices joined by an edge are said to be **adjacent**, and the edge is **incident** with each of its **end vertices**.

Only the connections matter. Where a vertex is placed on the page, and whether an edge is drawn straight or curved, carry no meaning whatsoever — the *same* graph may be drawn in many different-looking ways. This is why graphs model so much:

- a **social network**, whose vertices are people and whose edges join pairs who are friends;
- a **molecule**, whose vertices are atoms and whose edges are chemical bonds;
- a **utility network**, whose vertices are junctions and whose edges are the pipes or cables between them.




The diagram above is a small friendship network on five people $H,I,J,K,L$. Reading it off, $H$ is adjacent to $I$, $J$ and $K$; the edge $\{J,L\}$ is incident with $J$ and with $L$; and $H$ and $L$ are not adjacent.

**Loops and multiple edges.**

Two special features are allowed. An edge that joins a vertex to *itself* is a **loop**. A loop appears whenever an object is related to itself — a road that leaves a town and returns to it, or a chemical group bonded back to its own atom. When a pair of vertices is joined by *more than one* edge, those edges are called **multiple edges** (or **parallel edges**). Multiple edges are entirely natural: in a molecule a **double bond** joins two atoms by two edges, and a utility map may run two separate cables between the same pair of junctions.

A graph with **no loops and no multiple edges** is called a **simple graph**. In a simple graph every edge joins two *distinct* vertices, and any two vertices are joined by *at most one* edge, so an edge is completely determined by its ends and the name $\{u, v\}$ picks out exactly one edge. Once loops or multiple edges are admitted this is no longer so — two parallel edges have the same pair of ends, and a loop has both of its ends at one vertex — so in general an edge is an object in its own right, named by its ends only when those ends pick it out unambiguously. The friendship network above is simple; the double bonds and loops met later in this section are not.

**Representing a graph: diagram, edge list and adjacency matrix.**

A diagram is vivid but imprecise, and useless to a computer. The same graph can be recorded in two exact ways.

The **edge list** simply lists every edge as a pair of vertices. For the friendship network above,

$$E = \{HI, HJ, HK, IJ, IK, JL, KL\},$$

writing $HI$ for the edge $\{H, I\}$. The order of the list, and the order of the two letters in each pair, do not matter.

The **adjacency matrix** stores the same information as a square array of numbers. Label the vertices $1, 2, \ldots, n$ in a fixed order; for $i \neq j$ the entry in **row $i$, column $j$** is the *number of edges joining vertex $i$ to vertex $j$*. Because an edge between $i$ and $j$ is equally an edge between $j$ and $i$, the adjacency matrix of an undirected graph is always **symmetric**: it equals its own reflection in the leading diagonal, so the entry in row $i$, column $j$ equals the entry in row $j$, column $i$. Two conventions complete the definition and record the special features:

- a **loop** at vertex $i$ is written as a $2$ in the diagonal entry (row $i$, column $i$), and a vertex carrying no loop has a $0$ there — the reason for the 2 appears under *degree* below;
- $r$ **multiple edges** between $i$ and $j$ are recorded as the number $r$.

For the friendship network, ordering the vertices $H, I, J, K, L$,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Row $H$ reads $0, 1, 1, 1, 0$: no loop at $H$ (the diagonal $0$), and single edges to $I$, $J$ and $K$. A **simple** graph therefore has an adjacency matrix that is symmetric, with only $0$s on the leading diagonal and only $0$s and $1$s elsewhere.

**Reading a graph from its adjacency matrix** reverses the process. Given the array, a $1$ in row $i$, column $j$ says *draw one edge between vertices $i$ and $j$*; a $2$ on the diagonal says *draw a loop*; a number $r$ off the diagonal says *draw $r$ parallel edges*. The matrix and the drawing carry identical information — the matrix is just the drawing written down.

**The degree of a vertex.**

The **degree** of a vertex $v$, written $deg(v)$, is the number of **edge-ends** meeting at $v$. Each ordinary edge has two ends and so contributes 1 to the degree of each of its two end vertices. A **loop contributes** $2$ to the degree of its vertex, because *both* of its ends are attached there — this is exactly why a loop is recorded as a $2$ on the diagonal. A vertex of degree $0$, joined to nothing, is **isolated**.

In the adjacency matrix the degree is read straight off:

- the entries across **row $i$ add up to** $deg(v_i)$, since they count the edge-ends at vertex $i$ (a loop's diagonal $2$ correctly adds 2); and
- the sum of **all** the entries of the matrix equals $2e$.

For the friendship network the row sums are $3, 3, 3, 3, 2$, so $deg(H) = deg(I) = deg(J) = deg(K) = 3$ and $deg(L) = 2$. In a social network the degree is a person's number of friends; in a molecule it is an atom's **valence** (carbon $4$, oxygen $2$, hydrogen $1$); in a utility network it is the number of cables meeting at a junction.

**The handshaking lemma.**

Add up the degrees of *every* vertex of a graph. The result is always exactly twice the number of edges. This is the **handshaking lemma**.

> **Handshaking lemma.** For any graph with $e$ edges,

$$\sum_{v \in V} deg(v) = 2e.$$

*Proof.* Count, in two different ways, the total number of **edge-ends** in the graph. Every edge has two ends, each of them attached to a vertex, and the two ends of an edge are counted as two even when they are attached to the same vertex. Call this total $N$.

Count first **by vertices**. The number of edge-ends attached to $v$ is, by the definition of degree, exactly $deg(v)$ (a loop supplies two such ends at its vertex, and the degree already counts both). Every edge-end is attached to exactly one vertex, so adding these counts over all vertices counts each edge-end once and once only:

$$N = \sum_{v \in V} deg(v).$$

Count next **by edges**. Every edge has exactly two ends — an ordinary edge joining $u$ and $v$ has one end at $u$ and one end at $v$, while a loop at $w$ has both of its ends at $w$. Every edge-end belongs to exactly one edge, so adding these counts over all $e$ edges again counts each edge-end once and once only:

$$N = 2e.$$

The two counts are of the same quantity $N$, so they are equal: $\sum_{v \in V} deg(v) = 2e$, as required.

The lemma is worth its weight because of what it forbids. Since $2e$ is an even number, **the sum of all the degrees in any graph is even** — no graph can have degrees adding to an odd total. And since the degree sum is $2e$, **the number of edges is exactly half the degree sum**, a fact used constantly to count edges without drawing anything.

**A consequence: odd-degree vertices come in even numbers.**

Call a vertex **odd** if its degree is odd and **even** if its degree is even. The lemma pins down how many odd vertices a graph can have.

> **Consequence.** In any graph the number of odd-degree vertices is even.

*Proof.* Split the vertices into the set $V_{\text{odd}}$ of odd-degree vertices and the set $V_{\text{even}}$ of even-degree vertices. Splitting the degree sum the same way,

$$\sum_{v \in V_{\text{odd}}} deg(v) + \sum_{v \in V_{\text{even}}} deg(v) = \sum_{v \in V} deg(v) = 2e.$$

The right-hand side is even. In the left-hand side, every term of the second sum is an even number, so that sum is even; subtracting it from the even total $2e$ leaves

$$\sum_{v \in V_{\text{odd}}} deg(v) = 2e - \sum_{v \in V_{\text{even}}} deg(v),$$

which is even. But that first sum is a sum of **odd** numbers, and a sum of odd numbers is even exactly when there is an **even count** of them. Hence the number of vertices in $V_{\text{odd}}$ is even, as required.

This is the graph behind the old chestnut that *at any party the number of people who have shaken hands an odd number of times is even*. It is also a quick impossibility test: a proposed list of degrees with an odd number of odd entries — or with an odd total — describes no graph at all.

**The standard problems of graph theory.**

The questions this subject asks — here and in the sections that follow — fall into four broad kinds, and it is worth naming them at the outset. An **existence problem** asks *whether* something is possible: is there a path joining two given vertices, does a closed route exist that uses every edge exactly once (an **Euler circuit**), or could a graph with a prescribed list of degrees exist at all? The handshaking lemma just proved is precisely such a tool, settling the last of these without drawing anything. A **construction problem** asks us to *build* a required object — to construct a **spanning tree** that links every vertex, or to produce a **colouring** of the vertices in which no edge joins two of the same colour. A **counting problem** asks *how many*: how many edges a network has, how many distinct paths run between two vertices, or how many spanning trees it contains. For instance, in a graph on five vertices in which *every* pair is joined, each vertex is adjacent to the other four, so every degree is $4$; by the handshaking lemma the number of edges is then $\frac{5\times 4}{2}=10$. Finally, an **optimisation problem** asks for the *best* among many possibilities — the shortest route between two towns, or the **minimum spanning tree** that connects a network for the least total cost. Every later section of this chapter is, at heart, an instance of one of these four.

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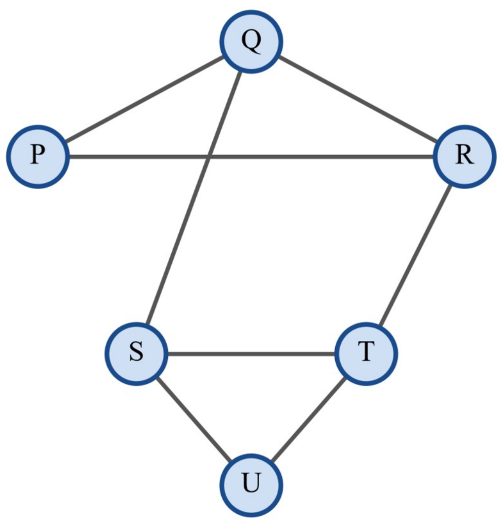
## Worked examples

### *Example 1 — scaffolded*

The diagram shows a reading app on which six members $P, Q, R, S, T, U$ have mutually "followed" one another (an edge joins two members who each follow the other).




Find the degree of each member, the sum of the degrees and the number of edges, verify the handshaking lemma, and confirm that the number of odd-degree vertices is even.

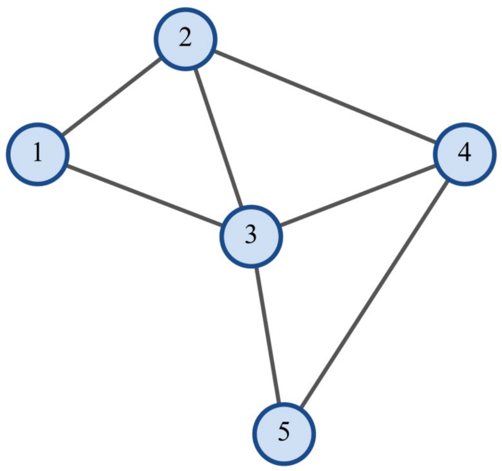
| Working | Comment |
|---|---|
| $deg(P)=2$, $deg(Q)=3$, $deg(R)=3$, $deg(S)=3$, $deg(T)=3$, $deg(U)=2$. | Count the edge-ends meeting each vertex in the diagram. |
| Sum of degrees $=2+3+3+3+3+2=16$. | Add the six degrees. |
| Number of edges $e=\frac{16}{2}=8$. | The handshaking lemma: $e$ is half the degree sum. |

| Working | Comment |
|---|---|
| The edges are $PQ$, $PR$, $QR$, $QS$, $RT$, $ST$, $SU$, $TU$ — exactly $8$. | Counting the edges confirms $\sum deg(v) = 2e = 16$. |
| Odd-degree vertices: $Q, R, S, T$ — that is $4$ of them. | Four is even, as the consequence guarantees. |

So the degrees are $2, 3, 3, 3, 3, 2$ across $8$ edges, the degree sum $16$ equals $2e$, and there are $4$ (an even number of) odd-degree vertices.

### *Example 2 — scaffolded*

A monitoring system links five sensors $1, 2, 3, 4, 5$ by cables as shown.




Write down the adjacency matrix (ordering the vertices $1, 2, 3, 4, 5$), use its row sums to state the degree of each sensor, and use the total of its entries to find the number of cables.

| Working | Comment |
|---|---|
| Row 1: sensor $1$ joins $2$ and $3$, giving $0, 1, 1, 0, 0$. | Row $i$ records the number of edges from vertex $i$; no loop, so the diagonal entry is $0$. |
| $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$. | The matrix is symmetric — each cable from $i$ to $j$ is also from $j$ to $i$. |
| Row sums: $deg(1) = 2$, $deg(2) = 3$, $deg(3) = 4$, $deg(4) = 3$, $deg(5) = 2$. | Each row sum is that sensor's degree. |
| Sum of all entries $= 2 + 3 + 4 + 3 + 2 = 14$, so $e = \dfrac{14}{2} = 7$. | The entries total $2e$; halving gives the number of cables. |

So sensor $3$ has the highest degree, $4$; the degrees are $2, 3, 4, 3, 2$; and there are $7$ cables (matching the seven lines in the diagram).

### *Example 3 — unscaffolded*

In a molecule of methanal (formaldehyde, HCHO) a central carbon atom $C$ is joined to an oxygen atom $O$ by a **double bond** and to two hydrogen atoms $H_1$, $H_2$ by single bonds. Model the molecule as a graph, give its edge list and adjacency matrix, state the degree (valence) of each atom, verify the handshaking lemma, and explain why the graph is not simple.

The double bond is a pair of **multiple edges** between $C$ and $O$, so the edge list is

$$CO, CO, CH_1, CH_2,$$

with $CO$ listed twice — once for each of the two edges making up the double bond, which share the same pair of ends. Ordering the vertices $C, O, H_1, H_2$ and recording the double bond as a $2$,

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The row sums give the valences $deg(C) = 4$, $deg(O) = 2$, $deg(H_1) = 1$ and $deg(H_2) = 1$ — exactly the chemical valences of carbon, oxygen and hydrogen. The degree sum is $4 + 2 + 1 + 1 = 8$, and there are $e = 4$ bonds (the two edges of the double bond and the two single bonds), so $\sum deg = 8 = 2 \times 4 = 2e$ and the handshaking lemma holds. The graph is not simple because the double bond is a pair of multiple edges between $C$ and $O$.

∴ the atoms have degrees $deg(C) = 4$, $deg(O) = 2$, $deg(H_1) = deg(H_2) = 1$ across $4$ bonds, with $\sum deg = 8 = 2e$; the graph is not simple because $C$ and $O$ carry multiple edges.

### *Example 4 — unscaffolded*

Five guests at a dinner shake hands with one another. Afterwards it is claimed that the numbers of hands each guest shook were $4, 3, 3, 2$ and $1$. Using the handshaking lemma and its consequence, decide whether this claim can be correct.

Model the guests as vertices and each handshake as an edge; then the number of hands a guest shook is that vertex's degree. Two independent checks both fail. The degree sum would be

$$4 + 3 + 3 + 2 + 1 = 13,$$

which is **odd**; but the handshaking lemma forces the degree sum to equal $2e$, an even number, so no graph can have these degrees. Equally, the odd degrees in the list are $3, 3$ and $1$ — **three** of them — whereas the consequence requires the number of odd-degree vertices to be even. Either check alone rules the claim out.

∴ the claim is impossible: the degrees sum to the odd number $13$ (and there is an odd number of odd degrees), contradicting the handshaking lemma.

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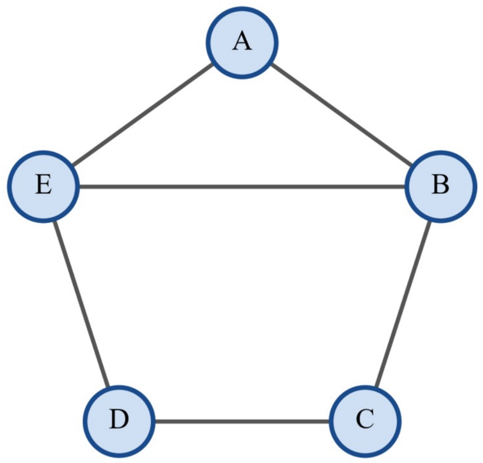
## Practice questions

### *Scaffolded*

**Q1.** The diagram shows a co-authorship network in which five researchers $A, B, C, D, E$ are joined whenever they have written a paper together.




(a) Write down the degree of each researcher.
(b) Find the sum of the degrees.
(c) Use the handshaking lemma to find the number of edges.
(d) List the odd-degree vertices and confirm that there is an even number of them.

**Q2.** A molecule of ammonia ($NH_3$) has a central nitrogen atom $N$ joined by single bonds to three hydrogen atoms $H_1$, $H_2$, $H_3$. (a) Write down the adjacency matrix, ordering the vertices $N, H_1, H_2, H_3$. (b) Use the row sums to state the valence (degree) of each atom. (c) Verify the handshaking lemma for the molecule. (d) Explain how the matrix shows that this graph is simple.

**Q3.** A molecule of carbon dioxide ($CO_2$) is the chain $O_1{=}C{=}O_2$, in which the carbon $C$ is joined to each oxygen by a **double bond**. (a) Explain why each double bond is a pair of multiple edges, and write down the adjacency matrix ordering the vertices $O_1, C, O_2$. (b) State the degree of each atom from the row sums. (c) Find the number of bonds and verify the handshaking lemma. (d) State, with a reason, whether the graph is simple, and how many odd-degree vertices it has.

**Q4.** Use the handshaking lemma throughout. (a) A graph has degrees $5, 4, 4, 3, 2$. How many edges does it have? (b) A student claims to have drawn a graph with degrees $3, 3, 2, 1$. Explain why no such graph exists. (c) For the valid graph in part (a), count the odd-degree vertices and confirm the count is even.

**Q5.** A road map on four outback towns $1, 2, 3, 4$ has adjacency matrix

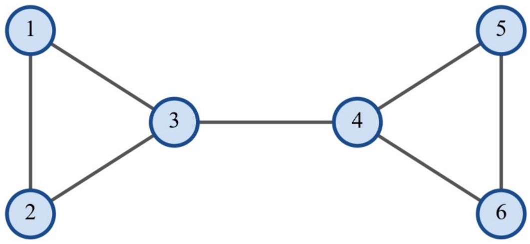
$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

(a) Is the graph simple? Identify any loop and any multiple edge. (b) Use the row sums to state the degree of each town. (c) Find the number of roads from the total of the entries. (d) Verify the handshaking lemma and confirm that the number of odd-degree towns is even.

**Q6.** In a research team of five people $A, B, C, D, E$, the pairs who have worked together are $AB$, $AC$, $AE$, $BC$, $BD$, $CD$ and $DE$. (a) Write down the adjacency matrix, ordering the vertices $A, B, C, D, E$. (b) State the degree of each person from the row sums. (c) Find the number of edges and verify the handshaking lemma. (d) List the odd-degree vertices and confirm there is an even number of them.

### *Unscaffolded*

**Q7.** The diagram shows six junctions $1, \ldots, 6$ of a town water-supply grid joined by pipes.




Find the degree of each junction, the sum of the degrees and the number of pipes, and verify the handshaking lemma.

**Q8.** A molecule of propane ($C_3H_8$) has its three carbon atoms $C_1, C_2, C_3$ joined in a chain $C_1 - C_2 - C_3$ by single bonds, with three hydrogen atoms bonded to each end carbon and two to the middle carbon, all by single bonds. Taking the atoms as vertices and the bonds as edges, state the degree of each atom, find the number of bonds, and verify the handshaking lemma. Hence explain why the number of hydrogen atoms must be even.

**Q9.** Prove that in any gathering of people who shake hands, the number of people who shake hands an odd number of times is even. (Model the gathering as a graph and use the handshaking lemma.)

**Q10.** In a warehouse, two automated tracks run between charging stations $X$ and $Y$, single tracks run $X - Z$ and $Y - Z$, and a maintenance loop leaves $Z$ and returns to $Z$. (a) Find the degree of each of $X$, $Y$, $Z$, remembering that a loop counts 2. (b) Write down the adjacency matrix, ordering the vertices $X, Y, Z$, and state the number of tracks. (c) Verify the handshaking lemma, and state why the graph is not simple.

**Q11.** A molecule of ethene ($C_2H_4$) has its two carbon atoms $C_1$, $C_2$ joined by a **double bond**, and each carbon is also joined by single bonds to two hydrogen atoms ($H_1, H_2$ on $C_1$; $H_3, H_4$ on $C_2$). Write down the adjacency matrix ordering the vertices $C_1, C_2, H_1, H_2, H_3, H_4$, state the degree of each atom, find the number of bonds, and identify the multiple edge.

**Q12.** A network engineer wants seven servers each connected directly to exactly three other servers. Using the handshaking lemma, explain why this is impossible.

**Q13.** A graph has $6$ vertices and $9$ edges. Five of the vertices have degrees $4, 3, 3, 2$ and $2$. Find the degree of the sixth vertex.

**Q14.** A communications network on five nodes $1, 2, 3, 4, 5$ has adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

List the edges the matrix describes, state the degree of each node, find the number of links, and state — with a reason from the matrix — whether the network is simple.

**Q15.** An electricity grid of six pylons has degree sequence $4, 3, 3, 3, 3, 2$. Find the number of cables, and confirm that the number of odd-degree pylons is even.

**Q16.** In a hydrocarbon, every carbon atom has degree $4$ and every hydrogen atom has degree $1$. Using the handshaking lemma, prove that the number of hydrogen atoms in the molecule must be even.

**Q17.** A graph has $6$ vertices and $8$ edges. Two of the vertices have degree $4$, and the remaining four vertices all have the same degree $d$. Find $d$.

**Q18.** (a) A club claims that its "plays-chess-with" graph on six members has degrees $4, 4, 4, 3, 2, 2$. Use the handshaking lemma to show that this is impossible. (b) Prove that if **every** vertex of a graph has odd degree, then the graph has an even number of vertices.

