Mathematical Methods Units 3 & 4

Chapter 10 — Anti-differentiation and integration

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Significant effort has been made to ensure this content is legal. Please send any areas of concern to admin@vwx.au with appropriate document/legal references so errors can be verified and changes be made.

Contents

Chapter 10 Anti-differentiation and integration — 10A Estimating areas and the trapezium rule

Theory

The **area under the graph** of a function $y=f(x)$ between $x=a$ and $x=b$ (and above the x-axis) can be **estimated** by dividing $[a,b]$ into n strips of equal width and adding the areas of simple shapes that approximate each strip.

Strip width. With n strips the width is $h=\frac{b-a}{n}$, and the strip boundaries are at $x_0=a, x_1=a+h, x_2=a+2h, \dots, x_n=b$.

Left-endpoint rectangles. Each strip is approximated by a rectangle whose height is the value of f at the **left** edge:

$$L=h\left[f(x_0)+f(x_1)+\cdots+f(x_{n-1})\right].$$

Right-endpoint rectangles. Use the value of f at the **right** edge of each strip:

$$R=h\left[f(x_1)+f(x_2)+\cdots+f(x_n)\right].$$

For a function that is **increasing** on $[a,b]$, the left rectangles sit below the curve so L **under**-estimates the area, while the right rectangles rise above it so R **over**-estimates it. For a **decreasing** function the roles are reversed.

The trapezium rule. Joining the tops of adjacent boundary points by straight lines makes each strip a trapezium. Adding their areas gives

$$T=\frac{h}{2}\left[f(x_0)+f(x_n)+2\left(f(x_1)+f(x_2)+\cdots+f(x_{n-1})\right)\right],$$

that is, *half the width times (first + last + twice each of the middle ordinates)*. The trapezium estimate is the **average** of the left and right estimates: $T=\frac{L+R}{2}$, so it is usually much more accurate.

The area as a limiting sum. As the number of strips increases, the estimate approaches the exact area:

$$\text{Area}=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\delta x.$$

This exact value is found using *integration* (Sections 10B onwards).

To estimate an area: (1) find $h=\frac{b-a}{n}$ and the ordinates $f(x_0),\dots,f(x_n)$; (2) apply the required rule (left, right or trapezium); (3) state whether it is an over- or under-estimate by comparing with the shape of the curve.

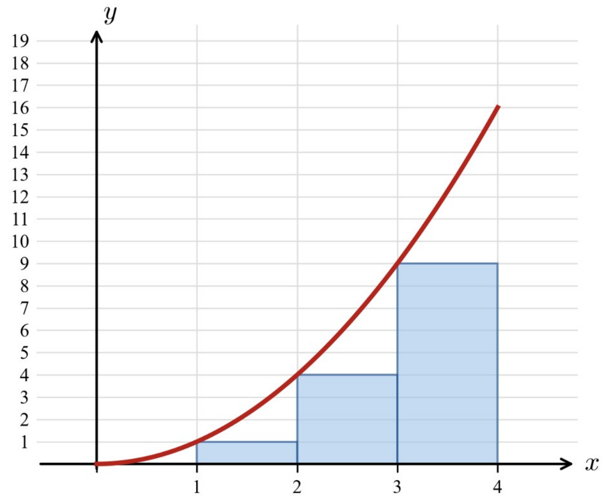
Worked examples

Example 1 — scaffolded

Estimate the area under $y=x^2$ from $x=0$ to $x=4$ using **four left-endpoint rectangles**.

Working	Comment
$h=\frac{4-0}{4}=1$; boundaries $x=0,1,2,3,4$.	Width $=\frac{b-a}{n}$.
Left heights: $f(0)=0, f(1)=1, f(2)=4, f(3)=9$.	Left rectangles use the value at the **left** of each strip.
$L=1\times(0+1+4+9)=14$.	$L=h\left[f(x_0)+\cdots+f(x_3)\right]$.

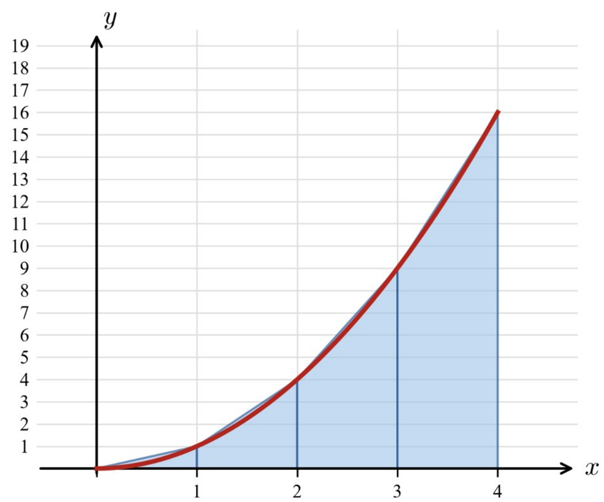
Working	Comment
The exact area is $\frac{64}{3} \approx 21.33$, so $L=14$ is an **under-estimate**.	$y=x^2$ is increasing, so left rectangles sit below the curve.



Example 2 — scaffolded

Estimate the area under $y=x^2$ from $x=0$ to $x=4$ using the **trapezium rule with four strips**.

Working	Comment
$h=1$; ordinates $f(0)=0, f(1)=1, f(2)=4, f(3)=9, f(4)=16$.	Same strips as Example 1, but now use every ordinate.
$T=\frac{1}{2}[0+16+2(1+4+9)]$ $=\frac{1}{2}(16+28)=\frac{1}{2}(44)=22.$	first + last + twice the middle ordinates.
$\therefore$ the estimate is 22 (exact $\frac{64}{3} \approx 21.33$, a slight over-estimate).	$T=\frac{L+R}{2}=\frac{14+30}{2}=22.$



Example 3 — unscaffolded

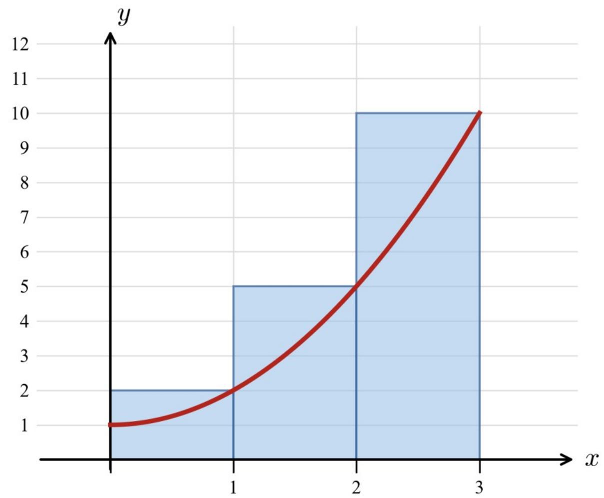
Estimate the area under $y=x^2+1$ from $x=0$ to $x=3$ using **three right-endpoint rectangles**.

$h = \frac{3-0}{3} = 1$; boundaries $x = 0, 1, 2, 3$.

Right heights: $f(1) = 2, f(2) = 5, f(3) = 10$.

$R = 1 \times (2+5+10) = 17$.

$\therefore$ the estimate is 17. The exact area is 12, so this **over**-estimates (the curve is increasing, so right rectangles rise above it).



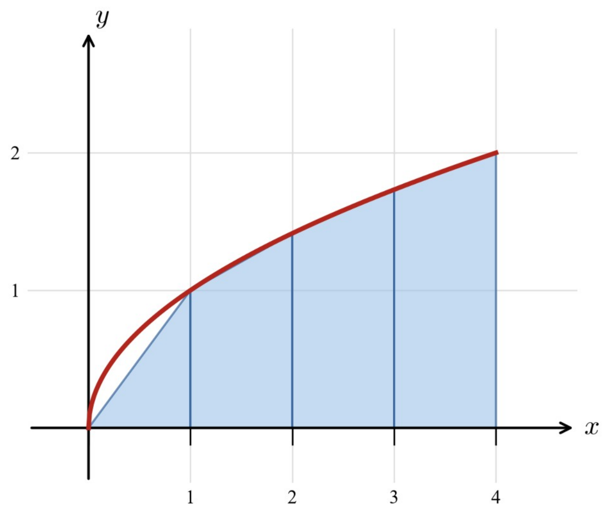
Example 4 — unscaffolded

Use the **trapezium rule with four strips** to estimate the area under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, giving an exact answer.

