Mathematical Methods Units 3 & 4

Chapter 13 — Continuous probability distributions

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Contents

Chapter 13 Continuous probability distributions — 13A Continuous random variables

Theory

A **continuous random variable** X can take any value in an interval of the real numbers (for example a time, length or mass). Its behaviour is described by a **probability density function** (pdf) f, and probabilities are found as **areas** under the graph of f.

Conditions for a pdf. A function f is a valid probability density function if

1. $f(x) \geq 0$ for all x (the graph never goes below the x-axis); and
2. the total area is 1: $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Usually f is given by a rule on a **support** interval $[a, b]$ and is 0 elsewhere, so condition 2 becomes $\int_a^b f(x)\,dx = 1$.

Probability is area. For $a \leq b$,

$$Pr(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

- Because a single point has no area, $Pr(X = c) = 0$. Hence ≤ and < give the **same** probability: $Pr(a \leq X \leq b) = Pr(a < X < b)$.
- $Pr(X \leq b) = \int_a^b f(x)\,dx$ and $Pr(X \geq a) = \int_a^b f(x)\,dx$ are computed over the part of the support to the left / right.

Finding an unknown constant. If f contains an unknown k, set the total area equal to 1 and solve: $\int_a^b f(x)\,dx = 1$.

Conditional probability. As for any events,

$$Pr(X \leq b \,/\, X \geq a) = \frac{Pr(a \leq X \leq b)}{Pr(X \geq a)}.$$

To work with a pdf: (1) if there is an unknown constant, use total area $= 1$ to find it; (2) write the required probability as a definite integral of f; (3) evaluate, giving an exact value.

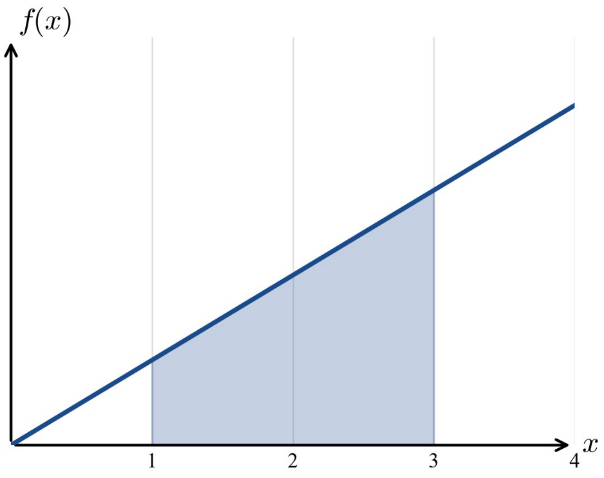
Worked examples

Example 1 — scaffolded

The continuous random variable X has pdf $f(x) = kx$ for $0 \leq x \leq 4$, and $f(x) = 0$ elsewhere. Find k, then find $Pr(1 \leq X \leq 3)$.

Working	Comment
Total area $= 1$: $\int_0^4 kx\,dx = 1$.	Use condition 2 for a pdf.
$k\left[\frac{x^2}{2}\right]_0^4 = k \cdot 8 = 1$, so $k = \frac{1}{8}$.	Evaluate the integral and solve for k.

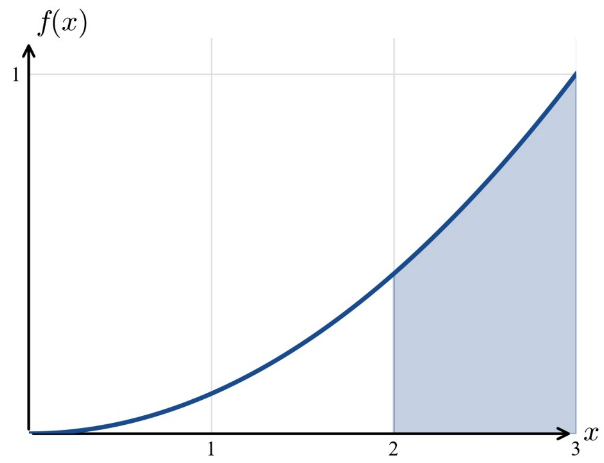
Working	Comment
$Pr(1 \le X \le 3) = \int_1^3 \frac{x}{8}\,dx = \left[\frac{x^2}{16}\right]_1^3.$	Probability is the area from 1 to 3.
$= \frac{9}{16} - \frac{1}{16} = \frac{8}{16} = \frac{1}{2}.$	$\therefore Pr(1 \le X \le 3) = \frac{1}{2}.$



Example 2 — scaffolded

The pdf of X is $f(x) = k\,x^2$ for $0 \le x \le 3$ (and 0 elsewhere). Find k, then find $Pr(2 \le X \le 3)$ and $Pr(1 \le X \le 2)$.

Working	Comment
$\int_0^3 k\,x^2\,dx = k\left[\frac{x^3}{3}\right]_0^3 = k \cdot 9 = 1$, so $k = \frac{1}{9}$.	Total area $= 1$.
$Pr(2 \le X \le 3) = \int_2^3 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_2^3 = \frac{27-8}{27} = \frac{19}{27}.$	Area from 2 to 3.
$Pr(1 \le X \le 2) = \left[\frac{x^3}{27}\right]_1^2 = \frac{8-1}{27} = \frac{7}{27}.$	Area from 1 to 2.



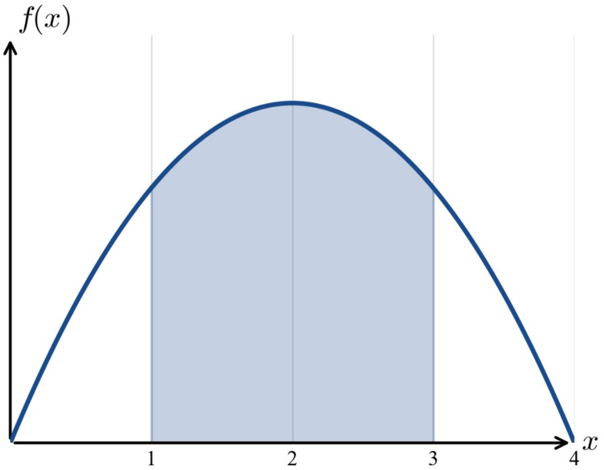
Example 3 — unscaffolded

The pdf of X is $f(x) = k\,x(4-x)$ for $0 \le x \le 4$ (and 0 elsewhere). Find k and $Pr(1 \le X \le 3)$.

Total area $= 1$: $\int_0^4 k(4x - x^2)\,dx = k\left[2x^2 - \frac{x^3}{3}\right]_0^4 = k\left(32 - \frac{64}{3}\right) = k \cdot \frac{32}{3} = 1$, so $k = \frac{3}{32}$.

$Pr(1 \le X \le 3) = \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_1^3 = \frac{3}{32}\left(9 - \frac{5}{3}\right) = \frac{3}{32} \cdot \frac{22}{3} = \frac{22}{32} = \frac{11}{16}.$

$\therefore k = \frac{3}{32}$ and $Pr(1 \le X \le 3) = \frac{11}{16}.$



Example 4 — unscaffolded

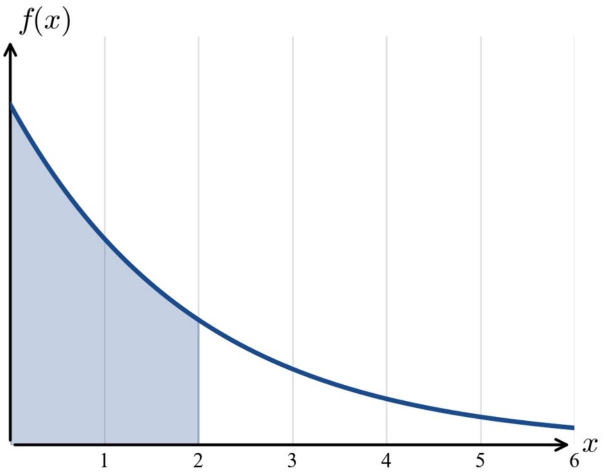
The pdf of X is $f(x) = k e^{-x/2}$ for $x \ge 0$ (and 0 elsewhere). Find k, then find $Pr(X \le 2)$ and $Pr(X > 1)$.

Total area $= 1$: $\int_0^\infty k e^{-x/2} dx = k\left[-2e^{-x/2}\right]_0^\infty = k(0+2) = 2k = 1$, so $k = \frac{1}{2}$.

$Pr(X \le 2) = \int_0^2 \frac{1}{2} e^{-x/2} dx = \left[-e^{-x/2}\right]_0^2 = 1 - e^{-1}.$

$Pr(X > 1) = \int_1^\infty \frac{1}{2} e^{-x/2} dx = \left[-e^{-x/2}\right]_1^\infty = e^{-1/2}.$

$\therefore k = \frac{1}{2}$, $Pr(X \le 2) = 1 - e^{-1}$ and $Pr(X > 1) = e^{-1/2}$.



Practice questions

Scaffolded

Q1. X has pdf $f(x) = kx$ for $0 \le x \le 2$ (and 0 elsewhere). (a) Find k; (b) find $Pr(X \le 1)$; (c) find $Pr(X \ge 1)$.

Q2. X has pdf $f(x)=k$ for $2\le x\le 6$ (a uniform distribution). (a) Find k; (b) sketch the graph of f; (c) find $Pr(3\le X\le 5)$.

Q3. X has pdf $f(x)=k\,x^2$ for $0\le x\le 2$. (a) Find k; (b) find $Pr(X\le 1)$.

Q4. X has pdf $f(x)=k\,x(3-x)$ for $0\le x\le 3$. (a) Find k; (b) find $Pr(1\le X\le 2)$.

Q5. X has pdf $f(x)=k\,e^{-x/3}$ for $x\ge 0$. (a) Find k; (b) find $Pr(X\le 3)$, giving an exact answer.

Q6. X has pdf $f(x)=\begin{cases} k\,x, & 0\le x\le 2\\ k(4-x), & 2<x\le 4\end{cases}$ (and 0 elsewhere). (a) Find k; (b) find $Pr(X\le 2)$; (c) what feature of the graph explains your answer to (b)?

Unscaffolded

For each pdf, find the constant where one is given, then find the required probability (exact values).

Q7. $f(x)=k\,x$, $0\le x\le 6$. Find k and $Pr(2\le X\le 4)$.

Q8. $f(x)=k\,x^2$, $0\le x\le 4$. Find k and $Pr(X\ge 2)$.

Q9. $f(x)=k\,x(1-x)$, $0\le x\le 1$. Find k and $Pr(X\le 1/2)$.

Q10. $f(x)=k$, $0\le x\le 5$. Find k and $Pr(1\le X\le 3)$.

Q11. $f(x)=k\,e^{-2x}$, $x\ge 0$. Find k and $Pr(X\le 1)$.

Q12. $f(x)=k(4-x^2)$, $0\le x\le 2$. Find k and $Pr(X\le 1)$.

Q13. $f(x)=\begin{cases} k, & 0\le x\le 1\\ k\,x, & 1<x\le 2\end{cases}$. Find k and $Pr(X\ge 3/2)$.

Q14. $f(x)=k\,x^3$, $0\le x\le 2$. Find k and $Pr(X\ge 1)$.

Q15. $f(x)=k\,e^{-x}$, $x\ge 0$. Find k and $Pr(1\le X\le 2)$.

Q16. $f(x)=k(x-1)(3-x)$, $1\le x\le 3$. Find k and $Pr(X\le 2)$.

Q17. X is uniform with $f(x)=\frac{1}{4}$, $0\le x\le 4$. Find $Pr(X\le 3\,/\,X\ge 1)$.

Q18. $f(x)=k\,x^{-1/2}$, $0<x\le 1$. Find k and $Pr(X\le 1/4)$.

Q19. The pdf is $f(x)=k\,x(6-x)$, $0\le x\le 6$. Show that $k=\frac{1}{36}$, and hence find $Pr(2\le X\le 4)$.

Q20. A random variable has pdf $f(x)=\begin{cases} \frac{x}{9}, & 0\le x\le 3\\ \frac{6-x}{9}, & 3<x\le 6\end{cases}$. Verify that the total area is 1, then find $Pr(2\le X\le 4)$.

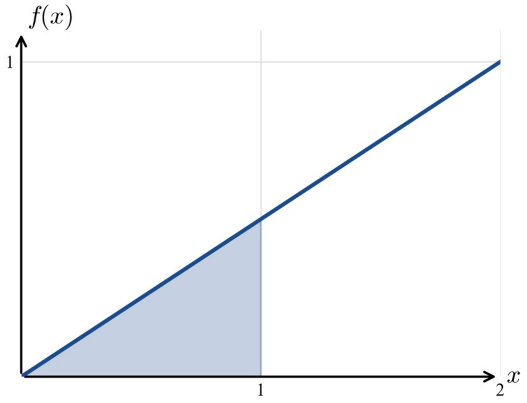
Q21. The waiting time X (minutes) has pdf $f(x)=\frac{1}{4}e^{-x/4}$, $x\ge 0$. Find $Pr(X>6\,/\,X>2)$.

Answers

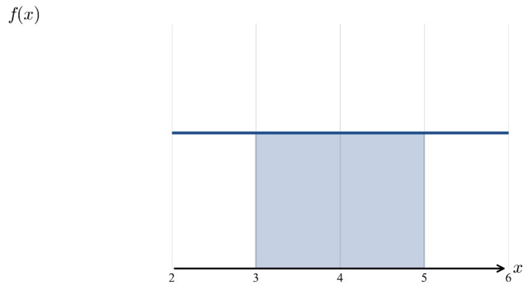
Q1. $k=\frac{1}{2}$; (b) $\frac{1}{4}$; (c) $\frac{3}{4}$. **Q2.** $k=\frac{1}{4}$; (c) $\frac{1}{2}$. **Q3.** $k=\frac{3}{8}$; (b) $\frac{1}{8}$. **Q4.** $k=\frac{2}{9}$; (b) $\frac{13}{27}$. **Q5.** $k=\frac{1}{3}$; (b) $1-e^{-1}$. **Q6.** $k=\frac{1}{4}$; (b) $\frac{1}{2}$. **Q7.** $k=\frac{1}{18}$; $\frac{1}{3}$. **Q8.** $k=\frac{3}{64}$; $\frac{7}{8}$. **Q9.** $k=6$; $\frac{1}{2}$. **Q10.** $k=\frac{1}{5}$; $\frac{2}{5}$. **Q11.** $k=2$; $1-e^{-2}$. **Q12.** $k=\frac{3}{16}$; $\frac{11}{16}$. **Q13.** $k=\frac{2}{5}$; $\frac{7}{20}$. **Q14.** $k=\frac{1}{4}$; $\frac{15}{16}$. **Q15.** $k=1$; $e^{-1}-e^{-2}$. **Q16.** $k=\frac{3}{4}$; $\frac{1}{2}$. **Q17.** $\frac{2}{3}$. **Q18.** $k=\frac{1}{2}$; $\frac{1}{2}$. **Q19.** $\frac{13}{27}$. **Q20.** $\frac{5}{9}$. **Q21.** e^{-1}.

Fully-worked solutions

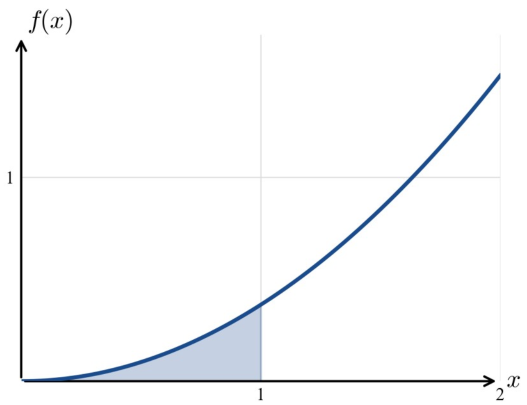
Q1. $\int_0^2 kx\,dx = k\left[\frac{x^2}{2}\right]_0^2 = 2k = 1 \Rightarrow k=\frac{1}{2}$. (b) $Pr(X\le 1)=\int_0^1 \frac{x}{2}\,dx=\left[\frac{x^2}{4}\right]_0^1=\frac{1}{4}$. (c) $Pr(X\ge 1)=1-\frac{1}{4}=\frac{3}{4}$.



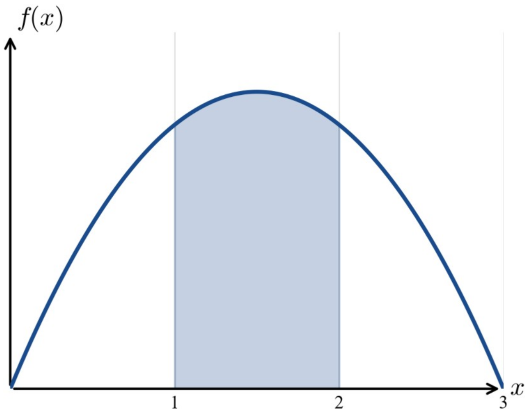
Q2. $\int_2^6 k\,dx = k(6-2)=4k=1\Rightarrow k=\frac{1}{4}$. (c) $Pr(3\le X\le 5)=\frac{1}{4}(5-3)=\frac{1}{2}$ (a rectangle of width 2, height $1/4$).