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### Answers

**Q1.** (a) $deg(A)=2$, $deg(B)=3$, $deg(C)=2$, $deg(D)=2$, $deg(E)=3$. (b) 12. (c) 6 edges. (d) Odd: $B, E$ — two, an even number. **Q2.** (a) $\begin{bmatrix}0&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\end{bmatrix}$. (b) $deg(N)=3$, $deg(H_1)=deg(H_2)=deg(H_3)=1$. (c) $\sum deg=6=2\times 3=2e$. (d) The diagonal is all 0 (no loops) and every entry is 0 or 1 (no multiple edges). **Q3.** (a) A double bond joins the same two atoms by two edges; $\begin{bmatrix}0&2&0\\2&0&2\\0&2&0\end{bmatrix}$. (b) $deg(O_1)=2$, $deg(C)=4$, $deg(O_2)=2$. (c) 4 bonds; $\sum deg=8=2\times 4$. (d) Not simple (multiple edges $C-O_1$ and $C-O_2$); 0 odd-degree vertices, which is even. **Q4.** (a) $e=9$. (b) The degrees sum to 9, which is odd, but every degree sum equals $2e$ (even). (c) Odd-degree vertices: 5 and 3 — two, an even number. **Q5.** (a) Not simple — a loop at town 4 (diagonal 2) and a multiple edge between towns 1 and 3 (entry 2). (b) $deg(1)=deg(2)=deg(3)=deg(4)=3$. (c) 6 roads. (d) $\sum deg=12=2\times 6$; four odd-degree towns, an even number. **Q6.** (a) $\begin{bmatrix}0&1&1&0&1\\1&0&1&1&0\\1&1&0&1&0\\0&1&1&0&1\\1&0&0&1&0\end{bmatrix}$. (b) $deg(A)=deg(B)=deg(C)=deg(D)=3$, $deg(E)=2$. (c) $e=7$; $\sum deg=14=2\times 7$. (d) Odd: $A, B, C, D$ — four, an even number. **Q7.** Degrees $2, 2, 3, 3, 2, 2$; sum 14; 7 pipes ($\sum deg=14=2\times 7$). **Q8.** $deg(C_1)=deg(C_2)=deg(C_3)=4$ and each of the eight hydrogens has degree 1; $\sum deg=20$, so $e=10$ bonds. The hydrogen atoms are the odd-degree (degree 1) vertices, and by the consequence their number is even — here 8. **Q9.** See solution — the odd-degree vertices number evenly by the handshaking lemma. **Q10.** (a) $deg(X)=3$, $deg(Y)=3$, $deg(Z)=4$. (b) $\begin{bmatrix}0&2&1\\2&0&1\\1&1&2\end{bmatrix}$; 5 tracks. (c) $\sum deg=10=2\times 5$; not simple (loop at $Z$, multiple edge $X-Y$). **Q11.** $\begin{bmatrix}0&2&1&1&0&0\\2&0&0&0&1&1\\1&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{bmatrix}$; $deg(C_1)=deg(C_2)=4$, each hydrogen degree 1; 6 bonds; the multiple edge is the double bond $C_1-C_2$. **Q12.** Impossible: $7\times 3=21$ is odd, but the degree sum must equal $2e$ (even). (Also, seven odd-degree vertices is an odd count.) **Q13.** Degree 4. **Q14.** Edges $1-2$, $1-4$, $1-5$, $2-3$ (double), $3-4$, $4-5$; degrees $3, 3, 3, 3, 2$; 7 links; not simple, because the entry 2 (nodes 2 and 3) is a multiple edge. **Q15.** 9 cables; the odd-degree pylons are the four of degree 3 — an even number. **Q16.** See solution — $4c+h=2e$ forces $h$ even. **Q17.** $d=2$. **Q18.** (a) $4+4+4+3+2+2=19$ is odd, but the degree sum equals $2e$ (even), so no such graph exists. (b) See solution.

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### Fully-worked solutions

**Q1.** Reading the diagram, $A$ joins $B$ and $E$; $B$ joins $A$, $C$ and $E$; $C$ joins $B$ and $D$; $D$ joins $C$ and $E$; $E$ joins $A$, $B$ and $D$. Hence $deg(A)=2$, $deg(B)=3$, $deg(C)=2$, $deg(D)=2$, $deg(E)=3$, with sum $2+3+2+2+3=12$. By the handshaking lemma $e=12/2=6$ (matching the six edges $AB$, $BC$, $CD$, $DE$, $EA$, $BE$). The odd-degree vertices are $B$ and $E$ — two of them, an even number, as the consequence requires.

**Q2.** (a) With $N$ joined to $H_1$, $H_2$, $H_3$ and no other edges,

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The row sums are $deg(N)=3$ and $deg(H_1)=deg(H_2)=deg(H_3)=1$. (c) The degree sum is $3+1+1+1=6$, and there are $e=3$ bonds, so $\sum deg=6=2\times 3=2e$. (d) The leading diagonal is all $0$, so there are no loops, and every off-diagonal entry is $0$ or $1$, so there are no multiple edges; the graph is therefore simple.

**Q3.** (a) A double bond joins the same pair of atoms by two separate bonds, so each is a pair of multiple edges; recording each double bond as a 2,

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}.$$

(b) The row sums give $deg(O_1)=2$, $deg(C)=4$, $deg(O_2)=2$ — the valences of oxygen and carbon. (c) There are $e=4$ bonds (two per double bond), and $\sum deg=2+4+2=8=2\times 4=2e$. (d) The graph is not simple, because $C$ carries multiple edges to each oxygen. Every degree is even, so there are $0$ odd-degree vertices — and $0$ is even, so the consequence still holds.

**Q4.** (a) By the handshaking lemma the degree sum equals $2e$; here $5+4+4+3+2=18$, so $e=18/2=9$. (b) The degrees $3, 3, 2, 1$ sum to 9, which is odd. Since every degree sum must equal $2e$, an even number, no graph can have these degrees. (c) In part (a) the odd degrees are 5 and 3 — two odd-degree vertices, which is an even number, consistent with the consequence.

**Q5.** (a) The graph is not simple: the diagonal entry 2 in row 4 is a loop at town $4$, and the entry 2 in row 1, column 3 (and its mirror) is a multiple edge — two roads between towns 1 and 3. (b) The row sums are $deg(1)=0+1+2+0=3$, $deg(2)=1+0+1+1=3$, $deg(3)=2+1+0+0=3$ and $deg(4)=0+1+0+2=3$. (c) The entries total $3+3+3+3=12=2e$, so $e=6$ roads (the double road $1-3$, single roads $1-2$, $2-3$, $2-4$, and the loop at 4). (d) The degree sum 12 equals $2\times 6$, verifying the lemma; all four towns are odd-degree, and four is even.

**Q6.** (a) The collaborations $AB$, $AC$, $AE$, $BC$, $BD$, $CD$, $DE$ give

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(b) The row sums are $deg(A)=3$, $deg(B)=3$, $deg(C)=3$, $deg(D)=3$, $deg(E)=2$. (c) The degree sum is $3+3+3+3+2=14$, so $e=14/2=7$, matching the seven listed pairs; thus $\sum deg=14=2\times 7=2e$. (d) The odd-degree vertices are $A, B, C, D$ — four of them, an even number.

**Q7.** From the diagram, junctions 1 and 2 each join two pipes, and 5 and 6 each join two pipes; junction 3 joins 1, 2 and 4, and junction 4 joins 3, 5 and 6. Hence $deg(1)=2$, $deg(2)=2$, $deg(3)=3$, $deg(4)=3$, $deg(5)=2$, $deg(6)=2$, with sum $2+2+3+3+2+2=14$. By the handshaking lemma the number of pipes is $e=14/2=7$, and $\sum deg=14=2\times 7=2e$ confirms the count.

**Q8.** Each carbon is bonded four times — $C_1$ to $C_2$ and to its three hydrogens, $C_2$ to $C_1$, $C_3$ and its two hydrogens, $C_3$ to $C_2$ and its three hydrogens — so $deg(C_1)=deg(C_2)=deg(C_3)=4$, while each of the eight hydrogens has degree 1. The degree sum is $3\times 4+8\times 1=12+8=20$, so the number of bonds is $e=20/2=10$ (two carbon–carbon bonds and eight carbon–hydrogen bonds), and $\sum deg=20=2\times 10=2e$. The hydrogen atoms are precisely the odd-degree (degree 1) vertices; by the consequence of the handshaking lemma their number must be even — and indeed there are 8.

**Q9.** Model the gathering as a graph: one vertex per person, and one edge for each handshake between two people. The number of times a person shakes hands is then the degree of the corresponding vertex. By the handshaking lemma the sum of all the degrees equals $2e$, where $e$ is the number of handshakes, so the degree sum is even. Splitting it into the sum over odd-degree vertices and the sum over even-degree vertices, the even-degree part is a sum of even numbers and so is even; subtracting it from the even total leaves the odd-degree part even as well. But that part is a sum of odd numbers, which is even only when there is an even number of them. Hence the number of people who shook hands an odd number of times is even, as required.

**Q10.** (a) At $X$ the two $X-Y$ tracks contribute 2 and the $X-Z$ track contributes 1, so $deg(X)=3$; similarly $deg(Y)=3$. At $Z$ the $X-Z$ and $Y-Z$ tracks contribute 1 each and the loop contributes 2, so $deg(Z)=4$. (b) Recording the double track as a 2 and the loop as a 2 on the diagonal,

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

and the tracks are the two $X-Y$, the $X-Z$, the $Y-Z$ and the loop — $e=5$ in all. (c) The degree sum is $3+3+4=10=2\times 5=2e$, verifying the lemma. The graph is not simple, because it has a loop (at $Z$) and multiple edges (between $X$ and $Y$).

**Q11.** The double bond $C_1=C_2$ is a pair of multiple edges, recorded as a 2, and each carbon has two single bonds to hydrogens:

$$\begin{bmatrix} 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The row sums give $deg(C_1)=2+1+1=4$ and likewise $deg(C_2)=4$, while each hydrogen has degree 1. The degree sum is $4+4+1+1+1+1=12$, so there are $e=6$ bonds. The only multiple edge is the double bond between $C_1$ and $C_2$ (the entry 2).

**Q12.** If seven servers each had degree 3, the degree sum would be $7\times 3=21$. By the handshaking lemma the degree sum must equal $2e$, which is even, but 21 is odd — a contradiction. (Equivalently, seven vertices would all be odd-degree, an odd count, whereas the number of odd-degree vertices must be even.) So no such network of servers can exist.

**Q13.** By the handshaking lemma the degree sum equals $2e=2\times 9=18$. The five known degrees add to $4+3+3+2+2=14$, so the sixth degree is $18-14=4$.

**Q14.** Reading the matrix, the off-diagonal 1s give the single edges $1-2$, $1-4$, $1-5$, $3-4$ and $4-5$, while the entry 2 in row 2, column 3 is a double edge $2-3$. The row sums are $deg(1)=3$, $deg(2)=3$, $deg(3)=3$, $deg(4)=3$, $deg(5)=2$, summing to 14, so the number of links is $e=14/2=7$. The network is not simple: the entry 2 (between nodes 2 and 3) is a multiple edge, so the matrix has an off-diagonal entry exceeding 1.

**Q15.** By the handshaking lemma the number of cables is half the degree sum: $\sum deg=4+3+3+3+3+2=18$, so $e=18/2=9$. The odd-degree pylons are the four with degree 3, and four is even — as the consequence of the lemma guarantees.

**Q16.** Let the molecule have $c$ carbon atoms, each of degree 4, and $h$ hydrogen atoms, each of degree 1 (lower-case letters, since these are counts of atoms and not the atoms themselves). By the handshaking lemma the degree sum equals $2e$, where $e$ is the number of bonds:

$$4c+h=2e.$$

The right-hand side $2e$ is even, and $4c$ is even, so $h = 2e - 4c$ is a difference of two even numbers and is therefore even. Hence the number of hydrogen atoms must be even, as required.

**Q17.** By the handshaking lemma the degree sum equals $2e = 2 \times 8 = 16$. The two degree-$4$ vertices contribute $2 \times 4 = 8$, leaving $16 - 8 = 8$ to be shared equally among the remaining four vertices, so $4d = 8$ and $d = 2$.

**Q18.** (a) The proposed degrees sum to $4 + 4 + 4 + 3 + 2 + 2 = 19$, which is odd. The handshaking lemma requires the degree sum to equal $2e$, an even number, so no graph can have these degrees and the claim is impossible. (b) Suppose every vertex of the graph has odd degree. By the consequence of the handshaking lemma, the number of odd-degree vertices is even; but here *every* vertex is odd-degree, so the number of odd-degree vertices is the total number of vertices. Therefore the number of vertices is even, as required.

## Theory

In 2A a graph was built from **vertices** and **edges**, the **degree** of a vertex was defined, and the **handshaking lemma** $d_1+d_2+\ldots+d_n=2e$ — the degree sum equals twice the number of edges — was established. This section is about the *pattern* of the edges rather than any single graph: which graphs are simple, how one graph sits inside another as a subgraph, when two differently drawn graphs are really the same, and several families — complete, bipartite and regular graphs — that reappear throughout the course. Only the connections matter, never the placement of a vertex on the page or whether an edge is drawn straight or curved.

**Simple graphs.**

Recall that a **loop** joins a vertex to itself and that **multiple edges** join the same pair of vertices more than once. A graph with **no loops and no multiple edges** is a **simple graph**. In a simple graph an edge is completely determined by its two distinct end vertices, so it may be named by the unordered pair it joins: the edge between $u$ and $v$ is written $uv$ (equally $vu$ — order carries no meaning; for numerical labels we write $1-2$ to avoid reading 12 as a single number). Two vertices joined by an edge are **adjacent**, or **neighbours**, and each edge is **incident** with its two ends. Because a vertex of a simple graph on $n$ vertices can be adjacent to at most the other $n-1$ vertices, every degree satisfies $0\le deg(v)\le n-1$. Every graph in this section is simple unless stated otherwise.

**Subgraphs.**

Write $V(G)$ for the **vertex set** of a graph $G$ and $E(G)$ for its **edge set**. A **subgraph** of $G$ is any graph $H$ whose vertices are all vertices of $G$ and whose edges are all edges of $G$; that is, $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$, with each edge of $H$ joining two vertices that are themselves in $H$. A subgraph is obtained by deleting some vertices and/or some edges of $G$ — and deleting a vertex forces the deletion of every edge meeting it, since an edge cannot survive without both of its ends. Two special kinds recur. A **spanning subgraph** retains *every* vertex of $G$ (so $V(H)=V(G)$) and discards only edges. The subgraph **induced by** a set $S$ of vertices keeps the vertices of $S$ together with *every* edge of $G$ that joins two of them.

**Walks, paths and cycles.**

A **walk** in a graph is a route along its edges: a sequence of vertices in which each vertex is joined to the next by an edge. The **length** of a walk is the **number of edges** it uses. A walk that repeats no vertex is a **path**, and a walk that finishes where it began is **closed**. A **cycle** is a closed walk of at least three edges that repeats no vertex apart from its start, which is also its finish — that is, distinct vertices $v_1,v_2,\ldots,v_k$ joined round in a single ring $v_1-v_2-\ldots-v_k-v_1$. Its length is that number of edges $k$, and a cycle of length $k$ is also called a **$k$-cycle**; a cycle is **odd** or **even** according as its length is odd or even. A **triangle** is a cycle of length 3. The graph consisting of a single ring of $n\ge 3$ vertices and nothing else is the **cycle graph** $C_n$.

**Connectedness and components.**

A graph is **connected** if it is possible to travel from any vertex to any other along a walk; otherwise it is **disconnected**. A **connected component** (or just **component**) of $G$ is a maximal connected subgraph — a connected piece that cannot be enlarged by adding a further vertex of $G$ while remaining connected. Every vertex lies in exactly one component, so the components partition the vertex set, and a graph is connected precisely when it consists of a single component.

**Isomorphism.**

Because only the pattern of connections matters, two graphs that look quite different may in fact be the *same* graph relabelled or redrawn. Graphs $G$ and $H$ are **isomorphic**, written $G\cong H$, if there is a **bijection** (a one-to-one and onto matching) $f:V(G)\to V(H)$ between their vertex sets for which

$$u,v\text{ are adjacent in }G\Leftrightarrow f(u),f(v)\text{ are adjacent in }H.$$

Such a matching is an **isomorphism**: it pairs the vertices so that edges correspond exactly to edges, and non-edges to non-edges. Isomorphic graphs are indistinguishable *as graphs*.

Any quantity that an isomorphism is forced to preserve is an **invariant**, and comparing invariants is the practical way to reason about isomorphism. Isomorphic graphs must have

- the **same number of vertices** and the **same number of edges**;
- the **same degree sequence** — the list of all vertex degrees written in decreasing order; and
- the **same number of components** (one cannot be connected while the other is not).

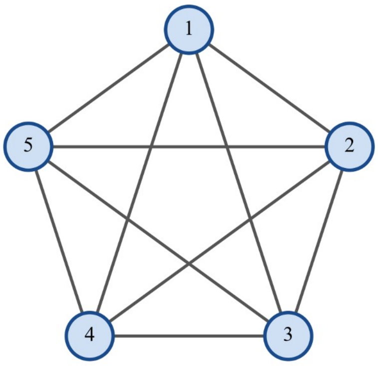
If any invariant differs, the graphs are **not** isomorphic. The converse fails: matching invariants do *not* guarantee isomorphism, since two graphs can agree in every count above and still differ in structure. Consequently, to **establish** an isomorphism you must actually exhibit a vertex matching that preserves adjacency, whereas to **disprove** one it is enough to name a single invariant — or a finer structural feature, such as containing a triangle — that the two graphs do not share.

**Complete graphs.**

A **complete graph** is a simple graph in which every pair of distinct vertices is joined by an edge. The complete graph on $n$ vertices is written $K_n$. Each vertex is adjacent to the other $n-1$ vertices, so **every vertex has degree** $n-1$, and the number of edges is the number of ways of choosing the two endpoints from $n$ vertices,

$$e(K_n)=\binom{n}{2}=\frac{n(n-1)}{2},$$

equally half the degree sum $1/2\,n(n-1)$ by the handshaking lemma.




The complete graph $K_5$ drawn above has every vertex of degree $4$ and $\binom{5}{2}=10$ edges.

**The complement.**

Given a simple graph $G$ on $n$ vertices, its **complement** $\acute{G}$ is the simple graph on the *same* vertex set in which two distinct vertices are joined **exactly when they are not joined in** $G$. Each unordered pair of vertices is an edge of precisely one of $G$ and $\acute{G}$, so their edge counts add to the total number of pairs,

$$e(G)+e(\acute{G})=\binom{n}{2},$$

and a vertex of degree $d$ in $G$ has degree $(n-1)-d$ in $\acute{G}$. Complementing twice restores the original graph, $\overline{\acute{G}}=G$. In particular the complement of $K_n$ has no edges at all — the **empty graph** (or **null graph**) $N_n$ of $n$ isolated vertices — and $\acute{N}_n=K_n$.

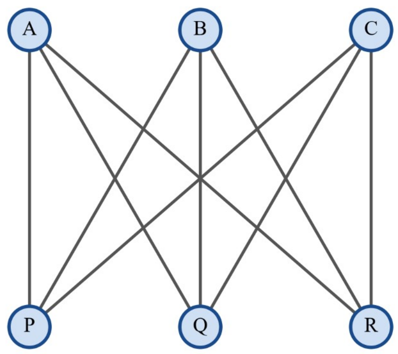
**Bipartite graphs.**

A graph is **bipartite** if its vertices can be split into two disjoint sets $X$ and $Y$ — a **bipartition** — so that *every edge joins a vertex of* $X$ *to a vertex of* $Y$; no edge lies within $X$ or within $Y$. Equivalently, the vertices can be coloured with two colours so that adjacent vertices always receive different colours (a $2$-**colouring**), the sets $X$ and $Y$ being the two colour classes. Tracing colours around a cycle gives a convenient test: **a graph is bipartite if and only if it contains no cycle of odd length**. In particular, any graph containing a triangle is not bipartite.

The **complete bipartite graph** $K_{m,n}$ is the bipartite graph with parts of sizes $m$ and $n$ in which *every* vertex of one part is joined to *every* vertex of the other. Each of the $m$ vertices of $X$ is joined to all $n$ vertices of $Y$, so

$$e(K_{m,n})=mn.$$

Every vertex of $X$ then has degree $n$ and every vertex of $Y$ has degree $m$.




The graph $K_{3,3}$ shown above joins each of its three upper vertices to each of its three lower vertices, giving $3\times3=9$ edges with every vertex of degree $3$.

**Regular graphs.**

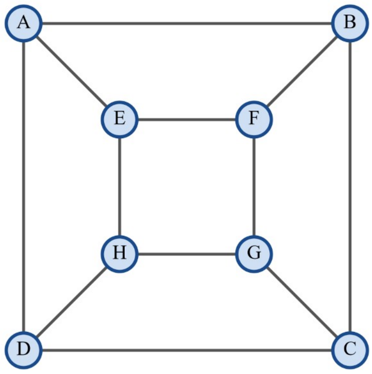
A graph is **regular** if all of its vertices have the same degree; when that common degree is $k$ the graph is $k$-**regular**. By the handshaking lemma a $k$-regular graph on $n$ vertices has degree sum $nk$, so its number of edges is

$$e=\frac{nk}{2}.$$

Since $e$ is a whole number, $nk$ must be **even**: there is no $k$-regular graph on $n$ vertices when $n$ and $k$ are both odd. Several families already met are regular: $K_n$ is $(n-1)$-regular, the cycle graph $C_n$ is 2-regular, and $K_{n,n}$ is $n$-regular (whereas $K_{m,n}$ with $m\neq n$ is **not** regular, its two parts having different degrees).

**Platonic graphs.**

The five **Platonic solids** — the tetrahedron, cube, octahedron, dodecahedron and icosahedron — are the regular convex polyhedra, and the graph formed by the corners (vertices) and edges of each is regular. These are the **Platonic graphs**.




The cube graph drawn above (its two square faces shown one inside the other) has 8 vertices, each of degree 3, and hence $8 \times 3/2 = 12$ edges. The table lists each Platonic graph with the edge count $nk/2$ that its regularity predicts.

| Platonic graph | vertices $n$ | degree $k$ | edges $nk/2$ |
|---|---|---|---|
| tetrahedron | 4 | 3 | 6 |
| cube | 8 | 3 | 12 |
| octahedron | 6 | 4 | 12 |
| dodecahedron | 20 | 3 | 30 |
| icosahedron | 12 | 5 | 30 |

Since the tetrahedron joins all 4 of its vertices in pairs, the tetrahedron graph is exactly $K_4$.

