Mathematical Methods Units 3 & 4

Chapter 14 — The normal distribution

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Contents

Theory

The **normal distribution** is the most important continuous distribution in statistics. A normally distributed random variable is written $X \sim N(\mu, \sigma^2)$, where μ is the **mean** and σ is the **standard deviation** (so σ^2 is the variance).

The bell-shaped curve. The probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

a smooth "bell" curve. You are **not** asked to integrate it by hand — its key features come from μ and σ.

Key features.

- The curve is **symmetric about the vertical line** $x = \mu$. Because of this symmetry the **mean, median and mode are all equal to** μ.
- The total area under the curve is 1, and by symmetry the area on each side of μ is $\frac{1}{2}$; that is,

  $Pr(X > \mu) = Pr(X < \mu) = \frac{1}{2}$.
- The curve has **points of inflection at** $x = \mu - \sigma$ **and** $x = \mu + \sigma$ — one standard deviation either side of the mean. (Reading the inflection points off a graph is a quick way to find σ.)
- The curve extends indefinitely in both directions, approaching the x-axis as an asymptote but never touching it.

The effect of μ and σ.

- μ fixes the **location** (centre) of the curve. Changing μ slides the whole curve left or right without changing its shape.
- σ fixes the **spread**. A **larger** σ gives a **wider, flatter** curve; a **smaller** σ gives a **taller, narrower** curve. (The total area stays 1, so a narrower curve must be taller.)

Using symmetry to find probabilities. Even before computing exact probabilities with technology (Section 14C), symmetry gives many results. If $a > 0$, then by the symmetry about μ:

$$Pr(X < \mu - a) = Pr(X > \mu + a), \quad Pr(\mu - a < X < \mu + a) = 1 - 2\,Pr(X < \mu - a).$$

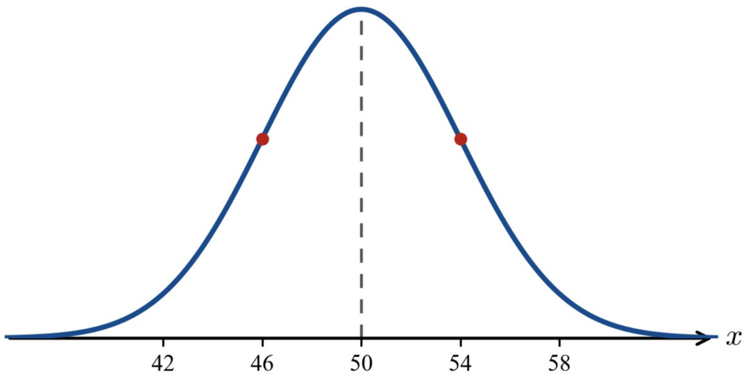
Worked examples

Example 1 — scaffolded

The random variable $X \sim N(50, 4^2)$. State the mean, median and mode, the coordinates (on the x-axis) of the points of inflection, and the value of $Pr(X > 50)$.

Working	Comment
Here $\mu = 50$ and $\sigma = 4$.	Read μ and σ from $N(\mu, \sigma^2)$; the second entry is $\sigma^2 = 16$.
Mean = median = mode = 50**.**	For a normal distribution all three central measures equal μ.
Points of inflection at $x = \mu \pm \sigma = 50 \pm 4$, i.e. $x = 46$ and $x = 54$.	The inflection points are one standard deviation either side of the mean.

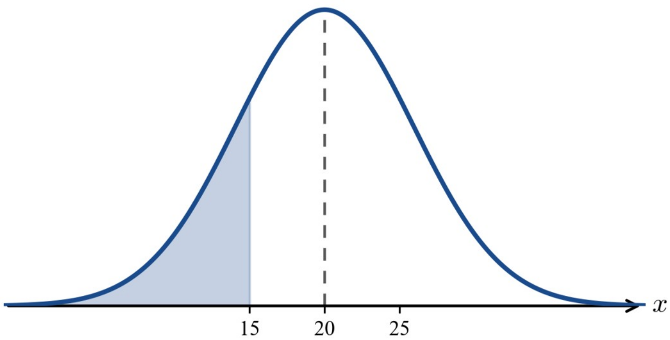
Working	Comment
$Pr(X>50)=\frac{1}{2}$.	The curve is symmetric about $x=\mu=50$, so half the area lies to the right.



Example 2 — scaffolded

X is normally distributed with mean $\mu=20$, and $Pr(X<15)=0.2$. Using symmetry, find (a) $Pr(X>25)$, (b) $Pr(15<X<25)$, and (c) $Pr(X>15)$.

Working	Comment
15 and 25 are equidistant from $\mu=20$ (each 5 units away).	Set up the symmetry: $15=\mu-5$ and $25=\mu+5$.
(a) $Pr(X>25)=Pr(X<15)=0.2$.	By symmetry, the right tail beyond $\mu+5$ equals the left tail below $\mu-5$.
(b) $Pr(15<X<25)=1-Pr(X<15)-Pr(X>25)=1-0$.	The middle area is 1 minus the two equal tails.
(c) $Pr(X>15)=1-Pr(X<15)=1-0.2=0.8$.	Complementary events.



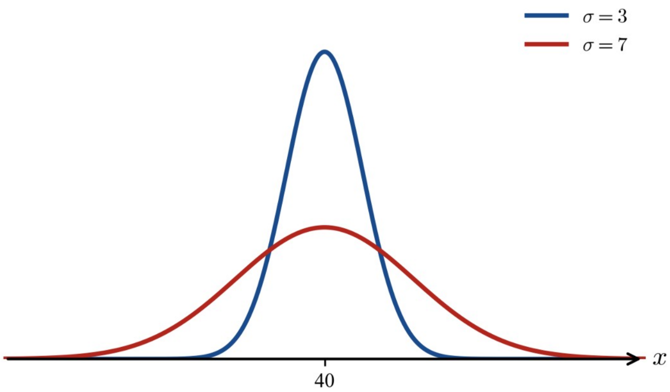
Example 3 — unscaffolded

Two classes sit the same test. The marks of class A are modelled by $A\sim N(65,5^2)$ and those of class B by $B\sim N(65,10^2)$. Compare the two distributions.

Both distributions have the **same mean**, $\mu=65$, so both curves are centred on 65 and have mean = median = mode = 65.

Class B has the larger standard deviation ($\sigma_B=10$ versus $\sigma_A=5$), so the marks of class B are **more spread out**: its bell curve is **wider and flatter**, while class A's is **taller and narrower** (its marks cluster more tightly around 65).

$\therefore$ same centre, but class B shows greater variability.



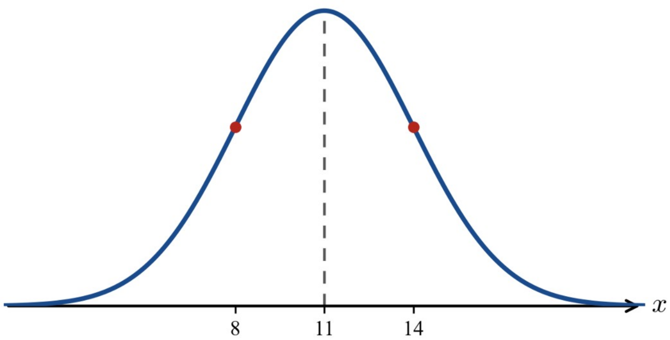
Example 4 — unscaffolded

The graph of a normal distribution has points of inflection at $x=8$ and $x=14$. Find the mean μ and the standard deviation σ.

The inflection points are at $\mu-\sigma$ and $\mu+\sigma$, so μ is midway between them and σ is half the distance between them:

$$\mu=\frac{8+14}{2}=11, \sigma=\frac{14-8}{2}=3.$$

$\therefore X\sim N(11,3^2)$.



Practice questions

Scaffolded

Q1. The heights (cm) of a large group are modelled by $X\sim N(100,15^2)$. (a) State the mean, median and mode. (b) Write the x-coordinates of the points of inflection. (c) State $Pr(X<100)$. (d) Between which two values does the "middle" one standard deviation either side of the mean lie?

Q2. $X\sim N(30,2^2)$. (a) State μ and σ. (b) Give the points of inflection. (c) State $Pr(X>30)$.

Q3. X is normal with mean 40 and $Pr(X<34)=0.15$. (a) By symmetry, state $Pr(X>46)$. (b) Find $Pr(34<X<46)$. (c) Find $Pr(X>34)$.

Q4. A normal curve has points of inflection at $x=27$ and $x=33$. (a) Find the mean. (b) Find the standard deviation. (c) Write the distribution in the form $N(\mu,\sigma^2)$.

Q5. $X\sim N(12,\sigma^2)$ and $Y\sim N(20,\sigma^2)$ have the same standard deviation. (a) How are the two curves related in shape? (b) How are they related in position?

Q6. For $X \sim N(50, 5^2)$, use symmetry to find: (a) $Pr(X<50)$; (b) $Pr(X>55 \text{ or } X<45)$ given $Pr(X>55)=0.16$; (c) $Pr(45<X<55)$.

Unscaffolded

Q7. For $X \sim N(25, 2^2)$, state the mean, the points of inflection, and $Pr(X>25)$.

Q8. A normal distribution has mean 70 and standard deviation 6. Between which two x-values are its points of inflection?

Q9. The points of inflection of a normal curve are at $x=4.5$ and $x=7.5$. Find μ and σ.

Q10. $X \sim N(60, 8^2)$. Given $Pr(60<X<68)=0.34$, use symmetry to find $Pr(X>68)$ and $Pr(52<X<68)$.

Q11. X is normal with mean 15 and $Pr(X<11)=0.25$. Find $Pr(X>19)$, $Pr(11<X<19)$ and $Pr(X>11)$.

Q12. Two normal curves have the same standard deviation but means $\mu_1=30$ and $\mu_2=45$. Describe, with reasons, how the graphs differ.

Q13. Two normal curves have the same mean 40 but standard deviations $\sigma_1=3$ and $\sigma_2=7$. Which curve is taller at its centre, and why?

Q14. A normal distribution is symmetric about $x=100$ and $Pr(X>118)=0.1$. Find $Pr(X<82)$ and $Pr(82<X<118)$.

Q15. For $X \sim N(\mu, \sigma^2)$, explain why $Pr(X=\mu)=0$ even though $x=\mu$ is the most likely region.

Q16. A machine fills bottles with a mean of 500 mL. The filling volume is normal with points of inflection at 498 mL and 502 mL. State the standard deviation and $Pr(X<500)$.

Q17. Sketch, on one set of axes, the curves $N(50, 4^2)$ and $N(50, 8^2)$, and describe how they differ.

Q18. X is normal with mean μ. Given $Pr(X<\mu-3)=0.12$, write down $Pr(X>\mu+3)$ and $Pr(\mu-3<X<\mu+3)$.

Q19. The marks in an exam are modelled by $N(62, 11^2)$. State the mean mark and the two marks at which the distribution has its points of inflection.

Q20. A normal curve has its maximum at $x=8$ and a point of inflection at $x=11$. Find μ and σ.

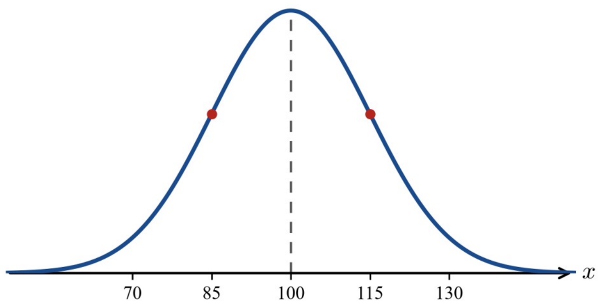
Q21. $X \sim N(20, \sigma^2)$ satisfies $Pr(X<14)=Pr(X>26)$. Explain why this must be true for **any** value of σ, and state the common value in terms of $p=Pr(X<14)$.

