Physics Units 3 & 4

Chapter 9 — Einstein’s special theory of relativity

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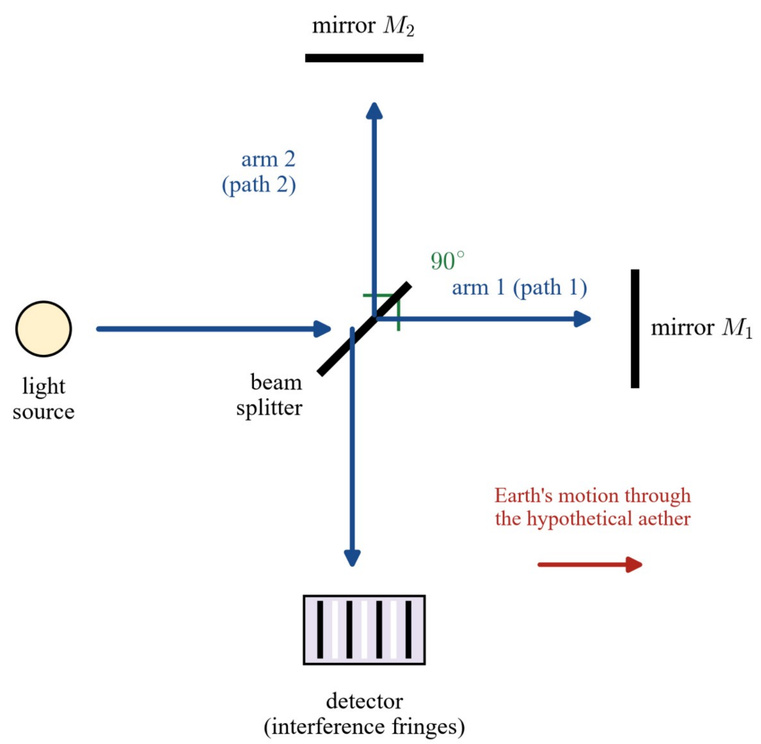
Contents

Theory

Classical mechanics and its limits. Newton's mechanics, developed in the earlier chapters, describes the motion of everyday objects extremely well: projectiles, orbits, collisions and circular motion all obey $\Sigma F = ma$. Built into this classical picture is the idea of **Galilean relativity** — that velocities simply add. If you walk forward at $2\,\mathrm{m\,s^{-1}}$ along a train moving at $30\,\mathrm{m\,s^{-1}}$, the ground sees you move at $32\,\mathrm{m\,s^{-1}}$. Classical physics also assumes that **time and space are absolute**: one second and one metre are the same for every observer, everywhere, whatever their motion. These assumptions work superbly at ordinary speeds, but they fail for objects moving at a large fraction of the speed of light $c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$.

The problem with light. In the nineteenth century, Maxwell's theory of electromagnetism predicted that light is an electromagnetic wave that travels through a vacuum at a single fixed speed, c. But a speed relative to *what*? Every other wave (sound, water waves) travels at a fixed speed relative to a medium, so physicists supposed light must travel through an all-pervading medium they called the **aether**, which would define an absolute frame of rest for light. If the aether existed, then Galilean addition should apply to light as well: an observer moving through the aether towards a source should measure light arriving faster than c, and one moving away should measure it slower.

The Michelson–Morley experiment. This experiment was designed to detect the Earth's motion through the aether. A beam of light from a source strikes a half-silvered **beam splitter**, which divides it into two beams that travel along **two arms set at right angles** — two perpendicular light paths of nearly equal length. Each beam reflects from a mirror at the end of its arm, returns to the beam splitter, and the two beams recombine at a **detector**, where they overlap and form a pattern of interference fringes.



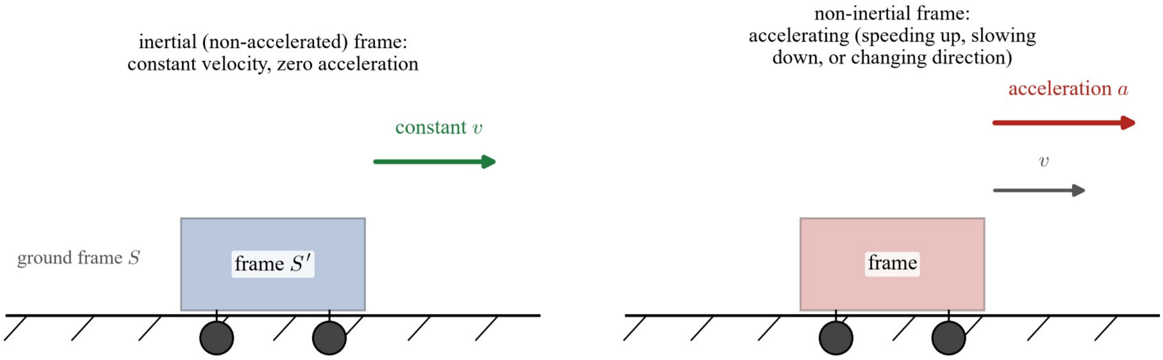
If the Earth were moving through a stationary aether, light would take different times to travel along the two perpendicular paths (one partly "with and against" the aether wind, the other "across" it). Slowly rotating the apparatus through 90° would then swap the roles of the two arms and shift the fringe pattern. Michelson and Morley looked for this shift with great precision, and repeated the experiment many times at different times of day and year. They found **no shift at all** — the famous **null result**. The two perpendicular paths always gave the same light-travel time, whatever the orientation of the apparatus, which means the speed of light was the same in every direction regardless of the Earth's motion. No aether wind could be detected.

Einstein's two postulates. In 1905 Einstein resolved the difficulty not by patching the aether idea but by discarding it. He built his **special theory of relativity** on two postulates:

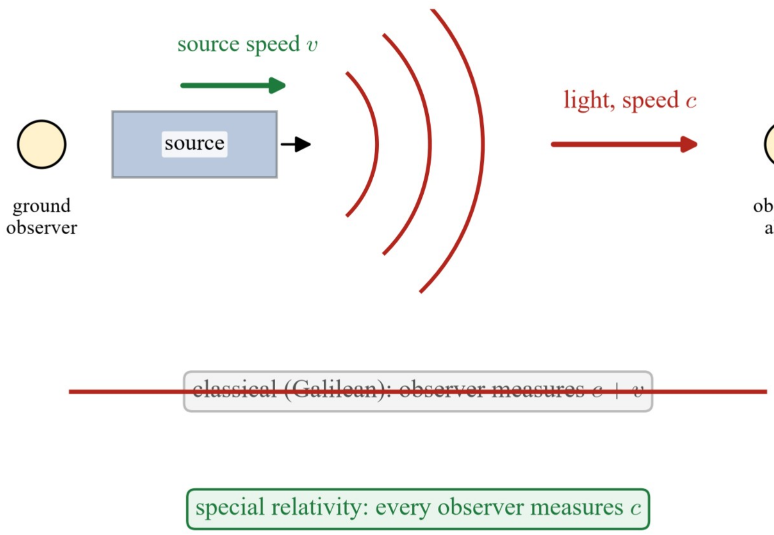
1. **The laws of physics are the same in all inertial (non-accelerated) frames of reference.**
2. **The speed of light has a constant value for all observers, regardless of their motion or the motion of the source.**

The first postulate is a statement that there is no privileged, absolute frame of rest: every non-accelerating observer is entitled to regard themselves as at rest and will find the same laws of physics. The second postulate is the radical one — it says that everyone measures light travelling at the same $c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$, no matter how fast the source or the observer is moving. The null result of the Michelson–Morley experiment is exactly what the second postulate predicts.

Inertial (non-accelerated) frames of reference. A **frame of reference** is a coordinate system, attached to an observer, from which positions and times are measured. An **inertial (non-accelerated) frame** is one that moves at constant velocity — it is neither speeding up, slowing down, nor changing direction, so an object left alone in it obeys Newton's first law. A train gliding at steady velocity along a straight track and a spacecraft coasting with its engines off are inertial. A car braking at a red light, a carriage rounding a bend, and a lift accelerating upwards are **non-inertial**, because each is accelerating. Einstein's postulates apply to inertial frames.



Why velocity addition fails for light. The second postulate directly contradicts Galilean velocity addition. Consider a spaceship moving at speed v that switches on a forward-pointing searchlight. Classical physics predicts that a stationary observer ahead would measure the light approaching at $c + v$. Special relativity insists that the observer — like every observer — measures the light at exactly c.



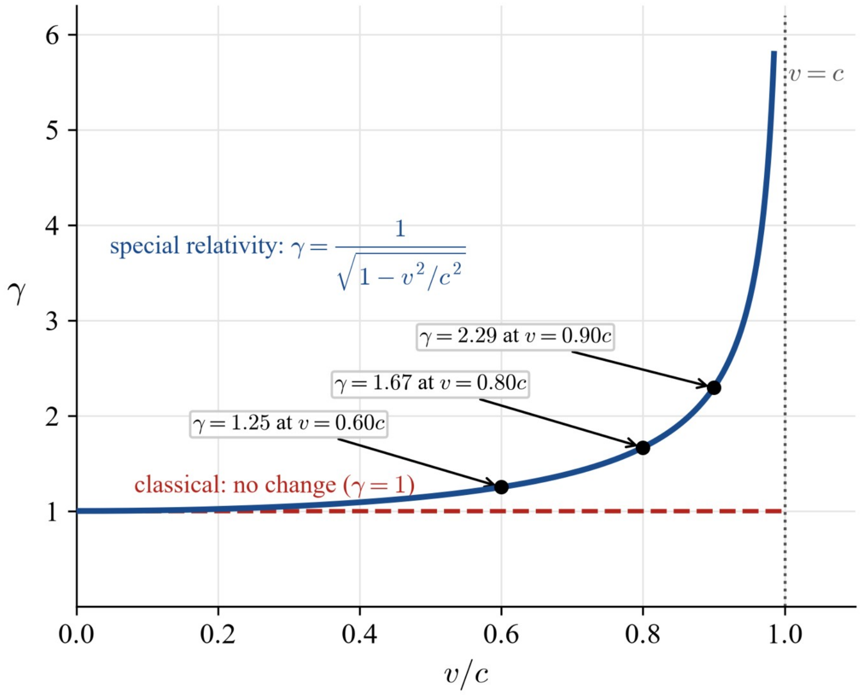
Experiment sides with Einstein: the measured speed of light does not depend on the motion of its source. The everyday rule that velocities add is only an excellent *approximation*, valid when all the speeds involved are tiny compared with c.

The Lorentz factor and the size of relativistic effects. How large the departure from classical physics becomes at a given speed is captured by a single dimensionless number, the **Lorentz factor**

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Classical physics is the special case $\gamma = 1$: no change, at every speed. The relativistic γ instead grows with speed, slowly at first and then without bound as $v \to c$:

v	$0.10\,c$	$0.50\,c$	$0.60\,c$	$0.80\,c$	$0.90\,c$	$0.99\,c$
γ	1.005	1.15	1.25	1.67	2.29	7.09



At everyday speeds γ is indistinguishable from 1 — for a jet at $250\,\text{m s}^{-1}$, $\gamma - 1 \approx 3.5 \times 10^{-13}$ — which is exactly why classical mechanics works so well in ordinary life and why the departures went unnoticed for so long. Only when v becomes a large fraction of c does γ climb steeply, so that relativistic effects become large and unavoidable. Because $\gamma \to \infty$ as $v \to c$, the energy needed to accelerate a massive object grows without limit as it nears the speed of light, and c acts as an unattainable speed limit for any object that has mass. In this subtopic γ is used only to gauge *how relativistic* a situation is; it is applied quantitatively to time, length and energy in the subtopics that follow.

Comparing special relativity with classical physics. The contrast can be summarised:

- *Speed of light.* Classical: adds to the motion of the source/observer ($c \pm v$). Relativity: the same value c for all observers (second postulate).
- *A preferred frame.* Classical: an absolute frame of rest set by the aether. Relativity: no preferred frame — all inertial frames are equivalent (first postulate).
- *Time and space.* Classical: absolute and the same for everyone. Relativity: time intervals and lengths depend on the relative motion of the observer (developed in 9B).
- *Speed limit.* Classical: none — any speed is possible. Relativity: c is a limit no massive object can reach.

Where all speeds are much smaller than c, the relativistic predictions collapse back onto the classical ones ($\gamma \to 1$), so special relativity contains classical mechanics as its low-speed approximation.

Throughout this subtopic the constant $c = 3.00 \times 10^8\,\text{m s}^{-1}$ is used.

Worked examples

Example 1 — scaffolded

A subatomic particle travels at a speed of $0.80\,c$. Calculate the Lorentz factor γ for the particle, and state what its value indicates about the size of relativistic effects at this speed.

Working	Comment
$v = 0.80\,c$, so $\dfrac{v}{c} = 0.80$.	Speeds near c are written as a fraction of c; only the ratio v/c enters γ.
$\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \dfrac{1}{\sqrt{1 - 0.80^2}}$.	The Lorentz factor from the formula sheet.
$= \dfrac{1}{\sqrt{1 - 0.64}} = \dfrac{1}{\sqrt{0.36}} = \dfrac{1}{0.60} = 1.67$.	Evaluate; γ is dimensionless.
$\gamma = 1.67 > 1$, so relativistic effects are large — about $67\,\%$ — at this speed.	γ measures how far predictions depart from the classical value $\gamma = 1$.

Example 2 — scaffolded

A spaceship moving at $0.60\,c$ relative to a space station switches on a searchlight pointing forwards, along the direction of travel. Calculate the speed of the emitted light measured by an observer on the station according to (i) classical (Galilean) velocity addition and (ii) special relativity, and state the size of the discrepancy.

Working	Comment
Source speed $v = 0.60\,c = 1.80 \times 10^8\ \text{m s}^{-1}$; light leaves the source at $c = 3.00 \times 10^8\ \text{m s}^{-1}$.	List the data and identify the two competing models.
Classical: measured speed $= c + v = 1.60\,c = 4.80 \times 10^8\ \text{m s}^{-1}$.	Galilean addition adds the source speed to the speed of the light.
Relativity: measured speed $= c = 3.00 \times 10^8\ \text{m s}^{-1}$.	Second postulate: c is the same for every observer, independent of source motion.
Discrepancy $= 4.80 \times 10^8 - 3.00 \times 10^8 = 1.80 \times 10^8\ \text{m s}^{-1}$.	The classical prediction exceeds c; experiment confirms the relativistic value.

Example 3 — unscaffolded

The Michelson–Morley experiment is often quoted as evidence for special relativity. Explain how the experiment supports Einstein's postulate that the speed of light is constant. Refer to the design of the apparatus and to the result obtained.

In the Michelson–Morley interferometer a beam of light is divided at a beam splitter into two beams that travel along **two arms set at right angles** — two perpendicular light paths of nearly equal length. Each beam reflects from a mirror, returns, and the two beams recombine at a detector, where they interfere to produce a fringe pattern. If light travelled through a stationary aether, the motion of the Earth through that aether would make light take different times along the two perpendicular paths, so rotating the apparatus should shift the fringes. When the experiment was performed — and repeated many times — **no fringe shift was observed**: this is the null result. The two perpendicular paths gave the same light-travel time whatever the orientation, so the speed of light was the same in every direction regardless of the Earth's motion.

∴ the null result of this two-perpendicular-path experiment supports Einstein's postulate that the speed of light has a constant value for all observers, independent of their motion or the motion of the source.

Example 4 — unscaffolded

A distant star moving rapidly towards the Earth emits light. State what classical physics and what special relativity each predict for the speed at which the light arrives at the Earth, and identify which of Einstein's postulates settles the disagreement.

Classical (Galilean) physics treats the speed of light like any other velocity, so it should add to the velocity of the source: light from a star approaching at speed v would be predicted to arrive at $c+v$, faster than light from a stationary star. Special relativity predicts instead that the light arrives at exactly $c=3.00\times10^{8}\,\mathrm{m\ s^{-1}}$ — the same speed as light from any other source — because the speed of light does not depend on the motion of the source or of the observer. Measurements of light from moving astronomical sources confirm the relativistic prediction.

∴ Einstein's second postulate — that the speed of light has a constant value for all observers regardless of their motion or the motion of the source — settles the disagreement in favour of special relativity.

Practice questions

Question 1 A particle physicist considers a particle accelerated to very high speeds. (a) Show that at a speed of $0.60\,c$ the Lorentz factor of the particle is $\gamma=1.25$. (b) Calculate the Lorentz factor of the particle at a speed of $0.90\,c$. (c) Using your answers to (a) and (b), describe what happens to γ as the particle's speed approaches c, and state what this indicates about accelerating a particle (which has mass) to the speed of light.

Question 2 The Michelson–Morley experiment is a key piece of evidence for special relativity. (a) State what the Michelson–Morley experiment was designed to detect. (b) Describe the apparatus, making clear the role of the two perpendicular light paths. (c) State the result of the experiment and explain, as a full chain of reasoning, how that result supports Einstein's postulate that the speed of light is constant.

Question 3 Einstein based special relativity on two postulates. (a) State Einstein's two postulates. (b) Define an inertial (non-accelerated) frame of reference, and give one example of an inertial frame and one example of a non-inertial frame. (c) A spacecraft coasts through deep space at constant velocity with its engines off and its windows covered. With reference to the first postulate, explain whether experiments performed inside the spacecraft could determine how fast it is moving.

Question 4 A spaceship travelling at $0.40\,c$ relative to a station switches on a headlight that shines forwards. (a) According to classical (Galilean) velocity addition, calculate the speed of the headlight beam that would be measured by an observer on the station. (b) According to special relativity, state the speed of the beam measured by that observer. (c) State which prediction is correct, and identify which of Einstein's postulates this demonstrates.

Question 5 A commercial jet cruises at $250\,\mathrm{m\ s^{-1}}$. (a) Calculate the ratio v/c for the jet. (b) Calculate the Lorentz factor γ for the jet (to three significant figures). (c) With reference to your answers, explain why classical mechanics gives accurate predictions for everyday speeds but fails for particles moving at speeds close to c.

Question 6 A student is comparing classical physics with special relativity. (a) State the classical (Newtonian) view of how the measured speed of a light beam depends on the motion of its source. (b) State the view of special relativity on the same question. (c) (Circle your answer.) In classical physics, the time interval between two events is: *absolute* / *relative*. Justify your choice, and state how the view of special relativity differs. No calculations are required.

Question 7 A proton in an accelerator reaches a speed of $0.95\,c$. Calculate the Lorentz factor γ for the proton.

Question 8 Determine the speed, as a fraction of c and in $\mathrm{m\ s^{-1}}$, at which the Lorentz factor is $\gamma=2.00$. Give your answer to three significant figures.

Question 9 A rapidly moving source travels at $0.30\,c$ relative to a laboratory and emits a pulse of light in its direction of motion. Calculate the speed of the pulse in the laboratory that classical velocity addition would predict, state the speed that special relativity predicts, and determine the difference between the two predictions.

Question 10 Before Einstein, physicists expected the Michelson–Morley experiment to produce a measurable shift in the interference fringes as the apparatus was rotated. Explain why a fringe shift was expected, and explain what the actual (null) result told physicists about the aether and about the speed of light.

Question 11 In an examination, a student explains the significance of the Michelson–Morley experiment by writing only: "The experiment shows that the speed of light is constant." Explain why this statement, on its own, is an incomplete account of how the experiment supports special relativity, and give the complete reasoning the student should have provided.

Question 12 A source of light moves at high speed towards a detector. Compare the account given by classical physics with the account given by special relativity for the speed at which the light reaches the detector, and identify the experimental evidence that distinguishes between them.

Question 13 A student claims: "A car braking to a stop at traffic lights is an inertial frame of reference, because it is still moving in a straight line along the road." Explain what is wrong with this claim, and state the condition a frame of reference must satisfy to be inertial.

Question 14 A physicist is sealed inside a windowless carriage that moves along a perfectly smooth, straight track at constant velocity. With reference to Einstein's first postulate, explain why no experiment carried out entirely inside the carriage can measure the carriage's velocity relative to the ground.

Question 15 A particle is accelerated until its Lorentz factor is $\gamma = 7.09$. (a) Show by calculation that the particle's speed is $0.99\,c$. (b) Explain, with reference to the behaviour of γ as $v \to c$, why no particle that has mass can ever be accelerated to the speed of light.

Question 16 A particle's speed is increased from $0.20\,c$ to $0.85\,c$. (a) (Circle your answer.) As the speed increases, the Lorentz factor γ: *decreases* / *stays the same* / *increases*. (b) Justify your choice by calculating γ at each speed.

Question 17 Describe the limitation of classical mechanics when it is applied to motion at speeds approaching the speed of light, and explain how special relativity resolves this limitation. In your answer, refer to at least two ways in which the relativistic description differs from the classical one.

Question 18 Write an account of how the understanding of the speed of light changed from the classical (aether) picture to Einstein's special relativity. Your account should describe: the aether hypothesis and what it was supposed to explain; the design of the Michelson–Morley experiment (including its two perpendicular light paths); the result obtained; and how Einstein's two postulates explained that result without an aether.

Answers

Question 1 (a) $\gamma = 1.25$ (shown); (b) $\gamma = 2.29$; (c) γ increases without bound as $v \to c$, so a massive particle cannot be accelerated to c. **Question 2** (a) the Earth's motion through the hypothetical aether; (b) light is split into two beams along two perpendicular arms and recombined at a detector; (c) null result (no fringe shift) $\Rightarrow$ same light speed along both perpendicular paths regardless of the Earth's motion $\Rightarrow$ constant c. **Question 3** (a) the two postulates (laws of physics the same in all inertial frames; c constant for all observers); (b) inertial = constant velocity (e.g. a coasting spacecraft); non-inertial = accelerating (e.g. a braking car); (c) no — every inertial frame is equivalent, so no internal experiment reveals the constant velocity. **Question 4** (a) $4.20 \times 10^8\ \mathrm{m\ s^{-1}}$; (b) $3.00 \times 10^8\ \mathrm{m\ s^{-1}}$; (c) the relativistic value (c) is correct — the second postulate. **Question 5** (a) $v / c = 8.33 \times 10^{-7}$; (b) $\gamma = 1.00$; (c) at everyday speeds $\gamma \approx 1$ so relativistic and classical predictions agree; near c, γ departs greatly from 1. **Question 6** (a) it adds to the source's motion ($c \pm v$); (b) it is the same value c for every observer; (c) absolute — classical time is the same for all observers, whereas in relativity the time interval depends on the observer's motion. **Question 7** $\gamma = 3.20$. **Question 8** $v = 0.866\,c = 2.60 \times 10^8\ \mathrm{m\ s^{-1}}$. **Question 9** classical $3.90 \times 10^8\ \mathrm{m\ s^{-1}}$; relativity $3.00 \times 10^8\ \mathrm{m\ s^{-1}}$; difference $9.00 \times 10^7\ \mathrm{m\ s^{-1}}$. **Question 10** a shift was expected because Earth's motion through the aether should make the two perpendicular paths differ in light-travel time; the null result showed no aether wind and that c is the same in all directions. **Question 11** it omits the two perpendicular paths and the null result — the reasoning must chain apparatus $\to$ null result $\to$ constant c. **Question 12** classical predicts $c + v$ (speed depends on source motion); relativity predicts c; measurements of moving sources confirm c. **Question 13** a braking car is decelerating, so it is accelerating and therefore non-inertial; an inertial frame must move at constant velocity (zero acceleration). **Question 14** by the first postulate all inertial frames are equivalent, so the laws of physics inside are identical to those at rest and reveal no absolute velocity. **Question 15** (a) $v = 0.99\,c$ (shown); (b) $\gamma \to \infty$ as $v \to c$, so the energy required grows without bound and c is unattainable for a massive particle. **Question 16** (a) increases; (b) $\gamma = 1.02$ at $0.20\,c$ and $\gamma = 1.90$ at $0.85\,c$. **Question 17** classically there is no speed limit and time/space are absolute; relativity keeps c constant for all observers, makes c a speed limit, and makes time and length depend on relative motion. **Question 18** account: aether as light's medium/absolute frame $\to$ Michelson–Morley two-perpendicular-path interferometer $\to$ null result $\to$ Einstein's postulates (relativity of inertial frames; constant c) remove the need for an aether.