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## Worked examples

### *Example 1 — scaffolded*

A network of six microclimate sensors $P, Q, R, S, T, U$ has a data link between the pairs $PQ$, $QR$, $RS$, $SP$, $PR$ and $TU$, and no others.

| Working | Comment |
|---|---|
| No link is a loop, and no pair is linked twice, so the graph is **simple**. | A simple graph has no loops and no multiple edges. |
| $deg\, P = 3$ (to $Q, S, R$), $deg\, Q = 2$, $deg\, R = 3$ (to $Q, S, P$), $deg\, S = 2$, $deg\, T = 1$, $deg\, U = 1$. | Count the links meeting each sensor. |
| Degree sequence $3, 3, 2, 2, 1, 1$; sum $= 12$, so $e = \frac{12}{2} = 6$. | Degrees in decreasing order; the handshaking lemma gives $e$ as half the degree sum. |
| $P, Q, R, S$ are all reachable from one another, but $T, U$ are joined only to each other. So the graph is **disconnected**, with components $\{P, Q, R, S\}$ and $\{T, U\}$. | A component is a maximal connected piece. |
| Subgraph induced by $\{P, Q, R\}$: keep those vertices and every link among them, namely $PQ$, $QR$ and $PR$. | Induced subgraph = the chosen vertices with all edges joining two of them. |

The induced subgraph on $\{P, Q, R\}$ joins all three pairs, so it is the complete graph $K_3$ (a triangle).

### ***Example 2 — scaffolded***

(a) For $K_8$, state the degree of each vertex, the value of $k$ for which it is $k$-regular, and the number of edges. (b) How many edges has a 4-regular graph on 10 vertices? (c) State the number of edges of the complete bipartite graph $K_{4,6}$ and the two degree values that occur in it.

| Working | Comment |
|---|---|
| (a) In $K_8$ each vertex joins the other 7, so every degree is 7; $K_8$ is 7-regular. | In $K_n$ every vertex has degree $n-1$. |
| $e(K_8)=\binom{8}{2}=\frac{8\times 7}{2}=28$. | Choose the two endpoints from 8 vertices. |
| (b) $e=\frac{nk}{2}=\frac{10\times 4}{2}=20$. | A $k$-regular graph on $n$ vertices has $nk/2$ edges. |
| (c) $e(K_{4,6})=4\times 6=24$. | $K_{m,n}$ has $mn$ edges. |
| Each of the 4 vertices in the first part has degree 6; each of the 6 in the second part has degree 4. | In $K_{m,n}$ one part has degree $n$, the other degree $m$. |

### ***Example 3 — unscaffolded***

Let $G$ be the simple graph on vertices $a,b,c,d,e$ whose edges are $ab$, $ac$, $bc$ and $cd$. Write down the complement $\acute{G}$, verify the identity $e(G)+e(\acute{G})=\binom{5}{2}$, and confirm that each vertex has degree $4-deg_G(v)$ in $\acute{G}$.

There are $\binom{5}{2}=10$ possible pairs among the five vertices. Removing the four edges of $G$ leaves the complement's edges as the remaining six pairs:

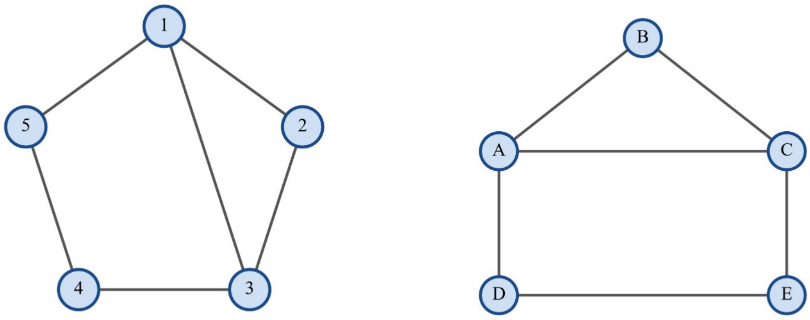
$$\acute{G}: ad, ae, bd, be, ce, de.$$

Thus $e(G)=4$ and $e(\acute{G})=6$, and indeed $4+6=10=\binom{5}{2}$. The degrees in $G$ are $deg_G a=2$, $deg_G b=2$, $deg_G c=3$, $deg_G d=1$ and $deg_G e=0$; subtracting each from $n-1=4$ predicts complement degrees $2,2,1,3,4$. Reading the degrees directly off $\acute{G}$ gives $deg_{\acute{G}} a=2$ (to $d,e$), $deg_{\acute{G}} b=2$ (to $d,e$), $deg_{\acute{G}} c=1$ (to $e$), $deg_{\acute{G}} d=3$ (to $a,b,e$) and $deg_{\acute{G}} e=4$ (to $a,b,c,d$), which agree.

$\therefore \acute{G}$ has the six edges $ad, ae, bd, be, ce, de$, with $e(G)+e(\acute{G})=4+6=10=\binom{5}{2}$, and each degree in $\acute{G}$ equals $4-deg_G(v)$.

### ***Example 4 — unscaffolded***

The graphs $G$ and $H$ below each have five vertices. Decide whether they are isomorphic, and if they are, give an explicit isomorphism.




Graph $G$ (left) has edges $1-2$, $2-3$, $3-4$, $4-5$, $5-1$ and $1-3$; graph $H$ (right) has edges $AB$, $BC$, $CA$, $AD$, $CE$ and $DE$.

Both graphs have 5 vertices and 6 edges. Their degrees are, in $G$, $deg\,1=3$, $deg\,3=3$, and $deg\,2=deg\,4=deg\,5=2$; in $H$, $deg\,A=3$, $deg\,C=3$, and $deg\,B=deg\,D=deg\,E=2$. So both have degree sequence $3,3,2,2,2$, both are connected, and each contains exactly one triangle — the vertices $1,2,3$ in $G$ (edges $1-2$, $2-3$, $1-3$) and $A,B,C$ in $H$ (edges $AB$, $BC$, $CA$). The invariants match, so an isomorphism is plausible; we look for one.

In each triangle the two degree-3 vertices are $1,3$ (in $G$) and $A,C$ (in $H$), and the degree-2 triangle vertex is 2 (in $G$) and $B$ (in $H$). Match $2 \leftrightarrow B$ and, say, $1 \leftrightarrow A$ and $3 \leftrightarrow C$. Outside the triangle, vertex 1 is joined to 5 and vertex 3 to 4, with $4-5$ an edge; correspondingly $A$ is joined to $D$ and $C$ to $E$, with $DE$ an edge. This forces $5 \leftrightarrow D$ and $4 \leftrightarrow E$. The matching

$$1 \leftrightarrow A, 2 \leftrightarrow B, 3 \leftrightarrow C, 4 \leftrightarrow E, 5 \leftrightarrow D$$

sends the edges of $G$ to $AB$, $BC$, $CE$, $ED$, $DA$ and $AC$ — exactly the six edges of $H$ — so adjacency is preserved throughout.

$\therefore G \cong H$, an isomorphism being $1 \leftrightarrow A$, $2 \leftrightarrow B$, $3 \leftrightarrow C$, $4 \leftrightarrow E$, $5 \leftrightarrow D$.

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## Practice questions

### *Scaffolded*

**Q1.** Six trading posts $A,B,C,D,E,F$ are linked by supply roads $AB$, $BC$, $CA$, $CD$, $DE$ and $EF$. (a) Is the graph simple? State the degree of each post. (b) Write down the degree sequence and use the handshaking lemma to find the number of roads. (c) Is the network connected? Justify your answer. (d) Write down the spanning subgraph obtained by deleting the road $CD$, and state whether it is connected, listing any components.

**Q2.** (a) Find the number of edges of $K_7$ and the degree of each vertex. (b) $K_n$ is $k$-regular; state $k$ in terms of $n$, and hence the common degree in $K_{12}$. (c) Find $n$ if $K_n$ has 45 edges. (d) A simple graph on 7 vertices currently has 15 edges. How many further edges are needed to turn it into $K_7$?

**Q3.** (a) How many edges has a 5-regular graph on 8 vertices? (b) Explain why no 3-regular graph on 9 vertices can exist. (c) The octahedron graph is $k$-regular on 6 vertices with 12 edges; find $k$. (d) Is $K_6$ regular? State its common degree and its number of edges.

**Q4.** (a) Show that the graph on $\{1,2,3,4,5,6\}$ with edges $1-2$, $2-3$, $3-4$, $4-5$, $5-6$, $6-1$ is bipartite by giving a bipartition. (b) Is the graph on $\{1,2,3,4,5\}$ with edges $1-2$, $2-3$, $3-4$, $4-5$, $5-1$ bipartite? Explain. (c) How many edges has $K_{4,7}$? (d) State the two degree values that occur in $K_{4,7}$.

**Q5.** Let $G$ be the graph on $\{1,2,3,4,5\}$ with edges $1-2$, $1-3$, $1-4$ and $1-5$. (a) List the edges of the complement $\acute{G}$. (b) State $e(G)$ and $e(\acute{G})$. (c) Verify that $e(G)+e(\acute{G})=\binom{5}{2}$. (d) State the degree of vertex 1 in $G$ and in $\acute{G}$, and name the complement as a familiar graph.

**Q6.** Graph $G$ has vertices $1,2,3,4$ and edges $1-2$, $2-3$, $3-4$, $4-1$. Graph $H$ has vertices $A,B,C,D$ and edges $AB$, $AC$, $BD$, $CD$. (a) Write down the degree sequence of each graph. (b) State the number of vertices and edges of each. (c) Find a matching of the vertices of $G$ with those of $H$ under which joined vertices stay joined. (d) Are $G$ and $H$ isomorphic? Explain.

### *Unscaffolded*

**Q7.** At a research symposium each of 11 delegates exchanges a contact card with every other delegate. Modelling the exchanges as $K_{11}$, find the degree of each vertex and the total number of card exchanges.

**Q8.** A 6-regular graph has 27 edges. How many vertices does it have?

**Q9.** A community choir pairs each of its 5 sopranos with each of its 8 tenors for a duet. Modelling the pairings as the complete bipartite graph $K_{5,8}$, find the number of duets and the degree of each singer.

**Q10.** Determine, with reasons, whether the graph on $\{1,2,3,4,5,6,7\}$ with edges $1-2$, $2-3$, $3-4$, $4-5$, $5-6$, $6-7$ and $7-1$ is bipartite.

**Q11.** Let $G$ be the 5-cycle on $\{1,2,3,4,5\}$ with edges $1-2$, $2-3$, $3-4$, $4-5$ and $5-1$. Find the complement $\acute{G}$, show that $\acute{G}$ is itself a 5-cycle, and deduce that $G \cong \acute{G}$.

**Q12.** Explain why a 3-regular graph must have an even number of vertices, and state the smallest number of vertices for which a 3-regular graph exists, naming that graph.

**Q13.** The dodecahedron graph is 3-regular and the icosahedron graph is 5-regular, and each has 30 edges. Find the number of vertices of each.

**Q14.** Graph $G$ is the complete bipartite graph $K_{2,3}$, with parts $\{a,b\}$ and $\{x,y,z\}$. Graph $H$ is the graph on $\{1,2,3,4,5\}$ with edges $1-2$, $2-3$, $3-4$, $4-1$, $1-5$ and $2-5$. Both have 5 vertices, 6 edges and degree sequence $3,3,2,2,2$. Determine, with justification, whether $G$ and $H$ are isomorphic.

**Q15.** A graph $G$ on $\{1,2,3,4,5,6,7\}$ has edges $1-2$, $1-3$, $2-3$, $4-5$, $4-6$, $5-6$ and $6-7$. State the number of connected components, identify the largest complete subgraph inside each component, and write down the degree sequence of $G$.

**Q16.** Show that $K_{4,4}$ is regular, stating the value of $k$, and find its number of edges. Explain why $K_{4,4}$ contains no triangle.

**Q17.** For which values of $m$ and $n$ is the complete bipartite graph $K_{m,n}$ regular? When it is regular and has $2n$ vertices in total, express its number of edges and its common degree in terms of $n$.

**Q18.** Each of the following 3-regular graphs has 6 vertices and 9 edges. Graph $G$, the triangular prism, has edges $1-2$, $2-3$, $3-1$, $4-5$, $5-6$, $6-4$, $1-4$, $2-5$ and $3-6$. Graph $H$ is $K_{3,3}$ with parts $\{1,3,5\}$ and $\{2,4,6\}$. Determine, with justification, whether $G$ and $H$ are isomorphic.

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## Answers

**Q1.** (a) Simple; $deg\,A=2$, $deg\,B=2$, $deg\,C=3$, $deg\,D=2$, $deg\,E=2$, $deg\,F=1$. (b) $3,2,2,2,2,1$; $e=6$. (c) Connected — every post is reachable. (d) Edges $AB$, $BC$, $CA$, $DE$, $EF$; disconnected, components $\{A,B,C\}$ and $\{D,E,F\}$. **Q2.** (a) 21 edges, degree 6. (b) $k=n-1$; degree 11 in $K_{12}$. (c) $n=10$. (d) 6 more edges. **Q3.** (a) 20. (b) $9\times3=27$ is odd, but the degree sum must equal $2e$ (even). (c) $k=4$. (d) Yes, 5-regular with 15 edges. **Q4.** (a) Bipartite; e.g. $X=\{1,3,5\}$, $Y=\{2,4,6\}$. (b) No — it is a 5-cycle, an odd cycle. (c) 28. (d) 7 and 4. **Q5.** (a) $2-3$, $2-4$, $2-5$, $3-4$, $3-5$, $4-5$. (b) $e(G)=4$, $e(\acute{G})=6$. (c) $4+6=10=\binom{5}{2}$. (d) Degree 4 in $G$, degree 0 in $\acute{G}$; $\acute{G}$ is $K_4$ on $\{2,3,4,5\}$ together with the isolated vertex 1 (and $G=K_{1,4}$). **Q6.** (a) Both $2,2,2,2$. (b) Each has 4 vertices and 4 edges. (c) $1\leftrightarrow A$, $2\leftrightarrow B$, $3\leftrightarrow D$, $4\leftrightarrow C$. (d) Yes — both are 4-cycles and the matching preserves every edge. **Q7.** Degree 10; 55 exchanges. **Q8.** 9 vertices. **Q9.** 40 duets; each soprano has degree 8, each tenor degree 5. **Q10.** Not bipartite — it is a 7-cycle, an odd cycle. **Q11.** $\acute{G}$ has edges $1-3$, $3-5$, $5-2$, $2-4$, $4-1$, the 5-cycle $1-3-5-2-4-1$; hence $G\cong\acute{G}$ ($G$ is self-complementary). **Q12.** For a 3-regular graph on $n$ vertices, $3n=2e$ is even, so $n$ is even; the smallest is $n=4$, giving $K_4$. **Q13.** Dodecahedron 20 vertices; icosahedron 12 vertices. **Q14.** Not isomorphic — $H$ contains a triangle $(1,2,5)$ whereas $K_{2,3}$ is bipartite and so triangle-free. **Q15.** 2 components, $\{1,2,3\}$ and $\{4,5,6,7\}$; the largest complete subgraph in each is $K_3$; degree sequence $3,2,2,2,2,2,1$. **Q16.** 4-regular, 16 edges; bipartite, so no odd cycle and hence no triangle. **Q17.** Regular iff $m=n$; then $K_{n,n}$ has $n^2$ edges and common degree $n$. **Q18.** Not isomorphic — $G$ contains triangles (e.g. $1,2,3$) whereas $H=K_{3,3}$ is bipartite and so triangle-free.

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## Fully-worked solutions

**Q1.** (a) No road is a loop and no pair is joined twice, so the graph is simple. Counting the roads at each post: $A$ joins $B,C$ so $deg\,A=2$; $B$ joins $A,C$ so $deg\,B=2$; $C$ joins $A,B,D$ so $deg\,C=3$; $D$ joins $C,E$ so $deg\,D=2$; $E$ joins $D,F$ so $deg\,E=2$; $F$ joins $E$ so $deg\,F=1$. (b) In decreasing order the degrees are $3,2,2,2,2,1$, summing to 12, so by the handshaking lemma $e=12/2=6$ (matching the six listed roads). (c) The graph is connected: from any post one can reach any other, for instance $A-C-D-E-F$ and $A-B$, so there is a single component. (d) Deleting the road $CD$ keeps all six vertices and leaves the edges $AB$, $BC$, $CA$, $DE$, $EF$. Now no road joins $\{A,B,C\}$ to $\{D,E,F\}$, so the spanning subgraph is disconnected, with components $\{A,B,C\}$ (a triangle) and $\{D,E,F\}$ (a path).

**Q2.** (a) In $K_7$ each vertex joins the other 6, so every degree is 6, and $e=\binom{7}{2}=7\times6/2=21$. (b) Every vertex of $K_n$ has degree $n-1$, so $K_n$ is $(n-1)$-regular, i.e. $k=n-1$; for $n=12$ the common degree is 11. (c) Solve $n(n-1)/2=45$, i.e. $n(n-1)=90=10\times9$, giving $n=10$. (d) $K_7$ has $\binom{7}{2}=21$ edges, so $21-15=6$ further edges are required.

**Q3.** (a) A 5-regular graph on 8 vertices has $e=nk/2=8\times5/2=20$ edges. (b) A 3-regular graph on 9 vertices would have degree sum $9\times3=27$, which is odd; but the degree sum must equal $2e$, an even number, so no such graph exists. (c) Regularity gives $k=2e/n=2\times12/6=4$, so the octahedron graph is 4-regular. (d) In $K_6$ every vertex has degree $6-1=5$, so it is regular (5-regular), with $e=\binom{6}{2}=15$ edges.

**Q4.** (a) Colour the vertices alternately around the six-cycle: $X=\{1,3,5\}$ and $Y=\{2,4,6\}$. Each edge $1-2$, $2-3$, $3-4$, $4-5$, $5-6$, $6-1$ joins an $X$-vertex to a $Y$-vertex, so the graph is bipartite. (b) The graph is the 5-cycle $1-2-3-4-5-1$. Attempting to 2-colour it around the cycle forces $1,3,5$ one colour and $2,4$ the other, but then the edge $5-1$ joins two same-coloured vertices. A cycle of odd length cannot be 2-coloured, so it is not bipartite. (c) $e(K_{4,7})=4\times7=28$. (d) Each of the 4 vertices in the first part has degree 7; each of the 7 vertices in the second part has degree 4. The two degree values are 7 and 4.

**Q5.** (a) The five vertices give $\binom{5}{2}=10$ pairs. Removing $G$'s edges $1-2$, $1-3$, $1-4$, $1-5$ leaves the complement's edges $2-3$, $2-4$, $2-5$, $3-4$, $3-5$, $4-5$. (b) $e(G)=4$ and $e(\acute{G})=6$. (c) $4+6=10=\binom{5}{2}$, as required. (d) Vertex 1 joins all others in $G$, so $deg_G 1=4$; it joins none of them in $\acute{G}$, so $deg_{\acute{G}} 1=0$ (consistent with $4-4=0$). The complement joins every pair among $2,3,4,5$, so it is $K_4$ on $\{2,3,4,5\}$ together with the isolated vertex 1. (The graph $G$ itself is the complete bipartite graph $K_{1,4}$, with parts $\{1\}$ and $\{2,3,4,5\}$.)

**Q6.** (a) In $G$ each of $1,2,3,4$ lies on the four-cycle, so every degree is 2 and the sequence is $2,2,2,2$. In $H$ the edges $AB$, $AC$, $BD$, $CD$ give $deg\,A=2$, $deg\,B=2$, $deg\,C=2$, $deg\,D=2$, so also $2,2,2,2$. (b) Both graphs have 4 vertices and 4 edges. (c) $H$ is the cycle $A-B-D-C-A$. Matching it to $G$'s cycle $1-2-3-4-1$ in order gives $1\leftrightarrow A$, $2\leftrightarrow B$, $3\leftrightarrow D$, $4\leftrightarrow C$, which sends $1-2\rightarrow AB$, $2-3\rightarrow BD$, $3-4\rightarrow DC$ and $4-1\rightarrow CA$ — every edge is preserved. (d) Yes: both are four-vertex cycles, and the matching in (c) is an isomorphism.

**Q7.** In $K_{11}$ each delegate exchanges a card with the other 10, so every vertex has degree 10. The number of exchanges is the number of edges, $e=\binom{11}{2}=11\times10/2=55$.

**Q8.** For a 6-regular graph on $n$ vertices, $e=nk/2=6n/2=3n$. Setting $3n=27$ gives $n=9$ vertices.

**Q9.** In $K_{5,8}$ every soprano is paired with every tenor, so the number of duets is $e=5\times8=40$. Each soprano is joined to all 8 tenors, giving degree 8, and each tenor is joined to all 5 sopranos, giving degree 5.

**Q10.** The graph is the 7-cycle $1-2-3-4-5-6-7-1$. A 2-colouring around the cycle forces the colours to alternate, but with an odd number of vertices the first and last (7 and 1) end up the same colour while joined by an edge. An odd cycle cannot be 2-coloured, so the graph is not bipartite.

**Q11.** Among the five vertices there are $\binom{5}{2}=10$ pairs. Removing $G$'s edges $1-2$, $2-3$, $3-4$, $4-5$, $5-1$ leaves the complement's edges $1-3$, $1-4$, $2-4$, $2-5$, $3-5$. Following these round, $1-3-5-2-4-1$, every vertex has degree 2 and the edges form a single cycle through all five vertices — so $\acute{G}$ is itself a 5-cycle. Two cycles of the same length are isomorphic (match them vertex-by-vertex in cycle order), so $G\cong\acute{G}$; that is, $G$ is self-complementary.