$h = 1$; ordinates $\sqrt{0} = 0, \sqrt{1} = 1, \sqrt{2}, \sqrt{3}, \sqrt{4} = 2$.

$T = \frac{1}{2}\left[0 + 2 + 2(1 + \sqrt{2} + \sqrt{3})\right] = \frac{1}{2}(2 + 2 + 2\sqrt{2} + 2\sqrt{3})$.

$\therefore T = 2 + \sqrt{2} + \sqrt{3} \approx 5.15$ (exact area $\frac{16}{3} \approx 5.33$).



Practice questions

Scaffolded

Q1. For the area under $y = x^2$ from $x = 0$ to $x = 3$ using **three left-endpoint rectangles**: (a) find the strip width h; (b) write the three left heights; (c) find the estimate L; (d) state whether it over- or under-estimates the exact area 9.

Q2. For the area under $y = x^2$ from $x = 1$ to $x = 4$ using **three right-endpoint rectangles**: (a) find h and the boundaries; (b) find the estimate R; (c) the exact area is 21 — is R an over- or under-estimate?

Q3. Use the **trapezium rule with four strips** to estimate the area under $y = x^3$ from $x = 0$ to $x = 2$: (a) state h and the five ordinates; (b) apply the rule; (c) compare with the exact area 4.

Q4. Use the **trapezium rule with four strips** to estimate the area under $y = \frac{1}{x}$ from $x = 1$ to $x = 3$: (a) state h and the ordinates; (b) find T; (c) the exact area is $\log_e 3$ — how good is the estimate?

Q5. For $y = x^2$ from $x = 0$ to $x = 2$ with **four strips**: (a) find the left estimate L; (b) find the right estimate R; (c) the exact area is $\frac{8}{3}$ — verify that it lies between L and R.

Q6. Use **two right-endpoint rectangles** to estimate the area under $y = \sqrt{x}$ from $x = 0$ to $x = 4$: (a) find h and the heights; (b) find R as an exact value; (c) sketch the rectangles.

Unscaffolded

Estimate the area under each curve as specified, giving exact values.

Q7. $y = x^2$, $x = 0$ to $x = 5$, **five left-endpoint rectangles**.

Q8. $y = x^2 + 2$, $x = 0$ to $x = 4$, **four right-endpoint rectangles**.

Q9. $y = x^3$, $x = 0$ to $x = 3$, **three left-endpoint rectangles**.

Q10. $y = \frac{1}{x}$, $x = 1$ to $x = 5$, **trapezium rule, four strips**.

Q11. $y = \sqrt{x}$, $x = 1$ to $x = 4$, **trapezium rule, three strips**.

Q12. $y = 4 - x^2$, $x = 0$ to $x = 2$, **four left-endpoint rectangles**. State whether it over- or under-estimates and explain using the shape of the curve.

Q13. $y = x^2$, $x = -2$ to $x = 2$, **trapezium rule, four strips**.

Q14. $y = e^x$, $x = 0$ to $x = 2$, **trapezium rule, four strips** (leave your answer in exact form and give a decimal to two places).

Q15. $y = x^3 + 1$, $x = 0$ to $x = 2$, **four right-endpoint rectangles**.

Q16. $y = x^2 + x$, $x = 0$ to $x = 4$, **four left-endpoint rectangles**.

Q17. $y = x^2$, $x = 0$ to $x = 6$, **three right-endpoint rectangles**.

Q18. $y = 2^x$, $x = 0$ to $x = 4$, **trapezium rule, four strips**.

Q19. $y = x(4 - x)$, $x = 0$ to $x = 4$, **four left-endpoint rectangles**. Comment on why the left and right estimates are equal here.

Q20. $y = \sqrt{x} + 1$, $x = 0$ to $x = 4$, **trapezium rule, four strips**.

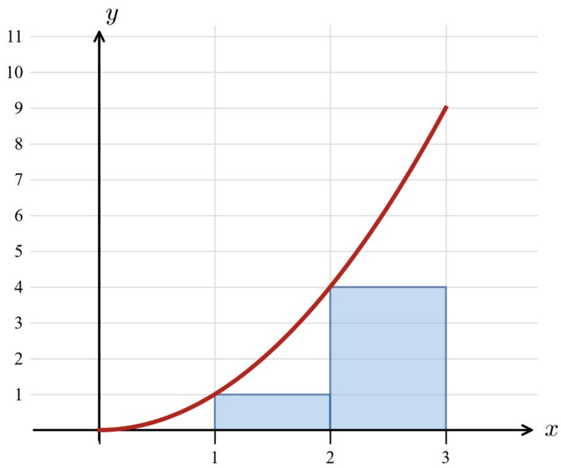
Q21. $y = x^2$, $x = 0$ to $x = 1$, **four strips**: find both the left estimate L and the right estimate R, and show the exact area $\frac{1}{3}$ lies between them.

Answers

Q1. (a) $h=1$; (b) $0,1,4$; (c) $L=5$; (d) under-estimate. **Q2.** (a) $h=1$, boundaries $1,2,3,4$; (b) $R=29$; (c) over-estimate. **Q3.** (a) $h=\frac{1}{2}$, ordinates $0,\frac{1}{8},1,\frac{27}{8},8$; (b) $T=\frac{17}{4}$; (c) slight over-estimate (exact 4). **Q4.** (a) $h=\frac{1}{2}$, ordinates $1,\frac{2}{3},\frac{1}{2},\frac{2}{5},\frac{1}{3}$; (b) $T=\frac{67}{60}$; (c) $\frac{67}{60}\approx 1.117$ vs $\log_e 3\approx 1.099$ — close over-estimate. **Q5.** (a) $L=\frac{7}{4}$; (b) $R=\frac{15}{4}$; (c) $\frac{7}{4}<\frac{8}{3}<\frac{15}{4}$. ✓ **Q6.** (a) $h=2$, heights $\sqrt{2},2$; (b) $R=4+2\sqrt{2}$; (c) see graph. **Q7.** $L=30$. **Q8.** $R=38$. **Q9.** $L=9$. **Q10.** $T=\frac{101}{60}$. **Q11.** $T=\frac{3}{2}+\sqrt{2}+\sqrt{3}$. **Q12.** $L=\frac{25}{4}$; over-estimate ($y=4-x^2$ is decreasing on $[0,2]$, so left rectangles rise above the curve). **Q13.** $T=6$. **Q14.** $T=\frac{1}{4}\left(1+2e^{1/2}+2e+2e^{3/2}+e^2\right)\approx 6.52$. **Q15.** $R=\frac{33}{4}$. **Q16.** $L=20$. **Q17.** $R=112$. **Q18.** $T=\frac{45}{2}$. **Q19.** $L=10$; $L=R$ because $f(0)=f(4)=0$ and the ordinates are symmetric about $x=2$. **Q20.** $T=6+\sqrt{2}+\sqrt{3}$. **Q21.** $L=\frac{7}{32}$, $R=\frac{15}{32}$; $\frac{7}{32}<\frac{1}{3}<\frac{15}{32}$. ✓

Fully-worked solutions

Q1. $h=\frac{3-0}{3}=1$; boundaries $0,1,2,3$. Left heights $f(0)=0, f(1)=1, f(2)=4$. $L=1(0+1+4)=5$. Since $y=x^2$ increases, the rectangles lie below the curve, so 5 **under**-estimates the exact area 9.



Q2. $h=\frac{4-1}{3}=1$; boundaries $1,2,3,4$. Right heights $f(2)=4, f(3)=9, f(4)=16$. $R=1(4+9+16)=29$. The curve increases, so right rectangles rise above it: 29 **over**-estimates the exact area 21.