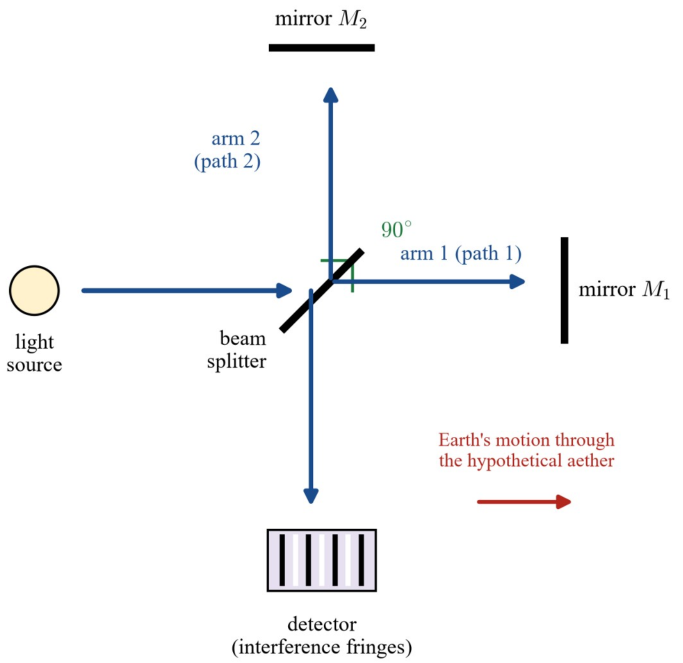
Fully-worked solutions

Question 1 (a) With $v = 0.60\,c$, $\dfrac{v}{c} = 0.60$, so

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.60^2}} = \frac{1}{\sqrt{1 - 0.36}} = \frac{1}{\sqrt{0.64}} = \frac{1}{0.80} = 1.25,$$

as required. (b) With $v = 0.90\,c$, $\gamma = \dfrac{1}{\sqrt{1 - 0.90^2}} = \dfrac{1}{\sqrt{1 - 0.81}} = \dfrac{1}{\sqrt{0.19}} = 2.29$. (c) As the speed rises from $0.60\,c$ to $0.90\,c$, γ rises from 1.25 to 2.29, and as v approaches c the denominator $\sqrt{1 - v^2/c^2}$ approaches zero, so γ increases without bound ($\gamma \to \infty$). Because the energy needed to speed the particle up further grows without limit, a particle that has mass can never actually be accelerated all the way to the speed of light.

Question 2 (a) It was designed to detect the motion of the Earth through the hypothetical **aether** — the absolute medium in which light was thought to travel. (b) Light from a source is divided at a beam splitter into two beams that travel along **two arms set at right angles to each other** (two perpendicular light paths of nearly equal length). Each beam reflects from a mirror at the end of its arm and returns, and the two beams recombine at a detector, where they interfere to form a fringe pattern.



(c) The result was a **null result**: rotating the apparatus produced **no shift in the interference fringes**. Had the Earth been moving through an aether, light would have taken different times to travel the two perpendicular paths, and rotation would have shifted the fringes. Because no shift occurred, the light-travel time was the same along both perpendicular paths whatever the orientation, so the speed of light was the same in every direction regardless of the Earth's motion. ∴ the experiment supports Einstein's postulate that the speed of light has a constant value for all observers, independent of their motion or the motion of the source.

Question 3 (a) (1) The laws of physics are the same in all inertial (non-accelerated) frames of reference. (2) The speed of light has a constant value for all observers, regardless of their motion or the motion of the source. (b) An inertial (non-accelerated) frame of reference is one that moves at constant velocity — it has zero acceleration, so an object left alone in it obeys Newton's first law. Example of an inertial frame: a spacecraft coasting at constant velocity with its engines off. Example of a non-inertial frame: a car braking at traffic lights (it is accelerating). (c) No. By the first postulate the laws of physics are identical in every inertial frame, so any experiment done inside the coasting spacecraft gives the same results as it would if the spacecraft were "at rest". There is no preferred, absolute frame against which to measure the constant velocity, so no internal experiment can reveal how fast the spacecraft is moving.

Question 4 (a) Classical (Galilean) velocity addition predicts the source speed adds to the speed of the light: $c+v=c+0.40\,c=1.40\,c=1.40\times3.00\times10^{8}=4.20\times10^{8}\,\mathrm{m\ s^{-1}}$. (b) Special relativity predicts the observer measures the beam at exactly $c=3.00\times10^{8}\,\mathrm{m\ s^{-1}}$. (c) The relativistic value is the correct one — measurements show the speed of light does not depend on the motion of the source. This demonstrates Einstein's **second postulate**, that the speed of light is constant for all observers regardless of the motion of the source.

Question 5 (a) $\dfrac{v}{c}=\dfrac{250}{3.00\times10^{8}}=8.33\times10^{-7}$. (b) $\gamma=\dfrac{1}{\sqrt{1-(8.33\times10^{-7})^{2}}}=1.00$ (to three significant figures); more precisely $\gamma-1\approx3.5\times10^{-13}$. (c) At $250\,\mathrm{m\ s^{-1}}$ the ratio v/c is so tiny that γ is indistinguishable from 1, so the relativistic and classical predictions agree to about thirteen decimal places — classical mechanics is therefore accurate for everyday speeds. For a particle moving at a large fraction of c, v/c is close to 1 and γ becomes much larger than 1 (for example $\gamma=2.29$ at $0.90\,c$), so the classical predictions depart greatly from the true relativistic ones and classical mechanics fails.

Question 6 (a) Classically, the measured speed of light depends on the motion of the source: the source's velocity adds to (or subtracts from) the speed of the light, giving $c\pm v$. (b) In special relativity the measured speed of light is the same value, c, for every observer, and does **not** depend on the motion of the source or the observer (the second postulate). (c) **Absolute.** In classical physics the time interval between two events is assumed to be the same for every observer, whatever their motion — a universal, absolute time. Special relativity differs: the time interval measured between two events depends on the observer's motion (time is relative), an effect developed quantitatively in the next subtopic.

Question 7 With $v = 0.95\,c$,

$$\gamma = \frac{1}{\sqrt{1-0.95^2}} = \frac{1}{\sqrt{1-0.9025}} = \frac{1}{\sqrt{0.0975}} = 3.20.$$

Question 8 Setting $\gamma = 2.00$ and solving for v:

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\gamma} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}.$$

$$\frac{v}{c} = \sqrt{1-\frac{1}{2.00^2}} = \sqrt{1-0.250} = \sqrt{0.750} = 0.866.$$

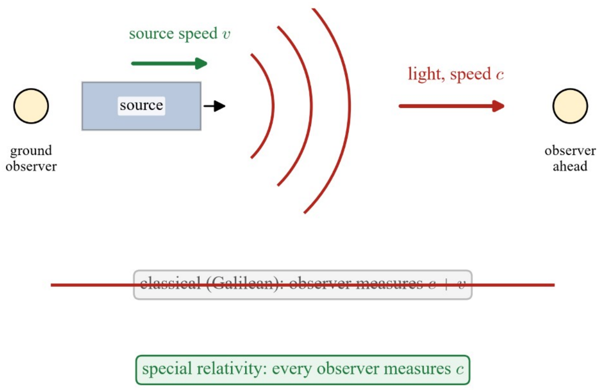
So $v = 0.866\,c = 0.866 \times 3.00 \times 10^8 = 2.60 \times 10^8\ \mathrm{m\ s^{-1}}$ (to three significant figures).

Question 9 Classical velocity addition predicts the source speed adds to the light: $c + v = 1.30\,c = 1.30 \times 3.00 \times 10^8 = 3.90 \times 10^8\ \mathrm{m\ s^{-1}}$. Special relativity predicts the pulse travels at $c = 3.00 \times 10^8\ \mathrm{m\ s^{-1}}$ in the laboratory. The difference between the predictions is

$$3.90 \times 10^8 - 3.00 \times 10^8 = 9.00 \times 10^7\ \mathrm{m\ s^{-1}}.$$

Experiment confirms the relativistic prediction: the pulse is measured at c.

Question 10 A fringe shift was **expected** because, if light travelled through a stationary aether, the Earth's motion through that aether (an "aether wind") would make light take different times to travel the two perpendicular arms; rotating the apparatus through 90° would swap the arms and change the difference in travel times, shifting the fringe pattern. The **actual result was null** — no shift was seen at any orientation, at any time of day or year.

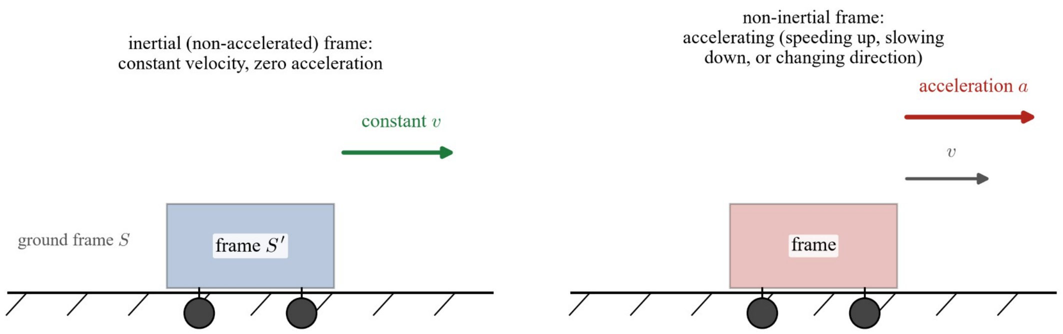


This told physicists two things: first, that no aether wind could be detected (so the aether either did not exist or could never be observed), and second, that the speed of light is the same in every direction and is unaffected by the motion of the Earth. The null result is what Einstein's second postulate — a constant speed of light for all observers — predicts.

Question 11 The bare statement gives the *conclusion* but none of the *evidence*, so it does not actually explain how the experiment supports relativity. A complete account must chain the reasoning: (1) the apparatus splits light into two beams that travel along **two perpendicular paths** and recombines them to form interference fringes; (2) if an aether existed, the Earth's motion through it would make the two perpendicular paths differ in light-travel time, so rotating the apparatus should **shift the fringes**; (3) the experiment gave a **null result** — no fringe shift — showing the light-travel time was the same along both perpendicular paths whatever the orientation; (4) ∴ the speed of light must be the same in all directions regardless of the Earth's motion, which is exactly Einstein's postulate of a constant c. Only with steps (1)–(3) does the conclusion in step (4) follow.

Question 12 Classical physics treats the speed of light like any other velocity, so it predicts that a light beam from a source approaching at speed v reaches the detector at $c+v$ — the measured speed depends on the motion of the source. Special relativity predicts instead that the light reaches the detector at exactly c, independent of the source's motion, because the speed of light is constant for all observers (second postulate). The two accounts therefore disagree, and the experimental evidence distinguishes them: measurements of light from fast-moving sources (and the null result of the Michelson–Morley experiment) show that the speed of light is always c, supporting special relativity and not the classical velocity-addition rule.

Question 13 The claim is wrong because a braking car is **decelerating** — its speed is changing, so it has a non-zero acceleration, even though it moves in a straight line. A frame of reference is inertial only if it has **zero acceleration**, i.e. it moves at constant velocity (constant speed *and* constant direction).



Since the braking car is accelerating, it is a non-inertial frame, and Einstein's postulates (which apply to inertial frames) do not apply directly to it.

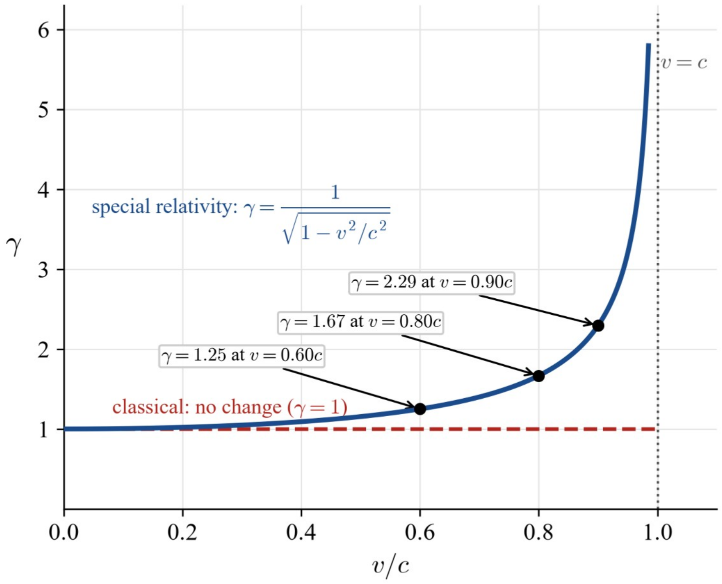
Question 14 By Einstein's first postulate, the laws of physics are identical in all inertial (non-accelerated) frames of reference. The carriage moves at constant velocity, so it is an inertial frame; every mechanical, electrical and optical experiment performed inside it therefore gives exactly the same result as it would if the carriage were "at rest". Because there is no absolute frame of rest and every inertial frame is equally valid, there is no internal measurement whose result depends on the carriage's velocity relative to the ground. ∴ the physicist cannot determine the carriage's velocity from inside; only by looking out at another frame (the ground) could relative motion be found.

Question 15 (a) Rearranging $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$ for v with $\gamma = 7.09$:

$$\frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}} = \sqrt{1-\frac{1}{7.09^2}} = \sqrt{1-0.0199} = \sqrt{0.980} = 0.990,$$

so $v = 0.99c$, as required. (b) As $v \to c$, the term $v^2/c^2 \to 1$, so $\sqrt{1-v^2/c^2} \to 0$ and $\gamma \to \infty$. Physically, the closer a massive particle gets to c, the more its Lorentz factor grows, and the energy that must be supplied to speed it up any further grows without bound. Since no finite energy can raise γ to infinity, a particle that has mass can approach c but can never reach it; c is an unattainable speed limit for matter.

Question 16 (a) **Increases.** (b) At $0.20c$: $\gamma = \dfrac{1}{\sqrt{1-0.20^2}} = \dfrac{1}{\sqrt{0.96}} = 1.02$. At $0.85c$: $\gamma = \dfrac{1}{\sqrt{1-0.85^2}} = \dfrac{1}{\sqrt{0.2775}} = 1.90$. Since γ rises from 1.02 to 1.90 as the speed increases, γ increases with speed.



Question 17 Classical mechanics assumes velocities simply add, that time and space are absolute, and that there is no upper limit to speed. These assumptions fail near the speed of light: they wrongly predict that the measured speed of light depends on the motion of the source (giving values above c), and they take no account of the way moving clocks and lengths behave at high speed. Special relativity resolves the limitation by replacing these assumptions with Einstein's two postulates. In the relativistic description: (1) the speed of light is the same value c for every observer, so velocities do not simply add and c becomes an unreachable speed limit for objects with mass; and (2) because c is fixed for all observers, time intervals and lengths are no longer absolute but depend on the observer's relative motion (through the factor γ). At speeds much less than c, $\gamma \rightarrow 1$ and the relativistic predictions reduce to the familiar classical ones, so special relativity contains classical mechanics as its low-speed limit.

Question 18 In the classical picture, light was assumed to be a wave that travelled through an all-pervading medium called the **aether**, which provided an absolute frame of rest for light and explained the single fixed speed c that Maxwell's theory predicted. If the aether existed, the Earth's motion through it should be detectable, because light's measured speed would then depend on the direction travelled relative to the aether. The **Michelson–Morley experiment** was built to detect this: a beam of light was split at a beam splitter into two beams travelling along **two arms set at right angles** — two perpendicular light paths — each reflecting from a mirror and recombining at a detector to form interference fringes. Motion through the aether would make the two perpendicular paths differ in light-travel time, so rotating the apparatus should have shifted the fringes. The result, repeated many times, was a **null result**: no fringe shift, meaning the speed of light was the same along both perpendicular paths whatever the Earth's motion. Einstein explained this without any aether by adopting two postulates: that the laws of physics are the same in all inertial (non-accelerated) frames of reference, and that the speed of light has a constant value for all observers regardless of their motion or the motion of the source. The second postulate makes the null result inevitable — there is no aether frame to move through, and c is simply the same for everyone — so the understanding of light shifted from a medium-based, absolute-frame picture to one in which the constancy of c is a fundamental law of nature.

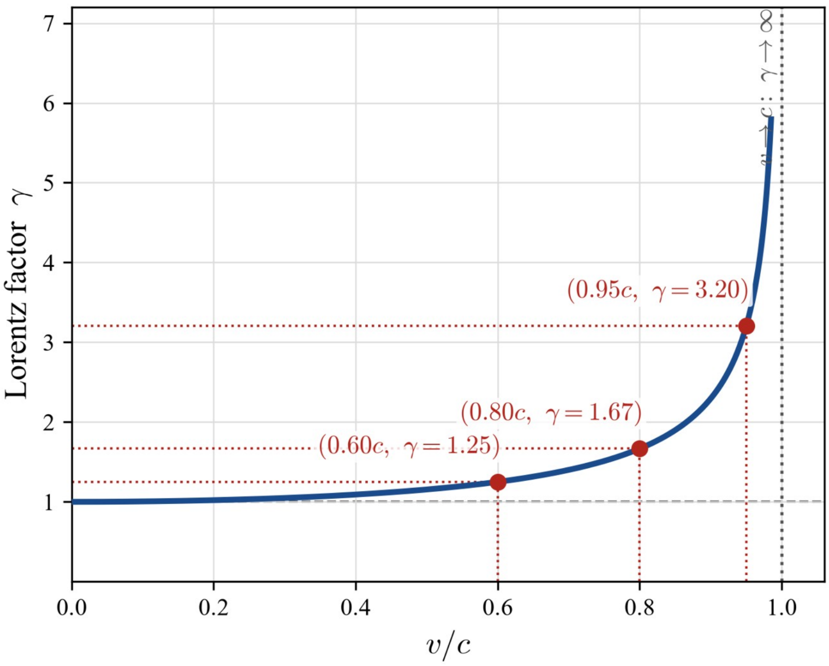
Theory

Special relativity begins with a puzzle about *motion measured from different frames of reference*. An **inertial (non-accelerating) frame** is any frame that moves at constant velocity. Einstein's two postulates — that the laws of physics are identical in every inertial frame, and that the speed of light c has the same value for every observer regardless of the motion of the source or the observer — are developed in 9A. This subtopic takes those postulates as given and works out their most striking consequences: **observers in relative motion do not agree on time intervals or on lengths.**

The Lorentz factor. For two frames in relative motion at speed v, the amount by which their measurements differ is set by a single dimensionless number, the **Lorentz factor**:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Because $v < c$ always, the term $\frac{v^2}{c^2}$ lies between 0 and 1, so γ is always **greater than or equal to 1**. At everyday speeds $v \ll c$ the factor γ is indistinguishable from 1 and relativistic effects are unmeasurably small; as $v \to c$, $\gamma \to \infty$. It is convenient to quote speeds as fractions of c: at $v = 0.60\,c$, $\gamma = 1.25$; at $v = 0.80\,c$, $\gamma = 5/3 = 1.67$; at $v = 0.95\,c$, $\gamma = 3.20$.

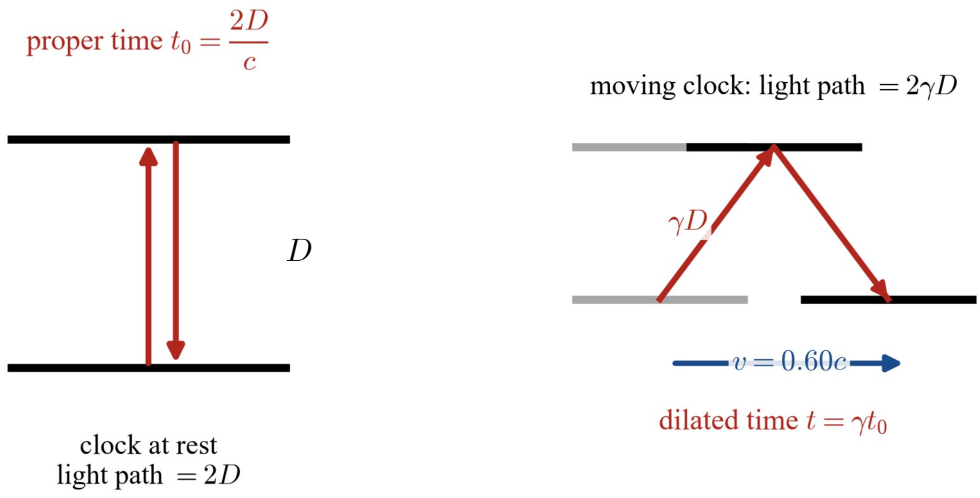


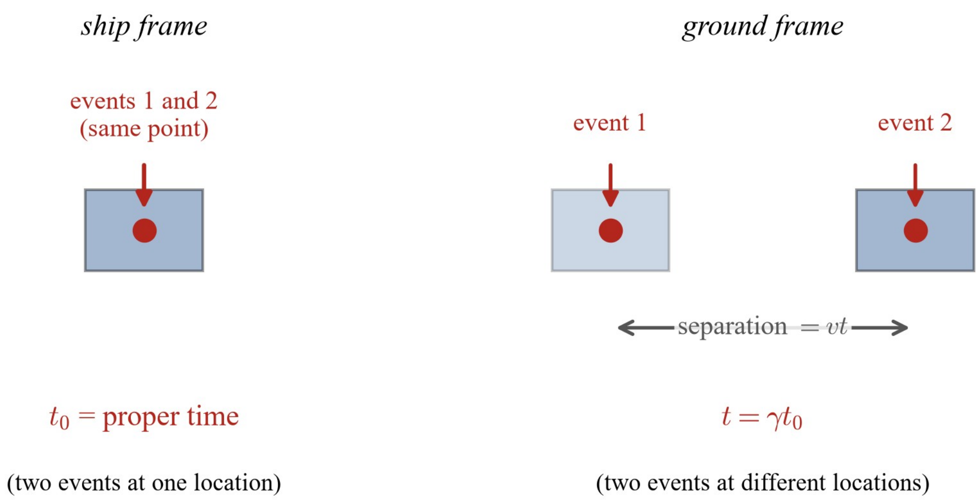
Proper time. The time interval between two events depends on the frame in which it is measured. The **proper time** t_0 is defined as the time interval between two events measured in the reference frame in which the two events occur **at the same point in space**. This "same-point" test is the key to every relativity problem: find the frame in which both events happen at one location, and the clock that is at rest at that location reads the proper time.

A **light clock** — light bouncing between two mirrors a fixed distance apart, one tick per round trip — makes this concrete. When the clock is at rest, the light travels straight up and down. When the same clock is seen moving, the light must travel a longer, *diagonal* path to keep up with the moving mirrors. But light travels at the same speed c in every frame (postulate 2), so a longer path at the same speed takes **more time**: the moving clock is measured to tick more slowly. Following the geometry through gives the **time dilation** relation

$$t = \gamma t_0,$$

where t_0 is the proper time (read on the single clock that is present at both events) and t is the interval measured in any frame that sees that clock moving. Since $\gamma \geq 1$, $t \geq t_0$: **the proper time is the shortest time any observer measures**, and a moving clock is observed to run slow.

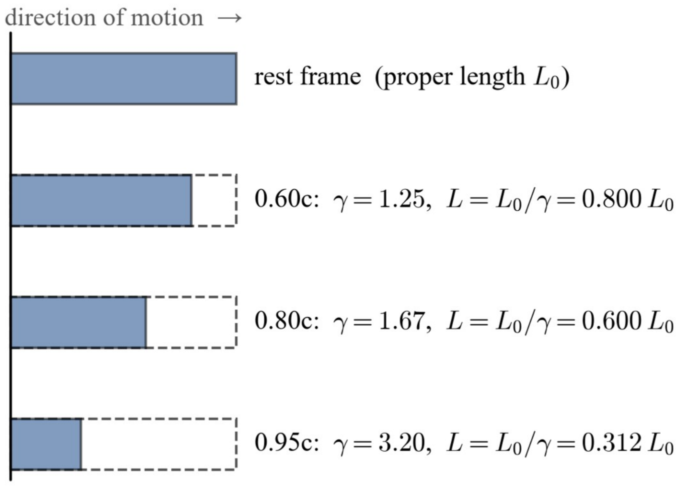




Proper length. Length, too, is frame-dependent. The **proper length** L_0 is the length of an object measured in the reference frame in which the object is **at rest**. Any observer who sees the object moving (at speed v along its length) measures a shorter length, given by **length contraction**:

$$L = \frac{L_0}{\gamma}.$$

Since $\gamma \geq 1$, $L \leq L_0$: **the proper length is the greatest length any observer measures**, and a moving object is measured to be contracted along its direction of motion.



Contraction acts only along the direction of motion. Length contraction shortens an object only in the direction in which it is travelling. Dimensions *perpendicular* to the motion are unchanged: a rod that is $2.0\,\text{m}$ long and $0.10\,\text{m}$ wide, flying lengthwise, is measured shorter than $2.0\,\text{m}$ along its length but still $0.10\,\text{m}$ across.

