Mathematical Methods Units 3 & 4

Chapter 11 — Discrete probability distributions

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Contents

Theory

An experiment has a **sample space** ξ — the set of all possible outcomes. An **event** is a subset of the sample space. When all outcomes are **equally likely**,

$$Pr(A)=\frac{\text{number of outcomes in } A}{\text{number of outcomes in } \xi}.$$

Every probability satisfies $0 \le Pr(A) \le 1$, with $Pr(\xi)=1$ and $Pr(\quad)=0$.

Complement. The complement A' is "A does not occur", and

$$Pr(A')=1-Pr(A).$$

Addition rule. For any two events A and B,

$$Pr(A \cup B)=Pr(A)+Pr(B)-Pr(A \cap B).$$

Here $A \cup B$ is "A **or** B (or both)" and $A \cap B$ is "A **and** B".

Mutually exclusive events cannot occur together, so $A \cap B= \quad$, $Pr(A \cap B)=0$, and the addition rule reduces to $Pr(A \cup B)=Pr(A)+Pr(B)$.

Probabilities are often organised in a **Venn diagram** or a **two-way table**. In a Venn diagram the four regions are "A only", "A and B", "B only" and "neither", and they add to 1.

Worked examples

Example 1 — scaffolded

A fair six-sided die is rolled. Let A be "an even number" and B be "a number greater than 3". Find $Pr(A)$, $Pr(B)$, $Pr(A \cap B)$, $Pr(A \cup B)$ and $Pr(A')$.

Working	Comment
$\xi=\{1,2,3,4,5,6\}$; $A=\{2,4,6\}$, $B=\{4,5,6\}$.	List the outcomes in each event.
$Pr(A)=\frac{3}{6}=\frac{1}{2}$, $Pr(B)=\frac{3}{6}=\frac{1}{2}$.	Equally likely outcomes.
$A \cap B=\{4,6\}$, so $Pr(A \cap B)=\frac{2}{6}=\frac{1}{3}$.	"Even **and** greater than 3".
$Pr(A \cup B)=\frac{1}{2}+\frac{1}{2}-\frac{1}{3}=\frac{2}{3}$.	Addition rule.
$Pr(A')=1-\frac{1}{2}=\frac{1}{2}$.	Complement.

Example 2 — scaffolded

Forty students were surveyed about whether they play a musical instrument (M) and whether they play sport (S). The results are shown.

	Sport	No sport	Total
Music	12	8	20
No music	15	5	20

	Sport	No sport	Total
Total	27	13	40

A student is chosen at random. Find $Pr(M)$, $Pr(M \cap S)$, $Pr(M \cup S)$ and the probability the student does neither.

Working	Comment
$Pr(M)=\frac{20}{40}=\frac{1}{2}$, $Pr(S)=\frac{27}{40}$.	Read the row/column totals.
$Pr(M \cap S)=\frac{12}{40}=\frac{3}{10}$.	The "Music **and** Sport" cell.
$Pr(M \cup S)=\frac{1}{2}+\frac{27}{40}-\frac{3}{10}=\frac{7}{8}$.	Addition rule.
Neither $=\frac{5}{40}=\frac{1}{8}$.	The "No music, No sport" cell (or $1-7/8$).

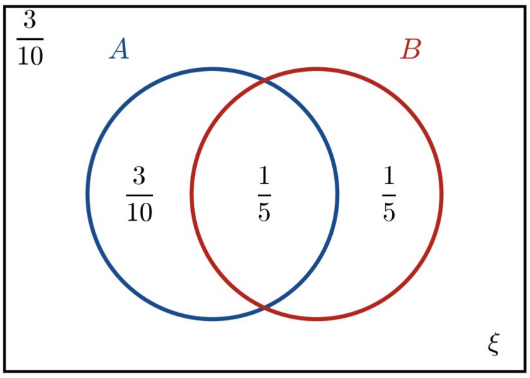
Example 3 — unscaffolded

For two events, $Pr(A)=\frac{1}{2}$, $Pr(B)=\frac{2}{5}$ and $Pr(A \cap B)=\frac{1}{5}$. Find $Pr(A \cup B)$ and the probability that neither event occurs.

$Pr(A \cup B)=\frac{1}{2}+\frac{2}{5}-\frac{1}{5}=\frac{7}{10}$.

A only $=\frac{1}{2}-\frac{1}{5}=\frac{3}{10}$; B only $=\frac{2}{5}-\frac{1}{5}=\frac{1}{5}$.

Neither $=1-Pr(A \cup B)=1-\frac{7}{10}=\frac{3}{10}$.



$\therefore Pr(A \cup B)=\frac{7}{10}$ and $Pr(\text{neither})=\frac{3}{10}$.

Example 4 — unscaffolded

A bag contains only red, blue and green counters. $Pr(\text{red})=\frac{3}{10}$, $Pr(\text{blue})=\frac{1}{5}$ and $Pr(\text{green})=\frac{1}{2}$. A counter is drawn at random. Find the probability it is red or blue, the probability it is not green, and the probability it is red or green.

The three colours are mutually exclusive and their probabilities sum to $\frac{3}{10}+\frac{1}{5}+\frac{1}{2}=1$, as required.

$Pr(\text{red or blue}) = \frac{3}{10} + \frac{1}{5} = \frac{1}{2}$ (mutually exclusive).

$Pr(\text{not green}) = 1 - \frac{1}{2} = \frac{1}{2}$.

$Pr(\text{red or green}) = \frac{3}{10} + \frac{1}{2} = \frac{4}{5}$.

$\therefore$ the required probabilities are $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{4}{5}$.

Practice questions

Scaffolded

Q1. One card is drawn from a standard pack of 52. Let A be "a heart" and B be "a face card (Jack, Queen or King)". (a) Find $Pr(A)$ and $Pr(B)$. (b) Find $Pr(A \cap B)$. (c) Hence find $Pr(A \cup B)$.

Q2. A fair die is rolled. Let A be "less than 3" and B be "an odd number". (a) List the outcomes in A and in B. (b) Find $Pr(A)$, $Pr(B)$ and $Pr(A \cap B)$. (c) Find $Pr(A \cup B)$ and $Pr(A')$.

Q3. For two events, $Pr(A) = 0.6$, $Pr(B) = 0.5$ and $Pr(A \cap B) = 0.3$. (a) Find $Pr(A \cup B)$. (b) Find the probability that neither occurs. (c) Find the probability that only A occurs. (d) Illustrate the information on a Venn diagram.

Q4. A spinner is equally likely to land on any number from 1 to 8. Let A be "a multiple of 3" and B be "greater than 5". (a) List the outcomes in each event. (b) Find $Pr(A)$, $Pr(B)$ and $Pr(A \cap B)$. (c) Find $Pr(A \cup B)$.

Q5. Two events A and B are mutually exclusive with $Pr(A) = 0.35$ and $Pr(B) = 0.4$. (a) Find $Pr(A \cap B)$. (b) Find $Pr(A \cup B)$. (c) Find the probability that neither occurs.

Q6. Of 100 households, 45 own a dog, 30 own a cat, and 12 own both. (a) Find the probability a randomly chosen household owns a dog or a cat. (b) Find the probability it owns neither.

Unscaffolded

Q7. Two fair dice are rolled. Find the probability that the sum of the two numbers is 7.

Q8. For two events, $Pr(A) = 0.7$, $Pr(B) = 0.45$ and $Pr(A \cup B) = 0.85$. Find $Pr(A \cap B)$.

Q9. For two events, $Pr(A) = 0.5$ and $Pr(A \cap B) = 0.15$. Find the probability that A occurs but B does not.

Q10. One card is drawn from a standard pack. Find the probability it is red or a King.

Q11. Two mutually exclusive events have $Pr(A \cup B) = 0.6$. Find the probability that neither occurs.

Q12. A fair die is rolled. Find the probability the result is not greater than 4.

Q13. Of 200 students, 80 study French, 60 study Music and 25 study both. Find the probability that a randomly chosen student studies French or Music.

Q14. For two events, $Pr(A) = \frac{1}{3}$, $Pr(B) = \frac{1}{4}$ and $Pr(A \cap B) = \frac{1}{12}$. Find $Pr(A \cup B)$.

Q15. A spinner is equally likely to show any integer from 1 to 10. Find the probability it shows a prime number.

Q16. For two events, $Pr(A) = 0.55$, $Pr(B) = 0.5$ and $Pr(A' \cap B') = 0.2$. **Show that** $Pr(A \cap B) = 0.25$.

Q17. A bag contains 3 red, 4 blue and 5 green counters. Find the probability that a counter drawn at random is not blue.

Q18. Two fair dice are rolled. Find the probability of getting at least one 6.

Q19. For three events, the probability that none of them occurs is 0.1. Find the probability that at least one of them occurs.

Q20. Two events are mutually exclusive with $Pr(A)=x$ and $Pr(B)=2x$. Given $Pr(A \cup B)=0.9$, find the value of x.

Q21. A fair die is rolled. Let A be "an even number" and B be "a multiple of 3". Find $Pr(A \cup B)$.

Answers

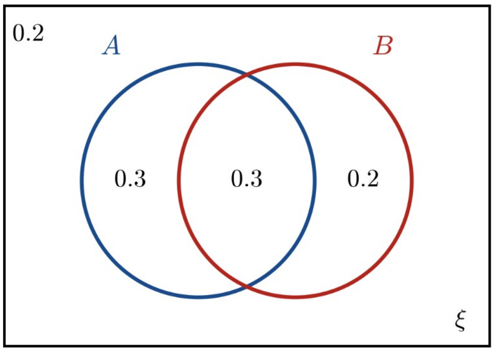
Q1. (a) $\frac{1}{4}, \frac{3}{13}$ (b) $\frac{3}{52}$ (c) $\frac{11}{26}$ **Q2.** (b) $\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$ (c) $\frac{2}{3}, \frac{2}{3}$ **Q3.** (a) $\frac{4}{5}$ (b) $\frac{1}{5}$ (c) $\frac{3}{10}$ **Q4.** (b) $\frac{1}{4}, \frac{3}{8}, \frac{1}{8}$ (c) $\frac{1}{2}$ **Q5.** (a) 0 (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ **Q6.** (a) $\frac{63}{100}$ (b) $\frac{37}{100}$ **Q7.** $\frac{1}{6}$ **Q8.** $\frac{3}{10}$ **Q9.** $\frac{7}{20}$ **Q10.** $\frac{7}{13}$ **Q11.** $\frac{2}{5}$ **Q12.** $\frac{2}{3}$ **Q13.** $\frac{23}{40}$ **Q14.** $\frac{1}{2}$ **Q15.** $\frac{2}{5}$ **Q16.** $\frac{1}{4}$ **Q17.** $\frac{2}{3}$ **Q18.** $\frac{11}{36}$ **Q19.** $\frac{9}{10}$ **Q20.** $x=\frac{3}{10}$ **Q21.** $\frac{2}{3}$

Fully-worked solutions

Q1. (a) $Pr(A)=\frac{13}{52}=\frac{1}{4}$; $Pr(B)=\frac{12}{52}=\frac{3}{13}$. (b) The three cards that are both hearts and face cards give $Pr(A\cap B)=\frac{3}{52}$. (c) $Pr(A\cup B)=\frac{1}{4}+\frac{3}{13}-\frac{3}{52}=\frac{13+12-3}{52}=\frac{22}{52}=\frac{11}{26}$.

Q2. (a) $A=\{1,2\}$, $B=\{1,3,5\}$. (b) $Pr(A)=\frac{2}{6}=\frac{1}{3}$, $Pr(B)=\frac{3}{6}=\frac{1}{2}$, $A\cap B=\{1\}$ so $Pr(A\cap B)=\frac{1}{6}$. (c) $Pr(A\cup B)=\frac{1}{3}+\frac{1}{2}-\frac{1}{6}=\frac{2}{3}$; $Pr(A')=1-\frac{1}{3}=\frac{2}{3}$.