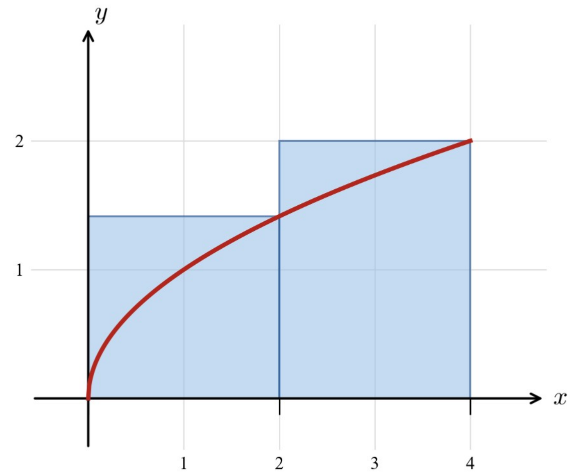
Q3. $h=\frac{2-0}{4}=\frac{1}{2}$; ordinates $f(0)=0, f(\frac{1}{2})=\frac{1}{8}, f(1)=1, f(\frac{3}{2})=\frac{27}{8}, f(2)=8$.

$$T=\frac{1/2}{2}\left[0+8+2\left(\frac{1}{8}+1+\frac{27}{8}\right)\right]=\frac{1}{4}\left[8+2(\frac{36}{8}+1)\right]=\frac{1}{4}[8+9]=\frac{17}{4}.$$ Exact area 4, so a slight over-estimate.

Q4. $h=\frac{1}{2}$; ordinates $1,\frac{2}{3},\frac{1}{2},\frac{2}{5},\frac{1}{3}$. $T=\frac{1}{4}\left[1+\frac{1}{3}+2\left(\frac{2}{3}+\frac{1}{2}+\frac{2}{5}\right)\right]=\frac{67}{60}\approx 1.117$. The exact area $\log_e 3\approx 1.099$, so the trapezium rule slightly over-estimates.

Q5. $h=\frac{1}{2}$; ordinates $f(0)=0, f(\frac{1}{2})=\frac{1}{4}, f(1)=1, f(\frac{3}{2})=\frac{9}{4}, f(2)=4$. $L=\frac{1}{2}(0+\frac{1}{4}+1+\frac{9}{4})=\frac{7}{4}$; $R=\frac{1}{2}(\frac{1}{4}+1+\frac{9}{4}+4)=\frac{15}{4}$. $\frac{7}{4}=1.75<\frac{8}{3}\approx 2.67<\frac{15}{4}=3.75$. ✓

Q6. $h=\frac{4-0}{2}=2$; boundaries $0, 2, 4$. Right heights $f(2)=\sqrt{2}, f(4)=2$. $R=2(\sqrt{2}+2)=4+2\sqrt{2}\approx 6.83$.



Q7. $h=1$; left heights $f(0),\dots,f(4)=0, 1, 4, 9, 16$. $L=1(0+1+4+9+16)=30$.

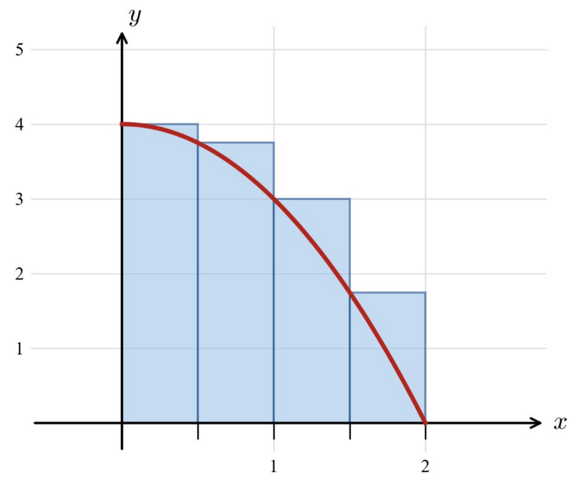
Q8. $h=1$; right heights $f(1),\dots,f(4)=3, 6, 11, 18$. $R=1(3+6+11+18)=38$.

Q9. $h=1$; left heights $f(0), f(1), f(2)=0, 1, 8$. $L=1(0+1+8)=9$.

Q10. $h=1$; ordinates $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. $T=\frac{1}{2}\left[1+\frac{1}{5}+2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\right]=\frac{101}{60}\approx 1.683$.

Q11. $h=1$; ordinates $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}=1, \sqrt{2}, \sqrt{3}, 2$. $T=\frac{1}{2}[1+2+2(\sqrt{2}+\sqrt{3})]=\frac{3}{2}+\sqrt{2}+\sqrt{3}\approx 4.65$.

Q12. $h=\frac{1}{2}$; left heights $f(0)=4, f(\frac{1}{2})=\frac{15}{4}, f(1)=3, f(\frac{3}{2})=\frac{7}{4}$. $L=\frac{1}{2}\left(4+\frac{15}{4}+3+\frac{7}{4}\right)=\frac{1}{2}\cdot\frac{50}{4}=\frac{25}{4}$. Since $y=4-x^2$ is **decreasing** on $[0, 2]$, the left rectangles rise above the curve, so this **over**-estimates the exact area $\frac{16}{3}$.



Q13. $h=1$; ordinates $f(-2), f(-1), f(0), f(1), f(2)=4, 1, 0, 1, 4$. $T=\frac{1}{2}[4+4+2(1+0+1)]=\frac{1}{2}(8+4)=6$.

Q14. $h=\frac{1}{2}$; ordinates $e^0, e^{1/2}, e^1, e^{3/2}, e^2$. $T=\frac{1}{4}\left[e^0+e^2+2\left(e^{1/2}+e+e^{3/2}\right)\right]=\frac{1}{4}\left(1+2e^{1/2}+2e+2e^{3/2}+e^2\right)\approx 6.52$. (Exact area $e^2-1\approx 6.39$.)

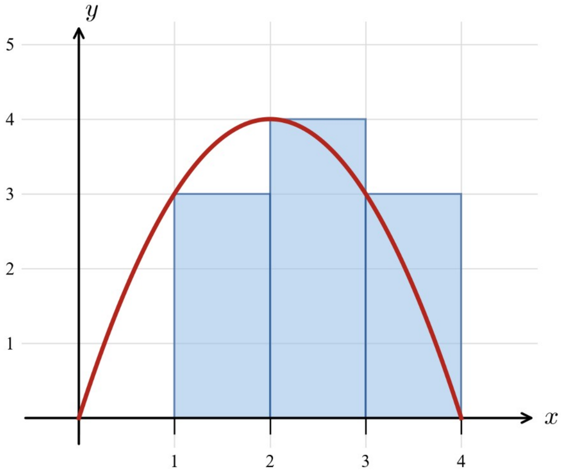
Q15. $h=\frac{1}{2}$; right heights $f\left(\frac{1}{2}\right)=\frac{9}{8}, f(1)=2, f\left(\frac{3}{2}\right)=\frac{35}{8}, f(2)=9$. $R=\frac{1}{2}\left(\frac{9}{8}+2+\frac{35}{8}+9\right)=\frac{1}{2}\cdot\frac{132}{8}=\frac{33}{4}$.

Q16. $h=1$; left heights $f(0), f(1), f(2), f(3)=0,2,6,12$. $L=1(0+2+6+12)=20$.

Q17. $h=2$; right heights $f(2), f(4), f(6)=4,16,36$. $R=2(4+16+36)=112$.

Q18. $h=1$; ordinates $2^0, 2^1, 2^2, 2^3, 2^4=1,2,4,8,16$. $T=\frac{1}{2}[1+16+2(2+4+8)]=\frac{1}{2}(17+28)=\frac{45}{2}$.

Q19. $h=1$; heights $f(0), f(1), f(2), f(3)=0,3,4,3$. $L=1(0+3+4+3)=10$. The right estimate uses $f(1), f(2), f(3), f(4)=3,4,3,0$, giving the same sum 10: the ordinates are symmetric about $x=2$ and $f(0)=f(4)=0$, so $L=R$.



Q20. $h=1$; ordinates $f(0)=1, f(1)=2, f(2)=\sqrt{2}+1, f(3)=\sqrt{3}+1, f(4)=3$.
$T=\frac{1}{2}\left[1+3+2\left(2+\sqrt{2}+1+\sqrt{3}+1\right)\right]=\frac{1}{2}\left(4+2\left(4+\sqrt{2}+\sqrt{3}\right)\right)=6+\sqrt{2}+\sqrt{3}\approx 9.15$.