Identifying the proper quantity — the discipline that avoids every trap. The most common error in relativity is to put a measured value on the wrong side of $t = \gamma t_0$ or $L = L_0/\gamma$. Before substituting anything, always decide:

- *Which frame measures the proper time?* — the frame in which the two events happen at the **same place** (usually the frame of the clock or object that is present at both events). That frame reads the shorter time t_0.
- *Which frame measures the proper length?* — the frame in which the object is **at rest**. That frame measures the longer length L_0.

The proper quantities t_0 and L_0 belong to the object's own (rest) frame; an observer who sees the object moving always measures a **longer time** and a **shorter length**, by the factor γ.

Reciprocity. Motion is relative, so the situation is symmetric. If ship A sees ship B fly past at $0.80\,c$, then B sees A fly past at $0.80\,c$ in the opposite direction. Each observer measures the *other's* clock to run slow and the *other's* ship to be contracted — and both are correct, because each is measuring a moving clock or object from their own rest frame, using a different pair of events and a different ruler. There is no contradiction, and no absolute answer to whose clock is "really" the slow one.

Throughout this subtopic the speed of light is $c = 3.00 \times 10^8\,\text{m s}^{-1}$, and the Lorentz factor γ is dimensionless.

Worked examples

Example 1 — scaffolded

A spacecraft flies past Earth at a constant $0.80\,c$. An astronaut on board measures the time between two ticks of a clock fixed on the spacecraft to be $5.0\,\text{s}$. Calculate the time between the same two ticks as measured by an observer on Earth.

Working	Comment
The two ticks occur at the same point in space in the **spacecraft** frame (at the fixed location of the clock), so $t_0 = 5.0\,\text{s}$ is the proper time, measured on board.	Identify the proper time first: apply the same-point-in-space test.
$v = 0.80\,c$: $\gamma = \dfrac{1}{\sqrt{1-0.80^2}} = \dfrac{1}{\sqrt{0.36}} = \dfrac{1}{0.60} = 1.67$.	Lorentz factor at $0.80\,c$ (exactly $5/3$).
$t = \gamma t_0 = 5/3 \times 5.0 = 8.33\,\text{s}$.	Earth sees the clock moving, so measures the dilated time $t = \gamma t_0$.

Working	Comment
The Earth observer measures $8.33\,\mathrm{s}$ — longer than $5.0\,\mathrm{s}$; the moving clock runs slow.	Check: $t > t_0$, as it must be.

Example 2 — scaffolded

A spacecraft has a length of $120\,\mathrm{m}$ when measured in the dock where it is built. It then flies past a space station at a constant $0.60\,c$. Calculate the length of the spacecraft as measured by an observer on the station.

Working	Comment
The spacecraft is at rest in the dock (its own frame), so the length measured there, $L_0 = 120\,\mathrm{m}$, is the proper length.	Identify the proper length first: the rest frame of the object.
$v = 0.60\,c$: $\gamma = \dfrac{1}{\sqrt{1-0.60^2}} = \dfrac{1}{\sqrt{0.64}} = \dfrac{1}{0.80} = 1.25$.	Lorentz factor at $0.60\,c$.
$L = \dfrac{L_0}{\gamma} = \dfrac{120}{1.25} = 96\,\mathrm{m}$.	The station sees the ship moving, so measures the contracted length $L = L_0/\gamma$.
The station observer measures $96\,\mathrm{m}$ — shorter than the proper length $120\,\mathrm{m}$.	Check: $L < L_0$, as it must be.

Example 3 — unscaffolded

An observer on Earth watches a clock on a passing spacecraft and finds that $2.5\,\mathrm{s}$ elapse on Earth for every $1.0\,\mathrm{s}$ that elapses on the spacecraft's clock. Calculate the speed of the spacecraft relative to Earth.

The two ticks of the spacecraft's clock occur at the same point in space in the **spacecraft** frame, so the $1.0\,\mathrm{s}$ read on that clock is the proper time t_0; the $2.5\,\mathrm{s}$ measured on Earth is the dilated time t. From $t = \gamma\, t_0$,

$$\gamma = \frac{t}{t_0} = \frac{2.5}{1.0} = 2.5\,.$$

Rearranging $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ gives $1 - \dfrac{v^2}{c^2} = \dfrac{1}{\gamma^2}$, so

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} = 1 - \frac{1}{2.5^2} = 1 - 0.16 = 0.84\,,$$

$$v = \sqrt{0.84}\,c = 0.917\,c = 0.917 \times 3.00 \times 10^8 = 2.75 \times 10^8\,\mathrm{m\ s^{-1}}\,.$$

∴ the spacecraft is moving at $0.917\,c$, or $2.75 \times 10^8\,\mathrm{m\ s^{-1}}$.

Example 4 — unscaffolded

As a rod flies lengthwise past a laboratory at a constant $0.80\,c$, an observer in the laboratory measures its length to be $0.68\,\mathrm{m}$. Calculate the proper length of the rod.

The rod is moving in the laboratory frame, so $0.68\,\mathrm{m}$ is the contracted length L, **not** the proper length. The proper length L_0 is the length in the rod's own rest frame, related to the measured length by $L = L_0/\gamma$. At $v = 0.80\,c$,

$$\gamma = \frac{1}{\sqrt{1-0.80^2}} = \frac{1}{\sqrt{0.36}} = \frac{1}{0.60} = \frac{5}{3}\,(=1.67)\,,$$

$$L_0 = \gamma\, L = \frac{5}{3} \times 0.68 = 1.13\,\mathrm{m}\,.$$

∴ the proper length of the rod is $1.13\,\mathrm{m}$ — longer than the measured $0.68\,\mathrm{m}$, since the proper length is the greatest length any observer measures.

Practice questions

Question 1 A spacecraft passes Earth at a constant $0.60\,c$. A clock fixed on the spacecraft measures the interval between two events, which occur at the same location on board, to be $12\,\mathrm{s}$. (a) Calculate the Lorentz factor γ for $v = 0.60\,c$. (b) State which of the two intervals is the proper time, and explain why. (c) Calculate the interval between the two events as measured by an observer on Earth.

Question 2 A spacecraft has a proper length of $150\,\mathrm{m}$ and travels past a space station at a constant $0.80\,c$. (a) Explain which frame of reference measures the proper length of the spacecraft. (b) Calculate the Lorentz factor γ for $v = 0.80\,c$. (c) Calculate the length of the spacecraft as measured by an observer on the station.

Question 3 A spacecraft travels at a constant $0.95\,c$ relative to Earth. (a) Show that the Lorentz factor for this speed is $\gamma = 3.20$ (to three significant figures). (b) A process on board the spacecraft is measured by the crew to last $5.0\,\mathrm{s}$. Calculate the duration of the process as measured by an observer on Earth. (c) State whether the crew's clock or the Earth observer records the proper time for the process, and justify your answer using the same-point-in-space test.

Question 4 A metal rod moves lengthwise past a laboratory at a constant $0.60\,c$. The laboratory measures the rod's length to be $3.2\,\mathrm{m}$. (a) Identify which frame measures the proper length of the rod. (b) Calculate the proper length of the rod. (c) At rest, the rod is $0.15\,\mathrm{m}$ wide (perpendicular to its length). State the width of the rod as measured in the laboratory as it flies past, and justify your answer.

Question 5 As a spacecraft flies past a planet, observers on the planet measure its length to be $80\,\%$ of its proper length. (a) Write down the equation relating the measured length L, the proper length L_0 and the Lorentz factor γ. (b) Show that $\gamma = 1.25$ for this spacecraft. (c) Calculate the speed of the spacecraft relative to the planet, both as a fraction of c and in $\mathrm{m\,s^{-1}}$. Give your answers to three significant figures.

Question 6 Two identical spacecraft, A and B, pass each other in deep space with a constant relative speed of $0.70\,c$. Each carries an identical clock, and each captain observes the other spacecraft's clock. (a) Captain A says that B's clock runs slow; Captain B says that A's clock runs slow. Explain how both statements can be correct, with reference to proper time and to the relative motion of the two frames. (b) In A's frame, which clock measures the proper time between two ticks of A's own clock? Justify your answer using the same-point-in-space test. (c) State how each captain measures the length of the other spacecraft compared with their own, and name the effect responsible.

Question 7 Calculate the Lorentz factor γ for a particle moving at (a) $0.50\,c$, (b) $0.90\,c$ and (c) $0.995\,c$. Give each value to three significant figures.

Question 8 A spacecraft moves at a constant $0.80\,c$ relative to Earth. A biological experiment on board runs for $24\,\mathrm{hours}$ as measured by the crew. Calculate the duration of the experiment as measured from Earth.

Question 9 A rocket flies past a planet at a constant $0.60\,c$. A beacon fixed on the rocket flashes at regular intervals, and observers on the planet measure successive flashes to be $5.0\,\mathrm{s}$ apart. Calculate the time between flashes as measured by an astronaut on the rocket.

Question 10 A spacecraft of proper length $85\,\mathrm{m}$ passes Earth at a constant $0.95\,c$. Calculate the length of the spacecraft as measured by an observer on Earth. Give your answer to three significant figures.

Question 11 A clock on a fast spacecraft is observed from Earth to run at half the rate of an identical Earth clock — that is, $2.0\,\mathrm{s}$ pass on Earth for every $1.0\,\mathrm{s}$ on the spacecraft. Calculate the speed of the spacecraft relative to Earth.

Question 12 As a rod flies lengthwise past a stationary observer, the observer measures its length to be exactly one-third of its proper length. Calculate the speed of the rod. Give your answer to three significant figures.

Question 13 A spacecraft travels at a constant $0.80\,c$ past a space station. On board it carries a rod of proper length $12\,\mathrm{m}$ aligned with the direction of motion, and a lamp that is switched on for $3.0\,\mathrm{s}$ as measured by the crew. As measured by an observer on the station, calculate (a) the length of the rod and (b) the length of time for which the lamp is on.

Question 14 A student says: “If I fly past you at very high speed, I will measure your clock to run slow, and you will measure my clock to run slow. That must be a contradiction — one of us has to be wrong.” Explain why there is no contradiction. Refer to proper time and to what each observer is actually measuring.

Question 15 Using the definitions of proper time and proper length, explain why the proper length of an object is the greatest length any observer measures, while the proper time between two events is the shortest time any observer measures.

Question 16 A cube has sides of length $1.0\,\mathrm{m}$ when at rest. It then moves at a constant high speed in a direction parallel to one set of its edges. Describe the shape the cube is measured to have by a stationary observer, and explain which of its dimensions change and which do not.

Question 17 A spacecraft travels at a constant $0.90\,c$ past a space station. The crew measures a certain task to take $8.0\,\mathrm{s}$, and measures the proper length of the spacecraft to be $60\,\mathrm{m}$. As measured by an observer on the station, calculate (a) the duration of the task and (b) the length of the spacecraft. Give your answers to three significant figures.

Question 18 A stationary observer measures the length of a passing spacecraft to be $60.0\,\mathrm{m}$, while its proper length is $100\,\mathrm{m}$. Determine (a) the speed of the spacecraft relative to the observer and (b) the factor by which the observer measures the spacecraft’s clock to run slow.

Question 19 A particle moves through a laboratory at a constant $0.99\,c$. In the particle’s own rest frame, a certain internal oscillation takes $1.0 \times 10^{-8}\,\mathrm{s}$. Calculate (a) the Lorentz factor for the particle and (b) the duration of the oscillation as measured in the laboratory.

Question 20 A spacecraft flies past a space station at a constant $0.80\,c$. (Circle your answer.) Compared with the proper length of the spacecraft, the length measured by the station is: *greater* / *the same* / *smaller*. Justify your choice, and calculate the ratio of the measured length to the proper length, L / L_0.

Question 21 Two markers on the ground are $900\,\mathrm{m}$ apart, as measured on the ground. A rocket flies from one marker to the other at a constant $0.60\,c$. (a) Calculate the time taken for the rocket to travel between the markers, as measured on the ground. (b) Calculate the distance between the markers as measured by the rocket’s crew. (c) Calculate the time taken to travel between the markers as measured by the rocket’s crew, and identify which of the two times, (a) or (c), is the proper time for the journey.

Answers

Question 1 (a) 1.25; (b) the $12\,\mathrm{s}$ on the spacecraft clock (the two events are at the same point on board); (c) $15\,\mathrm{s}$. **Question 2** (a) the spacecraft's own rest frame; (b) 1.67; (c) $90\,\mathrm{m}$. **Question 3** (a) 3.20; (b) $16.0\,\mathrm{s}$; (c) the crew's clock (both events at the same point on board). **Question 4** (a) the rod's own rest frame; (b) $4.0\,\mathrm{m}$; (c) $0.15\,\mathrm{m}$, unchanged (contraction acts only along the motion). **Question 5** (b) 1.25; (c) $0.600\,c = 1.80\times10^{8}\,\mathrm{m\,s^{-1}}$. **Question 6** (a) reciprocity — each measures the *other's* moving clock; (b) A's own clock (two ticks at one place in A's frame); (c) each measures the other's ship contracted (length contraction). **Question 7** (a) 1.15; (b) 2.29; (c) 10.0. **Question 8** $40\,\mathrm{hours}$. **Question 9** $4.0\,\mathrm{s}$. **Question 10** $26.5\,\mathrm{m}$. **Question 11** $0.866\,c = 2.60\times10^{8}\,\mathrm{m\,s^{-1}}$. **Question 12** $0.943\,c = 2.83\times10^{8}\,\mathrm{m\,s^{-1}}$. **Question 13** (a) $7.2\,\mathrm{m}$; (b) $5.0\,\mathrm{s}$. **Question 14** No contradiction — each observer measures the *other's* (moving) clock from their own rest frame. **Question 15** $\gamma \ge 1$, so $L = L_0/\gamma \le L_0$ (proper length greatest) and $t = \gamma t_0 \ge t_0$ (proper time shortest). **Question 16** A rectangular box: contracted to $1.0/\gamma\,\mathrm{m}$ along the motion; $1.0\,\mathrm{m}$ unchanged in the two perpendicular directions. **Question 17** (a) $18.4\,\mathrm{s}$; (b) $26.2\,\mathrm{m}$. **Question 18** (a) $0.80\,c = 2.40\times10^{8}\,\mathrm{m\,s^{-1}}$; (b) 1.67. **Question 19** (a) 7.09; (b) $7.09\times10^{-8}\,\mathrm{s}$. **Question 20** smaller; $L/L_0 = 0.60$. **Question 21** (a) $5.0\times10^{-6}\,\mathrm{s}$; (b) $720\,\mathrm{m}$; (c) $4.0\times10^{-6}\,\mathrm{s}$ — the rocket crew's time is the proper time.

Fully-worked solutions

Question 1 (a) $\gamma = \dfrac{1}{\sqrt{1-0.60^2}} = \dfrac{1}{\sqrt{0.64}} = \dfrac{1}{0.80} = 1.25$. (b) The proper time is the $12\,\mathrm{s}$ read on the spacecraft's clock. The two events occur at the same point in space in the spacecraft frame — at the fixed location of the clock on board — which is the defining condition for proper time. The Earth observer sees those two events happen at two different places (the ship moves between them), so the Earth interval is not the proper time. (c) The Earth observer sees the clock moving and measures the dilated time: $t = \gamma t_0 = 1.25\times12 = 15\,\mathrm{s}$.

Question 2 (a) The proper length is measured in the frame in which the spacecraft is at rest — the spacecraft's own frame. The stated $150\,\mathrm{m}$ is the proper length, i.e. the length an astronaut would measure on board. (b) $\gamma = \dfrac{1}{\sqrt{1-0.80^2}} = \dfrac{1}{\sqrt{0.36}} = \dfrac{1}{0.60} = 1.67$ (exactly $5/3$). (c) The station sees the spacecraft moving, so measures the contracted length: $L = \dfrac{L_0}{\gamma} = \dfrac{150}{5/3} = 150\times\dfrac{3}{5} = 90\,\mathrm{m}$.

Question 3 The two events (start and end of the process) occur at the same point in space in the spacecraft frame, so the crew's clock reads the proper time $t_0 = 5.0\,\mathrm{s}$. (a) $\gamma = \dfrac{1}{\sqrt{1-0.95^2}} = \dfrac{1}{\sqrt{1-0.9025}} = \dfrac{1}{\sqrt{0.0975}} = \dfrac{1}{0.3122} = 3.20$ (to three significant figures). (b) The Earth observer measures the dilated time: $t = \gamma t_0 = 3.20\times5.0 = 16.0\,\mathrm{s}$. (c) The crew's clock records the proper time. By the same-point-in-space test, in the spacecraft frame the process starts and finishes at the same location (on board); in the Earth frame the spacecraft moves between the start and the end, so those two events are at different places and the Earth interval is the longer, dilated time.

Question 4 (a) The proper length is measured in the rod's own rest frame — the frame in which the rod is not moving. The laboratory sees the rod moving, so its $3.2\,\mathrm{m}$ is the contracted length L, not the proper length. (b) $\gamma = \dfrac{1}{\sqrt{1-0.60^2}} = 1.25$. From $L = \dfrac{L_0}{\gamma}$, the proper length is $L_0 = \gamma L = 1.25\times3.2 = 4.0\,\mathrm{m}$. (c) The width is measured as $0.15\,\mathrm{m}$, unchanged. Length contraction acts only along the direction of motion; the width is perpendicular to the motion, so it is measured to be the same in the laboratory as in the rod's rest frame.

Question 5 The observers on the planet see the spacecraft moving, so they measure the contracted length L; the proper length L_0 is the length in the spacecraft's rest frame. (a) $L = \dfrac{L_0}{\gamma}$. (b) The measured length is $L = 0.80\,L_0$, so

$\dfrac{L_0}{\gamma}=0.80\,L_0$, giving $\dfrac{1}{\gamma}=0.80$ and $\gamma=\dfrac{1}{0.80}=1.25$. (c) $1-\dfrac{v^2}{c^2}=\dfrac{1}{\gamma^2}=\dfrac{1}{1.25^2}=\dfrac{1}{1.5625}=0.64$, so $\dfrac{v^2}{c^2}=0.36$ and $v=\sqrt{0.36}\,c=0.600\,c$. In $\mathrm{m\,s^{-1}}$: $v=0.600\times3.00\times10^8=1.80\times10^8\,\mathrm{m\,s^{-1}}$.

Question 6 (a) Both captains are correct. Motion is relative: in A's rest frame, B and B's clock move at $0.70\,c$, so A measures B's clock — a moving clock — to run slow by the factor γ. In B's rest frame it is A and A's clock that move at $0.70\,c$, so B measures A's clock to run slow by the same factor. Each is measuring the *other's* moving clock from their own rest frame, using a different pair of events, so the two statements are not about the same measurement and cannot conflict. Neither clock is "really" slow in any absolute sense, because there is no preferred frame. (b) A's own clock measures the proper time between two of its own ticks. By the same-point-in-space test, two successive ticks of A's clock happen at the same location in A's frame (the clock is at rest there), which is exactly the condition for proper time. B, who sees A's clock moving, measures those two ticks at different places and so records the longer, dilated time. (c) Each captain measures the other spacecraft to be shorter — contracted along the direction of motion — than their own identical spacecraft, by the factor γ. The effect is length contraction, and it too is symmetric: each sees the other's ship contracted.

Question 7 The Lorentz factor is $\gamma=\dfrac{1}{\sqrt{1-v^2/c^2}}$. (a) $0.50\,c$: $\gamma=\dfrac{1}{\sqrt{1-0.50^2}}=\dfrac{1}{\sqrt{0.75}}=1.15$. (b) $0.90\,c$: $\gamma=\dfrac{1}{\sqrt{1-0.90^2}}=\dfrac{1}{\sqrt{0.19}}=2.29$. (c) $0.995\,c$: $\gamma=\dfrac{1}{\sqrt{1-0.995^2}}=\dfrac{1}{\sqrt{0.009975}}=10.0$.

Question 8 The experiment starts and ends at the same point in space in the spacecraft frame (on board), so the crew's 24 hours is the proper time t_0. The Earth observer sees the spacecraft moving and measures the dilated time:

$$t=\gamma\,t_0=\frac{1}{\sqrt{1-0.80^2}}\times24=\frac{5}{3}\times24=40\,\text{hours}.$$

Question 9 The beacon is fixed on the rocket, so successive flashes occur at the same point in space in the rocket frame: the astronaut on the rocket measures the proper time t_0 between flashes. The planet sees the beacon moving and measures the longer, dilated interval $t=5.0\,\mathrm{s}$. With $\gamma=\dfrac{1}{\sqrt{1-0.60^2}}=1.25$, from $t=\gamma\,t_0$,

$$t_0=\frac{t}{\gamma}=\frac{5.0}{1.25}=4.0\,\mathrm{s}.$$

Question 10 The proper length $L_0=85\,\mathrm{m}$ is measured in the spacecraft's rest frame. The Earth observer sees the spacecraft moving, so measures the contracted length. With $\gamma=\dfrac{1}{\sqrt{1-0.95^2}}=\dfrac{1}{\sqrt{0.0975}}=3.203$,

$$L=\frac{L_0}{\gamma}=\frac{85}{3.203}=26.5\,\mathrm{m}.$$

Question 11 The $1.0\,\mathrm{s}$ on the spacecraft's clock is the proper time t_0 (the two ticks occur at the same point on board); the $2.0\,\mathrm{s}$ measured on Earth is the dilated time t. From $t=\gamma\,t_0$,

$$\gamma=\frac{t}{t_0}=\frac{2.0}{1.0}=2.0.$$

Then $1-\dfrac{v^2}{c^2}=\dfrac{1}{\gamma^2}=\dfrac{1}{4}=0.25$, so $\dfrac{v^2}{c^2}=0.75$ and

$$v=\sqrt{0.75}\,c=0.866\,c=0.866\times3.00\times10^8=2.60\times10^8\,\mathrm{m\,s^{-1}}.$$

Question 12 The observer sees the rod moving, so the measured length is the contracted length L; the proper length L_0 is measured in the rod's rest frame. Given $L=\frac{L_0}{3}$, comparison with $L=\frac{L_0}{\gamma}$ gives $\gamma=3$. Then

$$1-\frac{v^2}{c^2}=\frac{1}{\gamma^2}=\frac{1}{9}, \frac{v^2}{c^2}=1-\frac{1}{9}=\frac{8}{9}=0.8889,$$

$$v=\sqrt{0.8889}\,c=0.943\,c=0.943\times 3.00\times 10^8=2.83\times 10^8\,\mathrm{m\ s^{-1}}.$$

Question 13 The rod's proper length ($12\,\mathrm{m}$) and the lamp's on-time ($3.0\,\mathrm{s}$) are both measured on board, in the spacecraft's rest frame: the rod is at rest there, and the lamp switches on and off at the same point in space there (proper time). The station sees the spacecraft moving at $0.80\,c$, with $\gamma=\frac{5}{3}$. (a) $L=\frac{L_0}{\gamma}=\frac{12}{5/3}=12\times\frac{3}{5}=7.2\,\mathrm{m}$. (b) $t=\gamma\,t_0=\frac{5}{3}\times 3.0=5.0\,\mathrm{s}$.

Question 14 There is no contradiction. Motion is relative, so from the student's rest frame it is *my* clock that is moving, and the student measures it to run slow by the factor γ; from my rest frame it is the *student's* clock that is moving, and I measure it to run slow by the same factor. Each observer measures the *other's* (moving) clock relative to their own rest frame, using a different pair of events happening at a different pair of places, so the two statements are not two measurements of the same thing and cannot conflict. A true contradiction would require one clock to be *absolutely* slow, but "running slow" has meaning only relative to a chosen frame — there is no preferred frame in which either clock is really the slow one.

Question 15 The proper length L_0 is measured in the object's own rest frame. Any other observer sees the object moving and measures $L=\frac{L_0}{\gamma}$. Because $\gamma\geq 1$ for every relative speed, $L\leq L_0$: every moving observer measures a length no greater than L_0, so the proper length is the greatest length any observer measures. Likewise, the proper time t_0 is measured in the frame where the two events occur at the same place. Any other observer sees those events at different places and measures $t=\gamma\,t_0$. Because $\gamma\geq 1$, $t\geq t_0$: every such observer measures a time no shorter than t_0, so the proper time is the shortest time any observer measures. The two results run in opposite directions because γ *multiplies* the proper time but *divides* the proper length.

Question 16 To the stationary observer the object is measured as a rectangular box, not a cube. The dimension along the direction of motion is contracted to $L=\frac{1.0}{\gamma}\,\mathrm{m}$, shorter than $1.0\,\mathrm{m}$, because length contraction acts along the direction of motion. The two dimensions perpendicular to the motion are unchanged at $1.0\,\mathrm{m}$ each, because contraction does not affect lengths perpendicular to the velocity. The object is therefore measured to be foreshortened in one direction only, giving a box of dimensions $\frac{1.0}{\gamma}\times 1.0\times 1.0\,\mathrm{m}$.