Q3. (a) $Pr(A\cup B)=0.6+0.5-0.3=0.8=\frac{4}{5}$. (b) Neither $=1-\frac{4}{5}=\frac{1}{5}$. (c) Only $A=0.6-0.3=0.3=\frac{3}{10}$.



Q4. (a) $A=\{3,6\}$, $B=\{6,7,8\}$. (b) $Pr(A)=\frac{2}{8}=\frac{1}{4}$, $Pr(B)=\frac{3}{8}$, $A\cap B=\{6\}$ so $Pr(A\cap B)=\frac{1}{8}$. (c) $Pr(A\cup B)=\frac{1}{4}+\frac{3}{8}-\frac{1}{8}=\frac{1}{2}$.

Q5. (a) Mutually exclusive $\Rightarrow Pr(A\cap B)=0$. (b) $Pr(A\cup B)=0.35+0.4=0.75=\frac{3}{4}$. (c) Neither $=1-\frac{3}{4}=\frac{1}{4}$.

Q6. (a) $Pr(\text{dog}\cup\text{cat})=\frac{45}{100}+\frac{30}{100}-\frac{12}{100}=\frac{63}{100}$. (b) Neither $=1-\frac{63}{100}=\frac{37}{100}$.

Q7. The pairs summing to 7 are $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$: 6 of the 36 equally likely outcomes, so $Pr=\frac{6}{36}=\frac{1}{6}$.

Q8. From $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$: $0.85=0.7+0.45-Pr(A\cap B)$, so $Pr(A\cap B)=1.15-0.85=0.3=\frac{3}{10}$.

Q9. A but not $B = Pr(A) - Pr(A \cap B) = 0.5 - 0.15 = 0.35 = \frac{7}{20}$.

Q10. $Pr(\text{red}) = \frac{26}{52} = \frac{1}{2}$, $Pr(\text{King}) = \frac{4}{52} = \frac{1}{13}$, both (red Kings) $= \frac{2}{52} = \frac{1}{26}$.
$Pr = \frac{1}{2} + \frac{1}{13} - \frac{1}{26} = \frac{13+2-1}{26} = \frac{14}{26} = \frac{7}{13}$.

Q11. For mutually exclusive events $Pr(A \cup B) = Pr(A) + Pr(B) = 0.6$, so the probability neither occurs is $1 - 0.6 = 0.4 = \frac{2}{5}$.

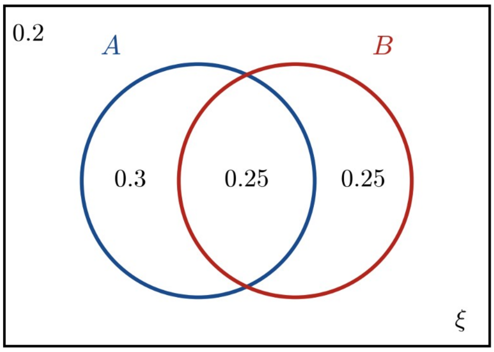
Q12. "Not greater than 4" $= \{1,2,3,4\}$, so $Pr = 1 - Pr(\{5,6\}) = 1 - \frac{2}{6} = \frac{2}{3}$.

Q13. $Pr = \frac{80}{200} + \frac{60}{200} - \frac{25}{200} = \frac{115}{200} = \frac{23}{40}$.

Q14. $Pr(A \cup B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{4+3-1}{12} = \frac{6}{12} = \frac{1}{2}$.

Q15. Primes in $\{1,\ldots,10\}$ are $\{2,3,5,7\}$, so $Pr = \frac{4}{10} = \frac{2}{5}$.

Q16. $Pr(A' \cap B') = Pr((A \cup B)') = 1 - Pr(A \cup B)$, so $Pr(A \cup B) = 1 - 0.2 = 0.8$. Then $Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B) = 0.55 + 0.5 - 0.8 = 0.25$. $\therefore Pr(A \cap B) = \frac{1}{4}$.



Q17. $Pr(\text{not blue}) = 1 - \frac{4}{12} = \frac{8}{12} = \frac{2}{3}$.

Q18. $Pr(\text{no 6 on either die}) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$, so $Pr(\text{at least one 6}) = 1 - \frac{25}{36} = \frac{11}{36}$.

Q19. At least one $= 1 - Pr(\text{none}) = 1 - 0.1 = \frac{9}{10}$.

Q20. Mutually exclusive: $Pr(A \cup B) = x + 2x = 3x = 0.9$, so $x = 0.3 = \frac{3}{10}$.

Q21. $A = \{2,4,6\}$, $B = \{3,6\}$, $A \cap B = \{6\}$: $Pr(A \cup B) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$.

Theory

Conditional probability. The probability of event A **given** that event B has occurred is

$$Pr(A/B)=\frac{Pr(A\cap B)}{Pr(B)},\ Pr(B)>0.$$

Conditioning on B restricts attention to the outcomes in B; among those, we ask what fraction also lie in A.

The multiplication rule. Rearranging gives

$$Pr(A\cap B)=Pr(A/B)Pr(B)=Pr(B/A)Pr(A).$$

Tree diagrams. For a two-stage experiment, draw a branch for each first-stage outcome (label it with its probability), then a branch for each second-stage outcome (label it with the **conditional** probability given the first stage). **Multiply along** a path to get the probability of that combined outcome; **add** the path probabilities of the outcomes that make up an event.

The law of total probability. If B and $B^{'}$ partition the sample space,

$$Pr(A)=Pr(A/B)Pr(B)+Pr(A/B^{'})Pr(B^{'}).$$

Reversing the conditioning. Combining the two forms of the multiplication rule,

$$Pr(B/A)=\frac{Pr(A/B)Pr(B)}{Pr(A)}.$$

Independence. Events A and B are **independent** if the occurrence of one does not affect the probability of the other:

$$A, B\ \text{independent} \Leftrightarrow Pr(A\cap B)=Pr(A)Pr(B) \Leftrightarrow Pr(A/B)=Pr(A).$$

To **test** independence, compute $Pr(A)Pr(B)$ and compare it with $Pr(A\cap B)$: equal ⇒ independent; unequal ⇒ not independent. (Do not assume independence from context — check it.)

Worked examples

Example 1 — scaffolded

In a class, $Pr(S)=\frac{3}{5}$ play sport, $Pr(M)=\frac{1}{2}$ play a musical instrument, and $Pr(S\cap M)=\frac{3}{10}$. Find $Pr(S/M)$ and determine whether S and M are independent.

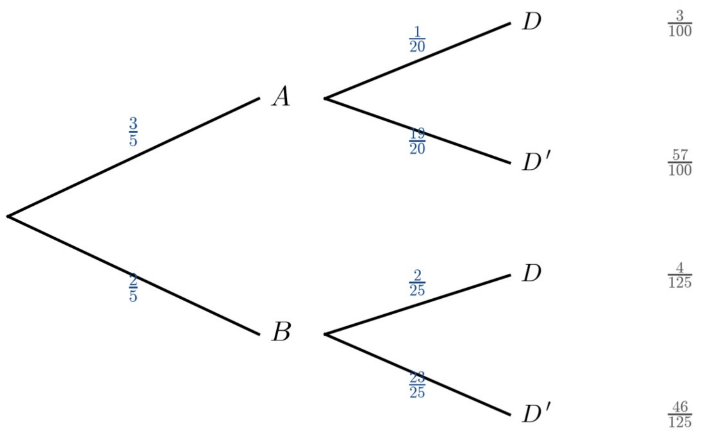
Working	Comment
$Pr(S/M)=\frac{Pr(S\cap M)}{Pr(M)}=\frac{3/10}{1/2}$	Apply the conditional-probability formula.
$=\frac{3}{10}\times\frac{2}{1}=\frac{3}{5}$	Divide by multiplying by the reciprocal.
$Pr(S)Pr(M)=\frac{3}{5}\times\frac{1}{2}=\frac{3}{10}=Pr(S\cap M)$	Test independence.
∴ S and M are **independent** (equivalently $Pr(S/M)=\frac{3}{5}=Pr(S)$).	The condition $Pr(A\cap B)=Pr(A)Pr(B)$ holds.

Example 2 — scaffolded

A factory uses machine A for $\frac{3}{5}$ of its output and machine B for $\frac{2}{5}$. Machine A produces a defective item with probability $\frac{1}{20}$; machine B with probability $\frac{2}{25}$. Find the probability that a randomly chosen item is defective, and the probability it came from A given that it is defective.

Working	Comment
Draw the tree (machine, then defective D or not).	See the diagram below.
$Pr(A \cap D) = \frac{3}{5} \times \frac{1}{20} = \frac{3}{100}$;	Multiply along each defective path.
$Pr(B \cap D) = \frac{2}{5} \times \frac{2}{25} = \frac{4}{125}$	
$Pr(D) = \frac{3}{100} + \frac{4}{125} = \frac{15}{500} + \frac{16}{500} = \frac{31}{500}$	Add the defective paths (total probability).
$Pr(A / D) = \frac{Pr(A \cap D)}{Pr(D)} = \frac{3/100}{31/500} = \frac{3}{100} \times \frac{500}{31} = \frac{15}{31}$	Reverse the conditioning.

Machine (A or B), then defective (D) or not



Example 3 — unscaffolded

A bag contains 3 red and 2 blue counters. Two are drawn **without replacement**. Find (a) $Pr(\text{both red})$, (b) $Pr(\text{second red} / \text{first red})$, and (c) $Pr(\text{second red})$.

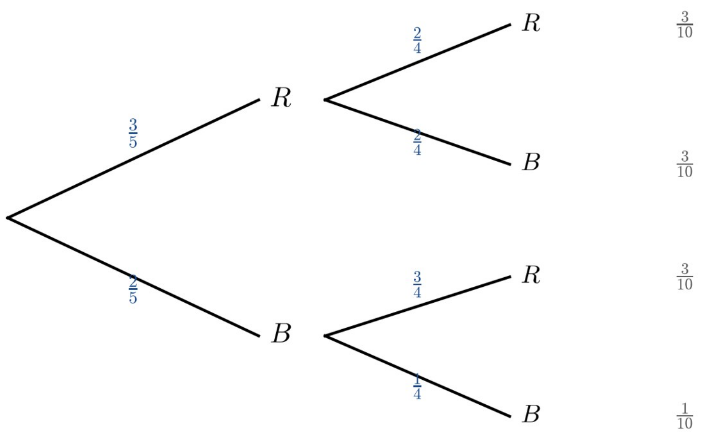
(a) $Pr(RR) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$.

(b) After a red is removed, 2 of the remaining 4 counters are red, so $Pr(\text{second red} / \text{first red}) = \frac{2}{4} = \frac{1}{2}$.

(c) By the law of total probability, $Pr(\text{second red}) = Pr(RR) + Pr(BR) = \frac{3}{10} + \frac{2}{5} \times \frac{3}{4} = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$.

$\therefore$ the counter drawn second is red with probability $\frac{3}{5}$ — the same as for the first draw (by symmetry).

Two draws without replacement (3R, 2B)



Example 4 — unscaffolded

Two fair dice are rolled. Let A be the event "the first die is even" and B the event "the sum is 7". Show that A and B are independent.

$Pr(A)=\frac{3}{6}=\frac{1}{2}$ (the first die is $2,4$ or 6).

$Pr(B)=\frac{6}{36}=\frac{1}{6}$ (the pairs $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$).

$A\cap B$ requires the first die even **and** the sum 7: $(2,5),(4,3),(6,1)$, so $Pr(A\cap B)=\frac{3}{36}=\frac{1}{12}$.