Answers

Q1. (a) 100; (b) 85 and 115; (c) $\frac{1}{2}$; (d) 85 to 115. **Q2.** (a) $\mu=30$, $\sigma=2$; (b) 28 and 32; (c) $\frac{1}{2}$. **Q3.** (a) 0.15; (b) 0.7; (c) 0.85. **Q4.** (a) 30; (b) 3; (c) $N(30,3^2)$. **Q5.** (a) identical shape (same σ); (b) Y's curve is Y translated 8 units right of X's. **Q6.** (a) 0.5; (b) 0.32; (c) 0.68. **Q7.** mean 25; inflection 23 and 27; $Pr(X>25)=\frac{1}{2}$. **Q8.** 64 and 76. **Q9.** $\mu=6$, $\sigma=1.5$. **Q10.** $Pr(X>68)=0.16$; $Pr(52<X<68)=0.68$. **Q11.** $Pr(X>19)=0.25$; $Pr(11<X<19)=0.5$; $Pr(X>11)=0.75$. **Q12.** identical shape; the second curve is the first translated 15 units to the right. **Q13.** the $\sigma_1=3$ curve is taller (smaller spread ⇒ narrower and, since total area $=1$, taller). **Q14.** $Pr(X<82)=0.1$; $Pr(82<X<118)=0.8$. **Q15.** for a continuous variable the probability of any single exact value is the area of a line, which is 0. **Q16.** $\sigma=2$ mL; $Pr(X<500)=\frac{1}{2}$. **Q17.** same centre 50; $N(50,8^2)$ is wider and flatter (twice the spread). **Q18.** $Pr(X>\mu+3)=0.12$; $Pr(\mu-3<X<\mu+3)=0.76$. **Q19.** mean 62; inflection at 51 and 73. **Q20.** $\mu=8$, $\sigma=3$. **Q21.** 14 and 26 are equidistant from $\mu=20$, so symmetry forces the tails equal for every σ; common value $=p$.

Fully-worked solutions

Q1. $X\sim N(100,15^2)$, so $\mu=100$, $\sigma=15$. (a) Mean = median = mode = 100. (b) Inflection at $\mu\pm\sigma=100\pm15$, i.e. 85 and 115. (c) By symmetry $Pr(X<100)=\frac{1}{2}$. (d) One standard deviation either side of the mean is from 85 to 115.



Q2. $\mu=30$, $\sigma=2$. (a) as stated. (b) Inflection at 30 ± 2, i.e. 28 and 32. (c) $Pr(X>30)=\frac{1}{2}$ by symmetry about $\mu=30$.

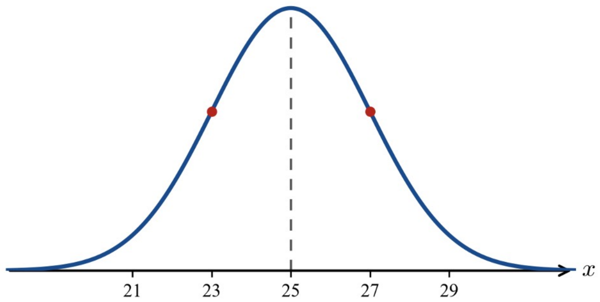
Q3. Mean 40; $34=\mu-6$ and $46=\mu+6$. (a) $Pr(X>46)=Pr(X<34)=0.15$ by symmetry. (b) $Pr(34<X<46)=1-0.15-0.15=0.7$. (c) $Pr(X>34)=1-0.15=0.85$.

Q4. Inflection at $\mu\pm\sigma$. (a) $\mu=\frac{27+33}{2}=30$. (b) $\sigma=\frac{33-27}{2}=3$. (c) $N(30,3^2)$.

Q5. Same σ, so (a) the curves have **identical shape**. (b) The means differ by $20-12=8$, so Y's curve is X's curve translated 8 units to the right.

Q6. $X\sim N(50,5^2)$. (a) $Pr(X<50)=\frac{1}{2}$. (b) $45=\mu-5$ and $55=\mu+5$; by symmetry $Pr(X<45)=Pr(X>55)=0.16$, so $Pr(X>55 \text{ or } X<45)=0.16+0.16=0.32$. (c) $Pr(45<X<55)=1-0.32=0.68$.

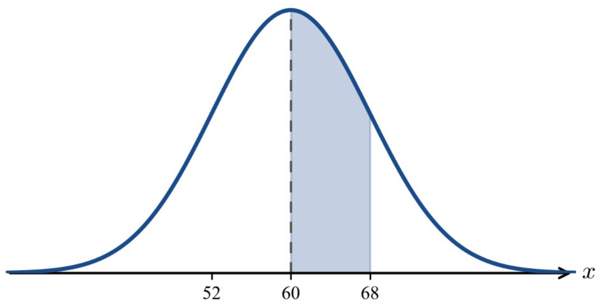
Q7. $\mu=25$, $\sigma=2$: mean 25; inflection $25\pm2=23$ and 27; $Pr(X>25)=\frac{1}{2}$.



Q8. Inflection at $\mu \pm \sigma = 70 \pm 6$, i.e. 64 and 76.

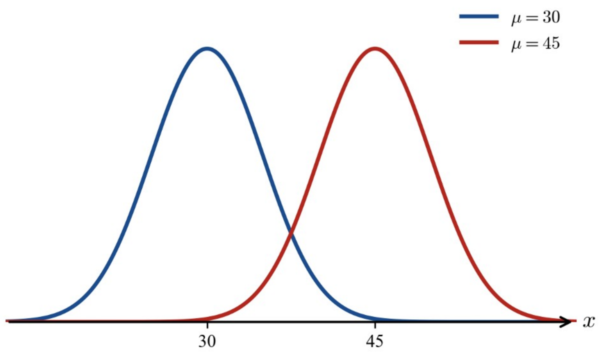
Q9. $\mu = \dfrac{4.5 + 7.5}{2} = 6$; $\sigma = \dfrac{7.5 - 4.5}{2} = 1.5$.

Q10. $X \sim N(60, 8^2)$, symmetric about 60. Since $Pr(X > 60) = 0.5$ and $Pr(60 < X < 68) = 0.34$, $Pr(X > 68) = 0.5 - 0.34 = 0.16$. By symmetry $Pr(52 < X < 60) = 0.34$, so $Pr(52 < X < 68) = 0.34 + 0.34 = 0.68$.



Q11. Mean 15; $11 = \mu - 4$, $19 = \mu + 4$. $Pr(X > 19) = Pr(X < 11) = 0.25$; $Pr(11 < X < 19) = 1 - 0.25 - 0.25 = 0.5$; $Pr(X > 11) = 1 - 0.25 = 0.75$.

Q12. Same $\sigma \Rightarrow$ identical shape; the means differ by $45 - 30 = 15$, so the second curve is the first translated 15 units to the right (same height and width).



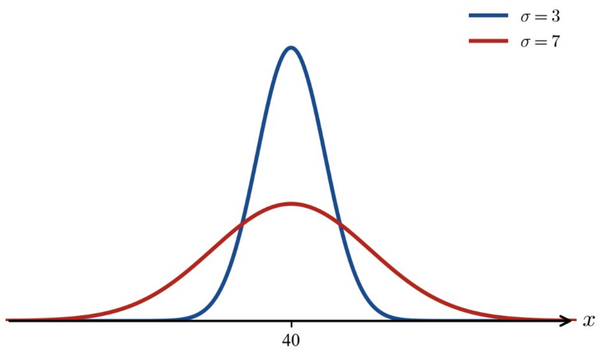
Q13. Both are centred on 40. The curve with $\sigma_1 = 3$ has the smaller spread, so it is narrower; because the area under each curve is 1, a narrower curve must be **taller**. Hence the $\sigma = 3$ curve is taller at its centre.

Q14. Symmetric about 100; $118 = \mu + 18$ and $82 = \mu - 18$. By symmetry $Pr(X < 82) = Pr(X > 118) = 0.1$; then $Pr(82 < X < 118) = 1 - 0.1 - 0.1 = 0.8$.

Q15. X is continuous, so probability is area under the density curve. The “area” above a single point $x = \mu$ is a line segment of zero width, hence zero area, so $Pr(X = \mu) = 0$; probability accrues only over an interval.

Q16. Inflection at $\mu \pm \sigma$: $\mu = \frac{498+502}{2} = 500$ and $\sigma = \frac{502-498}{2} = 2$ mL. $Pr(X < 500) = \frac{1}{2}$ by symmetry.

Q17. Both curves are centred on $x = 50$ (same mean), with inflection points at $46, 54$ for $N(50, 4^2)$ and at $42, 58$ for $N(50, 8^2)$. The $\sigma = 8$ curve is **wider and flatter** (twice the spread) while the $\sigma = 4$ curve is taller and narrower.



Q18. $\mu - 3$ and $\mu + 3$ are equidistant from μ, so by symmetry $Pr(X > \mu + 3) = Pr(X < \mu - 3) = 0.12$; then $Pr(\mu - 3 < X < \mu + 3) = 1 - 0.12 - 0.12 = 0.76$.

Q19. $\mu = 62$, $\sigma = 11$: mean 62; inflection at $62 \pm 11 = 51$ and 73.

Q20. The maximum of a normal curve is at the mean, so $\mu = 8$. A point of inflection is at $\mu + \sigma$ (or $\mu - \sigma$): $11 = 8 + \sigma \Rightarrow \sigma = 3$.

Q21. $14 = 20 - 6$ and $26 = 20 + 6$ are the same distance (6) from the mean $\mu = 20$. The normal curve is symmetric about $x = \mu$ for **every** σ, so the left tail $Pr(X < 20 - 6)$ always equals the right tail $Pr(X > 20 + 6)$, regardless of σ. The common value is $p = Pr(X < 14)$.

Theory

Every normal distribution is symmetric and bell-shaped, centred on its mean μ, with its spread governed by the standard deviation σ. We write $X \sim N(\mu, \sigma^2)$.

The standard normal. The normal distribution with mean 0 and standard deviation 1 is called the **standard normal distribution**, and its variable is written $Z \sim N(0, 1)$.

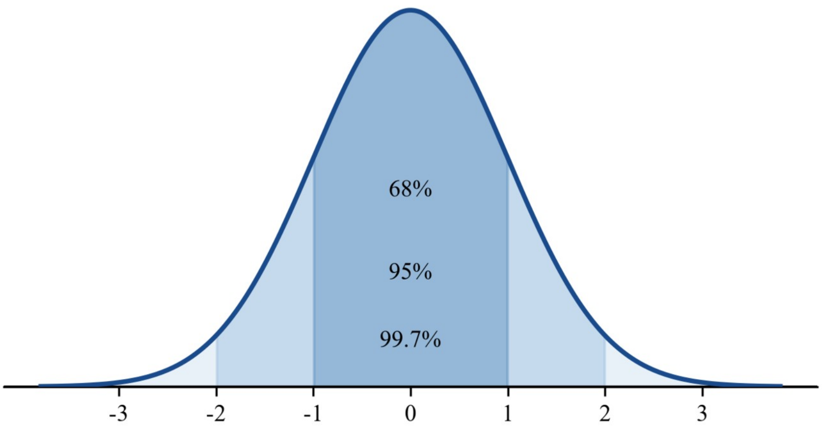
Standardisation. Any normal variable $X \sim N(\mu, \sigma^2)$ can be converted to the standard normal by

$$Z = \frac{X - \mu}{\sigma}.$$

The value z is the z**-score** (or standard score): it measures how many standard deviations x lies from the mean. A positive z is above the mean, a negative z below it. To go back, $x = \mu + z\sigma$.

The 68–95–99.7% rule. For any normal distribution, approximately:

- **68%** of values lie within **one** standard deviation of the mean ($\mu - \sigma$ to $\mu + \sigma$);
- **95%** lie within **two** standard deviations ($\mu - 2\sigma$ to $\mu + 2\sigma$);
- **99.7%** lie within **three** standard deviations ($\mu - 3\sigma$ to $\mu + 3\sigma$).



