Mathematical Methods Units 3 & 4

Practice Examinations

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Contents

Practice Exam 1 — Technology-free

*Reading time: 15 minutes. Writing time: 1 hour. Students are **not** permitted to bring any technology (calculators or software), or notes of any kind, into the examination room. Answer all questions in the spaces provided. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Total 40 marks.*

Question 1 (4 marks)

a. Differentiate $f(x)=x^3-5x^2+2x$ with respect to x. 1 mark

b. Differentiate $y=(3x+1)^4$ with respect to x. 1 mark

c. For $y=x^2e^x$, find $\frac{dy}{dx}$ and hence find the gradient of the graph at $x=1$. 2 marks

Question 2 (4 marks)

a. Find $\int(4x^3-6x+5)\,dx$. 2 marks

b. Given that $\frac{dy}{dx}=2e^x$ and $y=3$ when $x=0$, find y in terms of x. 2 marks

Question 3 (5 marks)

a. Solve $\sqrt{2}\cos(x)=1$ for $x\in[0,2\pi]$. 2 marks

b. State the amplitude and the period of $y=3\sin(2x)$. 2 marks

c. State the range of $y=3\sin(2x)-1$. 1 mark

Question 4 (5 marks)

a. Solve $2e^x=6$ for x, giving your answer as an exact value. 1 mark

b. Solve $\log_e(x-1)=2$ for x. 2 marks

c. Solve $e^{2x}-4e^x+3=0$ for x. 2 marks

Question 5 (4 marks)

The discrete random variable X has the probability distribution shown.

x	0	1	2	3
$Pr(X=x)$	0.2	0.3	k	0.1

a. Find the value of k. 1 mark

b. Find $E(X)$. 2 marks

c. Find $Pr(X\geq 2)$. 1 mark

Question 6 (5 marks)

Consider the function $f(x)=x^3-3x^2$.

a. Find the x-intercepts of the graph of $y=f(x)$. 1 mark

b. Find the coordinates of the stationary points. 2 marks

c. State the nature of each stationary point. 1 mark

d. Sketch the graph of $y = f(x)$, labelling the intercepts and stationary points. 1 mark

Question 7 (4 marks)

The random variable X has a binomial distribution with $n = 4$ and $p = \frac{1}{2}$.

a. Find $Pr(X = 2)$. 2 marks

b. Find $E(X)$. 1 mark

c. Find $Var(X)$. 1 mark

Question 8 (6 marks)

The continuous random variable X has probability density function

$$f(x) = \begin{cases} kx^2, & 0 \le x \le 3 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Show that $k = \frac{1}{9}$. 2 marks

b. Find $Pr(X \le 1)$. 2 marks

c. Find $E(X)$. 2 marks

Question 9 (3 marks)

The graph of $y = \frac{1}{x}$ is transformed to give the graph of $y = \frac{2}{x+1} - 3$.

a. State the equations of the two asymptotes. 2 marks

b. Find the coordinates of the y-intercept. 1 mark

End of examination questions

Practice Exam 1 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a trapezium $\frac{1}{2}(a+b)h$

curved surface area of a cylinder $2\pi r h$

volume of a cylinder $\pi r^2 h$

volume of a cone $\frac{1}{3}\pi r^2 h$

volume of a pyramid $\frac{1}{3}A h$

volume of a sphere $\frac{4}{3}\pi r^3$

area of a triangle $\frac{1}{2}b c \sin(A)$

Calculus

$\frac{d}{dx}(x^n)=n x^{n-1}$

$\frac{d}{dx}((ax+b)^n)=a n(ax+b)^{n-1}$

$\frac{d}{dx}(e^{ax})=a e^{ax}$

$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$

$\frac{d}{dx}(\sin(ax))=a\cos(ax)$

$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$

$\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}$

$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c, n\neq-1$

$\int (ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c, n\neq-1$

$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$

$\int \frac{1}{x}\,dx=\log_e(x)+c, x>0$

$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$

$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Newton's method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

trapezium rule approximation: $\text{Area}\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Probability

complement $Pr(A)=1-Pr(A')$

addition rule $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$

conditional probability	$Pr(A/B)=\dfrac{Pr(A\cap B)}{Pr(B)}$
binomial coefficient	$\dbinom{n}{x}={}^{n}C_{x}=\dfrac{n!}{x!(n-x)!}$
mean	$\mu=E(X)$
variance	$var(X)=\sigma^2=E\left((X-\mu)^2\right)=E\left(X^2\right)-\mu^2$

Probability distribution	Mean and variance
discrete: $Pr(X=x)=p(x)$	$\mu=\sum x\,p(x)$; $\sigma^2=\sum(x-\mu)^2\,p(x)$
binomial: $Pr(X=x)=\dbinom{n}{x}p^x(1-p)^{n-x}$	$\mu=np$; $\sigma^2=np(1-p)$
continuous: $Pr(a<X<b)=\displaystyle\int_a^b f(x)\,dx$	$\mu=\displaystyle\int_{-\infty}^{\infty} x f(x)\,dx$; $\sigma^2=\displaystyle\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Sample proportions

sample proportion	$\hat{P}=\dfrac{X}{n}$
mean	$E(\hat{P})=p$
standard deviation	$sd(\hat{P})=\sqrt{\dfrac{p(1-p)}{n}}$
approximate confidence interval	$\left(\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

End of Formula Sheet

Practice Exam 1 — solutions

Q1. (a) $f'(x)=3x^2-10x+2$. (b) $\frac{dy}{dx}=4(3x+1)^3\times 3=12(3x+1)^3$. (c) By the product rule, $\frac{dy}{dx}=2xe^x+x^2e^x=e^x(x^2+2x)$. At $x=1$: $e^1(1+2)=3e$.

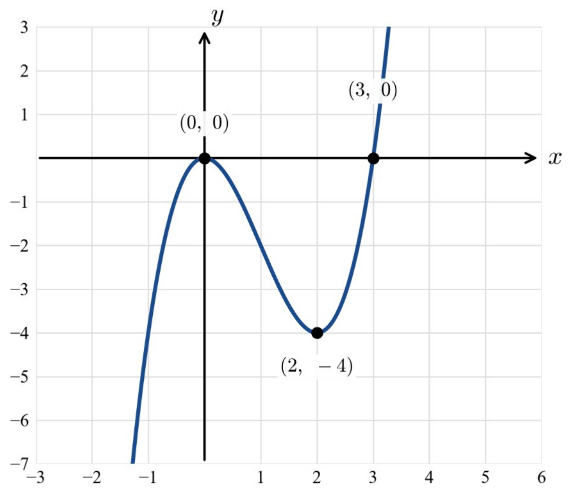
Q2. (a) $\int(4x^3-6x+5)\,dx=x^4-3x^2+5x+c$. (b) $y=\int 2e^x\,dx=2e^x+c$. When $x=0$, $y=3$: $3=2e^0+c=2+c$, so $c=1$. $\therefore y=2e^x+1$.

Q3. (a) $\cos(x)=\frac{1}{\sqrt{2}}$. In $[0,2\pi]$ the solutions are $x=\frac{\pi}{4}$ and $x=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$. (b) Amplitude $=3$; period $=\frac{2\pi}{2}=\pi$. (c) Since $3\sin(2x)\in[-3,3]$, the range of $3\sin(2x)-1$ is $[-4,2]$.

Q4. (a) $e^x=3$, so $x=\log_e 3$. (b) $x-1=e^2$, so $x=e^2+1$. (c) Let $u=e^x$: $u^2-4u+3=(u-1)(u-3)=0$, so $u=1$ or $u=3$. Then $e^x=1\Rightarrow x=0$, or $e^x=3\Rightarrow x=\log_e 3$.

Q5. (a) $k=1-(0.2+0.3+0.1)=0.4$. (b) $E(X)=0(0.2)+1(0.3)+2(0.4)+3(0.1)=0.3+0.8+0.3=1.4$. (c) $Pr(X\geq 2)=Pr(X=2)+Pr(X=3)=0.4+0.1=0.5$.

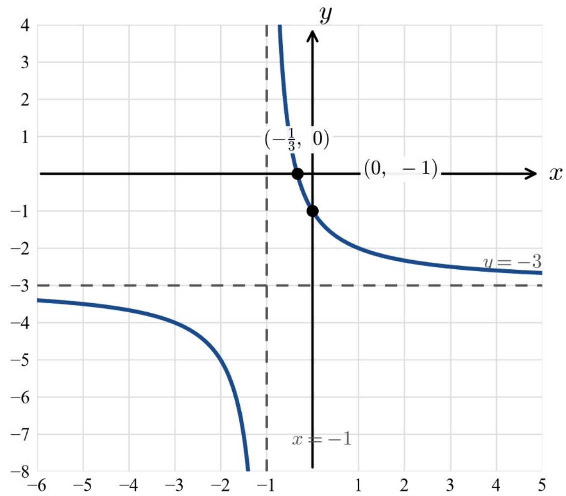
Q6. (a) $f(x)=x^2(x-3)=0$, so the x-intercepts are $x=0$ (repeated) and $x=3$. (b) $f'(x)=3x^2-6x=3x(x-2)=0$, so $x=0$ or $x=2$. $f(0)=0$ and $f(2)=8-12=-4$, giving stationary points $(0,0)$ and $(2,-4)$. (c) $f''(x)=6x-6$: $f''(0)=-6<0$, so $(0,0)$ is a local maximum; $f''(2)=6>0$, so $(2,-4)$ is a local minimum. (d)



Q7. (a) $Pr(X=2)=\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=6\times\frac{1}{16}=\frac{3}{8}$. (b) $E(X)=np=4\times\frac{1}{2}=2$. (c) $Var(X)=np(1-p)=4\times\frac{1}{2}\times\frac{1}{2}=1$.

Q8. (a) $\int_0^3 kx^2\,dx=k\left[\frac{x^3}{3}\right]_0^3=k\times 9=9k$. For a valid pdf $9k=1$, so $k=\frac{1}{9}$. (b) $Pr(X\leq 1)=\int_0^1\frac{1}{9}x^2\,dx=\frac{1}{9}\left[\frac{x^3}{3}\right]_0^1=\frac{1}{9}\times\frac{1}{3}=\frac{1}{27}$. (c) $E(X)=\int_0^3 x\cdot\frac{1}{9}x^2\,dx=\frac{1}{9}\left[\frac{x^4}{4}\right]_0^3=\frac{1}{9}\times\frac{81}{4}=\frac{9}{4}$.

Q9. (a) Comparing with $y=\frac{a}{x-h}+k$ (here $h=-1$, $k=-3$), the asymptotes are $x=-1$ and $y=-3$. (b) $x=0$: $y=\frac{2}{1}-3=-1$, giving $(0,-1)$.



y
x
$(-\frac{1}{3},\ 0)$
$(0,\ -1)$
$y=-3$
$x=-1$

Practice Exam 2 — Technology-free

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Question 1 (3 marks)

a. Differentiate $y=(2x^2+1)^3$ with respect to x. 1 mark

b. Let $f(x)=x\log_e x$. Find $f'(x)$. 2 marks

Question 2 (3 marks)

A curve has gradient $\frac{dy}{dx}=6x^2-4x$ and passes through the point $(1,3)$. Find the equation of the curve.