$Pr(A)Pr(B)=\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}=Pr(A\cap B)$.

$\therefore$ A and B are independent.

Practice questions

Scaffolded

Q1. For events with $Pr(C)=\frac{1}{2}$, $Pr(T)=\frac{2}{5}$ and $Pr(C\cap T)=\frac{1}{5}$: (a) find $Pr(C/T)$; (b) find $Pr(T/C)$; (c) determine whether C and T are independent.

Q2. A disease affects $\frac{1}{50}$ of a population. A test is positive with probability $\frac{19}{20}$ for someone with the disease, and with probability $\frac{1}{10}$ for someone without it. (a) Draw a tree diagram. (b) Find $Pr(\text{positive})$. (c) Find $Pr(\text{disease}/\text{positive})$.

Q3. A bag has 4 white and 3 black counters. Two are drawn without replacement. (a) Find $Pr(\text{both white})$. (b) Find $Pr(\text{both the same colour})$. (c) Find $Pr(\text{second black}/\text{first white})$.

Q4. A fair die is rolled. Let A be "the result is at most 3" and B be "the result is odd". (a) Find $Pr(A)$, $Pr(B)$ and $Pr(A\cap B)$. (b) Find $Pr(A/B)$. (c) Are A and B independent? Justify your answer.

Q5. A shop buys $\frac{7}{10}$ of its stock from supplier X and $\frac{3}{10}$ from supplier Y. Supplier X's items are defective with probability $\frac{1}{25}$; supplier Y's with probability $\frac{1}{10}$. (a) Draw a tree diagram. (b) Find $Pr(\text{defective})$. (c) Find $Pr(Y/\text{defective})$.

Q6. An email is spam with probability $\frac{3}{10}$. A spam email contains the word "free" with probability $\frac{3}{5}$; a non-spam email with probability $\frac{1}{10}$. (a) Find $Pr(\ free\ )$. (b) Find $Pr(\text{spam}/\ free\ )$.

Unscaffolded

Q7. A fair die is rolled. A = "even", B = "at least 4". Find $Pr(A/B)$ and state, with reasons, whether A and B are independent.

Q8. One card is drawn from a standard pack. H = "a heart", F = "a picture card (J, Q, K)". Find $Pr(H/F)$ and determine whether H and F are independent.

Q9. A box has 5 red and 3 blue pens. Two are taken without replacement. Find $Pr(\text{both red})$ and $Pr(\text{second blue}/\text{first red})$.

Q10. Bag 1 holds 2 red and 3 white balls; Bag 2 holds 4 red and 1 white. A bag is chosen at random and one ball drawn. Find $Pr(\text{red})$ and $Pr(\text{Bag 1}/\text{red})$.

Q11. On a rainy day (probability $\frac{3}{10}$) a train is late with probability $\frac{1}{2}$; on a dry day it is late with probability $\frac{1}{10}$. Find $Pr(\text{late})$ and $Pr(\text{rain}/\text{late})$.

Q12. Three fair coins are tossed. A = "the first is a head", B = "exactly two heads". Find $Pr(A/B)$ and determine whether A and B are independent.

Q13. In a group of 200 people, 100 own a car (A), 80 own a bike (B), and 40 own both. Find $Pr(A/B)$ and determine whether owning a car and owning a bike are independent.

Q14. A box contains 2 red, 3 green and 4 blue counters. Two are drawn without replacement. Find $Pr(\text{both the same colour})$.

Q15. Two fair dice are rolled. A = "at least one 6", B = "the sum is 8". Find $Pr(B/A)$ and state whether A and B are independent.

Q16. A family has three children, each equally likely to be a boy or a girl. Find $Pr(\text{all girls}/\text{at least one girl})$.

Q17. A drawer has 6 working and 2 faulty batteries. Two are chosen without replacement. Find $Pr(\text{at least one faulty})$.

Q18. Two cards are drawn without replacement from a standard pack. Find $Pr(\text{both kings})$.

Q19. A and B are independent events with $Pr(A)=\frac{2}{5}$ and $Pr(B)=\frac{1}{2}$. Find $Pr(A \cup B)$.

Q20. A student must pass two stages of a test. They pass stage 1 with probability $\frac{4}{5}$, and — given they passed stage 1 — pass stage 2 with probability $\frac{7}{10}$. Find $Pr(\text{pass both stages})$.

Q21. A screening test for a condition present in $\frac{1}{20}$ of a population is positive with probability $\frac{9}{10}$ for an affected person and $\frac{1}{20}$ for an unaffected person. Find $Pr(\text{condition} / \text{positive})$.

Answers

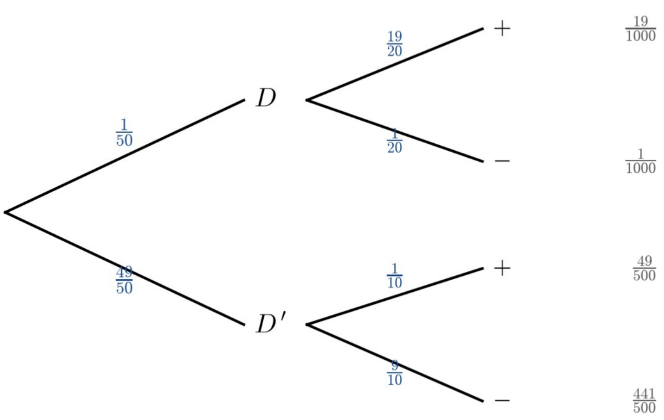
Q1. (a) $\frac{1}{2}$; (b) $\frac{2}{5}$; (c) independent ($Pr(C)Pr(T)=\frac{1}{5}=Pr(C\cap T)$). **Q2.** (b) $\frac{117}{1000}$; (c) $\frac{19}{117}$. **Q3.** (a) $\frac{2}{7}$; (b) $\frac{3}{7}$; (c) $\frac{1}{2}$. **Q4.** (a) $Pr(A)=\frac{1}{2}$, $Pr(B)=\frac{1}{2}$, $Pr(A\cap B)=\frac{1}{3}$; (b) $\frac{2}{3}$; (c) not independent ($\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}\neq\frac{1}{3}$). **Q5.** (b) $\frac{29}{500}$; (c) $\frac{15}{29}$. **Q6.** (a) $\frac{1}{4}$; (b) $\frac{18}{25}$. **Q7.** $Pr(A/B)=\frac{2}{3}$; not independent ($\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}\neq\frac{1}{3}$). **Q8.** $Pr(H/F)=\frac{1}{4}$; independent ($Pr(H)Pr(F)=\frac{1}{4}\times\frac{3}{13}=\frac{3}{52}=Pr(H\cap F)$). **Q9.** $Pr(\text{both red})=\frac{5}{14}$; $Pr(\text{second blue / first red})=\frac{3}{7}$. **Q10.** $Pr(\text{red})=\frac{3}{5}$; $Pr(\text{Bag 1 / red})=\frac{1}{3}$. **Q11.** $Pr(\text{late})=\frac{11}{50}$; $Pr(\text{rain / late})=\frac{15}{22}$. **Q12.** $Pr(A/B)=\frac{2}{3}$; not independent ($\frac{1}{2}\times\frac{3}{8}=\frac{3}{16}\neq\frac{1}{4}$). **Q13.** $Pr(A/B)=\frac{1}{2}$; independent ($\frac{1}{2}\times\frac{2}{5}=\frac{1}{5}=Pr(A\cap B)$). **Q14.** $\frac{5}{18}$. **Q15.** $Pr(B/A)=\frac{2}{11}$; not independent. **Q16.** $\frac{1}{7}$. **Q17.** $\frac{13}{28}$. **Q18.** $\frac{1}{221}$. **Q19.** $\frac{7}{10}$. **Q20.** $\frac{14}{25}$. **Q21.** $\frac{18}{37}$.

Fully-worked solutions

Q1. (a) $Pr(C/T)=\frac{Pr(C\cap T)}{Pr(T)}=\frac{1/5}{2/5}=\frac{1}{2}$. (b) $Pr(T/C)=\frac{1/5}{1/2}=\frac{2}{5}$. (c) $Pr(C)Pr(T)=\frac{1}{2}\times\frac{2}{5}=\frac{1}{5}=Pr(C\cap T)$, so C and T are **independent**.

Q2. (b) $Pr(+)=\frac{1}{50}\times\frac{19}{20}+\frac{49}{50}\times\frac{1}{10}=\frac{19}{1000}+\frac{49}{500}=\frac{19}{1000}+\frac{98}{1000}=\frac{117}{1000}$. (c) $Pr(D/+)=\frac{Pr(D\cap +)}{Pr(+)}=\frac{19/1000}{117/1000}=\frac{19}{117}$.

Disease (D), then test result

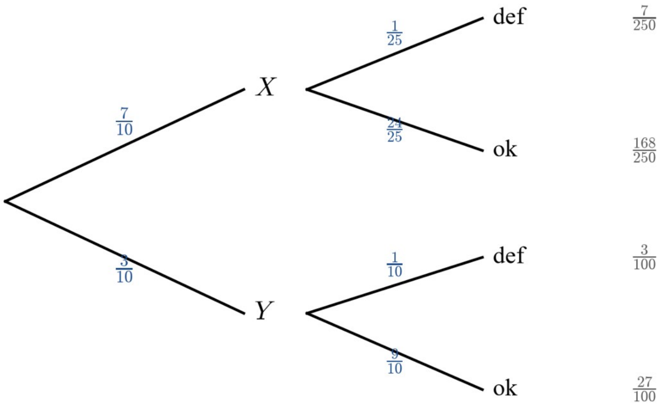


Q3. (a) $Pr(WW)=\frac{4}{7}\times\frac{3}{6}=\frac{12}{42}=\frac{2}{7}$. (b) $Pr(\text{same})=Pr(WW)+Pr(BB)=\frac{2}{7}+\frac{3}{7}\times\frac{2}{6}=\frac{2}{7}+\frac{1}{7}=\frac{3}{7}$. (c) After a white is removed, 3 of the 6 remaining are black, so $Pr(\text{second black / first white})=\frac{3}{6}=\frac{1}{2}$.

Q4. (a) $A=\{1,2,3\}$ so $Pr(A)=\frac{1}{2}$; $B=\{1,3,5\}$ so $Pr(B)=\frac{1}{2}$; $A\cap B=\{1,3\}$ so $Pr(A\cap B)=\frac{2}{6}=\frac{1}{3}$. (b) $Pr(A/B)=\frac{1/3}{1/2}=\frac{2}{3}$. (c) $Pr(A)Pr(B)=\frac{1}{4}\neq\frac{1}{3}=Pr(A\cap B)$, so A and B are **not** independent.

Q5. (b) $Pr(\text{def}) = \frac{7}{10} \times \frac{1}{25} + \frac{3}{10} \times \frac{1}{10} = \frac{7}{250} + \frac{3}{100} = \frac{14}{500} + \frac{15}{500} = \frac{29}{500}$. (c) $Pr(Y / \text{def}) = \frac{Pr(Y \cap \text{def})}{Pr(\text{def})} = \frac{3/100}{29/500} = \frac{3}{100} \times \frac{500}{29} = \frac{15}{29}$.

Supplier (X or Y), then defective



Q6. (a) $Pr(\ free\ ) = \frac{3}{10} \times \frac{3}{5} + \frac{7}{10} \times \frac{1}{10} = \frac{9}{50} + \frac{7}{100} = \frac{18}{100} + \frac{7}{100} = \frac{1}{4}$. (b) $Pr(\text{spam} /\ free\ ) = \frac{9/50}{1/4} = \frac{9}{50} \times 4 = \frac{18}{25}$.