**Q12.** For a 3-regular graph on $n$ vertices the degree sum is $3n$, and by the handshaking lemma $3n=2e$ is even. Since 3 is odd, $n$ must be even. The smallest even $n$ for which each vertex can reach the required degree 3 (a vertex needs at least 3 others to join) is $n=4$; the 3-regular graph on 4 vertices joins every pair, so it is $K_4$.

**Q13.** For a $k$-regular graph, $e=nk/2$, so $n=2e/k$. The dodecahedron is 3-regular with 30 edges, giving $n=2\times30/3=20$ vertices; the icosahedron is 5-regular with 30 edges, giving $n=2\times30/5=12$ vertices.

**Q14.** Both graphs have 5 vertices, 6 edges and degree sequence $3,3,2,2,2$, so the basic invariants match. However $H$ contains the triangle $1,2,5$ (the edges $1-2$, $1-5$ and $2-5$), a cycle of length 3. A bipartite graph has no odd cycle, and $K_{2,3}$ is bipartite by construction (every edge joins $\{a,b\}$ to $\{x,y,z\}$), so $K_{2,3}$ contains no triangle. Containing a triangle is preserved by any isomorphism, so $G$ and $H$ are not isomorphic.

**Q15.** The edges split into two groups with no edge between them: $1-2$, $1-3$, $2-3$ on $\{1,2,3\}$, and $4-5$, $4-6$, $5-6$, $6-7$ on $\{4,5,6,7\}$. Hence there are 2 connected components, $\{1,2,3\}$ and $\{4,5,6,7\}$. In the first, all three pairs are joined, so the whole component is $K_3$. In the second, $4,5,6$ are joined in every pair while 7 hangs off 6; the largest complete subgraph there is the triangle on $\{4,5,6\}$, again $K_3$. The degrees are $deg\,1=deg\,2=2$, $deg\,3=2$, $deg\,4=deg\,5=2$, $deg\,6=3$, $deg\,7=1$, so the degree sequence is $3,2,2,2,2,2,1$.

**Q16.** In $K_{4,4}$ each of the 4 vertices in one part is joined to all 4 vertices in the other, so every vertex has degree 4: the graph is 4-regular. Its number of edges is $e=nk/2=8\times4/2=16$ (equivalently $4\times4=16$). Being complete bipartite it is bipartite, so it contains no odd cycle; a triangle is a cycle of length 3 (odd), so $K_{4,4}$ contains no triangle.

**Q17.** In $K_{m,n}$ the vertices of one part have degree $n$ and those of the other have degree $m$. The graph is regular exactly when these are equal, that is when $m=n$. Then $K_{n,n}$ has $n+n=2n$ vertices, common degree $n$ (it is $n$-regular), and $e=n\times n=n^2$ edges.

**Q18.** Both $G$ and $H$ have 6 vertices, 9 edges, degree sequence $3,3,3,3,3,3$ and are connected, so these invariants do not separate them. But $G$ contains triangles — for example $1,2,3$, from the edges $1-2$, $2-3$, $3-1$ — whereas $H=K_{3,3}$ is bipartite (every edge joins $\{1,3,5\}$ to $\{2,4,6\}$) and so has no odd cycle, hence no triangle. Since any isomorphism preserves the presence of a triangle, $G$ and $H$ are not isomorphic.

## Theory

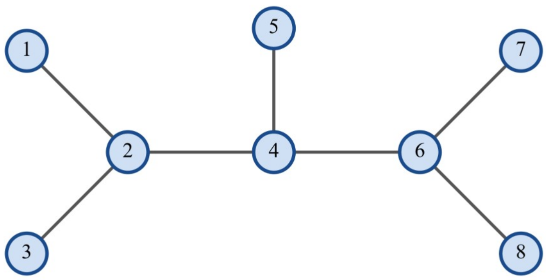
A graph can carry a great deal of structure — many edges, several cycles, vertices of high degree. This section studies graphs that carry the *least* structure needed to hold all their vertices together in one piece: the **trees**. Trees are the simplest connected graphs, and they sit at the heart of the subject because every connected graph contains one as a skeleton. The work here is deliberately proof-oriented: each fact is stated precisely and then justified from the definitions.

Throughout, "graph" means a **simple** graph — no loops and no repeated edges — with a finite vertex set $V$ and edge set $E$. We write $n=|V|$ for the number of vertices and $|E|$ for the number of edges. Recall the vocabulary from the previous sections: a **walk** is a sequence of vertices in which consecutive vertices are joined by an edge; a **path** is a walk that repeats no vertex; a **cycle** is a walk of at least three edges that returns to its start and repeats no other vertex; and a graph is **connected** when some path joins every pair of its vertices. Recall also the **handshaking lemma**: adding up the degrees of all the vertices counts every edge exactly twice, so

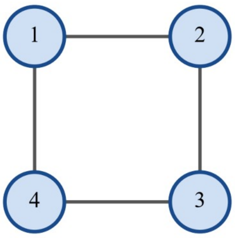
$$\sum_{v \in V} deg(v)=2|E|.$$

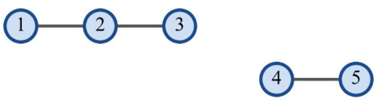
**Trees and forests.**

A **tree** is a **connected** graph that contains **no cycle**. A graph that contains no cycle but need not be connected is called a **forest**; each connected piece of a forest is a tree, so a forest is exactly a disjoint union of trees. The two defining properties of a tree — *connected* and *acyclic* — pull in opposite directions. Connectedness asks for *enough* edges to join everything up; being acyclic asks for *few enough* edges that no edge is redundant. A tree is the exact balance point between the two.




The graph above is a tree on the eight vertices $1,2,\ldots,8$. It is connected, and a quick check confirms there is no cycle. By contrast, the graph below on the left is connected but contains the cycle $1-2-3-4-1$, so it is **not** a tree; the graph on the right has no cycle but is in two separate pieces, so it is a forest but again **not** a tree.




**Leaves.**

A vertex of a tree that has degree 1 is called a **leaf** (or an *end vertex*). In the tree above, the leaves are the vertices $1, 3, 5, 7$ and 8; the remaining vertices $2, 4, 6$ each have degree greater than 1 and are called **internal** vertices. Leaves matter because they can be pruned without harming the tree, and this is the key to the first theorem.

**Lemma (a tree has a leaf).** Every tree with at least two vertices has at least one leaf.

*Proof.* Let $T$ be a tree with $n \geq 2$ vertices. Because $T$ is finite it has only finitely many paths, so we may choose a path $P$ containing as many edges as possible. Since $n \geq 2$ and $T$ is connected, $P$ has at least one edge; let $u$ be an end vertex of $P$. Suppose, for contradiction, that $deg(u) \geq 2$. Then $u$ has a neighbour $w$ different from its neighbour along $P$. If $w$ did not already lie on $P$, then attaching the edge $uw$ to $P$ would produce a strictly longer path, contradicting the maximality of $P$. If instead $w$ did lie on $P$, then the edge $uw$ together with the part of $P$ from $u$ to $w$ would form a cycle, contradicting that $T$ is acyclic. Both cases are impossible, so $deg(u) = 1$ and $u$ is a leaf, as required.

In fact the *same* argument applied to the other end vertex of $P$ produces a second leaf, so a tree with at least two vertices always has **at least two** leaves; this is left to the exercises.

**How many edges does a tree have?**

The single most useful fact about trees is a count of their edges. It is proved by induction, and pruning a leaf is exactly what makes the induction work.

**Theorem 1.** A tree with $n$ vertices has exactly $n-1$ edges.

*Proof (induction on $n$).* For the base case $n = 1$ the tree is a single vertex with no edges, and $0 = 1 - 1$, so the claim holds. Assume, as inductive hypothesis, that every tree with $k$ vertices has $k-1$ edges, and let $T$ be a tree with $k+1$ vertices. Since $k+1 \geq 2$, the Lemma gives a leaf $\ell$ of $T$. Delete $\ell$ together with its single incident edge, and call the result $T'$. Removing edges cannot create a cycle, so $T'$ is still acyclic; and because $\ell$ had degree 1, no path between two *other* vertices ever used $\ell$, so $T'$ is still connected. Hence $T'$ is a tree on $k$ vertices, and by the inductive hypothesis it has $k-1$ edges. Restoring $\ell$ and its one edge shows that $T$ has $(k-1)+1 = k$ edges, that is $(k+1)-1$ edges. By induction the theorem holds for all $n$, as required.

Combining Theorem 1 with the handshaking lemma gives a handy relation for any tree on $n$ vertices:

$$\sum_{v \in V} deg(v) = 2|E| = 2(n-1).$$

The degrees of a tree therefore add up to $2n-2$ — a fact that pins down, for example, how many leaves a tree with prescribed internal degrees must have.

**Spanning trees.**

Trees are important partly because every connected graph contains one that reaches all of its vertices. A **spanning tree** of a connected graph $G$ is a subgraph that

- includes **every** vertex of $G$, and
- is itself a **tree** (connected, with no cycle).

A spanning tree keeps just enough of the edges of $G$ to hold all the vertices together and discards the rest. By Theorem 1 a spanning tree of a graph on $n$ vertices has exactly $n-1$ edges.

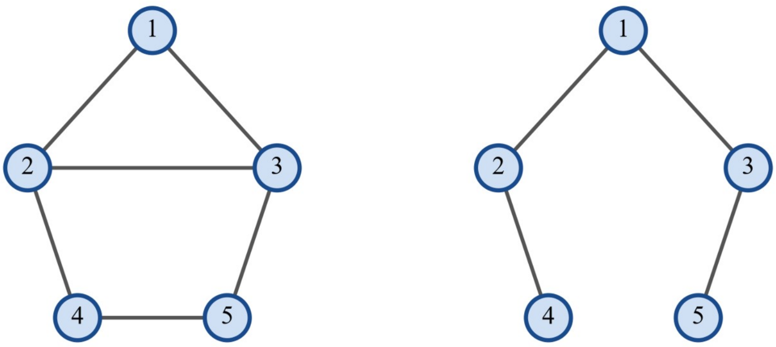
**Theorem 2.** Every connected graph has a spanning tree.

*Proof.* Let $G$ be connected with $n$ vertices. If $G$ has no cycle then $G$ is already a tree that reaches all of its vertices, so $G$ is its own spanning tree. Otherwise $G$ contains a cycle; choose any edge $e$ lying on that cycle and delete it. An edge on a cycle is never the only route between its endpoints — the rest of the cycle provides another — so deleting $e$ leaves the graph connected, on the same vertex set, with one fewer edge. Repeat this step. Each repetition removes one edge and destroys at least one cycle while keeping the graph connected and spanning; since $G$ has only finitely many edges, the process must stop. It stops precisely when no cycle remains — that is, at a connected, acyclic spanning subgraph, which is a spanning tree of $G$, as required.

Two consequences follow at once. First, because a spanning tree uses $n-1$ of the edges of $G$, **a connected graph on $n$ vertices has at least $n-1$ edges**, with equality exactly when the graph is itself a tree. Second, if a connected graph $G$ has $n$ vertices and $m$ edges, then forming a spanning tree keeps $n-1$ edges and deletes the rest, so the number of edges that must be removed is

$$m-(n-1)=m-n+1.$$

This number is the count of **independent cycles** in $G$; a tree is the case $m-n+1=0$. A connected graph usually has *many* spanning trees, obtained by different choices of which cycle edges to discard.




On the left is a connected graph $G$ with $n=5$ vertices and $m=6$ edges; on the right is one of its spanning trees, using the 4 edges $1-2$, $1-3$, $2-4$ and $3-5$. Exactly $m-n+1=6-5+1=2$ edges (here $2-3$ and $4-5$) were deleted.

**Equivalent descriptions of a tree.**

We defined a tree as a graph that is *connected and acyclic*, but for a graph on a fixed number of vertices several other conditions say exactly the same thing. Knowing these lets you recognise a tree from whichever information you happen to have.

> **Theorem 3 (characterisations of a tree).** Let $G$ be a simple graph with $n$ vertices. The following statements are equivalent.

1. $G$ is a tree (connected and acyclic).
2. $G$ is connected and has exactly $n-1$ edges.
3. $G$ is acyclic and has exactly $n-1$ edges.
4. Any two vertices of $G$ are joined by **exactly one** path.
5. $G$ is connected, but deleting any single edge disconnects it.

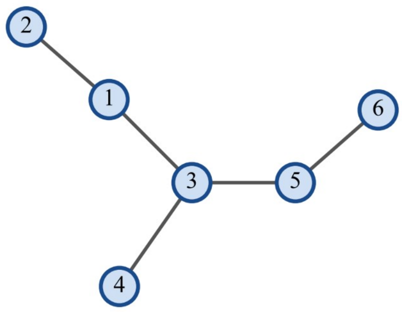
*Why these agree.* Condition (1) gives (2) by Theorem 1. For (2) $\Rightarrow$ (1), suppose $G$ is connected with $n-1$ edges but contains a cycle; deleting an edge of that cycle leaves $G$ connected on $n$ vertices with only $n-2$ edges, contradicting the fact that a connected graph on $n$ vertices needs at least $n-1$ edges — so $G$ is acyclic and therefore a tree. For (3) $\Rightarrow$ (1), note that a forest with $n$ vertices split into $c$ connected pieces has $n-c$ edges (apply Theorem 1 to each piece); if the forest has $n-1$ edges then $n-c=n-1$, so $c=1$ and the graph is connected, hence a tree. The unique-path condition (4) is proved equivalent to (1) in Worked Example 4. Finally (1) $\Leftrightarrow$ (5): in a tree every edge is the *only* path between its two endpoints, so removing any edge separates them; conversely, if $G$ is connected but has a cycle, then removing an edge of that cycle leaves the graph connected, so $G$ is not minimally connected. Thus a graph is a tree exactly when it is **minimally connected** — connected, with no edge to spare.

The unique-path property (4) is worth singling out: a tree is precisely a graph in which there is one and only one way to travel between any two vertices. This is the property that makes trees the natural model for hierarchies and for networks with no redundancy.

### Worked examples

#### *Example 1 — scaffolded*

The graph $T$ below has vertices $1, 2, \ldots, 6$.




(a) Explain why $T$ is a tree.

(b) State the number of vertices and edges, and confirm the relationship of Theorem 1.

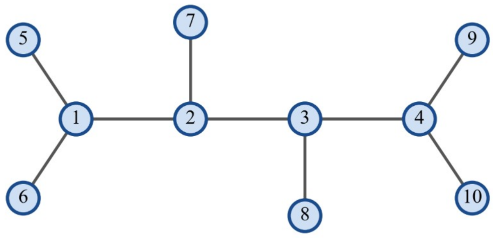
(c) List the leaves of $T$.

| Working | Comment |
|---|---|
| The edges are $1-2$, $1-3$, $3-4$, $3-5$, $5-6$. Every vertex can be reached from every other, so $T$ is connected. The degree-1 vertices 2, 4 and 6 can lie on no cycle, and deleting them leaves only the path $1-3-5$, so $T$ has no cycle. | A tree is a connected graph with no cycle — check both properties; every vertex of a cycle has degree at least 2. |
| Hence $T$ is connected and acyclic, so $T$ is a tree. | Both defining conditions hold. |
| There are $n=6$ vertices and 5 edges. | Count directly from the figure. |
| Theorem 1 predicts $n-1=6-1=5$ edges, which matches. | A tree on $n$ vertices has exactly $n-1$ edges. |
| The degrees are $deg(1)=2$, $deg(2)=1$, $deg(3)=3$, $deg(4)=1$, $deg(5)=2$, $deg(6)=1$. | A leaf is a vertex of degree 1. |
| The leaves are the degree-1 vertices 2, 4 and 6. | Vertices $1, 3, 5$ are internal. |

#### *Example 2 — scaffolded*

A tree $T$ has 10 vertices. Exactly 6 of them are leaves, and the remaining vertices all have the same degree $d$. Find $d$, and confirm that such a tree exists.

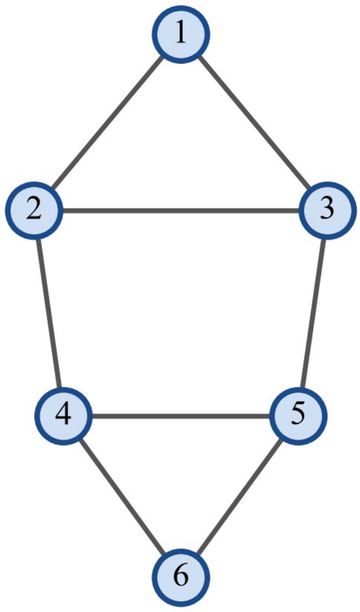
| Working | Comment |
|---|---|
| A tree on $n=10$ vertices has $n-1=9$ edges. | Theorem 1. |
| By the handshaking lemma the degrees add to $2\times 9=18$ . | $\sum deg(v)=2\|E\|$. |
| The 6 leaves contribute $6\times 1=6$ to this total. | Each leaf has degree 1. |
| The other $10-6=4$ vertices each have degree $d$, contributing $4d$. | The remaining vertices. |
| So $6+4d=18$, giving $4d=12$ and $d=3$. | Solve for $d$. |
| A tree with four degree-3 vertices in a row, each carrying enough leaves, realises this — see the figure. | The count is achievable, so the answer is consistent. |



$d=3$: the four internal vertices each have degree 3.

***Example 3 — unscaffolded***

The graph $G$ below has vertices $1,2,\dots,6$ and 8 edges.

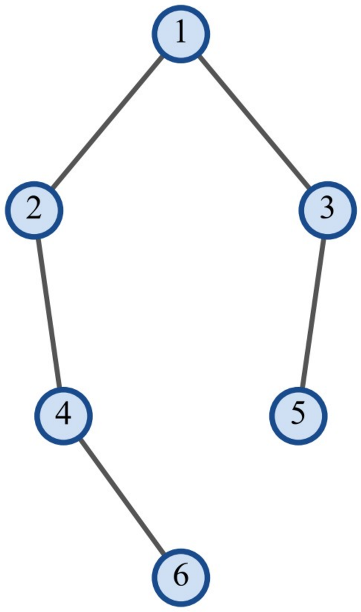


(a) Explain why $G$ is not a tree.
(b) Find a spanning tree of $G$, and state how many edges must be deleted to obtain it.

With $n=6$ vertices a tree would have exactly $n-1=5$ edges, but $G$ has 8 edges; since $8\neq 5$, Theorem 1 shows $G$ cannot be a tree. (Equivalently, $G$ contains cycles, such as $1-2-3-1$.) A spanning tree must keep every vertex but only $n-1=5$ edges, so

$$8-(6-1)=8-5=3$$

edges must be deleted. Deleting $2-3$, $4-5$ and $5-6$ leaves the edges $1-2$, $1-3$, $2-4$, $3-5$ and $4-6$, which connect all six vertices with no cycle.




$\therefore$ one spanning tree is $\{1-2, 1-3, 2-4, 3-5, 4-6\}$, obtained by deleting $3$ of the $8$ edges.

***Example 4 — unscaffolded***

Prove that a graph $G$ is a tree if and only if any two of its vertices are joined by exactly one path. (This establishes the equivalence of conditions (1) and (4) in Theorem 3.)

First suppose $G$ is a tree. Let $u$ and $v$ be any two vertices. Because $G$ is connected, at least one path joins $u$ to $v$. Suppose there were two *different* paths $P$ and $Q$ from $u$ to $v$. Following $P$ from $u$ and then returning along $Q$ to $u$ gives a closed walk; since $P \neq Q$ they first diverge at some vertex $x$ and afterwards meet again at some vertex $y$. The two portions from $x$ to $y$ meet only at $x$ and $y$ and are different, so at least one of them has two or more edges; together they therefore form a closed walk of at least three edges repeating no vertex but $x$ — a cycle. This contradicts $G$ being acyclic. Hence the path from $u$ to $v$ is unique.

Conversely, suppose every pair of vertices is joined by exactly one path. In particular every pair is joined by *at least* one path, so $G$ is connected. If $G$ contained a cycle, then any two distinct vertices $x$ and $y$ on that cycle would be joined by two different paths — one around each side of the cycle — contradicting uniqueness. So $G$ has no cycle. Being connected and acyclic, $G$ is a tree.

$\therefore$ $G$ is a tree exactly when every two vertices are joined by a unique path, as required.

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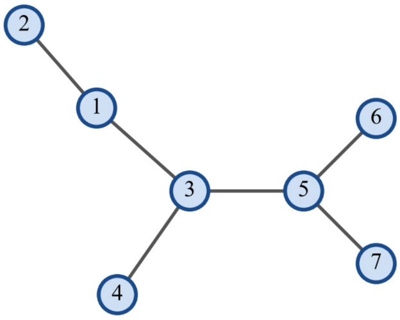
## Practice questions

### *Scaffolded*

**Q1.** The graph $T$ below has vertices $1, 2, \ldots, 7$.




(a) How many edges does $T$ have, and how does this compare with $n-1$?
(b) List the leaves of $T$.
(c) Use the handshaking lemma to state the sum of all the vertex degrees.
(d) A different tree has 30 vertices. How many edges does it have?

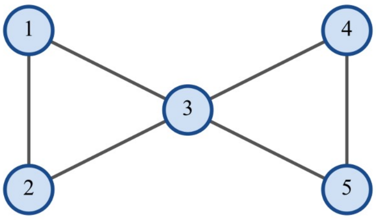
**Q2.** A tree has 16 vertices. It has 9 leaves, and every non-leaf vertex has degree 3. (a) How many edges does the tree have? (b) Using the handshaking lemma, find the number of non-leaf vertices, and check that your answer agrees with the total vertex count.

**Q3.** For each list of numbers, decide whether it can be the list of vertex degrees of some tree. Justify each answer using the fact that the degrees of a tree on $n$ vertices add to $2n-2$ (and each degree is at least 1). (a) Six vertices with degrees $1,1,1,1,3,3$. (b) Five vertices with degrees $2,2,2,2,2$. (c) Seven vertices with degrees $1,1,1,1,2,2,4$. (d) Six vertices with degrees $1,1,2,2,2,3$.