Question 3 (5 marks)

a. Solve $2\cos x+\sqrt{3}=0$ for $x\in[0,2\pi]$. 3 marks

b. For the function $y=2\cos(3x)+1$, state the amplitude, the period and the range. 2 marks

Question 4 (4 marks)

The discrete random variable X has the probability distribution shown.

x	1	2	3	4
$Pr(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	k	$\frac{1}{4}$

a. Find the value of k. 1 mark

b. Find $\mathrm{E}(X)$. 2 marks

c. Find $Pr(X\geq 3)$. 1 mark

Question 5 (4 marks)

a. Solve $2e^{2x}-5e^x+2=0$ for x. 3 marks

b. State the range of $f(x)=e^x-3$. 1 mark

Question 6 (4 marks)

The random variable X has a binomial distribution with $n=5$ and $p=\frac{1}{3}$.

a. Find $Pr(X=2)$, giving your answer as an exact fraction. 2 marks

b. Find $\mathrm{Var}(X)$. 2 marks

Question 7 (6 marks)

Let $f(x)=x^3-3x^2-9x+27$.

a. Show that $x=3$ is a solution of $f(x)=0$. 1 mark

b. Hence express $f(x)$ as a product of linear factors. 2 marks

c. Find the coordinates of the stationary points of the graph of $y = f(x)$. 2 marks

d. Sketch the graph of $y = f(x)$, labelling the axis intercepts and the stationary points with their coordinates. 1 mark

Question 8 (5 marks)

The continuous random variable X has probability density function

$$f(x) = \begin{cases} kx, & 0 \le x \le 4 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Show that $k = \frac{1}{8}$. 2 marks

b. Find $Pr(X \le 2)$. 2 marks

c. Find $\mathrm{E}(X)$. 1 mark

Question 9 (6 marks)

Consider the parabola $y = x^2 - 4x + 3$.

a. Find the coordinates of the turning point. 2 marks

b. Find the equation of the tangent to the parabola at the point where $x = 3$. 2 marks

c. Find the exact area of the region enclosed between the parabola and the x-axis. 2 marks

End of examination questions

Practice Exam 2 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a trapezium $\frac{1}{2}(a+b)h$

curved surface area of a cylinder $2\pi r h$

volume of a cylinder $\pi r^2 h$

volume of a cone $\frac{1}{3}\pi r^2 h$

volume of a pyramid $\frac{1}{3}A h$

volume of a sphere $\frac{4}{3}\pi r^3$

area of a triangle $\frac{1}{2}b c \sin(A)$

Calculus

$$\frac{d}{dx}(x^n)=n x^{n-1}$$

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$$

$$\frac{d}{dx}((ax+b)^n)=a n(ax+b)^{n-1}$$

$$\int (ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c, n\neq -1$$

$$\frac{d}{dx}(e^{ax})=a e^{ax}$$

$$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$$

$$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$$

$$\int \frac{1}{x}\,dx=\log_e(x)+c, x>0$$

$$\frac{d}{dx}(\sin(ax))=a\cos(ax)$$

$$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$$

$$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$$

$$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$$

$$\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}$$

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Newton's method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

trapezium rule approximation: $\text{Area}\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Probability

complement $Pr(A)=1-Pr(A')$

addition rule $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$

conditional probability $Pr(A/B)=\frac{Pr(A\cap B)}{Pr(B)}$

binomial coefficient $\binom{n}{x} = {}^{n}C_{x} = \frac{n!}{x!(n-x)!}$

mean $\mu=E(X)$

variance $var(X)=\sigma^2=E\left((X-\mu)^2\right)=E\left(X^2\right)-\mu^2$

Probability distribution	Mean and variance
discrete: $Pr(X=x)=p(x)$	$\mu=\sum x\,p(x);\ \sigma^2=\sum (x-\mu)^2\,p(x)$
binomial: $Pr(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$\mu=np;\ \sigma^2=np(1-p)$
continuous: $Pr(a<X<b)=\int_a^b f(x)\,dx$	$\mu=\int_{-\infty}^{\infty} x f(x)\,dx;\ \sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Sample proportions

sample proportion $\hat{P}=\frac{X}{n}$

mean $E(\hat{P})=p$

standard deviation $sd(\hat{P})=\sqrt{\frac{p(1-p)}{n}}$

approximate confidence interval $\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

End of Formula Sheet

Practice Exam 2 — solutions

Question 1. a. Using the chain rule, $\frac{dy}{dx}=3(2x^2+1)^2\times 4x=12x(2x^2+1)^2$. **b.** Using the product rule with $u=x$, $v=\log_e x$: $f'(x)=1\times\log_e x+x\times\frac{1}{x}=\log_e x+1$.

Question 2. Antidifferentiating, $y=\int(6x^2-4x)\,dx=2x^3-2x^2+c$. Substituting $(1,3)$: $3=2(1)^3-2(1)^2+c=2-2+c$, so $c=3$. $\therefore\ y=2x^3-2x^2+3$.

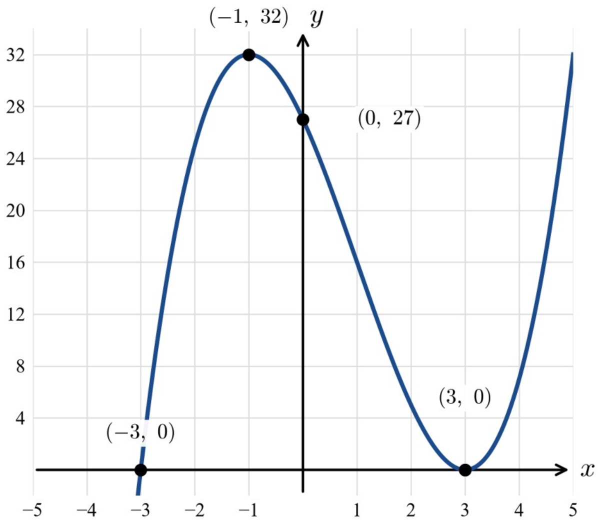
Question 3. a. $2\cos x+\sqrt{3}=0\Rightarrow\cos x=-\frac{\sqrt{3}}{2}$. The base angle is $\frac{\pi}{6}$, and cosine is negative in the second and third quadrants, so $x=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$ or $x=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$. **b.** Amplitude 2; period $\frac{2\pi}{3}$; range $[-1,3]$ (since $2\cos(3x)\in[-2,2]$, then add 1).

Question 4. a. The probabilities sum to 1: $\frac{1}{6}+\frac{1}{3}+k+\frac{1}{4}=1$, i.e. $\frac{2+4+3}{12}+k=\frac{9}{12}+k=1$, so $k=\frac{1}{4}$. **b.** $\mathrm{E}(X)=1\cdot\frac{1}{6}+2\cdot\frac{1}{3}+3\cdot\frac{1}{4}+4\cdot\frac{1}{4}=\frac{2+8+9+12}{12}=\frac{31}{12}$. **c.** $Pr(X\geq 3)=k+\frac{1}{4}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

Question 5. a. Let $u=e^x$ (so $u>0$). Then $2u^2-5u+2=0\Rightarrow(2u-1)(u-2)=0$, so $u=\frac{1}{2}$ or $u=2$. Both are positive, so $e^x=\frac{1}{2}$ or $e^x=2$, giving $x=-\log_e 2$ or $x=\log_e 2$. **b.** Since $e^x>0$, $f(x)=e^x-3>-3$. The range is $(-3,\infty)$.

Question 6. a. $Pr(X=2)=\binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3=10\times\frac{1}{9}\times\frac{8}{27}=\frac{80}{243}$. **b.** $\mathrm{Var}(X)=np(1-p)=5\times\frac{1}{3}\times\frac{2}{3}=\frac{10}{9}$.

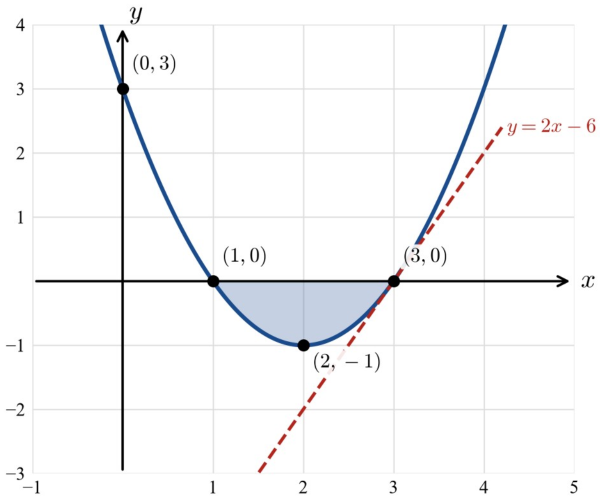
Question 7. a. $f(3)=27-27-27+27=0$, so $x=3$ is a solution. **b.** Since $x=3$ is a repeated factor (checking, $f'(3)=0$ below), dividing gives $f(x)=(x-3)(x^2-9)=(x-3)(x-3)(x+3)=(x-3)^2(x+3)$. **c.** $f'(x)=3x^2-6x-9=3(x^2-2x-3)=3(x-3)(x+1)$. Setting $f'(x)=0$ gives $x=3$ or $x=-1$. Then $f(3)=0$ and $f(-1)=-1-3+9+27=32$, so the stationary points are $(3,0)$ (a local minimum) and $(-1,32)$ (a local maximum). **d.** y-intercept $(0,27)$; x-intercepts $(-3,0)$ and $(3,0)$ (a turning point on the axis):



Question 8. a. For a valid pdf, $\int_0^4 kx\,dx=1$. Now $\int_0^4 kx\,dx=k\left[\frac{x^2}{2}\right]_0^4=k\times 8=8k$. So $8k=1$, giving $k=\frac{1}{8}$. **b.** $Pr(X\leq 2)=\int_0^2\frac{1}{8}x\,dx=\frac{1}{8}\left[\frac{x^2}{2}\right]_0^2=\frac{1}{8}\times 2=\frac{1}{4}$. **c.** $\mathrm{E}(X)=\int_0^4 x\cdot\frac{1}{8}x\,dx=\frac{1}{8}\left[\frac{x^3}{3}\right]_0^4=\frac{1}{8}\times\frac{64}{3}=\frac{8}{3}$.