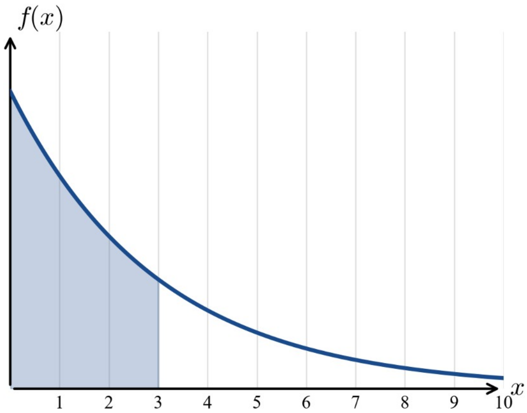
Q3. $\int_0^2 kx^2\,dx = k\left[\frac{x^3}{3}\right]_0^2=\frac{8k}{3}=1\Rightarrow k=\frac{3}{8}$. (b) $Pr(X\le 1)=\int_0^1 \frac{3}{8}x^2\,dx=\frac{3}{8}\cdot\frac{1}{3}=\frac{1}{8}$.



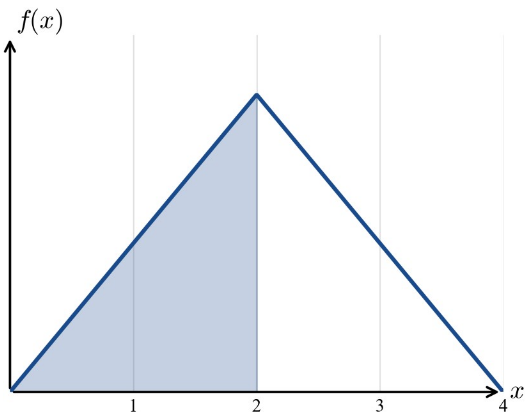
Q4. $\int_0^3 k(3x-x^2)\,dx = k\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3 = k\left(\frac{27}{2}-9\right)=\frac{9k}{2}=1 \Rightarrow k=\frac{2}{9}$. (b)
$Pr(1\le X\le 2)=\frac{2}{9}\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_1^2=\frac{2}{9}\cdot\frac{13}{6}=\frac{13}{27}$.



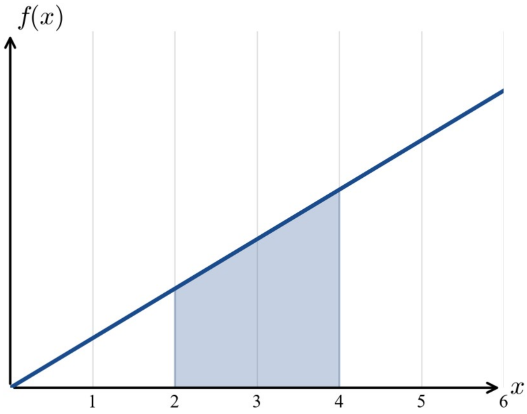
Q5. $\int_0^\infty k e^{-x/3}\,dx = k\left[-3e^{-x/3}\right]_0^\infty = 3k=1 \Rightarrow k=\frac{1}{3}$. (b) $Pr(X\le 3)=\left[-e^{-x/3}\right]_0^3=1-e^{-1}$.



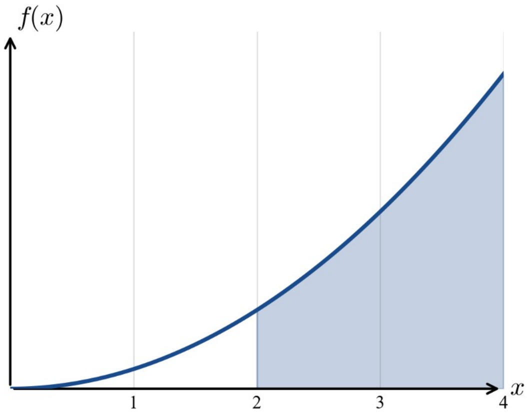
Q6. By symmetry each triangular half has area $\int_0^2 kx\,dx=2k$; total $4k=1\Rightarrow k=\frac{1}{4}$. (b)
$Pr(X\le 2)=\int_0^2 \frac{x}{4}\,dx=\left[\frac{x^2}{8}\right]_0^2=\frac{1}{2}$. (c) The graph is symmetric about $x=2$, so exactly half the area lies to each side.



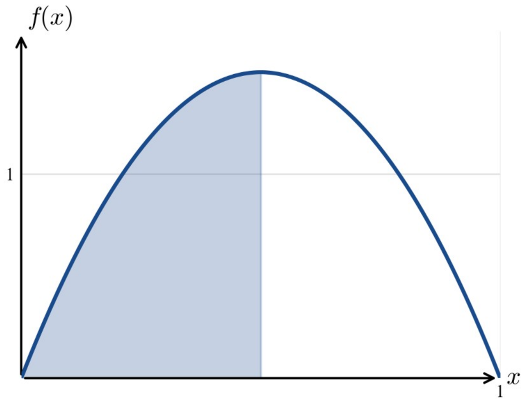
Q7. $\int_0^6 kx\,dx=18k=1\Rightarrow k=\frac{1}{18}$. $Pr(2\le X\le 4)=\left[\frac{x^2}{36}\right]_2^4=\frac{16-4}{36}=\frac{12}{36}=\frac{1}{3}$.



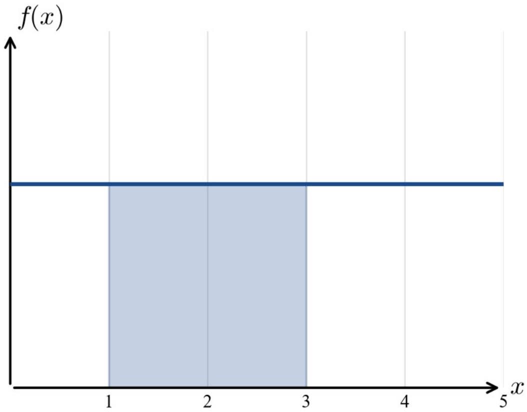
Q8. $\int_0^4 kx^2\,dx = \frac{64k}{3} = 1 \Rightarrow k = \frac{3}{64}$. $Pr(X \geq 2) = \left[\frac{x^3}{64}\right]_2^4 = \frac{64-8}{64} = \frac{56}{64} = \frac{7}{8}$.



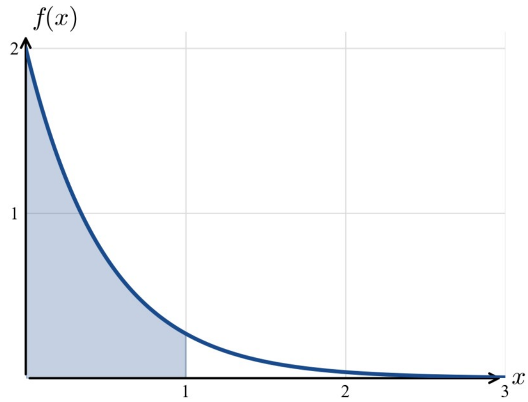
Q9. $\int_0^1 k(x - x^2)\,dx = k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6} = 1 \Rightarrow k = 6$. $Pr(X \leq 1/2) = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1/2} = 6\left(\frac{1}{8} - \frac{1}{24}\right) = 6 \cdot \frac{1}{12} = \frac{1}{2}$.



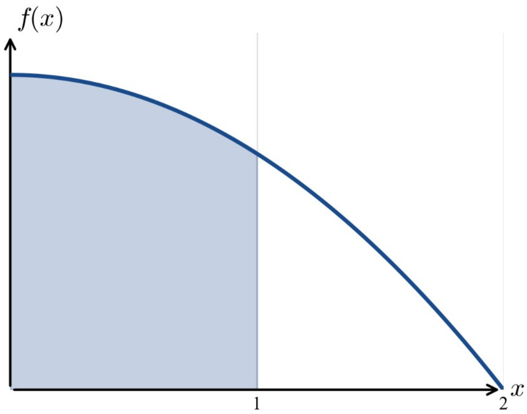
Q10. $5k = 1 \Rightarrow k = \frac{1}{5}$. $Pr(1 \leq X \leq 3) = \frac{1}{5}(3-1) = \frac{2}{5}$.



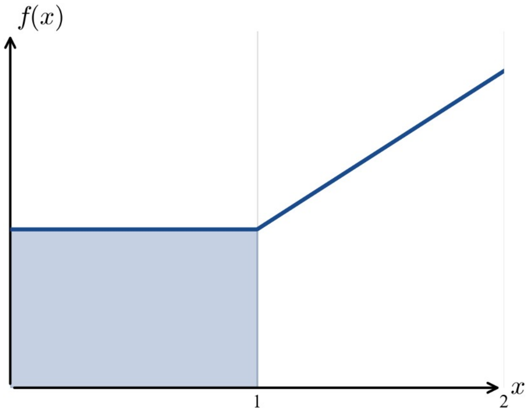
Q11. $\int_0^\infty k e^{-2x}\,dx = k\left[-\frac{1}{2}e^{-2x}\right]_0^\infty = \frac{k}{2} = 1 \Rightarrow k = 2$. $Pr(X \le 1) = \left[-e^{-2x}\right]_0^1 = 1 - e^{-2}$.



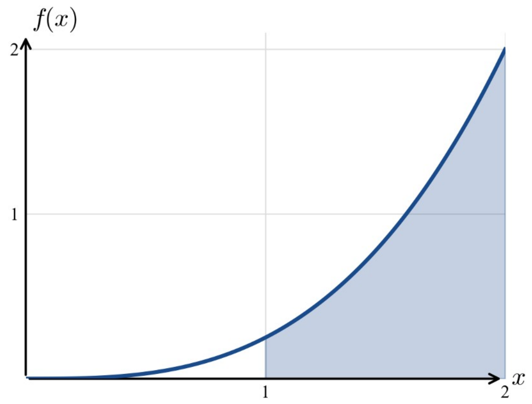
Q12. $\int_0^2 k(4 - x^2)\,dx = k\left[4x - \frac{x^3}{3}\right]_0^2 = k \cdot \frac{16}{3} = 1 \Rightarrow k = \frac{3}{16}$. $Pr(X \le 1) = \frac{3}{16}\left[4x - \frac{x^3}{3}\right]_0^1 = \frac{3}{16} \cdot \frac{11}{3} = \frac{11}{16}$.



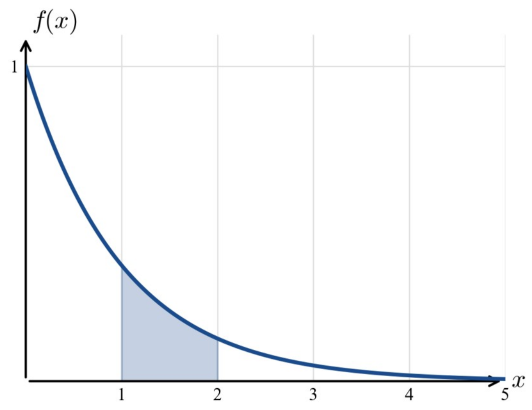
Q13. $\int_0^1 k\,dx + \int_1^2 kx\,dx = k + k \cdot \frac{3}{2} = \frac{5k}{2} = 1 \Rightarrow k = \frac{2}{5}$. $Pr(X \ge 3/2) = \int_{3/2}^2 \frac{2}{5}x\,dx = \frac{2}{5}\left[\frac{x^2}{2}\right]_{3/2}^2 = \frac{2}{5} \cdot \frac{7}{8} = \frac{7}{20}$.



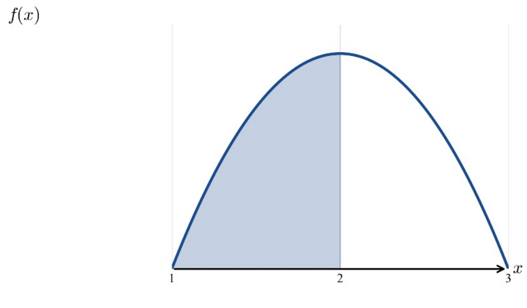
Q14. $\int_0^2 kx^3\,dx = k\left[\frac{x^4}{4}\right]_0^2 = 4k = 1 \Rightarrow k = \frac{1}{4}$. $Pr(X \geq 1) = \left[\frac{x^4}{16}\right]_1^2 = \frac{16-1}{16} = \frac{15}{16}$.



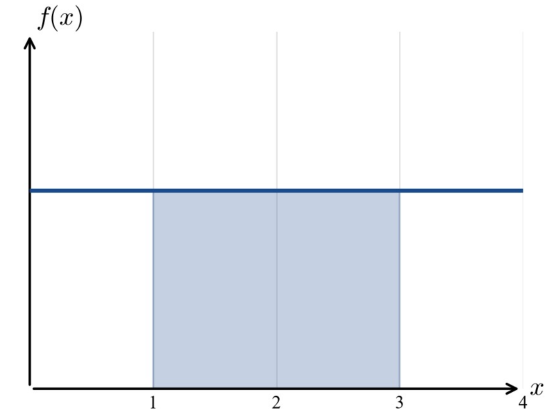
Q15. $\int_0^\infty ke^{-x}\,dx = k = 1 \Rightarrow k = 1$. $Pr(1 \leq X \leq 2) = \left[-e^{-x}\right]_1^2 = e^{-1} - e^{-2}$.



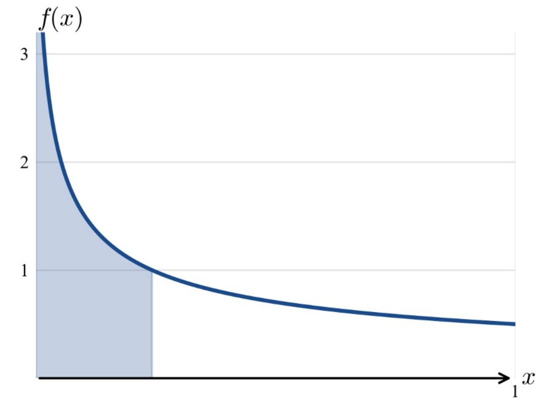
Q16. $\int_1^3 k(x-1)(3-x)\,dx = k\left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3 = k \cdot \frac{4}{3} = 1 \Rightarrow k = \frac{3}{4}$. By symmetry about $x = 2$, $Pr(X \leq 2) = \frac{1}{2}$.



Q17. $Pr(1 \le X \le 3) = \frac{1}{4}(3-1) = \frac{1}{2}$ and $Pr(X \ge 1) = \frac{1}{4}(4-1) = \frac{3}{4}$. So $Pr(X \le 3 / X \ge 1) = \frac{1/2}{3/4} = \frac{2}{3}$.

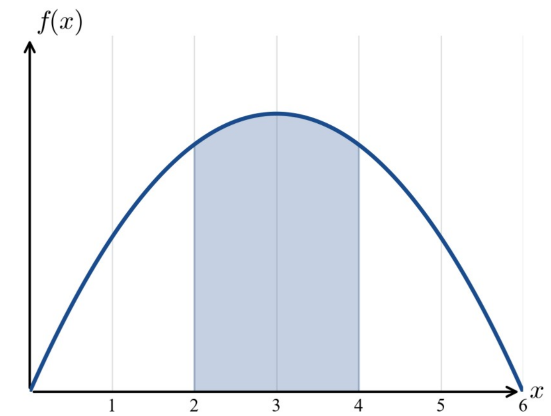


Q18. $\int_0^1 k x^{-1/2} \, dx = k \left[2\sqrt{x} \right]_0^1 = 2k = 1 \Rightarrow k = \frac{1}{2}$. $Pr(X \le 1/4) = \left[\sqrt{x} \right]_0^{1/4} = \frac{1}{2}$. (The graph is unbounded near $x = 0$ but the area is still finite.)



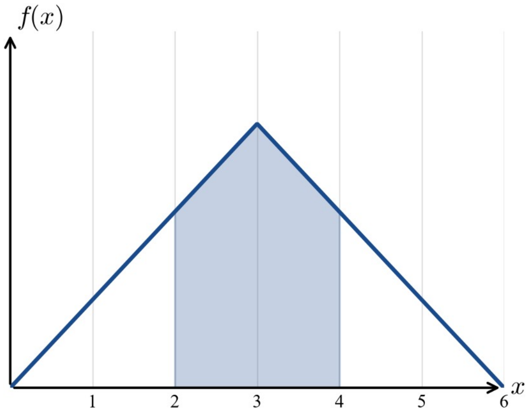
Q19. $\int_0^6 k(6x - x^2) \, dx = k\left[3x^2 - \frac{x^3}{3} \right]_0^6 = k(108 - 72) = 36k = 1$, so $k = \frac{1}{36}$. Then

$Pr(2 \le X \le 4) = \frac{1}{36}\left[3x^2 - \frac{x^3}{3} \right]_2^4 = \frac{1}{36} \cdot \frac{52}{3} = \frac{13}{27}$.

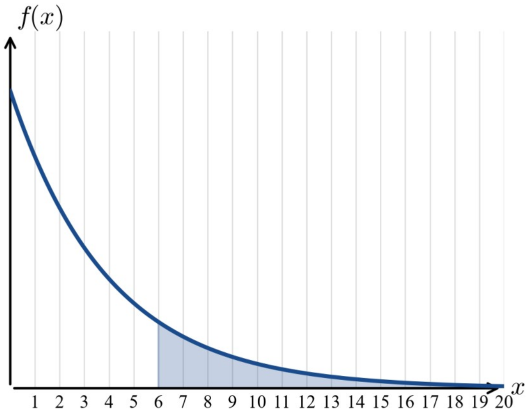


Q20. Total area = $\int_0^3 \frac{x}{9} \, dx + \int_3^6 \frac{6-x}{9} \, dx = \frac{1}{2} + \frac{1}{2} = 1$, so it is a valid pdf.