The 68–95–99.7% rule for the standard normal

Consequences (using symmetry). Because the curve is symmetric, the central percentages split evenly on each side. The most useful values are:

Region	Probability
$Pr(X < \mu)$	0.5
$Pr(\mu - \sigma < X < \mu + \sigma)$	0.68
$Pr(\mu < X < \mu + \sigma)$	0.34
$Pr(X > \mu + \sigma) = Pr(X < \mu - \sigma)$	0.16
$Pr(X < \mu + \sigma)$	0.84
$Pr(\mu + \sigma < X < \mu + 2\sigma)$	0.135
$Pr(X > \mu + 2\sigma) = Pr(X < \mu - 2\sigma)$	0.025
$Pr(X < \mu + 2\sigma)$	0.975

Region	Probability
$Pr(X > \mu + 3\sigma)$	0.0015

Combining these gives any probability whose boundaries fall on whole numbers of standard deviations — for example $Pr(\mu - \sigma < X < \mu + 2\sigma) = 0.34 + 0.475 = 0.815$.

Worked examples

Example 1 — scaffolded

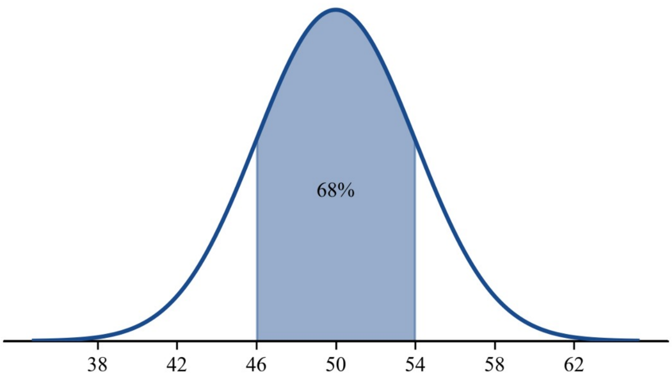
The heights of a large group have distribution $X \sim N(100, 15^2)$. Find the z-score of each value and interpret it: (a) $x = 115$; (b) $x = 70$; (c) $x = 130$.

Working	Comment
Here $\mu = 100$, $\sigma = 15$, and $z = \dfrac{x - \mu}{\sigma}$.	Identify the mean and standard deviation.
(a) $z = \dfrac{115 - 100}{15} = 1$.	115 is **1 standard deviation above** the mean.
(b) $z = \dfrac{70 - 100}{15} = -2$.	70 is **2 standard deviations below** the mean.
(c) $z = \dfrac{130 - 100}{15} = 2$.	130 is **2 standard deviations above** the mean.

Example 2 — scaffolded

For $X \sim N(50, 4^2)$, use the 68–95–99.7% rule to find: (a) $Pr(46 < X < 54)$; (b) $Pr(X > 58)$; (c) $Pr(42 < X < 58)$.

Working	Comment
$\mu = 50$, $\sigma = 4$.	Note the boundaries in terms of σ.
(a) $46 = \mu - \sigma$ and $54 = \mu + \sigma$, so $Pr(46 < X < 54) = 0.68$.	Within one standard deviation.
(b) $58 = \mu + 2\sigma$, so $Pr(X > 58) = Pr(X > \mu + 2\sigma) = 0.025$.	Upper tail beyond two standard deviations.
(c) $42 = \mu - 2\sigma$, $58 = \mu + 2\sigma$, so $Pr(42 < X < 58) = 0.95$.	Within two standard deviations.



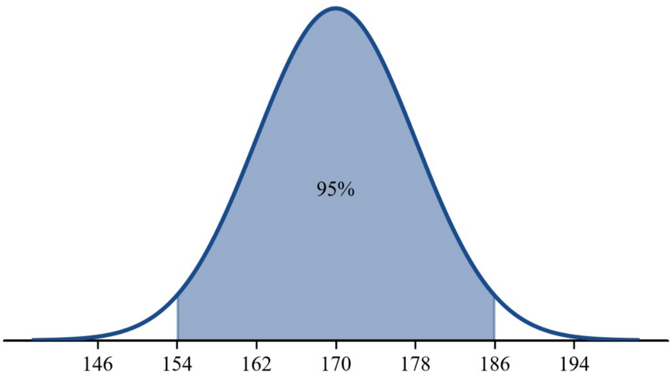
Example 3 — unscaffolded

The lengths (mm) of a component are $X \sim N(170, 8^2)$. Find $Pr(154 < X < 186)$ and $Pr(X < 162)$.

$\mu = 170$, $\sigma = 8$. Now $154 = 170 - 2(8) = \mu - 2\sigma$ and $186 = 170 + 2(8) = \mu + 2\sigma$, so $Pr(154 < X < 186) = 0.95$.

Also $162 = 170 - 8 = \mu - \sigma$, so $Pr(X < 162) = Pr(X < \mu - \sigma) = 0.16$.

$\therefore Pr(154 < X < 186) = 0.95$ and $Pr(X < 162) = 0.16$.

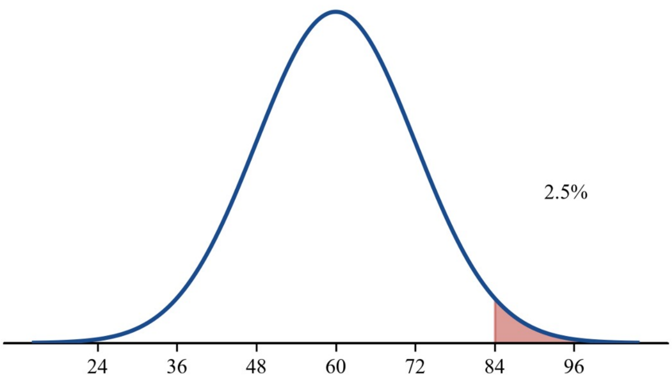


Example 4 — unscaffolded

Exam marks are $X \sim N(60, 12^2)$. The top 2.5% of students receive a distinction. Find the lowest mark that earns a distinction.

The top 2.5% satisfies $Pr(X > x) = 0.025$. Since $Pr(X > \mu + 2\sigma) = 0.025$, the cut-off is at $x = \mu + 2\sigma = 60 + 2(12) = 84$.

$\therefore$ a mark of 84 (or above) earns a distinction.



Practice questions

Scaffolded

Q1. Scores are $X \sim N(500, 100^2)$. (a) Find the z-score of $x = 600$. (b) Find the z-score of $x = 300$. (c) Find the score with a z-score of 1.5. (d) Interpret the z-score in part (a).

Q2. For $X \sim N(25, 3^2)$, use the 68–95–99.7% rule to find: (a) $Pr(22 < X < 28)$; (b) $Pr(19 < X < 31)$; (c) $Pr(X > 34)$.

Q3. For $X \sim N(1000, 50^2)$, find: (a) $Pr(950 < X < 1050)$; (b) $Pr(X < 900)$; (c) $Pr(850 < X < 1150)$.

Q4. For $X \sim N(20, 5^2)$, find the z-score of: (a) $x = 30$; (b) $x = 15$; (c) $x = 35$.

Q5. For $X \sim N(80, 10^2)$, find: (a) $Pr(80 < X < 90)$; (b) $Pr(90 < X < 100)$; (c) $Pr(X > 100)$.

Q6. For $X \sim N(200, 20^2)$: (a) find the value exceeded by 16% of the distribution; (b) find the value exceeded by 2.5% of the distribution; (c) between which two values does the middle 68% lie?

Unscaffolded

Q7. For $X \sim N(70, 6^2)$, find the z-scores of $x = 82$ and $x = 64$.

Q8. For $X \sim N(150, 25^2)$, find $Pr(125 < X < 175)$ and $Pr(X > 200)$.

Q9. For $X \sim N(40, 5^2)$, find $Pr(30 < X < 50)$ and $Pr(X < 35)$.

Q10. For $X \sim N(12, 2^2)$, find $Pr(X > 18)$ and $Pr(8 < X < 16)$.

Q11. Student A scored 75 on a test with $X \sim N(60, 10^2)$; student B scored 82 on a different test with $Y \sim N(70, 8^2)$. By comparing z-scores, determine who performed better relative to their group.

Q12. For $X \sim N(500, 80^2)$: (a) find the value with a z-score of -1.5; (b) find the value exceeded by 84% of the distribution.

Q13. For $X \sim N(100, 20^2)$, find $Pr(100 < X < 120)$, $Pr(60 < X < 80)$ and $Pr(X < 40)$.

Q14. For $X \sim N(25, 4^2)$, between which two values does the middle 95% lie, and what is $Pr(X > 29)$?

Q15. A normal variable X has $Pr(X < 50) = 0.16$ and $Pr(X > 80) = 0.16$. Find its mean μ and standard deviation σ.

Q16. For the standard normal $Z \sim N(0, 1)$, find $Pr(Z < 1)$, $Pr(Z > 2)$ and $Pr(-1 < Z < 2)$.

Q17. For $X \sim N(60, 15^2)$, find $Pr(45 < X < 90)$ and $Pr(X > 75)$.

Q18. Bags of sugar have mass $X \sim N(500, 10^2)$ grams. What percentage of bags have a mass between 490 g and 520 g, and what percentage are underweight (less than 480 g)?

Q19. A normal variable X has $Pr(X < 20) = 0.025$ and $Pr(X < 50) = 0.975$. Find μ and σ.

Q20. A normal variable X has mean 70 and $Pr(X > 85) = 0.0015$. Show that $\sigma = 5$.

Q21. The lifetimes of a brand of globe are $X \sim N(1000, 50^2)$ hours. (a) Between which two values do the middle 99.7% of lifetimes lie? (b) Find the proportion of globes that last more than 1100 hours. (c) Hence estimate how many of a batch of 2000 globes last less than 900 hours.

Answers

Q1. (a) 1 (b) -2 (c) 650 (d) 600 is one standard deviation above the mean. **Q2.** (a) 0.68 (b) 0.95 (c) 0.0015. **Q3.** (a) 0.68 (b) 0.025 (c) 0.997. **Q4.** (a) 2 (b) -1 (c) 3. **Q5.** (a) 0.34 (b) 0.135 (c) 0.025. **Q6.** (a) 220 (b) 240 (c) 180 to 220. **Q7.** 2 and -1. **Q8.** 0.68 and 0.025. **Q9.** 0.95 and 0.16. **Q10.** 0.0015 and 0.95. **Q11.** Both have $z=1.5$, so they performed **equally well** relative to their groups. **Q12.** (a) 380 (b) 420. **Q13.** 0.34, 0.135 and 0.0015. **Q14.** 17 to 33; $Pr(X>29)=0.16$. **Q15.** $\mu=65$, $\sigma=15$. **Q16.** 0.84, 0.025 and 0.815. **Q17.** 0.815 and 0.16. **Q18.** 81.5% between 490 and 520 g; 2.5% underweight. **Q19.** $\mu=35$, $\sigma=7.5$. **Q20.** $\sigma=5$. **Q21.** (a) 850 to 1150 hours (b) 0.025 (c) about 50 globes.

Fully-worked solutions

Q1. $\mu=500$, $\sigma=100$. (a) $z=\dfrac{600-500}{100}=1$. (b) $z=\dfrac{300-500}{100}=-2$. (c) $x=\mu+z\sigma=500+1.5(100)=650$. (d) A mark of 600 is one standard deviation above the mean.

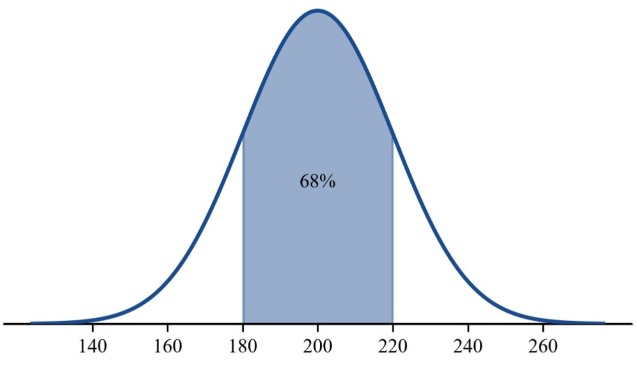
Q2. $\mu=25$, $\sigma=3$. (a) $22=\mu-\sigma$, $28=\mu+\sigma$, so $Pr(22<X<28)=0.68$. (b) $19=\mu-2\sigma$, $31=\mu+2\sigma$, so $Pr(19<X<31)=0.95$. (c) $34=\mu+3\sigma$, so $Pr(X>34)=Pr(X>\mu+3\sigma)=0.0015$.