Question 9. a. The axis of symmetry is $x=-\frac{-4}{2\times1}=2$, and $y(2)=4-8+3=-1$, so the turning point is $(2,-1)$. **b.** $\frac{dy}{dx}=2x-4$, so at $x=3$ the gradient is $2(3)-4=2$. Since $y(3)=9-12+3=0$, the tangent passes through $(3,0)$: $y-0=2(x-3)$, i.e. $y=2x-6$. **c.** The parabola meets the x-axis where $x^2-4x+3=(x-1)(x-3)=0$, i.e. at $x=1$ and $x=3$; the curve lies below the axis between these values, so the area is

$$A=-\int_1^3(x^2-4x+3)\,dx=-\left[\frac{x^3}{3}-2x^2+3x\right]_1^3=-\left(0-\frac{4}{3}\right)=\frac{4}{3}.$$



Practice Exam 3 — Technology-free

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Question 1 (4 marks)

a. Differentiate $y=(2x-1)^4$ with respect to x. 1 mark

b. For $f(x)=x^2e^{-x}$, find $f'(x)$ and hence the x-coordinates of the stationary points of the graph of f. 3 marks

Question 2 (4 marks)

a. Find an antiderivative of $\frac{3}{2x-1}$. 2 marks

b. Given that $f'(x)=4e^{2x}$ and $f(0)=3$, find $f(x)$. 2 marks

Question 3 (5 marks)

Let $f(x)=2\sin(2x)-1$ for $0\le x\le 2\pi$.

a. State the amplitude, the period and the range of f. 1 mark

b. Solve the equation $f(x)=0$ for $0\le x\le 2\pi$. 3 marks

c. Sketch the graph of $y=f(x)$ over $0\le x\le 2\pi$, labelling the endpoints with their coordinates. 1 mark

Question 4 (4 marks)

The random variable X has a binomial distribution with $n=6$ and $p=\frac{1}{4}$.

a. Find $Pr(X=2)$, giving your answer as an exact fraction. 2 marks

b. Find $Pr(X\ge 4)$. 2 marks

Question 5 (4 marks)

a. Solve $e^{2x}-5e^{x}+4=0$ for x. 2 marks

b. Solve $\log_e(x)+\log_e(x-3)=\log_e(10)$ for x. 2 marks

Question 6 (5 marks)

The graph of $y=\frac{1}{x^2}$ is transformed to produce the graph of $y=\frac{2}{(x-1)^2}-3$.

a. Describe a sequence of transformations that maps the graph of $y=\frac{1}{x^2}$ onto this graph. 2 marks

b. State the equations of the asymptotes. 1 mark

c. Find the exact coordinates of the axis intercepts. 2 marks

Question 7 (5 marks)

Consider the cubic function $f(x)=x^3+2x^2-4x-8$.

a. Show that $x=-2$ is a repeated root, and hence factorise $f(x)$ fully. 2 marks

b. Find the coordinates of the stationary points of the graph of f. 2 marks

c. Sketch the graph of $y = f(x)$, labelling the stationary points and the axis intercepts. 1 mark

Question 8 (4 marks)

The continuous random variable X has probability density function

$$f(x) = \begin{cases} kx(2-x), & 0 \le x \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Show that $k = \frac{3}{4}$. 2 marks

b. Find the mean $E(X)$. 2 marks

Question 9 (5 marks)

Let $f(x) = (x-a)e^x$, where a is a positive constant.

a. Find $f'(x)$. 1 mark

b. Show that the graph of f has exactly one stationary point, and find its coordinates in terms of a. 2 marks

c. Determine the nature of this stationary point, justifying your answer. 2 marks

End of examination questions

Practice Exam 3 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a trapezium $\frac{1}{2}(a+b)h$

curved surface area of a cylinder $2\pi r h$

volume of a cylinder $\pi r^2 h$

volume of a cone $\frac{1}{3}\pi r^2 h$

volume of a pyramid $\frac{1}{3}Ah$

volume of a sphere $\frac{4}{3}\pi r^3$

area of a triangle $\frac{1}{2}bc\sin(A)$

Calculus

$\frac{d}{dx}(x^n)=nx^{n-1}$

$\frac{d}{dx}((ax+b)^n)=an(ax+b)^{n-1}$

$\frac{d}{dx}(e^{ax})=ae^{ax}$

$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$

$\frac{d}{dx}(\sin(ax))=a\cos(ax)$

$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$

$\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}$

$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$\int (ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c, n\neq -1$

$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$

$\int \frac{1}{x}\,dx=\log_e(x)+c, x>0$

$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$

$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Newton's method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

trapezium rule approximation: $\text{Area}\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Probability

complement $Pr(A)=1-Pr(A')$

addition rule $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$

conditional probability	$Pr(A/B)=\dfrac{Pr(A\cap B)}{Pr(B)}$
binomial coefficient	$\binom{n}{x}={}^{n}C_{x}=\dfrac{n!}{x!(n-x)!}$
mean	$\mu=E(X)$
variance	$var(X)=\sigma^2=E\left((X-\mu)^2\right)=E\left(X^2\right)-\mu^2$

Probability distribution	Mean and variance
discrete: $Pr(X=x)=p(x)$	$\mu=\sum x\,p(x);\ \sigma^2=\sum(x-\mu)^2\,p(x)$
binomial: $Pr(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$\mu=np;\ \sigma^2=np(1-p)$
continuous: $Pr(a<X<b)=\int_a^b f(x)\,dx$	$\mu=\int_{-\infty}^{\infty}x f(x)\,dx;\ \sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Sample proportions

sample proportion	$\hat{P}=\dfrac{X}{n}$
mean	$E(\hat{P})=p$
standard deviation	$sd(\hat{P})=\sqrt{\dfrac{p(1-p)}{n}}$
approximate confidence interval	$\left(\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

End of Formula Sheet

Practice Exam 3 — solutions

Question 1. (a) $\frac{dy}{dx} = 4(2x-1)^3 \times 2 = 8(2x-1)^3$.

(b) By the product rule, $f'(x) = 2xe^{-x} + x^2(-e^{-x}) = xe^{-x}(2-x)$. Stationary points where $f'(x)=0$: since $e^{-x}>0$, $x(2-x)=0$, so $x=0$ or $x=2$.

Question 2. (a) $\int \frac{3}{2x-1}\,dx = \frac{3}{2}\log_e|2x-1| + c$.

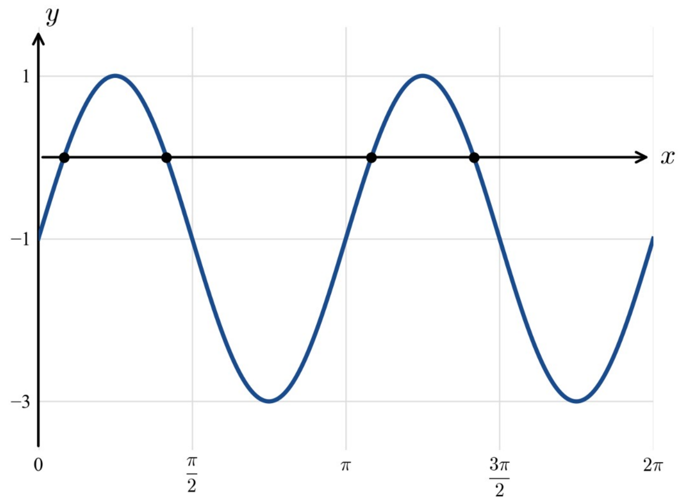
(b) $f(x) = \int 4e^{2x}\,dx = 2e^{2x}+c$. Since $f(0)=2+c=3$, $c=1$, so $f(x)=2e^{2x}+1$.

Question 3. (a) Amplitude 2, period $\frac{2\pi}{2}=\pi$, range $[-3,1]$.

(b) $2\sin(2x)-1=0 \Rightarrow \sin(2x)=\frac{1}{2}$. With $0\le x\le 2\pi$, let $\theta=2x\in[0,4\pi]$. Then $\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$, so

$$x=\frac{\pi}{12},\frac{5\pi}{12},\frac{13\pi}{12},\frac{17\pi}{12}.$$

(c) Endpoints $f(0)=-1$ and $f(2\pi)=-1$, so $(0,-1)$ and $(2\pi,-1)$; maximum value 1, minimum value -3.



Question 4. (a) $Pr(X=2)=\binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 = 15\times\frac{1}{16}\times\frac{81}{256}=\frac{1215}{4096}$.

(b) $Pr(X\ge 4)=\binom{6}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^2+\binom{6}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)+\left(\frac{1}{4}\right)^6=\frac{135}{4096}+\frac{18}{4096}+\frac{1}{4096}=\frac{77}{2048}$.

Question 5. (a) Let $u=e^x$: $u^2-5u+4=0\Rightarrow(u-1)(u-4)=0$, so $u=1$ or $u=4$. Then $e^x=1\Rightarrow x=0$, or $e^x=4\Rightarrow x=\log_e 4$. $\therefore x=0$ or $x=\log_e 4$.

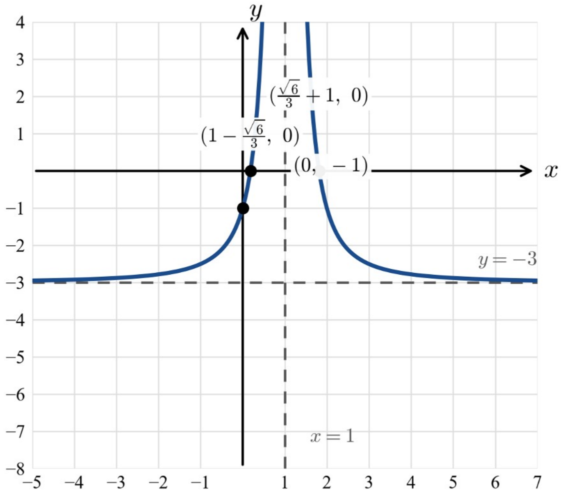
(b) $\log_e(x(x-3))=\log_e 10\Rightarrow x^2-3x-10=0\Rightarrow(x-5)(x+2)=0$. The domain requires $x>3$, so $x=5$ (rejecting $x=-2$).

Question 6. (a) From $y=\frac{1}{x^2}$: a dilation by factor 2 from the x-axis, then a translation of 1 unit to the right and 3 units down.

(b) Asymptotes $x=1$ and $y=-3$.

(c) y-intercept: $x=0 \Rightarrow y=\frac{2}{1}-3=-1$, giving $(0,-1)$. x-intercepts:

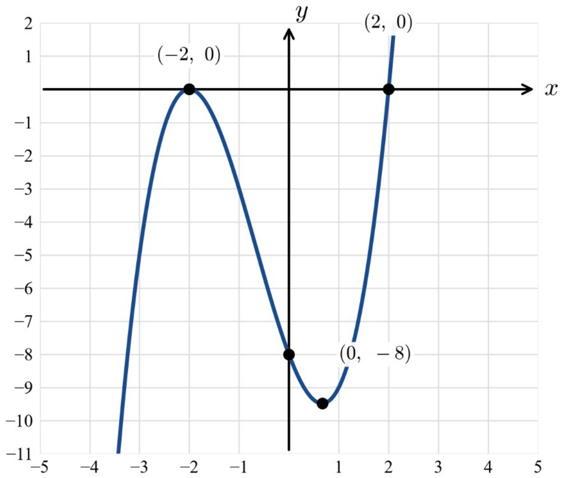
$\frac{2}{(x-1)^2}=3 \Rightarrow (x-1)^2=\frac{2}{3} \Rightarrow x-1=\pm\frac{\sqrt{6}}{3}$, giving $\left(1-\frac{\sqrt{6}}{3},0\right)$ and $\left(1+\frac{\sqrt{6}}{3},0\right)$.



Question 7. (a) $f(-2)=-8+8+8-8=0$, so $(x+2)$ is a factor. Dividing, $f(x)=(x+2)(x^2-4)=(x+2)(x-2)(x+2)=(x+2)^2(x-2)$, so $x=-2$ is a repeated (double) root.

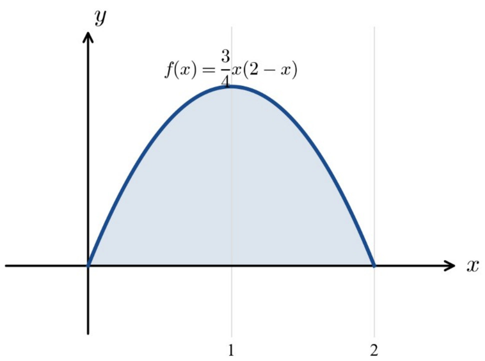
(b) $f'(x)=3x^2+4x-4=(3x-2)(x+2)$, so stationary points at $x=-2$ and $x=\frac{2}{3}$. $f(-2)=0$ and $f\left(\frac{2}{3}\right)=\frac{8}{27}+\frac{8}{9}-\frac{8}{3}-8=-\frac{256}{27}$, giving $(-2,0)$ and $\left(\frac{2}{3},-\frac{256}{27}\right)$.