$Pr(2 \le X \le 4) = \int_2^3 \frac{x}{9} \, dx + \int_3^4 \frac{6-x}{9} \, dx = \frac{5}{18} + \frac{5}{18} = \frac{10}{18} = \frac{5}{9}$.



Q21. $Pr(X > 6) = \int_6^\infty \frac{1}{4} e^{-x/4} \, dx = e^{-3/2}$ and $Pr(X > 2) = e^{-1/2}$. So $Pr(X > 6 / X > 2) = \frac{e^{-3/2}}{e^{-1/2}} = e^{-1}$.



Chapter 13 Continuous probability distributions — 13B Mean, median and percentiles

Theory

A continuous random variable X is described by a **probability density function** (pdf) f with $f(x) \geq 0$ and total area $\int_{-\infty}^{\infty} f(x)\,dx = 1$; probabilities are areas, $Pr(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$. This section finds the **centre** of such a distribution.

Mean (expected value). The mean is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx,$$

integrated over the domain where f is non-zero. More generally, for any function g, $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$ — for example $E(X^2) = \int x^2 f(x)\,dx$.

Median. The **median** m is the value that splits the area in half:

$$\int_{-\infty}^{m} f(x)\,dx = 1/2.$$

Half of the probability lies to the left of m and half to the right.

Percentiles. The p-th **percentile** is the value x_p with $\int_{-\infty}^{x_p} f(x)\,dx = \frac{p}{100}$. The median is the 50th percentile; the **lower** and **upper quartiles** are the 25th and 75th percentiles.

Mode. The **mode** is the value of x at which f is **greatest** — the highest point of the pdf. For a smooth pdf find it by solving $f'(x) = 0$ (and checking the endpoints); a distribution can have no single mode (for a uniform pdf every value is equally likely).

Shape. For a **symmetric** pdf the mean, median and mode all sit at the centre. If the distribution has a long tail to the **right** then $\mu > m$; a long tail to the **left** gives $\mu < m$.

Worked examples

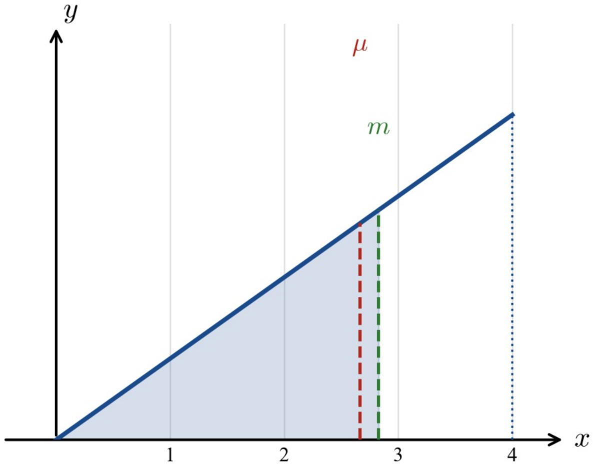
Example 1 — scaffolded

The continuous random variable X has pdf $f(x) = \frac{x}{8}$ for $0 \leq x \leq 4$ (and 0 elsewhere). Find (a) the mean, (b) the median, and (c) the mode.

Working	Comment
(a) $\mu = \int_{0}^{4} x \cdot \frac{x}{8}\,dx = \int_{0}^{4} \frac{x^2}{8}\,dx = \left[\frac{x^3}{24}\right]_0^4 = \frac{64}{24} = \frac{8}{3}$.	$E(X) = \int x f(x)\,dx$.
(b) Solve $\int_{0}^{m} \frac{x}{8}\,dx = \frac{1}{2}$: $\left[\frac{x^2}{16}\right]_0^m = \frac{m^2}{16} = \frac{1}{2}$.	The median has half the area to its left.
$m^2 = 8$, so $m = 2\sqrt{2}$ (taking the value in $[0, 4]$).	Reject the negative root — it is outside the domain.

Working	Comment
(c) f increases across $[0,4]$, so it is greatest at $x=4$: the mode is 4.	The pdf is largest at the right endpoint.

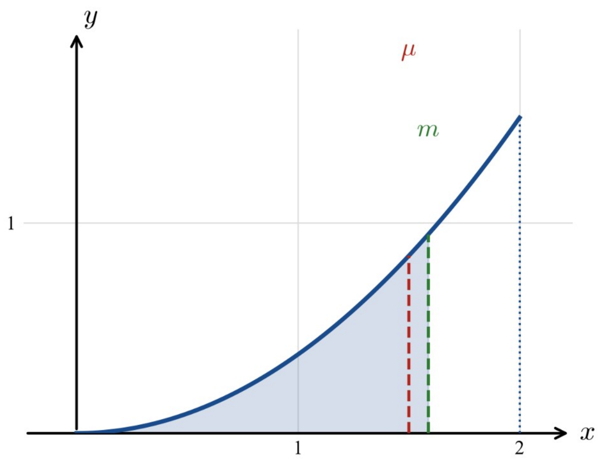
Since $\mu=\dfrac{8}{3}\approx 2.67<m=2\sqrt{2}\approx 2.83$, the distribution is slightly left-skewed.



Example 2 — scaffolded

X has pdf $f(x)=\dfrac{3x^2}{8}$ for $0\le x\le 2$. Find the mean, median and mode.

Working	Comment
$\mu=\int_0^2 x\cdot\dfrac{3x^2}{8}\,dx=\int_0^2\dfrac{3x^3}{8}\,dx=\left[\dfrac{3x^4}{32}\right]_0^2=\dfrac{48}{32}=\dfrac{3}{2}.$	$E(X)=\int x f\,dx.$
Median: $\int_0^m\dfrac{3x^2}{8}\,dx=\left[\dfrac{x^3}{8}\right]_0^m=\dfrac{m^3}{8}=\dfrac{1}{2}.$	Set the left area equal to $1/2$.
$m^3=4$, so $m=\sqrt[3]{4}$.	Exact value — a cube root.
f increases on $[0,2]$, so the mode is $x=2$.	Highest point of the pdf.



Example 3 — unscaffolded

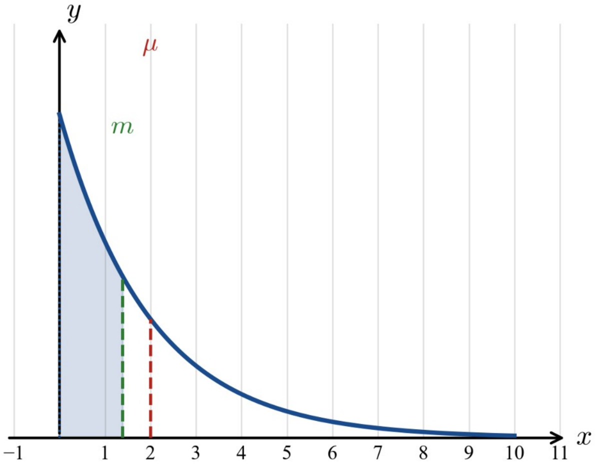
X has pdf $f(x)=\dfrac{1}{2}e^{-x/2}$ for $x\ge 0$. Find the mean, the median, and the mode.

Mean: $\mu=\int_0^\infty x \cdot 1/2\, e^{-x/2}\, dx=2$ (integration by parts).

Median: $\int_0^m 1/2\, e^{-x/2}\, dx=[-e^{-x/2}]_0^m=1-e^{-m/2}=\frac{1}{2}$, so $e^{-m/2}=\frac{1}{2}$, giving $-\frac{m}{2}=\log_e 1/2$ and $m=2\log_e 2$.

Mode: f is greatest at $x=0$ and decreases thereafter, so the mode is 0.

$\therefore\ \mu=2>m=2\log_e 2\approx 1.39$: the long right tail pulls the mean above the median.



Example 4 — unscaffolded

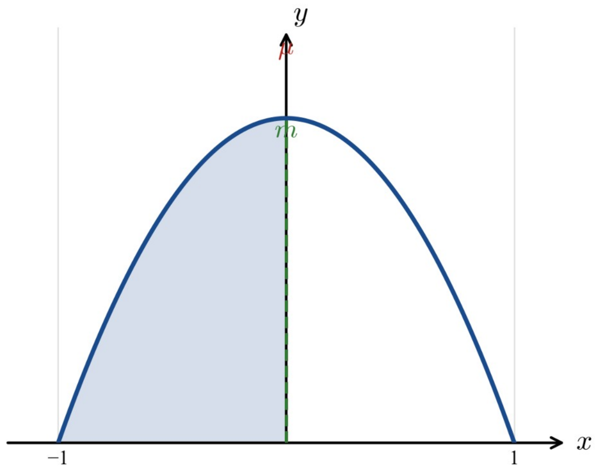
X has pdf $f(x)=\frac{3}{4}(1-x^2)$ for $-1\le x\le 1$. Find the mean, median and mode.

The pdf is **symmetric** about $x=0$ (it depends only on x^2). The mean is $\mu=\int_{-1}^{1} x \cdot 3/4(1-x^2)\, dx=0$, since the integrand is an odd function on a symmetric interval.

By symmetry the median is also $m=0$ (each half carries area $1/2$).

The mode is where f is greatest: $f(x)=3/4(1-x^2)$ is largest when x^2 is smallest, i.e. at $x=0$.

$\therefore\ \mu=m=\text{mode}=0$ — a symmetric distribution.



Practice questions

Scaffolded

Q1. X has pdf $f(x)=\frac{x}{2}$, $0\le x\le 2$. (a) find the mean; (b) find the median; (c) state the mode.

Q2. X has pdf $f(x)=\frac{1}{4}$, $1\le x\le 5$ (a uniform distribution). (a) find the mean; (b) find the median; (c) explain why there is no single mode.

Q3. X has pdf $f(x)=\frac{2x}{9}$, $0\le x\le 3$. (a) find the mean; (b) find the median; (c) find the lower quartile (the 25th percentile).

Q4. X has pdf $f(x)=\frac{3(x-1)^2}{8}$, $1\le x\le 3$. (a) find the mean; (b) find the median; (c) state the mode.

Q5. X has pdf $f(x)=\frac{3(4-x^2)}{32}$, $-2\le x\le 2$. (a) explain, using symmetry, why the mean and median are both 0; (b) confirm the mean is 0 by integration; (c) state the mode.

Q6. X has pdf $f(x)=\frac{3x^2}{64}$, $0\le x\le 4$. (a) find the mean; (b) find the median; (c) state the mode.

Unscaffolded

For each of the following, find the **mean**, the **median** and the **mode** (and any percentile requested). Give exact values.

Q7. $f(x)=\frac{x}{18}$, $0\le x\le 6$; also find the upper quartile (the 75th percentile).

Q8. $f(x)=\frac{1}{4}e^{-x/4}$, $x\ge 0$; also find the 90th percentile.

Q9. $f(x)=\frac{3(2-x)^2}{8}$, $0\le x\le 2$.

Q10. $f(x)=\frac{2(3-x)}{9}$, $0\le x\le 3$.

Q11. $f(x)=e^{-x}$, $x\ge 0$. Show that the median is $\log_e 2$.

Q12. $f(x)=4x^3$, $0\le x\le 1$.

Q13. $f(x)=\frac{1}{2}$, $2\le x\le 4$ (uniform): find the mean and median, and explain the mode.

Q14. $f(x)=3x^2$, $0\le x\le 1$.

Q15. $f(x)=\frac{x-1}{2}$, $1\le x\le 3$.

Q16. $f(x)=6x(1-x)$, $0\le x\le 1$. **Show that** the mean equals the median.

Q17. $f(x)=\frac{3x(10-x)}{500}$, $0\le x\le 10$.

Q18. $f(x)=\frac{1}{3}e^{-x/3}$, $x\geq 0$.

Q19. $f(x)=\frac{\pi}{2}\sin(\pi x)$, $0\leq x\leq 1$.

Q20. $f(x)=2e^{-2x}$, $x\geq 0$. Find the mean, mode, and show the median is $\frac{1}{2}\log_e 2$.

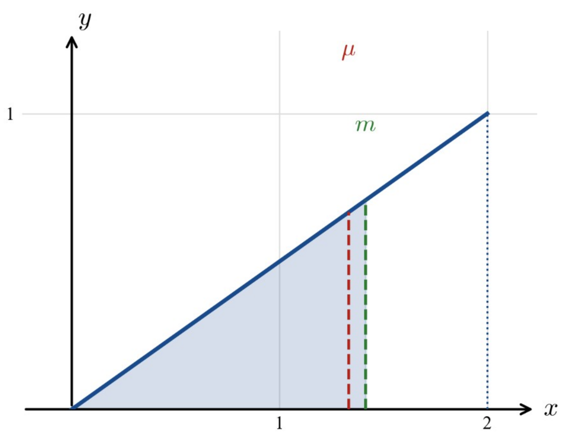
Q21. $f(x)=\frac{3\left(1-(x-1)^2\right)}{4}$, $0\leq x\leq 2$.

Answers

Q1. mean $\frac{4}{3}$, median $\sqrt{2}$, mode 2. **Q2.** mean 3, median 3, no single mode (uniform). **Q3.** mean 2, median $\frac{3\sqrt{2}}{2}$, lower quartile $\frac{3}{2}$; mode 3. **Q4.** mean $\frac{5}{2}$, median $1+\sqrt[3]{4}$, mode 3. **Q5.** mean 0, median 0, mode 0. **Q6.** mean 3, median $2\sqrt[3]{4}$, mode 4. **Q7.** mean 4, median $3\sqrt{2}$, upper quartile $3\sqrt{3}$, mode 6. **Q8.** mean 4, median $4\log_e 2$, 90th percentile $4\log_e 10$, mode 0. **Q9.** mean $\frac{1}{2}$, median $2-\sqrt[3]{4}$, mode 0. **Q10.** mean 1, median $3-\frac{3\sqrt{2}}{2}$, mode 0. **Q11.** mean 1, median $\log_e 2$, mode 0. **Q12.** mean $\frac{4}{5}$, median $\frac{\sqrt[4]{8}}{2}$, mode 1. **Q13.** mean 3, median 3, no single mode (uniform). **Q14.** mean $\frac{3}{4}$, median $\frac{\sqrt[3]{4}}{2}$, mode 1. **Q15.** mean $\frac{7}{3}$, median $1+\sqrt{2}$, mode 3. **Q16.** mean $\frac{1}{2}$, median $\frac{1}{2}$, mode $\frac{1}{2}$. **Q17.** mean 5, median 5, mode 5. **Q18.** mean 3, median $3\log_e 2$, mode 0. **Q19.** mean $\frac{1}{2}$, median $\frac{1}{2}$, mode $\frac{1}{2}$. **Q20.** mean $\frac{1}{2}$, median $\frac{1}{2}\log_e 2$, mode 0. **Q21.** mean 1, median 1, mode 1.

Fully-worked solutions

Q1. $f(x)=\frac{x}{2}$, $0\le x\le 2$. Mean $\mu=\int_0^2 \frac{x^2}{2}\,dx=\left[\frac{x^3}{6}\right]_0^2=\frac{8}{6}=\frac{4}{3}$. Median: $\int_0^m \frac{x}{2}\,dx=\frac{m^2}{4}=\frac{1}{2}\Rightarrow m^2=2\Rightarrow m=\sqrt{2}$. f increases, so mode $=2$.



Q2. Uniform $f(x)=\frac{1}{4}$, $1\le x\le 5$. Mean $=\int_1^5 \frac{x}{4}\,dx=\left[\frac{x^2}{8}\right]_1^5=\frac{25-1}{8}=3$ (the midpoint). Median: $\int_1^m \frac{1}{4}\,dx=\frac{m-1}{4}=\frac{1}{2}\Rightarrow m=3$. Every value of x in $[1,5]$ has the same density $1/4$, so no single value is most likely — there is **no unique mode**.

Q3. $f(x)=\frac{2x}{9}$, $0\le x\le 3$. Mean $=\int_0^3 \frac{2x^2}{9}\,dx=\left[\frac{2x^3}{27}\right]_0^3=2$. Median: $\int_0^m \frac{2x}{9}\,dx=\frac{m^2}{9}=\frac{1}{2}\Rightarrow m^2=\frac{9}{2}\Rightarrow m=\frac{3\sqrt{2}}{2}$. Lower quartile q: $\frac{q^2}{9}=\frac{1}{4}\Rightarrow q^2=\frac{9}{4}\Rightarrow q=\frac{3}{2}$. Mode $=3$.

Q4. $f(x)=\dfrac{3(x-1)^2}{8}$, $1\le x\le 3$. Mean $=\displaystyle\int_1^3 x\cdot\frac{3(x-1)^2}{8}\,dx=\frac{5}{2}$. Median:
$\displaystyle\int_1^m \frac{3(x-1)^2}{8}\,dx=\left[\frac{(x-1)^3}{8}\right]_1^m=\frac{(m-1)^3}{8}=\frac{1}{2}\Rightarrow(m-1)^3=4\Rightarrow m=1+\sqrt[3]{4}$. f increases on $[1,3]$, so mode $=3$.