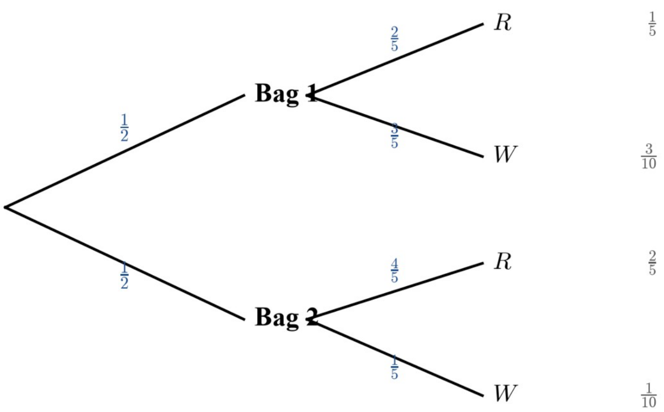
Q7. $A = \{2, 4, 6\}$, $B = \{4, 5, 6\}$, $A \cap B = \{4, 6\}$: $Pr(A) = \frac{1}{2}$, $Pr(B) = \frac{1}{2}$, $Pr(A \cap B) = \frac{1}{3}$. $Pr(A / B) = \frac{1/3}{1/2} = \frac{2}{3}$. Since $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq \frac{1}{3}$, they are not independent.

Q8. $Pr(H) = \frac{13}{52} = \frac{1}{4}$, $Pr(F) = \frac{12}{52} = \frac{3}{13}$, $Pr(H \cap F) = \frac{3}{52}$ (the J, Q, K of hearts). $Pr(H / F) = \frac{3/52}{3/13} = \frac{1}{4}$. Since $Pr(H)Pr(F) = \frac{1}{4} \times \frac{3}{13} = \frac{3}{52} = Pr(H \cap F)$, they are **independent**.

Q9. $Pr(\text{both red}) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \frac{5}{14}$. After a red is removed, 3 of the 7 remaining are blue, so $Pr(\text{second blue} / \text{first red}) = \frac{3}{7}$.

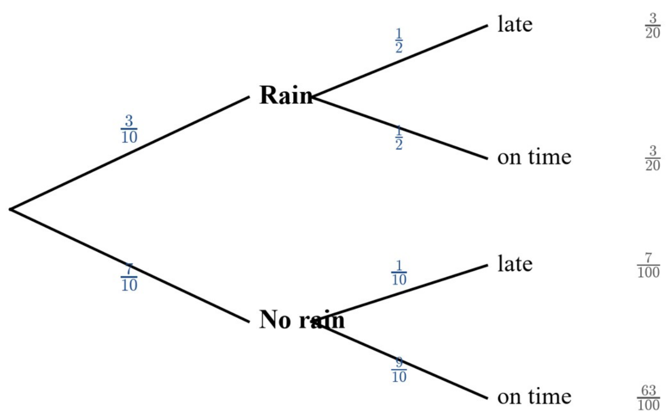
Q10. $Pr(\text{red}) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{5} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$. $Pr(\text{Bag 1} / \text{red}) = \frac{1/5}{3/5} = \frac{1}{3}$.

Choose a bag, then draw a ball



Q11. $Pr(\text{late}) = \frac{3}{10} \times \frac{1}{2} + \frac{7}{10} \times \frac{1}{10} = \frac{3}{20} + \frac{7}{100} = \frac{15}{100} + \frac{7}{100} = \frac{22}{100} = \frac{11}{50}$. $Pr(\text{rain} / \text{late}) = \frac{3/20}{11/50} = \frac{3}{20} \times \frac{50}{11} = \frac{15}{22}$.

Rain or not, then late or on time



Q12. The 8 equally likely outcomes give A (first head): 4 outcomes, $Pr(A) = \frac{1}{2}$; B (exactly two heads: HHT, HTH, THH): $Pr(B) = \frac{3}{8}$; $A \cap B$ (first head and exactly two heads: HHT, HTH): $Pr(A \cap B) = \frac{2}{8} = \frac{1}{4}$.
$Pr(A / B) = \frac{1/4}{3/8} = \frac{2}{3}$. Since $\frac{1}{2} \times \frac{3}{8} = \frac{3}{16} \neq \frac{1}{4}$, not independent.

Q13. $Pr(A) = \frac{100}{200} = \frac{1}{2}$, $Pr(B) = \frac{80}{200} = \frac{2}{5}$, $Pr(A \cap B) = \frac{40}{200} = \frac{1}{5}$. $Pr(A / B) = \frac{1/5}{2/5} = \frac{1}{2}$. Since $Pr(A)Pr(B) = \frac{1}{2} \times \frac{2}{5} = \frac{1}{5} = Pr(A \cap B)$, the two events are **independent**.

Q14. $Pr(\text{same}) = Pr(RR) + Pr(GG) + Pr(BB) = \frac{2}{9} \times \frac{1}{8} + \frac{3}{9} \times \frac{2}{8} + \frac{4}{9} \times \frac{3}{8} = \frac{2+6+12}{72} = \frac{20}{72} = \frac{5}{18}$.

Q15. $Pr(A) = \frac{11}{36}$ (the 11 pairs with at least one 6); B (sum 8): $(2,6),(3,5),(4,4),(5,3),(6,2)$, $Pr(B) = \frac{5}{36}$; $A \cap B = \{(2,6),(6,2)\}$, $Pr(A \cap B) = \frac{2}{36} = \frac{1}{18}$. $Pr(B / A) = \frac{1/18}{11/36} = \frac{2}{11}$. Since $Pr(A)Pr(B) = \frac{11}{36} \times \frac{5}{36} = \frac{55}{1296} \neq \frac{1}{18}$, not independent.

Q16. With 8 equally likely outcomes, “at least one girl” excludes only BBB, so $Pr(\text{at least one girl}) = \frac{7}{8}$, and “all girls” is GGG with probability $\frac{1}{8}$. Hence $Pr(\text{all girls} / \text{at least one girl}) = \frac{1/8}{7/8} = \frac{1}{7}$.

Q17. $Pr(\text{no faulty}) = \frac{6}{8} \times \frac{5}{7} = \frac{30}{56} = \frac{15}{28}$, so $Pr(\text{at least one faulty}) = 1 - \frac{15}{28} = \frac{13}{28}$.

Q18. $Pr(\text{both kings}) = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Q19. Independence gives $Pr(A \cap B) = Pr(A)Pr(B) = \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$. Then
$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = \frac{2}{5} + \frac{1}{2} - \frac{1}{5} = \frac{4}{10} + \frac{5}{10} - \frac{2}{10} = \frac{7}{10}$.

Q20. $Pr(\text{pass both}) = Pr(\text{pass 1})Pr(\text{pass 2 / pass 1}) = \frac{4}{5} \times \frac{7}{10} = \frac{28}{50} = \frac{14}{25}$.

Q21. $Pr(+) = \frac{1}{20} \times \frac{9}{10} + \frac{19}{20} \times \frac{1}{20} = \frac{9}{200} + \frac{19}{400} = \frac{18}{400} + \frac{19}{400} = \frac{37}{400}$.

$Pr(\text{condition} / +) = \frac{9/200}{37/400} = \frac{9}{200} \times \frac{400}{37} = \frac{18}{37}$.

Theory

A **random variable** assigns a number to each outcome of a random procedure. A random variable is **discrete** when it can take only a countable list of separate values (often $0, 1, 2, \ldots$).

Probability distribution. The distribution of a discrete random variable X lists each value x together with its probability $Pr(X=x)$. It may be given as a **table** or as a **probability mass function** (a rule for $Pr(X=x)$).

For any discrete probability distribution:

- every probability satisfies $0 \le Pr(X=x) \le 1$; and
- the probabilities sum to one: $\sum_x Pr(X=x)=1$.

The second condition is used to find an **unknown value** in a distribution: set the sum equal to 1 and solve.

Computing probabilities. From the distribution:

- $Pr(X=x)$ is read directly;
- $Pr(X \ge a)$, $Pr(X \le a)$ and $Pr(a \le X \le b)$ are found by **adding** the relevant probabilities;
- a **conditional** probability uses $Pr(A/B)=\dfrac{Pr(A \cap B)}{Pr(B)}$; for example $Pr(X=2/X \le 2)=\dfrac{Pr(X=2)}{Pr(X \le 2)}$.

A distribution can be shown graphically as a **probability distribution graph** — a vertical bar (or spike) of height $Pr(X=x)$ at each value x.

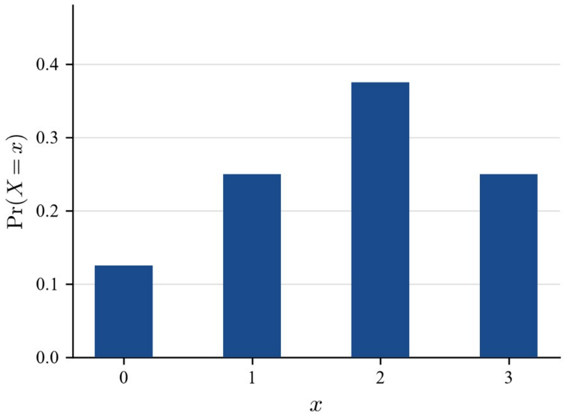
Worked examples

Example 1 — scaffolded

The random variable X has the distribution below. Find k, then $Pr(X \ge 2)$ and $Pr(X \le 1)$.

x	0	1	2	3
$Pr(X=x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	k

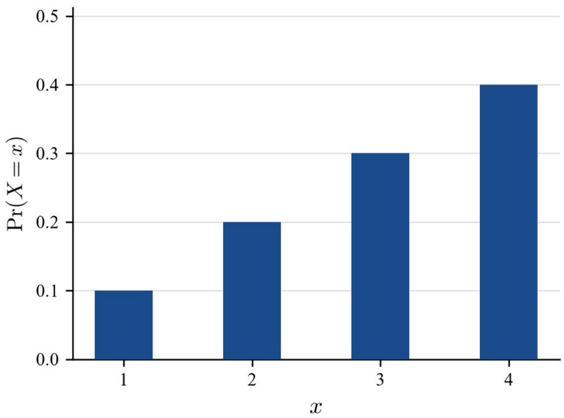
Working	Comment
$\frac{1}{8}+\frac{1}{4}+\frac{3}{8}+k=1$	The probabilities must sum to 1.
$\frac{1}{8}+\frac{2}{8}+\frac{3}{8}+k=1 \Rightarrow \frac{6}{8}+k=1$	Use a common denominator.
$k=\frac{2}{8}=\frac{1}{4}$	Solve for k.
$Pr(X \ge 2)=Pr(X=2)+Pr(X=3)=\frac{3}{8}+\frac{1}{4}=\frac{5}{8}$	Add the probabilities for $x=2, 3$.
$Pr(X \le 1)=\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$	Add the probabilities for $x=0, 1$.



Example 2 — scaffolded

A discrete random variable X takes the values $1,2,3,4$ with $Pr(X=x)=cx$. Find c, and hence $Pr(X\le 2)$ and $Pr(X\ge 3)$.

Working	Comment
$Pr(X=x)=cx$ for $x=1,2,3,4$, so $c(1+2+3+4)=1$	The probabilities must sum to 1.
$10c=1\Rightarrow c=\frac{1}{10}$	Solve for c.
Distribution: $\frac{1}{10},\frac{1}{5},\frac{3}{10},\frac{2}{5}$	$Pr(X=x)=\frac{x}{10}$.
$Pr(X\le 2)=\frac{1}{10}+\frac{1}{5}=\frac{3}{10}$	Add $x=1,2$.
$Pr(X\ge 3)=\frac{3}{10}+\frac{2}{5}=\frac{7}{10}$	Add $x=3,4$.