**Q4.** The graph $H$ has vertices $1,2,3,4$ and edges $1-2$, $2-3$, $3-4$, $4-1$ and $1-3$ (a four-cycle with one diagonal). (a) How many edges must be removed to obtain a spanning tree? (b) By listing systematically, find every spanning tree of $H$ and state how many there are.

**Q5.** Decide whether each statement is **true** or **false**, giving a brief reason based on Theorem 3. (a) A connected graph with 10 vertices and 9 edges must be a tree. (b) A graph with 7 vertices and 6 edges must be a tree. (c) An acyclic graph with 12 vertices and 11 edges must be a tree. (d) Every tree with at least two vertices has at least one leaf.

**Q6.** The graph $G$ below has vertices $1,2,3,4,5$; it is made of two triangles that share only the vertex 3.




(a) How many edges must be deleted to obtain a spanning tree?
(b) Write down the edge set of one spanning tree of $G$.
(c) Each triangle must lose exactly one of its three edges, and the two choices are independent. How many spanning trees does $G$ have altogether?

### *Unscaffolded*

**Q7.** A tree has 12 vertices, and every vertex has degree either 1 or 3. Find how many vertices there are of each degree.

**Q8.** A connected graph $G$ has 9 vertices and 11 edges. (a) Explain why $G$ is not a tree. (b) How many edges must be removed to obtain a spanning tree? (c) How many independent cycles does $G$ have?

**Q9.** Let $K_4$ be the complete graph on the four vertices $1,2,3,4$, in which every pair of vertices is joined by an edge (so it has $6$ edges). By counting the three-edge subsets that avoid forming a triangle, find the number of spanning trees of $K_4$.

**Q10.** Let $C_5$ be the cycle on vertices $1,2,3,4,5$ with edges $1-2$, $2-3$, $3-4$, $4-5$ and $5-1$. (a) How many spanning trees does $C_5$ have? (b) Generalise: how many spanning trees does the cycle $C_n$ have?

**Q11.** Prove that every tree with at least two vertices has **at least two** leaves. (Hint: consider a path with the greatest possible number of edges.)

**Q12.** A connected graph $G$ has $n$ vertices and $m$ edges. Express your answers in terms of $n$ and $m$ where appropriate. (a) How many edges are in any spanning tree of $G$? (b) What is the least number of edges that must be removed from $G$ to leave a spanning tree? (c) Hence, if $G$ has $10$ vertices and removing exactly $4$ edges yields a spanning tree, how many edges does $G$ have?

**Q13.** The graph $G$ has vertices $1,2,3,4,5$ and edges $1-3$, $1-4$, $1-5$, $2-3$, $2-4$ and $2-5$ — that is, each of 1 and 2 is joined to each of 3, 4 and 5, and there are no other edges. For each edge set below, state whether it is a spanning tree of $G$; if it is not, explain why. (i) $\{1-3,1-4,1-5,2-3\}$ (ii) $\{1-3,1-4,2-3,2-4\}$ (iii) $\{1-3,2-3,2-4,1-5\}$

**Q14.** Prove that a connected graph on $n$ vertices with exactly $n-1$ edges contains no cycle, and is therefore a tree. (You may use Theorem 2, that every connected graph has a spanning tree.)

**Q15.** A connected graph $G$ has $8$ vertices and $8$ edges. (a) How many edges must be removed to obtain a spanning tree? (b) Deduce that $G$ has exactly one cycle, and explain why the number of spanning trees of $G$ equals the number of edges in that cycle. (c) If the cycle has length 5, how many spanning trees does $G$ have?

**Q16.** A tree has $20$ vertices. Three of them have degree $4$, and every other vertex has degree 1 or degree 2. Let $L$ be the number of leaves and $t$ the number of degree-2 vertices. Write down two equations relating $L$ and $t$, and solve them.

**Q17.** Prove that if a new edge is added between two vertices of a tree that are not already joined by an edge, then exactly one cycle is created. (Use the unique-path property of Theorem 3.)

**Q18.** The graph $G$ has vertices $1,2,3,4,5,6$ and edges $1-2$, $2-3$, $3-4$, $4-5$, $5-6$, $6-1$ and $1-4$ — that is, the six-cycle $1-2-3-4-5-6-1$ together with the chord $1-4$. (a) How many edges must be removed to obtain a spanning tree? (b) The two branch vertices 1 and 4 are joined by three internally disjoint paths, of lengths 1, 3 and 3 edges. A spanning tree keeps one of these paths whole and drops a single edge from each of the other two; use this to find the number of spanning trees of $G$.

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### Answers

**Q1.** (a) 6 edges, equal to $n-1=6$. (b) Leaves $2,4,6,7$. (c) Sum of degrees $=12$. (d) 29 edges. **Q2.** (a) 15 edges. (b) 7 non-leaf vertices (and $9+7=16$ agrees). **Q3.** (a) Yes. (b) No. (c) Yes. (d) No. **Q4.** (a) 2 edges. (b) 8 spanning trees. **Q5.** (a) True. (b) False. (c) True. (d) True. **Q6.** (a) 2 edges. (b) e.g. $\{1-2, 2-3, 3-4, 4-5\}$. (c) 9 spanning trees. **Q7.** 7 vertices of degree 1 and 5 vertices of degree 3. **Q8.** (a) A tree on 9 vertices has 8 edges, but $G$ has 11. (b) 3 edges. (c) 3 independent cycles. **Q9.** 16 spanning trees. **Q10.** (a) 5. (b) $n$. **Q11.** Proof — see solutions. **Q12.** (a) $n-1$. (b) $m-n+1$. (c) 13 edges. **Q13.** (i) Spanning tree. (ii) Not — contains the cycle $1-3-2-4-1$ and omits vertex 5. (iii) Spanning tree. **Q14.** Proof — see solutions. **Q15.** (a) 1 edge. (b) The single cycle has $k$ edges; removing any one of them gives a spanning tree, and no other edge may be removed, so there are $k$ spanning trees. (c) 5. **Q16.** $L+t=17$ and $L+2t=26$; hence $t=9$ and $L=8$. **Q17.** Proof — see solutions. **Q18.** (a) 2 edges. (b) 15 spanning trees.

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### Fully-worked solutions

**Q1.** (a) The edges are $1-2$, $1-3$, $3-4$, $3-5$, $5-6$, $5-7$ — six edges. With $n=7$ vertices, $n-1=6$, so the count matches Theorem 1, as it must for a tree. (b) The degrees are $deg(1)=2$, $deg(3)=3$, $deg(5)=3$, and $deg(2)=deg(4)=deg(6)=deg(7)=1$; the leaves are the degree-1 vertices $2,4,6,7$. (c) By the handshaking lemma the degrees add to $2|E|=2\times 6=12$. (d) A tree on 30 vertices has $30-1=29$ edges.

**Q2.** (a) A tree on 16 vertices has $16-1=15$ edges. (b) The degrees add to $2\times 15=30$. The 9 leaves contribute 9, and if there are $k$ non-leaf vertices, each of degree 3, they contribute $3k$; so $9+3k=30$, giving $3k=21$ and $k=7$. This agrees with the vertex count $9+7=16$.

**Q3.** A list of positive integers is the degree list of some tree on $n$ vertices exactly when it has $n$ terms, each at least 1, adding to $2n-2$. (a) $n=6$, target $2n-2=10$; the degrees add to $1+1+1+1+3+3=10$. **Yes** (for instance two degree-3 vertices joined to each other and to the four leaves). (b) $n=5$, target $2n-2=8$; but $2+2+2+2+2=10\neq 8$. **No** — a connected graph in which every vertex has degree 2 is a cycle, not a tree. (c) $n=7$, target $2n-2=12$; the degrees add to $1+1+1+1+2+2+4=12$. **Yes**. (d) $n=6$, target $2n-2=10$; but $1+1+2+2+2+3=11\neq 10$. **No**.

**Q4.** (a) $H$ has $n=4$ vertices and $m=5$ edges, so a spanning tree keeps $n-1=3$ edges and $5-3=2$ edges must be removed. (b) Label the edges $a=1-2$, $b=2-3$, $c=3-4$, $d=4-1$, $e=1-3$. Any 3 of the 5 edges form a spanning tree unless those 3 edges contain a cycle. There are $\binom{5}{3}=10$ three-edge subsets. The only three-edge cycles are the triangles $\{a,b,e\}$ (that is $1-2-3-1$) and $\{c,d,e\}$ (that is $1-3-4-1$). Removing these two leaves $10-2=8$ spanning trees.

**Q5.** (a) **True.** Condition (2) of Theorem 3: a connected graph on $n=10$ vertices with $n-1=9$ edges is a tree. (b) **False.** Being a tree also requires connectedness. A graph with 7 vertices and 6 edges can be disconnected — for example a four-cycle on $\{1,2,3,4\}$ (four edges) together with the path $5-6-7$ (two edges) has 7 vertices and 6 edges but is in two pieces and contains a cycle, so it is not a tree. (c) **True.** Condition (3): an acyclic graph is a forest with $n-c$ edges on $c$ pieces; $11=12-c$ forces $c=1$, so it is connected, hence a tree. (d) **True.** By the Lemma, a longest path in the tree has a degree-1 vertex at its end, so at least one leaf exists.

**Q6.** (a) $G$ has $n=5$ vertices and $m=6$ edges, so a spanning tree keeps 4 edges and $6-4=2$ must be deleted. (b) One spanning tree is $\{1-2, 2-3, 3-4, 4-5\}$, the path through all five vertices (deleting $1-3$ and $3-5$). (c) The left triangle $1-2-3$ becomes acyclic when any one of its three edges is removed, giving 3 choices; independently the right triangle $3-4-5$ gives 3 choices; because the triangles meet only at 3, no other edge deletion is needed and the choices do not interfere. So there are $3\times 3=9$ spanning trees.

**Q7.** Let $a$ be the number of degree-1 vertices and $b$ the number of degree-3 vertices. The tree has 12 vertices, so $a+b=12$. It has $12-1=11$ edges, so by the handshaking lemma the degrees add to $2\times 11=22$, giving $a+3b=22$. Subtracting the first equation from the second, $2b=10$, so $b=5$ and $a=7$. Thus there are 7 leaves and 5 vertices of degree 3.

**Q8.** (a) A tree on 9 vertices would have exactly $9-1=8$ edges; $G$ has $11\neq 8$ edges, so by Theorem 1 it is not a tree. (b) A spanning tree keeps $9-1=8$ edges, so $11-8=3$ edges must be removed. (c) The number of independent cycles of a connected graph is $m-n+1=11-9+1=3$.

**Q9.** A spanning tree of $K_4$ uses $n-1=3$ of the 6 edges. There are $\binom{6}{3}=20$ three-edge subsets, and a set of 3 edges on 4 vertices is a spanning tree exactly when it contains no cycle. The only way 3 edges can contain a cycle is by forming a triangle, and $K_4$ has $\binom{4}{3}=4$ triangles (one for each choice of three vertices). Hence the number of spanning trees is $20-4=16$.

**Q10.** (a) $C_5$ has $n=5$ vertices and 5 edges, so a spanning tree keeps 4 edges and $5-4=1$ edge is removed. Removing any one of the 5 edges of the cycle leaves a path through all five vertices — a spanning tree — and different removed edges give different trees, so there are 5 spanning trees. (b) The same argument on $C_n$ gives one spanning tree for each of its $n$ edges, so $C_n$ has $n$ spanning trees.

**Q11.** Let $T$ be a tree with at least two vertices. As in the Lemma, choose a path $P$ with the greatest possible number of edges; since $T$ is connected and has at least two vertices, $P$ has at least one edge and therefore two distinct end vertices, $u$ and $v$. Consider $u$. If $deg(u)\geq 2$, then $u$ has a neighbour $w$ other than its neighbour on $P$; if $w\notin P$ we could lengthen $P$, and if $w\in P$ we would close a cycle — both impossible. Hence $deg(u)=1$, so $u$ is a leaf. The identical argument at the other end $v$ shows $deg(v)=1$, so $v$ is a second leaf, distinct from $u$. Therefore $T$ has at least two leaves, as required.

**Q12.** (a) A spanning tree of a graph on $n$ vertices is a tree on those $n$ vertices, so by Theorem 1 it has $n-1$ edges. (b) A spanning tree keeps $n-1$ of the $m$ edges, so the number removed is $m-(n-1)=m-n+1$. (c) Here $m-n+1=4$ with $n=10$, so $m=4+n-1=4+9=13$ edges.

**Q13.** A spanning tree of this 5-vertex graph must use exactly 4 edges, include all five vertices, and contain no cycle. (i) $\{1-3,1-4,1-5,2-3\}$: four edges reaching $1,2,3,4,5$ with no cycle — a **spanning tree**. (ii) $\{1-3,1-4,2-3,2-4\}$: these four edges close the cycle $1-3-2-4-1$, and vertex 5 is missing — **not** a spanning tree. (Note that $G$ has no triangle, so the cycle to look for here has length 4.) (iii) $\{1-3,2-3,2-4,1-5\}$: four edges forming the path $5-1-3-2-4$ through all five vertices, with no cycle — a **spanning tree**.

**Q14.** Let $G$ be connected with $n$ vertices and exactly $n-1$ edges. By Theorem 2, $G$ has a spanning tree $S$; being a tree on all $n$ vertices, $S$ has $n-1$ edges, and $S$ is a subgraph of $G$. But $G$ itself has only $n-1$ edges, so $S$ cannot be a *proper* subgraph — $S$ must use every edge of $G$. Therefore $G=S$ is a tree, and in particular $G$ contains no cycle, as required.

**Q15.** (a) A spanning tree keeps $8-1=7$ edges, so $8-7=1$ edge is removed. (b) Since only one edge is removed to break all cycles, and each independent cycle needs at least one edge removed, $G$ has exactly $m-n+1=8-8+1=1$ independent cycle — that is, exactly one cycle. A spanning tree is obtained by removing a single edge that lies on this cycle; removing an edge *not* on the cycle would disconnect $G$, while removing a cycle edge keeps it connected. Each of the cycle's edges gives a different spanning tree, so the number of spanning trees equals the number of edges on the cycle. (c) If the cycle has length 5, there are 5 spanning trees.

**Q16.** The tree has 20 vertices, so the three degree-4 vertices, the $L$ leaves and the $t$ degree-2 vertices account for all of them: $3+L+t=20$, that is $L+t=17$. It has $20-1=19$ edges, so the degrees add to $2\times 19=38$; the three degree-4 vertices contribute 12, the leaves contribute $L$ and the degree-2 vertices contribute $2t$, giving $12+L+2t=38$, that is $L+2t=26$. Subtracting the first equation from the second gives $t=9$, and then $L=17-9=8$. So the tree has 8 leaves and 9 vertices of degree 2.

**Q17.** Let $T$ be a tree and let $u,v$ be two vertices not joined by an edge. By the unique-path property (Theorem 3), there is exactly one path $P$ in $T$ from $u$ to $v$. Adding the new edge $uv$ creates the closed walk consisting of $P$ followed by the edge $vu$. Since $u$ and $v$ are not adjacent in $T$, the path $P$ has at least two edges, so this closed walk has at least three edges; and because $P$ repeats no vertex, the walk repeats no vertex but its start. It is therefore a cycle.

Suppose some cycle $C$ existed in $T+uv$ that did *not* use the new edge $uv$; then $C$ would lie entirely in $T$, contradicting that $T$ is acyclic. So every cycle must use $uv$. But any cycle through $uv$ consists of the edge $uv$ together with a path in $T$ from $u$ to $v$, and that path is unique — it is $P$. Hence there is exactly one cycle, as required.

**Q18.** (a) $G$ has $n=6$ vertices and $m=7$ edges, so a spanning tree keeps 5 edges and $7-5=2$ edges must be removed. (b) The branch vertices 1 and 4 are joined by three internally disjoint paths: $1-4$ (length 1), $1-2-3-4$ (length 3) and $1-6-5-4$ (length 3). A spanning tree keeps exactly one path complete and drops one edge from each of the other two. If the length-1 path is kept whole, the trees number $3\times3=9$; if a length-3 path is kept whole (two ways to choose which), the trees number $1\times3=3$ each. In total there are $9+3+3=15$ spanning trees.

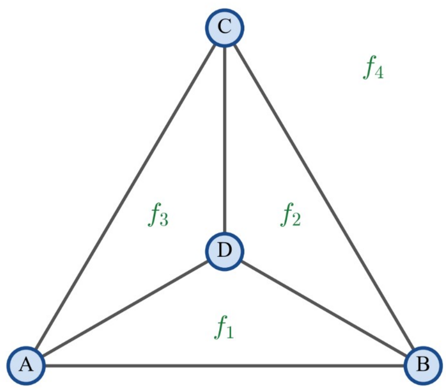
## Theory

A graph records only *which* vertices are joined, never *where* they are drawn, so the same graph can be pictured in many ways. Some of those pictures have edges that cross; others do not. A drawing with no crossings does more than look tidier — it slices the plane into regions, and those regions obey a single, astonishingly rigid numerical law. This section asks which graphs can be drawn without crossings (the **planar** graphs) and establishes that law, **Euler's formula**, together with its main consequences.

Everything here builds on the graph-theory language already developed in this chapter: **vertices** and **edges**, the **degree** of a vertex, the **handshaking lemma** (the degrees of all vertices sum to $2e$), **connected** graphs, **simple** graphs, the complete graph $K_n$, the complete bipartite graph $K_{m,n}$, and **trees**. These are assumed and are not re-derived.

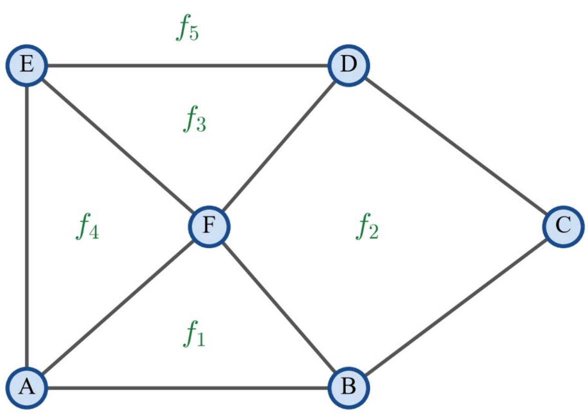
**Planar graphs and planar embeddings.**

A graph is **planar** if it can be drawn in the plane so that edges meet only at their shared endpoints — that is, with **no edge crossings**. Such a crossing-free drawing is called a **planar embedding** (or a plane drawing). The definition asks only whether *some* crossing-free drawing exists: a graph is still planar even if a first, careless attempt to draw it has crossings. The complete graph $K_4$ is the standard warning. Drawn as a square with both diagonals it shows one crossing, yet it *is* planar, because it can be redrawn as a triangle with a single central vertex, no two edges then meeting except at a vertex.




**Faces.**

Once a connected graph is embedded without crossings, its edges divide the plane into connected regions called **faces** (or **regions**). Exactly one of them is unbounded, reaching out to infinity in every direction; this is the **infinite face** or **outer face**. The rest are bounded. It is essential to the counting below that **the outer face is counted as a face**. The number of faces is written $f$.



The embedding above has $v=6$ vertices, $e=9$ edges and $f=5$ faces: the four bounded faces $f_1, f_2, f_3, f_4$ together with the outer face $f_5$.

Each face is enclosed by a closed walk of edges, and the number of edges along that walk is the **degree of the face**. An edge that lies between two different faces contributes 1 to the degree of each; a **bridge** (an edge on no cycle) has the same face on both sides and is traversed twice by that face's boundary walk. Either way every edge contributes exactly 2 to the total of all face degrees, giving the **face-sum relation**

$$(\text{sum of all face degrees})=2e,$$

the face-analogue of the handshaking lemma. It is the second ingredient — after Euler's formula — in the edge bounds proved below.

**Euler's formula.**

The central result is that the alternating count $v-e+f$ is always the same.

**Theorem (Euler's formula).** *In any planar embedding of a connected planar graph with $v$ vertices, $e$ edges and $f$ faces,*

$$v-e+f=2.$$

The value 2 depends on nothing but the graph being connected and planar — not on the particular embedding, nor on the shapes of the regions.

**Proof.** We argue by strong induction on the number of edges $e$.

*Base case* $(e=0)$. A connected graph with no edges consists of a single vertex, so $v=1$, and the whole plane is one region, so $f=1$. Then $v-e+f=1-0+1=2$.

*Inductive step.* Let $G$ be a connected planar graph with $e\geq 1$ edges, and assume the formula holds for every connected planar graph with fewer than $e$ edges. There are two cases.

*Case 1: $G$ contains a cycle.* Choose an edge $\varepsilon$ that lies on a cycle. Because $\varepsilon$ belongs to a cycle it is not a bridge, so deleting it leaves the graph connected. Moreover the cycle through $\varepsilon$ is a closed curve, so $\varepsilon$ has a *different* face on each side; deleting it removes the shared boundary and merges those two faces into one. Hence $G-\varepsilon$ is a connected planar graph with $v$ vertices, $e-1$ edges and $f-1$ faces. By the induction hypothesis $v-(e-1)+(f-1)=2$, which rearranges directly to $v-e+f=2$.

*Case 2: $G$ contains no cycle.* Then $G$ is connected and acyclic — a **tree** — and since $e\geq 1$ it has at least one **leaf**, a vertex of degree 1. Delete a leaf $u$ together with its single edge. The result $G-u$ is a smaller tree, still connected and planar, with $v-1$ vertices and $e-1$ edges; a tree encloses no region, so $G$ and $G-u$ each have exactly one face and $f$ is unchanged. By the induction hypothesis $(v-1)-(e-1)+f=2$, which is again $v-e+f=2$.

In both cases the formula holds for $G$, completing the induction — as required.

**Trees are planar.** A tree contains no cycle, so it can always be laid out without any crossing: **every tree is planar**, and every plane drawing of a tree has a single face. Putting $f=1$ into Euler's formula gives $v-e+1=2$, that is $e=v-1$, recovering the familiar edge count of a tree as a special case of the theorem.

**An upper bound on the edges.**

Euler's formula caps how many edges a *simple* planar graph can carry.