Q21. $h=\frac{1}{4}$; ordinates $f(0)=0, f\left(\frac{1}{4}\right)=\frac{1}{16}, f\left(\frac{1}{2}\right)=\frac{1}{4}, f\left(\frac{3}{4}\right)=\frac{9}{16}, f(1)=1$. $L=\frac{1}{4}\left(0+\frac{1}{16}+\frac{1}{4}+\frac{9}{16}\right)=\frac{7}{32}$;
$R=\frac{1}{4}\left(\frac{1}{16}+\frac{1}{4}+\frac{9}{16}+1\right)=\frac{15}{32}$. $\frac{7}{32}\approx 0.219<\frac{1}{3}\approx 0.333<\frac{15}{32}\approx 0.469$. ✓

Theory

Antidifferentiation is the reverse of differentiation. If $F'(x) = f(x)$, then F is called an **antiderivative** of f, and we write

$$\int f(x)\,dx = F(x) + c,$$

called the **indefinite integral** of f. Because the derivative of a constant is 0, every function of the form $F(x)+c$ is an antiderivative of f — so the **constant of integration** c must always be included.

The power rule for integration. For any rational $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c.$$

(The excluded case $n=-1$, that is $\int \frac{1}{x}\,dx = \log_e|x| + c$, is treated later.)

Linearity. For a constant k,

$$\int k f(x)\,dx = k\int f(x)\,dx, \int\left(f(x) \pm g(x)\right)dx = \int f(x)\,dx \pm \int g(x)\,dx,$$

so a sum is integrated term by term.

Rewrite before you integrate. The power rule needs each term as a single power of x:

- write surds as fractional powers: $\sqrt{x} = x^{\frac{1}{2}}$, $\sqrt[3]{x} = x^{\frac{1}{3}}$;
- write reciprocals as negative powers: $\frac{1}{x^2} = x^{-2}$;
- **expand** any products or powers of brackets, and **split** a single fraction over a common denominator into separate terms.

Boundary condition. An antiderivative is only known up to $+c$. If one value of the function is given (a *boundary condition* such as $f(a)=b$), substitute it to find c and so determine the function uniquely.

To find an indefinite integral: (1) rewrite each term as a power of x; (2) integrate term by term with the power rule; (3) add $+c$; (4) if a boundary condition is given, substitute it to find c.

Worked examples

Example 1 — scaffolded

Find $\int\left(3x^2 - 4x + 5\right)dx$.

Working	Comment
$\int 3x^2\,dx = 3\cdot\frac{x^3}{3} = x^3$	Power rule on each term: $\int x^n\,dx = \frac{x^{n+1}}{n+1}$.
$\int(-4x)\,dx = -4\cdot\frac{x^2}{2} = -2x^2$	The constant multiple stays out the front.
$\int 5\,dx = 5x$	$5 = 5x^0$, so $\int 5\,dx = 5x$.
$\therefore \int\left(3x^2 - 4x + 5\right)dx = x^3 - 2x^2 + 5x + c$	Combine and add the constant of integration.

Example 2 — scaffolded

Find $\int\left(2x^3-\frac{1}{x^2}+\sqrt{x}\right)dx$.

Working	Comment
Rewrite: $\frac{1}{x^2}=x^{-2}$ and $\sqrt{x}=x^{\frac{1}{2}}$.	Every term must be a power of x before integrating.
$\int 2x^3\,dx=\frac{x^4}{2}$	$2\cdot\frac{x^4}{4}=\frac{x^4}{2}$.
$\int -x^{-2}\,dx=-\frac{x^{-1}}{-1}=\frac{1}{x}$	$n=-2$, so $n+1=-1$; the two minus signs give $+\frac{1}{x}$.
$\int x^{\frac{1}{2}}\,dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}x^{\frac{3}{2}}$	$n=1/2$, so $n+1=3/2$; dividing by $3/2$ multiplies by $2/3$.
$\therefore \frac{x^4}{2}+\frac{1}{x}+\frac{2}{3}x^{\frac{3}{2}}+c$	Combine and add $+c$.

Example 3 — unscaffolded

A function f has $f'(x)=6x^2-2x$ and $f(1)=4$. Find $f(x)$.

The general antiderivative is $f(x)=\int(6x^2-2x)\,dx=2x^3-x^2+c$.

Substitute the boundary condition $f(1)=4$: $2(1)^3-(1)^2+c=4\Rightarrow 2-1+c=4\Rightarrow c=3$.

$\therefore f(x)=2x^3-x^2+3$.

Example 4 — unscaffolded

Find $\int(x-2)(x+3)\,dx$.

The power rule cannot be applied to a product, so **expand first**: $(x-2)(x+3)=x^2+x-6$.

$$\int(x^2+x-6)\,dx=\frac{x^3}{3}+\frac{x^2}{2}-6x+c.$$

$$\therefore \int(x-2)(x+3)\,dx=\frac{x^3}{3}+\frac{x^2}{2}-6x+c.$$

Practice questions

Scaffolded

Q1. For $\int(4x^3+3x^2-2)\,dx$: (a) integrate each term using the power rule; (b) write the indefinite integral, including the constant of integration.

Q2. For $\int\left(3x^2+\frac{4}{x^3}\right)dx$: (a) write $\frac{4}{x^3}$ in index form; (b) integrate each term; (c) state the indefinite integral.

Q3. For $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$: (a) write each term as a power of x; (b) integrate each term; (c) state the indefinite integral.

Q4. For $\int (2x-1)^2\,dx$: (a) expand $(2x-1)^2$; (b) hence find the indefinite integral.

Q5. For $\int \frac{x^2+1}{x^2}\,dx$: (a) split the fraction into two terms and simplify; (b) hence find the indefinite integral.

Q6. A function f has $f'(x)=3x^2+2x$ and $f(1)=5$. (a) Find the general antiderivative of $f'(x)$. (b) Use the condition $f(1)=5$ to find the constant. (c) State $f(x)$.

Unscaffolded

Find each of the following indefinite integrals (Q7–Q18) and functions (Q19–Q21).

Q7. $\int (6x^2-10x+3)\,dx$

Q8. $\int (x^4-3x^2+2x)\,dx$

Q9. $\int (5-2x)\,dx$

Q10. $\int (x^3-x)\,dx$

Q11. $\int (8x^3-6x+1)\,dx$

Q12. $\int (x^5-x^3)\,dx$

Q13. $\int x^{\frac{2}{3}}\,dx$

Q14. $\int (x+1)(x-3)\,dx$

Q15. $\int (3x-2)^2\,dx$

Q16. $\int \left(\frac{1}{x^2}-\frac{1}{x^3}\right)dx$

Q17. $\int (4\sqrt{x}-3)\,dx$

Q18. $\int (2x^3-5x)\,dx$

Q19. Find $f(x)$ given that $f'(x)=4x-1$ and $f(2)=3$.

Q20. Find $f(x)$ given that $f'(x)=6\sqrt{x}$ and $f(4)=10$.

Q21. Find $f(x)$ given that $f'(x)=\frac{1}{x^2}$ and $f(1)=2$.