Example 3 — unscaffolded

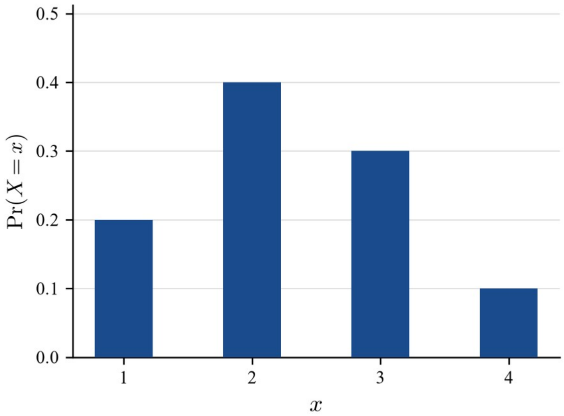
The distribution of X is given below. Find $Pr(X\ge 3)$ and $Pr(X=2/X\le 2)$.

x	1	2	3	4
$Pr(X=x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{10}$

$Pr(X\ge 3)=\frac{3}{10}+\frac{1}{10}=\frac{2}{5}$.

$Pr(X \le 2) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, so $Pr(X=2 / X \le 2) = \frac{Pr(X=2)}{Pr(X \le 2)} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$.

$\therefore Pr(X \ge 3) = \frac{2}{5}$ and $Pr(X=2 / X \le 2) = \frac{2}{3}$.



Example 4 — unscaffolded

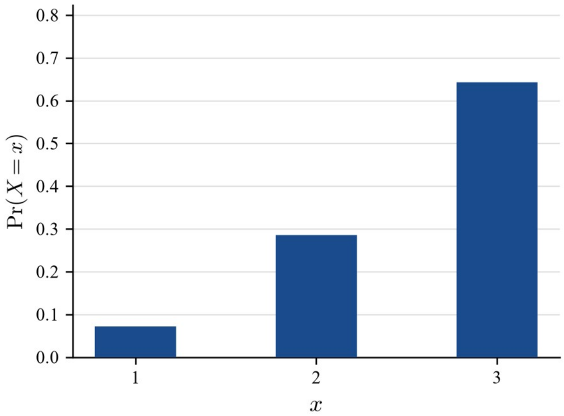
The random variable X takes the values $1, 2, 3$ with $Pr(X=x) = c x^2$. Find c and $Pr(X<3)$.

$c(1^2+2^2+3^2) = 1 \Rightarrow 14c = 1 \Rightarrow c = \frac{1}{14}$.

The distribution is $\frac{1}{14}, \frac{2}{7}, \frac{9}{14}$.

$Pr(X<3) = Pr(X=1) + Pr(X=2) = \frac{1}{14} + \frac{2}{7} = \frac{1}{14} + \frac{4}{14} = \frac{5}{14}$.

$\therefore c = \frac{1}{14}$ and $Pr(X<3) = \frac{5}{14}$.



Practice questions

Scaffolded

Q1. X has the distribution below.

x	1	2	3	4
$Pr(X=x)$	$\frac{1}{10}$	$\frac{3}{10}$	a	$\frac{1}{5}$

(a) Find a. (b) Find $Pr(X \le 2)$. (c) Find $Pr(X \ge 3)$. (d) Find $Pr(X=3 / X \ge 3)$.

Q2. X has the distribution below.

x	1	2	3	4
$Pr(X=x)$	$\frac{3}{20}$	$\frac{1}{4}$	k	$\frac{2}{5}$

(a) Find k. (b) Find $Pr(X \le 2)$. (c) Find $Pr(2 \le X \le 3)$.

Q3. X has the distribution below.

x	0	1	2	3
$Pr(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

(a) Verify that this is a valid probability distribution. (b) Find $Pr(1 \le X \le 2)$.

(b) Find $Pr(X \ge 1)$. (d) Comment on the symmetry of the distribution.

Q4. X takes the values $1,2,3,4,5$ with $Pr(X=x)=cx$. (a) Find c. (b) Write the distribution as a table. (c) Find $Pr(X \text{ is even})$.

Q5. X has the distribution below.

x	1	2	3	4
$Pr(X=x)$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$

(a) Find $Pr(X \ge 2)$. (b) Hence find $Pr(X=3 / X \ge 2)$.

Q6. X takes the values $1,2,3,4$ with $Pr(X=x)=c(2x-1)$. (a) Find c. (b) State the distribution. (c) Find $Pr(X \ge 3)$.

Unscaffolded

Q7. Find k, then $Pr(X \le 2)$, for the distribution with $x=0,1,2,3,4$ and probabilities $\frac{3}{20}, \frac{1}{4}, k, \frac{1}{5}, \frac{1}{10}$.

Q8. For $x=0,1,2,3$ with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{16}$, find $Pr(X \ge 1)$ and $Pr(X \le 1)$.

Q9. X takes the values $1,2,3$ with $Pr(X=x)=c\,2^x$. Find c and $Pr(X \ge 2)$.

Q10. A spinner scores $2,4,6,8$ with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}$. Find $Pr(X \le 4)$ and $Pr(X=8 / X \ge 6)$.

Q11. X takes the values $0,1,2$ with probabilities $\frac{1}{3}, \frac{4}{9}, \frac{2}{9}$. Verify the distribution is valid and find $Pr(X \ge 1)$.

Q12. For $x=1,2,3,4$ with probabilities $\frac{2}{5}, \frac{3}{10}, \frac{1}{5}, \frac{1}{10}$, find $Pr(X \le 2)$ and $Pr(X \ge 3 / X \ge 2)$.

Q13. For $x=0,1,2,3$ with probabilities $\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}$, find $Pr(X \text{ is odd})$.

Q14. Two fair dice are rolled and X is the sum of the two numbers shown. Find $Pr(X=7)$ and $Pr(X\geq10)$.

Q15. X takes the values $1,2,3,4,5$ with $Pr(X=x)=c(6-x)$. Find c and $Pr(X\leq2)$.

Q16. X takes the values $0,1,2,3$ with probabilities $\frac{8}{27},\frac{4}{9},\frac{2}{9},\frac{1}{27}$. Find $Pr(X\geq1)$ and $Pr(X=1/X\geq1)$.

Q17. X takes the values $1,2,3,4$ with $Pr(X=x)$ in the ratio $2:3:4:1$; that is, $Pr(X=x)=2k,3k,4k,k$. Find k and $Pr(X\text{ is even})$, and state the mode.

Q18. Find k for the distribution with $x=0,1,2,3,4$ and probabilities $\frac{1}{10},\frac{1}{5},k,\frac{1}{5},\frac{1}{5}$, then find $Pr(X\geq2)$.

Q19. For $x=0,1,2,3$ with probabilities $\frac{1}{5},\frac{2}{5},\frac{1}{4},\frac{3}{20}$, find $Pr(1\leq X\leq2)$.

Q20. X takes the values $1,2,3,4,5$ with probabilities $\frac{1}{10},\frac{1}{5},\frac{3}{10},\frac{1}{5},\frac{1}{5}$. State the mode and find $Pr(X\geq3)$.

Q21. X takes the values $1,2,3$ with $Pr(X=x)=k(x+1)$. Show that $k=\frac{1}{9}$, and hence find $Pr(X\leq2)$.

Answers

Q1. (a) $a=\frac{2}{5}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{2}{3}$ **Q2.** (a) $k=\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{2}{5}$ **Q3.** (a) sum $=1$, all in $[0,1]$ (b) $\frac{2}{3}$ (c) $\frac{5}{6}$ (d) symmetric about $x=1.5$ **Q4.** (a) $c=\frac{1}{15}$ (b) $\frac{1}{15},\frac{2}{15},\frac{1}{5},\frac{4}{15},\frac{1}{3}$ (c) $\frac{2}{5}$ **Q5.** (a) $\frac{7}{10}$ (b) $\frac{4}{7}$ **Q6.** (a) $c=\frac{1}{16}$ (b) $\frac{1}{16},\frac{3}{16},\frac{5}{16},\frac{7}{16}$ (c) $\frac{3}{4}$

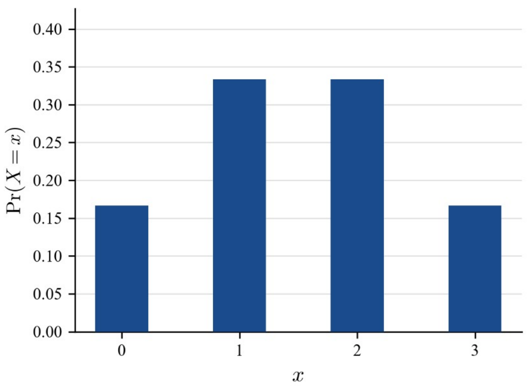
Q7. $k=\frac{3}{10}$; $Pr(X\le 2)=\frac{7}{10}$ **Q8.** $Pr(X\ge 1)=\frac{1}{2}$; $Pr(X\le 1)=\frac{3}{4}$ **Q9.** $c=\frac{1}{14}$; $Pr(X\ge 2)=\frac{6}{7}$ **Q10.** $Pr(X\le 4)=\frac{3}{4}$; $Pr(X=8/X\ge 6)=\frac{1}{2}$ **Q11.** valid; $Pr(X\ge 1)=\frac{2}{3}$ **Q12.** $Pr(X\le 2)=\frac{7}{10}$; $Pr(X\ge 3/X\ge 2)=\frac{1}{2}$ **Q13.** $\frac{1}{2}$ **Q14.** $Pr(X=7)=\frac{1}{6}$; $Pr(X\ge 10)=\frac{1}{6}$ **Q15.** $c=\frac{1}{15}$; $Pr(X\le 2)=\frac{3}{5}$ **Q16.** $Pr(X\ge 1)=\frac{19}{27}$; $Pr(X=1/X\ge 1)=\frac{12}{19}$ **Q17.** $k=\frac{1}{10}$; $Pr(X\text{ even})=\frac{2}{5}$; mode $x=3$ **Q18.** $k=\frac{3}{10}$; $Pr(X\ge 2)=\frac{7}{10}$ **Q19.** $\frac{13}{20}$ **Q20.** mode $x=3$; $Pr(X\ge 3)=\frac{7}{10}$ **Q21.** $k=\frac{1}{9}$; $Pr(X\le 2)=\frac{5}{9}$

Fully-worked solutions

Q1. Sum $=1$: $\frac{1}{10}+\frac{3}{10}+a+\frac{1}{5}=1\Rightarrow\frac{6}{10}+a=1\Rightarrow a=\frac{2}{5}$. (b) $Pr(X\le 2)=\frac{1}{10}+\frac{3}{10}=\frac{2}{5}$. (c) $Pr(X\ge 3)=\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.
(d) $Pr(X=3/X\ge 3)=\dfrac{Pr(X=3)}{Pr(X\ge 3)}=\dfrac{\frac{2}{5}}{\frac{3}{5}}=\frac{2}{3}$.

Q2. $\frac{3}{20}+\frac{1}{4}+k+\frac{2}{5}=1$. With denominator 20: $\frac{3}{20}+\frac{5}{20}+k+\frac{8}{20}=1\Rightarrow\frac{16}{20}+k=1\Rightarrow k=\frac{4}{20}=\frac{1}{5}$. (b) $Pr(X\le 2)=\frac{3}{20}+\frac{1}{4}=\frac{3}{20}+\frac{5}{20}=\frac{8}{20}=\frac{2}{5}$. (c) $Pr(2\le X\le 3)=\frac{1}{4}+\frac{1}{5}=\frac{5}{20}+\frac{4}{20}=\frac{9}{20}$.