**Result.** *For a simple connected planar graph with $v\geq 3$,*

$$e\leq 3v-6.$$

**Proof.** Suppose first that $G$ has a cycle, so that $e\geq 3$. In a simple graph no face can be bounded by fewer than three edges: a boundary of one or two edges would require a loop or a pair of parallel edges, both forbidden in a simple graph. Every face therefore has degree at least 3, and summing over all $f$ faces, $3f \le (\text{sum of face degrees}) = 2e$, so $f \le 2e/3$. Substituting into Euler's formula,

$$2 = v - e + f \le v - e + 2e/3 = v - e/3,$$

hence $e/3 \le v - 2$, giving $e \le 3v - 6$. If instead $G$ has no cycle it is a tree, and $e = v - 1 \le 3v - 6$ holds for every $v \ge 3$; so the bound holds in all cases — as required.

If a simple connected graph has **more** than $3v - 6$ edges it therefore cannot be planar. The converse fails: meeting the bound does not force planarity, as $K_{3,3}$ shows below.

**A sharper bound when there are no triangles.** If a simple graph additionally contains no triangle — no cycle of length 3, which holds for every bipartite graph — then each face is bounded by at least four edges. Repeating the argument with $4f \le 2e$ gives $f \le e/2$, and Euler's formula then yields

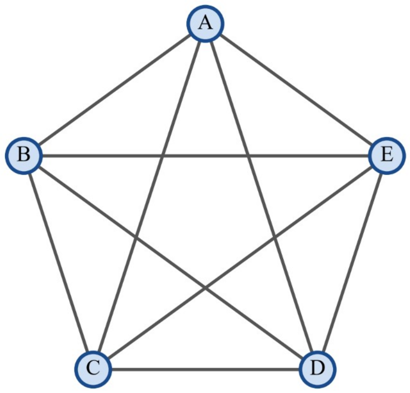
$$e \le 2v - 4 \, (v \ge 3, \text{triangle-free}).$$

**Complete and complete bipartite graphs.**

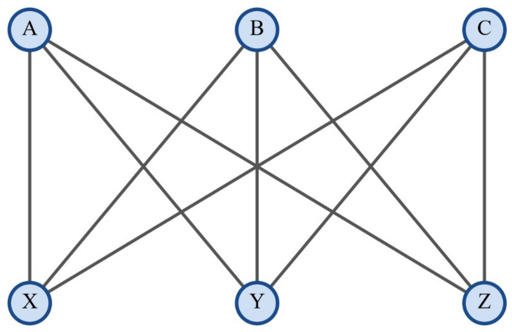
The relevant edge counts, established earlier in this chapter, are

$$K_n : e = \binom{n}{2} = \frac{n(n-1)}{2}, K_{m,n} : e = mn.$$

*The graph $K_5$ is not planar.* It is simple and connected with $v = 5$ and $e = \binom{5}{2} = 10$. Planarity would require $e \le 3v - 6 = 9$, yet $K_5$ has $10 > 9$ edges. No embedding can exist.




*The graph $K_{3,3}$ is not planar.* It is simple, connected and bipartite — hence triangle-free — with $v = 6$ and $e = 3 \times 3 = 9$. The triangle-free bound requires $e \le 2v - 4 = 8$, but $K_{3,3}$ has $9 > 8$ edges. Note that the weaker bound $3v - 6 = 12$ *is* satisfied here; it is simply not sharp enough to expose this graph, which is exactly why the triangle-free refinement is needed.




These two graphs are the basic obstructions to planarity, and within this course they are applied directly through the two conditions:

- **$K_n$ is planar exactly when $n \le 4$.** For $n \le 4$ an embedding exists (a point, an edge, a triangle, or the $K_4$ picture above); for $n \ge 5$ the graph contains $K_5$ as a subgraph and so cannot be planar.
- **$K_{m,n}$ is planar exactly when $m \le 2$ or $n \le 2$.** If one part has at most two vertices the graph can be drawn without crossings; if both parts have at least three vertices it contains $K_{3,3}$ and cannot be planar.

**Applying Euler's formula.** Because $v - e + f = 2$ locks the three counts together, any one of them can be recovered from the other two for a connected planar graph:

$$e = v + f - 2, f = 2 - v + e, v = 2 - f + e.$$

Paired with the handshaking lemma (which supplies $e$ from the vertex degrees) and with the planarity bounds (which test whether a drawing is even possible), this settles the counting problems that follow.

---

## Worked examples

### *Example 1 — scaffolded*

A connected planar graph has $v = 10$ vertices and $e = 15$ edges. Find the number of faces $f$ in a planar embedding, and check that the edge bound $e \le 3v - 6$ is satisfied.

| Working | Comment |
|---|---|
| The graph is connected and planar, so Euler's formula $v - e + f = 2$ applies. | Euler's formula holds for any connected planar embedding. |
| Make $f$ the subject: $f = 2 - v + e$. | Rearrange before substituting. |
| $f = 2 - 10 + 15 = 7$. | So the embedding has 7 faces (including the outer face). |
| Bound: $3v - 6 = 3(10) - 6 = 24$. | Compute the simple-planar limit. |
| Since $15 \le 24$, the edge count is consistent with a simple planar graph. | The bound is a necessary check, not a guarantee of planarity. |

### *Example 2 — scaffolded*

A connected planar graph has seven vertices, of degrees $3, 3, 3, 3, 2, 2, 2$. Find the number of edges, and hence the number of faces of a planar embedding.

| Working | Comment |
|---|---|
| Sum of degrees $= 3 + 3 + 3 + 3 + 2 + 2 + 2 = 18$. | Add the seven vertex degrees. |
| Handshaking lemma: the degrees sum to $2e$, so $2e = 18$ and $e = 9$. | Each edge contributes 2 to the degree total. |
| Euler's formula: $f = 2 - v + e = 2 - 7 + 9 = 4$. | Now $v = 7$ and $e = 9$ are both known. |

| Working | Comment |
| --- | --- |
| Check: $3v-6=3(7)-6=15$, and $9\leq 15$. | Edge count is within the simple-planar bound. |

***Example 3 — unscaffolded***

Show that the complete graph $K_5$ is not planar.

The graph $K_5$ is simple and connected, with $v=5$ vertices and

$$e=\binom{5}{2}=\frac{5\cdot 4}{2}=10$$

edges. If $K_5$ were planar, then, being simple and connected with $v=5\geq 3$, it would have to satisfy $e\leq 3v-6=3(5)-6=9$. But $K_5$ has $10$ edges, and $10>9$, contradicting the bound. No planar embedding of $K_5$ can exist.

$\therefore K_5$ is not planar.

***Example 4 — unscaffolded***

Prove that every simple connected planar graph with at least three vertices has a vertex of degree at most $5$.

Let $G$ be a simple connected planar graph with $v\geq 3$ vertices and $e$ edges, and suppose, for contradiction, that **every** vertex has degree at least $6$. Then the degrees sum to at least $6v$, and by the handshaking lemma that sum is exactly $2e$, so $2e\geq 6v$ and therefore $e\geq 3v$. On the other hand, because $G$ is simple, connected and planar with $v\geq 3$, the edge bound gives $e\leq 3v-6$. Combining the two inequalities,

$$3v\leq e\leq 3v-6,$$

so $3v\leq 3v-6$, that is $0\leq -6$, which is impossible. The supposition is false, so at least one vertex has degree $5$ or less.

$\therefore$ every simple connected planar graph with $v\geq 3$ has a vertex of degree at most $5$, as required.

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## Practice questions

### *Scaffolded*

**Q1.** A connected planar graph has $v=9$ vertices and $f=5$ faces. (a) State Euler's formula for a connected planar graph. (b) Rearrange it to make $e$ the subject. (c) Find the number of edges $e$. (d) Verify that $e\leq 3v-6$.

**Q2.** A graph has vertices $P,Q,R,S,T$ and edges $PQ,QR,RS,ST,TP,PR,PS$, and is drawn in the plane with no crossings. (a) Write down $v$ and $e$. (b) Use Euler's formula to find the number of faces $f$. (c) State how many of these faces are bounded. (d) Verify that $e\leq 3v-6$.

**Q3.** A connected planar graph has six vertices, of degrees $4,4,3,3,3,3$. (a) Find the sum of the degrees. (b) Use the handshaking lemma to find the number of edges $e$. (c) Use Euler's formula to find the number of faces $f$. (d) Check that $e\leq 3v-6$.

**Q4.** Consider the complete graphs $K_6$ and $K_7$. (a) Find the number of edges of $K_6$. (b) Find the number of edges of $K_7$. (c) Using the bound $e\leq 3v-6$, determine whether $K_6$ is planar.

**Q5.** Consider the complete bipartite graphs $K_{4,5}$ and $K_{2,7}$. (a) Find the number of edges of $K_{4,5}$. (b) Find the number of edges of $K_{2,7}$. (c) State which of the two graphs is planar, giving a brief reason for each.

**Q6.** A tree has 12 vertices. (a) State the number of edges of the tree. (b) State the number of faces in a plane drawing of the tree. (c) Verify Euler's formula for the tree.

### *Unscaffolded*

**Q7.** A connected planar graph has $15$ vertices and $20$ edges. How many faces does a planar embedding of it have?

**Q8.** A connected planar graph is embedded with $17$ edges and $10$ faces. How many vertices does it have?

**Q9.** Show that the complete bipartite graph $K_{3,3}$ is not planar.

**Q10.** Find the number of edges of $K_{10}$, and determine whether $K_{10}$ is planar.

**Q11.** What is the greatest number of edges that a simple connected planar graph on $6$ vertices can have?

**Q12.** Prove that the complete graph $K_5$ is not planar.

**Q13.** A connected planar graph is embedded so that every face, including the outer face, is bounded by exactly four edges, and the graph has $8$ vertices. Find the number of edges and the number of faces.

**Q14.** Determine, with justification, all values of $n$ for which the complete graph $K_n$ is planar.

**Q15.** A connected planar graph is embedded with $5$ vertices and $8$ edges. Find the number of faces, and state whether these values are consistent with the graph being simple.

**Q16.** Prove that every simple connected planar graph with $v \geq 3$ vertices satisfies $e \leq 3v - 6$.

**Q17.** The vertices and edges of a convex polyhedron form a connected planar graph. A certain polyhedron has $12$ vertices and $30$ edges. How many faces does it have?

**Q18.** A simple graph has $7$ vertices and $16$ edges. Explain why it cannot be planar.

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## Answers

**Q1.** (a) $v-e+f=2$; (b) $e=v+f-2$; (c) $e=12$; (d) $3v-6=21$, and $12\leq 21$. **Q2.** (a) $v=5$, $e=7$; (b) $f=4$; (c) 3 bounded faces; (d) $3v-6=9$, and $7\leq 9$. **Q3.** (a) 20; (b) $e=10$; (c) $f=6$; (d) $3v-6=12$, and $10\leq 12$. **Q4.** (a) 15; (b) 21; (c) not planar, since $15>3v-6=12$. **Q5.** (a) 20; (b) 14; (c) $K_{2,7}$ is planar (one part has only 2 vertices); $K_{4,5}$ is not planar (it contains $K_{3,3}$). **Q6.** (a) 11; (b) 1; (c) $12-11+1=2$. **Q7.** $f=7$. **Q8.** $v=9$. **Q9.** Not planar: $K_{3,3}$ is triangle-free with $e=9>2v-4=8$. **Q10.** 45 edges; not planar, since $45>3v-6=24$. **Q11.** 12. **Q12.** Not planar: $e=10>3v-6=9$. **Q13.** $e=12$ and $f=6$. **Q14.** $K_n$ is planar exactly when $n\leq 4$. **Q15.** $f=5$; consistent, since $8\leq 3v-6=9$. **Q16.** See fully-worked solution. **Q17.** $f=20$. **Q18.** $16>3v-6=15$, so it cannot be planar.

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## Fully-worked solutions

**Q1.** (a) For a connected planar graph, Euler's formula is $v-e+f=2$. (b) Making $e$ the subject, $e=v+f-2$. (c) With $v=9$ and $f=5$, $e=9+5-2=12$. (d) The bound is $3v-6=3(9)-6=21$, and $12\leq 21$, so the edge count is consistent with a simple planar graph.

**Q2.** (a) The five listed vertices give $v=5$, and the seven listed edges $PQ, QR, RS, ST, TP, PR, PS$ give $e=7$. (b) By Euler's formula, $f=2-v+e=2-5+7=4$. (c) One of the four faces is the outer face, so $4-1=3$ faces are bounded (the drawing splits into three triangular regions $PQR$, $PRS$, $PST$). (d) The bound is $3v-6=3(5)-6=9$, and $7\leq 9$.

**Q3.** (a) The degrees sum to $4+4+3+3+3+3=20$. (b) By the handshaking lemma the degrees sum to $2e$, so $2e=20$ and $e=10$. (c) By Euler's formula, $f=2-v+e=2-6+10=6$. (d) The bound is $3v-6=3(6)-6=12$, and $10\leq 12$, so the values are consistent.

**Q4.** (a) $K_6$ has $\binom{6}{2}=\frac{6\cdot 5}{2}=15$ edges. (b) $K_7$ has $\binom{7}{2}=\frac{7\cdot 6}{2}=21$ edges. (c) $K_6$ is simple and connected with $v=6$, so if it were planar it would satisfy $e\leq 3v-6=12$; but it has 15 edges and $15>12$, so $K_6$ is not planar.

**Q5.** (a) $K_{4,5}$ has $e=4\times 5=20$ edges. (b) $K_{2,7}$ has $e=2\times 7=14$ edges. (c) $K_{2,7}$ has a part with only two vertices, so by the condition "$K_{m,n}$ is planar when $m\leq 2$ or $n\leq 2$" it is planar. $K_{4,5}$ has both parts of size at least three, so it contains $K_{3,3}$ and is not planar (equivalently, it is triangle-free with $v=9$ and $e=20>2v-4=14$).

**Q6.** (a) A tree on $v$ vertices has $v-1$ edges, so the tree has $12-1=11$ edges. (b) A tree contains no cycle, so a plane drawing encloses no region and has just the outer face: $f=1$. (c) Euler's formula gives $v-e+f=12-11+1=2$, as required.

**Q7.** The graph is connected and planar, so $v-e+f=2$. With $v=15$ and $e=20$, $f=2-v+e=2-15+20=7$.

**Q8.** Rearranging Euler's formula for $v$ gives $v=2-f+e$. With $e=17$ and $f=10$, $v=2-10+17=9$.

**Q9.** $K_{3,3}$ is simple, connected and bipartite, so it is triangle-free, with $v=6$ vertices and $e=3\times 3=9$ edges. If it were planar, the triangle-free bound $e\leq 2v-4$ would give $e\leq 2(6)-4=8$. But $K_{3,3}$ has 9 edges and $9>8$, a contradiction. Hence $K_{3,3}$ is not planar. (The ordinary bound $3v-6=12$ is satisfied here, so it is the triangle-free bound that is needed.)

**Q10.** $K_{10}$ has $\binom{10}{2}=\frac{10\cdot 9}{2}=45$ edges. It is simple and connected with $v=10$, so planarity would require $e\leq 3v-6=3(10)-6=24$. Since $45>24$, $K_{10}$ is not planar.

**Q11.** For a simple connected planar graph the edge count satisfies $e\leq 3v-6$. With $v=6$ this gives $e\leq 3(6)-6=12$, so at most 12 edges are possible. The value 12 is attainable — for example the octahedron is a simple planar graph with 6 vertices and 12 edges (and 8 faces, since $6-12+8=2$). Hence the greatest possible number of edges is 12.

**Q12.** $K_5$ is simple and connected, with $v=5$ vertices and $e=\binom{5}{2}=\frac{5\cdot 4}{2}=10$ edges. Assume, for contradiction, that $K_5$ is planar. Then, as a simple connected planar graph with $v=5\geq 3$, it must satisfy $e\leq 3v-6=3(5)-6=9$. But $e=10$ and $10>9$, a contradiction. Therefore $K_5$ is not planar, as required.

**Q13.** Let the embedding have $e$ edges and $f$ faces. Every face is bounded by exactly four edges, so the sum of the face degrees is $4f$; by the face-sum relation this equals $2e$, giving $4f=2e$, i.e. $e=2f$. Euler's formula with $v=8$ then reads

$$8-e+f=2\Rightarrow 8-2f+f=2\Rightarrow f=6,$$

and hence $e=2f=12$. (As a check, $3v-6=18\geq 12$, and the triangle-free bound $2v-4=12=e$ is met with equality, consistent with every face being a quadrilateral.)

**Q14.** For $n\leq 4$ the graph $K_n$ can be drawn without crossings: $K_1$ is a single point, $K_2$ a single edge, $K_3$ a triangle, and $K_4$ a triangle with a central vertex. So $K_n$ is planar for $n=1,2,3,4$. For $n\geq 5$, $K_n$ contains $K_5$ as a subgraph; since $K_5$ is not planar (it is simple and connected with $e=\binom{5}{2}=10>3(5)-6=9$) and any subgraph of a planar graph is planar, $K_n$ cannot be planar. Therefore $K_n$ is planar exactly when $n\leq 4$.

**Q15.** By Euler's formula, $f=2-v+e=2-5+8=5$. For consistency with a simple planar graph the bound $e\leq 3v-6=3(5)-6=9$ must hold; since $8\leq 9$, the values are consistent with the graph being simple.

**Q16.** Let $G$ be a simple connected planar graph with $v\geq 3$ vertices and $e$ edges, embedded in the plane with $f$ faces. If $G$ has a cycle then $e\geq 3$, and because $G$ is simple every face is bounded by at least three edges, so each of the $f$ faces has degree at least 3. Summing the face degrees and using the face-sum relation,

$$3f\leq(\text{sum of face degrees})=2e,\ \text{so}\ f\leq 2e/3.$$

Substituting into Euler's formula $v-e+f=2$,

$$2=v-e+f\leq v-e+2e/3=v-e/3,$$

so $e/3\leq v-2$ and hence $e\leq 3v-6$. If instead $G$ has no cycle it is a tree, with $e=v-1\leq 3v-6$ for every $v\geq 3$. In all cases $e\leq 3v-6$, as required.

**Q17.** The polyhedron's vertices and edges form a connected planar graph, so Euler's formula applies: $f=2-v+e=2-12+30=20$. The polyhedron has 20 faces (it is an icosahedron).

**Q18.** A simple planar graph with $v\geq 3$ vertices must satisfy $e\leq 3v-6$. Here $v=7$, so a planar drawing would need $e\leq 3(7)-6=15$. But the graph has 16 edges, and $16>15$, so no planar embedding is possible: the graph cannot be planar.

## Theory

Once a graph models a real situation — regions joined by bridges, sites joined by walkways, joints joined by struts — the natural question is how one may **travel** through it: in what order can the edges, or the vertices, be visited? This section makes those route-types precise, then answers two famous existence questions: when can every **edge** be travelled exactly once, and when can every **vertex** be visited exactly once. The first has a complete and beautifully simple answer in terms of vertex degrees; the second, surprisingly, has no such simple answer at all.

Throughout we use the **degree** $deg(v)$ of a vertex $v$ — the number of edge-ends meeting at $v$, a loop counting twice — and the **handshaking lemma** from Section 2A: the degrees of all vertices sum to twice the number of edges,

$$\sum_{v} deg(v)=2|E|.$$

**Walks and their length.**

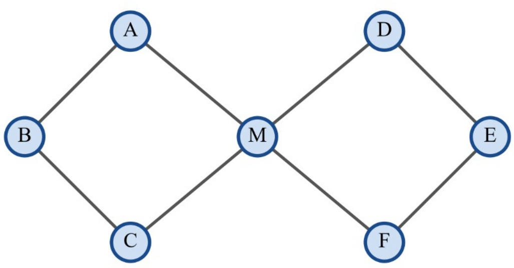
A **walk** is any route through a graph: a sequence of vertices in which each vertex is joined to the next by an edge. We record a walk by listing its vertices, such as $A-C-B-C$. In a walk, **both vertices and edges may be repeated** — nothing is forbidden, so *every* route is a walk. The **length** of a walk is the **number of edges** it uses, counting a re-used edge again each time; the length is always one less than the number of vertices listed. A walk whose first and last vertices are the **same** is **closed**; otherwise it is **open**. A graph is **connected** if there is a walk between every pair of its vertices; if not, it splits into separate **components**.

**Trails, paths, circuits and cycles.**

Restricting what may be repeated gives four increasingly strict route-types.

- A **trail** is a walk with **no repeated edge**. A trail may still revisit a vertex, provided it arrives along a new edge each time.
- A **path** is a walk with **no repeated vertex**. Since a path never returns to a vertex, it can never re-use an edge either, so **every path is automatically a trail**.
- A **circuit** is a **closed trail**: it starts and finishes at the same vertex and repeats no edge, but it *may* pass through an intermediate vertex more than once.
- A **cycle** is a **closed path**: a circuit that also repeats no vertex except that the first and last coincide.

Every path is a trail and every trail is a walk; every cycle is a circuit and every circuit is a closed walk. When asked to classify a route, give the **most precise** name that applies. The graph below is built from two squares sharing the single corner $M$, which therefore has degree $4$.




On it, $M-A-B-C-M$ is a **cycle** of length $4$ (closed, nothing repeated but the ends); $M-A-B-C-M-D-E-F-M$ repeats no edge and returns to $M$, but passes through $M$ a second time, so it is a **circuit that is not a cycle**, of length $8$. Only $M$, with its degree of $4$, has enough edge-ends for a circuit to pass through it twice.

**Euler trails and Euler circuits.**

An **Euler trail** is a trail that uses **every edge of the graph exactly once**; an **Euler circuit** is a closed Euler trail — an Euler trail that returns to its starting vertex. (These honour Leonhard Euler, who settled the question below in 1736.) A graph that has an Euler circuit is called **Eulerian**. The remarkable fact is that whether such a route exists can be decided *without any searching*, purely from the vertex degrees.

**The even-degree criterion.** Suppose a route is a trail through a vertex $v$ that $v$ is not an endpoint of. Each time the route visits $v$ it arrives along one edge and departs along another, and — the route being a trail — these edges are all different. So the edges at such a $v$ are used up in **arrive–depart pairs**, which means an Euler trail forces $deg(v)$ to be **even** at every vertex it merely passes through. This single observation yields Euler's theorem.