(c) $(-2,0)$ is a local maximum (a touch point), $\left(\frac{2}{3},-\frac{256}{27}\right)$ a local minimum; intercepts $(-2,0)$, $(2,0)$, $(0,-8)$.



Question 8. (a) $\int_0^2 kx(2-x)\,dx=k\int_0^2 (2x-x^2)\,dx=k\left[x^2-\frac{x^3}{3}\right]_0^2=k\left(4-\frac{8}{3}\right)=\frac{4k}{3}$. Setting this equal to 1 gives $k=\frac{3}{4}$.

(b) $E(X)=\int_0^2 x\cdot\frac{3}{4}x(2-x)\,dx=\frac{3}{4}\int_0^2 (2x^2-x^3)\,dx=\frac{3}{4}\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2=\frac{3}{4}\left(\frac{16}{3}-4\right)=\frac{3}{4}\times\frac{4}{3}=1$. (This agrees with the symmetry of f about $x=1$.)



Question 9. (a) By the product rule, $f'(x) = e^x + (x-a)e^x = e^x(x-a+1)$.

(b) $f'(x) = 0 \Rightarrow e^x(x-(a-1)) = 0$. Since $e^x > 0$ for all x, the only solution is $x = a-1$, so there is exactly one stationary point. $f(a-1) = (a-1-a)e^{a-1} = -e^{a-1}$, giving $(a-1, -e^{a-1})$.

(c) For $x < a-1$, $x-(a-1) < 0$ so $f'(x) < 0$; for $x > a-1$, $f'(x) > 0$. The gradient changes from negative to positive, so the stationary point is a **local minimum**. (Equivalently $f''(x) = e^x(x-a+2)$, and $f''(a-1) = e^{a-1} > 0$.)

Practice Exam 4 — Technology-active

Reading time: 15 minutes. Writing time: 2 hours. Section A: 20 multiple-choice questions (20 marks). Section B: 60 marks. Approved materials: protractors, set squares and aids for curve sketching; one bound reference; one approved CAS calculator or CAS software, and one scientific calculator. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Total 80 marks.

Section A — Multiple-choice questions

Choose the response that is correct for the question.

Question 1

The maximal domain of $f(x)=\sqrt{x-2}$ is

A. $[2,\infty)$

B. $(2,\infty)$

C. $(-\infty,2]$

D. $\mathbb{R}$

Question 2

The graph of $y=x^2$ is translated 3 units to the right and 2 units down. The rule for the new graph is

A. $y=(x+3)^2+2$

B. $y=(x-3)^2-2$

C. $y=(x-2)^2-3$

D. $y=(x+3)^2-2$

Question 3

If $f(x)=3x-5$, then $f(4)$ equals

A. 2

B. 7

C. 12

D. 17

Question 4

If $f(x)=x^2$ and $g(x)=x-1$, then $f(g(x))$ equals

A. x^2-1

B. $(x-1)^2$

C. x^2+1

D. $(x+1)^2$

Question 5

The period of $y=3\sin(2x)$ is

A. π

B. 2π

C. 3

D. $\frac{\pi}{2}$

Question 6

If $y=x^3-2x$, then $\frac{dy}{dx}$ equals

A. $3x^2-2$

B. $3x^3-2$

C. $\frac{x^4}{4}-x^2$

D. $3x^2-2x$

Question 7

$\int_0^1 2x\,dx$ equals

A. 1

B. 2

C. $\frac{1}{2}$

D. 4

Question 8

$\log_2 8$ equals

A. 3

B. 4

C. 2

D. 16

Question 9

The inverse of $f(x)=2x+4$ is $f^{-1}(x)=$

A. $\frac{x-4}{2}$

B. $\frac{x+4}{2}$

C. $2x-4$

D. $\frac{1}{2x+4}$

Question 10

The gradient of the tangent to $y=x^2$ at the point $(1,1)$ is

A. 1

B. 2

C. 3

D. $\frac{1}{2}$

Question 11

If $X \sim Bi\left(3,\frac{1}{2}\right)$, then $Pr(X=2)$ equals

A. $\frac{1}{8}$

B. $\frac{3}{8}$

C. $\frac{1}{2}$

D. $\frac{3}{4}$

Question 12

A discrete random variable has $Pr(X=1)=0.2$, $Pr(X=2)=0.5$, $Pr(X=3)=0.3$. The mean $E(X)$ is

A. 2

B. 2.1

C. 2.5

D. 6

Question 13

If $X \sim N(10, 2^2)$, then $Pr(X < 10)$ equals

A. 0

B. 0.25

C. 0.5

D. 1

Question 14

If $f(x) = kx$ is a probability density function on $[0, 2]$, then k equals

A. 1

B. $\frac{1}{2}$

C. $\frac{1}{4}$

D. 2

Question 15

The number of solutions of $\sin x = \frac{1}{2}$ for $x \in [0, 2\pi]$ is

A. 1

B. 2

C. 3

D. 4

Question 16

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected in the

A. y-axis

B. x-axis

C. line $y = x$

D. origin

Question 17

In a random sample of 200 items, 40 are defective. The sample proportion of defective items is

A. 0.2

B. 0.4

C. 40

D. 0.02

Question 18

An antiderivative of e^{2x} is

A. e^{2x}

B. $\frac{1}{2}e^{2x}$

C. $2e^{2x}$

D. $\frac{e^{2x}}{x}$

Question 19

The equation $x^2+bx+9=0$ has exactly one solution when b equals

A. 3

B. 6

C. 9

D. 18

Question 20

Consider the algorithm: $s \leftarrow 0$; for i from 1 to 3: $s \leftarrow s+i$. The final value of s is

A. 3

B. 6

C. 9

D. 0

End of Section A

Section B

Question 1 (13 marks)

Consider the function $f(x)=x^3-8x^2+16x$.

a. Find the coordinates of the axis intercepts. 2 marks

b. Find the coordinates of the stationary points and state the nature of each. 4 marks

c. Sketch the graph of $y=f(x)$, labelling the intercepts and the stationary points. 3 marks

d. Find the equation of the tangent to the curve at the point where $x=1$. 2 marks

e. Find the area of the region enclosed between the curve and the x-axis. 2 marks

Question 2 (12 marks)

Let $g(x)=e^x-1$.

a. State the equation of the horizontal asymptote and the coordinates of the point where the graph crosses the axes. 2 marks

b. Find $g'(x)$ and hence the gradient of the graph at $x=0$. 2 marks

c. Sketch the graph of $y=g(x)$. 2 marks

d. Find the rule for $g^{-1}(x)$ and state its domain. 3 marks

e. Solve $g(x)=e^2-1$ for x. 1 mark

f. Find the area of the region enclosed between $y=g(x)$, the x-axis and the line $x=\log_e 2$. 2 marks

Question 3 (14 marks)

A biased spinner lands on red with probability $\frac{2}{5}$ on each spin. Let X be the number of reds obtained in 5 independent spins.

a. State the distribution of X. 1 mark

b. Find $Pr(X=2)$. 2 marks

c. Find $Pr(X \geq 1)$. 2 marks

d. Find $E(X)$ and $Var(X)$. 3 marks

The number of coins Y that a child has in their pocket has the distribution below.

y	0	1	2	3
$Pr(Y=y)$	0.1	0.3	0.4	0.2

e. Find $E(Y)$. 2 marks

f. Find $Pr(Y \geq 2)$. 1 mark

g. Find $Pr(Y \geq 2 / Y \geq 1)$. 3 marks

Question 4 (13 marks)

A continuous random variable X has probability density function

$$f(x)=\begin{cases} \frac{3}{8}x^2, & 0 \leq x \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Verify that f is a valid probability density function. 2 marks

b. Find the mean $E(X)$. 2 marks

c. Find $Pr(X \leq 1)$. 2 marks

The mass M grams of a manufactured item is normally distributed with mean 50 and standard deviation 4.

d. Find $Pr(M>54)$, correct to four decimal places. 2 marks

e. Find $Pr(46<M<54)$, correct to four decimal places. 2 marks

f. A sample of 200 items is taken. Find the expected number of items with a mass greater than 54 grams. 1 mark

g. Find the value of m for which $Pr(M<m)=0.9$, correct to two decimal places. 2 marks

Question 5 (8 marks)

Consider $f(x) = 2\sin x + 1$ for $x \in [0, 2\pi]$.

a. State the amplitude, the period and the range of f. 3 marks

b. Solve $f(x) = 2$ for $x \in [0, 2\pi]$. 2 marks

c. Find the area of the region enclosed between the graph of $y = f(x)$ and the line $y = 1$ over $[0, \pi]$. 3 marks

End of examination questions

Practice Exam 4 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a trapezium $\frac{1}{2}(a+b)h$

curved surface area of a cylinder $2\pi r h$

volume of a cylinder $\pi r^2 h$

volume of a cone $\frac{1}{3}\pi r^2 h$

volume of a pyramid $\frac{1}{3}A h$

volume of a sphere $\frac{4}{3}\pi r^3$

area of a triangle $\frac{1}{2}b c\sin(A)$

Calculus

$$\frac{d}{dx}\left(x^n\right)=n x^{n-1}$$

$$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$$

$$\frac{d}{dx}\left(e^{ax}\right)=a e^{ax}$$

$$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$$

$$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$$

$$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$$

$$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}$$

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c,\, n\neq -1$$

$$\int (ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c,\, n\neq -1$$

$$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$$

$$\int \frac{1}{x}\,dx=\log_e(x)+c,\, x>0$$

$$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$$

$$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$$

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Newton's method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

trapezium rule approximation: $\text{Area}\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Probability

complement $Pr(A)=1-Pr(A')$

addition rule $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$

conditional probability $Pr(A/B)=\dfrac{Pr(A\cap B)}{Pr(B)}$

binomial coefficient $\binom{n}{x} = {}^{n}C_{x} = \dfrac{n!}{x!(n-x)!}$

mean $\mu=E(X)$

variance $var(X)=\sigma^2=E\left((X-\mu)^2\right)=E\left(X^2\right)-\mu^2$

Probability distribution	Mean and variance
discrete: $Pr(X=x)=p(x)$	$\mu=\sum x\,p(x)$; $\sigma^2=\sum(x-\mu)^2\,p(x)$
binomial: $Pr(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$\mu=np$; $\sigma^2=np(1-p)$
continuous: $Pr(a<X<b)=\int_a^b f(x)\,dx$	$\mu=\int_{-\infty}^{\infty} x f(x)\,dx$; $\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Sample proportions

sample proportion $\hat{P}=\dfrac{X}{n}$

mean $E(\hat{P})=p$

standard deviation $sd(\hat{P})=\sqrt{\dfrac{p(1-p)}{n}}$

approximate confidence interval $\left(\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