Q3. (a) $\frac{1}{6}+\frac{1}{3}+\frac{1}{3}+\frac{1}{6}=1$ and each probability lies in $[0,1]$, so it is valid. (b) $Pr(1\le X\le 2)=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$. (c) $Pr(X\ge 1)=\frac{1}{3}+\frac{1}{3}+\frac{1}{6}=\frac{5}{6}$. (d) The probabilities read the same forwards and backwards, so the distribution is symmetric about $x=1.5$.



Q4. $c(1+2+3+4+5)=1 \Rightarrow 15c=1 \Rightarrow c=\frac{1}{15}$. (b) $\frac{1}{15},\frac{2}{15},\frac{1}{5},\frac{4}{15},\frac{1}{3}$. (c) $Pr(X\text{ even})=Pr(X=2)+Pr(X=4)=\frac{2}{15}+\frac{4}{15}=\frac{6}{15}=\frac{2}{5}$.

Q5. (a) $Pr(X\ge 2)=\frac{1}{5}+\frac{2}{5}+\frac{1}{10}=\frac{2}{10}+\frac{4}{10}+\frac{1}{10}=\frac{7}{10}$. (b) $Pr(X=3\,/\,X\ge 2)=\frac{Pr(X=3)}{Pr(X\ge 2)}=\frac{\frac{2}{5}}{\frac{7}{10}}=\frac{4}{10}\times\frac{10}{7}=\frac{4}{7}$.

Q6. $c(1+3+5+7)=1 \Rightarrow 16c=1 \Rightarrow c=\frac{1}{16}$. (b) $\frac{1}{16},\frac{3}{16},\frac{5}{16},\frac{7}{16}$. (c) $Pr(X\ge 3)=\frac{5}{16}+\frac{7}{16}=\frac{12}{16}=\frac{3}{4}$.

Q7. $\frac{3}{20}+\frac{1}{4}+k+\frac{1}{5}+\frac{1}{10}=1$. Denominator 20: $\frac{3}{20}+\frac{5}{20}+k+\frac{4}{20}+\frac{2}{20}=1 \Rightarrow \frac{14}{20}+k=1 \Rightarrow k=\frac{6}{20}=\frac{3}{10}$. $Pr(X\le 2)=\frac{3}{20}+\frac{1}{4}+\frac{3}{10}=\frac{3}{20}+\frac{5}{20}+\frac{6}{20}=\frac{14}{20}=\frac{7}{10}$.

Q8. $Pr(X\ge 1)=\frac{1}{4}+\frac{3}{16}+\frac{1}{16}=\frac{4}{16}+\frac{3}{16}+\frac{1}{16}=\frac{8}{16}=\frac{1}{2}$; $Pr(X\le 1)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$.

Q9. $c(2^1+2^2+2^3)=1 \Rightarrow 14c=1 \Rightarrow c=\frac{1}{14}$; the probabilities are $\frac{2}{14},\frac{4}{14},\frac{8}{14}=\frac{1}{7},\frac{2}{7},\frac{4}{7}$. $Pr(X\ge 2)=\frac{2}{7}+\frac{4}{7}=\frac{6}{7}$.

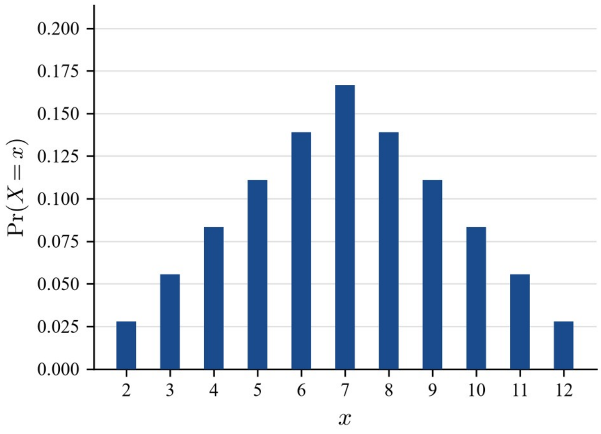
Q10. $Pr(X\le 4)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$. $Pr(X\ge 6)=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$, so $Pr(X=8\,/\,X\ge 6)=\frac{\frac{1}{8}}{\frac{1}{4}}=\frac{1}{2}$.

Q11. $\frac{1}{3}+\frac{4}{9}+\frac{2}{9}=\frac{3}{9}+\frac{4}{9}+\frac{2}{9}=1$, valid. $Pr(X\ge 1)=\frac{4}{9}+\frac{2}{9}=\frac{6}{9}=\frac{2}{3}$.

Q12. $Pr(X\le 2)=\frac{2}{5}+\frac{3}{10}=\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$. $Pr(X\ge 2)=\frac{3}{10}+\frac{1}{5}+\frac{1}{10}=\frac{3}{10}+\frac{2}{10}+\frac{1}{10}=\frac{6}{10}=\frac{3}{5}$; $Pr(X\ge 3)=\frac{1}{5}+\frac{1}{10}=\frac{3}{10}$, so $Pr(X\ge 3\,/\,X\ge 2)=\frac{\frac{3}{10}}{\frac{3}{5}}=\frac{3}{10}\times\frac{5}{3}=\frac{1}{2}$.

Q13. $Pr(X\text{ odd})=Pr(X=1)+Pr(X=3)=\frac{3}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$.

Q14. There are 36 equally likely outcomes. A sum of 7 occurs in 6 ways, so $Pr(X=7)=\frac{6}{36}=\frac{1}{6}$. Sums of $10, 11, 12$ occur in $3+2+1=6$ ways, so $Pr(X\ge 10)=\frac{6}{36}=\frac{1}{6}$.



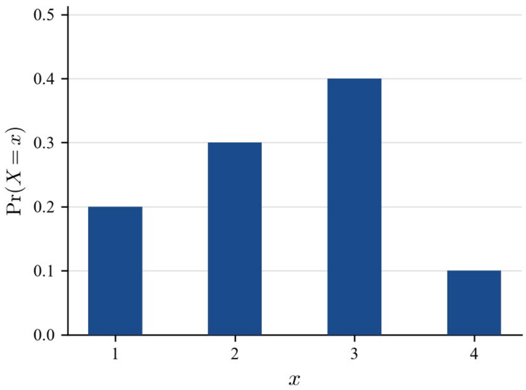
Q15. $c((6-1)+(6-2)+(6-3)+(6-4)+(6-5))=c(5+4+3+2+1)=15c=1 \Rightarrow c=\frac{1}{15}$.

$Pr(X \le 2)=\frac{5}{15}+\frac{4}{15}=\frac{9}{15}=\frac{3}{5}$.

Q16. $Pr(X \ge 1)=\frac{4}{9}+\frac{2}{9}+\frac{1}{27}=\frac{12}{27}+\frac{6}{27}+\frac{1}{27}=\frac{19}{27}$. $Pr(X=1/X \ge 1)=\frac{Pr(X=1)}{Pr(X \ge 1)}=\frac{\frac{4}{9}}{\frac{19}{27}}=\frac{12}{27}\times\frac{27}{19}=\frac{12}{19}$.

Q17. $2k+3k+4k+k=10k=1 \Rightarrow k=\frac{1}{10}$; the distribution is $\frac{1}{5},\frac{3}{10},\frac{2}{5},\frac{1}{10}$.

$Pr(X \text{ even})=Pr(X=2)+Pr(X=4)=\frac{3}{10}+\frac{1}{10}=\frac{4}{10}=\frac{2}{5}$. The largest probability is at $x=3$, so the **mode** is 3.



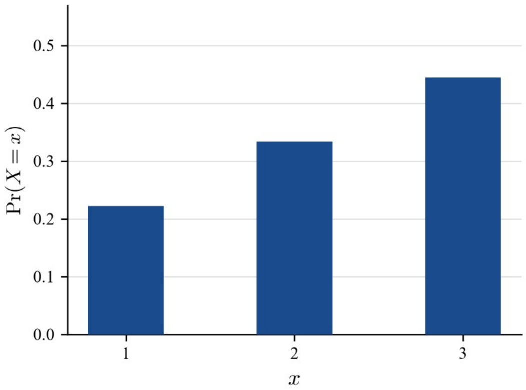
Q18. $\frac{1}{10}+\frac{1}{5}+k+\frac{1}{5}+\frac{1}{5}=1 \Rightarrow \frac{1}{10}+\frac{6}{10}+k=1 \Rightarrow k=\frac{3}{10}$. $Pr(X \ge 2)=\frac{3}{10}+\frac{1}{5}+\frac{1}{5}=\frac{3}{10}+\frac{2}{10}+\frac{2}{10}=\frac{7}{10}$.

Q19. $Pr(1 \le X \le 2)=\frac{2}{5}+\frac{1}{4}=\frac{8}{20}+\frac{5}{20}=\frac{13}{20}$.

Q20. The largest probability, $\frac{3}{10}$, is at $x=3$, so the mode is 3. $Pr(X \ge 3)=\frac{3}{10}+\frac{1}{5}+\frac{1}{5}=\frac{3}{10}+\frac{2}{10}+\frac{2}{10}=\frac{7}{10}$.

Q21. $k(1+1)+k(2+1)+k(3+1)=k(2+3+4)=9k=1 \Rightarrow k=\frac{1}{9}$; the distribution is $\frac{2}{9},\frac{3}{9},\frac{4}{9}$.

$Pr(X \le 2)=\frac{2}{9}+\frac{3}{9}=\frac{5}{9}$.



0.5
0.4
0.3
0.2
0.1
0.0
$\Pr(X = x)$
1
2
3
x

Theory

For a discrete random variable X with probability distribution $Pr(X=x)$, three numbers summarise the distribution: its **centre** (the mean) and its **spread** (the variance and standard deviation).

Sigma notation. The symbol ∑ (capital sigma) means "add up". $\sum_x x\,Pr(X=x)$ instructs us to form the product $x\,Pr(X=x)$ for every value x that X can take, and add the results. When a starting and an ending value are attached, written

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

the counter k runs from the first value to the last.

Expected value (mean). The expected value is the long-run average value of X:

$$E(X) = \mu = \sum_x x\,Pr(X=x).$$

Expected value of a function of X. For any function g,

$$E(g(X)) = \sum_x g(x)\,Pr(X=x).$$

In particular $E(X^2) = \sum_x x^2\,Pr(X=x)$. Note that in general $E(g(X)) \neq g(E(X))$ — for example $E(X^2) \neq (E(X))^2$.

Variance. The variance measures the average squared distance from the mean:

$$Var(X) = E((X-\mu)^2) = E(X^2) - (E(X))^2 = E(X^2) - \mu^2.$$

The second form $E(X^2) - \mu^2$ is almost always the quicker one to use.

Standard deviation. The standard deviation has the same units as X and is the square root of the variance:

$$sd(X) = \sigma = \sqrt{Var(X)}.$$

> **Watch the distinction.** The variance and the standard deviation are *different* numbers: $\sigma = \sqrt{Var(X)}$. A very common error is to compute the variance and then quote it as the standard deviation. Always take the square root for σ — and, conversely, square σ to get the variance.

Linear transformations. If a and b are constants, then

$$E(aX+b) = aE(X)+b,\ Var(aX+b) = a^2 Var(X),\ sd(aX+b) = |a|\sigma.$$

Adding a constant b shifts the mean but leaves the spread unchanged; multiplying by a scales the mean by a and the standard deviation by $|a|$ (so the variance by a^2).

Method (mean, variance, standard deviation from a table):

1. Compute $E(X) = \sum x\,Pr(X=x)$.
2. Compute $E(X^2) = \sum x^2\,Pr(X=x)$.
3. $Var(X) = E(X^2) - (E(X))^2$.

4. $sd(X)=\sqrt{Var(X)}$ (leave it as an exact surd unless a decimal is asked).