> **Theorem (Euler).** Let $G$ be a connected graph with at least one edge. 1. $G$ has an **Euler circuit** if and only if **every vertex has even degree**. 2. $G$ has an **Euler trail that is not a circuit** if and only if **exactly two vertices have odd degree**; every such trail must **begin at one** odd vertex and **end at the other**.

*Why the degree conditions are necessary.* Take an Euler circuit. It has no special start in the sense that arrival and departure are paired even at the start/finish vertex (the first departure pairs with the final arrival). So at *every* vertex the incident edges pair up, and each degree is even. For an open Euler trail from $s$ to $t \neq s$, every vertex except $s$ and $t$ is only ever passed through, so has even degree; but the very first departure from $s$ and the very last arrival at $t$ are each unpaired, making $deg(s)$ and $deg(t)$ odd. So exactly two vertices — the two ends — are of odd degree.

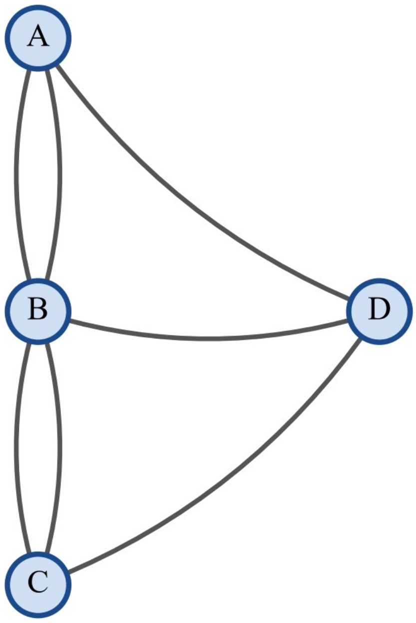
*Why they are also sufficient.* If a connected graph has all even degrees, a route may be built greedily: start anywhere, keep walking along unused edges; because every degree is even you can never become stranded except back at the start, and any edges missed form smaller all-even pieces that can be spliced in. This constructs an Euler circuit. If instead there are exactly two odd vertices, begin at one of them; the same process ends at the other, giving an open Euler trail.

A useful check comes from the handshaking lemma. Since $\sum_v deg(v) = 2|E|$ is even, the odd-degree vertices must sum to an even total, so **the number of odd-degree vertices is always even** — never one, never three. The only possibilities are therefore:

| Odd-degree vertices | 0 | 2 | $4, 6, \ldots$ |
|---|---|---|---|
| Best route using every edge once | Euler **circuit** | Euler **trail** (open) | **neither** |

**The Königsberg bridge problem.**

The city of Königsberg lay on the river Pregel, which split the land into four regions: the **north bank** $A$, the **south bank** $C$, and two islands $B$ and $D$, joined by seven bridges. The townsfolk asked whether one could take a stroll that crossed **every bridge exactly once**. Modelled as a graph, each region is a vertex and each bridge an edge — producing a **multigraph**, since some regions are joined by two bridges.



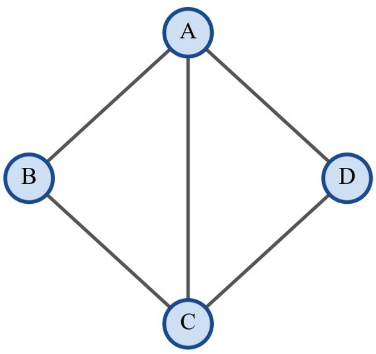

Here two bridges join $A$ to $B$ and two join $B$ to $C$, with single bridges $B-D$, $A-D$ and $C-D$. The degrees are

$$deg(A)=3, deg(B)=5, deg(C)=3, deg(D)=3,$$

so **all four vertices have odd degree**. A stroll crossing every bridge once would be an Euler trail, which needs 0 or 2 odd vertices; with four, none can exist. This is exactly what Euler proved — no such walk is possible — and in doing so he founded graph theory.

**Hamiltonian paths and cycles.**

The companion question replaces *edges* by *vertices*. A **Hamiltonian path** is a path that visits **every vertex of the graph exactly once**; a **Hamiltonian cycle** is a cycle that visits every vertex exactly once and returns to the start. A graph with a Hamiltonian cycle is called **Hamiltonian**. The distinction from Euler routes is essential: an Euler route accounts for every **edge**, a Hamiltonian route for every **vertex**.



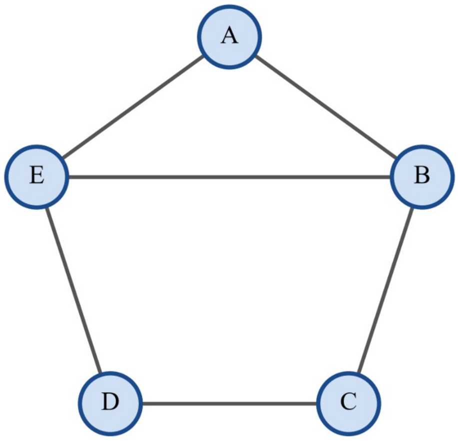

The graph above (two triangles sharing the edge $A-C$) has $deg(A)=deg(C)=3$ and $deg(B)=deg(D)=2$: exactly two odd vertices, so it has an Euler trail — for instance $A-B-C-A-D-C$ — running from $A$ to $C$, but no Euler circuit. It is also Hamiltonian, with cycle $A-B-C-D-A$.

Here the contrast with Euler routes becomes sharp: **there is no simple degree condition that decides whether a graph is Hamiltonian.** A graph may have every vertex of high degree yet no Hamiltonian cycle, or low degrees yet possess one. Deciding Hamiltonicity in general is genuinely hard — no efficient rule like the even-degree criterion is known — so a Hamiltonian cycle must be found by construction, and its non-existence argued case by case (often by finding a vertex whose removal breaks the graph into too many pieces). The two properties are logically independent: as Worked Example 4 shows, a graph can be Eulerian without being Hamiltonian, and Hamiltonian without being Eulerian.

## Worked examples

### *Example 1 — scaffolded*

The ground floor of a natural-history museum has five galleries $A, B, C, D, E$ joined by the internal doorways $A-B$, $B-C$, $C-D$, $D-E$, $E-A$ (around the floor) and $B-E$ (across the middle), as shown.

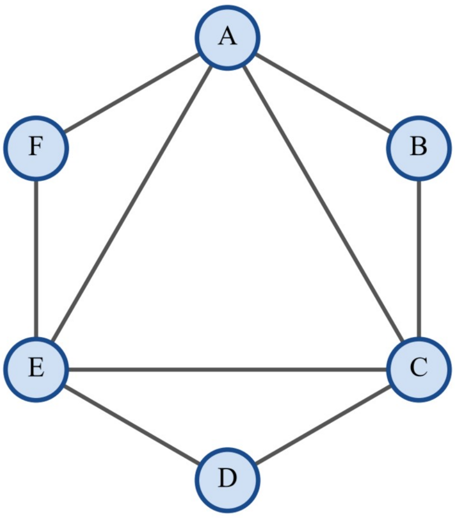



Classify each visitor route below, giving the most precise name — walk, trail, path, circuit or cycle — and state its length. (a) $A-B-C-D-E-A$ (b) $A-B-E-D-C-B$ (c) $A-B-E-B-C$ (d) $A-E-B-C-D$

| Working | Comment |
|---|---|
| (a) Uses doorways $AB, BC, CD, DE, EA$ — all different — and returns to $A$ with no other repeated vertex. **Cycle**, length $5$. | Closed, no repeated edge or vertex: a cycle. |
| (b) Uses $AB, BE, DE, CD, BC$ — five different doorways — but visits $B$ twice and finishes away from the start. **Trail (not a path)**, length $5$. | No repeated edge makes it a trail; the repeated vertex $B$ stops it being a path. |
| (c) Steps $A-B$, $B-E$, then $E-B$: doorway $BE$ is used twice. **Walk only**, length $4$. | A repeated edge means it is not even a trail. |
| (d) Uses $EA, BE, BC, CD$ and repeats no vertex at all. **Path**, length $4$. | No repeated vertex is the strictest condition. |

### *Example 2 — scaffolded*

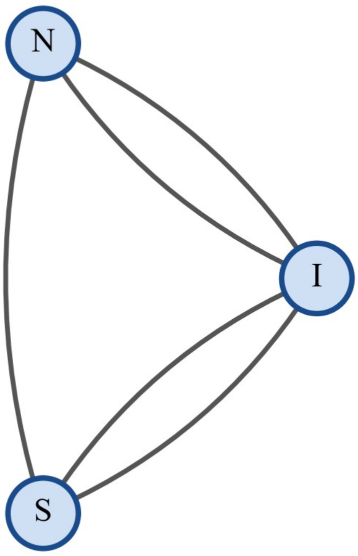
A street-sweeping vehicle is based at a depot at intersection $A$ of a small estate. The streets join intersections as $A-B$, $B-C$, $C-D$, $D-E$, $E-F$, $F-A$ around the outside, together with $A-C$, $C-E$, $E-A$ across the middle. Determine whether the sweeper can travel along every street exactly once and return to the depot, and if so give such a route.




| Working | Comment |
|---|---|
| Degrees: $deg(A)=4$, $deg(B)=2$, $deg(C)=4$, $deg(D)=2$, $deg(E)=4$, $deg(F)=2$. | Count the streets meeting each intersection. |
| Every degree is even, and the network is connected. | The Euler-circuit criterion: connected with all degrees even. |
| So an **Euler circuit** exists — a closed route using every street once. | The criterion is satisfied, so such a route exists. |
| Starting and finishing at the depot: $A-B-C-D-E-F-A-C-E-A$. | Take the outer ring, then the inner triangle $A-C-E$, back to $A$. |
| The route uses all $9$ streets once and returns to $A$. | Length $9$; the depot is regained. |

***Example 3 — unscaffolded***

The river town of Marlowe has a north bank $N$, a south bank $S$, and a mill island $I$ in mid-stream. Five footbridges cross the water: two between $N$ and $I$, two between $S$ and $I$, and one directly between $N$ and $S$. A ranger wishes to inspect every footbridge by walking across each exactly once. Decide whether this is possible, whether she can return to her starting bank, and give a suitable route.




Modelling the banks and island as vertices and the footbridges as edges gives a multigraph with degrees

$$deg(N)=3, deg(I)=4, deg(S)=3.$$

The island has even degree but both banks are odd, so there are **exactly two odd-degree vertices**. By Euler's theorem the network has an Euler trail — a walk crossing every footbridge once — but no Euler circuit, and the trail must **begin at one bank and end at the other**. Starting on the north bank,

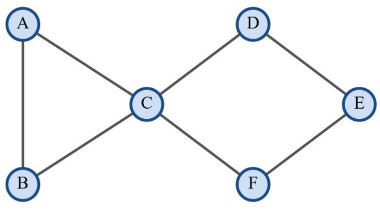
$$N-I-S-I-N-S$$

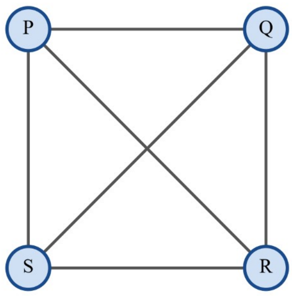
crosses the first $N-I$ bridge, a first $S-I$ bridge, the second $S-I$ bridge, the second $N-I$ bridge and finally the $N-S$ bridge: five bridges, each once.

∴ The ranger can cross every footbridge exactly once, for example along $N-I-S-I-N-S$, but she cannot return to her starting bank, because the two odd-degree banks force any such walk to start at one and finish at the other.

### *Example 4 — unscaffolded*

Show that being **Eulerian** and being **Hamiltonian** are independent properties, by examining the two networks below. Network 1 is a triangle $A-B-C$ and a square $C-D-E-F-C$ sharing the single vertex $C$. Network 2 is the complete graph on the four vertices $P, Q, R, S$ (every pair joined).




In Network 1 the degrees are $deg(A)=deg(B)=deg(D)=deg(E)=deg(F)=2$ and $deg(C)=4$, all even, and the graph is connected, so it has an **Euler circuit**, for instance $A-B-C-D-E-F-C-A$. It has **no Hamiltonian cycle**: any Hamiltonian cycle would visit $C$ exactly once, yet to include both the triangle vertices $A, B$

and the square vertices $D, E, F$ the cycle must pass from one side to the other and back, and the only vertex joining the two sides is $C$ — so $C$ would have to be used twice, which a cycle forbids.

In Network 2 every vertex has degree 3, so all four vertices are odd; with more than two odd vertices it has **neither an Euler trail nor an Euler circuit**. Yet $P-Q-R-S-P$ visits every vertex once and returns to the start, so Network 2 **is Hamiltonian**.

∴ Network 1 is Eulerian but not Hamiltonian and Network 2 is Hamiltonian but not Eulerian, so neither property implies the other — as required.

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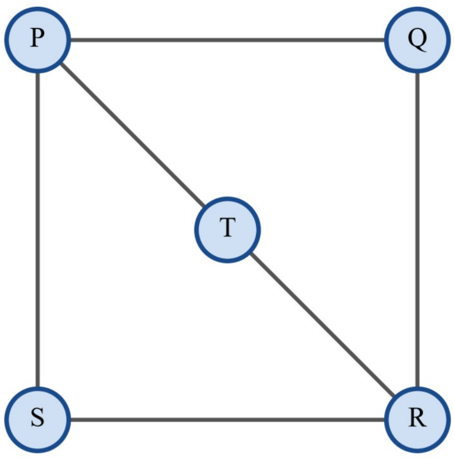
## Practice questions

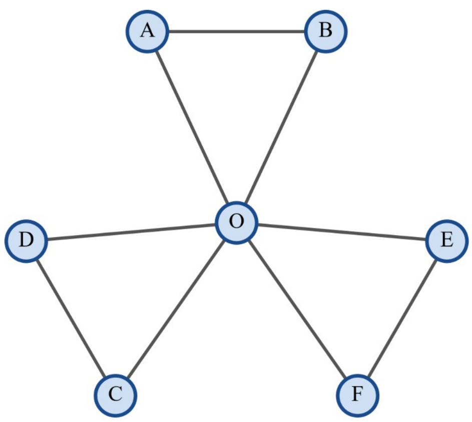
### *Scaffolded*

**Q1.** Four courtyards $P, Q, R, S$ of a college are joined by cloisters $P-Q$, $Q-R$, $R-S$, $S-P$ around a square, and a central lawn $T$ is joined to $P$ and to $R$.




Classify each route below, giving the most precise name, and state its length. (a) $P-Q-R-S-P$ (b) $Q-P-T-R-S$ (c) $S-P-Q-R-Q$ (d) $T-P-Q-R-S-P$

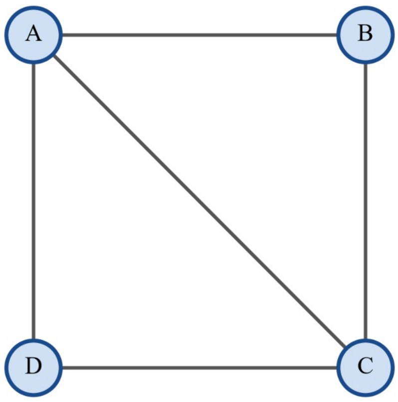
**Q2.** A mountain-bike park has three tracks that meet only at a single start-and-finish gate $O$: the first runs $O-A-B-O$, the second runs $O-C-D-O$ and the third runs $O-E-F-O$, so the network below has nine sections of track as its edges. A course marshal wants to ride along every section exactly once and return to the gate.




(a) State the degree of each vertex.

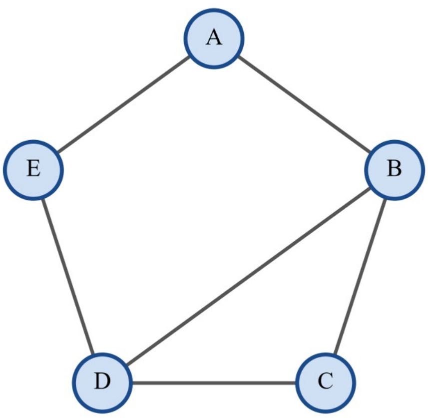
(b) Are all the degrees even?
(c) State whether an Euler circuit exists, giving the reason.
(d) Write down such a circuit, starting and finishing at $O$.

**Q3.** A lagoon has four islets $A, B, C, D$ joined by five footbridges: $A-B$, $B-C$, $C-D$, $D-A$ around the lagoon and $A-C$ across it.



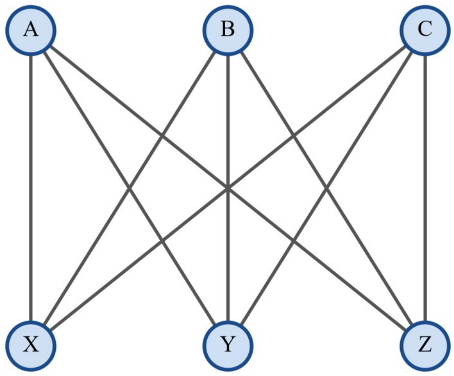

(a) State the degree of each islet.
(b) How many islets have odd degree?
(c) Decide whether a walk crossing every footbridge exactly once exists; if so, give one and state where it starts and finishes.
(d) State which single extra footbridge could be built so that such a walk could also return to its starting islet.

**Q4.** Five campus buildings $A, B, C, D, E$ are linked by covered walkways $A-B$, $B-C$, $C-D$, $D-E$, $E-A$ and $B-D$.




(a) State the degree of each building.
(b) State how many buildings have odd degree, and hence which of an Euler circuit or an Euler trail (if either) the network has.
(c) Give such a route, stating the buildings where it begins and ends.
(d) Explain why no route using every walkway once can return to its starting building.

**Q5.** Three cottages $A, B, C$ are each connected to three services — water $X$, gas $Y$ and electricity $Z$ — so that every cottage is joined to every service and no two cottages (and no two services) are joined directly.



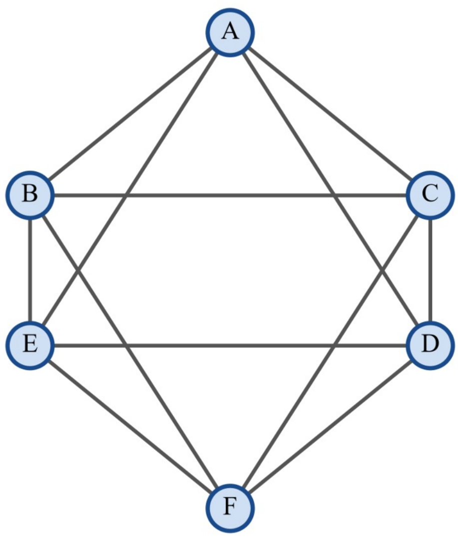

(a) State the degree of each vertex.
(b) Find a Hamiltonian cycle (a route visiting every vertex exactly once and returning to the start).
(c) Find a Hamiltonian path.
(d) Decide whether the network is Eulerian, giving a reason.

**Q6.** Each network below is connected. For each, write its degree sequence, count its odd-degree vertices, and state whether it has an Euler circuit, an Euler trail (but not a circuit), or neither. (a) $G_1$: vertices $A, B, C, D$; edges $A-B$, $B-C$, $C-D$, $D-A$. (b) $G_2$: vertices $A, B, C, D, E$; edges $A-B$, $B-C$, $C-D$, $D-E$. (c) $G_3$: vertices $A, B, C, D$; edges $A-B$, $A-C$, $A-D$, $B-C$, $B-D$, $C-D$. (d) Explain why no connected network can have exactly one vertex of odd degree.

### *Unscaffolded*

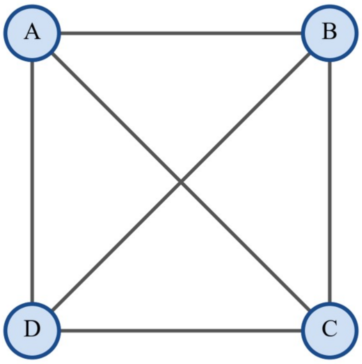
**Q7.** A network has vertices $A, B, C, D, E$ and edges $A-B$, $B-C$, $C-D$, $D-E$, $E-A$, $A-C$ and $C-E$. Classify each route precisely and state its length. (i) $A-B-C-D-E-A$ (ii) $A-C-E-D-C-B$ (iii) $A-C-E-C-D$ (iv) $B-A-E-C-D$

**Q8.** A geodesic play-dome has six joints $A, B, C, D, E, F$ and twelve struts: the four struts $A-B$, $A-C$, $A-D$, $A-E$ from the apex $A$; the four struts $F-B$, $F-C$, $F-D$, $F-E$ to the base joint $F$; and the square $B-C-D-E-B$.



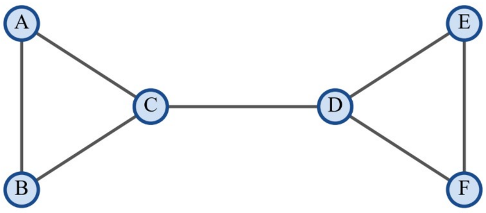

(a) State the degree of each joint.
(b) Explain why a safety inspector can walk along every strut exactly once and finish where she began.
(c) Give such a route (an Euler circuit).

**Q9.** Four rocky islets $A, B, C, D$ are joined by six rope bridges, one between each pair.




(a) Find the degree of each islet and hence explain why no walk can cross every bridge exactly once.
(b) One bridge, $A-B$, is washed away. Show that a walk crossing each of the five remaining bridges exactly once is now possible, and state where it must start and finish.
(c) After the loss of bridge $A-B$, can such a walk also return to its starting islet? Explain.