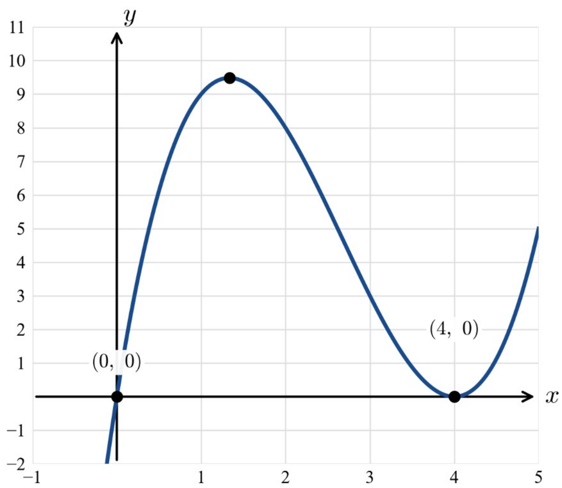
End of Formula Sheet

Practice Exam 4 — solutions

Section A — answer key

Q	Ans	Reason	Q	Ans	Reason
1	A	$x-2\geq 0 \Rightarrow x\geq 2$	11	B	$\binom{3}{2}(1/2)^3=3/8$
2	B	right h, down k: $(x-3)^2-2$	12	B	$0.2+1.0+0.9=2.$
3	B	$3(4)-5=7$	13	C	symmetry about the mean
4	B	g then square: $(x-1)^2$	14	B	$\int_0^2 kx\,dx=2k=1$
5	A	$2\pi/2=\pi$	15	B	$x=\pi/6, 5\pi/6$
6	A	$3x^2-2$	16	B	$y\mapsto -y$
7	A	$[x^2]_0^1=1$	17	A	$40/200=0.2$
8	A	$2^3=8$	18	B	$d/dx\,1/2\,e^{2x}=e^{2.}$
9	A	swap & solve: $x-4/2$	19	B	$\Delta=b^2-36=0\Rightarrow l$
10	B	$d/dx\,x^2\|_1=2$	20	B	$1+2+3=6$

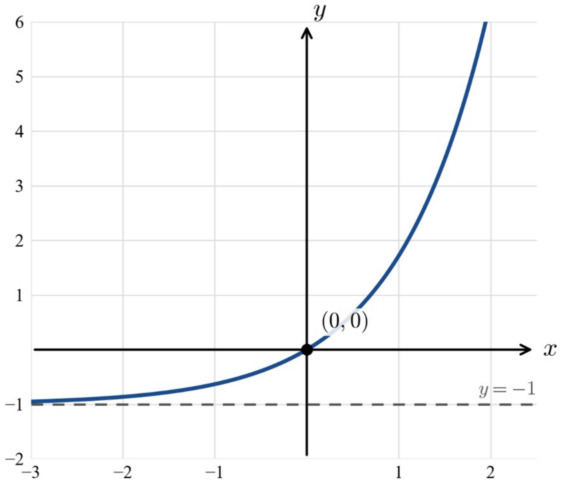
Section B — Question 1. $f(x)=x^3-8x^2+16x=x(x-4)^2$. (a) x-intercepts: $x(x-4)^2=0\Rightarrow(0,0)$ and $(4,0)$; the y-intercept is also $(0,0)$. (b) $f'(x)=3x^2-16x+16=(3x-4)(x-4)=0\Rightarrow x=\frac{4}{3},4$. $f\left(\frac{4}{3}\right)=\frac{256}{27}$, $f(4)=0$. Sign of f': positive, then negative, then positive, so $\left(\frac{4}{3},\frac{256}{27}\right)$ is a **local maximum** and $(4,0)$ is a **local minimum**.



(c) (d) $f(1)=1-8+16=9$, $f'(1)=(3-4)(1-4)=3$.

Tangent: $y-9=3(x-1)\Rightarrow y=3x+6$. (e) The curve is above the x-axis on $[0,4]$ (touching at $x=4$), so area $=\int_0^4 (x^3-8x^2+16x)\,dx=\left[\frac{x^4}{4}-\frac{8x^3}{3}+8x^2\right]_0^4=64-\frac{512}{3}+128=\frac{64}{3}$ square units.

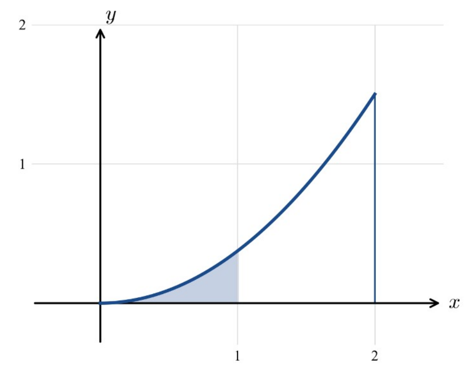
Section B — Question 2. $g(x)=e^x-1$. (a) Horizontal asymptote $y=-1$; the graph passes through $(0,0)$ (both intercepts). (b) $g'(x)=e^x$, so the gradient at $x=0$ is $e^0=1$. (c)



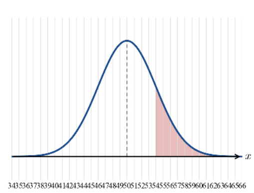
(d) Swap x and y: $x=e^y-1 \Rightarrow e^y=x+1 \Rightarrow y=\log_e(x+1)$. So $g^{-1}(x)=\log_e(x+1)$, domain $(-1,\infty)$ (= range of g). (e) $e^x-1=e^2-1 \Rightarrow e^x=e^2 \Rightarrow x=2$. (f) Area = $\int_0^{\log_e 2}(e^x-1)\,dx=\left[e^x-x\right]_0^{\log_e 2}=(2-\log_e 2)-(1-0)=1-\log_e 2$ square units.

Section B — Question 3. (a) $X \sim Bi\left(5,\frac{2}{5}\right)$. (b) $Pr(X=2)=\binom{5}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^3=\frac{216}{625}$. (c) $Pr(X\geq 1)=1-Pr(X=0)=1-\left(\frac{3}{5}\right)^5=1-\frac{243}{3125}=\frac{2882}{3125}$. (d) $E(X)=np=5\times\frac{2}{5}=2$; $Var(X)=np(1-p)=5\times\frac{2}{5}\times\frac{3}{5}=\frac{6}{5}$. (e) $E(Y)=0(0.1)+1(0.3)+2(0.4)+3(0.2)=1.7$. (f) $Pr(Y\geq 2)=0.4+0.2=0.6$. (g) $Pr(Y\geq 2 / Y\geq 1)=\frac{Pr(Y\geq 2)}{Pr(Y\geq 1)}=\frac{0.6}{0.9}=\frac{2}{3}$.

Section B — Question 4. (a) $\int_0^2 \frac{3}{8}x^2\,dx=\frac{3}{8}\left[\frac{x^3}{3}\right]_0^2=\frac{3}{8}\cdot\frac{8}{3}=1$, and $f(x)\geq 0$ on $[0,2]$, so f is a valid pdf.

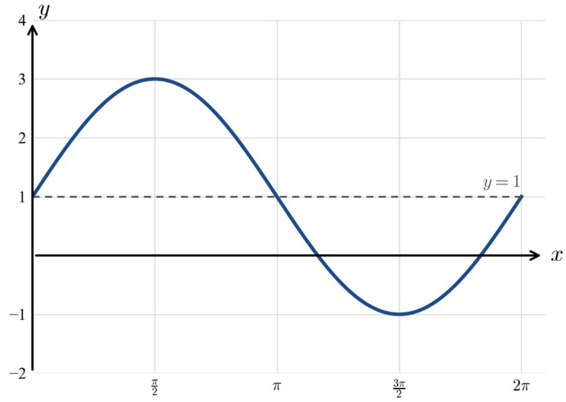


(b) $E(X)=\int_0^2 x\cdot\frac{3}{8}x^2\,dx=\frac{3}{8}\left[\frac{x^4}{4}\right]_0^2=\frac{3}{8}\cdot 4=\frac{3}{2}$. (c) $Pr(X\leq 1)=\int_0^1 \frac{3}{8}x^2\,dx=\frac{3}{8}\cdot\frac{1}{3}=\frac{1}{8}$. (d) $Pr(M>54)=0.1587$.



(e) $Pr(46<M<54)=0.6827$ (within one standard deviation). (f) Expected number $=200\times 0.1587=31.74\approx 32$ items. (g) $Pr(M<m)=0.9 \Rightarrow m=50+4\times 1.2816=55.13$ (2 d.p.).

Section B — Question 5. $f(x) = 2\sin x + 1$. (a) Amplitude 2; period $\frac{2\pi}{1} = 2\pi$; range $[-1, 3]$. (b) $2\sin x + 1 = 2 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$. (c) On $[0, \pi]$, $f(x) \geq 1$, so area $= \int_0^{\pi} ((2\sin x + 1) - 1)\, dx = \int_0^{\pi} 2\sin x\, dx = [-2\cos x]_0^{\pi} = 2 - (-2) = 4$ square units.



Practice Exam 5 — Technology-active

Reading time: 15 minutes. Writing time: 2 hours. Section A consists of 20 multiple-choice questions (20 marks). Section B consists of extended-response questions (60 marks). Approved materials: protractors, set squares and aids for curve sketching; one bound reference; one approved CAS calculator or CAS software, and one scientific calculator. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Total 80 marks.

Section A — Multiple-choice questions

Choose the response that is correct for the question.

Question 1

The range of $y = 2\cos x + 1$ is

A. $[-2, 2]$

B. $[1, 3]$

C. $[-1, 3]$

D. $[-3, 3]$

Question 2

The graph of $y = \frac{1}{x-2} + 3$ has asymptotes

A. $x = 2,\ y = 3$

B. $x = 3,\ y = 2$

C. $x = -2,\ y = 3$

D. $x = 2,\ y = -3$

Question 3

If $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, then $f(g(x))$ equals

A. $2x^2 - 5$

B. $4x^2 - 2$

C. $4x^2 + 4x + 1$

D. $4x^2 + 4x - 2$

Question 4

The derivative of e^{3x} is

A. $3e^{3x}$

B. e^{3x}

C. $3e^{3x-1}$

D. $\frac{1}{3}e^{3x}$

Question 5

$\int_0^1 2x\,dx$ equals

A. 2

B. 1

C. $\frac{1}{2}$

D. 0

Question 6

$\frac{d}{dx}\left(\log_e(2x)\right)$ equals

A. $\frac{2}{x}$

B. $\frac{1}{2x}$

C. $\frac{2}{2x}$

D. $\frac{1}{x}$

Question 7

The period of $y=\sin(2x)$ is

A. 2π

B. π

C. $\frac{\pi}{2}$

D. 4π

Question 8

The graph of $y=x^2$ is mapped to $y=(x-3)^2+1$ by a translation of

A. 3 left and 1 up

B. 1 right and 3 up

C. 3 right and 1 up

D. 3 right and 1 down

Question 9

If $X \sim Bi(5, 0.2)$, then $Pr(X=1)$ equals

A. 0.4096

B. 0.32768

C. 0.2

D. 0.08192

Question 10

If $X \sim Bi(20, 0.3)$, then $E(X)$ equals

A. 3

B. 4.2

C. 20

D. 6

Question 11

If $Pr(A)=0.6$, $Pr(B)=0.5$ and $Pr(A \cap B)=0.3$, then $Pr(A \cup B)$ equals

A. 1.1

B. 0.8

C. 0.9

D. 0.3

Question 12

The solution of $2^x=8$ is $x=$

A. 3

B. 4

C. 2

D. $\log_e 8$

Question 13

A continuous random variable has probability density function $f(x)=kx$ on $[0,2]$ (and 0 elsewhere). The value of k is