Question 17 The task duration ($8.0\,\mathrm{s}$) and the proper length ($60\,\mathrm{m}$) are both measured on board: the two events marking the task occur at the same place on the spacecraft (proper time), and the spacecraft is at rest in its own frame (proper length). The station sees the spacecraft moving at $0.90\,c$, with $\gamma=\frac{1}{\sqrt{1-0.90^2}}=\frac{1}{\sqrt{0.19}}=2.294$. (a) Dilated time: $t=\gamma\,t_0=2.294\times 8.0=18.4\,\mathrm{s}$. (b) Contracted length: $L=\frac{L_0}{\gamma}=\frac{60}{2.294}=26.2\,\mathrm{m}$.

Question 18 The proper length $L_0=100\,\mathrm{m}$ is the spacecraft's length in its own rest frame; the observer sees it moving and measures the contracted length $L=60.0\,\mathrm{m}$. (a) From $L=\frac{L_0}{\gamma}$, $\gamma=\frac{L_0}{L}=\frac{100}{60.0}=\frac{5}{3}=1.67$. Then $1-\frac{v^2}{c^2}=\frac{1}{\gamma^2}=\left(\frac{3}{5}\right)^2=0.36$, so $\frac{v^2}{c^2}=0.64$ and $v=0.80\,c=0.80\times 3.00\times 10^8=2.40\times 10^8\,\mathrm{m\ s^{-1}}$. (b) A moving clock is measured to run slow by the factor γ, so the observer measures the spacecraft's clock to run slow by a factor of 1.67 (that is, $5/3$).

Question 19 In the particle's own rest frame the oscillation begins and ends at the same point in space (at the particle), so 1.0×10^{-8} s is the proper time t_0. The laboratory sees the particle moving at $0.99\,c$. (a) $\gamma=\dfrac{1}{\sqrt{1-0.99^2}}=\dfrac{1}{\sqrt{0.0199}}=7.09$. (b) Dilated time: $t=\gamma\,t_0=7.09\times1.0\times10^{-8}=7.09\times10^{-8}$ s.

Question 20 Smaller. The station sees the spacecraft moving, so it measures the contracted length $L=\dfrac{L_0}{\gamma}$, which is smaller than the proper length L_0 because $\gamma>1$. (The proper length is measured only in the spacecraft's own rest frame.) With $\gamma=\dfrac{1}{\sqrt{1-0.80^2}}=\dfrac{5}{3}$,

$$\frac{L}{L_0}=\frac{1}{\gamma}=\frac{3}{5}=0.60.$$

Question 21 (a) On the ground the two markers are at rest and $900\,\mathrm{m}$ apart, so this is the distance measured in the ground frame. Ground clocks measure the time for the rocket to cover it at $0.60\,c$:

$$t_{\text{ground}}=\frac{900}{0.60\times3.00\times10^8}=\frac{900}{1.80\times10^8}=5.0\times10^{-6}\,\mathrm{s}.$$

(b) The rocket's crew see the ground — and the marker separation — rushing past at $0.60\,c$, so they measure the contracted distance, with $\gamma=1.25$:

$$L=\frac{L_0}{\gamma}=\frac{900}{1.25}=720\,\mathrm{m}.$$

(c) In the rocket frame the two events "rocket passes marker 1" and "rocket passes marker 2" both occur at the same point in space — at the rocket itself — so the rocket's crew measure the proper time t_0. Using the contracted distance and the relative speed,

$$t_0=\frac{720}{1.80\times10^8}=4.0\times10^{-6}\,\mathrm{s}.$$

Equivalently $t_0=\dfrac{t_{\text{ground}}}{\gamma}=\dfrac{5.0\times10^{-6}}{1.25}=4.0\times10^{-6}$ s. The rocket crew's time in (c) is the proper time, because the two events happen at one location in the rocket frame; the ground time in (a) is the longer, dilated interval.

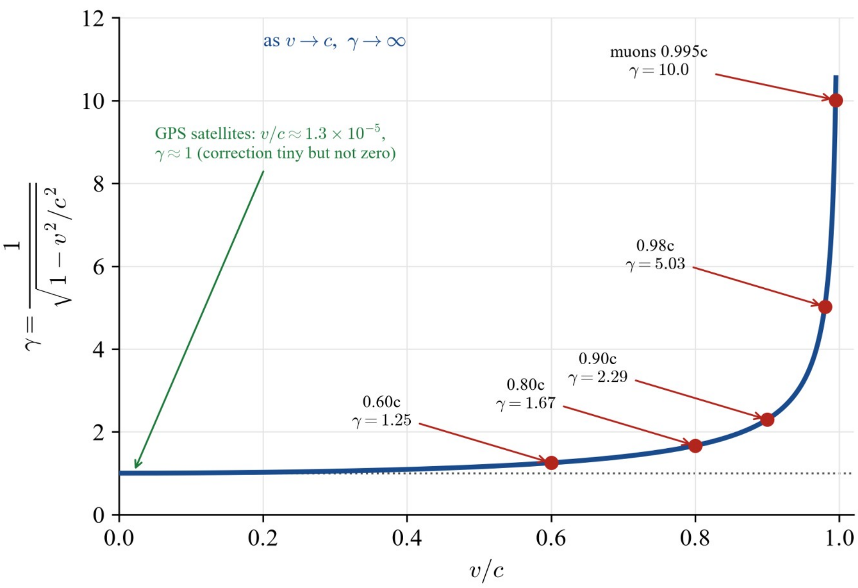
Theory

Special relativity is not just an abstract set of ideas about clocks and rulers. Its two measurable predictions — **time dilation** $t = \gamma t_0$ and **length contraction** $L = L_0 / \gamma$, with the **Lorentz factor**

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

developed in 9B — are needed to make correct sense of three real situations: cosmic-ray **muons** that reach the ground, the design of **particle accelerators**, and the timing signals sent by **GPS satellites**. In every one of these, a calculation that ignores relativity gives an answer that plainly disagrees with what is observed, and a calculation that mixes relativistic ideas *carelessly* gives a self-contradictory one. This subtopic is about applying $t = \gamma t_0$ and $L = L_0 / \gamma$ correctly.

Why γ decides everything. The Lorentz factor is 1 at rest and grows without bound as $v \to c$. It stays extremely close to 1 until the speed is a large fraction of c: $\gamma = 1.005$ at $0.10c$, but $\gamma = 2.29$ at $0.90c$ and $\gamma = 10.0$ at $0.995c$. This is why relativistic effects are dramatic for near-light-speed muons and accelerator particles, yet almost undetectable for everyday objects — and why they must nevertheless be *accounted for* in a system as precise as GPS.



The proper quantities. Two definitions from 9B are used throughout. The **proper time** t_0 is the time between two events measured in the frame where the events happen at the same place; every other observer measures a *longer* time $t = \gamma t_0$. The **proper length** L_0 is the length measured in the frame in which the object is at rest; an observer who sees it moving measures a *shorter* length $L = L_0 / \gamma$. The single most important discipline in this subtopic is to decide **which frame you are working in first**, and then take *every* quantity from that one frame. Time dilation and length contraction are two views of the *same* physics seen from two different frames — they are never applied together to the same object.

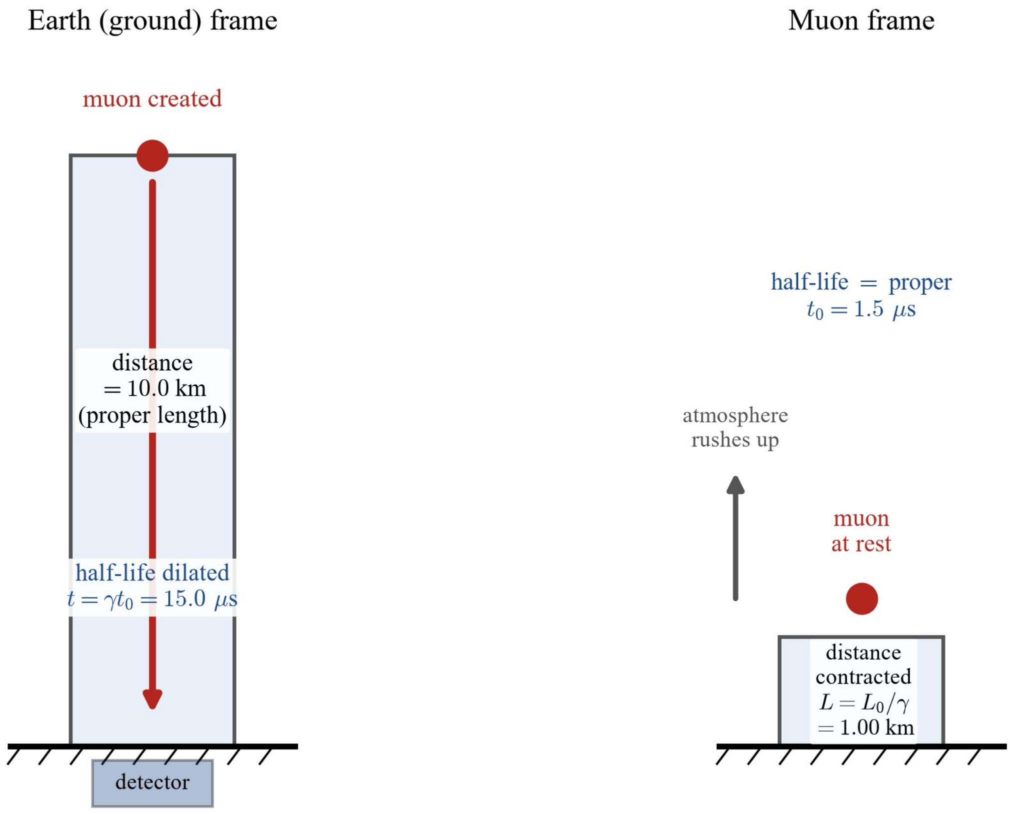
Application 1 — muons reaching the ground

Muons are unstable particles created about 10 km up when cosmic rays strike the upper atmosphere. A muon at rest has a **mean lifetime** of $\tau_0 = 2.2\,\mu\text{s}$ — the average time an individual muon survives. Its **half-life**, the time for *half* of a population to decay, is the shorter quantity

$$t_{1/2} = \tau_0 \ln 2 = 2.2 \times 0.693 = 1.52 \approx 1.5\,\mu\text{s},$$

and that is the value used throughout this subtopic. The two are easily confused, and the confusion is worth a mark: $(1/2)^n$ counts **half-lives**, so read each stem carefully and check which of the two it has given you. After each half-life, half of a population of muons has decayed, so after n half-lives the surviving fraction is $(1/2)^n$.

The classical puzzle. Suppose a muon travels down at $0.995c$. Ignoring relativity, the time to fall $10\,\text{km}$ is $t = \dfrac{10\,000}{0.995 \times 3.00 \times 10^8} = 3.35 \times 10^{-5}\,\text{s}$, that is $33.5\,\mu\text{s}$ — about 22 half-lives. Non-relativistic physics therefore predicts a surviving fraction of $(1/2)^{22} \approx 2 \times 10^{-7}$: almost no muons should reach the ground. Yet large numbers *are* detected at sea level. Special relativity resolves the puzzle, and it can be resolved from either frame — provided you stay in one.



Both frames: $\gamma = 10.0$ at $v = 0.995c$ $\rightarrow$ 2.23 half-lives elapse $\rightarrow$ $\approx 21\%$ of muons reach the ground

Resolved in the Earth (ground) frame. Here the atmosphere really is $10\,\text{km}$ thick (it is at rest in this frame, so $10\,\text{km}$ is its proper length). What changes is the muon's clock: its half-life is **dilated** to $t = \gamma t_0 = 10.0 \times 1.5\,\mu\text{s} = 15.0\,\mu\text{s}$. The descent still takes $33.5\,\mu\text{s}$, but that is now only

$$n = \frac{t_{\text{travel}}}{\gamma t_0} = \frac{33.5}{15.0} = 2.23\,\text{half-lives}\,,$$

so a fraction $(1/2)^{2.23} \approx 0.21$, about $21\,\%$, survives.

Resolved in the muon's frame. Here the muon's clock is the ordinary proper clock, so its half-life is just $1.5\,\mu\text{s}$. What changes is the distance: the atmosphere rushes past at $0.995c$ and is **length-contracted** to $L = L_0/\gamma = \dfrac{10\,000}{10.0} = 1.00 \times 10^3\,\text{m}$. The muon crosses this in $t' = \dfrac{1.00 \times 10^3}{0.995 \times 3.00 \times 10^8} = 3.35 \times 10^{-6}\,\text{s} = 3.35\,\mu\text{s}$, which is

$$n = \frac{t'}{t_0} = \frac{3.35}{1.5} = 2.23\,\text{half-lives}$$

— the *same* number, and the *same* 21% survival. Both frames agree, because in either case

$$n=\frac{H}{\gamma v t_0},$$

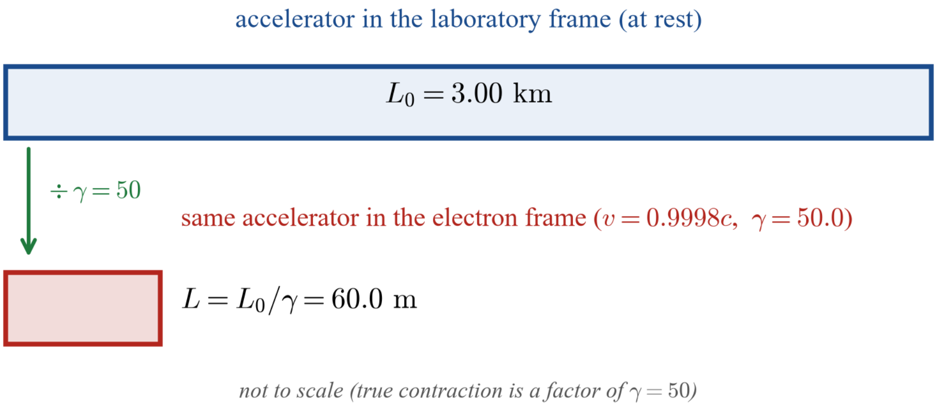
where H is the ground-frame thickness. The two frames are simply two descriptions of one reality.

The one-frame discipline (critical). It is wrong to say "the half-life is dilated to $15.0\,\mu\mathrm{s}$ *and* the distance is contracted to $1.00\,\mathrm{km}$, so the muon easily survives." That statement double-counts the effect: it takes the dilated time from the ground frame and the contracted distance from the muon frame and combines them, which corresponds to *no real observer*. Time dilation belongs to the ground frame; length contraction belongs to the muon frame. Use one or the other — never both for the same muon.

Application 2 — particle-accelerator lengths

In a linear accelerator (linac), charged particles are pushed to speeds extremely close to c along a straight machine of length L_0 measured in the laboratory. Because the particles move so fast, relativity governs the geometry the machine must have.

Seen from the particle's frame, the whole accelerator is length-contracted to $L=L_0/\gamma$. A $3.00\,\mathrm{km}$ linac in which electrons reach $\gamma=50.0$ is only $\dfrac{3000}{50.0}=60.0\,\mathrm{m}$ long in the electrons' own frame.



Seen from the laboratory frame, the same physics appears as time dilation: the internal clocks of the particles — including the decay clocks of unstable particles such as muons or pions — run slow by a factor γ, so a particle survives γ times longer and travels γ times further than a non-relativistic calculation would allow. Designers exploit this: beam lines, storage rings and detectors are built long enough to use short-lived particles that classically would decay almost immediately. Either way, an accelerator "designed with Newtonian physics" would have the wrong length; its dimensions must be chosen to take special relativity into account.

Application 3 — GPS time-signal corrections

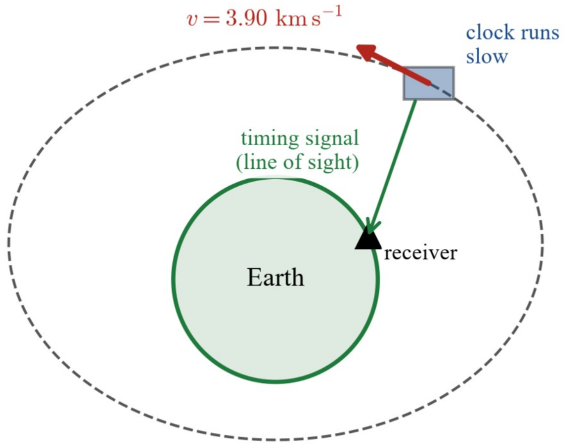
A GPS receiver finds its position by timing signals from several satellites and multiplying each travel time by c to get a distance. Because c is so large, a tiny timing error becomes a large distance error: light travels about $300\,\mathrm{m}$ in a single microsecond. The satellites' atomic clocks must therefore keep near-perfect time — and special relativity says a moving clock does not.

A GPS satellite orbits at about $v=3.9\,\mathrm{km\,s^{-1}}$. Its clock is time-dilated relative to the ground, so it **runs slow**. The speed is so small compared with c that a calculator returns $\gamma=1$ exactly; to see the effect we use the accurate approximation (valid for $v \ll c$)

$$\gamma\approx 1+1/2\left(\frac{v}{c}\right)^2,$$

so that the fractional rate at which the clock loses time is $\gamma - 1 \approx 1/2(v/c)^2$. Over one day this accumulates to several microseconds, which — multiplied by c — would corrupt positions by kilometres per day if it were not corrected. GPS satellite clocks are adjusted in advance so that, once in orbit, they agree with ground clocks.

orbital velocity $\Rightarrow$ time dilation $\Rightarrow$ satellite clock runs slow



$\gamma - 1 \approx \frac{1}{2}(v/c)^2 = 8.45 \times 10^{-11}$ $\rightarrow$ $7.30\ \mu$s per day
$\rightarrow$ position error $c\,\Delta t \approx 2.19$ km per day if uncorrected

A GPS satellite's clock is in fact affected by two separate relativistic effects: the special-relativistic slowing due to its orbital velocity, treated here, and a gravitational effect described by general relativity, which lies beyond this course. The study design asks only for the correction due to **orbital velocity**, and that is what every calculation below addresses.

Throughout this subtopic, $c = 3.00 \times 10^8\,\mathrm{m\ s^{-1}}$ and, where a muon appears, its half-life is the stated in-stem value of $1.5\,\mu\mathrm{s}$ (equivalently, a mean lifetime of $2.2\,\mu\mathrm{s}$).

Worked examples

Example 1 — scaffolded

A muon is created $10\,\mathrm{km}$ above the ground and travels straight down at $0.995c$. Take its half-life to be $1.5\,\mu\mathrm{s}$. Working **entirely in the Earth (ground) frame**, calculate the Lorentz factor, the muon's half-life as seen from the ground, the time it takes to reach the ground, and the number of half-lives that elapse during the descent.

Working	Comment
$v = 0.995c,\ H = 1.00 \times 10^4\,\mathrm{m},\ t_0 = 1.5 \times 10^{-6}\,\mathrm{s}$.	List the data; choose the ground frame and stay in it.
$\gamma = \dfrac{1}{\sqrt{1 - 0.995^2}} = 10.0$.	Lorentz factor at $0.995c$.
$t_{1/2} = \gamma t_0 = 10.0 \times 1.5\,\mu\mathrm{s} = 15.0\,\mu\mathrm{s}$.	In the ground frame the muon's clock is **dilated**; the distance is unchanged.
$t_{\text{travel}} = \dfrac{H}{v} = \dfrac{1.00 \times 10^4}{0.995 \times 3.00 \times 10^8} = 3.35 \times 10^{-5}\,\mathrm{s}$ (= $33.5\,\mu\mathrm{s}$).	Ground-frame travel time over the full $10\,\mathrm{km}$.
$n = \dfrac{t_{\text{travel}}}{\gamma t_0} = \dfrac{H}{\gamma v t_0} = \dfrac{1.00 \times 10^4}{10.0 \times 0.995 \times 3.00 \times 10^8 \times 1.5 \times 10^{-}}$.	Number of dilated half-lives in the descent.

Working	Comment
Surviving fraction $= (1/2)^{2.23} \approx 0.21$, i.e. about 21%.	A substantial fraction reaches the ground — as observed.

Example 2 — scaffolded

A linear accelerator is $3.00\,\text{km}$ long in the laboratory. Electrons travelling through it reach a speed of $0.9998\,c$. Calculate the Lorentz factor of the electrons, the length of the accelerator in the electrons' rest frame, and the time an electron takes to travel the full length as measured in the laboratory.

Working	Comment
$v = 0.9998\,c$, $L_0 = 3.00 \times 10^3\,\text{m}$ (laboratory = proper length of the machine).	The accelerator is at rest in the laboratory, so $3.00\,\text{km}$ is its proper length.
$\gamma = \dfrac{1}{\sqrt{1 - 0.9998^2}} = 50.0$.	Lorentz factor at $0.9998\,c$.
$L = \dfrac{L_0}{\gamma} = \dfrac{3.00 \times 10^3}{50.0} = 60.0\,\text{m}$.	In the electrons' frame the machine is **length-contracted**.
$t = \dfrac{L_0}{v} = \dfrac{3.00 \times 10^3}{0.9998 \times 3.00 \times 10^8} = 1.00 \times 10^{-5}\,\text{s}$ (= $10.0\,\mu\text{s}$).	Laboratory-frame traverse time uses the laboratory length L_0.

Example 3 — unscaffolded

Repeat the muon problem of Example 1 — a muon created $10\,\text{km}$ up, moving at $0.995\,c$, with a half-life of $1.5\,\mu\text{s}$ — but now work **entirely in the muon's own reference frame**. Find the thickness of the atmosphere the muon must cross, the time the crossing takes, and the number of half-lives that elapse. Compare your result with Example 1.

In the muon's frame the muon is at rest, so its half-life is simply the proper value $t_0 = 1.5\,\mu\text{s}$; there is no dilation to apply. Instead it is the atmosphere that moves, at $0.995\,c$, so its thickness is length-contracted:

$$L = \frac{L_0}{\gamma} = \frac{1.00 \times 10^4}{10.0} = 1.00 \times 10^3\,\text{m}.$$

The muon crosses this contracted distance in

$$t' = \frac{L}{v} = \frac{1.00 \times 10^3}{0.995 \times 3.00 \times 10^8} = 3.35 \times 10^{-6}\,\text{s} = 3.35\,\mu\text{s},$$

which is

$$n = \frac{t'}{t_0} = \frac{H}{\gamma v t_0} = \frac{1.00 \times 10^4}{10.0 \times 0.995 \times 3.00 \times 10^8 \times 1.5 \times 10^{-6}} = 2.23\,\text{half-lives}.$$

This is exactly the number found in Example 1, and gives the same surviving fraction of about 21%. The ground frame (dilated time, full distance) and the muon frame (proper time, contracted distance) are two descriptions of one event and must agree.

∴ in the muon frame the atmosphere is only $1.00 \times 10^3\,\text{m}$ thick, crossed in $3.35\,\mu\text{s}$ — 2.23 half-lives, identical to the ground-frame result.

Example 4 — unscaffolded

A GPS satellite orbits at $v = 3.90\,\text{km s}^{-1}$. Using the approximation $\gamma \approx 1 + 1/2\,(v/c)^2$, valid because $v \ll c$, determine by how much the satellite's clock falls behind an identical ground clock in one day ($86\,400\,\text{s}$), and the position error that this would cause if left uncorrected.

The speed ratio is

$$\frac{v}{c}=\frac{3.90\times10^{3}}{3.00\times10^{8}}=1.30\times10^{-5},$$

so the fractional rate at which the moving clock loses time is

$$\gamma-1\approx1/2\left(\frac{v}{c}\right)^{2}=1/2\left(1.30\times10^{-5}\right)^{2}=8.45\times10^{-11}.$$

Over one day the clock falls behind by

$$\Delta t=(\gamma-1)T=8.45\times10^{-11}\times86\,400=7.30\times10^{-6}\,\mathrm{s}=7.30\,\mu\mathrm{s}.$$

Because the receiver turns signal-travel time into distance by multiplying by c, an uncorrected error of Δt becomes a ranging error of

$$c\,\Delta t=3.00\times10^{8}\times7.30\times10^{-6}=2.19\times10^{3}\,\mathrm{m}\approx2.19\,\mathrm{km}.$$

∴ the satellite clock loses about $7.30\,\mu\mathrm{s}$ per day, which would corrupt positions by roughly $2.19\,\mathrm{km}$ each day if the velocity-based correction were not made.

Practice questions

Question 1 A muon in the upper atmosphere travels straight down at $0.98\,c$. Take its half-life to be $1.5\,\mu\mathrm{s}$. (a) Show that the Lorentz factor for the muon is $\gamma=5.03$. (b) Working in the Earth (ground) frame, calculate the muon's half-life as measured by an observer on the ground. (c) The muon is created at an altitude of $8.0\,\mathrm{km}$. Working entirely in the ground frame, calculate the time it takes to reach the ground and hence the number of half-lives that elapse during the descent.