Q3. $\mu=1000$, $\sigma=50$. (a) $950=\mu-\sigma$, $1050=\mu+\sigma$: $Pr=0.68$. (b) $900=\mu-2\sigma$: $Pr(X<900)=0.025$. (c) $850=\mu-3\sigma$, $1150=\mu+3\sigma$: $Pr=0.997$.

Q4. $\mu=20$, $\sigma=5$. (a) $z=\dfrac{30-20}{5}=2$. (b) $z=\dfrac{15-20}{5}=-1$. (c) $z=\dfrac{35-20}{5}=3$.

Q5. $\mu=80$, $\sigma=10$. (a) $80=\mu$, $90=\mu+\sigma$, so $Pr(80<X<90)=\dfrac{0.68}{2}=0.34$. (b) $90=\mu+\sigma$, $100=\mu+2\sigma$, so $Pr=\dfrac{0.95-0.68}{2}=0.135$. (c) $100=\mu+2\sigma$, so $Pr(X>100)=0.025$.

Q6. $\mu=200$, $\sigma=20$. (a) $Pr(X>x)=0.16$ at $x=\mu+\sigma=220$. (b) $Pr(X>x)=0.025$ at $x=\mu+2\sigma=240$. (c) The middle 68% lies between $\mu-\sigma=180$ and $\mu+\sigma=220$.



Q7. $\mu=70$, $\sigma=6$. $z=\dfrac{82-70}{6}=2$; $z=\dfrac{64-70}{6}=-1$.

Q8. $\mu=150$, $\sigma=25$. $125=\mu-\sigma$, $175=\mu+\sigma$: $Pr(125<X<175)=0.68$. $200=\mu+2\sigma$: $Pr(X>200)=0.025$.

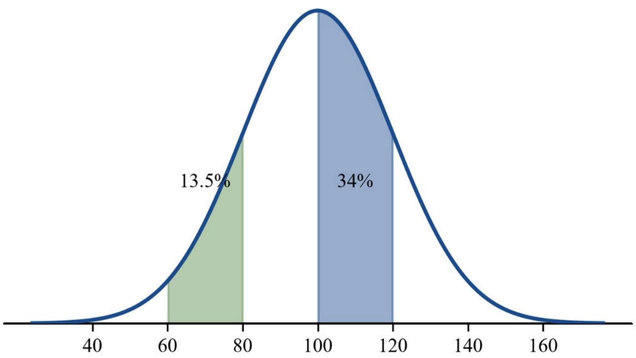
Q9. $\mu=40$, $\sigma=5$. $30=\mu-2\sigma$, $50=\mu+2\sigma$: $Pr=0.95$. $35=\mu-\sigma$: $Pr(X<35)=0.16$.

Q10. $\mu=12$, $\sigma=2$. $18=\mu+3\sigma$: $Pr(X>18)=0.0015$. $8=\mu-2\sigma$, $16=\mu+2\sigma$: $Pr(8<X<16)=0.95$.

Q11. Student A: $z=\frac{75-60}{10}=1.5$. Student B: $z=\frac{82-70}{8}=1.5$. The z-scores are equal, so relative to their respective groups the two students performed **equally well**.

Q12. $\mu=500$, $\sigma=80$. (a) $x=\mu+z\sigma=500+(-1.5)(80)=380$. (b) $Pr(X>x)=0.84$ means x is below the mean at $Pr(X>\mu-\sigma)=0.84$, so $x=\mu-\sigma=420$.

Q13. $\mu=100$, $\sigma=20$. $Pr(100<X<120)=Pr(\mu<X<\mu+\sigma)=0.34$.
$Pr(60<X<80)=Pr(\mu-2\sigma<X<\mu-\sigma)=0.135$. $40=\mu-3\sigma$, so $Pr(X<40)=0.0015$.



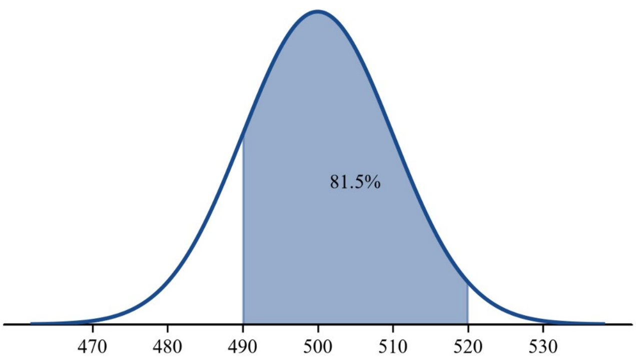
Q14. $\mu=25$, $\sigma=4$. The middle 95% lies within 2σ: from $\mu-2\sigma=17$ to $\mu+2\sigma=33$. Also $29=\mu+\sigma$, so $Pr(X>29)=0.16$.

Q15. By symmetry, $Pr(X<50)=Pr(X>80)=0.16$ means $50=\mu-\sigma$ and $80=\mu+\sigma$. Adding: $2\mu=130$, so $\mu=65$; subtracting: $2\sigma=30$, so $\sigma=15$.

Q16. For $Z\sim N(0,1)$: $Pr(Z<1)=0.84$; $Pr(Z>2)=0.025$;
$Pr(-1<Z<2)=Pr(-1<Z<0)+Pr(0<Z<2)=0.34+0.475=0.815$.

Q17. $\mu=60$, $\sigma=15$. $45=\mu-\sigma$, $90=\mu+2\sigma$, so $Pr(45<X<90)=0.34+0.475=0.815$. Also $75=\mu+\sigma$, so $Pr(X>75)=0.16$.

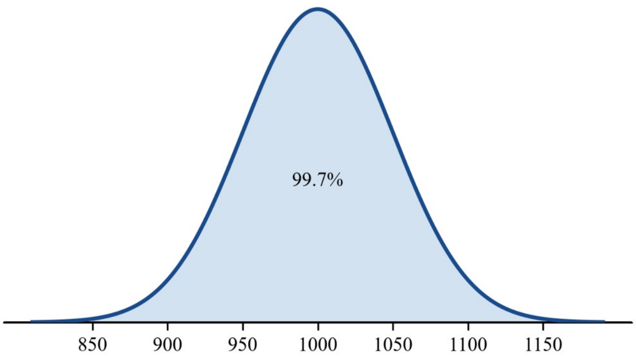
Q18. $\mu=500$, $\sigma=10$. $490=\mu-\sigma$, $520=\mu+2\sigma$, so $Pr(490<X<520)=0.34+0.475=0.815=81.5\%$.
$480=\mu-2\sigma$, so $Pr(X<480)=0.025=2.5\%$ underweight.



Q19. $Pr(X<20)=0.025=Pr(X<\mu-2\sigma)$, so $20=\mu-2\sigma$. $Pr(X<50)=0.975=Pr(X<\mu+2\sigma)$, so $50=\mu+2\sigma$. Adding: $2\mu=70$, $\mu=35$; subtracting: $4\sigma=30$, $\sigma=7.5$.

Q20. $Pr(X>85)=0.0015=Pr(X>\mu+3\sigma)$, so $85=\mu+3\sigma=70+3\sigma$. Then $3\sigma=15$, so $\sigma=5$.

Q21. $\mu=1000$, $\sigma=50$. (a) The middle 99.7% lies within 3σ: from $1000-150=850$ to $1000+150=1150$ hours. (b) $1100=\mu+2\sigma$, so $Pr(X>1100)=0.025$. (c) $900=\mu-2\sigma$, so $Pr(X<900)=0.025$; of 2000 globes, about $2000\times0.025=50$ last less than 900 hours.


99.7%
850
900
950
1000
1050
1100
1150

Theory

If a continuous random variable X is **normally distributed** with mean μ and standard deviation σ, we write $X \sim N(\mu, \sigma^2)$. Its graph is the symmetric bell-shaped curve centred on $x = \mu$, and a probability $Pr(a \leq X \leq b)$ is the **area** under the curve between $x = a$ and $x = b$.

Because there is no elementary antiderivative for the normal density, these areas are found with **technology** (a CAS calculator) — this is a technology-active skill. Using the CAS command $normalcdf(\text{lower}, \text{upper}, \mu, \sigma)$:

- $Pr(X < a) = normalcdf(-\infty, a, \mu, \sigma)$;
- $Pr(X > a) = normalcdf(a, \infty, \mu, \sigma)$;
- $Pr(a < X < b) = normalcdf(a, b, \mu, \sigma)$.

Three facts often let you find a probability more quickly, or check a CAS answer:

- **Symmetry.** The mean splits the area in half: $Pr(X < \mu) = Pr(X > \mu) = 0.5$. Also $Pr(\mu - k < X < \mu) = Pr(\mu < X < \mu + k)$.
- **Complement.** $Pr(X > a) = 1 - Pr(X < a)$, since the total area is 1.
- **Interval.** $Pr(a < X < b) = Pr(X < b) - Pr(X < a)$.

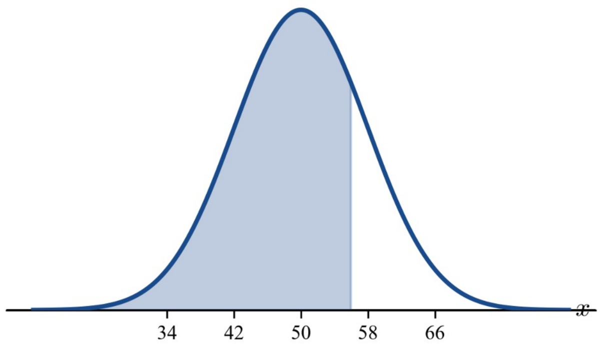
For a continuous variable $Pr(X = a) = 0$, so $<$ and $\leq$ give the same probability. Unless an exact form is available, state normal probabilities **correct to four decimal places**.

Worked examples

Example 1 — scaffolded

The lengths X mm of a machine part are normally distributed with $\mu = 50$ and $\sigma = 8$. Find $Pr(X < 56)$, correct to four decimal places.

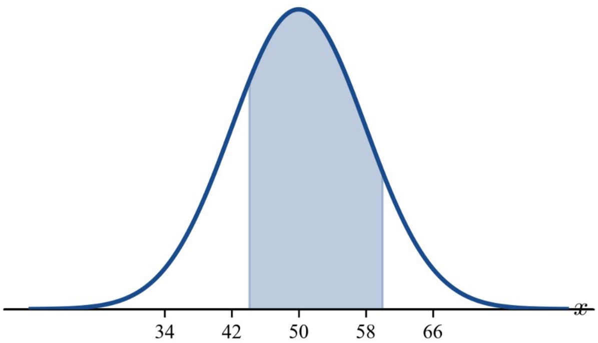
Working	Comment
$X \sim N(50, 8^2)$.	Identify $\mu = 50$, $\sigma = 8$.
Required area: $Pr(X < 56)$ — the area to the **left** of 56.	56 is above the mean, so expect a probability above 0.5.
$Pr(X < 56) = normalcdf(-\infty, 56, 50, 8)$	Set the lower limit to $-\infty$ (a very large negative value on the CAS).
$= 0.7734$.	Read the CAS value to four decimal places.



Example 2 — scaffolded

For the same variable $X \sim N(50, 8^2)$, find (a) $Pr(X > 60)$ and (b) $Pr(44 < X < 60)$.

Working	Comment
(a) $Pr(X > 60) = normalcdf(60, \infty, 50, 8) = 0.1056.$	The area to the **right** of 60; equivalently $1 - Pr(X < 60)$.
(b) $Pr(44 < X < 60) = normalcdf(44, 60, 50, 8)$ $= 0.6677.$	The area **between** 44 and 60. Equivalently $Pr(X < 60) - Pr(X < 44)$.



Example 3 — unscaffolded

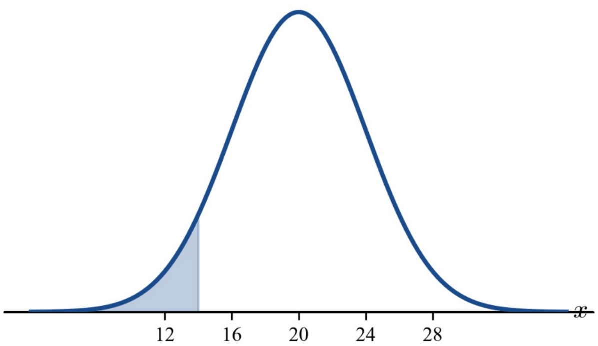
The mass X kg of newborn lambs is $X \sim N(20, 4^2)$. Find $Pr(X < 14)$ and $Pr(X > 26)$, and comment.