Answers

Q1. x^4+x^3-2x+c **Q2.** $x^3-\frac{2}{x^2}+c$ **Q3.** $\frac{2}{3}x^{\frac{3}{2}}+2\sqrt{x}+c$ **Q4.** $\frac{4}{3}x^3-2x^2+x+c$ **Q5.** $x-\frac{1}{x}+c$ **Q6.** $f(x)=x^3+x^2+3$ **Q7.** $2x^3-5x^2+3x+c$ **Q8.** $\frac{x^5}{5}-x^3+x^2+c$ **Q9.** $5x-x^2+c$ **Q10.** $\frac{x^4}{4}-\frac{x^2}{2}+c$ **Q11.** $2x^4-3x^2+x+c$ **Q12.** $\frac{x^6}{6}-\frac{x^4}{4}+c$ **Q13.** $\frac{3}{5}x^{\frac{5}{3}}+c$ **Q14.** $\frac{x^3}{3}-x^2-3x+c$ **Q15.** $3x^3-6x^2+4x+c$ **Q16.** $-\frac{1}{x}+\frac{1}{2x^2}+c$ **Q17.** $\frac{8}{3}x^{\frac{3}{2}}-3x+c$ **Q18.** $\frac{x^4}{2}-\frac{5x^2}{2}+c$ **Q19.** $f(x)=2x^2-x-3$ **Q20.** $f(x)=4x^{\frac{3}{2}}-22$ **Q21.** $f(x)=3-\frac{1}{x}$

Fully-worked solutions

Q1. $\int(4x^3+3x^2-2)\,dx=4\cdot\frac{x^4}{4}+3\cdot\frac{x^3}{3}-2x+c=x^4+x^3-2x+c.$

Q2. $\frac{4}{x^3}=4x^{-3}$, so $\int(3x^2+4x^{-3})\,dx=3\cdot\frac{x^3}{3}+4\cdot\frac{x^{-2}}{-2}+c=x^3-2x^{-2}+c=x^3-\frac{2}{x^2}+c.$

Q3. $\sqrt{x}=x^{\frac{1}{2}}$ and $\frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}$, so $\int\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c=\frac{2}{3}x^{\frac{3}{2}}+2\sqrt{x}+c.$

Q4. $(2x-1)^2=4x^2-4x+1$, so $\int(4x^2-4x+1)\,dx=\frac{4}{3}x^3-2x^2+x+c.$

Q5. $\frac{x^2+1}{x^2}=1+x^{-2}$, so $\int(1+x^{-2})\,dx=x+\frac{x^{-1}}{-1}+c=x-\frac{1}{x}+c.$

Q6. (a) $f(x)=\int(3x^2+2x)\,dx=x^3+x^2+c$. (b) $f(1)=1+1+c=5\Rightarrow c=3$. (c) $f(x)=x^3+x^2+3.$

Q7. $\int(6x^2-10x+3)\,dx=2x^3-5x^2+3x+c.$

Q8. $\int(x^4-3x^2+2x)\,dx=\frac{x^5}{5}-x^3+x^2+c.$

Q9. $\int(5-2x)\,dx=5x-x^2+c.$

Q10. $\int(x^3-x)\,dx=\frac{x^4}{4}-\frac{x^2}{2}+c.$

Q11. $\int(8x^3-6x+1)\,dx=2x^4-3x^2+x+c.$

Q12. $\int(x^5-x^3)\,dx=\frac{x^6}{6}-\frac{x^4}{4}+c.$

Q13. $\int x^{\frac{2}{3}}\,dx=\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+c=\frac{3}{5}x^{\frac{5}{3}}+c.$

Q14. $(x+1)(x-3)=x^2-2x-3$, so $\int(x^2-2x-3)\,dx=\frac{x^3}{3}-x^2-3x+c.$

Q15. $(3x-2)^2 = 9x^2-12x+4$, so $\int(9x^2-12x+4)\,dx = 3x^3-6x^2+4x+c$.

Q16. $\int(x^{-2}-x^{-3})\,dx = \frac{x^{-1}}{-1}-\frac{x^{-2}}{-2}+c = -\frac{1}{x}+\frac{1}{2x^2}+c$.

Q17. $4\sqrt{x} = 4x^{\frac{1}{2}}$, so $\int\left(4x^{\frac{1}{2}}-3\right)dx = 4\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-3x+c = \frac{8}{3}x^{\frac{3}{2}}-3x+c$.

Q18. $\int(2x^3-5x)\,dx = \frac{x^4}{2}-\frac{5x^2}{2}+c$.

Q19. $f(x) = \int(4x-1)\,dx = 2x^2-x+c$. $f(2) = 8-2+c = 3 \Rightarrow c = -3$. $\therefore f(x) = 2x^2-x-3$.

Q20. $f(x) = \int 6x^{\frac{1}{2}}\,dx = 6\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c = 4x^{\frac{3}{2}}+c$. $f(4) = 4(4)^{\frac{3}{2}}+c = 4(8)+c = 32+c = 10 \Rightarrow c = -22$.

$\therefore f(x) = 4x^{\frac{3}{2}}-22$.

Q21. $f(x) = \int x^{-2}\,dx = \frac{x^{-1}}{-1}+c = -\frac{1}{x}+c$. $f(1) = -1+c = 2 \Rightarrow c = 3$. $\therefore f(x) = 3-\frac{1}{x}$.

Theory

Antidifferentiation reverses differentiation. To differentiate $(ax+b)^{n+1}$ we use the chain rule and obtain $a(n+1)(ax+b)^n$. Reversing this gives the key rule of this section.

The rule. For $a \neq 0$ and $n \neq -1$,

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

In words: **raise the power by one, divide by the new power *and* by the coefficient a of x**. The extra division by a is what "undoes" the chain rule. This holds whether n is a positive integer, a negative integer, or a rational number.

The special case $n = -1$. The rule above fails when $n = -1$ (it would divide by 0). Instead,

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \log_e |ax+b| + c.$$

This is the antiderivative of a reciprocal of a linear expression; the modulus signs keep the logarithm defined on both sides of the vertical asymptote $x = -\frac{b}{a}$.

Rewrite first. Before integrating, rewrite roots and reciprocals as powers:

$$\sqrt{ax+b} = (ax+b)^{1/2}, \frac{1}{(ax+b)^k} = (ax+b)^{-k}.$$

Constant multiples and sums. $\int k f \, dx = k \int f \, dx$, and a sum is integrated term by term.

Definite integrals. By the fundamental theorem of calculus, $\int_p^q f(x) \, dx = [F(x)]_p^q = F(q) - F(p)$, where F is any antiderivative. The constant c cancels, so it is omitted in a definite integral.

Boundary conditions. If a rate $f'(x)$ and one value $f(x_0) = y_0$ are known, antidifferentiate to get $f(x) = F(x) + c$, then substitute the known point to find c.

To antidifferentiate a power of a linear expression:

1. Rewrite any root or reciprocal as a power $(ax+b)^n$.
2. If $n \neq -1$: add one to the power, then divide by the new power and by a.
3. If $n = -1$: the answer is $\frac{1}{a} \log_e |ax+b| + c$.
4. Add $+c$ for an indefinite integral; evaluate the bracket for a definite integral.

Worked examples

Example 1 — scaffolded

Find $\int (2x+3)^4 \, dx$.

Working	Comment
Here $a=2$, $b=3$, $n=4$.	Identify the linear expression and the power.

Working	Comment
$\int (2x+3)^4\,dx = \dfrac{(2x+3)^5}{2\times 5} + c$	Add one to the power (to 5), then divide by the new power 5 and by $a=2$.
$= \dfrac{(2x+3)^5}{10} + c$	Simplify $2\times 5 = 10$.

$$\therefore \int (2x+3)^4\,dx = \frac{(2x+3)^5}{10} + c.$$

Example 2 — scaffolded

Find $\int \dfrac{1}{(3x-1)^2}\,dx$.

Working	Comment
$\dfrac{1}{(3x-1)^2} = (3x-1)^{-2}$	Rewrite the reciprocal as a power before integrating.
$\int (3x-1)^{-2}\,dx = \dfrac{(3x-1)^{-1}}{3\times(-1)} + c$	Add one to the power ($-2 \to -1$), divide by the new power -1 and by $a=3$.
$= -\dfrac{1}{3(3x-1)} + c$	$(3x-1)^{-1} = \dfrac{1}{3x-1}$; the sign comes from dividing by -1.

$$\therefore \int \frac{1}{(3x-1)^2}\,dx = -\frac{1}{3(3x-1)} + c.$$

Example 3 — unscaffolded

Find $\int \sqrt{4x+1}\,dx$.

Rewrite the surd as a power: $\sqrt{4x+1} = (4x+1)^{1/2}$, so $n = \dfrac{1}{2}$ and $a=4$.

$$\int (4x+1)^{1/2}\,dx = \frac{(4x+1)^{3/2}}{4\times 3/2} + c = \frac{(4x+1)^{3/2}}{6} + c.$$

$$\therefore \int \sqrt{4x+1}\,dx = \frac{(4x+1)^{3/2}}{6} + c.$$

Example 4 — unscaffolded

Find $\int \dfrac{3}{3x+2}\,dx$.