A. $\frac{1}{4}$

B. 1

C. $\frac{1}{2}$

D. 2

Question 14

The gradient of $y = x^3$ at $x = 2$ is

A. 8

B. 12

C. 6

D. 4

Question 15

The inverse of $f(x) = 3x - 6$ is $f^{-1}(x) =$

A. $3x + 6$

B. $\frac{x-6}{3}$

C. $\frac{x}{3} - 6$

D. $\frac{x+6}{3}$

Question 16

If $Z \sim N(0, 1)$, then $Pr(Z < 0)$ equals

A. 0

B. 1

C. 0.5

D. 0.16

Question 17

The average value of $f(x) = x^2$ over $[0, 3]$ is

A. 9

B. 3

C. 27

D. 1

Question 18

$\cos\left(\frac{2\pi}{3}\right)$ equals

A. $-\frac{1}{2}$

B. $\frac{1}{2}$

C. $-\frac{\sqrt{3}}{2}$

D. $\frac{\sqrt{3}}{2}$

Question 19

The number of solutions of $\sin x = \frac{1}{2}$ for $x \in [0, 2\pi]$ is

A. 1

B. 3

C. 4

D. 2

Question 20

In a sample of 50 people, 20 have a certain characteristic. The sample proportion $\hat{p}$ is

A. 0.2

B. 20

C. 0.4

D. 0.5

End of Section A

Section B

Question 1 (15 marks)

Consider the function $f(x) = x^3 - 6x^2 + 9x$.

a. Find $f'(x)$ and hence the coordinates of the stationary points of f. 3 marks

b. Find the coordinates of the point of inflection. 2 marks

c. Sketch the graph of $y = f(x)$ for $x \in [0, 4]$, labelling the stationary points, the point of inflection and the axis intercepts. 3 marks

d. Find the equation of the tangent to the curve at $x = 2$. 3 marks

e. Find the exact area of the region enclosed by the curve and the x-axis between $x = 0$ and $x = 3$. 4 marks

Question 2 (14 marks)

Let $g(x)=e^{2x}-5e^{x}+4$.

a. Show that $g(x)=0$ when $e^{x}=1$ or $e^{x}=4$, and hence state the exact x-intercepts. 3 marks

b. Find $g'(x)$ and the exact coordinates of the stationary point. 4 marks

c. State the range of g. 2 marks

d. Sketch the graph of $y=g(x)$, labelling the intercepts and the stationary point. 3 marks

e. Find the exact value(s) of x for which $g(x)=4$. 2 marks

Question 3 (16 marks)

A machine produces components, of which 8% are defective.

a. A random sample of 20 components is selected. Let X be the number of defective components. Write down the distribution of X. 1 mark

b. Find $Pr(X=2)$, correct to four decimal places. 2 marks

c. Find $Pr(X\geq 1)$, correct to four decimal places. 2 marks

d. Find the mean and standard deviation of X, the standard deviation correct to four decimal places. 2 marks

The mass, M grams, of a component is normally distributed with mean 50 and standard deviation 2.

e. Find $Pr(48<M<53)$, correct to four decimal places. 3 marks

f. Find the value of m such that $Pr(M>m)=0.1$, correct to two decimal places. 2 marks

g. In a separate quality check, 30 of a random sample of 400 components are defective. Determine an approximate 95% confidence interval for the population proportion of defective components, giving the endpoints correct to four decimal places. 4 marks

Question 4 (15 marks)

The temperature T (in °C) inside a greenhouse t hours after midnight is modelled by

$$T(t)=18+6\sin\left(\frac{\pi t}{12}\right), 0\leq t\leq 24.$$

a. State the maximum and minimum temperatures, and the times at which they first occur. 3 marks

b. Find the exact temperature at $t=0$ and at $t=3$. 2 marks

c. Find the times during the 24 hours at which the temperature is $21^{\circ}C$. 4 marks

d. Find $\frac{dT}{dt}$ and hence the exact rate of change of temperature at $t=3$. 3 marks

e. Find the exact average temperature over the first 12 hours. 3 marks

End of examination questions

Practice Exam 5 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a trapezium $\frac{1}{2}(a+b)h$

curved surface area of a cylinder $2\pi r h$

volume of a cylinder $\pi r^2 h$

volume of a cone $\frac{1}{3}\pi r^2 h$

volume of a pyramid $\frac{1}{3}Ah$

volume of a sphere $\frac{4}{3}\pi r^3$

area of a triangle $\frac{1}{2}bc\sin(A)$

Calculus

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

$$\frac{d}{dx}((ax+b)^n)=an(ax+b)^{n-1}$$

$$\frac{d}{dx}(e^{ax})=ae^{ax}$$

$$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$$

$$\frac{d}{dx}(\sin(ax))=a\cos(ax)$$

$$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$$

$$\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}$$

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c,\ n\neq-1$$

$$\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c,\ n\neq-1$$

$$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$$

$$\int\frac{1}{x}\,dx=\log_e(x)+c,\ x>0$$

$$\int\sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$$

$$\int\cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$$

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Newton's method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

trapezium rule approximation: $\text{Area}\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Probability

complement $Pr(A)=1-Pr(A')$

addition rule $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$

conditional probability $Pr(A/B) = \dfrac{Pr(A \cap B)}{Pr(B)}$

binomial coefficient $\binom{n}{x} = {}^{n}C_{x} = \dfrac{n!}{x!(n-x)!}$

mean $\mu = E(X)$

variance $var(X) = \sigma^2 = E\left((X-\mu)^2\right) = E\left(X^2\right) - \mu^2$

Probability distribution	Mean and variance
discrete: $Pr(X=x) = p(x)$	$\mu = \sum x\, p(x)$; $\sigma^2 = \sum (x-\mu)^2 p(x)$
binomial: $Pr(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$	$\mu = np$; $\sigma^2 = np(1-p)$
continuous: $Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$; $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Sample proportions

sample proportion $\hat{P} = \dfrac{X}{n}$

mean $E(\hat{P}) = p$

standard deviation $sd(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$

approximate confidence interval $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

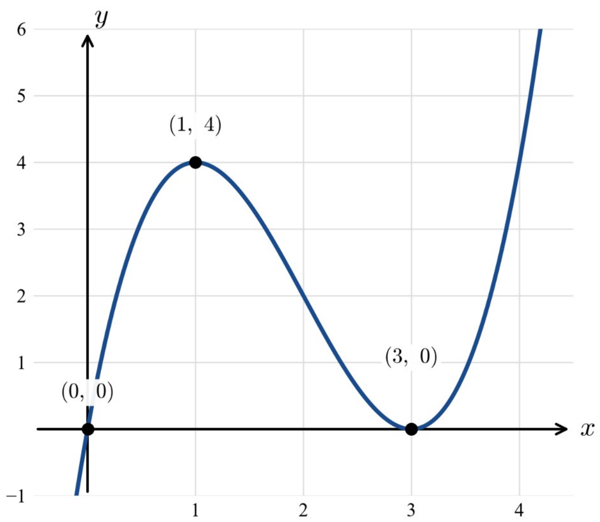
End of Formula Sheet

Practice Exam 5 — solutions

Section A

Q	Ans	Working	Q	Ans	Working
1	C	amplitude 2, centre 1: $[-1,3]$	11	B	$0.6+0.5-0.3=0$
2	A	$x=h=2$, $y=k=3$	12	A	$2^x=2^3 \Rightarrow x=3$
3	D	$(2x+1)^2-3=4x$	13	C	$\int_0^2 kx=2k=1 \Rightarrow k$
4	A	chain rule: $3e^{3x}$	14	B	$3x^2$ at $2=12$
5	B	$[x^2]_0^1=1$	15	D	$x=3y-6 \Rightarrow y=x$
6	D	$\log_e 2+\log_e x$, deriv $1/x$	16	C	symmetry: 0.5
7	B	$2\pi/2=\pi$	17	B	$1/3\int_0^3 x^2=1/3(9)$
8	C	$h=3, k=1$: right 3, up 1	18	A	$-\cos\pi/3=-1/2$
9	A	$\binom{5}{1}(0.2)(0.8)^4=$	19	D	$\pi/6, 5\pi/6$: 2
10	D	$np=20(0.3)=6$	20	C	$20/50=0.4$

Question 1. (a) $f'(x)=3x^2-12x+9=3(x-1)(x-3)$. Stationary points where $f'(x)=0$: $x=1$, $f(1)=4$, and $x=3$, $f(3)=0$. $\therefore$ stationary points $(1,4)$ (local maximum) and $(3,0)$ (local minimum). (b) $f''(x)=6x-12=0 \Rightarrow x=2$; $f(2)=2$. Point of inflection $(2,2)$. (c) x-intercepts: $x(x-3)^2=0 \Rightarrow (0,0)$ and $(3,0)$.

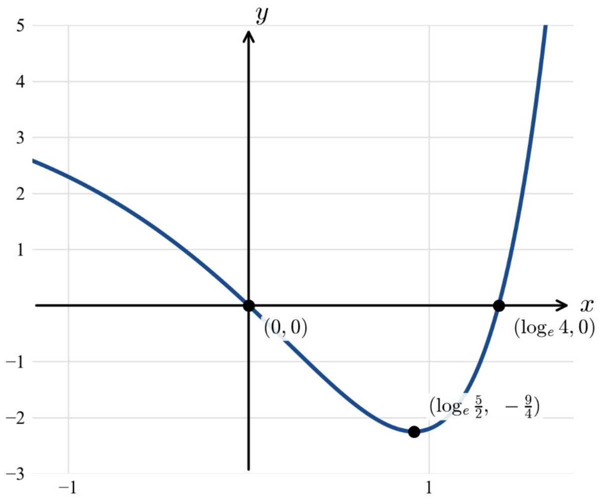


(d) $f'(2)=3(2)^2-12(2)+9=-3$; $f(2)=2$. Tangent: $y-2=-3(x-2)$, i.e. $y=-3x+8$.

(e) On $[0,3]$, $f(x)=x(x-3)^2\geq 0$, so the area is

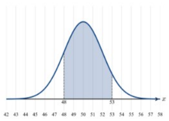
$$\int_0^3 (x^3-6x^2+9x)\,dx=\left[\frac{x^4}{4}-2x^3+\frac{9x^2}{2}\right]_0^3=\frac{81}{4}-54+\frac{81}{2}=\frac{27}{4}$$ square units.

Question 2. (a) $g(x)=e^{2x}-5e^{x}+4$. Let $u=e^{x}$: $u^{2}-5u+4=(u-1)(u-4)=0$, so $e^{x}=1$ or $e^{x}=4$. $\therefore$ x-intercepts $(0,0)$ and $(\log_e 4,0)$. (b) $g'(x)=2e^{2x}-5e^{x}=e^{x}(2e^{x}-5)=0 \Rightarrow e^{x}=\frac{5}{2}$, so $x=\log_e \frac{5}{2}$. Then $u=\frac{5}{2}$: $g=\frac{25}{4}-\frac{25}{2}+4=-\frac{9}{4}$. Stationary point $\left(\log_e \frac{5}{2},-\frac{9}{4}\right)$ (minimum). (c) Range $\left[-\frac{9}{4},\infty\right)$. (d)



(e) $e^{2x}-5e^{x}+4=4 \Rightarrow e^{x}(e^{x}-5)=0 \Rightarrow e^{x}=5$, so $x=\log_e 5$.