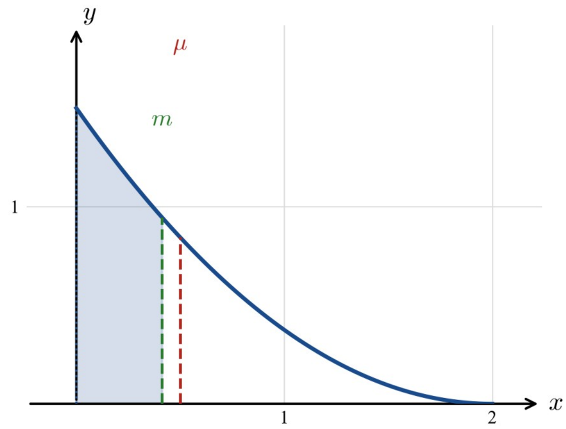
Q5. $f(x)=\dfrac{3(4-x^2)}{32}$, $-2\le x\le 2$. (a) f depends only on x^2, so it is symmetric about $x=0$; the two halves carry equal area, so $m=0$, and the balance point is $\mu=0$. (b) $\mu=\displaystyle\int_{-2}^{2} x\cdot\frac{3(4-x^2)}{32}\,dx=0$ (odd integrand over a symmetric interval). (c) f is greatest where x^2 is least, i.e. mode $=0$.

Q6. $f(x)=\dfrac{3x^2}{64}$, $0\le x\le 4$. Mean $=\displaystyle\int_0^4 \frac{3x^3}{64}\,dx=\left[\frac{3x^4}{256}\right]_0^4=\frac{768}{256}=3$. Median: $\dfrac{m^3}{64}=\dfrac{1}{2}\Rightarrow m^3=32\Rightarrow m=\sqrt[3]{32}=2\sqrt[3]{4}$. Mode $=4$.

Q7. $f(x)=\dfrac{x}{18}$, $0\le x\le 6$. Mean $=\displaystyle\int_0^6 \frac{x^2}{18}\,dx=\left[\frac{x^3}{54}\right]_0^6=4$. Median: $\dfrac{m^2}{36}=\dfrac{1}{2}\Rightarrow m^2=18\Rightarrow m=3\sqrt{2}$. Upper quartile q: $\dfrac{q^2}{36}=\dfrac{3}{4}\Rightarrow q^2=27\Rightarrow q=3\sqrt{3}$. Mode $=6$.

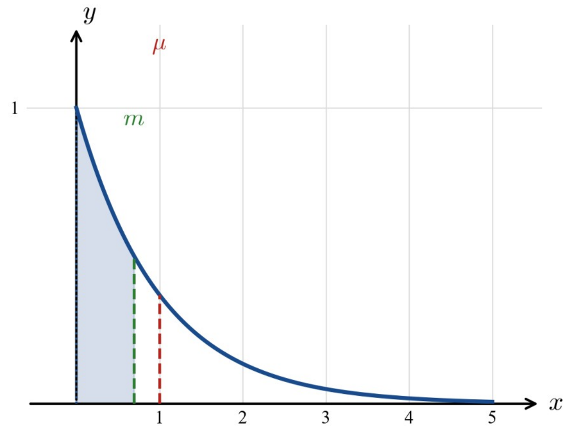
Q8. $f(x)=\dfrac{1}{4}e^{-x/4}$, $x\ge 0$. Mean $=\displaystyle\int_0^\infty \frac{x}{4}e^{-x/4}\,dx=4$. Median: $1-e^{-m/4}=\dfrac{1}{2}\Rightarrow e^{-m/4}=\dfrac{1}{2}\Rightarrow m=4\log_e 2$. 90th percentile x_{90}: $1-e^{-x_{90}/4}=0.9\Rightarrow e^{-x_{90}/4}=\dfrac{1}{10}\Rightarrow x_{90}=4\log_e 10$. Mode $=0$.

Q9. $f(x)=\dfrac{3(2-x)^2}{8}$, $0\le x\le 2$. Mean $=\displaystyle\int_0^2 x\cdot\frac{3(2-x)^2}{8}\,dx=\frac{1}{2}$. Median:
$\displaystyle\int_0^m \frac{3(2-x)^2}{8}\,dx=\left[-\frac{(2-x)^3}{8}\right]_0^m=\frac{8-(2-m)^3}{8}=\frac{1}{2}\Rightarrow(2-m)^3=4\Rightarrow m=2-\sqrt[3]{4}$. f is greatest at $x=0$: mode $=0$.



Q10. $f(x)=\dfrac{2(3-x)}{9}$, $0\le x\le 3$. Mean $=\displaystyle\int_0^3 x\cdot\frac{2(3-x)}{9}\,dx=1$. Median:
$\displaystyle\int_0^m \frac{2(3-x)}{9}\,dx=\frac{6m-m^2}{9}=\frac{1}{2}\Rightarrow m^2-6m+\frac{9}{2}=0\Rightarrow m=3-\frac{3\sqrt{2}}{2}$ (the root in $[0,3]$). Mode $=0$.

Q11. $f(x)=e^{-x}$, $x \geq 0$. Mean $= \int_0^{\infty} x e^{-x}\,dx = 1$. Median: $\int_0^m e^{-x}\,dx = 1-e^{-m} = \frac{1}{2} \Rightarrow e^{-m} = \frac{1}{2} \Rightarrow m = \log_e 2$. Mode $=0$ (density greatest at $x=0$).



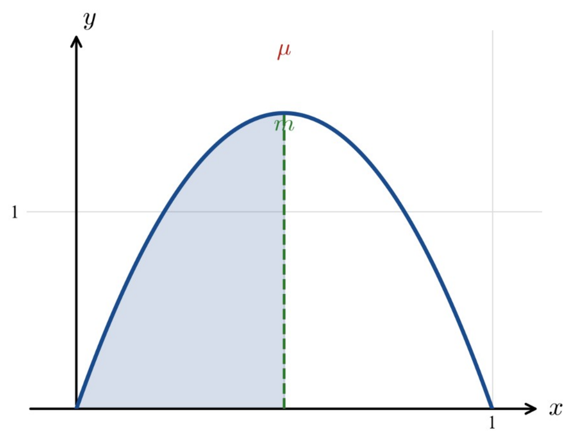
Q12. $f(x)=4x^3$, $0 \leq x \leq 1$. Mean $= \int_0^1 4x^4\,dx = \left[\frac{4x^5}{5}\right]_0^1 = \frac{4}{5}$. Median: $\int_0^m 4x^3\,dx = m^4 = \frac{1}{2} \Rightarrow m = \left(\frac{1}{2}\right)^{1/4} = \frac{\sqrt[4]{8}}{2}$. Mode $=1$.

Q13. Uniform $f(x)=\frac{1}{2}$, $2 \leq x \leq 4$. Mean $=3$ (midpoint); median: $\frac{m-2}{2} = \frac{1}{2} \Rightarrow m=3$. As the density is constant, no single value is most likely — **no unique mode**.

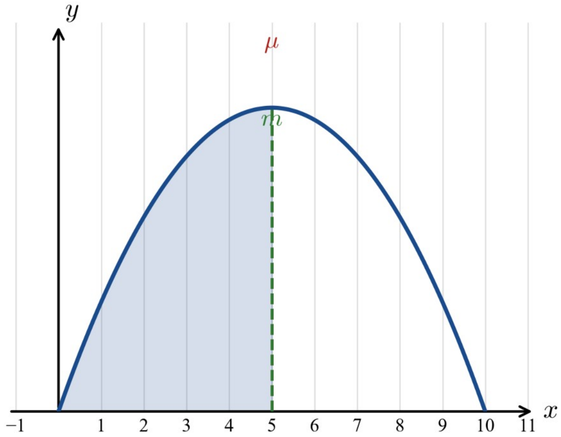
Q14. $f(x)=3x^2$, $0 \leq x \leq 1$. Mean $= \int_0^1 3x^3\,dx = \frac{3}{4}$. Median: $m^3 = \frac{1}{2} \Rightarrow m = \frac{1}{\sqrt[3]{2}} = \frac{\sqrt[3]{4}}{2}$. Mode $=1$.

Q15. $f(x)=\frac{x-1}{2}$, $1 \leq x \leq 3$. Mean $= \int_1^3 x \cdot \frac{x-1}{2}\,dx = \frac{7}{3}$. Median: $\int_1^m \frac{x-1}{2}\,dx = \frac{(m-1)^2}{4} = \frac{1}{2} \Rightarrow (m-1)^2 = 2 \Rightarrow m = 1+\sqrt{2}$. Mode $=3$.

Q16. $f(x)=6x(1-x)$, $0 \leq x \leq 1$. This is symmetric about $x=1/2$ (replacing x by $1-x$ leaves f unchanged), so the mean equals the median. Mean $= \int_0^1 6x^2(1-x)\,dx = 6\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{2}$. Median: $\int_0^m 6x(1-x)\,dx = 3m^2 - 2m^3 = \frac{1}{2}$, and $m=\frac{1}{2}$ satisfies this ($3 \cdot 1/4 - 2 \cdot 1/8 = 3/4 - 1/4 = 1/2$). $\therefore$ mean = median $= \frac{1}{2}$. Mode: $f'(x) = 6-12x = 0 \Rightarrow x = \frac{1}{2}$.

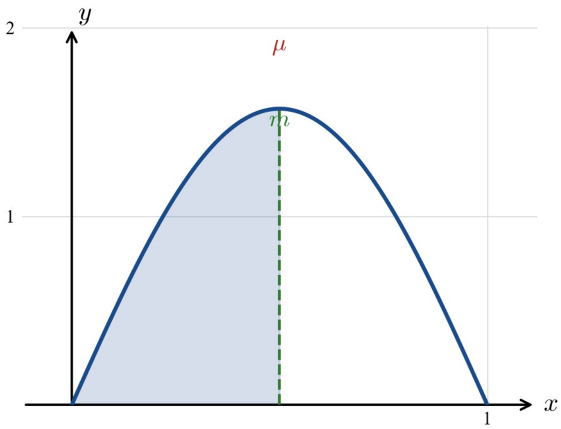


Q17. $f(x)=\frac{3x(10-x)}{500}$, $0\le x\le 10$. Symmetric about $x=5$, so mean = median $=5$. (Mean $=\int_0^{10} x\cdot\frac{3x(10-x)}{500}\,dx=5$.) Mode: $f'(x)=\frac{3(10-2x)}{500}=0\Rightarrow x=5$.



Q18. $f(x)=\frac{1}{3}e^{-x/3}$, $x\ge 0$. Mean $=3$. Median: $1-e^{-m/3}=\frac{1}{2}\Rightarrow m=3\log_e 2$. Mode $=0$.

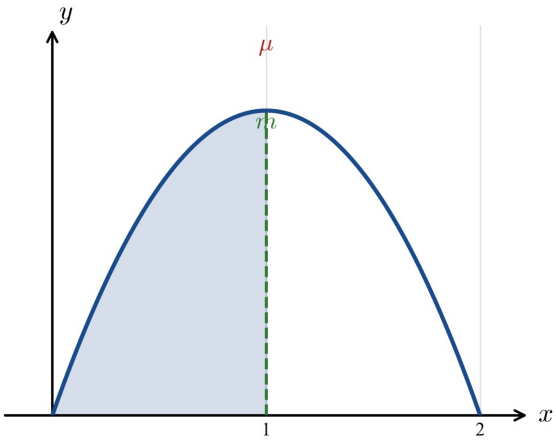
Q19. $f(x)=\frac{\pi}{2}\sin(\pi x)$, $0\le x\le 1$. Symmetric about $x=1/2$, so mean = median $=\frac{1}{2}$. (Mean $=\int_0^{1} x\cdot\frac{\pi}{2}\sin(\pi x)\,dx=\frac{1}{2}$.) Mode: $\sin(\pi x)$ is greatest at $x=\frac{1}{2}$.



Q20. $f(x)=2e^{-2x}$, $x\ge 0$. Mean $=\int_0^{\infty} 2xe^{-2x}\,dx=\frac{1}{2}$. Mode $=0$. Median:

$\int_0^{m} 2e^{-2x}\,dx=1-e^{-2m}=\frac{1}{2}\Rightarrow e^{-2m}=\frac{1}{2}\Rightarrow -2m=\log_e 1/2\Rightarrow m=\frac{1}{2}\log_e 2$.

Q21. $f(x)=\frac{3\left(1-(x-1)^2\right)}{4}$, $0\le x\le 2$. Symmetric about $x=1$, so mean = median $=1$. (Mean $=\int_0^{2} x\cdot\frac{3\left(1-(x-1)^2\right)}{4}\,dx=1$.) Mode: f is greatest where $(x-1)^2$ is least, i.e. $x=1$.



y
μ
m
x
1
2

Theory

For a continuous random variable X with probability density function f on $[a, b]$, the **mean** (expected value) is

$$\mu = E(X) = \int_a^b x f(x)\, dx.$$

The **variance** measures how spread out the distribution is about the mean:

$$Var(X) = E\left((X-\mu)^2\right) = E(X^2) - \mu^2, \text{where } E(X^2) = \int_a^b x^2 f(x)\, dx.$$

The second form $Var(X) = E(X^2) - \mu^2$ is almost always the easier one to use.

- The variance is measured in the **square** of the units of X. The **standard deviation** returns to the original units:
$$sd(X) = \sigma = \sqrt{Var(X)}.$$
- Because it involves $(X-\mu)^2$, the variance is unchanged by a **translation** of the distribution and is always ≥ 0.
- **Take care:** the question may ask for the **variance** or the **standard deviation** — they are not the same. Always take the square root for the standard deviation.

The interquartile range. The **quartiles** Q_1 and Q_3 and the **median** m are the values that divide the area under the pdf into quarters:

$$\int_a^{Q_1} f(x)\, dx = 1/4, \int_a^{m} f(x)\, dx = 1/2, \int_a^{Q_3} f(x)\, dx = 3/4.$$

The **interquartile range** is $\text{IQR} = Q_3 - Q_1$; it is the width of the middle 50% of the distribution and, unlike the standard deviation, is not affected by the tails.

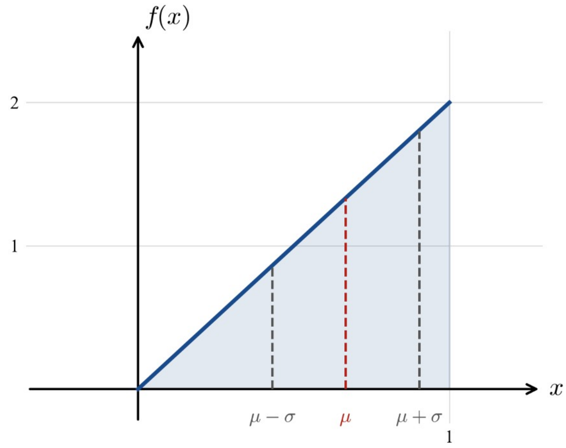
To find the spread: (1) find $\mu = E(X)$; (2) find $E(X^2)$; (3) $Var(X) = E(X^2) - \mu^2$; (4) $\sigma = \sqrt{Var(X)}$. For the IQR, solve the two area equations for Q_1 and Q_3.

Worked examples

Example 1 — scaffolded

The continuous random variable X has pdf $f(x) = 2x$ for $0 \leq x \leq 1$ (and 0 elsewhere). Find the variance and the standard deviation of X.

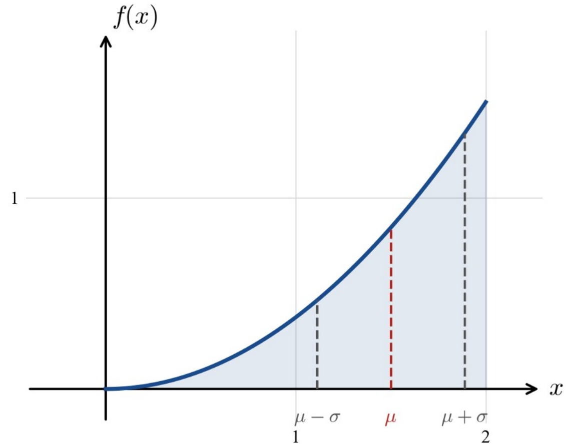
Working	Comment
$\mu = E(X) = \int_0^1 x(2x)\, dx = \int_0^1 2x^2\, dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$	Mean $= \int x f\, dx$.
$E(X^2) = \int_0^1 x^2(2x)\, dx = \int_0^1 2x^3\, dx = \left[\frac{x^4}{2}\right]_0^1 = \frac{1}{2}$	Now integrate $x^2 f$.
$Var(X) = E(X^2) - \mu^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$	Use $Var = E(X^2) - \mu^2$.
$\sigma = \sqrt{Var(X)} = \sqrt{\frac{1}{18}} = \frac{\sqrt{2}}{6}$	Standard deviation $= \sqrt{Var}$. Keep it exact.



Example 2 — scaffolded

The pdf of X is $f(x)=\dfrac{3x^2}{8}$ for $0\le x\le 2$. Find $Var(X)$ and $sd(X)$.