The power is -1, so this is the logarithm case with $a=3$:

$$\int \frac{3}{3x+2}\,dx = 3\times\frac{1}{3}\log_e|3x+2| + c = \log_e|3x+2| + c.$$

$$\therefore \int \frac{3}{3x+2}\,dx = \log_e|3x+2| + c.$$

Practice questions

Scaffolded

Q1. For $\int (x+5)^3 dx$: (a) state a, b and n; (b) add one to the power and divide by the new power and a; (c) write the antiderivative, including $+c$.

Q2. Find $\int (2x-1)^5 dx$, showing the division by the new power and by a.

Q3. For $\int (5-2x)^4 dx$: (a) identify a (note $a=-2$); (b) hence find the antiderivative.

Q4. For $\int \frac{1}{(2x+1)^3} dx$: (a) rewrite the integrand as a power of $(2x+1)$; (b) hence find the antiderivative.

Q5. For $\int \sqrt{3x-2}\, dx$: (a) write $\sqrt{3x-2}$ as a power; (b) hence find the antiderivative.

Q6. For $\int \frac{4}{4x-3} dx$: (a) explain why this is the logarithm case; (b) hence find the antiderivative.

Unscaffolded

Find each of the following (include $+c$ for indefinite integrals; give exact values for definite integrals).

Q7. $\int (3x+1)^6 dx$

Q8. $\int (1-4x)^3 dx$

Q9. $\int (2x+7)^{-4} dx$

Q10. $\int \sqrt{6x-5}\, dx$

Q11. $\int \frac{1}{(3x+4)^2} dx$

Q12. $\int \frac{5}{5x+1} dx$

Q13. $\int (2-x)^5 dx$

Q14. $\int \frac{2}{(x+3)^3} dx$

Q15. $\int \frac{1}{\sqrt{2x+1}} dx$

Q16. $\int_0^1 (2x+1)^3 dx$

Q17. $\int_1^2 \frac{1}{(2x-1)^2} dx$

Q18. $\int_1^3 \frac{1}{x+1} dx$, giving your answer in exact form.

Q19. A function has $f'(x)=(2x-1)^2$ and $f(1)=3$. Find $f(x)$.

Q20. A function has $f'(x)=\frac{1}{2x+1}$ and $f(0)=1$. Find $f(x)$.

Q21. Find the exact value of $\int_0^2 \sqrt{2x+1}\,dx$.

Answers

Q1. $\frac{(x+5)^4}{4}+c$ **Q2.** $\frac{(2x-1)^6}{12}+c$ **Q3.** $-\frac{(5-2x)^5}{10}+c$ **Q4.** $-\frac{1}{4(2x+1)^2}+c$ **Q5.** $\frac{2(3x-2)^{3/2}}{9}+c$ **Q6.** $\log_e|4x-3|+c$ **Q7.** $\frac{(3x+1)^7}{21}+c$ **Q8.** $-\frac{(1-4x)^4}{16}+c$ **Q9.** $-\frac{1}{6(2x+7)^3}+c$ **Q10.** $\frac{(6x-5)^{3/2}}{9}+c$ **Q11.** $-\frac{1}{3(3x+4)}+c$ **Q12.** $\log_e|5x+1|+c$ **Q13.** $-\frac{(2-x)^6}{6}+c$ **Q14.** $-\frac{1}{(x+3)^2}+c$ **Q15.** $\sqrt{2x+1}+c$ **Q16.** 10 **Q17.** $\frac{1}{3}$ **Q18.** $\log_e 2$ **Q19.** $f(x)=\frac{(2x-1)^3}{6}+\frac{17}{6}$ **Q20.** $f(x)=\frac{1}{2}\log_e|2x+1|+1$ **Q21.** $\frac{5\sqrt{5}-1}{3}$

Fully-worked solutions

Q7. $\int(3x+1)^6\,dx=\frac{(3x+1)^7}{3\times7}+c=\frac{(3x+1)^7}{21}+c$.

Q8. $\int(1-4x)^3\,dx=\frac{(1-4x)^4}{(-4)\times4}+c=-\frac{(1-4x)^4}{16}+c$ (here $a=-4$).

Q9. $\int(2x+7)^{-4}\,dx=\frac{(2x+7)^{-3}}{2\times(-3)}+c=-\frac{1}{6(2x+7)^3}+c$.

Q10. $\sqrt{6x-5}=(6x-5)^{1/2}$, so $\int(6x-5)^{1/2}\,dx=\frac{(6x-5)^{3/2}}{6\times3/2}+c=\frac{(6x-5)^{3/2}}{9}+c$.

Q11. $\frac{1}{(3x+4)^2}=(3x+4)^{-2}$, so $\int(3x+4)^{-2}\,dx=\frac{(3x+4)^{-1}}{3\times(-1)}+c=-\frac{1}{3(3x+4)}+c$.

Q12. Power -1, logarithm case with $a=5$: $\int\frac{5}{5x+1}\,dx=5\times\frac{1}{5}\log_e|5x+1|+c=\log_e|5x+1|+c$.

Q13. $\int(2-x)^5\,dx=\frac{(2-x)^6}{(-1)\times6}+c=-\frac{(2-x)^6}{6}+c$ (here $a=-1$).

Q14. $\frac{2}{(x+3)^3}=2(x+3)^{-3}$, so $\int2(x+3)^{-3}\,dx=2\times\frac{(x+3)^{-2}}{1\times(-2)}+c=-\frac{1}{(x+3)^2}+c$.

Q15. $\frac{1}{\sqrt{2x+1}}=(2x+1)^{-1/2}$, so $\int(2x+1)^{-1/2}\,dx=\frac{(2x+1)^{1/2}}{2\times1/2}+c=(2x+1)^{1/2}+c=\sqrt{2x+1}+c$.

Q16. $\int_0^1(2x+1)^3\,dx=\left[\frac{(2x+1)^4}{8}\right]_0^1=\frac{3^4}{8}-\frac{1^4}{8}=\frac{81-1}{8}=\frac{80}{8}=10$.

Q17. $\int_1^2(2x-1)^{-2}\,dx=\left[-\frac{1}{2(2x-1)}\right]_1^2=-\frac{1}{2(3)}+\frac{1}{2(1)}=-\frac{1}{6}+\frac{1}{2}=\frac{1}{3}$.

Q18. $\int_1^3\frac{1}{x+1}\,dx=[\log_e|x+1|]_1^3=\log_e4-\log_e2=\log_e\frac{4}{2}=\log_e2$.

Q19. $f(x)=\int(2x-1)^2dx=\frac{(2x-1)^3}{2\times3}+c=\frac{(2x-1)^3}{6}+c$. Using $f(1)=3$: $\frac{(1)^3}{6}+c=3\Rightarrow c=3-\frac{1}{6}=\frac{17}{6}$.

$\therefore f(x)=\frac{(2x-1)^3}{6}+\frac{17}{6}$.

Q20. $f(x)=\int\frac{1}{2x+1}dx=\frac{1}{2}\log_e|2x+1|+c$. Using $f(0)=1$: $\frac{1}{2}\log_e 1+c=1\Rightarrow c=1$. $\therefore f(x)=\frac{1}{2}\log_e|2x+1|+1$.

Q21. $\int_0^2\sqrt{2x+1}\,dx=\left[\frac{(2x+1)^{3/2}}{3}\right]_0^2=\frac{5^{3/2}}{3}-\frac{1^{3/2}}{3}=\frac{5\sqrt{5}-1}{3}$.

Chapter 10 Anti-differentiation and integration — 10D Antiderivatives of exponential functions

Theory

Because $\frac{d}{dx}\left(e^{kx}\right)=ke^{kx}$, reversing the process gives the antiderivative of an exponential function.

The basic rule.

$$\int e^{kx}\,dx=\frac{1}{k}e^{kx}+c,\,k\neq 0.$$

Check: $\frac{d}{dx}\left(\frac{1}{k}e^{kx}\right)=\frac{1}{k}\cdot ke^{kx}=e^{kx}$.