Question 2 A muon moves toward the ground at $0.99\,c$. Take its half-life as $1.5\,\mu\mathrm{s}$. The layer of atmosphere it must cross is $9.0\,\mathrm{km}$ thick as measured from the ground. (a) Calculate the Lorentz factor γ. (b) Working in the muon's rest frame, calculate the thickness of this atmospheric layer. (c) Hence calculate the time the crossing takes in the muon's frame, and the number of half-lives that elapse.

Question 3 A linear accelerator (linac) is $2.5\,\mathrm{km}$ long in the laboratory. Electrons in it reach a speed of $0.999\,c$. (a) Calculate the Lorentz factor γ for these electrons. (b) Calculate the length of the linac in the reference frame of the electrons. Give your answer to three significant figures. (c) Working in the laboratory frame, calculate the time an electron takes to travel the full length of the linac.

Question 4 A GPS satellite moves along its orbit at $3.87\,\mathrm{km\,s^{-1}}$. For a speed this small compared with c, the Lorentz factor is accurately given by $\gamma\approx1+1/2\left(v/c\right)^{2}$. (a) Calculate the ratio v/c for the satellite. (b) Using the approximation above, calculate $\gamma-1$. (c) Hence calculate the amount by which the satellite's clock falls behind an identical ground clock in one day ($86\,400\,\mathrm{s}$), and state the position error $c\,\Delta t$ that would result if this were not corrected.

Question 5 A student is asked to explain how muons created $10\,\mathrm{km}$ up, travelling at $0.995\,c$ (for which $\gamma=10.0$), reach the ground when their half-life is only $1.5\,\mu\mathrm{s}$. The student writes: *"In the ground frame the half-life is dilated to* $15.0\,\mu\mathrm{s}$*, and the* $10\,\mathrm{km}$ *of atmosphere is length-contracted to* $1.00\times10^{3}\,\mathrm{m}$*, so the muon easily survives the trip."* (a) Identify the error in the student's reasoning. (b) Give a correct explanation that works entirely in the Earth (ground) frame. (c) Give a correct explanation that works entirely in the muon's frame.

Question 6 GPS satellites carry atomic clocks that must be corrected for special-relativistic effects due to their orbital velocity. (a) Explain why the clock on a GPS satellite runs slow compared with an identical clock on the ground, with reference to the satellite's orbital velocity. (b) A timing error of only a few microseconds per day might seem negligible. Explain why it cannot be ignored for a positioning system. (c) (Circle your answer.) If the satellite orbited at a higher speed, the daily time correction required would be: *larger / the same / smaller.* Justify your choice. No calculations are required.

Question 7 A muon travels vertically downward at $0.96c$ and is created at an altitude of $6.0\ \mathrm{km}$. Taking its half-life as $1.5\ \mu\mathrm{s}$ and working entirely in the ground frame, determine the number of half-lives that elapse before it reaches the ground.

Question 8 A muon moves at $0.90c$ toward the ground. Taking its half-life as $1.5\ \mu\mathrm{s}$ and working in the muon's rest frame, calculate the number of half-lives that elapse while it crosses a $3.0\ \mathrm{km}$ layer of atmosphere (thickness measured from the ground).

Question 9 A muon travelling at $0.866c$ has a half-life of $1.5\ \mu\mathrm{s}$. (a) Show that its Lorentz factor is $\gamma = 2.00$. (b) In the ground frame the muon is observed to take $15\ \mu\mathrm{s}$ to reach a detector. Calculate the number of half-lives that elapse and hence the fraction of such muons expected to survive the trip.

Question 10 In a proton linac $4.0\ \mathrm{km}$ long (laboratory length), protons reach $0.95c$. Calculate the length of the accelerator as measured in the protons' rest frame.

Question 11 In a muon storage ring, muons circulate at $0.9994c$. Taking the proper half-life of a muon as $1.5\ \mu\mathrm{s}$, calculate (i) the muon half-life measured in the laboratory, and (ii) the distance a muon travels in the laboratory during one such half-life. Comment briefly on why this matters for the scale of the facility.

Question 12 Explain why the length of a particle accelerator, and the spacing of its components, must be designed with special relativity in mind. Refer to what happens in the reference frame of the accelerated particles.

Question 13 Working entirely in the muon's own reference frame, explain the chain of reasoning by which a muon with a half-life of only $1.5\ \mu\mathrm{s}$ is able to reach the ground from high in the atmosphere.

Question 14 (Circle your answer.) Two muons are created at the same height in the atmosphere. Muon A travels at $0.90c$ and muon B at $0.99c$. The muon more likely to reach the ground is: *A / B / equally likely.* Justify your answer with reference to special relativity. No calculations are required.

Question 15 A satellite in a low orbit travels at $7.66\ \mathrm{km\ s^{-1}}$, nearly twice the orbital speed of a GPS satellite. Using $\gamma \approx 1 + 1/2\left(v/c\right)^2$, calculate the amount by which its clock falls behind a ground clock in one day ($86\,400\ \mathrm{s}$), and the corresponding uncorrected position error $c\,\Delta t$.

Question 16 Two navigation satellites orbit at different altitudes; the lower one moves faster. Explain which satellite requires the larger velocity-based time correction, and why.

Question 17 A muon travels at $0.60c$ through a $3.6\ \mathrm{km}$ layer of atmosphere (thickness measured from the ground); its half-life is $1.5\ \mu\mathrm{s}$. By calculating the number of half-lives that elapse in **both** the ground frame and the muon's frame, show that the two frames predict the same outcome.

Question 18 Electrons in a linac $3.2\ \mathrm{km}$ long (laboratory length) are accelerated to $0.9999c$. Calculate (i) the length of the linac in the electrons' frame, (ii) the time the electrons take to traverse the linac as measured in the laboratory, and (iii) the time for the same traversal as measured in the electrons' own frame.

Question 19 A colleague argues that because the velocity-based relativistic correction for a GPS satellite is only about $7\ \mu\mathrm{s}$ per day, it is far too small to affect everyday navigation and can safely be ignored. Explain, with a supporting estimate, why this argument is wrong.

Question 20 Explain how the detection of muons at the Earth's surface provides experimental evidence for time dilation. In your answer, contrast the prediction made using non-relativistic (classical) physics with what is actually observed.

Question 21 Relativistic effects are crucial when designing particle accelerators and when interpreting muon observations, yet they are completely negligible for a passenger jet cruising at about $250\ \mathrm{m\ s^{-1}}$. Explain why, with reference to how γ depends on speed.

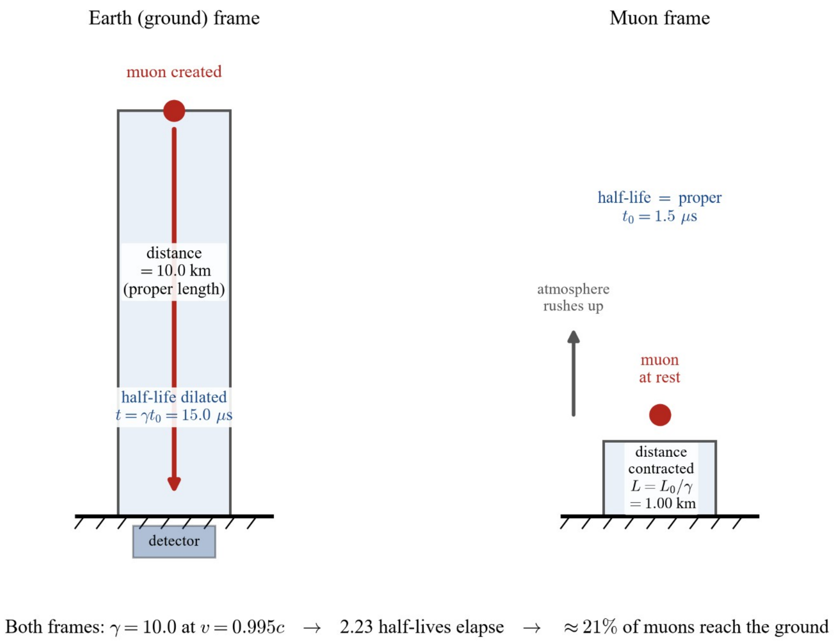
Answers

Question 1 (a) $\gamma = 5.03$; (b) $7.54\,\mu\text{s}$; (c) $27.2\,\mu\text{s}$, 3.61 half-lives. **Question 2** (a) $\gamma = 7.09$; (b) $1.27\,\text{km}$ ($1.27 \times 10^3\,\text{m}$); (c) $4.27\,\mu\text{s}$, 2.85 half-lives. **Question 3** (a) $\gamma = 22.4$; (b) $112\,\text{m}$; (c) $8.34\,\mu\text{s}$. **Question 4** (a) 1.29×10^{-5}; (b) 8.32×10^{-11}; (c) $7.19\,\mu\text{s}$ per day, position error $\approx 2.16\,\text{km}$. **Question 5** (a) The explanation illegitimately combines a ground-frame result (dilation) with a muon-frame result (contraction), double-counting the effect; (b), (c) single-frame explanations below. **Question 6** (a), (b) qualitative; (c) *larger*. **Question 7** 3.89 half-lives. **Question 8** 3.23 half-lives. **Question 9** (a) $\gamma = 2.00$; (b) 5.00 half-lives, about $3.1\,\%$ survive. **Question 10** $1.25\,\text{km}$ ($1.25 \times 10^3\,\text{m}$). **Question 11** (i) $43.3\,\mu\text{s}$; (ii) $13.0\,\text{km}$. **Question 12** Qualitative — see solution. **Question 13** Qualitative — see solution. **Question 14** *B* (the faster muon). **Question 15** $28.2\,\mu\text{s}$ per day; position error $\approx 8.45\,\text{km}$. **Question 16** The lower, faster satellite. **Question 17** 10.7 half-lives in each frame — the two frames agree. **Question 18** (i) $45.3\,\text{m}$; (ii) $10.7\,\mu\text{s}$; (iii) $0.151\,\mu\text{s}$. **Question 19** Qualitative — an uncorrected $7\,\mu\text{s}$ per day is about $2\,\text{km}$ per day of position error. **Question 20** Qualitative — see solution. **Question 21** Qualitative — see solution.

Fully-worked solutions

Question 1 $v = 0.98\,c$, $t_0 = 1.5 \times 10^{-6}\,\text{s}$, $H = 8.0 \times 10^3\,\text{m}$. (a) $\gamma = \dfrac{1}{\sqrt{1 - 0.98^2}} = \dfrac{1}{\sqrt{1 - 0.9604}} = \dfrac{1}{\sqrt{0.0396}} = 5.03$, as required. (b) In the ground frame the muon's clock is dilated: $t_{1/2} = \gamma\, t_0 = 5.025 \times 1.5 \times 10^{-6} = 7.54 \times 10^{-6}\,\text{s} = 7.54\,\mu\text{s}$. (c) Ground-frame travel time over the full $8.0\,\text{km}$: $t_{\text{travel}} = \dfrac{H}{v} = \dfrac{8.0 \times 10^3}{0.98 \times 3.00 \times 10^8} = 2.72 \times 10^{-5}\,\text{s} = 27.2\,\mu\text{s}$. The number of (dilated) half-lives is $n = \dfrac{t_{\text{travel}}}{\gamma\, t_0} = \dfrac{H}{\gamma\, v\, t_0} = \dfrac{8.0 \times 10^3}{5.025 \times 0.98 \times 3.00 \times 10^8 \times 1.5 \times 10^{-6}} = 3.61$. All quantities are taken from the one (ground) frame.

Question 2 $v = 0.99\,c$, $t_0 = 1.5 \times 10^{-6}\,\text{s}$, $H = 9.0 \times 10^3\,\text{m}$.

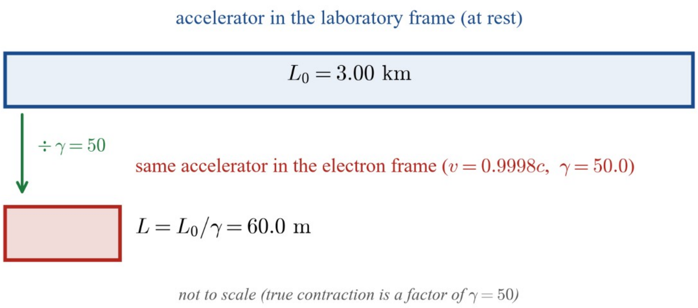


(a) $\gamma = \dfrac{1}{\sqrt{1 - 0.99^2}} = \dfrac{1}{\sqrt{0.0199}} = 7.09$.

(b) In the muon's frame the atmosphere is length-contracted: $L = \dfrac{L_0}{\gamma} = \dfrac{9.0 \times 10^3}{7.089} = 1.27 \times 10^3\,\text{m}$ (1.27 km).

(c) The muon's own clock reads the proper half-life, so no dilation is applied. Crossing time $t' = \frac{L}{v} = \frac{1.27 \times 10^3}{0.99 \times 3.00 \times 10^8} = 4.27 \times 10^{-6}\,\text{s} = 4.27\,\mu\text{s}$. Number of half-lives $n = \frac{t'}{t_0} = \frac{H}{\gamma v t_0} = \frac{9.0 \times 10^3}{7.089 \times 0.99 \times 3.00 \times 10^8 \times 1.5 \times 10^{-6}} = 2.85$.

Question 3 $v = 0.999\,c$, $L_0 = 2.5 \times 10^3\,\text{m}$.



(a) $\gamma = \frac{1}{\sqrt{1 - 0.999^2}} = \frac{1}{\sqrt{0.001999}} = 22.4$.

(b) In the electrons' frame the machine is contracted: $L = \frac{L_0}{\gamma} = \frac{2.5 \times 10^3}{22.37} = 112\,\text{m}$ (to three significant figures).

(c) The laboratory-frame traverse uses the laboratory length: $t = \frac{L_0}{v} = \frac{2.5 \times 10^3}{0.999 \times 3.00 \times 10^8} = 8.34 \times 10^{-6}\,\text{s} = 8.34\,\mu\text{s}$.

Question 4 $v = 3.87 \times 10^3\,\text{m s}^{-1}$, $T = 86\,400\,\text{s}$. (a) $\frac{v}{c} = \frac{3.87 \times 10^3}{3.00 \times 10^8} = 1.29 \times 10^{-5}$. (b) $\gamma - 1 \approx 1/2\left(\frac{v}{c}\right)^2 = 1/2(1.29 \times 10^{-5})^2 = 8.32 \times 10^{-11}$. (c) $\Delta t = (\gamma - 1)T = 8.32 \times 10^{-11} \times 86\,400 = 7.19 \times 10^{-6}\,\text{s} = 7.19\,\mu\text{s}$ per day. Uncorrected, this is a ranging error of $c\,\Delta t = 3.00 \times 10^8 \times 7.19 \times 10^{-6} = 2.16 \times 10^3\,\text{m} \approx 2.16\,\text{km}$.

Question 5 (a) The two effects belong to two *different* frames and cannot both be applied to the same muon. Dilation of the half-life to $15.0\,\mu\text{s}$ is a ground-frame statement (in which the atmosphere is the full $10\,\text{km}$ thick); contraction of the atmosphere to $1.00 \times 10^3\,\text{m}$ is a muon-frame statement (in which the half-life is the proper $1.5\,\mu\text{s}$). Combining the dilated time with the contracted distance double-counts the relativistic effect and describes no real observer. A correct explanation uses one frame only. (b) *Ground frame.* The atmosphere is $10\,\text{km}$ thick. The muon's clock is dilated, so its half-life is $\gamma t_0 = 10.0 \times 1.5 = 15.0\,\mu\text{s}$. The descent takes $\frac{10\,000}{0.995 \times 3.00 \times 10^8} = 33.5\,\mu\text{s}$, which is $n = \frac{33.5}{15.0} = 2.23$ half-lives — so about $(1/2)^{2.23} \approx 21\,\%$ survive. (c) *Muon frame.* The half-life is the proper $1.5\,\mu\text{s}$. The atmosphere rushes past and is contracted to $\frac{10\,000}{10.0} = 1.00 \times 10^3\,\text{m}$, crossed in $\frac{1.00 \times 10^3}{0.995 \times 3.00 \times 10^8} = 3.35\,\mu\text{s}$, which is $n = \frac{3.35}{1.5} = 2.23$ half-lives — the same $21\,\%$ survival. Each frame, used consistently, gives the same answer.

Question 6 (a) In the ground frame the satellite is moving at its orbital velocity, so its clock is time-dilated: the time between ticks measured from the ground, $t = \gamma t_0$, is longer than the proper time between ticks the clock itself keeps. A dilated interval between ticks means fewer ticks per second as seen from the ground, i.e. the moving clock runs slow. The larger γ becomes, the more slowly it runs; for orbital speeds γ exceeds 1 by only a tiny amount, so the slowing is small but non-zero. (b) A GPS receiver converts each signal's travel time to a distance by multiplying by c. Since

$c \approx 3.00 \times 10^8\,\mathrm{m\ s^{-1}}$, light covers about $300\,\mathrm{m}$ in one microsecond, so an error of only a few microseconds corresponds to a distance error of order a kilometre — and the error grows every day. Left uncorrected, positions would drift by kilometres within a day, far exceeding the metre-level accuracy GPS is built to provide. (c) **Larger.** The fractional slowing is $\gamma - 1 \approx 1/2(v/c)^2$, which increases with v. A higher orbital speed therefore makes the clock run slower and demands a larger correction.

Question 7 $v = 0.96c$, $H = 6.0 \times 10^3\,\mathrm{m}$, $t_0 = 1.5 \times 10^{-6}\,\mathrm{s}$. $\gamma = \dfrac{1}{\sqrt{1-0.96^2}} = \dfrac{1}{\sqrt{0.0784}} = 3.57$. In the ground frame the dilated half-life is $\gamma t_0 = 3.571 \times 1.5 = 5.36\,\mu\mathrm{s}$ and the travel time is $\dfrac{H}{v} = \dfrac{6.0 \times 10^3}{0.96 \times 3.00 \times 10^8} = 2.08 \times 10^{-5}\,\mathrm{s} = 20.8\,\mu\mathrm{s}$, so

$$n = \frac{H}{\gamma v t_0} = \frac{6.0 \times 10^3}{3.571 \times 0.96 \times 3.00 \times 10^8 \times 1.5 \times 10^{-6}} = 3.89\ \text{half-lives.}$$

Question 8 $v = 0.90c$, $H = 3.0 \times 10^3\,\mathrm{m}$, $t_0 = 1.5 \times 10^{-6}\,\mathrm{s}$. $\gamma = \dfrac{1}{\sqrt{1-0.90^2}} = \dfrac{1}{\sqrt{0.19}} = 2.29$. In the muon frame the atmosphere contracts to $L = \dfrac{H}{\gamma} = \dfrac{3.0 \times 10^3}{2.294} = 1.31 \times 10^3\,\mathrm{m}$, crossed in $t' = \dfrac{L}{v} = \dfrac{1.31 \times 10^3}{0.90 \times 3.00 \times 10^8} = 4.84 \times 10^{-6}\,\mathrm{s} = 4.84\,\mu\mathrm{s}$, so

$$n = \frac{t'}{t_0} = \frac{H}{\gamma v t_0} = \frac{3.0 \times 10^3}{2.294 \times 0.90 \times 3.00 \times 10^8 \times 1.5 \times 10^{-6}} = 3.23\ \text{half-lives.}$$

Question 9 $v = 0.866c$, $t_0 = 1.5\,\mu\mathrm{s}$. (a) $\gamma = \dfrac{1}{\sqrt{1-0.866^2}} = \dfrac{1}{\sqrt{1-0.750}} = \dfrac{1}{\sqrt{0.250}} = 2.00$, as required. (b) In the ground frame the dilated half-life is $\gamma t_0 = 2.00 \times 1.5 = 3.00\,\mu\mathrm{s}$. In the observed travel time of $15\,\mu\mathrm{s}$ the number of half-lives is $n = \dfrac{15}{2.00 \times 1.5} = \dfrac{15}{3.00} = 5.00$. The surviving fraction is $(1/2)^{5.00} \approx 0.031$, i.e. about 3.1%.

Question 10 $v = 0.95c$, $L_0 = 4.0 \times 10^3\,\mathrm{m}$. $\gamma = \dfrac{1}{\sqrt{1-0.95^2}} = \dfrac{1}{\sqrt{0.0975}} = 3.20$. In the protons' frame the machine is contracted to $L = \dfrac{L_0}{\gamma} = \dfrac{4.0 \times 10^3}{3.203} = 1.25 \times 10^3\,\mathrm{m}$ $(1.25\,\mathrm{km})$.

Question 11 $v = 0.9994c$, $t_0 = 1.5 \times 10^{-6}\,\mathrm{s}$. $\gamma = \dfrac{1}{\sqrt{1-0.9994^2}} = \dfrac{1}{\sqrt{1.200 \times 10^{-3}}} = 28.9$. (i) In the laboratory the muon half-life is dilated: $t_{1/2} = \gamma t_0 = 28.87 \times 1.5 \times 10^{-6} = 4.33 \times 10^{-5}\,\mathrm{s} = 43.3\,\mu\mathrm{s}$. (ii) In one such half-life the muon travels $d = v t_{1/2} = 0.9994 \times 3.00 \times 10^8 \times 4.331 \times 10^{-5} = 1.30 \times 10^4\,\mathrm{m} = 13.0\,\mathrm{km}$. The muons persist for tens of microseconds and cover kilometres in the laboratory — nearly 30 times the $v t_0 \approx 450\,\mathrm{m}$ a non-relativistic half-life would give — so the ring and its instruments must be built on that larger scale to make use of the beam.

Question 12 In the reference frame of the accelerated particles, the accelerator and the spacing of its components are **length-contracted** by the factor γ, which becomes very large near c; equivalently, viewed from the laboratory, the particles' internal clocks (including the decay clocks of unstable particles) are **time-dilated**, so the particles survive far longer and travel far further than non-relativistic physics predicts. Either description shows that the dimensions and timing that make the machine work — the length over which particles are accelerated, the spacing of accelerating structures, the length of beam lines and storage rings — depend on γ and therefore on the relativistic speed reached. A machine sized using non-relativistic physics would have the wrong length and would not keep the accelerating pushes in step with the particles; the design must take special relativity into account.

Question 13 In the muon's own frame the muon is at rest, so its clock keeps the ordinary proper half-life of $1.5\,\mu\mathrm{s}$ — there is no time dilation to help it. What helps instead is that the atmosphere is rushing toward the muon at close to c, so the thickness of atmosphere between the muon and the ground is **length-contracted** to a small fraction of its ground-frame value. Because that contracted distance is short, the muon covers it in only a couple of proper half-lives, and so a significant fraction of muons survive the crossing and reach the ground. The reasoning stays entirely within the muon's frame: proper half-life, contracted distance, few half-lives, appreciable survival.

Question 14 B. Muon B is faster, so it has the larger Lorentz factor γ. In the ground frame a larger γ means a more strongly dilated half-life, so fewer half-lives elapse during the descent from the same starting height; equivalently, in each muon's own frame the faster muon sees the atmosphere contracted to a shorter distance, again crossed in fewer half-lives. Either way, the faster muon loses a smaller fraction of its population and is more likely to reach the ground.

Question 15 $v=7.66\times10^3\,\mathrm{m\,s^{-1}}$, $T=86\,400\,\mathrm{s}$. $\dfrac{v}{c}=\dfrac{7.66\times10^3}{3.00\times10^8}=2.55\times10^{-5}$, so $\gamma-1\approx1/2(v/c)^2=1/2(2.55\times10^{-5})^2=3.26\times10^{-10}$. Over one day, $\Delta t=(\gamma-1)T=3.26\times10^{-10}\times86\,400=2.82\times10^{-5}\,\mathrm{s}=28.2\,\mu\mathrm{s}$. The uncorrected position error is $c\,\Delta t=3.00\times10^8\times2.82\times10^{-5}=8.45\times10^3\,\mathrm{m}\approx8.45\,\mathrm{km}$. (The effect is about four times that of the slower GPS satellite, because it scales as v^2.)

Question 16 The velocity-based correction depends on the fractional clock-slowing $\gamma-1\approx1/2(v/c)^2$, which grows with orbital speed. The lower satellite moves faster, so it has the larger v, the larger $(v/c)^2$, the larger $\gamma-1$, and therefore the larger daily time correction. The higher, slower satellite needs a smaller velocity correction.

Question 17 $v=0.60\,c$, $H=3.6\times10^3\,\mathrm{m}$, $t_0=1.5\times10^{-6}\,\mathrm{s}$, and $\gamma=\dfrac{1}{\sqrt{1-0.60^2}}=\dfrac{1}{\sqrt{0.64}}=1.25$.