**Q10.** The network below is two triangles $A-B-C$ and $D-E-F$ joined by the single connector $C-D$.



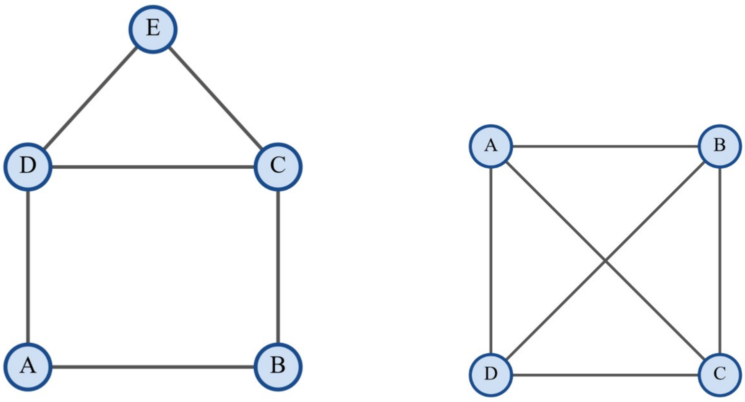

(a) Determine whether the network has an Euler circuit, an Euler trail, or neither, and give a route where one exists.
(b) Prove that the network has no Hamiltonian cycle.
(c) Find a Hamiltonian path.

**Q11.** Prove that in **any** graph the number of vertices of odd degree is even. (You may use the handshaking lemma $\sum_v deg(v) = 2|E|$.)

**Q12.** Prove that if a connected graph has an Euler circuit, then every vertex has even degree.

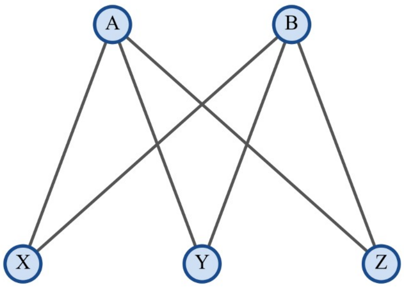
**Q13.** Recall the Königsberg bridges: the north bank $A$, south bank $C$ and islands $B, D$ are joined by two bridges $A-B$, two bridges $B-C$, and one bridge each of $B-D$, $A-D$ and $C-D$. (a) Write the degree of each region and hence prove that no stroll can cross every bridge exactly once. (b) A new bridge is built directly between the two banks $A$ and $C$. Determine whether a stroll crossing every one of the now eight bridges exactly once is possible, and if so between which two regions it must run.

**Q14.** Two connect-the-dots puzzles are shown. Puzzle (i) has vertices $A, B, C, D, E$ and edges $A-B$, $B-C$, $C-D$, $D-A$, $D-E$, $C-E$. Puzzle (ii) has vertices $A, B, C, D$ with edges $A-B$, $B-C$, $C-D$, $D-A$, $A-C$, $B-D$.




(a) State which puzzle can be drawn without lifting the pen and without retracing any line.
(b) For that puzzle, identify the odd-degree vertices and give a suitable route, stating where it starts and finishes.
(c) For the other puzzle, explain why it cannot be drawn in that way.

**Q15.** Two lifeguard towers $A, B$ are each linked to three flag stations $X, Y, Z$, so that every tower is joined to every station and there are no other links.

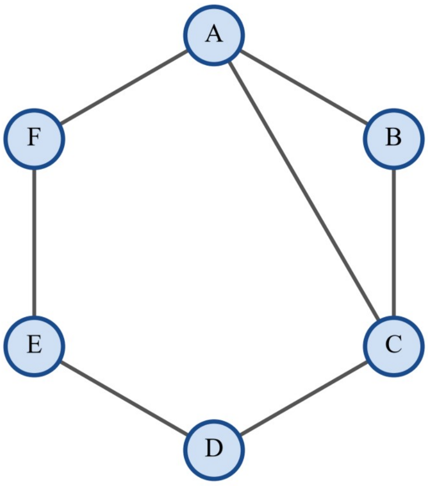



(a) Prove that this network has no Hamiltonian cycle.
(b) Find a Hamiltonian path.

**Q16.** Let $K_n$ be the complete graph on $n$ vertices ($n \geq 3$), in which every pair of vertices is joined by an edge. (a) Determine, with proof, exactly which $K_n$ have an Euler circuit. (b) Show that $K_n$ has a Hamiltonian cycle for every $n \geq 3$, and comment on the contrast with your answer to (a).

**Q17.** Prove that if a connected graph has an open Euler trail (one that is not a circuit), then it has exactly two vertices of odd degree, and the trail begins at one of them and ends at the other.

**Q18.** Six heritage sites $A, B, C, D, E, F$ lie on a ring road $A-B$, $B-C$, $C-D$, $D-E$, $E-F$, $F-A$, with one shortcut $A-C$.




(a) Write the degree sequence and identify the odd-degree sites.

(b) Decide whether the roads support an Euler circuit or an Euler trail, and give a route travelling every road exactly once.

(c) Give a Hamiltonian cycle for the sites.

(d) State briefly what this shows about a graph possessing both an Euler route and a Hamiltonian cycle without being complete or having all even degrees.

## Answers

**Q1.** (a) cycle, length 4; (b) path, length 4; (c) walk only (cloister $QR$ repeated), length 4; (d) trail but not a path ($P$ repeated), length 5. **Q2.** (a) $deg(O)=6$, all others 2; (b) yes; (c) connected with every degree even, so an Euler circuit exists; (d) $O-A-B-O-C-D-O-E-F-O$. **Q3.** (a) $deg(A)=3$, $deg(B)=2$, $deg(C)=3$, $deg(D)=2$; (b) two; (c) yes, an Euler trail $A-B-C-D-A-C$ from $A$ to $C$; (d) a second $A-C$ bridge. **Q4.** (a) $deg(A)=2$, $deg(B)=3$, $deg(C)=2$, $deg(D)=3$, $deg(E)=2$; (b) two odd, so an Euler trail (not a circuit); (c) $B-A-E-D-C-B-D$ from $B$ to $D$; (d) the two odd-degree buildings force any such route to start at one and end at the other. **Q5.** (a) every vertex has degree 3; (b) $A-X-B-Y-C-Z-A$; (c) $A-X-B-Y-C-Z$; (d) no — all six vertices are odd, so it has neither an Euler trail nor a circuit. **Q6.** (a) $2,2,2,2$; 0 odd; Euler circuit. (b) $2,2,2,1,1$; 2 odd; Euler trail. (c) $3,3,3,3$; 4 odd; neither. (d) the number of odd-degree vertices is always even, so it can never be one. **Q7.** (i) cycle, length 5; (ii) trail but not a path, length 5; (iii) walk only ($CE$ repeated), length 4; (iv) path, length 4. **Q8.** (a) every joint has degree 4; (b) all degrees are even and the frame is connected, so an Euler circuit exists; (c) $A-B-C-A-D-C-F-B-E-D-F-E-A$. **Q9.** (a) every islet has degree 3; all four are odd, so no Euler trail or circuit exists — no such walk. (b) After removing $A-B$: $deg(A)=deg(B)=2$, $deg(C)=deg(D)=3$; two odd, so an Euler trail $C-A-D-B-C-D$ from $C$ to $D$. (c) No — a round trip needs all degrees even, but $C, D$ are odd. **Q10.** (a) two odd vertices $C, D$, so an Euler trail $C-A-B-C-D-E-F-D$ from $C$ to $D$; no circuit. (b) see solution ($C-D$ is the only link between the two triangles). (c) $A-B-C-D-E-F$. **Q11.** See solution. **Q12.** See solution. **Q13.** (a) $deg(A)=3$, $deg(B)=5$, $deg(C)=3$, $deg(D)=3$; four odd vertices, so no Euler trail exists — the stroll is impossible. (b) After adding $A-C$: $deg(A)=deg(C)=4$, $deg(B)=5$, $deg(D)=3$; exactly two odd, so a stroll is now possible, running between $B$ and $D$ (e.g. $B-A-D-C-A-B-C-B-D$). **Q14.** (a) puzzle (i); (b) odd vertices $C, D$, route $C-B-A-D-E-C-D$ from $C$ to $D$; (c) puzzle (ii) has four odd-degree vertices, so it cannot be drawn without lifting the pen or retracing. **Q15.** (a) see solution (a bipartite cycle needs equal-sized sides, but the sides have 2 and 3 vertices); (b) $X-A-Y-B-Z$. **Q16.** (a) $K_n$ has an Euler circuit if and only if $n$ is odd; (b) $K_n$ is Hamiltonian for every $n \geq 3$ (see solution). **Q17.** See solution. **Q18.** (a) degree sequence $3,3,2,2,2,2$; odd sites $A, C$. (b) two odd vertices, so an Euler trail (not a circuit): $A-B-C-A-F-E-D-C$ from $A$ to $C$. (c) $A-B-C-D-E-F-A$. (d) it shows the two properties are independent and neither requires completeness nor all-even degrees.

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## Fully-worked solutions

**Q1.** The cloisters are $PQ, QR, RS, SP, PT, RT$. (a) $P-Q-R-S-P$ uses $PQ, QR, RS, SP$, all different, and returns to $P$ with no other repeated vertex: a **cycle** of length 4. (b) $Q-P-T-R-S$ uses $QP, PT, TR, RS$ and repeats no vertex: a **path** of length 4. (c) $S-P-Q-R-Q$ steps $S-P$, $P-Q$, $Q-R$, then $R-Q$ — the cloister $QR$ is used twice — so it is a **walk** of length 4, not a trail. (d) $T-P-Q-R-S-P$ uses $TP, PQ, QR, RS, SP$, all different, so it is a trail; but it visits $P$ twice and ends away from its start, so it is a **trail but not a path**, of length 5.

**Q2.** (a) The gate $O$ meets two sections of each of the three tracks, so $deg(O)=6$; each of $A, B, C, D, E, F$ lies on one track only and has degree 2. (b) Yes: 6 and 2 are both even. (c) The network is connected and every vertex has even degree, so by Euler's theorem an **Euler circuit exists**. (d) Riding each track in turn from the gate, $O-A-B-O-C-D-O-E-F-O$ uses all nine sections once and returns to $O$.

**Q3.** The footbridges are $AB, BC, CD, DA, AC$. (a) $deg(A)=3$ (bridges $AB, DA, AC$), $deg(B)=2$, $deg(C)=3$ (bridges $BC, CD, AC$), $deg(D)=2$. (b) Two islets, $A$ and $C$, have odd degree. (c) With exactly two odd-degree vertices, an Euler trail exists but no Euler circuit, and it must run from $A$ to $C$. One such walk is $A-B-C-D-A-C$, crossing $AB, BC, CD, DA, AC$ once each; it **starts at $A$ and finishes at $C$**. (d) A round trip needs every degree even. Building a **second $A-C$ bridge** raises $deg(A)$ and $deg(C)$ to 4, making all degrees even, so an Euler circuit would then exist.

**Q4.** The walkways are $AB, BC, CD, DE, EA, BD$. (a) $deg(A)=2$, $deg(B)=3$ (walkways $AB, BC, BD$), $deg(C)=2$, $deg(D)=3$ (walkways $CD, DE, BD$), $deg(E)=2$. (b) Two buildings ($B$ and $D$) have odd degree, so the network has an **Euler trail but no Euler circuit**. (c) The trail runs between the odd-degree buildings: $B-A-E-D-C-B-D$ uses $BA, AE, ED, DC, CB, BD$ once each, **beginning at $B$ and ending at $D$**. (d) Any route using every walkway once is an Euler trail; with exactly two odd-degree vertices such a trail must start at one of them and end at the other, so it can never return to its starting building.

**Q5.** This is the complete bipartite graph $K_{3,3}$ with cottages $\{A, B, C\}$ and services $\{X, Y, Z\}$; every one of the nine edges joins a cottage to a service. (a) Each cottage joins all three services and each service joins all three cottages, so every vertex has degree 3. (b) $A-X-B-Y-C-Z-A$ alternates cottage–service and visits all six vertices once before closing at $A$: a Hamiltonian cycle. (c) Dropping the final edge, $A-X-B-Y-C-Z$ is a Hamiltonian path. (d) All six vertices have odd degree 3. Since more than two vertices are odd, the network has neither an Euler trail nor an Euler circuit, so it is **not Eulerian**.

**Q6.** (a) $G_1$ is a 4-cycle: degree sequence $2, 2, 2, 2$; 0 odd vertices; connected with all even degrees, so it has an **Euler circuit**. (b) $G_2$ is a path: $deg(A)=1$, $deg(E)=1$, and $deg(B)=deg(C)=deg(D)=2$; degree sequence $2, 2, 2, 1, 1$; two odd vertices ($A, E$), so it has an **Euler trail** (from $A$ to $E$) but no circuit. (c) $G_3$ is $K_4$: every vertex has degree 3; degree sequence $3, 3, 3, 3$; four odd vertices, so it has **neither**. (d) By the handshaking lemma the degrees sum to $2|E|$, which is even; the even-degree vertices contribute an even total, so the odd-degree vertices must also contribute an even total, which is only possible if there is an even number of them. Hence a connected network can never have exactly one odd-degree vertex.

**Q7.** The edges are $AB, BC, CD, DE, EA, AC, CE$. (i) $A-B-C-D-E-A$ uses $AB, BC, CD, DE, EA$, all different, and returns to $A$ with no other repeated vertex: a **cycle** of length 5. (ii) $A-C-E-D-C-B$ uses $AC, CE, DE, CD, BC$ — five different edges — but visits $C$ twice and ends away from its start: a **trail but not a path**, of length 5. (iii) $A-C-E-C-D$ steps $A-C$, $C-E$, then $E-C$ — edge $CE$ used twice — so it is a **walk** of length 4. (iv) $B-A-E-C-D$ uses $AB, EA, CE, CD$ and repeats no vertex: a **path** of length $4$.

**Q8.** The struts are $AB, AC, AD, AE, FB, FC, FD, FE, BC, CD, DE, EB$ (twelve in all). (a) Apex $A$ meets $AB, AC, AD, AE$, so $deg(A)=4$; base $F$ meets $FB, FC, FD, FE$, so $deg(F)=4$; each of $B, C, D, E$ meets one strut to $A$, one to $F$ and two along the square, so also has degree $4$. (b) Every joint has even degree and the frame is connected, so by Euler's theorem an Euler circuit exists — a route along every strut once, finishing where it began. (c) One such circuit is $A-B-C-A-D-C-F-B-E-D-F-E-A$, which uses each of the twelve struts exactly once and returns to $A$.

**Q9.** With all six bridges the graph is $K_4$. (a) Each islet is joined to the other three, so every degree is 3; all four vertices are odd. An Euler trail needs 0 or 2 odd vertices, so with four odd vertices no walk can cross every bridge exactly once. (b) Removing $A-B$ leaves bridges $AC, AD, BC, BD, CD$, with $deg(A)=deg(B)=2$ and $deg(C)=deg(D)=3$. Exactly two vertices are now odd, so an Euler trail exists and must run between them: $C-A-D-B-C-D$ crosses each remaining bridge once, **starting at $C$ and finishing at $D$**. (c) No. Returning to the start would require an Euler circuit, which needs every degree even; but $C$ and $D$ remain odd, so no round trip using every remaining bridge once is possible.

**Q10.** The edges are $AB, BC, CA$ (first triangle), $DE, EF, FD$ (second triangle) and the connector $CD$. (a) $deg(C)=3$ (edges $BC, CA, CD$) and $deg(D)=3$ (edges $DE, FD, CD$); the other four vertices have degree 2. With exactly two odd-degree vertices there is an Euler trail but no circuit, running from $C$ to $D$: $C-A-B-C-D-E-F-D$ uses every edge once. (b) The connector $C-D$ is the **only** edge joining the vertices $\{A, B, C\}$ to $\{D, E, F\}$. A Hamiltonian cycle visits every vertex once and is closed, so it would have to pass from the first set to the second and back again; each crossing must use $C-D$, so the cycle would use that single edge at least twice — impossible in a cycle. Hence no Hamiltonian cycle exists. (c) A Hamiltonian path need not be closed, so it may cross the connector just once: $A-B-C-D-E-F$ visits every vertex exactly once.

**Q11.** Let the graph have vertex set $V$ and edge set $E$. Each edge contributes $1$ to the degree of each of its two endpoints, so summing degrees counts every edge twice:

$$\sum_{v \in V} deg(v) = 2|E|,$$

an even number. Split the sum into vertices of even degree and vertices of odd degree:

$$\sum_{deg\text{ even}} deg(v) + \sum_{deg\text{ odd}} deg(v) = 2|E|.$$

The first sum is a sum of even numbers, hence even, so the second sum equals $2|E|$ minus an even number and is therefore even too. But a sum of odd numbers is even only when there is an even number of terms. Hence the number of odd-degree vertices is even, as required.

**Q12.** Let $G$ be connected with an Euler circuit $C$ — a closed trail using every edge exactly once. Fix any vertex $v$ and follow $C$ from its start. Every time $C$ arrives at $v$ it must immediately leave along a different, so-far-unused edge, because a trail never repeats an edge and the walk only halts on returning to its start with all edges spent. Thus each arrival at $v$ is matched with a departure, pairing the edges at $v$; even the start/finish vertex is paired, its opening departure matching its closing arrival. Every edge incident with $v$ therefore belongs to exactly one such arrive–depart pair, so $deg(v)$ is even. As $v$ was arbitrary, every vertex has even degree, as required.

**Q13.** (a) The bridges give $deg(A)=3$ (two to $B$, one to $D$), $deg(B)=5$ (two to $A$, two to $C$, one to $D$), $deg(C)=3$ (two to $B$, one to $D$) and $deg(D)=3$ (one each to $A$, $B$, $C$). All four regions have odd degree. A stroll crossing every bridge once would be an Euler trail, which requires $0$ or $2$ odd-degree vertices; with four, none can exist, so the stroll is impossible. (b) Adding a bridge $A-C$ raises $deg(A)$ and $deg(C)$ from 3 to 4, leaving $deg(B)=5$ and $deg(D)=3$ odd. Now exactly two vertices are odd, so an Euler trail exists and must run between $B$ and $D$. For instance $B-A-D-C-A-B-C-B-D$ crosses all eight bridges once, from $B$ to $D$.

**Q14.** (a) In puzzle (i) the degrees are $deg(A)=2$, $deg(B)=2$, $deg(C)=3$, $deg(D)=3$, $deg(E)=2$ — exactly two odd vertices — so it has an Euler trail and can be drawn in one stroke. In puzzle (ii) all four vertices have degree $3$, so it cannot. So **puzzle (i)** can be drawn without lifting the pen. (b) The odd-degree vertices of puzzle (i) are $C$ and $D$; the drawing must start at one and finish at the other. One route is $C-B-A-D-E-C-D$, using $BC, AB, DA, DE, CE, CD$ once each, **from** $C$ **to** $D$. (c) Puzzle (ii) is $K_4$, with all four vertices of odd degree. A one-stroke drawing is an Euler trail, which needs $0$ or $2$ odd vertices; with four, no such trail exists, so the puzzle cannot be drawn without lifting the pen or retracing a line.

**Q15.** This is the complete bipartite graph $K_{2,3}$ with sides $\{A, B\}$ and $\{X, Y, Z\}$; every edge joins the two sides. (a) In any cycle, consecutive vertices are joined by an edge, and every edge here crosses from one side to the other, so the vertices of a cycle must alternate between the two sides. A cycle visiting all vertices would therefore use equal numbers from each side. But the sides have 2 and 3 vertices, which are unequal, so no Hamiltonian cycle can exist. (b) A Hamiltonian path may begin and end on the larger side: $X-A-Y-B-Z$ visits all five vertices exactly once.

**Q16.** (a) In $K_n$ each vertex is joined to the other $n-1$ vertices, so every vertex has degree $n-1$, and $K_n$ is connected. By Euler's theorem it has an Euler circuit exactly when every degree is even, that is when $n-1$ is even, that is when $n$ is **odd**. So $K_3, K_5, K_7, \ldots$ are Eulerian, while $K_4, K_6, \ldots$ are not (for those, all $n$ vertices are odd). (b) Label the vertices $v_1, v_2, \ldots, v_n$. Since every pair is joined, each consecutive pair $v_i v_{i+1}$ and the closing pair $v_n v_1$ is an edge, so $v_1 - v_2 - \cdots - v_n - v_1$ is a cycle through all $n$ vertices: a Hamiltonian cycle. This works for every $n \geq 3$. The contrast is striking: $K_n$ is Hamiltonian for **all** $n \geq 3$, yet Eulerian **only** for odd $n$ — so the two properties are governed by quite different conditions.

**Q17.** Let $G$ be connected with an open Euler trail $T$ from $s$ to $t$, where $t \neq s$, using every edge exactly once. Take any vertex $v$ other than $s$ and $t$. Because $T$ neither begins nor ends at $v$, each visit to $v$ consists of an arrival along one edge and a departure along another, and since $T$ is a trail these edges are all distinct; so the edges at $v$ pair up and $deg(v)$ is even. At the start $s$, the first departure has no earlier arrival to pair with, while every later visit to $s$ pairs an arrival with a departure; so $s$ has exactly one unpaired edge and $deg(s)$ is odd. Symmetrically the final arrival at $t$ is unpaired, so $deg(t)$ is odd. Thus $s$ and $t$ are the only odd-degree vertices — exactly two — and $T$ begins and ends at them, as required.

**Q18.** The roads are $AB, BC, CD, DE, EF, FA$ and the shortcut $AC$. (a) $deg(A)=3$ (roads $AB, FA, AC$), $deg(C)=3$ (roads $BC, CD, AC$), and $deg(B)=deg(D)=deg(E)=deg(F)=2$. Written in decreasing order the degree sequence is $3, 3, 2, 2, 2, 2$ and the odd-degree sites are $A$ and $C$. (b) With exactly two odd-degree vertices the network has an Euler trail but no Euler circuit, running from $A$ to $C$. One such route is $A - B - C - A - F - E - D - C$, using each of the seven roads once. (c) Ignoring the shortcut, the ring road itself gives the Hamiltonian cycle $A - B - C - D - E - F - A$, visiting every site once. (d) The graph is neither complete nor all-even, yet it has both an Euler trail and a Hamiltonian cycle. This confirms that the two route-types are independent of each other and of completeness: possessing one says nothing about possessing the other.