$Pr(X < 14) = normalcdf(-\infty, 14, 20, 4) = 0.0668.$

$Pr(X > 26) = normalcdf(26, \infty, 20, 4) = 0.0668.$

The values 14 and 26 are each $6 = 3/2\,\sigma$ from the mean 20, on opposite sides, so by the **symmetry** of the normal curve the two tail areas are equal.

$\therefore Pr(X < 14) = Pr(X > 26) = 0.0668.$



Example 4 — unscaffolded

IQ scores are modelled by $X \sim N(100, 15^2)$. Given that a person has an IQ above 100, find the probability that it is above 115.

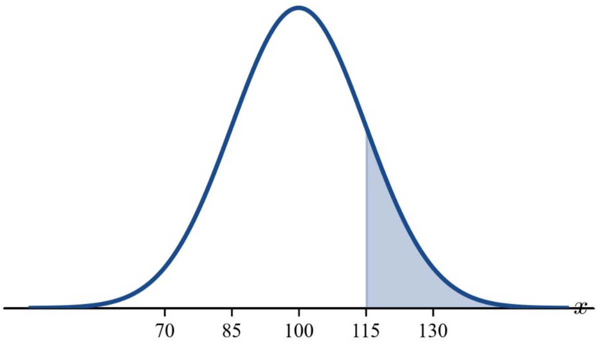
This is the conditional probability $Pr(X > 115 / X > 100)$. Since $\{X > 115\} \subseteq \{X > 100\}$,

$$Pr(X > 115 / X > 100) = \frac{Pr(X > 115)}{Pr(X > 100)}.$$

$Pr(X>115)=normalcdf(115,\infty,100,15)=0.158655$, and $Pr(X>100)=0.5$ (symmetry).

$$Pr(X>115 / X>100)=\frac{0.158655}{0.5}=0.3173.$$

$\therefore$ the conditional probability is 0.3173.



Practice questions

Give all probabilities correct to four decimal places.

Scaffolded

Q1. $X\sim N(70,10^2)$. Find: (a) $Pr(X<85)$; (b) $Pr(X>60)$; (c) $Pr(65<X<80)$.

Q2. $X\sim N(25,5^2)$. Find: (a) $Pr(X<25)$ (use symmetry — no CAS needed); (b) $Pr(X>35)$; (c) $Pr(20<X<30)$.

Q3. $Z\sim N(0,1^2)$ (the standard normal). Find: (a) $Pr(Z<1.5)$; (b) $Pr(Z>-1)$; (c) $Pr(-1<Z<2)$.

Q4. The contents X mL of a drink bottle are $X\sim N(500,50^2)$. Find: (a) $Pr(X>600)$; (b) $Pr(450<X<550)$; (c) hence write down $Pr(X<450 \text{ or } X>550)$.

Q5. $X\sim N(12,2^2)$. Find: (a) $Pr(X>12)$; (b) $Pr(10<X<14)$; (c) what does the answer to (b) tell you about the proportion of values within one standard deviation of the mean?

Q6. The diameters X cm of ball bearings are $X\sim N(1.6,0.1^2)$. Given that a bearing has diameter greater than 1.6 cm, find the probability that it is greater than 1.75 cm.

Unscaffolded

State each probability correct to four decimal places.

Q7. $X\sim N(60,12^2)$. Find $Pr(X<75)$.

Q8. $X\sim N(60,12^2)$. Find $Pr(X>80)$.

Q9. $X\sim N(60,12^2)$. Find $Pr(50<X<78)$.

Q10. The heights X cm of adult males are $X\sim N(170,8^2)$. Find the probability that a randomly chosen man is taller than 180 cm.

Q11. For the heights in **Q10**, find $Pr(160<X<175)$.

Q12. The masses X kg of bags of sugar are $X \sim N(3.2, 0.4^2)$. Find the probability that a bag has mass less than 3 kg.

Q13. Battery lifetimes X hours are $X \sim N(500, 75^2)$. Find $Pr(X > 620)$.

Q14. Exam marks X are $X \sim N(72, 9^2)$. A mark below 50 is a fail. Find the probability that a randomly chosen student fails.

Q15. For the marks in **Q14**, find $Pr(65 < X < 90)$.

Q16. $X \sim N(250, 20^2)$. Find $Pr(X > 300)$.

Q17. $Z \sim N(0, 1^2)$. Find $Pr(Z < -0.5)$.

Q18. $X \sim N(20, 3^2)$. Find $Pr(17 < X < 26)$.

Q19. Waiting times X minutes are $X \sim N(15, 2.5^2)$. Given that a wait is longer than 15 minutes, find the probability that it is longer than 18 minutes.

Q20. $X \sim N(100, 15^2)$. Find $Pr(85 < X < 130)$.

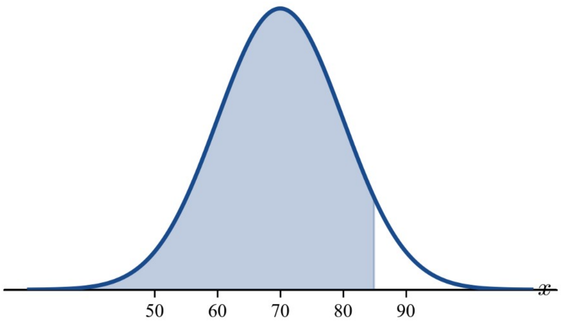
Q21. $X \sim N(50, 10^2)$. Given that $X > 60$, find the probability that $X > 70$, correct to four decimal places.

Answers

Q1. (a) 0.9332 (b) 0.8413 (c) 0.5328. **Q2.** (a) 0.5 (b) 0.0228 (c) 0.6827. **Q3.** (a) 0.9332 (b) 0.8413 (c) 0.8186. **Q4.** (a) 0.0228 (b) 0.6827 (c) 0.3173. **Q5.** (a) 0.5 (b) 0.6827 (c) about 68.27 % of values lie within one standard deviation of the mean. **Q6.** 0.1336. **Q7.** 0.8944. **Q8.** 0.0478. **Q9.** 0.7309. **Q10.** 0.1056. **Q11.** 0.6284. **Q12.** 0.3085 . **Q13.** 0.0548. **Q14.** 0.0073. **Q15.** 0.7589. **Q16.** 0.0062. **Q17.** 0.3085. **Q18.** 0.8186. **Q19.** 0.2301. **Q20.** 0.7377. **Q21.** 0.1434.

Fully-worked solutions

Q1. $X \sim N(70, 10^2)$. (a) $Pr(X<85) = normalcdf(-\infty, 85, 70, 10) = 0.9332$. (b) $Pr(X>60) = normalcdf(60, \infty, 70, 10) = 0.8413$. (c) $Pr(65<X<80) = normalcdf(65, 80, 70, 10) = 0.5328$.



Q2. $X \sim N(25, 5^2)$. (a) By symmetry about the mean, $Pr(X<25) = 0.5$. (b) $Pr(X>35) = normalcdf(35, \infty, 25, 5) = 0.0228$. (c) $Pr(20<X<30) = normalcdf(20, 30, 25, 5) = 0.6827$ (20 and 30 are each one standard deviation from the mean).

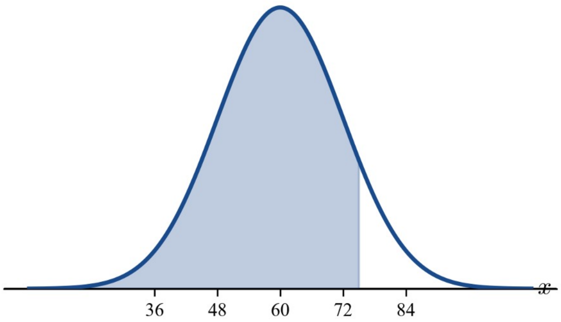
Q3. $Z \sim N(0, 1^2)$. (a) $Pr(Z<1.5) = 0.9332$. (b) $Pr(Z>-1) = 0.8413$. (c) $Pr(-1<Z<2) = normalcdf(-1, 2, 0, 1) = 0.8186$ (equivalently $Pr(Z<2) - Pr(Z<-1)$).

Q4. $X \sim N(500, 50^2)$. (a) $Pr(X>600) = normalcdf(600, \infty, 500, 50) = 0.0228$. (b) $Pr(450<X<550) = normalcdf(450, 550, 500, 50) = 0.6827$. (c) The remaining area is $Pr(X<450 \text{ or } X>550) = 1 - 0.6827 = 0.3173$.

Q5. $X \sim N(12, 2^2)$. (a) $Pr(X>12) = 0.5$ (symmetry). (b) $Pr(10<X<14) = normalcdf(10, 14, 12, 2) = 0.6827$. (c) 10 and 14 are exactly one standard deviation either side of the mean, so about 68.27 % of the values lie within one standard deviation of the mean.

Q6. $X \sim N(1.6, 0.1^2)$. $Pr(X>1.75 / X>1.6) = \dfrac{Pr(X>1.75)}{Pr(X>1.6)}$. Now $Pr(X>1.75) = normalcdf(1.75, \infty, 1.6, 0.1) = 0.066807$ and $Pr(X>1.6) = 0.5$, so the probability is $\dfrac{0.066807}{0.5} = 0.1336$.

Q7. $Pr(X<75) = normalcdf(-\infty, 75, 60, 12) = 0.8944$.



Q8. $Pr(X>80)=normalcdf(80,\infty,60,12)=0.0478$.

Q9. $Pr(50<X<78)=normalcdf(50,78,60,12)=0.7309$.

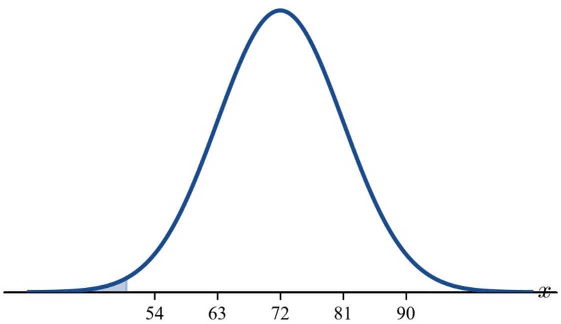
Q10. $X\sim N(170,8^2)$. $Pr(X>180)=normalcdf(180,\infty,170,8)=0.1056$.

Q11. $Pr(160<X<175)=normalcdf(160,175,170,8)=0.6284$.

Q12. $X\sim N(3.2,0.4^2)$. $Pr(X<3)=normalcdf(-\infty,3,3.2,0.4)=0.3085$.

Q13. $Pr(X>620)=normalcdf(620,\infty,500,75)=0.0548$.

Q14. $X\sim N(72,9^2)$. $Pr(X<50)=normalcdf(-\infty,50,72,9)=0.0073$.



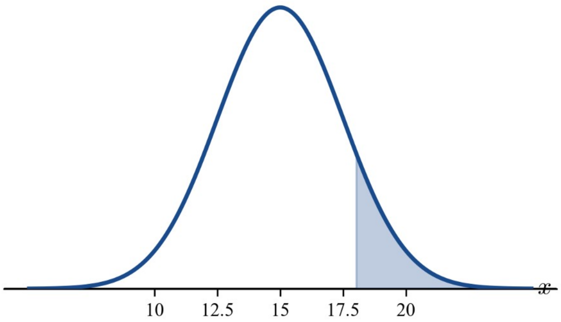
Q15. $Pr(65<X<90)=normalcdf(65,90,72,9)=0.7589$.

Q16. $Pr(X>300)=normalcdf(300,\infty,250,20)=0.0062$.

Q17. $Pr(Z<-0.5)=normalcdf(-\infty,-0.5,0,1)=0.3085$.

Q18. $Pr(17<X<26)=normalcdf(17,26,20,3)=0.8186$.