Worked examples

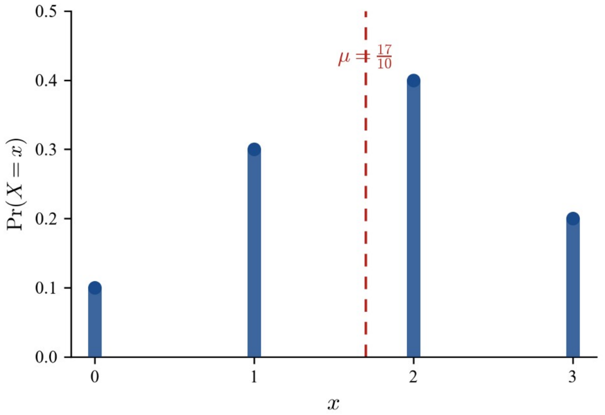
Example 1 — scaffolded

The random variable X has the distribution below. Find $E(X)$, $Var(X)$ and $sd(X)$.

x	0	1	2	3
$Pr(X=x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{2}{10}$

Working	Comment
$E(X)=0\cdot\frac{1}{10}+1\cdot\frac{3}{10}+2\cdot\frac{4}{10}+3\cdot\frac{2}{10}=\frac{17}{10}$	$E(X)=\sum x\,Pr(X=x)$.
$E(X^2)=0^2\cdot\frac{1}{10}+1^2\cdot\frac{3}{10}+2^2\cdot\frac{4}{10}+3^2\cdot\frac{2}{10}=\frac{37}{10}$	$E(X^2)=\sum x^2\,Pr(X=x)$.
$Var(X)=\frac{37}{10}-\left(\frac{17}{10}\right)^2=\frac{370}{100}-\frac{289}{100}=\frac{81}{100}$	$Var(X)=E(X^2)-\mu^2$.
$sd(X)=\sqrt{\frac{81}{100}}=\frac{9}{10}$	Take the square root of the variance.

$\therefore E(X)=\frac{17}{10}$, $Var(X)=\frac{81}{100}$, $sd(X)=\frac{9}{10}$.



Example 2 — scaffolded

A fair spinner is equally likely to show 1, 2, 3 or 4. Let X be the number shown. Find $E(X)$ and $sd(X)$, and hence find $E(2X+1)$ and $Var(2X+1)$.

Working	Comment
Each outcome has probability $\frac{1}{4}$, so $E(X)=\frac{1+2+3+4}{4}=\frac{5}{2}$.	Uniform distribution: average the values.
$E(X^2)=\frac{1+4+9+16}{4}=\frac{30}{4}=\frac{15}{2}$	$E(X^2)=\sum x^2\,Pr(X=x)$.
$Var(X)=\frac{15}{2}-\left(\frac{5}{2}\right)^2=\frac{30}{4}-\frac{25}{4}=\frac{5}{4}$, so $sd(X)=\frac{\sqrt{5}}{2}$.	Variance then square root.

Working	Comment
$E(2X+1)=2E(X)+1=2\cdot\frac{5}{2}+1=6$	$E(aX+b)=aE(X)+b$.
$Var(2X+1)=2^2 Var(X)=4\cdot\frac{5}{4}=5$	$Var(aX+b)=a^2 Var(X)$ — the $+1$ has no effect on spread.

$\therefore E(2X+1)=6$ and $Var(2X+1)=5$.

Example 3 — unscaffolded

The distribution of X is given below. Find the mean and standard deviation.

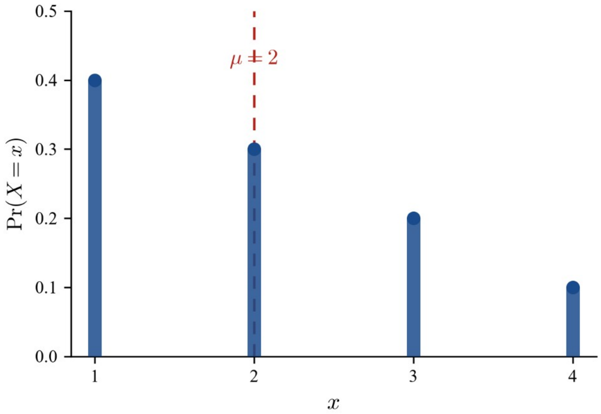
x	1	2	3	4
$Pr(X=x)$	0.4	0.3	0.2	0.1

$E(X)=1(0.4)+2(0.3)+3(0.2)+4(0.1)=0.4+0.6+0.6+0.4=2$.

$E(X^2)=1(0.4)+4(0.3)+9(0.2)+16(0.1)=0.4+1.2+1.8+1.6=5$.

$Var(X)=5-2^2=1$, so $sd(X)=\sqrt{1}=1$.

$\therefore$ the mean is 2 and the standard deviation is 1.



Example 4 — unscaffolded

A random variable X has $E(X)=5$ and $Var(X)=4$. Find $E(3X-2)$, $Var(3X-2)$ and $sd(3X-2)$.

$E(3X-2)=3E(X)-2=3(5)-2=13$.

$Var(3X-2)=3^2 Var(X)=9(4)=36$.

$sd(3X-2)=\sqrt{36}=6$ (equivalently $|3|\cdot sd(X)=3\cdot 2=6$).

$\therefore E(3X-2)=13$, $Var(3X-2)=36$, $sd(3X-2)=6$.

Practice questions

Scaffolded

Q1. X has $Pr(X=1)=\frac{1}{2}$, $Pr(X=2)=\frac{1}{3}$, $Pr(X=3)=\frac{1}{6}$. (a) Find $E(X)$. (b) Find $E(X^2)$. (c) Hence find $Var(X)$ and $sd(X)$.

Q2. For the distribution below:

x	0	1	2	3
$Pr(X=x)$	0.1	0.2	0.4	0.3

(a) find $E(X)$; (b) find $E(X^2)$; (c) find $Var(X)$ and $sd(X)$.

Q3. For the symmetric distribution below:

x	-1	0	1	2
$Pr(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(a) find $E(X)$; (b) find $Var(X)$; (c) state $sd(X)$ as an exact value.

Q4. X takes the values 2, 4, 6 with $Pr=\frac{1}{4},\frac{1}{2},\frac{1}{4}$. (a) Find $E(X)$ and $Var(X)$. (b) Let $Y=X-4$. Write down $E(Y)$ and $Var(Y)$, explaining the effect of subtracting 4.

Q5. For the distribution with $Pr(X=1)=0.1$, $Pr(X=2)=0.2$, $Pr(X=3)=0.3$, $Pr(X=4)=0.4$: (a) find $E(X)$; (b) find $Var(X)$ and $sd(X)$.

Q6. X takes the values 0, 1, 2 with $Pr=\frac{1}{4},\frac{1}{2},\frac{1}{4}$. (a) Find $E(X)$ and $Var(X)$. (b) Hence find $E(10X+5)$ and $sd(10X+5)$.

Unscaffolded

For each distribution, find $E(X)$, $Var(X)$ and $sd(X)$ (exact values), unless otherwise directed.

Q7. X is uniform on $\{0,1,2,3,4\}$.

Q8. $Pr(X=1)=0.5, Pr(X=2)=0.3, Pr(X=3)=0.2$.

Q9. $x=-2,-1,0,1,2$ with $Pr=\frac{1}{16},\frac{4}{16},\frac{6}{16},\frac{4}{16},\frac{1}{16}$.

Q10. $x=0,1,2,3$ with $Pr=0.4,0.3,0.2,0.1$.

Q11. $x=5,10,15,20$ with $Pr=0.4,0.3,0.2,0.1$. (Compare your answer with Q10 — what is the effect of the scaling $X\rightarrow 5X$?)

Q12. X is the number shown on a fair six-sided die.

Q13. $x=0,1,2$ with $Pr=0.49,0.42,0.09$.

Q14. $Pr(X=1)=k, Pr(X=2)=2k, Pr(X=3)=3k$. First find k, then $E(X)$, $Var(X)$ and $sd(X)$.

Q15. $x=-3,0,3$ each with probability $\frac{1}{3}$.

Q16. $x=0,1,2,3,4$ with $Pr=\frac{1}{16},\frac{4}{16},\frac{6}{16},\frac{4}{16},\frac{1}{16}$. Also find $E(2X)$ and $Var(2X)$.

Q17. $x=2,3,5,7$ with $Pr=0.3,0.3,0.3,0.1$.

Q18. $x=10,20,30$ with $Pr=0.2,0.5,0.3$.

Q19. $x=0,1,2,3$ with $Pr=\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8}$.

Q20. $x=1,2,4,8$ with $Pr=0.4,0.3,0.2,0.1$.

Q21. X takes the values -1 and 1, each with probability $\frac{1}{2}$. Find $E(X)$ and $Var(X)$, and hence find $E(4X+3)$ and $sd(4X+3)$.

Answers

Q1. $E(X)=\frac{5}{3}$; $E(X^2)=\frac{10}{3}$; $Var(X)=\frac{5}{9}$, $sd(X)=\frac{\sqrt{5}}{3}$. **Q2.** $E(X)=\frac{19}{10}$; $E(X^2)=\frac{9}{2}$; $Var(X)=\frac{89}{100}$, $sd(X)=\frac{\sqrt{89}}{10}$. **Q3.** $E(X)=\frac{1}{2}$; $Var(X)=\frac{3}{4}$; $sd(X)=\frac{\sqrt{3}}{2}$. **Q4.** $E(X)=4$, $Var(X)=2$; $E(Y)=0$, $Var(Y)=2$ (subtracting 4 shifts the mean by -4 but leaves the spread unchanged). **Q5.** $E(X)=3$; $Var(X)=1$, $sd(X)=1$. **Q6.** $E(X)=1$, $Var(X)=\frac{1}{2}$; $E(10X+5)=15$, $sd(10X+5)=5\sqrt{2}$. **Q7.** $E(X)=2$, $Var(X)=2$, $sd(X)=\sqrt{2}$. **Q8.** $E(X)=\frac{17}{10}$, $Var(X)=\frac{61}{100}$, $sd(X)=\frac{\sqrt{61}}{10}$. **Q9.** $E(X)=0$, $Var(X)=1$, $sd(X)=1$. **Q10.** $E(X)=1$, $Var(X)=1$, $sd(X)=1$. **Q11.** $E(X)=10$, $Var(X)=25$, $sd(X)=5$ (the values are 5 × those of Q10, so the mean and sd scale by 5 and the variance by $5^2=25$). **Q12.** $E(X)=\frac{7}{2}$, $Var(X)=\frac{35}{12}$, $sd(X)=\frac{\sqrt{105}}{6}$. **Q13.** $E(X)=\frac{3}{5}$, $Var(X)=\frac{21}{50}$, $sd(X)=\frac{\sqrt{42}}{10}$. **Q14.** $k=\frac{1}{6}$; $E(X)=\frac{7}{3}$, $Var(X)=\frac{5}{9}$, $sd(X)=\frac{\sqrt{5}}{3}$. **Q15.** $E(X)=0$, $Var(X)=6$, $sd(X)=\sqrt{6}$. **Q16.** $E(X)=2$, $Var(X)=1$, $sd(X)=1$; $E(2X)=4$, $Var(2X)=4$. **Q17.** $E(X)=\frac{37}{10}$, $Var(X)=\frac{261}{100}$, $sd(X)=\frac{3\sqrt{29}}{10}$. **Q18.** $E(X)=21$, $Var(X)=49$, $sd(X)=7$. **Q19.** $E(X)=\frac{3}{2}$, $Var(X)=\frac{3}{4}$, $sd(X)=\frac{\sqrt{3}}{2}$. **Q20.** $E(X)=\frac{13}{5}$, $Var(X)=\frac{111}{25}$, $sd(X)=\frac{\sqrt{111}}{5}$. **Q21.** $E(X)=0$, $Var(X)=1$; $E(4X+3)=3$, $sd(4X+3)=4$.