Ground frame. The dilated half-life is $\gamma t_0=1.25\times1.5=1.875\,\mu\mathrm{s}$, and the travel time over the full $3.6\,\mathrm{km}$ is $\dfrac{H}{v}=\dfrac{3.6\times10^3}{0.60\times3.00\times10^8}=2.00\times10^{-5}\,\mathrm{s}=20.0\,\mu\mathrm{s}$, so $n=\dfrac{20.0}{1.875}=10.7$ half-lives.

Muon frame. The atmosphere contracts to $L=\dfrac{H}{\gamma}=\dfrac{3.6\times10^3}{1.25}=2880\,\mathrm{m}$, crossed in $\dfrac{L}{v}=\dfrac{2880}{0.60\times3.00\times10^8}=1.60\times10^{-5}\,\mathrm{s}=16.0\,\mu\mathrm{s}$, so $n=\dfrac{16.0}{1.5}=10.7$ half-lives.

Both frames give $n=10.7$ half-lives (and hence the same surviving fraction, about $0.062\,\%$), because in each case $n=\dfrac{H}{\gamma v t_0}$. The dilated-time/full-distance and proper-time/contracted-distance descriptions are consistent.

Question 18 $v=0.9999\,c$, $L_0=3.2\times10^3\,\mathrm{m}$. $\gamma=\dfrac{1}{\sqrt{1-0.9999^2}}=\dfrac{1}{\sqrt{2.000\times10^{-4}}}=70.7$. (i) In the electrons' frame the linac is contracted to $L=\dfrac{L_0}{\gamma}=\dfrac{3.2\times10^3}{70.71}=45.3\,\mathrm{m}$. (ii) In the laboratory frame the traverse takes $t=\dfrac{L_0}{v}=\dfrac{3.2\times10^3}{0.9999\times3.00\times10^8}=1.07\times10^{-5}\,\mathrm{s}=10.7\,\mu\mathrm{s}$. (iii) In the electrons' frame the same traverse covers the contracted length at the same relative speed: $t'=\dfrac{L}{v}=\dfrac{45.25}{0.9999\times3.00\times10^8}=1.51\times10^{-7}\,\mathrm{s}=0.151\,\mu\mathrm{s}$ (equivalently $t'=t/\gamma$).

Question 19 The argument confuses "small time" with "small consequence." A GPS receiver turns time into distance by multiplying by c, and $c\approx3.00\times10^8\,\mathrm{m\,s^{-1}}$ means light travels about $300\,\mathrm{m}$ per microsecond. An uncorrected error of $7\,\mu\mathrm{s}$ therefore maps to $c\,\Delta t\approx3.00\times10^8\times7\times10^{-6}\approx2\times10^3\,\mathrm{m}$, i.e. about $2\,\mathrm{km}$ — and it accumulates each day. Within a single day positions would be wrong by kilometres, and within an hour by roughly $100\,\mathrm{m}$, both far larger than the metre-level accuracy required. A "few microseconds per day" is enormous by GPS standards and must be corrected.

Question 20 Non-relativistic physics predicts almost no muons at ground level. Using the proper half-life of $1.5\,\mu\mathrm{s}$ with the full $\sim 10\,\mathrm{km}$ of atmosphere, a muon at $0.995\,c$ takes about $33.5\,\mu\mathrm{s}$ to descend — roughly 22 half-lives — so classically only a fraction $(1/2)^{22} \approx 2 \times 10^{-7}$ should survive. In fact muons are detected at the surface in large numbers, far more than this. The discrepancy is removed once time dilation is included: in the ground frame the muon's half-life is dilated by γ, so only about 2.2 half-lives elapse during the descent and roughly a fifth survive. The measured surface muon flux therefore agrees with the relativistic prediction and disagrees with the classical one, providing direct experimental evidence for time dilation. (Equivalently, in the muon's frame the atmosphere is length-contracted — the same evidence viewed from the other frame.)

Question 21 The Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-(v/c)^2}}$ depends on the *square* of v/c, and it stays extremely close to 1 until v is a sizeable fraction of c. For a jet at $250\,\mathrm{m\ s^{-1}}$, $v/c \approx 8.3 \times 10^{-7}$, so $\gamma - 1 \approx 1/2(v/c)^2 \approx 3 \times 10^{-13}$ — a completely negligible departure from 1, meaning clocks and lengths on the jet differ from ground values by less than a part in a trillion. Muons and accelerator particles, by contrast, move at $0.9\,c$ to $0.9999\,c$, where $(v/c)^2$ is close to 1 and γ ranges from about 2 to many tens, so time dilation and length contraction become large and must be taken into account. The difference is entirely one of speed relative to c.

Theory

One of the most far-reaching consequences of Einstein's special theory of relativity is that **mass and energy are equivalent** — they are two measures of the same thing, related by the single most famous equation in physics,

$$E_0 = mc^2.$$

Here m is the mass of an object measured in the frame in which it is at rest, and E_0 is its **rest energy**: the energy an object possesses purely by virtue of having mass, even when it is not moving. Because $c = 3.00 \times 10^8 \text{ m s}^{-1}$ is so large, a very small mass is equivalent to an enormous energy. For an electron, $m_e = 9.11 \times 10^{-31}$ kg, so

$$E_0 = m_e c^2 = 9.11 \times 10^{-31} \times \left(3.00 \times 10^8\right)^2 = 8.20 \times 10^{-14} \text{ J}.$$

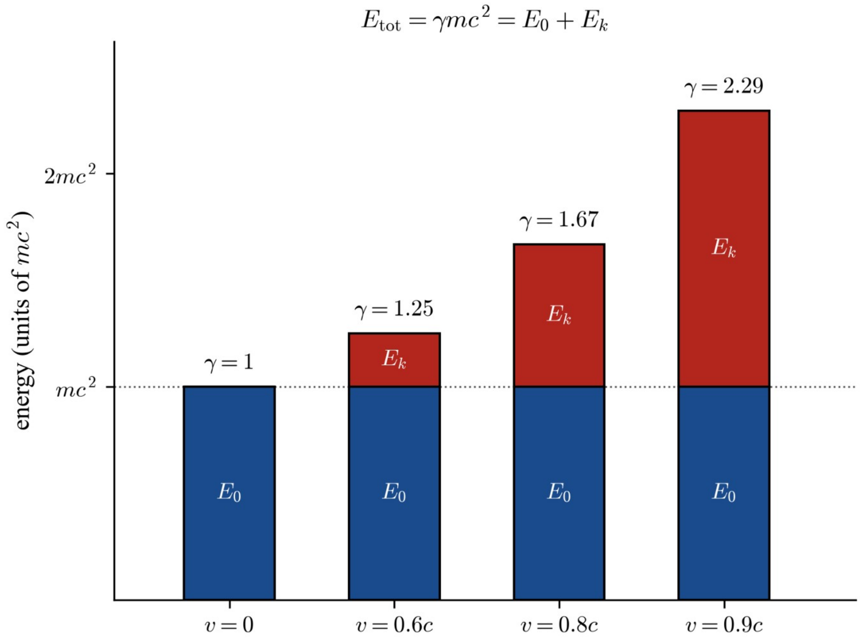
Total energy and kinetic energy. When an object moves, it has kinetic energy in addition to its rest energy. Special relativity gives the **total energy** of an object moving at speed v as

$$E_{\text{tot}} = \gamma mc^2, \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},$$

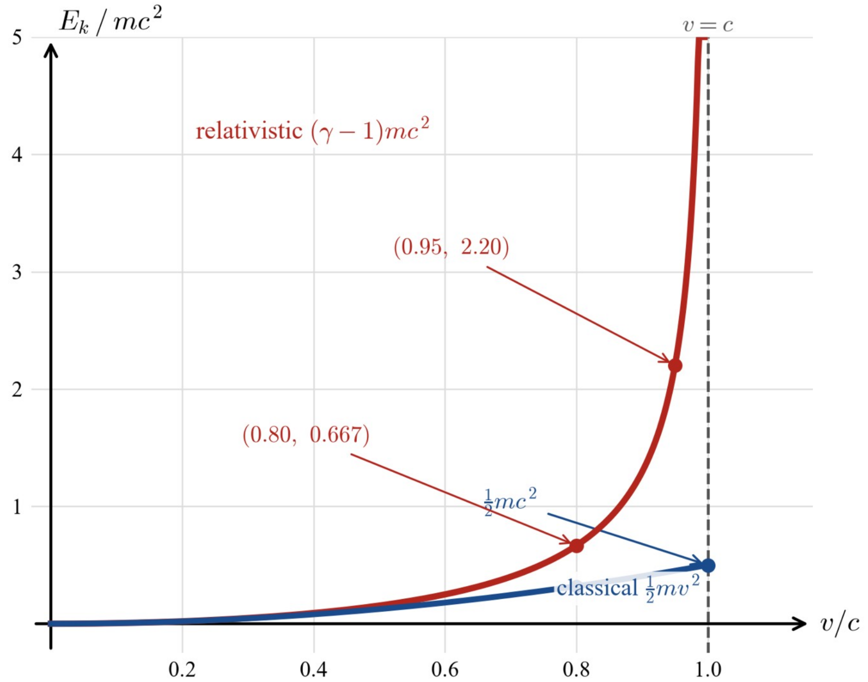
where γ is the Lorentz factor introduced in 9A. Since the total energy is the sum of the kinetic energy and the rest energy, $E_{\text{tot}} = E_k + E_0$, the **relativistic kinetic energy** is the total energy minus the rest energy:

$$E_k = E_{\text{tot}} - E_0 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2.$$

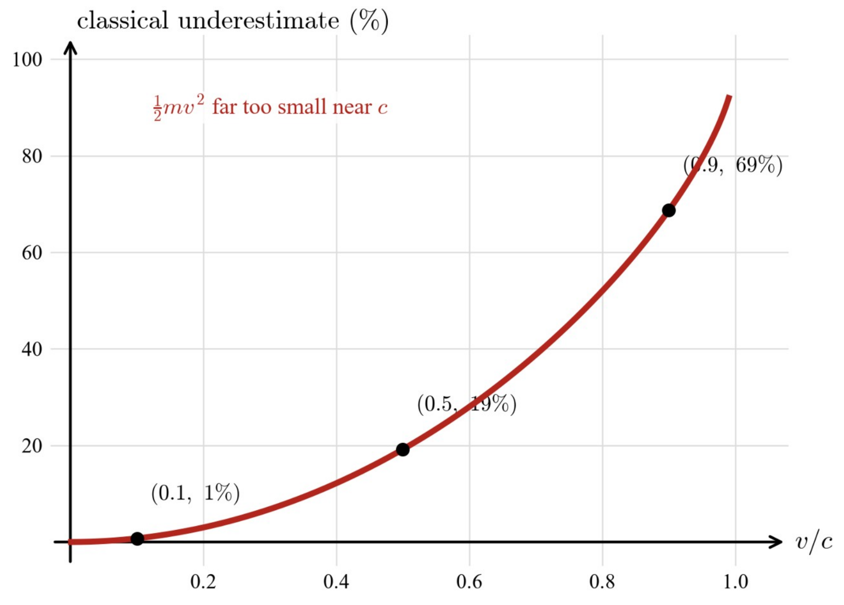
At rest, $v = 0$ and $\gamma = 1$, so $E_k = 0$ and $E_{\text{tot}} = E_0 = mc^2$: a stationary object still has energy mc^2. As v increases, γ grows, and both the total energy and the kinetic energy grow with it. The bar chart below shows how, for a fixed rest energy $E_0 = mc^2$, the kinetic energy $E_k = (\gamma - 1)mc^2$ stacks on top of it to give the total energy $E_{\text{tot}} = \gamma mc^2$: at $v = 0.6c$, $\gamma = 1.25$; at $v = 0.8c$, $\gamma = 1.67$; and at $v = 0.9c$, $\gamma = 2.29$.



Why the classical formula fails near c. In earlier chapters the kinetic energy of a moving object was written $1/2\,mv^2$. That expression is only the **low-speed approximation** of the correct relativistic result $E_k = (\gamma - 1)mc^2$; the two agree when $v \ll c$ but diverge dramatically as v approaches c. Measuring kinetic energy in units of the rest energy mc^2, the classical formula becomes $1/2(v/c)^2$ — a parabola that reaches only 0.5 (that is, $1/2\,mc^2$) even at $v=c$. The relativistic curve $(\gamma-1)$, by contrast, rises without limit as $v\to c$, because $\gamma\to\infty$.



At $v=0.10\,c$ the two curves agree almost exactly (both give about $0.005\,mc^2$). By $v=0.80\,c$ the classical value of $0.320\,mc^2$ has fallen well short of the true value of $0.667\,mc^2$, and by $v=0.95\,c$ the classical value $0.451\,mc^2$ is less than a quarter of the true value $2.20\,mc^2$. The percentage by which $1/2\,mv^2$ **underestimates** the true kinetic energy is negligible at low speed but becomes severe near c: about $19.2\,\%$ at $0.50\,c$ and about $68.7\,\%$ at $0.90\,c$.



The lesson is that **classical mechanics cannot be used to find the kinetic energy of a particle moving at a speed close to** c; the relativistic expression $E_k=(\gamma-1)mc^2$ must be used instead. It also follows that a massive object can never be accelerated to the speed of light: doing so would require $\gamma\to\infty$, and hence an infinite amount of energy.

Energy units: joules, electron-volts and mega-electron-volts. Because the energies of individual particles are tiny in joules, they are usually quoted in **electron-volts**. One electron-volt is the energy gained by a charge q_e moving through a potential difference of 1 V:

$$1\,\text{eV}=q_e\times1\,\text{V}=1.60\times10^{-19}\,\text{J}\,,1\,\text{MeV}=10^6\,\text{eV}=1.60\times10^{-13}\,\text{J}\,.$$

In these units the rest energy of the electron is $E_0 = 8.20 \times 10^{-14}\,\text{J} \div 1.60 \times 10^{-13}\,\text{J MeV}^{-1} = 0.512\,\text{MeV}$. Converting fluently between J, eV and MeV is an essential skill for this subtopic.

Mass conversion in the Sun. The Sun radiates energy into space at a huge rate — its **radiated power** (luminosity) is about $3.85 \times 10^{26}\,\text{W}$. Every joule of energy that leaves the Sun carries away an equivalent mass $m = E/c^2$, so the Sun continually loses mass. If it radiates energy at a power P, the **rate of mass loss** follows directly from $E_0 = mc^2$:

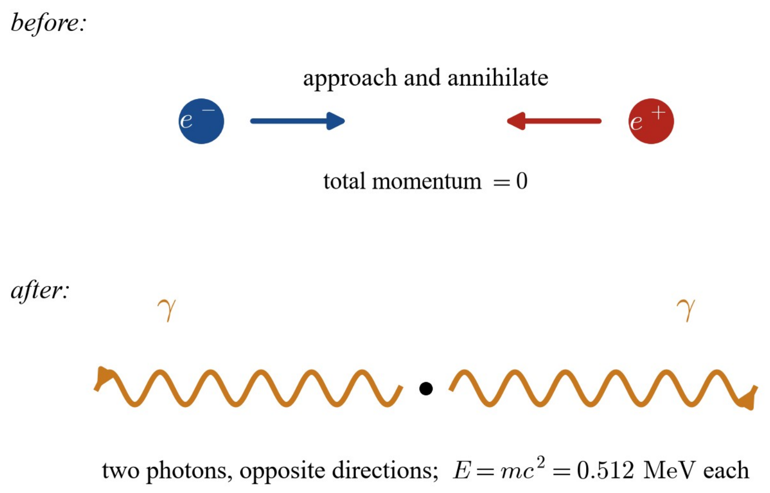
$$P = \frac{\Delta E}{\Delta t} = \frac{\Delta m}{\Delta t}c^2 \Rightarrow \frac{\Delta m}{\Delta t} = \frac{P}{c^2}.$$

This is the **mass-loss-rate** result: divide the radiated power by c^2 to find how many kilograms of mass are converted to radiated energy each second. The same idea applies to the nuclear reactions in the Sun's core, in which the total rest mass of the products is slightly less than that of the reactants; the "missing" mass appears as the energy that is ultimately radiated (the details of the particular nuclear processes are not required).

Positron–electron annihilation. When a particle meets its antiparticle they can **annihilate**, their entire mass being converted into electromagnetic energy. When an electron and a positron (the electron's antiparticle, with the same mass m_e) meet at rest, they are converted into two photons. Each photon carries an energy equal to the rest energy of one particle,

$$E = m_e c^2 = 8.20 \times 10^{-14}\,\text{J} = 0.512\,\text{MeV},$$

so the total energy released is $2m_e c^2 = 1.64 \times 10^{-13}\,\text{J} = 1.02\,\text{MeV}$. **Two** photons are produced, not one, because momentum must be conserved: the electron and positron have zero total momentum before annihilation, and a single photon would carry momentum $p = E/c$ away in one direction. Two photons of equal energy travelling in **opposite directions** have momenta that cancel, giving zero total momentum, in agreement with the initial state.



Nuclear transformations in particle accelerators. The reverse process also occurs. In a particle accelerator, particles are given enormous kinetic energy and made to collide. In the collision that kinetic energy can be converted into the rest mass of **new particles** according to $E_0 = mc^2$: to create a particle of rest mass m, at least an energy mc^2 must be supplied from the kinetic energy of the colliding particles. This is how accelerators produce particles far more massive than the protons or electrons that were fired in (the details of the particular nuclear processes are not required). Mass and energy are freely interconverted, always conserving the total mass–energy.

Throughout this subtopic $m_e = 9.11 \times 10^{-31}\,\text{kg}$, $c = 3.00 \times 10^8\,\text{m s}^{-1}$ and $q_e = 1.60 \times 10^{-19}\,\text{C}$ are used, with $1\,\text{eV} = 1.60 \times 10^{-19}\,\text{J}$ and $1\,\text{MeV} = 1.60 \times 10^{-13}\,\text{J}$. Masses of particles other than the electron are supplied where they are needed.

Worked examples

Example 1 — scaffolded

An electron ($m_e = 9.11 \times 10^{-31}$ kg) travels at $0.80\,c$. Calculate its rest energy, its Lorentz factor, its total energy and its kinetic energy, giving each energy in both joules and MeV.

Working	Comment
$E_0 = m_e c^2 = 9.11 \times 10^{-31} \times \left(3.00 \times 10^8\right)^2 = 8.20 \times 10^{-14}$ J .	Rest energy from $E_0 = mc^2$.
$E_0 = \dfrac{8.20 \times 10^{-14}}{1.60 \times 10^{-13}} = 0.512\,\text{MeV}$.	Convert to MeV: divide joules by 1.60×10^{-13}.
$\gamma = \dfrac{1}{\sqrt{1 - 0.80^2}} = \dfrac{1}{\sqrt{0.36}} = \dfrac{1}{0.60} = 1.67$.	Lorentz factor at $v = 0.80\,c$.
$E_{\text{tot}} = \gamma m c^2 = 1.667 \times 8.20 \times 10^{-14} = 1.37 \times 10^{-13}\,\text{J} = 0.8$.	Total energy $= \gamma \times$ rest energy.
$E_k = (\gamma - 1) m c^2 = 0.667 \times 8.20 \times 10^{-14} = 5.47 \times 10^{-14}\,\text{J} =$.	Kinetic energy $= (\gamma - 1) \times$ rest energy.

Example 2 — scaffolded

The Sun radiates energy into space at a power of $3.85 \times 10^{26}\,\text{W}$. Calculate the rate at which the Sun is losing mass, and the mass converted to radiated energy in one day ($1\,\text{day} = 86\,400\,\text{s}$).

Working	Comment
Energy radiated carries away mass: $P = \dfrac{\Delta m}{\Delta t} c^2$.	Apply $E_0 = mc^2$ to the radiated energy.
$\dfrac{\Delta m}{\Delta t} = \dfrac{P}{c^2} = \dfrac{3.85 \times 10^{26}}{\left(3.00 \times 10^8\right)^2} = \dfrac{3.85 \times 10^{26}}{9.00 \times 10^{16}}$.	Rearrange for the mass-loss rate.
$\dfrac{\Delta m}{\Delta t} = 4.28 \times 10^9\,\text{kg s}^{-1}$.	Mass converted each second.
$\Delta m = \dfrac{\Delta m}{\Delta t} \times \Delta t = 4.28 \times 10^9 \times 86\,400 = 3.70 \times 10^{14}\,\text{kg}$.	Multiply the rate by the number of seconds in a day.

Example 3 — unscaffolded

A positron and an electron, both effectively at rest, meet and annihilate, producing two identical photons. Calculate the energy of each photon (in joules and MeV) and explain why two photons are produced rather than one.

Each particle has rest energy $m_e c^2$, and the entire mass–energy is carried away by the photons. By symmetry the two identical photons share the energy equally, so each photon has energy

$$E = m_e c^2 = 9.11 \times 10^{-31} \times \left(3.00 \times 10^8\right)^2 = 8.20 \times 10^{-14}\,\text{J} = 0.512\,\text{MeV},$$

and the total energy released is $2 m_e c^2 = 1.64 \times 10^{-13}\,\text{J} = 1.02\,\text{MeV}$.

Two photons are required by conservation of momentum. Before annihilation the electron and positron are at rest, so the total momentum of the system is zero. A single photon would have to carry momentum $p = E/c$ in some direction, which could not be zero. Two photons of equal energy emitted in **opposite directions** have equal and opposite momenta that sum to zero, matching the initial momentum.

$\therefore$ each photon has energy $8.20 \times 10^{-14}\,\text{J} = 0.512\,\text{MeV}$, and two are produced (in opposite directions) so that momentum is conserved.

Example 4 — unscaffolded

A proton (mass 1.67×10^{-27} kg) travels at $0.90\,c$. Calculate its kinetic energy using (a) the classical formula $1/2\,m\,v^2$ and (b) the relativistic formula $(\gamma - 1)\,m\,c^2$, and find the percentage by which the classical value underestimates the true kinetic energy.

The speed is $v = 0.90\,c = 0.90 \times 3.00 \times 10^8 = 2.70 \times 10^8$ m s^{-1}.

Classical estimate:

$$1/2\,m\,v^2 = 1/2 \times 1.67 \times 10^{-27} \times \left(2.70 \times 10^8\right)^2 = 6.09 \times 10^{-11}\,\mathrm{J}.$$

Relativistic value, with $\gamma = \dfrac{1}{\sqrt{1 - 0.90^2}} = \dfrac{1}{\sqrt{0.19}} = 2.294$:

$$E_k = (\gamma - 1)\,m\,c^2 = 1.294 \times 1.67 \times 10^{-27} \times \left(3.00 \times 10^8\right)^2 = 1.95 \times 10^{-10}\,\mathrm{J}.$$

The classical estimate is smaller than the true value by

$$\left(1 - \frac{6.087 \times 10^{-11}}{1.945 \times 10^{-10}}\right) \times 100 = 68.7\,\%.$$

∴ the classical formula gives 6.09×10^{-11} J, whereas the true kinetic energy is 1.95×10^{-10} J — the classical value underestimates it by $68.7\,\%$, showing that $1/2\,m\,v^2$ cannot be used at this speed.

Practice questions

Question 1 An electron has mass $9.11 \times 10^{-31}\,\mathrm{kg}$. (a) Calculate the rest energy of the electron in joules. (b) Express this rest energy in MeV. (c) A proton has a rest energy of $938\,\mathrm{MeV}$. Calculate the proton's rest energy in joules, and hence its mass.

Question 2 An electron travels at $0.90\,c$. (a) Calculate the Lorentz factor γ for the electron. (b) Calculate the kinetic energy of the electron, in joules and in MeV. (c) Calculate the total energy of the electron, in joules and in MeV.

Question 3 An electron travels at $0.50\,c$. (a) Calculate the electron's kinetic energy using the classical formula $1/2\,m\,v^2$. (b) Calculate the electron's kinetic energy using the relativistic formula $(\gamma - 1)\,m\,c^2$. (c) State which value is correct, and determine the percentage by which the classical value underestimates the true kinetic energy.

Question 4 The Sun radiates energy at a power of $3.85 \times 10^{26}\,\mathrm{W}$. (a) Calculate the rate at which the Sun loses mass. (b) Calculate the mass converted to radiated energy in one year. Take $1\,\mathrm{year} = 3.15 \times 10^7\,\mathrm{s}$. (c) Explain, in terms of mass–energy equivalence, why the Sun's mass steadily decreases.

Question 5 An electron and a positron, both at rest, meet and annihilate, producing two identical photons. (a) Calculate the total energy released, in MeV and in joules. (b) Calculate the energy of each photon. (c) Explain what would happen to the energy of the two photons if, instead of being at rest, the electron and positron had significant kinetic energy before they annihilated.