Q19. $X\sim N(15,2.5^2)$. $Pr(X>18/X>15)=\dfrac{Pr(X>18)}{Pr(X>15)}$. Here $Pr(X>18)=normalcdf(18,\infty,15,2.5)=0.115070$ and $Pr(X>15)=0.5$, so the probability is $\dfrac{0.115070}{0.5}=0.2301$.



Q20. $Pr(85 < X < 130) = normalcdf(85, 130, 100, 15) = 0.7377$.

Q21. $X \sim N(50, 10^2)$. $Pr(X > 70 / X > 60) = \dfrac{Pr(X > 70)}{Pr(X > 60)}$. Now $Pr(X > 70) = normalcdf(70, \infty, 50, 10) = 0.022750$ and $Pr(X > 60) = normalcdf(60, \infty, 50, 10) = 0.158655$, so the probability is $\dfrac{0.022750}{0.158655} = 0.1434$.

Theory

In 14C you found probabilities for a normal random variable $X \sim N(\mu, \sigma^2)$. This section solves the **inverse** problems: given a probability, find a value; or find an unknown **mean** μ or **standard deviation** σ. These are technology-active — the CAS **inverse normal** command does the numerical work — but you must set up the equation by hand.

Standardisation. If $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X - \mu}{\sigma} \sim N(0, 1)$. A probability statement about X becomes one about Z, and the boundary value x corresponds to $z = \dfrac{x - \mu}{\sigma}$.

The inverse normal. For a given probability p, the value z with $Pr(Z < z) = p$ is written $z = \Phi^{-1}(p)$ and read from the CAS *inverse normal*. Useful values (know these):

p	0.10	0.20	0.80	0.90	0.95	0.975	0.99
z	-1.2816	-0.8416	0.8416	1.2816	1.6449	1.9600	2.3263

The three inverse problems.

- **Find a value** a with $Pr(X < a) = p$: compute $a = \mu + \sigma\Phi^{-1}(p)$ (directly, with CAS inverse normal on $N(\mu, \sigma^2)$).
- **Find** μ (given σ): from $\dfrac{a - \mu}{\sigma} = z$ where $z = \Phi^{-1}(p)$, so $\mu = a - \sigma z$.
- **Find** σ (given μ): $\sigma = \dfrac{a - \mu}{z}$.
- **Find both** μ **and** σ: two probability statements give two equations $\dfrac{a_1 - \mu}{\sigma} = z_1$ and $\dfrac{a_2 - \mu}{\sigma} = z_2$; solve simultaneously.

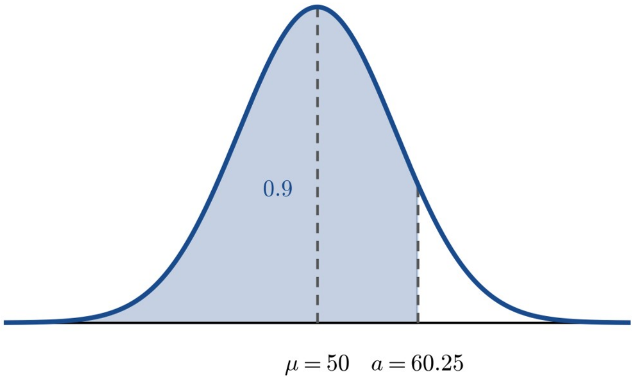
Watch the tail. For $Pr(X > a) = p$, convert to $Pr(X < a) = 1 - p$ first. Always sketch a bell curve and shade the given area — it tells you whether the required z is positive or negative and prevents sign errors.

Worked examples

Example 1 — scaffolded

$X \sim N(50, 8^2)$. Find the value a such that $Pr(X < a) = 0.9$, correct to two decimal places.

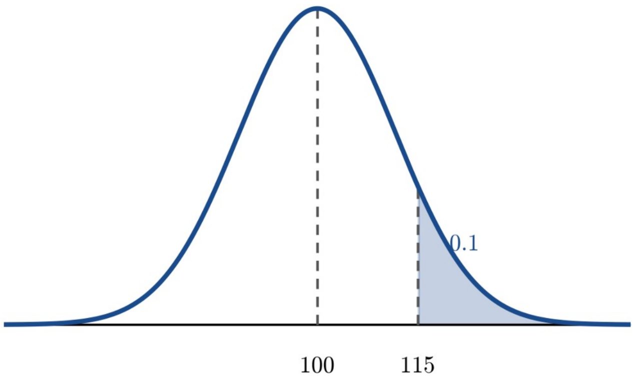
Working	Comment
$Z = \dfrac{X - 50}{8}$; require $Pr(Z < z) = 0.9$.	Standardise: the left area is 0.9.
Inverse normal: $z = \Phi^{-1}(0.9) = 1.2816$.	Read from CAS (or the table).
$\dfrac{a - 50}{8} = 1.2816 \Rightarrow a = 50 + 8(1.2816)$.	Un-standardise: $a = \mu + \sigma z$.
$a = 60.25$.	To two decimal places.



Example 2 — scaffolded

The lifetime X (hours) of a component is $X \sim N(100, \sigma^2)$. Given $Pr(X > 115) = 0.10$, find σ.

Working	Comment
$Pr(X > 115) = 0.10 \Rightarrow Pr(X < 115) = 0.90$.	Convert the right tail to a left area.
$z = \Phi^{-1}(0.90) = 1.2816$; $\dfrac{115 - 100}{\sigma} = 1.2816$.	Standardise the boundary $x = 115$.
$\sigma = \dfrac{15}{1.2816} = 11.70$.	Solve for σ.



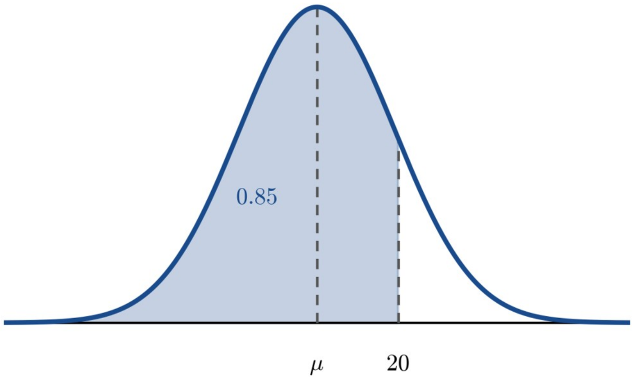
Example 3 — unscaffolded

$X \sim N(\mu, 5^2)$ and $Pr(X < 20) = 0.85$. Find μ.

Standardise: $\dfrac{20 - \mu}{5} = \Phi^{-1}(0.85) = 1.0364$.

So $20 - \mu = 5(1.0364) = 5.182$, giving $\mu = 20 - 5.182 = 14.82$.

$\therefore \mu = 14.82$.



Example 4 — unscaffolded

$X \sim N(\mu, \sigma^2)$ with $Pr(X<60)=0.2$ and $Pr(X<80)=0.9$. Find μ and σ.

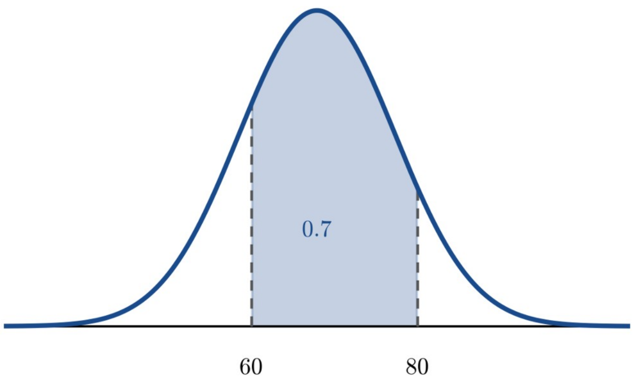
Two standardised equations, using $\Phi^{-1}(0.2)=-0.8416$ and $\Phi^{-1}(0.9)=1.2816$:

$$\frac{60-\mu}{\sigma}=-0.8416, \frac{80-\mu}{\sigma}=1.2816.$$

Subtracting: $\frac{80-60}{\sigma}=1.2816-(-0.8416)=2.1232$, so $\sigma=\frac{20}{2.1232}=9.42$.

Then $\mu=60+0.8416\sigma=60+0.8416(9.42)=67.93$.

$\therefore \mu=67.93, \sigma=9.42$.



Practice questions

Scaffolded

Q1. $X \sim N(70, 10^2)$. (a) State the z-value with $Pr(Z<z)=0.95$. (b) Hence find a such that $Pr(X<a)=0.95$, to two decimal places.

Q2. $X \sim N(25, \sigma^2)$ with $Pr(X<30)=0.8$. (a) Write the standardised equation for σ. (b) Find σ.

Q3. $X \sim N(\mu, 4^2)$ with $Pr(X>50)=0.05$. (a) Convert to a left-tail probability and state the z-value. (b) Find μ.

Q4. $X \sim N(500, 50^2)$. Find the value a exceeded with probability 0.9 (i.e. $Pr(X>a)=0.9$).

Q5. $X \sim N(\mu, \sigma^2)$ with $Pr(X<10)=0.1$ and $Pr(X<25)=0.7$. (a) Write the two standardised equations. (b) Solve for μ and σ.

Q6. Exam scores are $X \sim N(65, 12^2)$. The top 15% of students receive an A. Find the minimum score (to two decimal places) needed for an A.

Unscaffolded

Q7. $X \sim N(40, 6^2)$. Find a with $Pr(X < a) = 0.975$.

Q8. $X \sim N(200, \sigma^2)$ with $Pr(X > 230) = 0.025$. Find σ.

Q9. $X \sim N(\mu, 8^2)$ with $Pr(X < 100) = 0.6$. Find μ.

Q10. $X \sim N(15, \sigma^2)$ with $Pr(X < 12) = 0.2$. Find σ.

Q11. $X \sim N(\mu, \sigma^2)$ with $Pr(X < 5) = 0.15$ and $Pr(X < 9) = 0.75$. Find μ and σ.

Q12. The charge time X (minutes) of a battery is $X \sim N(50, \sigma^2)$ and $Pr(X > 60) = 0.1$. (a) Find σ. (b) Hence find $Pr(X > 45)$, to four decimal places.

Q13. For the standard normal $Z \sim N(0, 1)$, find the value a such that $Pr(-a < Z < a) = 0.9$.

Q14. $X \sim N(500, 20^2)$. Find the first and third quartiles Q_1 and Q_3, and hence the interquartile range.

Q15. The heights (cm) of a population are $X \sim N(170, 7^2)$. Find the height below which the shortest 10% lie.

Q16. $X \sim N(\mu, 5^2)$ with $Pr(X > 18) = 0.2$. Find μ.

Q17. $X \sim N(80, \sigma^2)$ with $Pr(74 < X < 86) = 0.6$. Find σ. *(Use the symmetry of the interval about the mean.)*

Q18. $X \sim N(\mu, \sigma^2)$ with $Pr(X < 30) = 0.3$ and $Pr(X > 50) = 0.1$. Find μ and σ.

Q19. A machine fills bottles with $X \sim N(\mu, 3^2)$ mL. Only 1% of bottles may contain less than 500 mL. Find the smallest mean μ (to two decimal places) that achieves this.

Q20. $X \sim N(100, 15^2)$. Find the 60th percentile.

Q21. $X \sim N(\mu, \sigma^2)$ with $Pr(X < 45) = 0.2$ and $Pr(X < 65) = 0.9$. (a) Find μ and σ. (b) Hence find $Pr(X > 70)$, to four decimal places.