Working	Comment
$\mu=\int_0^2 x\cdot\frac{3x^2}{8}\,dx=\int_0^2\frac{3x^3}{8}\,dx=\left[\frac{3x^4}{32}\right]_0^2=\frac{3}{2}$	Find the mean.
$E(X^2)=\int_0^2 x^2\cdot\frac{3x^2}{8}\,dx=\left[\frac{3x^5}{40}\right]_0^2=\frac{96}{40}=\frac{12}{5}$	Integrate $x^2 f$.
$Var(X)=\frac{12}{5}-\left(\frac{3}{2}\right)^2=\frac{12}{5}-\frac{9}{4}=\frac{48-45}{20}=\frac{3}{20}$	Subtract μ^2.
$\sigma=\sqrt{\frac{3}{20}}=\frac{\sqrt{15}}{10}$	$\sqrt{3/20}=\sqrt{3}/2\sqrt{5}=\sqrt{15}/10$.



Example 3 — unscaffolded

The waiting time X minutes is uniformly distributed with pdf $f(x)=\dfrac{1}{4}$ for $1\le x\le 5$. Find the standard deviation and the interquartile range of X.

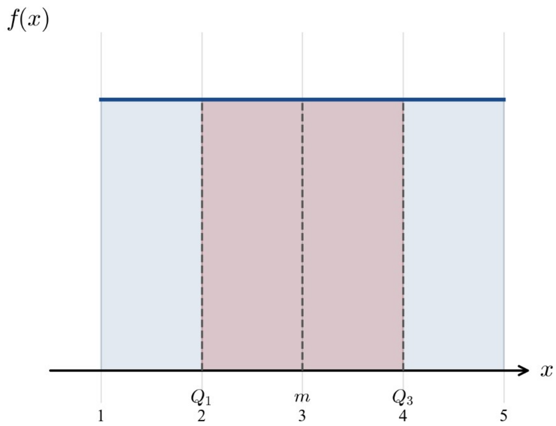
$$\mu=\int_1^5 x\cdot 1/4\,dx=\left[\frac{x^2}{8}\right]_1^5=\frac{25-1}{8}=3.$$

$$E(X^2)=\int_1^5 x^2\cdot 1/4\,dx=\left[\frac{x^3}{12}\right]_1^5=\frac{125-1}{12}=\frac{31}{3}.$$

$Var(X) = \frac{31}{3} - 3^2 = \frac{31}{3} - 9 = \frac{4}{3}, \sigma = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$.

For the quartiles, solve $\int_1^{Q_1} 1/4\, dx = 1/4 \Rightarrow \frac{Q_1 - 1}{4} = 1/4 \Rightarrow Q_1 = 2$, and $\frac{Q_3 - 1}{4} = 3/4 \Rightarrow Q_3 = 4$.

$\therefore \sigma = \frac{2\sqrt{3}}{3}$ minutes and IQR $= Q_3 - Q_1 = 2$ minutes.



Example 4 — unscaffolded

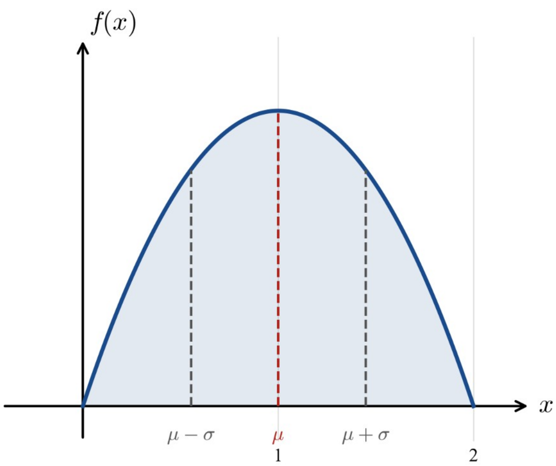
The pdf of X is $f(x) = \frac{3}{4}x(2-x)$ for $0 \le x \le 2$. Find the variance and standard deviation.

$$\mu = \int_0^2 x \cdot 3/4\, x(2-x)\, dx = 3/4 \int_0^2 (2x^2 - x^3)\, dx = 3/4 \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = 3/4 \cdot \frac{4}{3} = 1.$$

$$E(X^2) = \int_0^2 x^2 \cdot 3/4\, x(2-x)\, dx = 3/4 \int_0^2 (2x^3 - x^4)\, dx = 3/4 \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = 3/4 \cdot \frac{8}{5} = \frac{6}{5}.$$

$Var(X) = \frac{6}{5} - 1^2 = \frac{1}{5}, \sigma = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$.

$\therefore Var(X) = \frac{1}{5}$ and $\sigma = \frac{\sqrt{5}}{5}$.



Practice questions

Scaffolded

Q1. For $f(x)=3x^2$, $0\le x\le 1$: (a) find $E(X)$; (b) find $E(X^2)$; (c) hence find $Var(X)$; (d) find the standard deviation.

Q2. The uniform pdf $f(x)=\frac{1}{2}$, $0\le x\le 2$: (a) find $E(X)$; (b) find $Var(X)$ and $sd(X)$; (c) find the quartiles Q_1 and Q_3; (d) hence state the interquartile range.

Q3. For $f(x)=\frac{x}{8}$, $0\le x\le 4$: (a) find $E(X)$; (b) find $E(X^2)$; (c) find $Var(X)$; (d) find $sd(X)$.

Q4. For $f(x)=\frac{x^2}{9}$, $0\le x\le 3$: (a) find $E(X)$; (b) find $E(X^2)$; (c) find $Var(X)$ and $sd(X)$.

Q5. For $f(x)=\frac{3}{2}(1-x^2)$, $0\le x\le 1$: (a) find $E(X)$; (b) find $E(X^2)$; (c) hence find $Var(X)$ and $sd(X)$.

Q6. The uniform pdf $f(x)=\frac{1}{3}$, $2\le x\le 5$: (a) find $E(X)$ and $Var(X)$; (b) find $sd(X)$; (c) find the quartiles and the interquartile range.

Unscaffolded

For **Q7–Q13** and **Q15–Q17, Q19–Q21**, find the **variance** and the **standard deviation** of X. For **Q14** and **Q18**, find the quartiles and the interquartile range.

Q7. $f(x)=4x^3$, $0\le x\le 1$.

Q8. $f(x)=\frac{x}{2}$, $0\le x\le 2$.

Q9. $f(x)=\frac{3x^2}{64}$, $0\le x\le 4$.

Q10. $f(x)=\frac{2x}{25}$, $0\le x\le 5$.

Q11. $f(x)=6x(1-x)$, $0\le x\le 1$.

Q12. $f(x)=\frac{1}{6}$, $-1\le x\le 5$. Also find Q_1, Q_3 and the IQR.

Q13. $f(x)=\frac{x^3}{4}$, $0\le x\le 2$.

Q14. $f(x)=\frac{2x}{9}$, $0\le x\le 3$. Find the exact quartiles and interquartile range.

Q15. $f(x)=12x^2(1-x)$, $0\le x\le 1$.

Q16. $f(x)=\frac{x}{18}$, $0\le x\le 6$.

Q17. $f(x)=\frac{3}{2}\sqrt{x}$, $0\le x\le 1$. Show that $E(X^2)=\frac{3}{7}$, and hence find $Var(X)$.

Q18. $f(x)=\frac{1}{5}$, $0\le x\le 5$. Find Q_1, the median, Q_3 and the interquartile range.

Q19. $f(x)=\frac{3}{4}-\frac{3x^2}{16}$, $0\le x\le 2$.

Q20. $f(x)=5x^4$, $0\le x\le 1$.

Q21. $f(x)=\frac{3}{32}x(4-x)$, $0\le x\le 4$. Given that $E(X)=2$, find the standard deviation, and mark $\mu\pm\sigma$ on a sketch of the pdf.

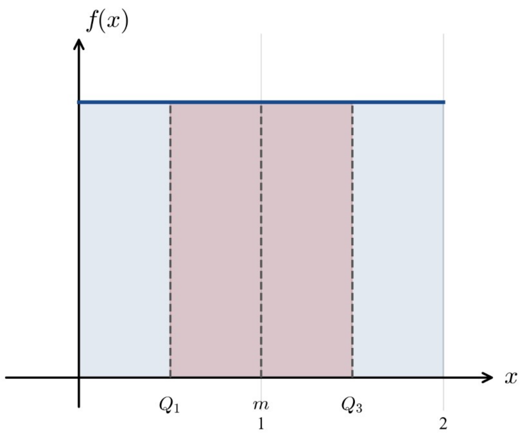
Answers

Q1. (a) $3/4$ (b) $3/5$ (c) $3/80$ (d) $\frac{\sqrt{15}}{20}$. **Q2.** (a) 1 (b) $Var=1/3$, $\sigma=\frac{\sqrt{3}}{3}$ (c) $Q_1=1/2$, $Q_3=3/2$ (d) IQR $=1$. **Q3.** (a) $8/3$ (b) 8 (c) $8/9$ (d) $\frac{2\sqrt{2}}{3}$. **Q4.** (a) $9/4$ (b) $27/5$ (c) $Var=27/80$, $\sigma=\frac{3\sqrt{15}}{20}$. **Q5.** (a) $3/8$ (b) $1/5$ (c) $Var=19/320$, $\sigma=\frac{\sqrt{95}}{40}$. **Q6.** (a) $E(X)=7/2$, $Var=3/4$ (b) $\sigma=\frac{\sqrt{3}}{2}$ (c) $Q_1=11/4$, $Q_3=17/4$, IQR $=3/2$. **Q7.** $Var=2/75$, $\sigma=\frac{\sqrt{6}}{15}$. **Q8.** $Var=2/9$, $\sigma=\frac{\sqrt{2}}{3}$. **Q9.** $Var=3/5$, $\sigma=\frac{\sqrt{15}}{5}$. **Q10.** $Var=25/18$, $\sigma=\frac{5\sqrt{2}}{6}$. **Q11.** $Var=1/20$, $\sigma=\frac{\sqrt{5}}{10}$. **Q12.** $Var=3$, $\sigma=\sqrt{3}$; $Q_1=1/2$, $Q_3=7/2$, IQR $=3$. **Q13.** $Var=8/75$, $\sigma=\frac{2\sqrt{6}}{15}$. **Q14.** $Q_1=3/2$, median $=\frac{3\sqrt{2}}{2}$, $Q_3=\frac{3\sqrt{3}}{2}$, IQR $=\frac{3\sqrt{3}-3}{2}$. **Q15.** $Var=1/25$, $\sigma=1/5$. **Q16.** $Var=2$, $\sigma=\sqrt{2}$. **Q17.** $Var=12/175$, $\sigma=\frac{2\sqrt{21}}{35}$. **Q18.** $Q_1=5/4$, median $=5/2$, $Q_3=15/4$, IQR $=5/2$. **Q19.** $Var=19/80$, $\sigma=\frac{\sqrt{95}}{20}$. **Q20.** $Var=5/252$, $\sigma=\frac{\sqrt{35}}{42}$. **Q21.** $Var=4/5$, $\sigma=\frac{2\sqrt{5}}{5}$.

Fully-worked solutions

Q1. $\mu=\int_0^1 x(3x^2)\,dx=\left[\frac{3x^4}{4}\right]_0^1=\frac{3}{4}$; $E(X^2)=\int_0^1 x^2(3x^2)\,dx=\left[\frac{3x^5}{5}\right]_0^1=\frac{3}{5}$; $Var=\frac{3}{5}-\frac{9}{16}=\frac{48-45}{80}=\frac{3}{80}$; $\sigma=\sqrt{\frac{3}{80}}=\frac{\sqrt{15}}{20}$.

Q2. $\mu=\int_0^2 \frac{x}{2}\,dx=1$; $E(X^2)=\int_0^2 \frac{x^2}{2}\,dx=\frac{4}{3}$; $Var=\frac{4}{3}-1=\frac{1}{3}$, $\sigma=\frac{\sqrt{3}}{3}$. Quartiles: $\int_0^{Q_1} 1/2\,dx=1/4 \Rightarrow \frac{Q_1}{2}=1/4 \Rightarrow Q_1=1/2$; similarly $Q_3=3/2$. IQR $=1$.

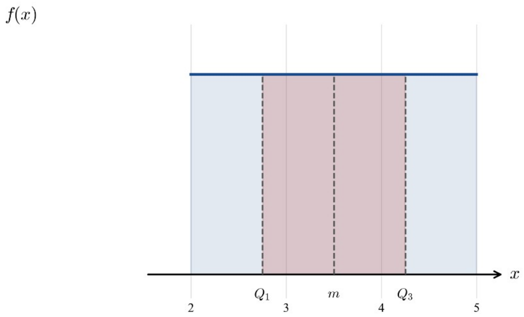


Q3. $\mu=\int_0^4 x\cdot\frac{x}{8}\,dx=\left[\frac{x^3}{24}\right]_0^4=\frac{64}{24}=\frac{8}{3}$; $E(X^2)=\int_0^4 \frac{x^3}{8}\,dx=\left[\frac{x^4}{32}\right]_0^4=8$; $Var=8-\frac{64}{9}=\frac{8}{9}$; $\sigma=\frac{2\sqrt{2}}{3}$.

Q4. $\mu=\int_0^3 \frac{x^3}{9}\,dx=\left[\frac{x^4}{36}\right]_0^3=\frac{81}{36}=\frac{9}{4}$; $E(X^2)=\int_0^3 \frac{x^4}{9}\,dx=\left[\frac{x^5}{45}\right]_0^3=\frac{243}{45}=\frac{27}{5}$; $Var=\frac{27}{5}-\frac{81}{16}=\frac{432-405}{80}=\frac{27}{80}$; $\sigma=\sqrt{\frac{27}{80}}=\frac{3\sqrt{15}}{20}$.

Q5. $\mu=\int_0^1 x\cdot 3/2(1-x^2)\,dx=3/2\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1=3/2\cdot\frac{1}{4}=\frac{3}{8}$;
$E(X^2)=3/2\int_0^1 (x^2-x^4)\,dx=3/2\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1=3/2\cdot\frac{2}{15}=\frac{1}{5}$; $Var=\frac{1}{5}-\frac{9}{64}=\frac{64-45}{320}=\frac{19}{320}$; $\sigma=\frac{\sqrt{95}}{40}$.

Q6. $\mu=\int_2^5 \frac{x}{3}\,dx=\left[\frac{x^2}{6}\right]_2^5=\frac{25-4}{6}=\frac{7}{2}$; $E(X^2)=\int_2^5 \frac{x^2}{3}\,dx=\left[\frac{x^3}{9}\right]_2^5=\frac{125-8}{9}=13$; $Var=13-\frac{49}{4}=\frac{3}{4}$, $\sigma=\frac{\sqrt{3}}{2}$.
Quartiles: $\frac{Q_1-2}{3}=1/4\Rightarrow Q_1=11/4$; $\frac{Q_3-2}{3}=3/4\Rightarrow Q_3=17/4$; IQR $=3/2$.



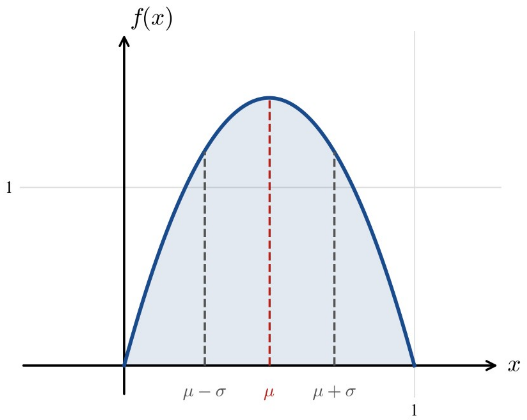
Q7. $\mu=\int_0^1 4x^4\,dx=\frac{4}{5}$; $E(X^2)=\int_0^1 4x^5\,dx=\frac{2}{3}$; $Var=\frac{2}{3}-\frac{16}{25}=\frac{50-48}{75}=\frac{2}{75}$; $\sigma=\frac{\sqrt{6}}{15}$.

Q8. $\mu=\int_0^2 \frac{x^2}{2}\,dx=\frac{4}{3}$; $E(X^2)=\int_0^2 \frac{x^3}{2}\,dx=2$; $Var=2-\frac{16}{9}=\frac{2}{9}$; $\sigma=\frac{\sqrt{2}}{3}$.

Q9. $\mu=\int_0^4 \frac{3x^3}{64}\,dx=\left[\frac{3x^4}{256}\right]_0^4=3$; $E(X^2)=\int_0^4 \frac{3x^4}{64}\,dx=\frac{48}{5}$; $Var=\frac{48}{5}-9=\frac{3}{5}$; $\sigma=\frac{\sqrt{15}}{5}$.

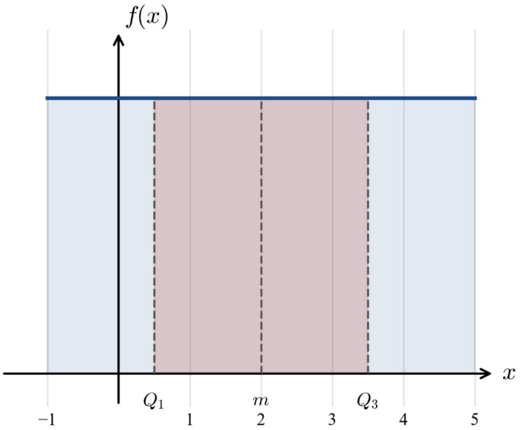
Q10. $\mu=\int_0^5 \frac{2x^2}{25}\,dx=\left[\frac{2x^3}{75}\right]_0^5=\frac{10}{3}$; $E(X^2)=\int_0^5 \frac{2x^3}{25}\,dx=\frac{25}{2}$; $Var=\frac{25}{2}-\frac{100}{9}=\frac{225-200}{18}=\frac{25}{18}$; $\sigma=\frac{5\sqrt{2}}{6}$.