Question 6 In a particle accelerator the kinetic energy of colliding particles is converted into the rest mass of new particles (the details of the nuclear process are not required). In one event a new particle is created with a rest mass of $2.4 \times 10^{-28}\,\mathrm{kg}$. (a) Calculate the rest energy of the new particle, in joules and in MeV. (b) State the minimum total kinetic energy, in MeV, that the colliding particles must have had. (c) Explain, in terms of mass–energy equivalence, how kinetic energy can be converted into mass.

Question 7 A small object has a mass of $2.0\,\mathrm{g}$. Calculate the total rest energy equivalent to this mass. Give your answer in joules to three significant figures.

Question 8 An electron travels at $0.99\,c$. Calculate its kinetic energy and its total energy, each in joules and in MeV.

Question 9 (a) Show that the rest energy of an electron is 8.20×10^{-14} J. (b) An electron is accelerated until its total energy is three times its rest energy. Calculate its kinetic energy in joules, and determine its speed as a fraction of c.

Question 10 A proton (mass 1.67×10^{-27} kg) is accelerated until its kinetic energy is $500\,\mathrm{MeV}$. Calculate the proton's total energy in joules, its Lorentz factor γ, and its speed as a fraction of c.

Question 11 In the core of the Sun, nuclear reactions convert mass into energy. In one such reaction the total rest mass of the products is 4.8×10^{-29} kg less than that of the reactants. (a) Calculate the energy released in this single reaction. (b) If the Sun completes 9.0×10^{37} such reactions per second, calculate the total power radiated by the Sun.

Question 12 An electron and a positron annihilate at rest, producing two identical photons. (a) Calculate the energy of each photon in joules. (b) Calculate the frequency of each photon. (Use $E=hf$ with $h=6.63\times10^{-34}$ J s.)

Question 13 Using the relationship $E_{\mathrm{tot}}=\gamma m c^2$, explain why an object with mass can never be accelerated to the speed of light. No calculations are required.

Question 14 A student calculates the kinetic energy of an electron moving at $0.95\,c$ using $1/2\,m v^2$ and obtains a value far smaller than the accepted value. Explain why the classical formula fails at this speed, and describe how the correct relativistic expression behaves as v approaches c.

Question 15 Explain why the minimum total energy of the two photons produced when an electron and a positron annihilate from rest is $1.02\,\mathrm{MeV}$, and state what determines any energy above this minimum.

Question 16 A muon has a mass of 1.88×10^{-28} kg. (a) Show that the rest energy of a muon is $106\,\mathrm{MeV}$. (b) A muon is produced in an accelerator with a total energy of $530\,\mathrm{MeV}$. Calculate its kinetic energy and its Lorentz factor γ.

Question 17 In a particle accelerator, two particles collide and their kinetic energy creates a single new particle of rest mass 5.4×10^{-27} kg. Calculate the minimum total kinetic energy needed to create this particle, in joules and in MeV.

Question 18 An electron and a proton (mass 1.67×10^{-27} kg) each have a kinetic energy of $2.00\,\mathrm{MeV}$. For each particle, calculate its Lorentz factor γ and its speed as a fraction of c. Hence explain why, for the same kinetic energy, the electron moves much closer to the speed of light than the proton.

Question 19 The Sun has a mass of 1.99×10^{30} kg and radiates at a power of $3.85\times10^{26}\,\mathrm{W}$. Assuming this power stays constant, estimate the time required for the Sun to lose $0.10\,\%$ of its mass. Give your answer in years ($1\,\mathrm{year}=3.15\times10^{7}$ s).

Question 20 Compare the total energy, rest energy and kinetic energy of a particle when it is (i) at rest and (ii) moving at a speed close to c. In your answer, refer to how each quantity depends on γ, and explain the meaning of the equation $E_{\mathrm{tot}}=E_k+E_0$.

Answers

Question 1 (a) 8.20×10^{-14} J; (b) 0.512 MeV; (c) 1.50×10^{-10} J, 1.67×10^{-27} kg. **Question 2** (a) 2.29; (b) 1.06×10^{-13} J $=0.663$ MeV; (c) 1.88×10^{-13} J $=1.18$ MeV. **Question 3** (a) 1.02×10^{-14} J; (b) 1.27×10^{-14} J; (c) the relativistic value is correct; classical underestimates by 19.2%. **Question 4** (a) 4.28×10^{9} kg s^{-1}; (b) 1.35×10^{17} kg; (c) radiated energy carries away an equivalent mass $\Delta m=\Delta E/c^2$. **Question 5** (a) 1.02 MeV $=1.64\times10^{-13}$ J; (b) 0.512 MeV $=8.20\times10^{-14}$ J; (c) the photons would be more energetic. **Question 6** (a) 2.16×10^{-11} J $=135$ MeV; (b) 135 MeV; (c) mass is a form of energy, so kinetic energy can appear as rest mass $m=E/c^2$. **Question 7** 1.80×10^{14} J. **Question 8** $E_k=4.99\times10^{-13}$ J $=3.12$ MeV; $E_{\text{tot}}=5.81\times10^{-13}$ J $=3.63$ MeV. **Question 9** (a) 8.20×10^{-14} J; (b) $E_k=1.64\times10^{-13}$ J, $v=0.943c$. **Question 10** $E_{\text{tot}}=2.30\times10^{-10}$ J (1.44×10^{3} MeV); $\gamma=1.53$; $v=0.758c$. **Question 11** (a) 4.32×10^{-12} J; (b) 3.89×10^{26} W. **Question 12** (a) 8.20×10^{-14} J; (b) 1.24×10^{20} Hz. **Question 13** As $v\to c$, $\gamma\to\infty$, so $E_{\text{tot}}=\gamma mc^2\to\infty$; infinite energy would be needed to reach c. **Question 14** $1/2\,mv^2$ is only the low-speed approximation; the true value $(\gamma-1)mc^2$ diverges as $v\to c$ (underestimate $\approx 80\%$ at $0.95c$). **Question 15** The combined rest energy is $2m_ec^2=1.02$ MeV; any excess comes from the particles' kinetic energy. **Question 16** (a) 106 MeV; (b) $E_k=424$ MeV, $\gamma=5.01$. **Question 17** 4.86×10^{-10} J $=3.04\times10^{3}$ MeV. **Question 18** electron $\gamma=4.90$, $v=0.979c$; proton $\gamma=1.00$, $v=0.0652c$. **Question 19** 1.48×10^{10} years. **Question 20** At rest $\gamma=1$, $E_k=0$, $E_{\text{tot}}=E_0$; near c, $\gamma\gg1$, E_k dominates, $E_{\text{tot}}\gg E_0$.

Fully-worked solutions

Question 1 $m_e=9.11\times10^{-31}$ kg. (a) $E_0=m_ec^2=9.11\times10^{-31}\times(3.00\times10^{8})^2=8.20\times10^{-14}$ J. (b) $E_0=\dfrac{8.20\times10^{-14}}{1.60\times10^{-13}}=0.512$ MeV. (c) In joules, $E_0=938\times1.60\times10^{-13}=1.50\times10^{-10}$ J. The mass follows from $E_0=mc^2$: $m=\dfrac{E_0}{c^2}=\dfrac{1.50\times10^{-10}}{9.00\times10^{16}}=1.67\times10^{-27}$ kg.

Question 2 $v=0.90c$, so $\gamma=\dfrac{1}{\sqrt{1-0.90^2}}=\dfrac{1}{\sqrt{0.19}}=2.294$. (a) $\gamma=2.29$. (b) $E_k=(\gamma-1)m_ec^2=1.294\times8.20\times10^{-14}=1.06\times10^{-13}$ J. In MeV, $\dfrac{1.06\times10^{-13}}{1.60\times10^{-13}}=0.663$ MeV. (c) $E_{\text{tot}}=\gamma m_ec^2=2.294\times8.20\times10^{-14}=1.88\times10^{-13}$ J $=1.18$ MeV.

Question 3 $v=0.50c=1.50\times10^{8}$ m s^{-1}; $\gamma=\dfrac{1}{\sqrt{1-0.50^2}}=1.155$. (a) $1/2\,m_ev^2=1/2\times9.11\times10^{-31}\times(1.50\times10^{8})^2=1.02\times10^{-14}$ J. (b) $E_k=(\gamma-1)m_ec^2=0.155\times8.20\times10^{-14}=1.27\times10^{-14}$ J. (c) The relativistic value 1.27×10^{-14} J is correct. The classical value is smaller by $\left(1-\dfrac{1.025\times10^{-14}}{1.268\times10^{-14}}\right)\times100=19.2\%$.

Question 4 $P=3.85\times10^{26}$ W. (a) $\dfrac{\Delta m}{\Delta t}=\dfrac{P}{c^2}=\dfrac{3.85\times10^{26}}{9.00\times10^{16}}=4.28\times10^{9}$ kg s^{-1}. (b) In one year, $\Delta m=4.28\times10^{9}\times3.15\times10^{7}=1.35\times10^{17}$ kg. (c) The Sun radiates energy (as light) into space. By $E_0=mc^2$, energy and mass are equivalent, so every amount of energy ΔE that leaves the Sun carries with it an equivalent mass $\Delta m=\Delta E/c^2$. Because this radiated energy is continuously lost to space, the corresponding mass is continuously removed from the Sun, and so the Sun's mass steadily decreases.

Question 5 (a) Both particles are converted entirely into photon energy, so the total energy released is $2m_ec^2=2\times8.20\times10^{-14}=1.64\times10^{-13}$ J $=1.02$ MeV. (b) By symmetry the two identical photons share this energy equally, so each has energy $m_e c^2 = 8.20 \times 10^{-14}\,\text{J} = 0.512\,\text{MeV}$. (c) Total energy is conserved. If the electron and positron had kinetic energy before annihilating, the total energy available would be their combined rest energy ($1.02\,\text{MeV}$) **plus** their kinetic energy. All of this is shared between the two photons, so each photon would be more energetic than $0.512\,\text{MeV}$; the extra energy above the $1.02\,\text{MeV}$ rest-energy minimum comes from the particles' kinetic energy.

Question 6 $m = 2.4 \times 10^{-28}\,\text{kg}$. (a) $E_0 = mc^2 = 2.4 \times 10^{-28} \times 9.00 \times 10^{16} = 2.16 \times 10^{-11}\,\text{J}$; in MeV, $\dfrac{2.16 \times 10^{-11}}{1.60 \times 10^{-13}} = 135\,\text{MeV}$. (b) The minimum kinetic energy is the amount that just supplies the new particle's rest energy, so the colliding particles must have had at least $135\,\text{MeV}$ of kinetic energy. (c) Mass and energy are equivalent ($E_0 = mc^2$), so mass is simply one form that energy can take. In the collision total energy is conserved, and the kinetic energy of the incoming particles can be transformed into the rest energy of a new particle — that is, it appears as new mass $m = E / c^2$. Provided enough kinetic energy is available to supply the rest energy mc^2, the particle can be created.

Question 7 $E = mc^2 = 2.0 \times 10^{-3} \times \left(3.00 \times 10^8\right)^2 = 2.0 \times 10^{-3} \times 9.00 \times 10^{16} = 1.80 \times 10^{14}\,\text{J}$.

Question 8 $v = 0.99c$, so $\gamma = \dfrac{1}{\sqrt{1 - 0.99^2}} = \dfrac{1}{\sqrt{0.0199}} = 7.089$.
$E_k = (\gamma - 1) m_e c^2 = 6.089 \times 8.20 \times 10^{-14} = 4.99 \times 10^{-13}\,\text{J} = 3.12\,\text{MeV}$.
$E_{\text{tot}} = \gamma m_e c^2 = 7.089 \times 8.20 \times 10^{-14} = 5.81 \times 10^{-13}\,\text{J} = 3.63\,\text{MeV}$.

Question 9 (a) $E_0 = m_e c^2 = 9.11 \times 10^{-31} \times \left(3.00 \times 10^8\right)^2 = 9.11 \times 10^{-31} \times 9.00 \times 10^{16} = 8.20 \times 10^{-14}\,\text{J}$, as required. (b) If $E_{\text{tot}} = 3E_0$ then $\gamma = 3$, and $E_k = E_{\text{tot}} - E_0 = 2E_0 = 2 \times 8.20 \times 10^{-14} = 1.64 \times 10^{-13}\,\text{J}$. From $\gamma = \dfrac{1}{\sqrt{1 - v^2 / c^2}} = 3$, $1 - \dfrac{v^2}{c^2} = \dfrac{1}{9}$, so $\dfrac{v^2}{c^2} = \dfrac{8}{9}$ and $v = \sqrt{8/9}\,c = 0.943c$.

Question 10 $m = 1.67 \times 10^{-27}\,\text{kg}$, $E_k = 500\,\text{MeV}$. Rest energy:
$E_0 = mc^2 = 1.67 \times 10^{-27} \times 9.00 \times 10^{16} = 1.503 \times 10^{-10}\,\text{J} = 939\,\text{MeV}$. Kinetic energy in joules:
$E_k = 500 \times 1.60 \times 10^{-13} = 8.00 \times 10^{-11}\,\text{J}$. Total energy: $E_{\text{tot}} = E_0 + E_k = 1.503 \times 10^{-10} + 8.00 \times 10^{-11} = 2.30 \times 10^{-10}\,\text{J}$ ($= 1.44 \times 10^3\,\text{MeV}$). Lorentz factor: $\gamma = \dfrac{E_{\text{tot}}}{E_0} = \dfrac{2.303 \times 10^{-10}}{1.503 \times 10^{-10}} = 1.532$. Speed: from $\gamma = \dfrac{1}{\sqrt{1 - v^2 / c^2}}$,
$\dfrac{v^2}{c^2} = 1 - \dfrac{1}{\gamma^2} = 1 - \dfrac{1}{1.532^2} = 0.574$, so $v = 0.758c$.

Question 11 (a) $E = \Delta m c^2 = 4.8 \times 10^{-29} \times 9.00 \times 10^{16} = 4.32 \times 10^{-12}\,\text{J}$. (b)
$P = (\text{reactions per second}) \times (\text{energy per reaction}) = 9.0 \times 10^{37} \times 4.32 \times 10^{-12} = 3.89 \times 10^{26}\,\text{W}$.

Question 12 (a) Each photon carries the rest energy of one particle: $E = m_e c^2 = 8.20 \times 10^{-14}\,\text{J}$. (b)
$f = \dfrac{E}{h} = \dfrac{8.20 \times 10^{-14}}{6.63 \times 10^{-34}} = 1.24 \times 10^{20}\,\text{Hz}$.

Question 13 The Lorentz factor is $\gamma = \dfrac{1}{\sqrt{1 - v^2 / c^2}}$. As $v \to c$, the ratio $v^2 / c^2 \to 1$, so the term under the square root $\to 0$ and $\gamma \to \infty$. The total energy $E_{\text{tot}} = \gamma mc^2$ and the kinetic energy $E_k = (\gamma - 1) mc^2$ therefore both grow without limit as the speed approaches c. To actually reach c would require an infinite amount of energy to be supplied to the object. Since only a finite amount of energy can ever be supplied, an object with mass can be accelerated arbitrarily close to c but can never reach it.

Question 14 The formula $1/2\,m\,v^2$ is only the low-speed approximation of the true relativistic kinetic energy $E_k=(\gamma-1)\,m\,c^2$; the two agree only when $v \ll c$. At $0.95\,c$ the Lorentz factor is large, $\gamma=3.20$, so the true kinetic energy $(\gamma-1)\,m\,c^2=2.20\,m\,c^2$ is more than four times the classical value $1/2\,m\,v^2=0.451\,m\,c^2$ — the classical formula underestimates the true value by about $80\,\%$. As v approaches c, $\gamma \to \infty$, so the relativistic kinetic energy increases without limit, whereas $1/2\,m\,v^2$ rises only towards $1/2\,m\,c^2$. The student's value is far too small because $1/2\,m\,v^2$ is simply the wrong expression to use at this speed.

Question 15 Each particle has a rest energy $m_e\,c^2=0.512\,\text{MeV}$, so the electron and positron together have a combined rest energy $2\,m_e\,c^2=1.02\,\text{MeV}$. By conservation of energy, the photons must carry away at least this rest energy, even when the particles start from rest; hence the minimum total photon energy is $1.02\,\text{MeV}$. Any energy above $1.02\,\text{MeV}$ comes from the kinetic energy the electron and positron possessed before they annihilated, since that energy is also conserved and is shared between the photons.

Question 16 $m=1.88\times10^{-28}\,\text{kg}$. (a) $E_0=m\,c^2=1.88\times10^{-28}\times\left(3.00\times10^8\right)^2=1.692\times10^{-11}$ J; in MeV, $\dfrac{1.692\times10^{-11}}{1.60\times10^{-13}}=106\,\text{MeV}$, as required. (b) $E_k=E_{\text{tot}}-E_0=530-106=424\,\text{MeV}$. In joules, $E_{\text{tot}}=530\times1.60\times10^{-13}=8.48\times10^{-11}$ J, so $\gamma=\dfrac{E_{\text{tot}}}{E_0}=\dfrac{8.48\times10^{-11}}{1.692\times10^{-11}}=5.01$.

Question 17 The minimum kinetic energy just supplies the rest energy of the new particle: $E=m\,c^2=5.4\times10^{-27}\times9.00\times10^{16}=4.86\times10^{-10}$ J. In MeV, $\dfrac{4.86\times10^{-10}}{1.60\times10^{-13}}=3.04\times10^3\,\text{MeV}$.

Question 18 Each particle has $E_k=2.00\,\text{MeV}=2.00\times1.60\times10^{-13}=3.20\times10^{-13}$ J. Electron ($E_0=8.20\times10^{-14}$ J): $\gamma=1+\dfrac{E_k}{E_0}=1+\dfrac{3.20\times10^{-13}}{8.20\times10^{-14}}=4.90$, so $\dfrac{v^2}{c^2}=1-\dfrac{1}{4.90^2}=0.958$ and $v=0.979\,c$. Proton ($E_0=m\,c^2=1.503\times10^{-10}$ J): $\gamma=1+\dfrac{3.20\times10^{-13}}{1.503\times10^{-10}}=1.00$, so $\dfrac{v^2}{c^2}=1-\dfrac{1}{1.00213^2}=0.00425$ and $v=0.0652\,c$. For the same kinetic energy the electron reaches a far higher speed because its rest energy is much smaller than the proton's ($0.512\,\text{MeV}$ against $939\,\text{MeV}$): the $2.00\,\text{MeV}$ of kinetic energy is several times the electron's rest energy, making γ large and pushing v very close to c, whereas for the proton the same $2.00\,\text{MeV}$ is only a tiny fraction of its rest energy, so γ is barely above 1 and the proton remains far below c.

Question 19 The mass to be lost is $\Delta m=0.10\,\%\times1.99\times10^{30}=1.0\times10^{-3}\times1.99\times10^{30}=1.99\times10^{27}\,\text{kg}$. The mass-loss rate is $\dfrac{\Delta m}{\Delta t}=\dfrac{P}{c^2}=\dfrac{3.85\times10^{26}}{9.00\times10^{16}}=4.28\times10^9\,\text{kg s}^{-1}$. The time required is $\Delta t=\dfrac{\Delta m}{P/c^2}=\dfrac{1.99\times10^{27}}{4.28\times10^9}=4.65\times10^{17}\,\text{s}$. In years, $\Delta t=\dfrac{4.65\times10^{17}}{3.15\times10^7}=1.48\times10^{10}\,\text{years}$.

Question 20 When the particle is at rest, $v=0$ and $\gamma=1$: its kinetic energy is $E_k=(\gamma-1)\,m\,c^2=0$, and its total energy is $E_{\text{tot}}=\gamma\,m\,c^2=m\,c^2=E_0$. The particle still possesses energy $E_0=m\,c^2$, its rest energy, purely because it has mass. When the particle moves at a speed close to c, $\gamma \gg 1$: the kinetic energy $E_k=(\gamma-1)\,m\,c^2$ becomes large and the total energy $E_{\text{tot}}=\gamma\,m\,c^2$ is many times the rest energy, so the kinetic energy dominates. The equation $E_{\text{tot}}=E_k+E_0$ states that a particle's total energy is the sum of its kinetic energy (its energy of motion) and its rest energy (the energy equivalent of its mass, $m\,c^2$). The essential new idea of mass–energy equivalence is that the rest-energy term E_0 is present even when the particle is completely at rest.

Topic 5 Test A — Special relativity (multiple choice)

Time allowed: 25 minutes · 20 marks · One scientific calculator permitted · The formula sheet (back of the Practice Examinations booklet) may be used.

Answer **all** questions. Choose the response that is **correct** or that **best answers** the question. A correct answer scores 1; an incorrect answer scores 0. Marks will **not** be deducted for incorrect answers. Where a numerical answer is required, take $c = 3.00 \times 10^8\ \mathrm{m\ s^{-1}}$. Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Which one of the following is one of Einstein's two postulates of special relativity?

A. The speed of light measured by an observer depends on the motion of the light source.

B. Light travels through an absolute medium, the aether, which defines a frame of rest.

C. The time interval between two events is the same for all observers.

D. The laws of physics are the same in all inertial (non-accelerated) frames of reference.

Question 2

Which one of the following is an inertial (non-accelerated) frame of reference?

A. A car rounding a bend at a constant speed of $15\ \mathrm{m\ s^{-1}}$.

B. A lift accelerating upwards from rest.

C. A spacecraft coasting through deep space at constant velocity with its engines switched off.

D. A satellite moving in a circular orbit at constant speed.

Question 3

Which one of the following best explains how the Michelson–Morley experiment supports Einstein's second postulate?

A. It measured the speed of the aether wind produced by the Earth's motion through space.

B. Rotating its two perpendicular light paths produced no shift in the interference fringes, showing that the speed of light is the same in all directions.

C. It showed that light travels slightly faster in the direction of the Earth's motion than across it.

D. It demonstrated that light needs a medium in order to travel from the source to the detector.

Question 4

A distant star moves directly towards the Earth at $0.30\,c$ and emits light. According to special relativity, the speed at which this light arrives at the Earth is closest to:

A. $3.00 \times 10^8\ \mathrm{m\ s^{-1}}$

B. $3.90 \times 10^8\ \mathrm{m\ s^{-1}}$

C. $2.10 \times 10^8\ \mathrm{m\ s^{-1}}$

D. $0.90 \times 10^8\ \mathrm{m\ s^{-1}}$

Question 5

A proton in an accelerator travels at $0.70\,c$. The Lorentz factor γ for the proton is closest to:

A. 1.40

B. 1.83

C. 1.96

D. 0.714

Question 6

A spacecraft travels past the Earth at a constant $0.80\,c$. The crew measure the time between two beats of an astronaut's heart to be $0.90\,\mathrm{s}$. The time between these two beats as measured by an observer on the Earth is closest to:

A. $0.54\,\mathrm{s}$

B. $0.90\,\mathrm{s}$

C. $1.13\,\mathrm{s}$

D. $1.50\,\mathrm{s}$

Question 7

A spacecraft has a proper length of $200\,\mathrm{m}$. It passes a space station at a constant $0.60\,c$. The length of the spacecraft measured by an observer on the station is closest to:

A. $250\,\mathrm{m}$

B. $200\,\mathrm{m}$

C. $160\,\mathrm{m}$

D. $120\,\mathrm{m}$

Question 8

A spacecraft flies past the Earth at high speed. A lamp fixed on the spacecraft is switched on and then off. Which one of the following measures the **proper time** for which the lamp is on?

A. An observer on the Earth, because the Earth is at rest.

B. The crew on the spacecraft, because the two events occur at the same point in space in their frame.

C. Whichever observer measures the longer time interval.

D. Both observers measure the same proper time, because time is absolute.

Question 9

Two spacecraft, P and Q, pass each other in deep space at a constant relative speed of $0.60\,c$. Each crew observes the other craft's identical clock. Which one of the following is correct?

A. Each crew measures the other craft's clock to run slow.

B. Each crew measures the other craft's clock to run fast.

C. Only the crew of the faster craft measure the other clock to run slow.

D. Both crews agree that P's clock is the one that really runs slow.

Question 10

A cube has sides of length $1.0\,\mathrm{m}$ when at rest. It then moves at $0.90\,c$ in a direction parallel to one set of its edges. As measured by a stationary observer, the cube is:

A. contracted equally in all three dimensions, so that it remains a cube.

B. contracted only in the two directions perpendicular to its motion.

C. unchanged in shape, because length contraction affects only time intervals.

D. contracted only along its direction of motion, so it is measured as a rectangular box.

Question 11

Cosmic-ray muons created high in the atmosphere are detected in large numbers at ground level, even though their proper half-life is only about $2.2\,\mu\mathrm{s}$. A student writes: *"In the Earth's frame the muon's half-life is dilated, and at the same time the atmosphere is length-contracted, so the muon easily survives."* Which one of the following best identifies the error in this explanation?