Fully-worked solutions

Q1. $E(X)=1\cdot\frac{1}{2}+2\cdot\frac{1}{3}+3\cdot\frac{1}{6}=\frac{1}{2}+\frac{2}{3}+\frac{1}{2}=\frac{5}{3}$. $E(X^2)=1\cdot\frac{1}{2}+4\cdot\frac{1}{3}+9\cdot\frac{1}{6}=\frac{1}{2}+\frac{4}{3}+\frac{3}{2}=\frac{10}{3}$.
$Var(X)=\frac{10}{3}-\left(\frac{5}{3}\right)^2=\frac{30}{9}-\frac{25}{9}=\frac{5}{9}$, so $sd(X)=\frac{\sqrt{5}}{3}$.

Q2. $E(X)=0(0.1)+1(0.2)+2(0.4)+3(0.3)=\frac{19}{10}$. $E(X^2)=0+0.2+1.6+2.7=4.5=\frac{9}{2}$.
$Var(X)=\frac{9}{2}-\left(\frac{19}{10}\right)^2=\frac{450}{100}-\frac{361}{100}=\frac{89}{100}$, so $sd(X)=\frac{\sqrt{89}}{10}$.

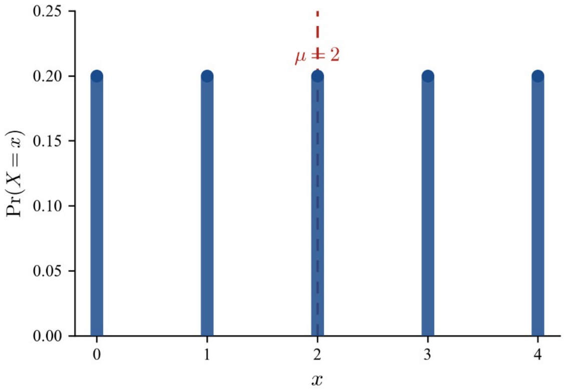
Q3. $E(X)=-1\cdot\frac{1}{8}+0\cdot\frac{3}{8}+1\cdot\frac{3}{8}+2\cdot\frac{1}{8}=\frac{-1+3+2}{8}=\frac{1}{2}$. $E(X^2)=1\cdot\frac{1}{8}+0+1\cdot\frac{3}{8}+4\cdot\frac{1}{8}=\frac{1+3+4}{8}=1$.
$Var(X)=1-\frac{1}{4}=\frac{3}{4}$, $sd(X)=\frac{\sqrt{3}}{2}$.

Q4. $E(X)=2\cdot\frac{1}{4}+4\cdot\frac{1}{2}+6\cdot\frac{1}{4}=4$. $E(X^2)=4\cdot\frac{1}{4}+16\cdot\frac{1}{2}+36\cdot\frac{1}{4}=1+8+9=18$. $Var(X)=18-16=2$. (b) $Y=X-4$: $E(Y)=E(X)-4=0$ and $Var(Y)=Var(X)=2$ — subtracting a constant shifts the distribution left by 4 (so the mean drops by 4) but does not change how spread out it is.

Q5. $E(X)=1(0.1)+2(0.2)+3(0.3)+4(0.4)=0.1+0.4+0.9+1.6=3$. $E(X^2)=0.1+0.8+2.7+6.4=10$.
$Var(X)=10-9=1$, $sd(X)=1$.

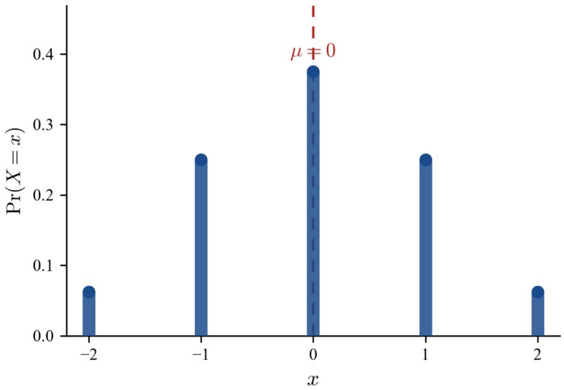
Q6. $E(X)=0\cdot\frac{1}{4}+1\cdot\frac{1}{2}+2\cdot\frac{1}{4}=1$. $E(X^2)=0+\frac{1}{2}+4\cdot\frac{1}{4}=\frac{3}{2}$. $Var(X)=\frac{3}{2}-1=\frac{1}{2}$. (b) $E(10X+5)=10(1)+5=15$; $Var(10X+5)=10^2\cdot\frac{1}{2}=50$, so $sd(10X+5)=\sqrt{50}=5\sqrt{2}$.

Q7. Uniform on $0,1,2,3,4$: $E(X)=\frac{0+1+2+3+4}{5}=2$; $E(X^2)=\frac{0+1+4+9+16}{5}=6$; $Var(X)=6-4=2$, $sd(X)=\sqrt{2}$.



Q8. $E(X)=1(0.5)+2(0.3)+3(0.2)=\frac{17}{10}$; $E(X^2)=0.5+1.2+1.8=3.5=\frac{7}{2}$;
$Var(X)=\frac{7}{2}-\left(\frac{17}{10}\right)^2=\frac{350}{100}-\frac{289}{100}=\frac{61}{100}$, $sd(X)=\frac{\sqrt{61}}{10}$.

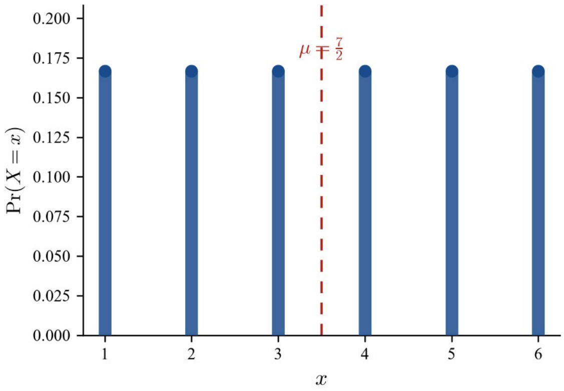
Q9. By symmetry $E(X)=0$. $E(X^2)=4\cdot\frac{1}{16}+1\cdot\frac{4}{16}+0+1\cdot\frac{4}{16}+4\cdot\frac{1}{16}=\frac{4+4+4+4}{16}=1$. $Var(X)=1-0=1$, $sd(X)=1$.



Q10. $E(X)=0(0.4)+1(0.3)+2(0.2)+3(0.1)=1$; $E(X^2)=0+0.3+0.8+0.9=2$; $Var(X)=2-1=1$, $sd(X)=1$.

Q11. The values are $5\times$ those in Q10 with the same probabilities, so $X_{11}=5X_{10}$. Hence $E(X)=5(1)=10$, $Var(X)=5^2(1)=25$ and $sd(X)=5(1)=5$. (Directly: $E(X)=5(0.4)+10(0.3)+15(0.2)+20(0.1)=10$, $E(X^2)=25(0.4)+100(0.3)+225(0.2)+400(0.1)=125$, $Var=125-100=25$.)

Q12. $E(X)=\frac{1+2+3+4+5+6}{6}=\frac{21}{6}=\frac{7}{2}$; $E(X^2)=\frac{1+4+9+16+25+36}{6}=\frac{91}{6}$;
$Var(X)=\frac{91}{6}-\frac{49}{4}=\frac{182}{12}-\frac{147}{12}=\frac{35}{12}$, $sd(X)=\frac{\sqrt{105}}{6}$.



Q13. $E(X)=0(0.49)+1(0.42)+2(0.09)=0.6=\frac{3}{5}$; $E(X^2)=0+0.42+4(0.09)=0.78=\frac{39}{50}$;
$Var(X)=\frac{39}{50}-\frac{9}{25}=\frac{39}{50}-\frac{18}{50}=\frac{21}{50}$, $sd(X)=\sqrt{\frac{21}{50}}=\frac{\sqrt{42}}{10}$.

Q14. Probabilities sum to 1: $k+2k+3k=6k=1 \Rightarrow k=\frac{1}{6}$. $E(X)=1\cdot\frac{1}{6}+2\cdot\frac{2}{6}+3\cdot\frac{3}{6}=\frac{1+4+9}{6}=\frac{14}{6}=\frac{7}{3}$.
$E(X^2)=1\cdot\frac{1}{6}+4\cdot\frac{2}{6}+9\cdot\frac{3}{6}=\frac{1+8+27}{6}=6$. $Var(X)=6-\left(\frac{7}{3}\right)^2=6-\frac{49}{9}=\frac{5}{9}$, $sd(X)=\frac{\sqrt{5}}{3}$.

Q15. By symmetry $E(X)=0$. $E(X^2)=9\cdot\frac{1}{3}+0+9\cdot\frac{1}{3}=6$. $Var(X)=6-0=6$, $sd(X)=\sqrt{6}$.

Q16. $E(X)=0\cdot\frac{1}{16}+1\cdot\frac{4}{16}+2\cdot\frac{6}{16}+3\cdot\frac{4}{16}+4\cdot\frac{1}{16}=\frac{4+12+12+4}{16}=2$.
$E(X^2)=0+\frac{4}{16}+4\cdot\frac{6}{16}+9\cdot\frac{4}{16}+16\cdot\frac{1}{16}=\frac{4+24+36+16}{16}=5$. $Var(X)=5-4=1$, $sd(X)=1$. Then $E(2X)=2(2)=4$ and $Var(2X)=2^2(1)=4$.

Q17. $E(X)=2(0.3)+3(0.3)+5(0.3)+7(0.1)=0.6+0.9+1.5+0.7=\frac{37}{10}$.
$E(X^2)=4(0.3)+9(0.3)+25(0.3)+49(0.1)=1.2+2.7+7.5+4.9=16.3=\frac{163}{10}$.
$Var(X)=\frac{163}{10}-\left(\frac{37}{10}\right)^2=\frac{1630}{100}-\frac{1369}{100}=\frac{261}{100}$, $sd(X)=\frac{\sqrt{261}}{10}=\frac{3\sqrt{29}}{10}$.

Q18. $E(X)=10(0.2)+20(0.5)+30(0.3)=2+10+9=21$.
$E(X^2)=100(0.2)+400(0.5)+900(0.3)=20+200+270=490$. $Var(X)=490-441=49$, $sd(X)=7$.

Q19. $E(X)=0\cdot\frac{1}{8}+1\cdot\frac{3}{8}+2\cdot\frac{3}{8}+3\cdot\frac{1}{8}=\frac{3+6+3}{8}=\frac{12}{8}=\frac{3}{2}$. $E(X^2)=0+\frac{3}{8}+4\cdot\frac{3}{8}+9\cdot\frac{1}{8}=\frac{3+12+9}{8}=3$.
$Var(X)=3-\frac{9}{4}=\frac{3}{4}$, $sd(X)=\frac{\sqrt{3}}{2}$.

Q20. $E(X)=1(0.4)+2(0.3)+4(0.2)+8(0.1)=0.4+0.6+0.8+0.8=\frac{13}{5}$.
$E(X^2)=1(0.4)+4(0.3)+16(0.2)+64(0.1)=0.4+1.2+3.2+6.4=11.2=\frac{56}{5}$.
$Var(X)=\frac{56}{5}-\left(\frac{13}{5}\right)^2=\frac{280}{25}-\frac{169}{25}=\frac{111}{25}$, $sd(X)=\frac{\sqrt{111}}{5}$.

Q21. $E(X) = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$. $E(X^2) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$. $Var(X) = 1 - 0 = 1$. Then $E(4X+3) = 4(0) + 3 = 3$; $Var(4X+3) = 4^2(1) = 16$, so $sd(4X+3) = \sqrt{16} = 4$.