Q11. $\mu=\int_0^1 6x^2(1-x)\,dx=6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{2}$; $E(X^2)=6\int_0^1 (x^3-x^4)\,dx=6\cdot\frac{1}{20}=\frac{3}{10}$; $Var=\frac{3}{10}-\frac{1}{4}=\frac{1}{20}$; $\sigma=\frac{\sqrt{5}}{10}$
.



Q12. $\mu=\int_{-1}^{5}\frac{x}{6}\,dx=\left[\frac{x^2}{12}\right]_{-1}^{5}=\frac{25-1}{12}=2$; $E(X^2)=\int_{-1}^{5}\frac{x^2}{6}\,dx=\left[\frac{x^3}{18}\right]_{-1}^{5}=\frac{125+1}{18}=7$; $Var=7-4=3$, $\sigma=\sqrt{3}$.

Quartiles: $\frac{Q_1+1}{6}=1/4\Rightarrow Q_1=1/2$; $\frac{Q_3+1}{6}=3/4\Rightarrow Q_3=7/2$; IQR $=3$.

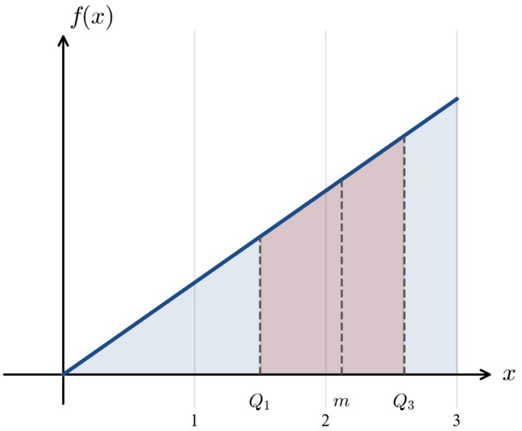


Q13. $\mu=\int_{0}^{2}\frac{x^4}{4}\,dx=\left[\frac{x^5}{20}\right]_{0}^{2}=\frac{32}{20}=\frac{8}{5}$; $E(X^2)=\int_{0}^{2}\frac{x^5}{4}\,dx=\left[\frac{x^6}{24}\right]_{0}^{2}=\frac{64}{24}=\frac{8}{3}$; $Var=\frac{8}{3}-\frac{64}{25}=\frac{200-192}{75}=\frac{8}{75}$; $\sigma=\frac{2\sqrt{6}}{15}$

.

Q14. Solve $\int_{0}^{Q}\frac{2x}{9}\,dx=\left[\frac{x^2}{9}\right]_{0}^{Q}=\frac{Q^2}{9}$. Then $\frac{Q_1^2}{9}=1/4\Rightarrow Q_1^2=9/4\Rightarrow Q_1=\frac{3}{2}$; median: $\frac{m^2}{9}=1/2\Rightarrow m=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$;

$\frac{Q_3^2}{9}=3/4\Rightarrow Q_3=\frac{3\sqrt{3}}{2}$. IQR $=Q_3-Q_1=\frac{3\sqrt{3}-3}{2}$.



Q15. $\mu=\int_{0}^{1}12x^3(1-x)\,dx=12\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_{0}^{1}=\frac{3}{5}$; $E(X^2)=12\int_{0}^{1}(x^4-x^5)\,dx=12\cdot\frac{1}{30}=\frac{2}{5}$; $Var=\frac{2}{5}-\frac{9}{25}=\frac{1}{25}$;

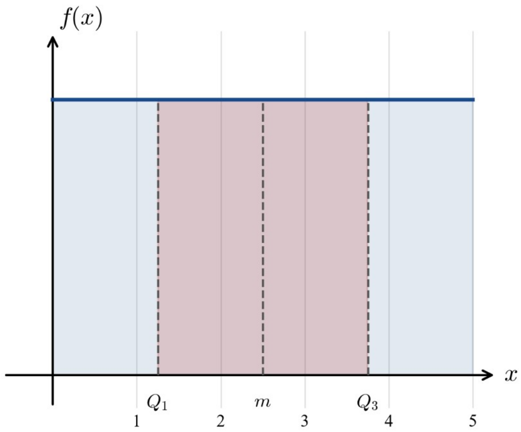
$\sigma=\frac{1}{5}$.

Q16. $\mu=\int_{0}^{6}\frac{x^2}{18}\,dx=\left[\frac{x^3}{54}\right]_{0}^{6}=4$; $E(X^2)=\int_{0}^{6}\frac{x^3}{18}\,dx=\left[\frac{x^4}{72}\right]_{0}^{6}=18$; $Var=18-16=2$; $\sigma=\sqrt{2}$.

Q17. $\mu=\int_{0}^{1}x\cdot 3/2\,x^{1/2}\,dx=3/2\int_{0}^{1}x^{3/2}\,dx=3/2\cdot\frac{2}{5}=\frac{3}{5}$; $E(X^2)=3/2\int_{0}^{1}x^{5/2}\,dx=3/2\cdot\frac{2}{7}=\frac{3}{7}$ (as required);

$Var=\frac{3}{7}-\frac{9}{25}=\frac{75-63}{175}=\frac{12}{175}$; $\sigma=\sqrt{\frac{12}{175}}=\frac{2\sqrt{21}}{35}$.

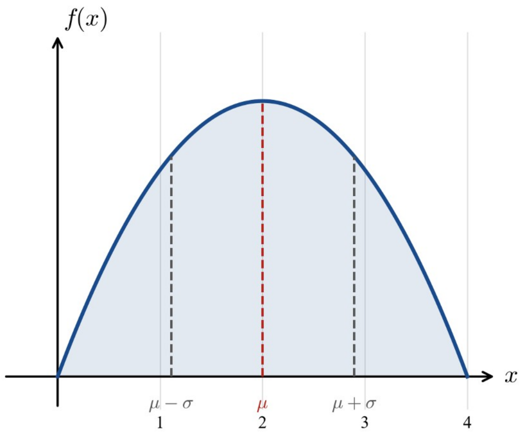
Q18. For the uniform pdf, \$ = \$ (the required fraction). $Q_1 = 5/4$, median $= 5/2$, $Q_3 = 15/4$; IQR $= 15/4 - 5/4 = 5/2$.



Q19. $\mu = \int_0^2 x(3/4 - 3x^2/16)\,dx = \left[\frac{3x^2}{8} - \frac{3x^4}{64}\right]_0^2 = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$;
$E(X^2) = \int_0^2 (3/4x^2 - 3x^4/16)\,dx = \left[\frac{x^3}{4} - \frac{3x^5}{80}\right]_0^2 = 2 - \frac{6}{5} = \frac{4}{5}$; $Var = \frac{4}{5} - \frac{9}{16} = \frac{64-45}{80} = \frac{19}{80}$; $\sigma = \frac{\sqrt{95}}{20}$.

Q20. $\mu = \int_0^1 5x^5\,dx = \frac{5}{6}$; $E(X^2) = \int_0^1 5x^6\,dx = \frac{5}{7}$; $Var = \frac{5}{7} - \frac{25}{36} = \frac{180-175}{252} = \frac{5}{252}$; $\sigma = \frac{\sqrt{35}}{42}$.

Q21. With $\mu = 2$: $E(X^2) = \int_0^4 x^2 \cdot 3/32\,x(4-x)\,dx = 3/32 \int_0^4 (4x^3 - x^4)\,dx = 3/32\left[x^4 - \frac{x^5}{5}\right]_0^4 = 3/32 \cdot \frac{256}{5} = \frac{24}{5}$;
$Var = \frac{24}{5} - 4 = \frac{4}{5}$; $\sigma = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5} \approx 0.894$. The interval $\mu \pm \sigma$ captures the central spread about the mean:



Chapter 13 Continuous probability distributions — 13D Mean and variance under linear change

Theory

Once the **mean** $\mu = \mathrm{E}(X)$ and **variance** $\mathrm{Var}(X)$ of a random variable X are known, the mean and variance of any **linear transformation** $aX+b$ can be found *without re-integrating*.

Linearity of expectation. For constants a and b,

$$\mathrm{E}(aX+b) = a\,\mathrm{E}(X)+b.$$

Expectation is a "balance point", so scaling X by a scales the mean by a, and shifting by b shifts the mean by b.

Variance under a linear transformation.

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X).$$

Adding a constant b slides the whole distribution along without changing its spread, so b has **no effect** on the variance. Scaling by a stretches the spread, and because variance is measured in *squared* units the factor is a^2. Taking the square root,

$$\mathrm{sd}(aX+b) = |a|\,\mathrm{sd}(X).$$

The computing formula for variance. Rearranging $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ gives the frequently-needed identity

$$\mathrm{E}(X^2) = \mathrm{Var}(X) + \mu^2.$$

This lets you evaluate $\mathrm{E}(X^2)$ (and hence E of any quadratic in X) directly from μ and $\mathrm{Var}(X)$.

Cautions.

- $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$ — do **not** write $a\,\mathrm{Var}(X)$, and the $+b$ disappears.
- Keep **variance and standard deviation** distinct: $\mathrm{sd} = \sqrt{\mathrm{Var}}$, and $\mathrm{sd}(aX+b) = |a|\,\mathrm{sd}(X)$ uses $|a|$ (a standard deviation is never negative).
- $\mathrm{E}(X^2) \neq [\mathrm{E}(X)]^2$ in general — the difference is exactly $\mathrm{Var}(X)$.

Worked examples

Example 1 — scaffolded

A random variable X has $\mathrm{E}(X)=5$ and $\mathrm{Var}(X)=4$. Find $\mathrm{E}(3X+2)$, $\mathrm{Var}(3X+2)$ and $\mathrm{sd}(3X+2)$.

Working	Comment
$\mathrm{E}(3X+2) = 3\,\mathrm{E}(X)+2 = 3(5)+2 = 17$	Use $\mathrm{E}(aX+b) = a\,\mathrm{E}(X)+b$ with $a=3$, $b=2$.
$\mathrm{Var}(3X+2) = 3^2\,\mathrm{Var}(X) = 9\times 4 = 36$	Use $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$; the $+2$ has no effect.
$\mathrm{sd}(3X+2) = \sqrt{36} = 6$	$\mathrm{sd} = \sqrt{\mathrm{Var}}$ (equivalently $|a|\,\mathrm{sd}(X) = 3\times 2$).

Example 2 — scaffolded

A random variable X has mean 10 and standard deviation 3. Find $\mathrm{E}(2X-1)$, $\mathrm{Var}(2X-1)$, $\mathrm{sd}(2X-1)$ and $\mathrm{E}(X^2)$.

Working	Comment
$\mathrm{E}(2X-1)=2(10)-1=19$	$a=2, b=-1$.
$\mathrm{Var}(X)=3^2=9$	Variance is the square of the standard deviation.
$\mathrm{Var}(2X-1)=2^2\times 9=36$, so $\mathrm{sd}(2X-1)=6$	$a^2\,\mathrm{Var}(X)$; then square-root.
$\mathrm{E}(X^2)=\mathrm{Var}(X)+\mu^2=9+10^2=109$	Use $\mathrm{E}(X^2)=\mathrm{Var}(X)+\mu^2$.

Example 3 — unscaffolded

The continuous random variable X has probability density function $f(x)=\dfrac{x}{2}$ for $0\le x\le 2$ (and 0 elsewhere). Find $\mathrm{Var}(X)$, and hence $\mathrm{E}(4X+1)$ and $\mathrm{Var}(4X+1)$.

$$\mathrm{E}(X)=\int_0^2 x\cdot\frac{x}{2}\,dx=\int_0^2\frac{x^2}{2}\,dx=\left[\frac{x^3}{6}\right]_0^2=\frac{8}{6}=\frac{4}{3}.$$

$$\mathrm{E}(X^2)=\int_0^2 x^2\cdot\frac{x}{2}\,dx=\left[\frac{x^4}{8}\right]_0^2=2.$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=2-\left(\frac{4}{3}\right)^2=2-\frac{16}{9}=\frac{2}{9}.$$

$$\mathrm{E}(4X+1)=4\times\frac{4}{3}+1=\frac{19}{3}.$$

$$\mathrm{Var}(4X+1)=4^2\times\frac{2}{9}=\frac{32}{9}.$$

$\therefore\ \mathrm{E}(4X+1)=\dfrac{19}{3}$, $\mathrm{Var}(4X+1)=\dfrac{32}{9}$ (and $\mathrm{sd}(4X+1)=\dfrac{4\sqrt{2}}{3}$).

Example 4 — unscaffolded

A random variable X has $\mathrm{E}(X)=2$ and $\mathrm{E}(X^2)=7$. Find $\mathrm{sd}(X)$, and hence $\mathrm{E}(2X-3)$ and $\mathrm{sd}(2X-3)$.

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=7-2^2=3$, so $\mathrm{sd}(X)=\sqrt{3}$.

$\mathrm{E}(2X-3)=2(2)-3=1$.

$\mathrm{Var}(2X-3)=2^2\times 3=12$, so $\mathrm{sd}(2X-3)=\sqrt{12}=2\sqrt{3}$.

$\therefore\ \mathrm{E}(2X-3)=1$ and $\mathrm{sd}(2X-3)=2\sqrt{3}$.

Practice questions

Scaffolded

Q1. X has $\mathrm{E}(X)=8$ and $\mathrm{Var}(X)=5$. (a) Find $\mathrm{E}(2X+3)$; (b) find $\mathrm{Var}(2X+3)$; (c) find $\mathrm{sd}(2X+3)$.

Q2. X has mean 4 and standard deviation 2. (a) Find $\mathrm{E}(5X)$; (b) find $\mathrm{sd}(5X)$; (c) find $\mathrm{Var}(5X)$; (d) find $\mathrm{E}(X^2)$.

Q3. X has $\mathrm{E}(X)=6$ and $\mathrm{Var}(X)=9$. (a) Find $\mathrm{E}(10-X)$; (b) find $\mathrm{Var}(10-X)$; (c) find $\mathrm{sd}(10-X)$; (d) explain why the variance is unchanged by the reflection.

Q4. X has $\mathrm{E}(X)=3$ and $\mathrm{Var}(X)=2$. (a) Find $\mathrm{E}(X^2)$; (b) find $\mathrm{E}(3X-1)$; (c) find $\mathrm{Var}(3X-1)$.

Q5. The continuous random variable X has pdf $f(x)=\frac{3x^2}{8}$ for $0\le x\le 2$. (a) Find $\mathrm{E}(X)$; (b) find $\mathrm{E}(X^2)$; (c) hence find $\mathrm{Var}(X)$; (d) find $\mathrm{E}(2X)$ and $\mathrm{Var}(2X)$.

Q6. A random variable X satisfies $\mathrm{E}(3X+1)=16$ and $\mathrm{Var}(3X+1)=45$. (a) Find $\mathrm{E}(X)$; (b) find $\mathrm{Var}(X)$.

Unscaffolded

Q7. X has $\mathrm{E}(X)=12$ and $\mathrm{Var}(X)=16$. Find $\mathrm{E}(X^2)$.

Q8. X has mean 7 and standard deviation 2. Find $\mathrm{Var}(3X)$ and $\mathrm{sd}(3X)$.

Q9. X has $\mathrm{E}(X)=5$ and $\mathrm{Var}(X)=8$. Find $\mathrm{E}(2X-4)$, $\mathrm{Var}(2X-4)$ and $\mathrm{sd}(2X-4)$.

Q10. X has $\mathrm{E}(X)=-3$ and $\mathrm{Var}(X)=5$. Find $\mathrm{E}(1-2X)$, $\mathrm{Var}(1-2X)$ and $\mathrm{sd}(1-2X)$.

Q11. X has $\mathrm{E}(X)=10$ and $\mathrm{E}(X^2)=125$. Find $\mathrm{Var}(X)$ and $\mathrm{sd}(X)$.

Q12. A random variable X satisfies $\mathrm{E}(4X-2)=10$ and $\mathrm{sd}(4X-2)=8$. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

Q13. The continuous random variable X is uniform with pdf $f(x)=\frac{1}{2}$ for $0\le x\le 2$. Find $\mathrm{Var}(X)$, and hence $\mathrm{E}(6X)$ and $\mathrm{Var}(6X)$.

Q14. A temperature X (in °C) has mean 20 and standard deviation 4. The temperature in °F is $Y=\frac{9}{5}X+32$. Find $\mathrm{E}(Y)$ and $\mathrm{sd}(Y)$.

Q15. The cost of a job is $C=5X+20$ dollars, where the number of hours X has $\mathrm{E}(X)=8$ and $\mathrm{Var}(X)=3$. Find $\mathrm{E}(C)$, $\mathrm{Var}(C)$ and $\mathrm{sd}(C)$.

Q16. X has $\mathrm{E}(X)=2$ and $\mathrm{Var}(X)=6$. Find $\mathrm{E}\left((X+1)^2\right)$.