Answers

Q1. (a) 1.6449 (b) 86.45 **Q2.** $\sigma=5.94$ **Q3.** $\mu=43.42$ **Q4.** $a=435.92$ **Q5.** $\mu=20.64$, $\sigma=8.31$ **Q6.** 77.44 **Q7.** 51.76 **Q8.** $\sigma=15.31$ **Q9.** $\mu=97.97$ **Q10.** $\sigma=3.56$ **Q11.** $\mu=7.42$, $\sigma=2.34$ **Q12.** (a) $\sigma=7.80$ (b) 0.7392 **Q13.** $a=1.64$ **Q14.** $Q_1=486.51$, $Q_3=513.49$, IQR $=26.98$ **Q15.** 161.03 cm **Q16.** $\mu=13.79$ **Q17.** $\sigma=7.13$ **Q18.** $\mu=35.81$, $\sigma=11.07$ **Q19.** $\mu=506.98$ mL **Q20.** 103.80 **Q21.** (a) $\mu=52.93$, $\sigma=9.42$ (b) 0.0350

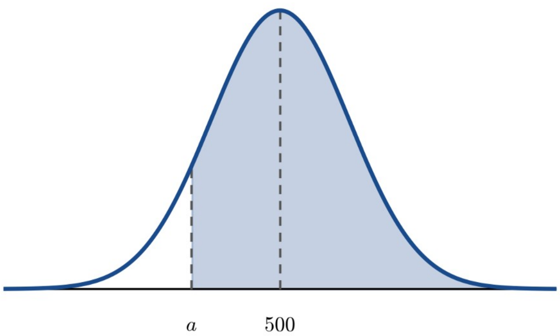
Fully-worked solutions

Q1. (a) $\Phi^{-1}(0.95)=1.6449$. (b) $a=70+10(1.6449)=86.45$.

Q2. $Pr(X<30)=0.8 \Rightarrow \dfrac{30-25}{\sigma}=\Phi^{-1}(0.8)=0.8416$, so $\sigma=\dfrac{5}{0.8416}=5.94$.

Q3. $Pr(X>50)=0.05 \Rightarrow Pr(X<50)=0.95$, $z=\Phi^{-1}(0.95)=1.6449$. Then $\dfrac{50-\mu}{4}=1.6449 \Rightarrow \mu=50-4(1.6449)=43.42$.

Q4. $Pr(X>a)=0.9 \Rightarrow Pr(X<a)=0.1$, $z=\Phi^{-1}(0.1)=-1.2816$. So $a=500+50(-1.2816)=435.92$.



Q5. $\dfrac{10-\mu}{\sigma}=\Phi^{-1}(0.1)=-1.2816$ and $\dfrac{25-\mu}{\sigma}=\Phi^{-1}(0.7)=0.5244$. Subtracting: $\dfrac{15}{\sigma}=0.5244-(-1.2816)=1.8060$, so $\sigma=8.31$; then $\mu=10+1.2816(8.31)=20.64$.

Q6. Top 15% means $Pr(X<a)=0.85$, $z=\Phi^{-1}(0.85)=1.0364$. $a=65+12(1.0364)=77.44$.

Q7. $a=40+6\Phi^{-1}(0.975)=40+6(1.9600)=51.76$.

Q8. $Pr(X>230)=0.025 \Rightarrow Pr(X<230)=0.975$, $z=1.9600$. $\dfrac{230-200}{\sigma}=1.96 \Rightarrow \sigma=\dfrac{30}{1.96}=15.31$.

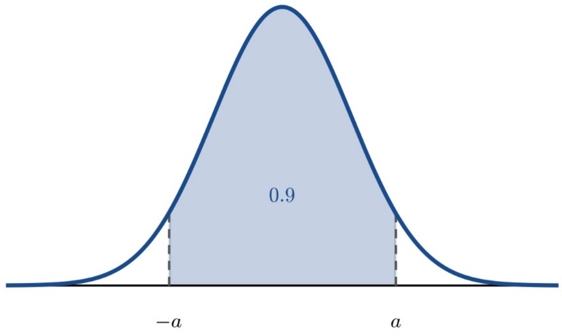
Q9. $\dfrac{100-\mu}{8}=\Phi^{-1}(0.6)=0.2533 \Rightarrow \mu=100-8(0.2533)=97.97$.

Q10. $\dfrac{12-15}{\sigma}=\Phi^{-1}(0.2)=-0.8416 \Rightarrow \sigma=\dfrac{-3}{-0.8416}=3.56$.

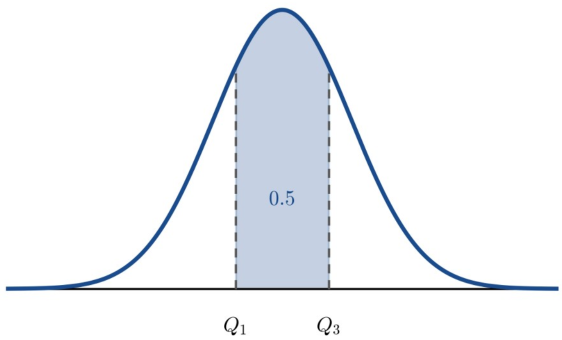
Q11. $\dfrac{5-\mu}{\sigma}=\Phi^{-1}(0.15)=-1.0364$ and $\dfrac{9-\mu}{\sigma}=\Phi^{-1}(0.75)=0.6745$. Subtracting: $\dfrac{4}{\sigma}=1.7109 \Rightarrow \sigma=2.34$; $\mu=5+1.0364(2.34)=7.42$.

Q12. (a) $Pr(X>60)=0.1 \Rightarrow Pr(X<60)=0.9$, $\dfrac{60-50}{\sigma}=1.2816 \Rightarrow \sigma=7.80$. (b) $Pr(X>45)=Pr\left(Z>\dfrac{45-50}{7.80}\right)=Pr(Z>-0.641)=0.7392$.

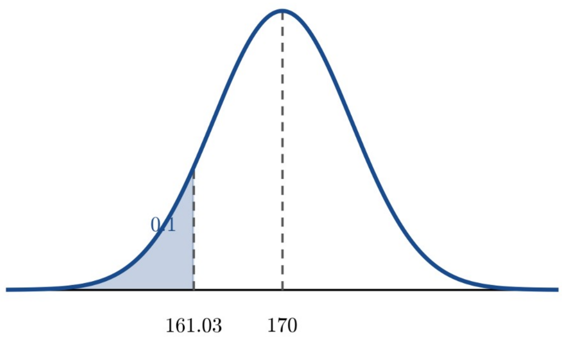
Q13. By symmetry $Pr(Z<a)=0.5+\dfrac{0.9}{2}=0.95$, so $a=\Phi^{-1}(0.95)=1.64$.



Q14. Q_1: $Pr(X<Q_1)=0.25$, $z=-0.6745$, $Q_1=500+20(-0.6745)=486.51$. Q_3: $z=0.6745$, $Q_3=513.49$. IQR $=513.49-486.51=26.98$.



Q15. $Pr(X<a)=0.1$, $z=-1.2816$, $a=170+7(-1.2816)=161.03$ cm.



Q16. $Pr(X>18)=0.2 \Rightarrow Pr(X<18)=0.8$, $z=0.8416$. $\dfrac{18-\mu}{5}=0.8416 \Rightarrow \mu=18-5(0.8416)=13.79$.

Q17. The interval $74<X<86$ is symmetric about $\mu=80$, so $Pr(X<86)=0.5+\dfrac{0.6}{2}=0.8$, $z=0.8416$. $\dfrac{86-80}{\sigma}=0.8416 \Rightarrow \sigma=\dfrac{6}{0.8416}=7.13$.

Q18. $\frac{30-\mu}{\sigma}=\Phi^{-1}(0.3)=-0.5244$; $Pr(X>50)=0.1 \Rightarrow Pr(X<50)=0.9$, $\frac{50-\mu}{\sigma}=1.2816$. Subtracting: $\frac{20}{\sigma}=1.8060 \Rightarrow \sigma=11.07$; $\mu=30+0.5244(11.07)=35.81$.

Q19. $Pr(X<500)=0.01$, $z=\Phi^{-1}(0.01)=-2.3263$. $\frac{500-\mu}{3}=-2.3263 \Rightarrow \mu=500+3(2.3263)=506.98$ mL.

Q20. The 60th percentile is a with $Pr(X<a)=0.6$: $a=100+15(0.2533)=103.80$.

Q21. (a) $\frac{45-\mu}{\sigma}=\Phi^{-1}(0.2)=-0.8416$ and $\frac{65-\mu}{\sigma}=\Phi^{-1}(0.9)=1.2816$. Subtracting: $\frac{20}{\sigma}=2.1232 \Rightarrow \sigma=9.42$; $\mu=45+0.8416(9.42)=52.93$. (b) $Pr(X>70)=Pr\left(Z>\frac{70-52.93}{9.42}\right)=Pr(Z>1.812)=0.0350$.

Chapter 14 The normal distribution — 14E The normal approximation to the binomial distribution

Theory

When the number of trials n is large, the binomial distribution $X \sim \mathrm{Bi}(n, p)$ has a graph that is close to a bell shape, and it can be **approximated by a normal distribution**.

The approximating normal. If $X \sim \mathrm{Bi}(n, p)$ then X is approximately

$$X \approx N(\mu, \sigma^2), \mu = np, \sigma^2 = np(1-p),$$

that is, the approximating normal has the **same mean and the same variance** as the binomial.

When is the approximation reasonable? The rule of thumb is that **both** np and $n(1-p)$ should be reasonably large — a common guide is that each is at least 5 (better, at least 10). The approximation is best when p is close to $\frac{1}{2}$ (so the binomial is nearly symmetric) and improves as n increases.

Using the approximation. To estimate a binomial probability such as $Pr(X \le k)$:

1. Find $\mu = np$ and $\sigma^2 = np(1-p)$ (so $\sigma = \sqrt{np(1-p)}$).
2. Treat X as $N(\mu, \sigma^2)$ and find the required probability — either directly with technology, or by **standardising** $Z = \dfrac{X-\mu}{\sigma}$ and using the standard normal.

For example, $Pr(X \le k) \approx Pr\left(Z \le \dfrac{k-\mu}{\sigma}\right)$.

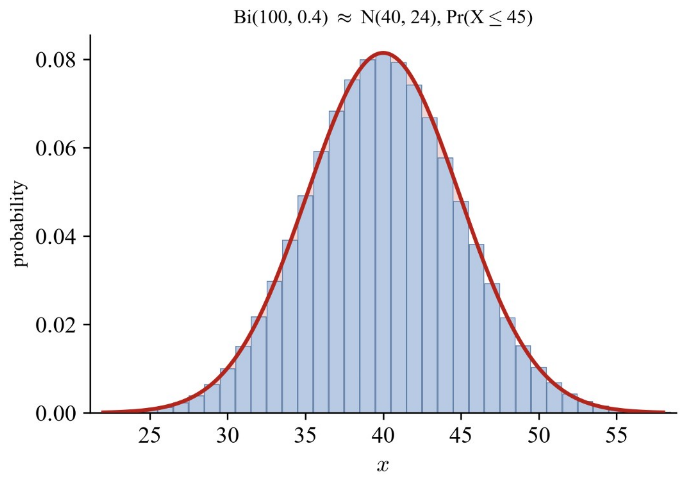
A note on accuracy. The approximate value is *close to* — but not equal to — the exact binomial value. In Mathematical Methods no continuity correction is applied; the normal approximation is used as a quick estimate, and the exact answer can always be found directly from the binomial.

Worked examples

Example 1 — scaffolded

The number of successes in 100 independent trials, each with probability 0.4 of success, is $X \sim \mathrm{Bi}(100, 0.4)$. Use a normal approximation to estimate $Pr(X \le 45)$.

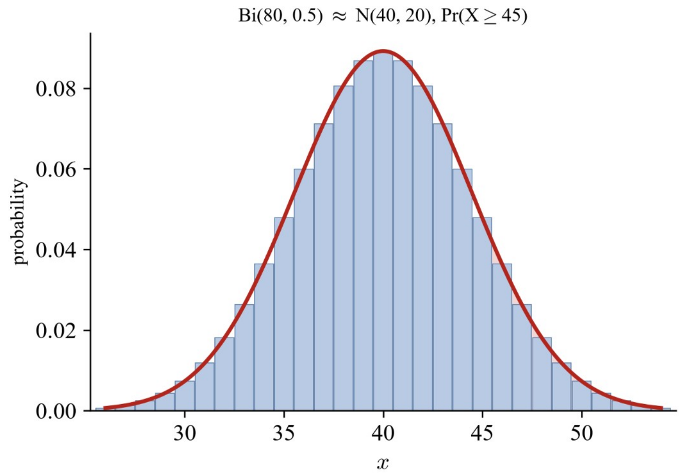
Working	Comment
$\mu = np = 100 \times 0.4 = 40$.	The approximating normal has mean np.
$\sigma^2 = np(1-p) = 100 \times 0.4 \times 0.6 = 24$, so $\sigma = 2\sqrt{6} \approx 4.899$.	…and variance $np(1-p)$.
Check: $np = 40$ and $n(1-p) = 60$ are both large, so the approximation is reasonable.	Both ≥ 10.
Approximate X by $N(40, 24)$. Standardise: $z = \dfrac{45-40}{2\sqrt{6}} \approx 1.0206$.	Convert to the standard normal.
$Pr(X \le 45) \approx Pr(Z \le 1.0206) \approx 0.8463$.	Read the standard-normal probability.
(The exact binomial value is 0.8689, so the estimate is close.)	Sense-check against the exact value.