A. Length contraction and time dilation do not occur at speeds below c.

B. In the Earth's frame the muon's half-life should be contracted, not dilated.

C. It combines a result from the Earth's frame (time dilation) with a result from the muon's frame (length contraction), which double-counts the effect.

D. The atmosphere is at rest, so it can never be length-contracted in any frame.

Question 12

A muon travelling towards the ground at $0.80\,c$ has a proper half-life of $2.2\,\mu\mathrm{s}$. As measured by an observer in the Earth (ground) frame, the muon's half-life is closest to:

A. $1.32\,\mu\mathrm{s}$

B. $3.67\,\mu\mathrm{s}$

C. $2.2\,\mu\mathrm{s}$

D. $4.40\,\mu\mathrm{s}$

Question 13

A linear accelerator is $1.8\,\mathrm{km}$ long in the laboratory. Electrons in it reach a speed for which $\gamma = 50.0$. In the electrons' rest frame, the length of the accelerator is closest to:

A. $180\,\mathrm{m}$

B. $1.8\,\mathrm{km}$

C. $90\,\mathrm{km}$

D. $36\,\mathrm{m}$

Question 14

The atomic clock on a GPS satellite would, if left uncorrected, lose about $8.0\,\mu\mathrm{s}$ over one day. Because a receiver multiplies each signal's travel time by c to obtain a distance, the position error that this timing offset would produce after one day is closest to:

A. $240\,\mathrm{m}$

B. $2.4\times10^{6}\,\mathrm{km}$

C. $2.4\,\mathrm{km}$

D. $24\,\mathrm{km}$

Question 15

A particle has a rest mass of $3.0\times10^{-28}\,\mathrm{kg}$. Its rest energy is closest to:

A. $2.7\times10^{-11}\,\mathrm{J}$

B. $9.0\times10^{-20}\,\mathrm{J}$

C. $1.8\times10^{-19}\,\mathrm{J}$

D. $3.3\times10^{-45}\,\mathrm{J}$

Question 16

As the speed of a particle approaches c, the classical formula $1/2\,m\,v^2$ for its kinetic energy:

A. gives a value that also approaches infinity, like the relativistic formula.

B. increasingly underestimates the true kinetic energy, which is $(\gamma-1)\,m\,c^2$ and grows without bound.

C. increasingly overestimates the true kinetic energy.

D. stays exactly equal to the relativistic value at every speed.

Question 17

A proton has a rest energy of $938\,\mathrm{MeV}$. It is accelerated until its Lorentz factor is $\gamma=1.50$. Its kinetic energy is closest to:

A. $313\,\mathrm{MeV}$

B. $938\,\mathrm{MeV}$

C. $1410\,\mathrm{MeV}$

D. $469\,\mathrm{MeV}$

Question 18

When an electron and a positron annihilate from rest, two photons are produced rather than one. The main reason two photons are produced is that:

A. the electron and positron carry equal and opposite charges.

B. a single photon could not carry away all of the energy released.

C. the total momentum before annihilation is zero, so two photons moving in opposite directions are required to conserve momentum.

D. energy must be shared equally between matter and antimatter.

Question 19

In an experiment measuring the flight times of particles along a beam line, a detector's internal clock is found to run consistently fast, so that every recorded time is too large by the same fraction. This is an example of:

A. a random error, whose effect is reduced by averaging repeated readings.

B. a systematic error, whose effect is not reduced by averaging repeated readings.

C. a mistake, which should simply be discarded as an outlier.

D. an uncertainty, which sets the resolution of the detector.

Question 20

In an experiment, the total energy E_{tot} of electrons is measured for several values of the Lorentz factor γ. To obtain a straight-line graph whose gradient is the electron's rest energy E_0, a student should plot:

A. E_{tot} against γ, giving a straight line through the origin with gradient E_0.

B. E_{tot} against γ^2, giving a straight line with gradient E_0.

C. E_{tot} against $1/\gamma$, giving a straight line with gradient E_0.

D. γ against E_{tot}, giving a straight line with gradient E_0.

Topic 5 Test A — solutions

Question	Answer	Question	Answer
1	D	11	C
2	C	12	B
3	B	13	D
4	A	14	C
5	A	15	A
6	D	16	B
7	C	17	D
8	B	18	C
9	A	19	B
10	D	20	A

Question 1 — D. Einstein's postulates are the equivalence of all inertial frames (D) and the constancy of c for all observers. A is classical (Galilean) velocity addition; B is the discarded aether model; C is the classical assumption of absolute time — all rejected by special relativity.

Question 2 — C. An inertial frame moves at constant velocity (zero acceleration). A coasting spacecraft qualifies. A (changing direction), B (speeding up) and D (centripetal acceleration in a circular orbit) are all accelerating, hence non-inertial.

Question 3 — B. The key evidence is the **null result** from the **two perpendicular paths**: no fringe shift on rotation means c is the same in all directions. A and C assume a detectable aether wind (the opposite of what was found); D revives the aether that the null result removed.

Question 4 — A. By the second postulate the light arrives at exactly $c = 3.00 \times 10^8\ \mathrm{m\ s^{-1}}$, independent of the source's motion. B is the classical $c + v = 1.30c$; C is $c - v = 0.70c$; D is the source speed $0.30c$ alone.

Question 5 — A. $\gamma = 1/\sqrt{1-0.70^2} = 1/\sqrt{0.51} = 1.40$. B omits the square on v/c: $1/\sqrt{1-0.70} = 1/\sqrt{0.30} = 1.83$. C omits the square root entirely: $1/(1-0.49) = 1.96$. D is the reciprocal $\sqrt{1-v^2/c^2} = 0.714$ (forgetting to invert).

Question 6 — D. The $0.90\ \mathrm{s}$ is the proper time (both events at the clock on board), so Earth measures $t = \gamma t_0 = 5/3 \times 0.90 = 1.50\ \mathrm{s}$. A divides by γ ($0.54\ \mathrm{s}$); B assumes no dilation; C uses the wrong $\gamma = 1.25$ (for $0.60c$).

Question 7 — C. The proper length is $200\ \mathrm{m}$, so the station measures $L = L_0/\gamma = 200/1.25 = 160\ \mathrm{m}$. A multiplies by γ ($250\ \mathrm{m}$); B assumes no contraction; D uses $\gamma = 5/3$ for $0.80c$ ($120\ \mathrm{m}$).

Question 8 — B. The proper time is measured in the frame where the two events (lamp on, lamp off) occur at the **same point in space** — the spacecraft. A wrongly privileges the Earth; C confuses proper time (the *shorter* interval) with the longer, dilated one; D asserts absolute time, which relativity rejects.

Question 9 — A. Motion is relative: each crew sees the *other* clock moving and so measures it to run slow — both are correct, using different events. B reverses the effect; C invents an absolute "faster" craft; D asserts an absolute slow clock, which does not exist (no preferred frame).

Question 10 — D. Length contraction acts only along the direction of motion, so the edge along the velocity shortens while the two perpendicular edges are unchanged — a rectangular box. A, B and C misplace or deny the contraction.

Question 11 — C. Time dilation belongs to the ground frame (full-thickness atmosphere) and length contraction to the muon frame (proper half-life); applying both to one muon double-counts the effect and describes no real observer. A and D are false; B inverts the ground-frame effect.

Question 12 — B. In the ground frame the muon's clock is dilated: $t = \gamma t_0 = 5/3 \times 2.2 = 3.67\ \mu\mathrm{s}$. A divides by γ ($1.32\ \mu\mathrm{s}$); C is the proper half-life (no dilation); D uses $\gamma = 2.0$ (for $0.866c$).

Question 13 — D. In the electrons' frame the machine is contracted: $L = L_0 / \gamma = 1800 / 50.0 = 36\,\mathrm{m}$. A uses $\gamma = 10$; B assumes no contraction; C multiplies by γ ($90\,\mathrm{km}$).

Question 14 — C. Light travels about $300\,\mathrm{m}$ per microsecond, so $c\,\Delta t = 3.00 \times 10^{8} \times 8.0 \times 10^{-6} = 2400\,\mathrm{m} = 2.4\,\mathrm{km}$. A uses $30\,\mathrm{m}$ per $\mu\mathrm{s}$; D uses $3000\,\mathrm{m}$ per $\mu\mathrm{s}$; B never converts $\mu\mathrm{s}$ to s.

Question 15 — A. $E_0 = m c^2 = 3.0 \times 10^{-28} \times (3.00 \times 10^{8})^2 = 2.7 \times 10^{-11}\,\mathrm{J}$. B uses $m c$ (one factor of c); C uses $2c$ instead of c^2; D divides by c^2.

Question 16 — B. $1/2\, m v^2$ is only the low-speed approximation; near c it falls further and further below the true $(\gamma - 1) m c^2$, which diverges as $\gamma \to \infty$. A and D wrongly claim agreement; C reverses the direction of the error.

Question 17 — D. $E_k = (\gamma - 1) m c^2 = (1.50 - 1) \times 938 = 469\,\mathrm{MeV}$. A uses $(1 - 1/\gamma) m c^2 = 313\,\mathrm{MeV}$; B is the rest energy alone; C is the total energy $\gamma m c^2 = 1410\,\mathrm{MeV}$ (omitting the -1).

Question 18 — C. The initial momentum is zero, so two photons with equal and opposite momenta are required to conserve momentum; a single photon would carry net momentum. A is irrelevant to the photon count; B is true of energy but is not the reason for *two* photons; D is not a conservation law.

Question 19 — B. A constant fractional offset on every reading is a systematic error, which shifts the mean and is **not** removed by averaging. A describes a random error; C a one-off mistake; D confuses error with the uncertainty/resolution.

Question 20 — A. Since $E_{\text{tot}} = \gamma m c^2 = E_0 \gamma$, a plot of E_{tot} against γ is linear through the origin with gradient E_0. B and C use the wrong horizontal variable; D swaps the axes, giving a gradient of $1 / E_0$.

Topic 5 Test B — Special relativity (written responses)

Time allowed: 50 minutes · 40 marks · One scientific calculator permitted · The formula sheet (back of the Practice Examinations booklet) may be used. In questions where more than one mark is available, appropriate working must be shown.

Answer **all** questions in the spaces provided. Where a numerical answer is required, take $c = 3.00 \times 10^{8}\ \mathrm{m\ s^{-1}}$. Give final answers to an appropriate number of significant figures. Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (5 marks)

Einstein built his special theory of relativity on two postulates, and the Michelson–Morley experiment is often cited as evidence for the second of them.

a. State Einstein's two postulates of special relativity. 2 marks

b. With reference to the design of the Michelson–Morley apparatus and the result it produced, explain how the experiment supports the postulate that the speed of light is constant. 3 marks

Question 2 (6 marks)

A spacecraft travels past the Earth at a constant $0.80\,c$.

a. Show that the Lorentz factor for the spacecraft is $\gamma = 1.67$ (to three significant figures). 1 mark

b. The crew measure a chemical reaction on board to take $9.0\,\mathrm{s}$. Calculate the duration of the reaction as measured by an observer on the Earth, and state which observer records the proper time. Give your answer in s. 3 marks

c. The spacecraft has a proper length of $75\,\mathrm{m}$. Calculate its length as measured by the observer on the Earth. Give your answer in m. 2 marks

Question 3 (4 marks)

Two probes, A and B, pass each other in deep space at a constant relative speed of $0.60\,c$. Each probe carries an identical clock.

a. Define the *proper time* t_0 between two events. 1 mark

b. The crew of A claim that B's clock runs slow, while the crew of B claim that A's clock runs slow. Explain, as a full chain of reasoning, why both claims are correct and there is no contradiction. 3 marks

Question 4 (5 marks)

A muon is created high in the atmosphere and travels straight down towards the ground at $0.995\,c$. Take its proper half-life to be $2.2\,\mu\mathrm{s}$.

a. Show that the Lorentz factor for the muon is $\gamma = 10.0$ (to three significant figures). 1 mark

b. The muon is created at an altitude of $12\,\mathrm{km}$. Working **entirely in the Earth (ground) frame**, calculate the muon's half-life as measured from the ground and the number of half-lives that elapse during the descent. 3 marks

c. A student adds: *"The atmosphere is also length-contracted to* $1.2\,\mathrm{km}$*, so combining this with the dilated half-life the muon survives even more easily."* Explain the error in combining the two effects. 1 mark

Question 5 (4 marks)

The atomic clock carried by a GPS satellite must be corrected for the special-relativistic effect arising from the satellite's orbital velocity.

a. Explain why the satellite's clock runs slow compared with an identical clock on the ground. Refer to the satellite's orbital velocity. 2 marks

b. (Circle your answer.) If the satellite were placed in a lower, faster orbit, the daily velocity-based time correction required would be:

greater / the same / smaller.

Justify your choice. No calculations are required. 2 marks

Question 6 (5 marks)

a. A star radiates energy into space at a power of 5.0×10^{26} W. Calculate the rate at which the star loses mass. Give your answer in kg s^{-1}. 2 marks

b. A proton has a rest energy of 938 MeV. It is accelerated until its total energy is 2.5×10^{3} MeV. Calculate its kinetic energy (in MeV) and its Lorentz factor γ. 3 marks

Question 7 (3 marks)

A research team measures the flight time of particles along a beam line several times under the same conditions. The readings scatter above and below their mean. Separately, the team later discovers that the timing gate adds a constant 3 ns to every recorded time.

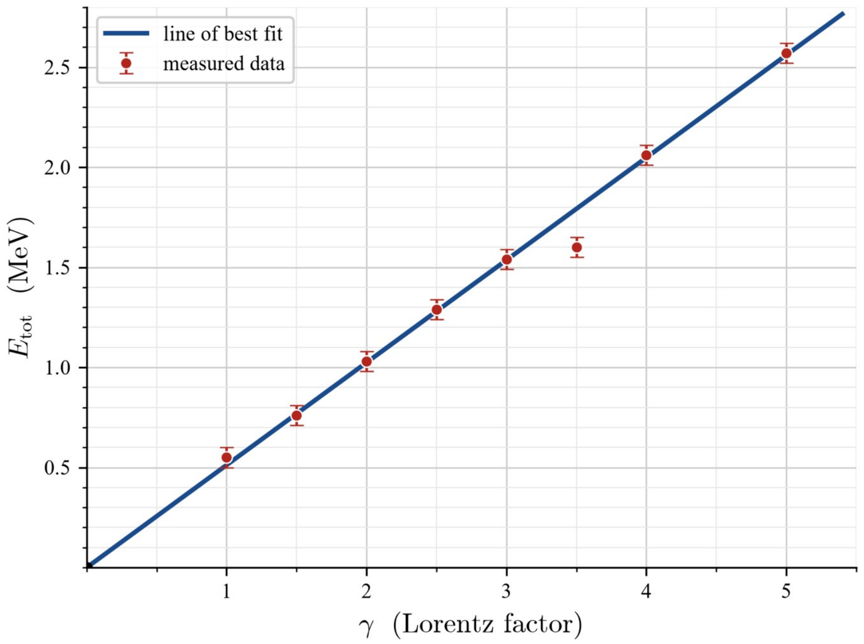
a. Name the type of error responsible for the scatter of the readings about their mean, and state whether averaging repeated readings reduces its effect. 2 marks

b. Name the type of error responsible for the constant 3 ns excess, and state whether averaging repeated readings reduces its effect. 1 mark

Question 8 (8 marks)

Electrons are accelerated to a range of speeds and, for each, the Lorentz factor γ and the total energy E_{tot} are found. The measured total energies are listed below, each with an uncertainty of ±0.05 MeV, and are plotted against γ in the graph that follows.

γ	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0
E_{tot} (MeV)	0.55	0.76	1.03	1.29	1.54	1.60	2.06	2.57



a. State the independent and the dependent variable, and explain why a graph of E_{tot} against γ is expected to be a straight line through the origin. Refer to $E_{tot} = \gamma m c^2$. 2 marks

b. Using two well-separated points that lie **on the line of best fit**, determine the gradient of the line, and hence the rest energy E_0 of the electron in MeV. 3 marks

c. Hence determine the rest mass of the electron, and compare it with the accepted value $m_e = 9.11 \times 10^{-31}$ kg. 2 marks

d. The point at $\gamma = 3.5$ lies noticeably below the line of best fit. State how a correctly drawn line of best fit treats such a point, and state what the uncertainty bar on a plotted point represents. 1 mark

Topic 5 Test B — solutions

Question 1 (5 marks)

a. (2 marks — 1 each)

- Postulate 1: the laws of physics are the same in all inertial (non-accelerated) frames of reference.
- Postulate 2: the speed of light has a constant value for all observers, regardless of their motion or the motion of the source.

b. (3 marks) In the Michelson–Morley interferometer a beam of light is split at a beam splitter into two beams that travel along **two arms set at right angles** — two perpendicular light paths of nearly equal length — each reflecting from a mirror and recombining at a detector to form interference fringes [1]. If light travelled through a stationary aether, the Earth's motion through it would make the light take different times along the two perpendicular paths, so rotating the apparatus should shift the fringes; instead the experiment gave a **null result** — no fringe shift at any orientation [1]. The light-travel time was therefore the same along both perpendicular paths whatever the direction, so the speed of light is the same in all directions regardless of the Earth's motion, which is exactly Einstein's second postulate [1].

∴ the two-perpendicular-path design together with the null result supports a constant speed of light for all observers.

Question 2 (6 marks)

a. (1 mark) $\gamma=\dfrac{1}{\sqrt{1-0.80^2}}=\dfrac{1}{\sqrt{1-0.64}}=\dfrac{1}{\sqrt{0.36}}=\dfrac{1}{0.60}=1.67$ (to three significant figures), as required.

b. (3 marks) The two events (start and end of the reaction) occur at the same point in space in the spacecraft frame, so the crew's $9.0\,\text{s}$ is the proper time t_0 [1]. The Earth observer sees the spacecraft moving and measures the dilated time:

$$t=\gamma\, t_0=1.67\times 9.0=15.0\,\text{s}\,[1].$$

The crew (on board) record the proper time; the Earth observer records the longer, dilated interval [1].

c. (2 marks) The proper length is $L_0=75\,\text{m}$ (measured in the spacecraft's rest frame). Using the unrounded value $\gamma=5/3$ from part **a**, the Earth observer measures the contracted length [1]:

$$L=\frac{L_0}{\gamma}=\frac{75}{5/3}=75\times 0.60=45.0\,\text{m}\,[1].$$

∴ the reaction lasts $15.0\,\text{s}$ from Earth, and the spacecraft is measured to be $45.0\,\text{m}$ long.

Question 3 (4 marks)

a. (1 mark) The proper time t_0 is the time interval between two events measured in the reference frame in which the two events occur at the **same point in space**.

b. (3 marks) Motion is relative, so there is no preferred frame [1]. In A's rest frame it is B (and B's clock) that moves at $0.60\,c$, so A measures B's moving clock to run slow by the factor γ; in B's rest frame it is A (and A's clock) that moves at $0.60\,c$, so B measures A's moving clock to run slow by the same factor [1]. Each crew is measuring the *other's* moving clock from their own rest frame, using a different pair of events at a different pair of places, so the two statements are not measurements of the same thing and cannot conflict [1].

∴ both claims are correct; neither clock is "really" the slow one, because "running slow" only has meaning relative to a chosen frame.

Question 4 (5 marks)

a. (1 mark) $\gamma=\dfrac{1}{\sqrt{1-0.995^2}}=\dfrac{1}{\sqrt{1-0.990025}}=\dfrac{1}{\sqrt{0.009975}}=10.0$ (to three significant figures), as required.

b. (3 marks) In the ground frame the atmosphere is the full $12\,\text{km}$ thick and it is the muon's clock that is dilated:

$$t_{1/2}=\gamma t_0=10.0\times 2.2=22.0\,\mu\text{s}[1].$$

The ground-frame travel time over $H=1.2\times 10^4\,\text{m}$ is

$$t_{\text{travel}}=\frac{H}{v}=\frac{1.2\times 10^4}{0.995\times 3.00\times 10^8}=4.02\times 10^{-5}\,\text{s}=40.2\,\mu\text{s}[1].$$

The number of (dilated) half-lives that elapse is

$$n=\frac{t_{\text{travel}}}{\gamma t_0}=\frac{40.2}{22.0}=1.83\,\text{half-lives}[1].$$

c. (1 mark) The dilated half-life ($22.0\,\mu\text{s}$) is a **ground-frame** result, in which the atmosphere is the full $12\,\text{km}$ thick; the contracted thickness ($1.2\,\text{km}$) is a **muon-frame** result, in which the half-life is the proper $2.2\,\mu\text{s}$. Combining the dilated time with the contracted distance double-counts the relativistic effect and describes no real observer — each frame, used on its own, already gives the same $n=1.83$ half-lives.

Question 5 (4 marks)

a. (2 marks) In the ground frame the satellite moves at its orbital velocity, so its clock is time-dilated: the interval between ticks measured from the ground, $t=\gamma t_0$, is longer than the proper interval the clock itself keeps [1]. A longer interval between ticks means fewer ticks per second as seen from the ground, so the moving clock is observed to run slow [1].

b. (2 marks) **Greater** [1]. The fractional slowing is $\gamma-1\approx 1/2(v/c)^2$, which increases with the orbital speed v; a lower, faster orbit therefore has a larger v, a larger $\gamma-1$, and so requires a larger daily time correction [1].

Question 6 (5 marks)

a. (2 marks) Radiated energy carries away an equivalent mass, $P=\frac{\Delta m}{\Delta t}c^2$, so

$$\frac{\Delta m}{\Delta t}=\frac{P}{c^2}=\frac{5.0\times 10^{26}}{(3.00\times 10^8)^2}=\frac{5.0\times 10^{26}}{9.00\times 10^{16}}=5.56\times 10^9\,\text{kg s}^{-1}.$$

[1 for P/c^2; 1 for the value]

b. (3 marks) The kinetic energy is the total energy minus the rest energy:

$$E_k=E_{\text{tot}}-E_0=2.5\times 10^3-938=1.56\times 10^3\,\text{MeV}[1].$$

The Lorentz factor follows from $E_{\text{tot}}=\gamma mc^2=\gamma E_0$:

$$\gamma=\frac{E_{\text{tot}}}{E_0}=\frac{2.5\times 10^3}{938}=2.67[2].$$

$\therefore$ the proton has $E_k=1.56\times 10^3\,\text{MeV}$ and $\gamma=2.67$.

Question 7 (3 marks)

a. (2 marks) The scatter of readings above and below their mean is caused by **random error** [1]. Averaging repeated readings **does** reduce its effect, because the positive and negative deviations tend to cancel, so the mean lies closer to the true value [1].

b. (1 mark) The constant $3\,\text{ns}$ excess on every reading is a **systematic error**, and averaging does **not** reduce its effect (every reading, and hence the mean, is shifted by the same amount).

Question 8 (8 marks)

a. (2 marks) The independent variable is the Lorentz factor γ (horizontal axis) and the dependent variable is the total energy E_{tot} (vertical axis) [1]. Since $E_{\text{tot}} = \gamma m c^2 = (m c^2)\gamma = E_0 \gamma$, the relationship has the form $y = m x$ with gradient E_0 and no constant term, so a graph of E_{tot} against γ is a straight line through the origin [1].

b. (3 marks) Reading two well-separated points **on the line of best fit**, $(1.0, 0.51)$ and $(5.0, 2.56)$ (with E_{tot} in MeV) [1]:

$$\text{gradient} = \frac{\Delta E_{\text{tot}}}{\Delta \gamma} = \frac{2.56 - 0.51}{5.0 - 1.0} = \frac{2.05}{4.0} = 0.5125\,\text{MeV}\,[1].$$

The gradient is the rest energy, so $E_0 = 0.5125\,\text{MeV}$ [1]. (Points are read off the drawn line, not taken from the scattered data; the gradient is kept unrounded for use in part **c**.)

c. (2 marks) Converting the rest energy to joules and using $E_0 = m_e c^2$:

$$E_0 = 0.5125 \times 1.60 \times 10^{-13} = 8.20 \times 10^{-14}\,\text{J},$$

$$m_e = \frac{E_0}{c^2} = \frac{8.20 \times 10^{-14}}{9.00 \times 10^{16}} = 9.11 \times 10^{-31}\,\text{kg}\,[1].$$

This agrees with the accepted value $m_e = 9.11 \times 10^{-31}\,\text{kg}$, so the experiment is consistent with the known electron mass [1].

d. (1 mark) A correct line of best fit is drawn to follow the trend of **all** the points, with them scattered as evenly as possible above and below it; the low point at $\gamma = 3.5$ is weighed like every other point and is **not** forced onto the line or simply deleted. The uncertainty bar on a point represents the range ($\pm 0.05\,\text{MeV}$) within which the true value of E_{tot} for that point is expected to lie.