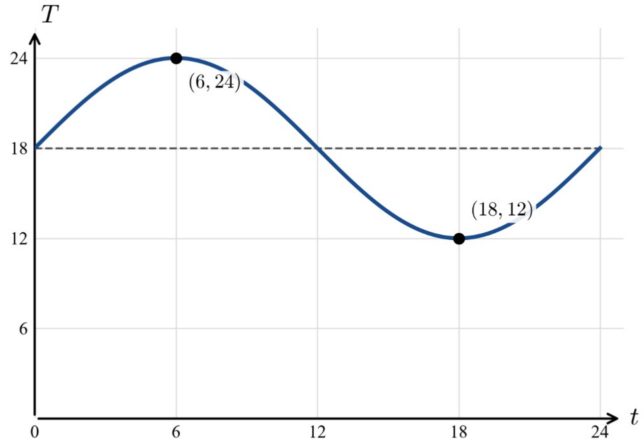
Question 3. (a) $X \sim Bi(20,0.08)$. (b) $Pr(X=2)=\binom{20}{2}(0.08)^{2}(0.92)^{18}=0.2711$. (c) $Pr(X\geq 1)=1-(0.92)^{20}=0.8113$. (d) $E(X)=20\times 0.08=1.6$; $sd(X)=\sqrt{20\times 0.08\times 0.92}=\sqrt{1.472}=1.2133$. (e) $M \sim N(50,2^{2})$: $Pr(48<M<53)=0.7745$.



(f) $Pr(M>m)=0.1 \Rightarrow Pr(M<m)=0.9$; inverse normal gives $m=52.56$.

(g) $\hat{p}=\frac{30}{400}=0.075$. A 95% interval is $\hat{p}\pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=0.075\pm 1.96\sqrt{\frac{0.075\times 0.925}{400}}$, giving $(0.0492,0.1008)$.

Question 4. (a) sin ranges over $[-1,1]$, so T ranges over $[12,24]$. Maximum $24^{\circ}C$ when $\sin(\pi t/12)=1$, i.e. $\pi t/12=\pi/2$, $t=6$; minimum $12^{\circ}C$ when $\pi t/12=3\pi/2$, $t=18$. (b) $T(0)=18+6\sin 0=18^{\circ}C$; $T(3)=18+6\sin \pi/4=18+6\cdot\frac{\sqrt{2}}{2}=18+3\sqrt{2}$ °C. (c) $18+6\sin(\pi t/12)=21 \Rightarrow \sin(\pi t/12)=\frac{1}{2}$. For $\pi t/12\in[0,2\pi]$: $\pi t/12=\pi/6$ or $5\pi/6$, so $t=2$ or $t=10$.



(d) $\frac{dT}{dt} = 6 \cdot \frac{\pi}{12} \cos(\pi t/12) = \frac{\pi}{2} \cos(\pi t/12)$. At $t = 3$: $\frac{\pi}{2} \cos \pi/4 = \frac{\pi}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}$ °C/h.

(e) Average $= \frac{1}{12} \int_0^{12} (18 + 6 \sin \pi t/12)\, dt = 18 + \frac{6}{12} \left[-\frac{12}{\pi} \cos \pi t/12 \right]_0^{12} = 18 + \frac{1}{2} \cdot \frac{24}{\pi} = 18 + \frac{12}{\pi}$ °C.

Practice Exam 6 — Technology-active

Reading time: 15 minutes. Writing time: 2 hours. Section A: 20 multiple-choice questions (20 marks). Section B: 60 marks. Approved materials: protractors, set squares and aids for curve sketching; one bound reference; one approved CAS calculator or CAS software, and one scientific calculator. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Total 80 marks.

Section A — Multiple-choice questions

Choose the response that is correct for the question. Each correct answer is worth 1 mark.

Question 1

The maximal domain of $f(x)=\sqrt{x-2}+\log_e(5-x)$ is

A. $(2,5)$

B. $[2,5)$

C. $[2,5]$

D. $(2,5]$

Question 2

The graph of $y=x^2$ is transformed to the graph of $y=-2(x-3)^2+1$ by

A. a dilation of factor 2 from the x-axis, a reflection in the x-axis, then a translation 3 units right and 1 unit up

B. a dilation of factor $1/2$ from the x-axis, a reflection in the x-axis, then a translation 3 units left and 1 unit up

C. a dilation of factor 2 from the y-axis, a reflection in the x-axis, then a translation 3 units right and 1 unit up

D. a reflection in the x-axis, then a translation 3 units left and 1 unit down

Question 3

If $f(x)=x^2$ and $g(x)=\sqrt{x-1}$, the maximal domain of $f(g(x))$ is

A. $\mathbb{R}$

B. $(1,\infty)$

C. $[1,\infty)$

D. $[0,\infty)$

Question 4

For $f(x)=2e^x-1$, the inverse function $f^{-1}(x)$ is

A. $\frac{1}{2}\log_e(x+1)$

B. $\dfrac{\log_e(x+1)}{2}$

C. $\log_e(2x+2)$

D. $\log_e\left(\dfrac{x+1}{2}\right)$

Question 5

The range of $y=\frac{2}{x-1}+3$ is

A. $\mathbb{R}\setminus\{1\}$

B. $\mathbb{R}\setminus\{0\}$

C. $(3,\infty)$

D. $\mathbb{R}\setminus\{3\}$

Question 6

The solution of $2\log_e x-\log_e(x+3)=0$ is

A. $\frac{1-\sqrt{13}}{2}$

B. $\frac{1+\sqrt{13}}{2}$

C. 3

D. $\frac{1\pm\sqrt{13}}{2}$

Question 7

The number of solutions of $3\cos(2x)-1=-1$ for $0\le x\le 2\pi$ is

A. 2

B. 3

C. 4

D. 8

Question 8

Given that $P(x)=x^3-2x^2-5x+6$ has a factor $(x-1)$, the solutions of $P(x)=0$ are

A. $x=-2,1,3$

B. $x=-3,1,2$

C. $x=-1,2,3$

D. $x=1,2,3$

Question 9

$\frac{d}{dx}\left(x^2 e^{3x}\right)=$

A. $2xe^{3x}$

B. $6xe^{3x}$

C. $3x^2e^{3x}$

D. $xe^{3x}(3x+2)$

Question 10

The function $f(x)=x^3-3x^2+4$ has

A. a local minimum at $(0,4)$ and a local maximum at $(2,0)$

B. a local maximum at $(0,4)$ and a local minimum at $(2,0)$

C. a local maximum at $(0,4)$ and a local minimum at $(2,4)$

D. a stationary point of inflection at $(1,2)$

Question 11

The gradient of the tangent to $y=\log_e(2x)$ at $x=1$ is

A. $\frac{1}{2}$

B. $\frac{1}{e}$

C. 1

D. 2

Question 12

$\int_0^1 (3x^2+2x)\,dx=$

A. 2

B. 3

C. 4

D. 5

Question 13

The average value of $f(x)=4-x^2$ over the interval $[0,2]$ is

A. $\frac{16}{3}$

B. 4

C. $\frac{8}{3}$

D. $\frac{4}{3}$

Question 14

Using the trapezium rule with two strips of width 2 and the values $f(0)=1$, $f(2)=3$, $f(4)=9$, the estimate of $\int_0^4 f(x)\,dx$ is

A. 13

B. 16

C. 26

D. 32

Question 15

A discrete random variable X has $Pr(X=0)=0.1$, $Pr(X=1)=0.3$, $Pr(X=2)=0.4$, $Pr(X=3)=0.2$. Then $E(X)=$

A. 1.5

B. 2

C. 0.4

D. 1.7

Question 16

If $X \sim Bi(5,1/3)$, then $Pr(X=2)=$

A. $\frac{80}{243}$

B. $\frac{40}{243}$

C. $\frac{10}{243}$

D. $\frac{80}{81}$

Question 17

The function $f(x) = kx^2$ is a probability density function on $[0, 3]$. The value of k is

A. $\frac{1}{3}$

B. $\frac{1}{9}$

C. $\frac{1}{27}$

D. 3

Question 18

If $W \sim N(60, 25)$, then $Pr(W > 65)$, correct to four decimal places, is

A. 0.8413

B. 0.0228

C. 0.1587

D. 0.3085

Question 19

For a sample of size 100 with sample proportion $\hat{p} = 0.6$, the standard deviation of $\hat{p}$ is closest to

A. 0.049

B. 0.24

C. 0.0024

D. 0.006

Question 20

Consider the algorithm: $s \leftarrow 0$; $i \leftarrow 1$; **while** $i \leq 4$ **do** $s \leftarrow s + i^2$; $i \leftarrow i + 1$. The value of s when the algorithm terminates is

A. 10

B. 20

C. 30

D. 16

End of Section A

Section B

Answer all questions. 60 marks.

Question 1 (14 marks)

Let $f(x) = (2 - x)e^x$.

a. Show that $f'(x) = (1 - x)e^x$. 2 marks

b. Find the coordinates of the stationary point of f and determine its nature. 3 marks

c. Find the equation of the tangent to the graph of f at $x=0$. 2 marks

d. Find the exact area of the region enclosed by the graph of f, the x-axis and the lines $x=0$ and $x=2$. 4 marks

e. Find the average value of f over the interval $[0,2]$. 3 marks

Question 2 (15 marks)

The depth of water, d metres, at a jetty t hours after midnight is modelled by $d(t)=3+2\sin\left(\frac{\pi t}{6}\right)$, $0\le t\le 12$.

a. State the amplitude and period of d, and the maximum and minimum depths. 3 marks

b. Find $d(3)$. 1 mark

c. Find the times during the first 12 hours at which the depth is 4 metres. 3 marks

d. Find $d'(t)$ and hence the instantaneous rate of change of the depth at $t=3$. 3 marks

e. Find the exact value of $\int_0^6 d(t)\,dt$ and interpret it. 5 marks

Question 3 (16 marks)

A machine fills bottles.

a. The number of faulty seals X in a randomly chosen crate has the distribution below. Find $E(X)$ and $Var(X)$.
4 marks

x	0	1	2	3
$Pr(X=x)$	0.5	0.3	0.15	0.05

b. Each bottle is faulty independently with probability 0.05. In a box of 20 bottles, find, correct to four decimal places, the probability that at least one bottle is faulty. 3 marks

c. The fill volume W mL is normally distributed with mean 500 and standard deviation 8.

i. Find $Pr(W<490)$, correct to four decimal places. 2 marks

ii. Find the value w, correct to two decimal places, such that $Pr(W>w)=0.10$. 3 marks

d. In a sample of 200 bottles, 9 are faulty. Determine an approximate 95% confidence interval for the population proportion of faulty bottles, giving endpoints correct to four decimal places. 4 marks

Question 4 (15 marks)

Let $f(x)=x^3-3a^2x$, where $a>0$.

a. Find $f'(x)$ and the coordinates of the stationary points of f in terms of a. 4 marks

b. Show that both stationary points lie on the curve $y=-2x^3$. 3 marks

c. For $a=2$, find the exact x-axis intercepts of f. 3 marks

d. For $a=2$, apply one iteration of Newton's method, starting from $x_0=4$, to approximate the largest positive root of $f(x)=0$. 3 marks

e. For $a=2$, find the exact area of the region enclosed by the graph of f and the x-axis for $0\le x\le 2\sqrt{3}$. 2 marks