Q17. A random variable X satisfies $\mathrm{E}(2X+1)=9$ and $\mathrm{Var}(2X+1)=16$. Find $\mathrm{E}(X^2)$.

Q18. The continuous random variable X has pdf $f(x)=\frac{2x}{9}$ for $0\le x\le 3$. Find $\mathrm{Var}(X)$, and hence $\mathrm{E}(3X-2)$ and $\mathrm{Var}(3X-2)$.

Q19. A random variable X has $\mathrm{Var}(3X-1)=45$. Find $\mathrm{Var}(X)$.

Q20. A random variable X has mean μ and standard deviation σ. The **standardised** variable is $Z=\frac{X-\mu}{\sigma}$. Show that $\mathrm{E}(Z)=0$ and $\mathrm{Var}(Z)=1$.

Q21. X has $\mathrm{E}(X)=5$ and $\mathrm{Var}(X)=9$. Find constants $a>0$ and b so that $Y=aX+b$ has $\mathrm{E}(Y)=0$ and $\mathrm{Var}(Y)=1$.

Answers

Q1. (a) 19 (b) 20 (c) $2\sqrt{5}$ **Q2.** (a) 20 (b) 10 (c) 100 (d) 20 **Q3.** (a) 4 (b) 9 (c) 3 (d) adding/reflecting shifts the distribution without changing its spread **Q4.** (a) 11 (b) 8 (c) 18 **Q5.** (a) $\frac{3}{2}$ (b) $\frac{12}{5}$ (c) $\frac{3}{20}$ (d) $\mathrm{E}(2X)=3$, $\mathrm{Var}(2X)=\frac{3}{5}$ **Q6.** (a) 5 (b) 5 **Q7.** 160 **Q8.** $\mathrm{Var}(3X)=36$, $\mathrm{sd}(3X)=6$ **Q9.** 6, 32, $4\sqrt{2}$ **Q10.** 7, 20, $2\sqrt{5}$ **Q11.** $\mathrm{Var}(X)=25$, $\mathrm{sd}(X)=5$ **Q12.** $\mathrm{E}(X)=3$, $\mathrm{Var}(X)=4$ **Q13.** $\mathrm{Var}(X)=\frac{1}{3}$; $\mathrm{E}(6X)=6$, $\mathrm{Var}(6X)=12$ **Q14.** $\mathrm{E}(Y)=68$, $\mathrm{sd}(Y)=\frac{36}{5}=7.2$ **Q15.** $\mathrm{E}(C)=60$, $\mathrm{Var}(C)=75$, $\mathrm{sd}(C)=5\sqrt{3}$ **Q16.** 15 **Q17.** 20 **Q18.** $\mathrm{Var}(X)=\frac{1}{2}$; $\mathrm{E}(3X-2)=4$, $\mathrm{Var}(3X-2)=\frac{9}{2}$ **Q19.** 5 **Q20.** $\mathrm{E}(Z)=0$, $\mathrm{Var}(Z)=1$ (see solution) **Q21.** $a=\frac{1}{3}$, $b=-\frac{5}{3}$

Fully-worked solutions

Q1. $\mathrm{E}(2X+3)=2(8)+3=19$; $\mathrm{Var}(2X+3)=2^2(5)=20$; $\mathrm{sd}(2X+3)=\sqrt{20}=2\sqrt{5}$.

Q2. $\mathrm{E}(5X)=5(4)=20$; $\mathrm{Var}(X)=2^2=4$, so $\mathrm{Var}(5X)=5^2(4)=100$ and $\mathrm{sd}(5X)=10$; $\mathrm{E}(X^2)=\mathrm{Var}(X)+\mu^2=4+16=20$.

Q3. With $a=-1$, $b=10$: $\mathrm{E}(10-X)=-6+10=4$; $\mathrm{Var}(10-X)=(-1)^2(9)=9$; $\mathrm{sd}(10-X)=3$. (d) The transformation $10-X$ reflects and shifts the distribution but does not stretch it, so the spread — and hence the variance — is unchanged.

Q4. $\mathrm{E}(X^2)=\mathrm{Var}(X)+\mu^2=2+9=11$; $\mathrm{E}(3X-1)=3(3)-1=8$; $\mathrm{Var}(3X-1)=3^2(2)=18$.

Q5. $\mathrm{E}(X)=\int_0^2 x\cdot\frac{3x^2}{8}\,dx=\left[\frac{3x^4}{32}\right]_0^2=\frac{48}{32}=\frac{3}{2}$; $\mathrm{E}(X^2)=\int_0^2 x^2\cdot\frac{3x^2}{8}\,dx=\left[\frac{3x^5}{40}\right]_0^2=\frac{96}{40}=\frac{12}{5}$; $\mathrm{Var}(X)=\frac{12}{5}-\left(\frac{3}{2}\right)^2=\frac{12}{5}-\frac{9}{4}=\frac{3}{20}$. Then $\mathrm{E}(2X)=2\times\frac{3}{2}=3$ and $\mathrm{Var}(2X)=2^2\times\frac{3}{20}=\frac{3}{5}$.

Q6. $\mathrm{E}(3X+1)=3\,\mathrm{E}(X)+1=16\Rightarrow\mathrm{E}(X)=\frac{15}{3}=5$. $\mathrm{Var}(3X+1)=9\,\mathrm{Var}(X)=45\Rightarrow\mathrm{Var}(X)=5$.

Q7. $\mathrm{E}(X^2)=\mathrm{Var}(X)+\mu^2=16+144=160$.

Q8. $\mathrm{Var}(X)=2^2=4$, so $\mathrm{Var}(3X)=3^2(4)=36$ and $\mathrm{sd}(3X)=6$.

Q9. $\mathrm{E}(2X-4)=2(5)-4=6$; $\mathrm{Var}(2X-4)=2^2(8)=32$; $\mathrm{sd}(2X-4)=\sqrt{32}=4\sqrt{2}$.

Q10. $\mathrm{E}(1-2X)=1-2(-3)=7$; $\mathrm{Var}(1-2X)=(-2)^2(5)=20$; $\mathrm{sd}(1-2X)=\sqrt{20}=2\sqrt{5}$.

Q11. $\mathrm{Var}(X)=\mathrm{E}(X^2)-\mu^2=125-100=25$; $\mathrm{sd}(X)=5$.

Q12. $\mathrm{E}(4X-2)=4\,\mathrm{E}(X)-2=10\Rightarrow\mathrm{E}(X)=\frac{12}{4}=3$. $\mathrm{sd}(4X-2)=|4|\,\mathrm{sd}(X)=8\Rightarrow\mathrm{sd}(X)=2$, so $\mathrm{Var}(X)=4$.

Q13. $\mathrm{E}(X)=\int_0^2 x\cdot\frac{1}{2}\,dx=\left[\frac{x^2}{4}\right]_0^2=1$; $\mathrm{E}(X^2)=\int_0^2 x^2\cdot\frac{1}{2}\,dx=\left[\frac{x^3}{6}\right]_0^2=\frac{4}{3}$; $\mathrm{Var}(X)=\frac{4}{3}-1=\frac{1}{3}$. Then $\mathrm{E}(6X)=6$ and $\mathrm{Var}(6X)=6^2\times\frac{1}{3}=12$.

Q14. $\mathrm{E}(Y)=\frac{9}{5}(20)+32=36+32=68$; $\mathrm{sd}(Y)=\left|\frac{9}{5}\right|\times 4=\frac{36}{5}=7.2$.

Q15. $\mathrm{E}(C)=5(8)+20=60$; $\mathrm{Var}(C)=5^2(3)=75$; $\mathrm{sd}(C)=\sqrt{75}=5\sqrt{3}$.

Q16. Expand: $\mathrm{E}\left((X+1)^2\right)=\mathrm{E}(X^2+2X+1)=\mathrm{E}(X^2)+2\,\mathrm{E}(X)+1$. Now $\mathrm{E}(X^2)=\mathrm{Var}(X)+\mu^2=6+4=10$, so $\mathrm{E}\left((X+1)^2\right)=10+2(2)+1=15$.

Q17. $\mathrm{E}(2X+1)=2\mu+1=9\Rightarrow\mu=4$; $\mathrm{Var}(2X+1)=4\,\mathrm{Var}(X)=16\Rightarrow\mathrm{Var}(X)=4$. Then $\mathrm{E}(X^2)=\mathrm{Var}(X)+\mu^2=4+16=20$.

Q18. $\mathrm{E}(X)=\int_0^3 x\cdot\frac{2x}{9}\,dx=\left[\frac{2x^3}{27}\right]_0^3=2$; $\mathrm{E}(X^2)=\int_0^3 x^2\cdot\frac{2x}{9}\,dx=\left[\frac{x^4}{18}\right]_0^3=\frac{9}{2}$; $\mathrm{Var}(X)=\frac{9}{2}-2^2=\frac{1}{2}$. Then $\mathrm{E}(3X-2)=3(2)-2=4$ and $\mathrm{Var}(3X-2)=3^2\times\frac{1}{2}=\frac{9}{2}$.

Q19. $\mathrm{Var}(3X-1)=3^2\,\mathrm{Var}(X)=45\Rightarrow\mathrm{Var}(X)=\frac{45}{9}=5$.

Q20. Writing $Z=\frac{X-\mu}{\sigma}=\frac{1}{\sigma}X-\frac{\mu}{\sigma}$ (a linear transformation with $a=\frac{1}{\sigma}$, $b=-\frac{\mu}{\sigma}$): $\mathrm{E}(Z)=\frac{1}{\sigma}\mathrm{E}(X)-\frac{\mu}{\sigma}=\frac{\mu}{\sigma}-\frac{\mu}{\sigma}=0$; $\mathrm{Var}(Z)=\left(\frac{1}{\sigma}\right)^2\mathrm{Var}(X)=\frac{\sigma^2}{\sigma^2}=1$. $\therefore\mathrm{E}(Z)=0$ and $\mathrm{Var}(Z)=1$.

Q21. For $Y=aX+b$: $\mathrm{Var}(Y)=a^2\,\mathrm{Var}(X)=9a^2=1\Rightarrow a^2=\frac{1}{9}\Rightarrow a=\frac{1}{3}$ (taking $a>0$). Then $\mathrm{E}(Y)=a\,\mathrm{E}(X)+b=\frac{1}{3}(5)+b=0\Rightarrow b=-\frac{5}{3}$. $\therefore a=\frac{1}{3}$, $b=-\frac{5}{3}$ (so $Y=\frac{X-5}{3}$, the standardisation of X).

Chapter 13 Continuous probability distributions — 13E Cumulative distribution functions

Note — optional section. The study design states that cumulative distribution functions *may be used but are not required*. This section is optional: it supports the percentile work of 13B, and CDF-specific techniques are not assessed in the graded topic tests or practice examinations.

Theory

For a continuous random variable X with probability density function f, the **cumulative distribution function** (cdf) is

$$F(x) = Pr(X \le x) = \int_{-\infty}^{x} f(t)\,dt.$$

Properties.

- F is **non-decreasing**, with $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to \infty$.
- Since F is an antiderivative of f, $F'(x) = f(x)$ wherever f is continuous — so the pdf can be recovered from the cdf by differentiating.
- For a pdf supported on $[a, b]$: $F(x) = 0$ for $x < a$, $F(x) = \int_{a}^{x} f(t)\,dt$ for $a \le x \le b$, and $F(x) = 1$ for $x > b$.

Using the cdf.

- **Probabilities:** $Pr(a < X \le b) = F(b) - F(a)$, and $Pr(X > a) = 1 - F(a)$.
- **Median** m**:** the solution of $F(m) = 1/2$.
- **Percentiles/quantiles:** the p-quantile is the solution of $F(x) = p$ (e.g. the lower quartile solves $F(x) = 1/4$).

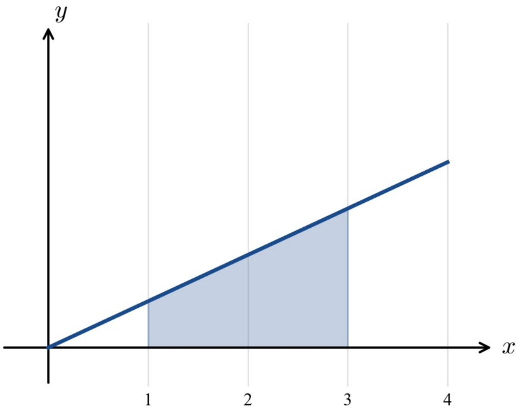
To construct F from f: integrate f from the left end of the support up to x, and state the three pieces (0, the integral, 1). **To recover f from F:** differentiate, $f = F'$.

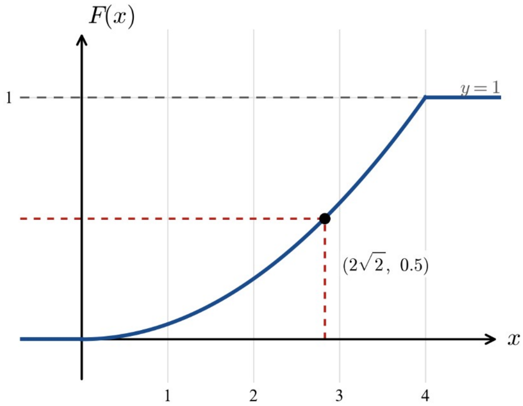
Worked examples

Example 1 — scaffolded

A continuous random variable X has pdf $f(x) = \dfrac{x}{8}$ for $0 \le x \le 4$ (and 0 elsewhere). Find the cdf F, hence $Pr(1 \le X \le 3)$ and the median.

Working	Comment
For $0 \le x \le 4$: $F(x) = \int_{0}^{x} \frac{t}{8}\,dt = \left[\frac{t^2}{16}\right]_0^x = \frac{x^2}{16}$.	Integrate the pdf from the left end 0 up to x.
$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{16}, & 0 \le x \le 4 \\ 1, & x > 4 \end{cases}$	State all three pieces. Check: $F(4) = \frac{16}{16} = 1$.
$Pr(1 \le X \le 3) = F(3) - F(1) = \frac{9}{16} - \frac{1}{16} = \frac{8}{16} = \frac{1}{2}$.	Use $Pr(a \le X \le b) = F(b) - F(a)$.
Median: $F(m) = \frac{1}{2} \Rightarrow \frac{m^2}{16} = \frac{1}{2} \Rightarrow m^2 = 8 \Rightarrow m = 2\sqrt{2}$.	Solve $F(m) = 1/2$; take the value in $[0, 4]$.





Example 2 — scaffolded

X has pdf $f(x)=\frac{3}{8}x^2$ for $0\leq x\leq 2$. Find F, hence $Pr(X\leq 1)$ and the median.

Working	Comment
$F(x)=\int_0^x \frac{3}{8}t^2\,dt=\left[\frac{t^3}{8}\right]_0^x=\frac{x^3}{8}$ for $0\leq x\leq 2$.	Integrate; $F(2)=\frac{8}{8}=1$. ✓
$Pr(X\leq 1)=F(1)=\frac{1}{8}$.	$Pr(X\leq a)=F(a)$.
Median: $\frac{m^3}{8}=\frac{1}{2}\Rightarrow m^3=4\Rightarrow m=\sqrt[3]{4}$.	Solve $F(m)=1/2$.

Example 3 — unscaffolded

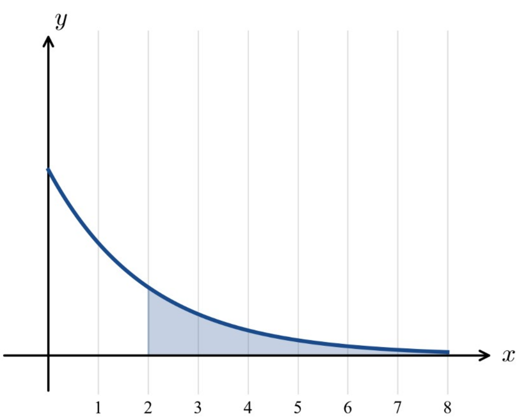
X has pdf $f(x)=\frac{1}{2}e^{-x/2}$ for $x\geq 0$. Find F, hence $Pr(X>2)$ and the median.

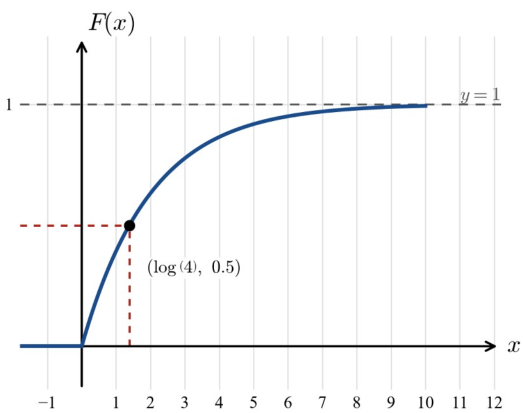
For $x\geq 0$: $F(x)=\int_0^x \frac{1}{2}e^{-t/2}\,dt=\left[-e^{-t/2}\right]_0^x=1-e^{-x/2}$.

$Pr(X>2)=1-F(2)=1-(1-e^{-1})=e^{-1}$.

Median: $1-e^{-m/2}=\frac{1}{2}\Rightarrow e^{-m/2}=\frac{1}{2}\Rightarrow -\frac{m}{2}=\log_e 1/2\Rightarrow m=2\log_e 2$.