Linear index. More generally, for the linear index $ax+b$,

$$\int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+c,\,a\neq 0.$$

Linearity. Antidifferentiation is linear, so a sum is done term by term and constants factor out:

$$\int\left(pe^{kx}+q\right)dx=\frac{p}{k}e^{kx}+qx+c.$$

Always include the **constant of integration** $+c$ for an indefinite integral.

Definite integrals. For a definite integral there is no $+c$; evaluate the antiderivative at the terminals:

$$\int_a^b e^{kx}\,dx=\left[\frac{1}{k}e^{kx}\right]_a^b=\frac{1}{k}\left(e^{kb}-e^{ka}\right).$$

Finding a function from its rate of change. If $\frac{dy}{dx}=f'(x)$ is known and one point (a *boundary condition*) is given, antidifferentiate to get the family $y=F(x)+c$, then substitute the point to find c.

Useful before integrating: expand brackets (e.g. $\left(e^x+1\right)^2=e^{2x}+2e^x+1$) so every term is a power of e or of x.

Worked examples

Example 1 — scaffolded

Find $\int e^{3x}\,dx$.

Working	Comment
The index is kx with $k=3$.	Identify k.
$\int e^{3x}\,dx=\frac{1}{3}e^{3x}+c$	Apply $\int e^{kx}\,dx=\frac{1}{k}e^{kx}+c$.
Check: $\frac{d}{dx}\left(\frac{1}{3}e^{3x}\right)=e^{3x}$.	Differentiate back to confirm.

Example 2 — scaffolded

Find $\int\left(4e^{2x}-3\right)dx$.

Working	Comment
Integrate term by term.	Linearity: split the sum, keep the constant factor.
$\int 4e^{2x}\,dx = 4\cdot\frac{1}{2}e^{2x} = 2e^{2x}$	$k=2$, and the factor 4 stays.
$\int(-3)\,dx = -3x$	The antiderivative of a constant.
$\int\left(4e^{2x}-3\right)dx = 2e^{2x}-3x+c$	Combine, with a single $+c$.

Example 3 — unscaffolded

Find $\int e^{2x+1}\,dx$.

The index is linear, $ax+b$ with $a=2$, $b=1$, so

$$\int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1}+c.$$

$\therefore \int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1}+c$.

Example 4 — unscaffolded

Evaluate $\int_0^1 e^{2x}\,dx$, giving the exact value.

$$\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}e^2 - \frac{1}{2}e^0 = \frac{1}{2}\left(e^2-1\right).$$

$\therefore \int_0^1 e^{2x}\,dx = \frac{e^2-1}{2}$.

Practice questions

Scaffolded

Q1. Find each antiderivative: (a) $\int e^{2x}\,dx$; (b) $\int e^{-4x}\,dx$; (c) $\int e^{x/3}\,dx$.

Q2. Find each antiderivative (linear index): (a) $\int e^{3x+1}\,dx$; (b) $\int e^{2-x}\,dx$; (c) $\int e^{4x-3}\,dx$.

Q3. Find each antiderivative: (a) $\int\left(e^{2x}+e^{3x}\right)dx$; (b) $\int\left(2e^{x}-e^{-x}\right)dx$; (c) $\int\left(e^{2x}-5\right)dx$.

Q4. For $\int_0^1 e^{3x}\,dx$: (a) write an antiderivative of e^{3x}; (b) hence evaluate the definite integral exactly.

Q5. For $\int\left(e^{x}-1\right)^2 dx$: (a) expand $\left(e^{x}-1\right)^2$; (b) hence find the antiderivative.

Q6. A function satisfies $\frac{dy}{dx}=e^{2x}$ and passes through $(0,2)$. (a) Write the general antiderivative $y=F(x)+c$; (b) use the point to find c; (c) state the rule for y.

Unscaffolded

Find each antiderivative (Q7–Q15), giving your answer with $+c$.

Q7. $\int e^{-x}\,dx$

Q8. $\int e^{x/2}\,dx$

Q9. $\int 6e^{-3x}\,dx$

Q10. $\int e^{5-x}\,dx$

Q11. $\int \left(e^{2x}+e^{-2x}\right)dx$

Q12. $\int \left(3e^{2x}-2x\right)dx$

Q13. $\int \left(e^{x}+1\right)^{2}dx$

Q14. $\int e^{3x-2}\,dx$

Q15. $\int \left(e^{4x}-e^{2x}\right)dx$

Evaluate each definite integral exactly (Q16–Q19).

Q16. $\int_{0}^{2} e^{-x}\,dx$

Q17. $\int_{-1}^{1} e^{x}\,dx$

Q18. Show that $\int_{0}^{1} \left(e^{2x}+1\right)dx=\dfrac{e^{2}+1}{2}$.

Q19. Evaluate $\int_{0}^{\log_e 2} e^{x}\,dx$.

Q20. The gradient of a curve is given by $\dfrac{dy}{dx}=e^{2x}$ and the curve passes through $(0,1)$. Find the rule for y.

Q21. A function has $f'(x)=e^{-x}+2$ and $f(0)=0$. Find $f(x)$.

Answers

Q1. (a) $\frac{1}{2}e^{2x}+c$ (b) $-\frac{1}{4}e^{-4x}+c$ (c) $3e^{x/3}+c$ **Q2.** (a) $\frac{1}{3}e^{3x+1}+c$ (b) $-e^{2-x}+c$ (c) $\frac{1}{4}e^{4x-3}+c$ **Q3.** (a) $\frac{1}{2}e^{2x}+\frac{1}{3}e^{3x}+c$ (b) $2e^{x}+e^{-x}+c$ (c) $\frac{1}{2}e^{2x}-5x+c$ **Q4.** (a) $\frac{1}{3}e^{3x}$ (b) $\frac{e^3-1}{3}$ **Q5.** (a) $e^{2x}-2e^{x}+1$ (b) $\frac{1}{2}e^{2x}-2e^{x}+x+c$ **Q6.** (a) $y=\frac{1}{2}e^{2x}+c$ (b) $c=\frac{3}{2}$ (c) $y=\frac{1}{2}e^{2x}+\frac{3}{2}$ **Q7.** $-e^{-x}+c$ **Q8.** $2e^{x/2}+c$ **Q9.** $-2e^{-3x}+c$ **Q10.** $-e^{5-x}+c$ **Q11.** $\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+c$ **Q12.** $\frac{3}{2}e^{2x}-x^2+c$ **Q13.** $\frac{1}{2}e^{2x}+2e^{x}+x+c$ **Q14.** $\frac{1}{3}e^{3x-2}+c$ **Q15.** $\frac{1}{4}e^{4x}-\frac{1}{2}e^{2x}+c$ **Q16.** $1-e^{-2}$ **Q17.** $e-\frac{1}{e}$ **Q18.** shown **Q19.** 1 **Q20.** $y=\frac{1}{2}e^{2x}+\frac{1}{2}$ **Q21.** $f(x)=-e^{-x}+2x+1$

Fully-worked solutions

Q1. (a) $\frac{1}{2}e^{2x}+c$. (b) $\frac{1}{-4}e^{-4x}=-\frac{1}{4}e^{-4x}+c$. (c) $k=\frac{1}{3}$, so $\frac{1}{1/3}e^{x/3}=3e^{x/3}+c$.

Q2. (a) $a=3$: $\frac{1}{3}e^{3x+1}+c$. (b) $a=-1$: $\frac{1}{-1}e^{2-x}=-e^{2-x}+c$. (c) $a=4$: $\frac{1}{4}e^{4x-3}+c$.

Q3. (a) $\frac{1}{2}e^{2x}+\frac{1}{3}e^{3x}+c$. (b) $2e^{x}-\left(-e^{-x}\right)=2e^{x}+e^{-x}+c$. (c) $\frac{1}{2}e^{2x}-5x+c$.

Q4. (a) $\frac{1}{3}e^{3x}$. (b) $\int_0^1 e^{3x}\,dx=\left[\frac{1}{3}e^{3x}\right]_0^1=\frac{1}{3}e^3-\frac{1}{3}=\frac{e^3-1}{3}$.