End of examination questions

Practice Exam 6 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a trapezium $\frac{1}{2}(a+b)h$

curved surface area of a cylinder $2\pi r h$

volume of a cylinder $\pi r^2 h$

volume of a cone $\frac{1}{3}\pi r^2 h$

volume of a pyramid $\frac{1}{3}Ah$

volume of a sphere $\frac{4}{3}\pi r^3$

area of a triangle $\frac{1}{2}bc\sin(A)$

Calculus

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

$$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$$

$$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$$

$$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$$

$$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$$

$$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$$

$$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}$$

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c,\, n\neq -1$$

$$\int (ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c,\, n\neq -1$$

$$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$$

$$\int \frac{1}{x}\,dx=\log_e(x)+c,\, x>0$$

$$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$$

$$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$$

product rule $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Newton's method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

trapezium rule approximation: $\text{Area}\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Probability

complement $Pr(A)=1-Pr(A')$

addition rule $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$

conditional probability	$Pr(A/B)=\dfrac{Pr(A\cap B)}{Pr(B)}$
binomial coefficient	$\binom{n}{x}={}^{n}C_{x}=\dfrac{n!}{x!(n-x)!}$
mean	$\mu=E(X)$
variance	$var(X)=\sigma^2=E\left((X-\mu)^2\right)=E\left(X^2\right)-\mu^2$

Probability distribution	Mean and variance
discrete: $Pr(X=x)=p(x)$	$\mu=\sum x\,p(x)$; $\sigma^2=\sum(x-\mu)^2\,p(x)$
binomial: $Pr(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$\mu=np$; $\sigma^2=np(1-p)$
continuous: $Pr(a<X<b)=\int_a^b f(x)\,dx$	$\mu=\int_{-\infty}^{\infty} x f(x)\,dx$; $\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Sample proportions

sample proportion	$\hat{P}=\dfrac{X}{n}$
mean	$E(\hat{P})=p$
standard deviation	$sd(\hat{P})=\sqrt{\dfrac{p(1-p)}{n}}$
approximate confidence interval	$\left(\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

End of Formula Sheet

Practice Exam 6 — solutions

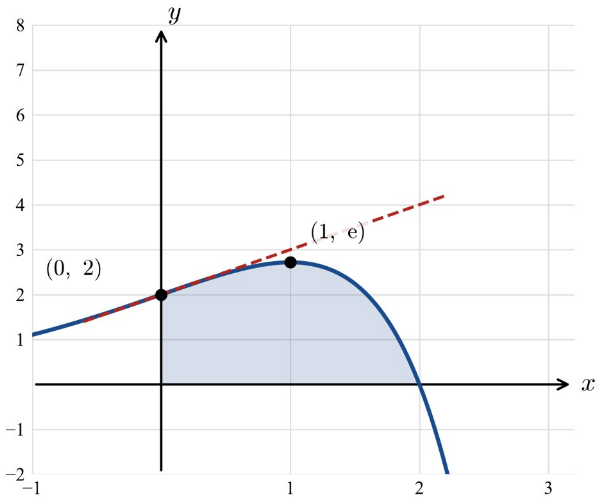
Section A — answer key

Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	**B** — $x \geq 2$ and $x < 5$	6	**B** — $x^2 = x + 3$, reject the negative root	11	**C** — $\frac{d}{dx}\log_e(2x$, at $x = 1$	16	**A** — $\binom{5}{2}(1/3)^2(2$
2	**A** — read $a = -2, h = 3$	7	**C** — $\cos 2x = 0$: $2x = \pi/2, \ldots$	12	**A** — $[x^3 + x^2]_0^1$	17	**B** — $\int_0^3 kx^2\,dx = 9$
3	**C** — $f(g) = x - 1$ on $x \geq 1$	8	**A** — $(x-1)(x-3$	13	**C** — $1/2\int_0^2 (4 - x^2$	18	**C** — $Pr(Z > 1)$
4	**D** — swap & solve	9	**D** — product rule	14	**B** — $2/2(1 + 2(3$	19	**A** — $\sqrt{0.6 \cdot 0.4/1}$
5	**D** — horizontal asymptote $y = 3$	10	**B** — $f' = 3x(x-2$	15	**D** — $\sum x\,Pr(X = x$	20	**C** — $1 + 4 + 9 + 16$

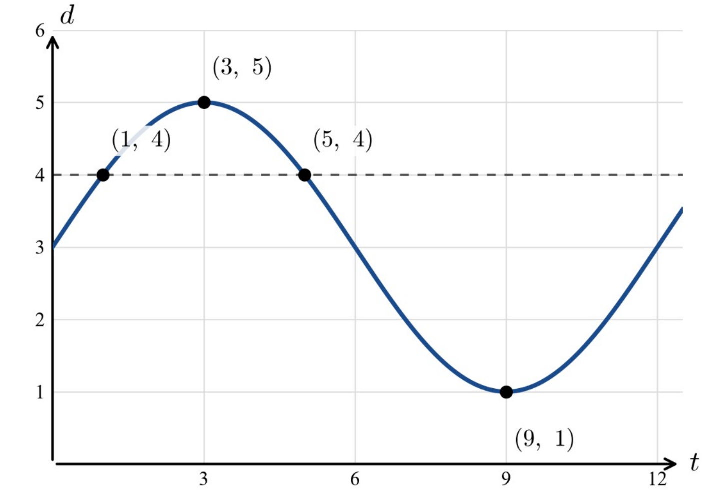
Question 1. (a) By the product rule with $u = 2 - x$, $v = e^x$: $f'(x) = (-1)e^x + (2-x)e^x = e^x(-1 + 2 - x) = (1-x)e^x$. (b) $f'(x) = 0 \Rightarrow x = 1$ (as $e^x > 0$). $f(1) = (2-1)e^1 = e$, so the stationary point is $(1, e)$. For $x < 1$, $f' > 0$; for $x > 1$, $f' < 0$: it is a **local maximum**. (c) $f(0) = 2$ and $f'(0) = (1-0)e^0 = 1$, so the tangent is $y - 2 = 1(x - 0)$, i.e. $y = x + 2$. (d) On $[0, 2]$, $f(x) \geq 0$. Using integration by parts (or recognising the antiderivative $(3-x)e^x$):

$$\int_0^2 (2-x)e^x\,dx = [(3-x)e^x]_0^2 = (1)e^2 - (3)e^0 = e^2 - 3.$$

$\therefore$ the area is $e^2 - 3$ square units. (e) Average value $= \dfrac{1}{2-0}\displaystyle\int_0^2 f(x)\,dx = \dfrac{e^2 - 3}{2}$.

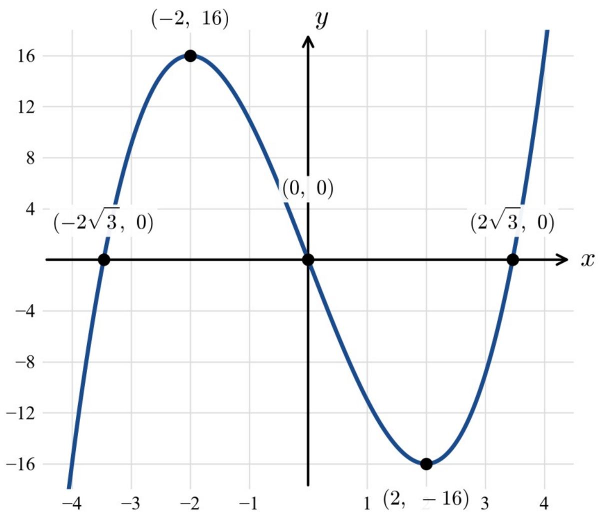


Question 2. (a) Amplitude 2; period $\frac{2\pi}{\pi/6}=12$ hours; maximum depth $3+2=5$ m, minimum depth $3-2=1$ m. (b) $d(3)=3+2\sin\left(\frac{\pi}{2}\right)=3+2=5$ m. (c) $3+2\sin\left(\frac{\pi t}{6}\right)=4 \Rightarrow \sin\left(\frac{\pi t}{6}\right)=\frac{1}{2} \Rightarrow \frac{\pi t}{6}=\frac{\pi}{6}$ or $\frac{5\pi}{6} \Rightarrow t=1$ or $t=5$ hours. (d) $d'(t)=2\cdot\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)=\frac{\pi}{3}\cos\left(\frac{\pi t}{6}\right)$. At $t=3$: $d'(3)=\frac{\pi}{3}\cos\left(\frac{\pi}{2}\right)=0$ m/h (the depth is momentarily stationary at high tide). (e) $\int_0^6 (3+2\sin \pi t/6)\,dt=\left[3t-\frac{12}{\pi}\cos \pi t/6\right]_0^6=\left(18+\frac{12}{\pi}\right)-\left(-\frac{12}{\pi}\right)=18+\frac{24}{\pi}$. This is the "depth–hours" over the first 6 hours (its average $\div 6$ gives the mean depth).



Question 3. (a) $E(X)=0(0.5)+1(0.3)+2(0.15)+3(0.05)=0.75$. $E(X^2)=0+0.3+0.6+0.45=1.35$, so $Var(X)=1.35-0.75^2=1.35-0.5625=0.7875$. (b) $Pr(\text{at least one faulty})=1-Pr(\text{none})=1-(0.95)^{20}\approx 0.6415$. (c) (i) $Pr(W<490)=Pr\left(Z<\frac{490-500}{8}\right)=Pr(Z<-1.25)\approx 0.1056$. (ii) $Pr(W>w)=0.10 \Rightarrow \frac{w-500}{8}=1.2816 \Rightarrow w=500+8(1.2816)\approx 510.25$ mL. (d) $\hat{p}=\frac{9}{200}=0.045$. Margin $=1.96\sqrt{\frac{0.045(0.955)}{200}}\approx 0.0287$. The 95% confidence interval is $0.045\pm 0.0287=(0.0163, 0.0737)$.

Question 4. (a) $f'(x)=3x^2-3a^2=3(x^2-a^2)$. $f'(x)=0 \Rightarrow x=\pm a$. $f(a)=a^3-3a^3=-2a^3$ and $f(-a)=-a^3+3a^3=2a^3$, so the stationary points are $(a,-2a^3)$ (minimum) and $(-a,2a^3)$ (maximum). (b) At $(a,-2a^3)$: $-2(a)^3=-2a^3$ ✓. At $(-a,2a^3)$: $-2(-a)^3=2a^3$ ✓. Both satisfy $y=-2x^3$. (c) For $a=2$: $f(x)=x^3-12x=x(x^2-12)=x(x-2\sqrt{3})(x+2\sqrt{3})$, so the x-intercepts are $x=0,\pm 2\sqrt{3}$. (d) $f(x)=x^3-12x$, $f'(x)=3x^2-12$. $x_1=x_0-\frac{f(x_0)}{f'(x_0)}=4-\frac{64-48}{48-12}=4-\frac{16}{36}=\frac{32}{9}\approx 3.556$ (the root is $2\sqrt{3}\approx 3.464$). (e) On $[0,2\sqrt{3}]$, $f(x)\le 0$, so the area is $-\int_0^{2\sqrt{3}}(x^3-12x)\,dx=-\left[\frac{x^4}{4}-6x^2\right]_0^{2\sqrt{3}}=-(36-72)=36$ square units.


$(-2,\ 16)$
y
16
12
8
4
$(-2\sqrt{3},\ 0)$
$(0,\ 0)$
$(2\sqrt{3},\ 0)$
x
−4
−8
−12
−16
−4
−3
−2
−1
1
$(2,\ -16)$
3
4