$\therefore F(x)=1-e^{-x/2}$ $(x\geq 0)$, $Pr(X>2)=e^{-1}$, median $2\log_e 2$.





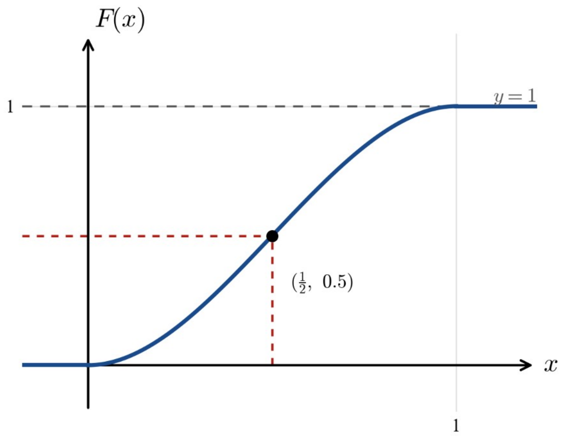
Example 4 — unscaffolded

A continuous random variable X has cdf $F(x)=3x^2-2x^3$ for $0\le x\le 1$ (with $F(x)=0$ for $x<0$ and $F(x)=1$ for $x>1$). Find the pdf f, and $Pr(X\le 1/2)$.

Recover f by differentiating: $f(x)=F'(x)=6x-6x^2=6x(1-x)$ for $0\le x\le 1$ (and 0 elsewhere).

$Pr(X\le 1/2)=F(1/2)=3\cdot 1/4-2\cdot 1/8=3/4-1/4=\frac{1}{2}$.

$\therefore f(x)=6x(1-x)$ on $[0,1]$, and $Pr(X\le 1/2)=\frac{1}{2}$ (so $1/2$ is the median).



Practice questions

Scaffolded

Q1. X has pdf $f(x)=\frac{x}{2}$ for $0\le x\le 2$. (a) Find the cdf $F(x)$ for $0\le x\le 2$. (b) Find $Pr(X\le 1)$. (c) Find the median. (d) Find $Pr(1/2\le X\le 3/2)$.

Q2. X has pdf $f(x)=3x^2$ for $0\le x\le 1$. (a) Find $F(x)$. (b) Find $Pr(X\le 1/2)$. (c) Find the median.

Q3. X is uniform with pdf $f(x)=\frac{1}{5}$ for $0\le x\le 5$. (a) Find $F(x)$. (b) State the median. (c) Find the 10th and 90th percentiles.

Q4. X has pdf $f(x)=\frac{x-1}{2}$ for $1\le x\le 3$. (a) Find $F(x)$. (b) Find $Pr(1\le X\le 2)$. (c) Find the median.

Q5. X has pdf $f(x)=e^{-x}$ for $x\ge 0$. (a) Find $F(x)$. (b) Find $Pr(X>1)$. (c) Find the median.

Q6. X has cdf $F(x)=\sin x$ for $0\le x\le \frac{\pi}{2}$. (a) Find the pdf $f(x)$. (b) Find $Pr(0\le X\le \pi/6)$. (c) State the median.

Unscaffolded

For each variable, find the cdf F (or the pdf f where the cdf is given), and the requested quantities.

Q7. $f(x)=\frac{x^2}{9}$, $0\le x\le 3$. Find F and $Pr(1\le X\le 2)$.

Q8. $f(x)=\frac{x}{18}$, $0\le x\le 6$. Find F, $Pr(2\le X\le 4)$ and the median.

Q9. $f(x)=4x^3$, $0\le x\le 1$. Find F and $Pr(X\le 1/2)$.

Q10. $f(x)=\frac{2}{9}(3-x)$, $0\le x\le 3$. Find F, $Pr(X\le 1)$ and the median.

Q11. $f(x)=\frac{1}{3}e^{-x/3}$, $x\ge 0$. Find F, $Pr(X>3)$ and the median.

Q12. $f(x)=5x^4$, $0\le x\le 1$. Find F and $Pr(X\le 1/2)$.

Q13. X is uniform with $f(x)=\frac{1}{3}$, $2\le x\le 5$. Find F, the median and $Pr(3\le X\le 4)$.

Q14. X has cdf $F(x)=1-e^{-x/4}$ for $x\ge 0$. Find the pdf f, and show that the median is $4\log_e 2$.

Q15. $f(x)=\frac{1}{2}\cos\left(\frac{x}{2}\right)$, $0\le x\le \pi$. Find F, the median and $Pr(0\le X\le \pi/2)$.

Q16. $f(x)=2x$, $0\le x\le 1$. Find F, $Pr(X\le 1/2)$ and the median.

Q17. X is uniform with $f(x)=\frac{1}{4}$, $1\le x\le 5$. Find F, the median and $Pr(2\le X\le 3)$.

Q18. $f(x)=\frac{1}{5}e^{-x/5}$, $x\ge 0$. Find F, $Pr(X>5)$ and the median.

Q19. X has cdf $F(x)=x^2(2-x^2)$ for $0\le x\le 1$. **Show that** $f(x)=4x(1-x^2)$ on $[0,1]$, and find $Pr(X\le 1/2)$.

Q20. X has cdf $F(x)=1-e^{-2x}$ for $x\ge 0$. Find the pdf f, $Pr(X>1)$ and the median.

Q21. A continuous random variable has pdf $f(x)=\frac{3}{32}x(4-x)$ for $0\le x\le 4$. Find the cdf $F(x)$ for $0\le x\le 4$, and hence find $Pr(0\le X\le 2)$. State the median (use the symmetry of f).

Answers

Q1. $F=\dfrac{x^2}{4}$; (b) $\dfrac{1}{4}$; (c) $\sqrt{2}$; (d) $\dfrac{1}{2}$. **Q2.** $F=x^3$; $\dfrac{1}{8}$; median $\dfrac{\sqrt[3]{4}}{2}$. **Q3.** $F=\dfrac{x}{5}$; median $\dfrac{5}{2}$; 10th $=\dfrac{1}{2}$, 90th $=\dfrac{9}{2}$. **Q4.** $F=\dfrac{(x-1)^2}{4}$; $\dfrac{1}{4}$; median $1+\sqrt{2}$. **Q5.** $F=1-e^{-x}$; e^{-1}; median $\log_e 2$. **Q6.** $f=\cos x$; $\dfrac{1}{2}$; median $\dfrac{\pi}{6}$. **Q7.** $F=\dfrac{x^3}{27}$; $Pr=\dfrac{7}{27}$. **Q8.** $F=\dfrac{x^2}{36}$; $\dfrac{1}{3}$; median $3\sqrt{2}$. **Q9.** $F=x^4$; $\dfrac{1}{16}$. **Q10.** $F=\dfrac{x(6-x)}{9}$; $\dfrac{5}{9}$; median $3-\dfrac{3\sqrt{2}}{2}$. **Q11.** $F=1-e^{-x/3}$; e^{-1}; median $3\log_e 2$. **Q12.** $F=x^5$; $\dfrac{1}{32}$. **Q13.** $F=\dfrac{x-2}{3}$; median $\dfrac{7}{2}$; $Pr=\dfrac{1}{3}$. **Q14.** $f=\dfrac{1}{4}e^{-x/4}$; median $4\log_e 2$. **Q15.** $F=\sin\left(\dfrac{x}{2}\right)$; median $\dfrac{\pi}{3}$; $Pr=\dfrac{\sqrt{2}}{2}$. **Q16.** $F=x^2$; $\dfrac{1}{4}$; median $\dfrac{\sqrt{2}}{2}$. **Q17.** $F=\dfrac{x-1}{4}$; median 3; $Pr=\dfrac{1}{4}$. **Q18.** $F=1-e^{-x/5}$; e^{-1}; median $5\log_e 2$. **Q19.** $f=4x(1-x^2)$; $Pr=\dfrac{7}{16}$. **Q20.** $f=2e^{-2x}$; e^{-2}; median $\dfrac{\log_e 2}{2}$. **Q21.** $F=\dfrac{x^2(6-x)}{32}$; $Pr=\dfrac{1}{2}$; median 2.

Fully-worked solutions

Q1. $F(x)=\int_0^x \dfrac{t}{2}\,dt=\dfrac{x^2}{4}$ ($0\le x\le 4$… check $F(2)=1$ ✓). (b) $F(1)=\dfrac{1}{4}$. (c) $\dfrac{m^2}{4}=\dfrac{1}{2}\Rightarrow m^2=2\Rightarrow m=\sqrt{2}$. (d) $F(3/2)-F(1/2)=\dfrac{9}{16}-\dfrac{1}{16}=\dfrac{1}{2}$.

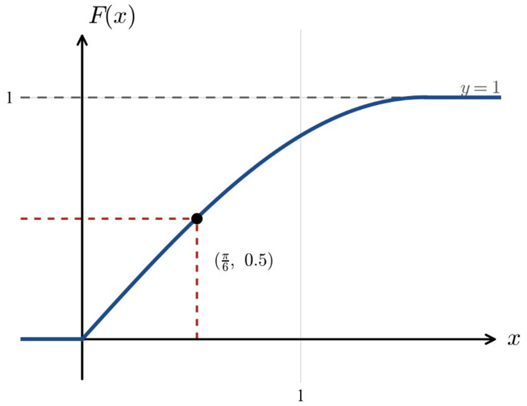
Q2. $F(x)=\int_0^x 3t^2\,dt=x^3$. (b) $F(1/2)=\dfrac{1}{8}$. (c) $m^3=\dfrac{1}{2}\Rightarrow m=\dfrac{1}{\sqrt[3]{2}}=\dfrac{\sqrt[3]{4}}{2}$.

Q3. $F(x)=\int_0^x \dfrac{1}{5}\,dt=\dfrac{x}{5}$. (b) $\dfrac{m}{5}=\dfrac{1}{2}\Rightarrow m=\dfrac{5}{2}$. (c) $\dfrac{x}{5}=\dfrac{1}{10}\Rightarrow x=\dfrac{1}{2}$; $\dfrac{x}{5}=\dfrac{9}{10}\Rightarrow x=\dfrac{9}{2}$.

Q4. $F(x)=\int_1^x \dfrac{t-1}{2}\,dt=\left[\dfrac{(t-1)^2}{4}\right]_1^x=\dfrac{(x-1)^2}{4}$ ($F(3)=1$ ✓). (b) $F(2)-F(1)=\dfrac{1}{4}-0=\dfrac{1}{4}$. (c) $\dfrac{(m-1)^2}{4}=\dfrac{1}{2}\Rightarrow(m-1)^2=2\Rightarrow m=1+\sqrt{2}$.

Q5. $F(x)=\int_0^x e^{-t}\,dt=1-e^{-x}$. (b) $Pr(X>1)=1-F(1)=e^{-1}$. (c) $1-e^{-m}=\dfrac{1}{2}\Rightarrow e^{-m}=\dfrac{1}{2}\Rightarrow m=\log_e 2$.

Q6. (a) $f(x)=F'(x)=\cos x$ for $0\le x\le\dfrac{\pi}{2}$. (b) $F(\pi/6)-F(0)=\sin\dfrac{\pi}{6}-0=\dfrac{1}{2}$. (c) $\sin m=\dfrac{1}{2}\Rightarrow m=\dfrac{\pi}{6}$.

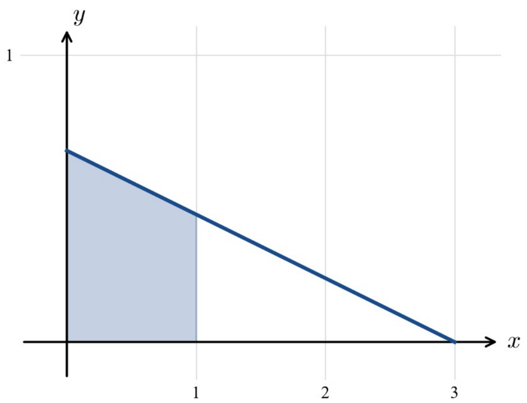


Q7. $F(x)=\int_0^x \frac{t^2}{9}\,dt=\frac{x^3}{27}$ ($F(3)=1$ ✓). $Pr(1\le X\le 2)=\frac{8}{27}-\frac{1}{27}=\frac{7}{27}$.

Q8. $F(x)=\frac{x^2}{36}$ ($F(6)=1$ ✓). $Pr(2\le X\le 4)=\frac{16}{36}-\frac{4}{36}=\frac{12}{36}=\frac{1}{3}$. Median $\frac{m^2}{36}=\frac{1}{2}\Rightarrow m^2=18\Rightarrow m=3\sqrt{2}$.

Q9. $F(x)=x^4$. $Pr(X\le 1/2)=(1/2)^4=\frac{1}{16}$.

Q10. $F(x)=\int_0^x \frac{2}{9}(3-t)\,dt=\frac{2}{9}\left[3t-\frac{t^2}{2}\right]_0^x=\frac{x(6-x)}{9}$ ($F(3)=\frac{9}{9}=1$ ✓). $Pr(X\le 1)=\frac{1\cdot 5}{9}=\frac{5}{9}$. Median $\frac{m(6-m)}{9}=\frac{1}{2}\Rightarrow m^2-6m+\frac{9}{2}=0\Rightarrow m=3-\frac{3\sqrt{2}}{2}$ (taking the root in $[0,3]$).



Q11. $F(x)=1-e^{-x/3}$. $Pr(X>3)=e^{-1}$. Median $e^{-m/3}=\frac{1}{2}\Rightarrow m=3\log_e 2$.

Q12. $F(x)=x^5$. $Pr(X\le 1/2)=\frac{1}{32}$.

Q13. $F(x)=\int_2^x \frac{1}{3}\,dt=\frac{x-2}{3}$ ($F(5)=1$ ✓). Median $\frac{m-2}{3}=\frac{1}{2}\Rightarrow m=\frac{7}{2}$. $Pr(3\le X\le 4)=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$.

Q14. $f(x)=F'(x)=\frac{1}{4}e^{-x/4}$. Median: $1-e^{-m/4}=\frac{1}{2}\Rightarrow e^{-m/4}=\frac{1}{2}\Rightarrow \frac{m}{4}=\log_e 2\Rightarrow m=4\log_e 2$, as required.

Q15. $F(x)=\int_0^x \frac{1}{2}\cos\left(\frac{t}{2}\right)dt=\left[\sin\left(\frac{t}{2}\right)\right]_0^x=\sin\left(\frac{x}{2}\right)$ ($F(\pi)=\sin\frac{\pi}{2}=1$ ✓). Median $\sin\frac{m}{2}=\frac{1}{2}\Rightarrow\frac{m}{2}=\frac{\pi}{6}\Rightarrow m=\frac{\pi}{3}$.
$Pr(0\le X\le \pi/2)=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$.

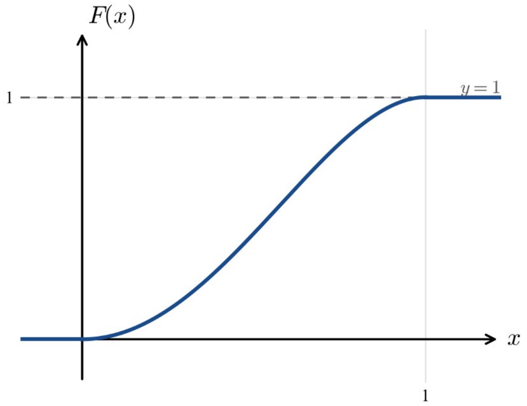
Q16. $F(x)=x^2$. $Pr(X\le 1/2)=\frac{1}{4}$. Median $m^2=\frac{1}{2}\Rightarrow m=\frac{\sqrt{2}}{2}$.

Q17. $F(x)=\frac{x-1}{4}$ ($F(5)=1$ ✓). Median $\frac{m-1}{4}=\frac{1}{2}\Rightarrow m=3$. $Pr(2\le X\le 3)=\frac{2}{4}-\frac{1}{4}=\frac{1}{4}$.

Q18. $F(x)=1-e^{-x/5}$. $Pr(X>5)=e^{-1}$. Median $e^{-m/5}=\frac{1}{2}\Rightarrow m=5\log_e 2$.

Q19. $f(x)=F'(x)=\frac{d}{dx}(2x^2-x^4)=4x-4x^3=4x(1-x^2)$ on $[0,1]$, as required.

$Pr(X\le 1/2)=F(1/2)=1/4(2-1/4)=1/4\cdot 7/4=\frac{7}{16}$.



Q20. $f(x)=F'(x)=2e^{-2x}$ for $x\ge 0$. $Pr(X>1)=1-F(1)=e^{-2}$. Median $e^{-2m}=\frac{1}{2}\Rightarrow 2m=\log_e 2\Rightarrow m=\frac{\log_e 2}{2}$.

Q21. $F(x)=\int_0^x \frac{3}{32}t(4-t)\,dt=\frac{3}{32}\left[2t^2-\frac{t^3}{3}\right]_0^x=\frac{3}{32}\left(2x^2-\frac{x^3}{3}\right)=\frac{x^2(6-x)}{32}$ ($F(4)=\frac{16\cdot 2}{32}=1$ ✓).

$Pr(0\le X\le 2)=F(2)=\frac{4\cdot 4}{32}=\frac{1}{2}$. Since f is symmetric about $x=2$ and $Pr(X\le 2)=1/2$, the median is 2.