Example 2 — scaffolded

For $X \sim \mathrm{Bi}(80, 0.5)$, use a normal approximation to estimate $Pr(X \geq 45)$.

Working	Comment
$\mu = 80 \times 0.5 = 40$; $\sigma^2 = 80 \times 0.5 \times 0.5 = 20$, so $\sigma = 2\sqrt{5} \approx 4.472$.	Same mean and variance as the binomial.
$np = n(1-p) = 40$ — both large, and $p = 0.5$, so the approximation is good.	Symmetric case.
$z = \dfrac{45-40}{2\sqrt{5}} \approx 1.1180$.	Standardise the value 45.
$Pr(X \geq 45) \approx Pr(Z \geq 1.1180) \approx 0.1318$.	Upper-tail probability.
(Exact binomial: 0.1572.)	Sense-check.



Example 3 — unscaffolded

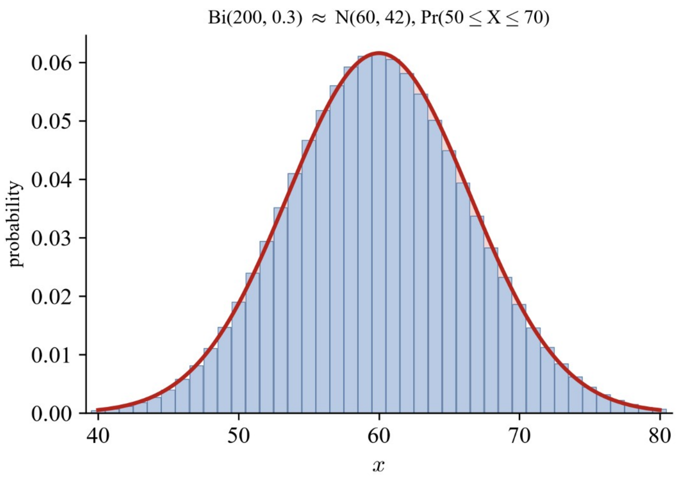
For $X \sim \mathrm{Bi}(200, 0.3)$, estimate $Pr(50 \leq X \leq 70)$ using a normal approximation.

$\mu = 200 \times 0.3 = 60$ and $\sigma^2 = 200 \times 0.3 \times 0.7 = 42$, so $\sigma = \sqrt{42} \approx 6.481$. Both $np = 60$ and $n(1-p) = 140$ are large, so $X \approx N(60, 42)$.

Standardising the endpoints: $z_1 = \dfrac{50-60}{\sqrt{42}} \approx -1.5430$ and $z_2 = \dfrac{70-60}{\sqrt{42}} \approx 1.5430$.

$Pr(50 \leq X \leq 70) \approx Pr(-1.5430 \leq Z \leq 1.5430) \approx 0.8772$.

$\therefore$ the required probability is about 0.8772 (exact binomial 0.8952).



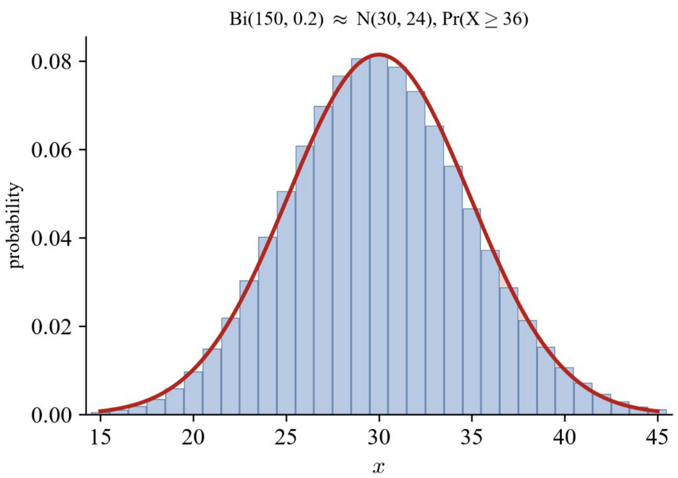
Example 4 — unscaffolded

A fair spinner shows a particular colour with probability 0.2. In 150 spins, let X be the number of times that colour appears. Estimate $Pr(X \geq 36)$ using a normal approximation.

$X \sim \text{Bi}(150, 0.2)$, so $\mu = 150 \times 0.2 = 30$ and $\sigma^2 = 150 \times 0.2 \times 0.8 = 24$, giving $\sigma = 2\sqrt{6} \approx 4.899$. Here $np = 30$ and $n(1-p) = 120$ are both large, so $X \approx N(30, 24)$.

$z = \dfrac{36-30}{2\sqrt{6}} \approx 1.2247$, so $Pr(X \geq 36) \approx Pr(Z \geq 1.2247) \approx 0.1103$.

$\therefore$ the colour appears at least 36 times with probability about 0.1103 (exact 0.1317).



Practice questions

Scaffolded

For each, give probabilities correct to four decimal places.

Q1. For $X \sim \text{Bi}(100, 0.5)$: (a) state μ and σ^2 for the approximating normal; (b) confirm the approximation is reasonable; (c) estimate $Pr(X \leq 55)$.

Q2. For $X \sim \text{Bi}(64, 0.25)$: (a) find μ and σ; (b) estimate $Pr(X \geq 20)$.

Q3. For $X \sim \text{Bi}(400, 0.5)$: (a) find μ and σ; (b) estimate $Pr(190 \leq X \leq 210)$; (c) which two standardised values are the endpoints?

Q4. For $X \sim \text{Bi}(50, 0.4)$: (a) find μ and σ^2; (b) estimate $Pr(X \leq 18)$.

Q5. For $X \sim \text{Bi}(300, 1/3)$: (a) find μ and σ^2; (b) estimate $Pr(X \geq 110)$.

Q6. For $X \sim \text{Bi}(144, 0.5)$: (a) find μ and σ; (b) estimate $Pr(66 \leq X \leq 81)$.

Unscaffolded

Use a normal approximation to estimate each probability, correct to four decimal places, and state the mean and variance of the approximating normal.

Q7. $X \sim \text{Bi}(100, 0.3)$; find $Pr(X \leq 35)$.

Q8. $X \sim \text{Bi}(250, 0.4)$; find $Pr(X \geq 110)$.

Q9. In a large batch, 50% of components pass a test. For a sample of 324 components, let X be the number that pass. Estimate $Pr(X \leq 174)$, and comment on whether the normal approximation is suitable here.

Answers

Q1. (a) $\mu = 50$, $\sigma^2 = 25$; (c) 0.8413. **Q2.** (a) $\mu = 16$, $\sigma = 2\sqrt{3} \approx 3.4641$; (b) 0.1241. **Q3.** (a) $\mu = 200$, $\sigma = 10$; (b) 0.6827; (c) $z = \pm 1$. **Q4.** (a) $\mu = 20$, $\sigma^2 = 12$; (b) 0.2819. **Q5.** (a) $\mu = 100$, $\sigma^2 = \frac{200}{3}$; (b) 0.1103. **Q6.** (a) $\mu = 72$, $\sigma = 6$; (b) 0.7745. **Q7.** $N(30, 21)$; 0.8624. **Q8.** $N(100, 60)$; 0.0984. **Q9.** $N(162, 81)$; 0.9088; suitable ($np = n(1-p) = 162$, both large, $p = 0.5$).

Fully-worked solutions

Q1. $\mu = 100(0.5) = 50$, $\sigma^2 = 100(0.5)(0.5) = 25$, $\sigma = 5$. Both np and $n(1-p)$ equal 50, so the approximation is reasonable. $z = \frac{55-50}{5} = 1$, so $Pr(X \le 55) \approx Pr(Z \le 1) \approx 0.8413$. (Exact 0.8644.)

Q2. $\mu = 64(0.25) = 16$, $\sigma^2 = 64(0.25)(0.75) = 12$, $\sigma = 2\sqrt{3} \approx 3.4641$. $z = \frac{20-16}{2\sqrt{3}} \approx 1.1547$, so $Pr(X \ge 20) \approx Pr(Z \ge 1.1547) \approx 0.1241$. (Exact 0.1561.)

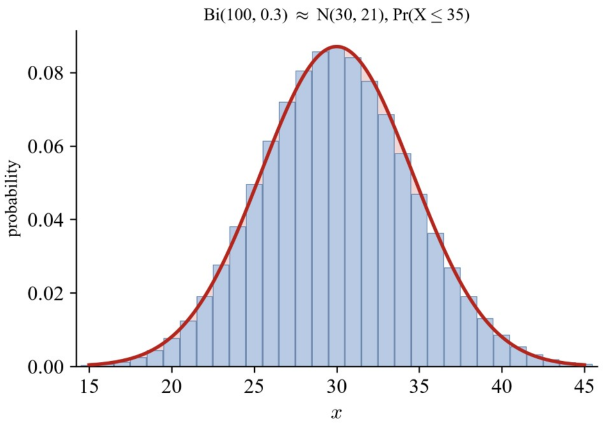
Q3. $\mu = 200$, $\sigma^2 = 400(0.5)(0.5) = 100$, $\sigma = 10$. Endpoints standardise to $z = \frac{190-200}{10} = -1$ and $z = \frac{210-200}{10} = 1$, so $Pr(190 \le X \le 210) \approx Pr(-1 \le Z \le 1) \approx 0.6827$. (Exact 0.7063.)

Q4. $\mu = 20$, $\sigma^2 = 50(0.4)(0.6) = 12$, $\sigma = 2\sqrt{3} \approx 3.4641$. $z = \frac{18-20}{2\sqrt{3}} \approx -0.5774$, so $Pr(X \le 18) \approx Pr(Z \le -0.5774) \approx 0.2819$. (Exact 0.3356.)

Q5. $\mu = 100$, $\sigma^2 = 300 \cdot 1/3 \cdot 2/3 = \frac{200}{3}$, $\sigma \approx 8.165$. $z = \frac{110-100}{8.165} \approx 1.2247$, so $Pr(X \ge 110) \approx Pr(Z \ge 1.2247) \approx 0.1103$. (Exact 0.1228.)

Q6. $\mu = 72$, $\sigma^2 = 144(0.5)(0.5) = 36$, $\sigma = 6$. Endpoints: $z_1 = \frac{66-72}{6} = -1$, $z_2 = \frac{81-72}{6} = 1.5$, so $Pr(66 \le X \le 81) \approx Pr(-1 \le Z \le 1.5) \approx 0.7745$. (Exact 0.8042.)

Q7. $\mu = 30$, $\sigma^2 = 100(0.3)(0.7) = 21$, $\sigma \approx 4.5826$. $z = \frac{35-30}{\sqrt{21}} \approx 1.0911$, so $Pr(X \le 35) \approx 0.8624$. (Exact 0.8839.)



Q8. $\mu = 100$, $\sigma^2 = 250(0.4)(0.6) = 60$, $\sigma \approx 7.746$. $z = \frac{110-100}{\sqrt{60}} \approx 1.2910$, so $Pr(X \ge 110) \approx 0.0984$. (Exact 0.1104.)

Q9. $\mu = 324(0.5) = 162$, $\sigma^2 = 324(0.5)(0.5) = 81$, $\sigma = 9$. Here $np = n(1-p) = 162$ are both large and $p = 0.5$, so the normal approximation is very suitable. $z = \dfrac{174 - 162}{9} \approx 1.3333$, so $Pr(X \leq 174) \approx 0.9088$. (Exact 0.9176.)