Q5. (a) $\left(e^{x}-1\right)^2=e^{2x}-2e^{x}+1$. (b) $\int\left(e^{2x}-2e^{x}+1\right)dx=\frac{1}{2}e^{2x}-2e^{x}+x+c$.

Q6. (a) $y=\frac{1}{2}e^{2x}+c$. (b) $2=\frac{1}{2}e^{0}+c=\frac{1}{2}+c$, so $c=\frac{3}{2}$. (c) $y=\frac{1}{2}e^{2x}+\frac{3}{2}$.

Q7. $k=-1$: $\frac{1}{-1}e^{-x}=-e^{-x}+c$.

Q8. $k=\frac{1}{2}$: $\frac{1}{1/2}e^{x/2}=2e^{x/2}+c$.

Q9. $6\cdot\frac{1}{-3}e^{-3x}=-2e^{-3x}+c$.

Q10. Index $5-x$, $a=-1$: $\frac{1}{-1}e^{5-x}=-e^{5-x}+c$.

Q11. $\frac{1}{2}e^{2x}+\frac{1}{-2}e^{-2x}=\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+c$.

Q12. $3\cdot\frac{1}{2}e^{2x}-2\cdot\frac{x^2}{2}=\frac{3}{2}e^{2x}-x^2+c$.

Q13. Expand: $\left(e^{x}+1\right)^2=e^{2x}+2e^{x}+1$. Then $\int=\frac{1}{2}e^{2x}+2e^{x}+x+c$.

Q14. Index $3x-2$, $a=3$: $\frac{1}{3}e^{3x-2}+c$.

Q15. $\frac{1}{4}e^{4x}-\frac{1}{2}e^{2x}+c$.

Q16. $\int_0^2 e^{-x}\,dx=\left[-e^{-x}\right]_0^2=-e^{-2}-(-e^0)=1-e^{-2}$.

Q17. $\int_{-1}^1 e^x\,dx=\left[e^x\right]_{-1}^1=e^1-e^{-1}=e-\frac{1}{e}$.

Q18. $\int_0^1 (e^{2x}+1)\,dx=\left[\frac{1}{2}e^{2x}+x\right]_0^1=\left(\frac{1}{2}e^2+1\right)-\left(\frac{1}{2}+0\right)=\frac{1}{2}e^2+\frac{1}{2}=\frac{e^2+1}{2}$. ∴ shown.

Q19. $\int_0^{\log_e 2} e^x\,dx=\left[e^x\right]_0^{\log_e 2}=e^{\log_e 2}-e^0=2-1=1$.

Q20. $y=\int e^{2x}\,dx=\frac{1}{2}e^{2x}+c$. At $(0,1)$: $1=\frac{1}{2}+c$, so $c=\frac{1}{2}$. $\therefore y=\frac{1}{2}e^{2x}+\frac{1}{2}$.

Q21. $f(x)=\int(e^{-x}+2)\,dx=-e^{-x}+2x+c$. At $x=0$: $f(0)=-1+0+c=0$, so $c=1$. $\therefore f(x)=-e^{-x}+2x+1$.

Theory

The definite integral. For a function f defined on $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ measures the **signed area** between the graph of $y = f(x)$ and the x-axis from $x = a$ to $x = b$: area **above** the axis counts as positive, area **below** as negative.

The fundamental theorem of calculus (FTC). If F is any antiderivative of f (that is, $F'(x) = f(x)$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The constant of integration is not needed, because it cancels: $(F(b)+c) - (F(a)+c) = F(b) - F(a)$.

We write the evaluation using square brackets:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Properties of the definite integral. For functions f, g and constant k:

- $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$ and $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (linearity);
- $\int_a^a f(x)\,dx = 0$;
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (reversing the limits changes the sign);
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ (splitting the interval).

To evaluate a definite integral: find an antiderivative F, then compute $F(b) - F(a)$. Give an **exact** value (fractions, surds, or in terms of e and $\log_e$) unless a decimal is asked for.

Worked examples

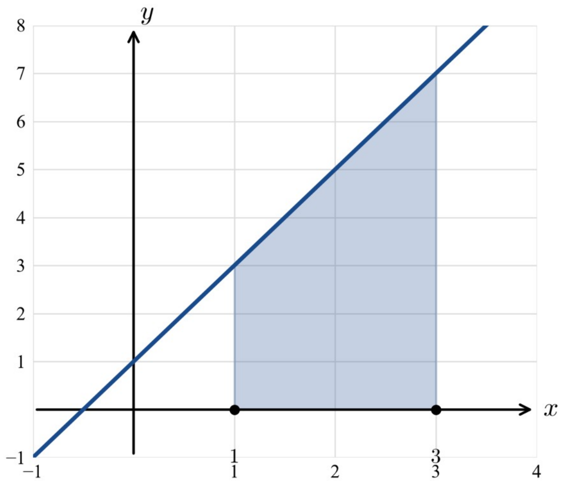
Example 1 — scaffolded

Evaluate $\int_1^3 (2x+1)\,dx$.

Working	Comment
An antiderivative of $2x+1$ is x^2+x.	Reverse the power rule term by term; no $+c$ is needed for a definite integral.
$\int_1^3 (2x+1)\,dx = [x^2+x]_1^3$	Write the antiderivative in square brackets with the limits.
$=(3^2+3)-(1^2+1)$	Substitute the upper limit, then the lower limit: $F(3)-F(1)$.

Working	Comment
$=12-2=10$	The definite integral is a number.

The region is the area under $y=2x+1$ from $x=1$ to $x=3$:



Example 2 — scaffolded

Evaluate $\int_0^2 (3x^2-4x+1)\,dx$.

Working	Comment
Antiderivative: x^3-2x^2+x.	Integrate each term.
$=[x^3-2x^2+x]_0^2$	Insert the limits 0 and 2.
$=(8-8+2)-(0)$	$F(2)-F(0)$; the lower limit gives 0.
$=2$	

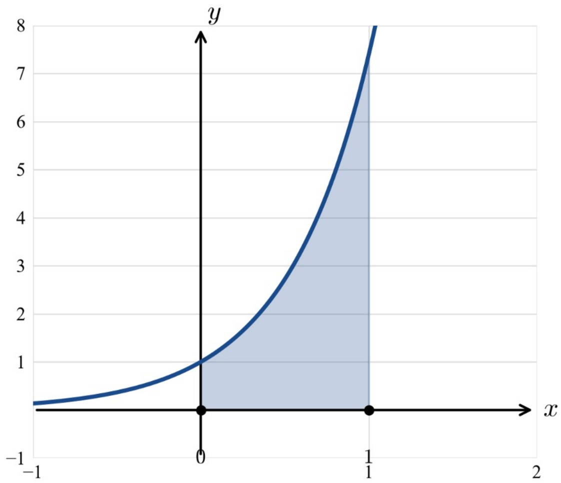
Example 3 — unscaffolded

Evaluate $\int_0^1 e^{2x}\,dx$.

An antiderivative of e^{2x} is $1/2\,e^{2x}$.

$$\int_0^1 e^{2x}\,dx=[1/2\,e^{2x}]_0^1=1/2\,e^2-1/2\,e^0=1/2(e^2-1).$$

$\therefore \int_0^1 e^{2x}\,dx=\dfrac{e^2-1}{2}$ (the shaded area below).



Example 4 — unscaffolded

It is given that $\int_2^5 f(x)\,dx = 8$ and $\int_2^5 g(x)\,dx = 3$. Find (a) $\int_2^5 (2f(x) - g(x))\,dx$ and (b) $\int_5^2 f(x)\,dx$.

(a) By linearity, $\int_2^5 (2f(x) - g(x))\,dx = 2\int_2^5 f(x)\,dx - \int_2^5 g(x)\,dx = 2(8) - 3 = 13$.

(b) Reversing the limits changes the sign: $\int_5^2 f(x)\,dx = -\int_2^5 f(x)\,dx = -8$.

∴ the answers are 13 and -8.

Practice questions

Scaffolded

Q1. Evaluate $\int_1^2 6x^2\,dx$: (a) write an antiderivative of $6x^2$; (b) evaluate it at $x=2$ and $x=1$; (c) hence state the value of the integral.

Q2. Evaluate $\int_0^3 (x^2 - 2x)\,dx$: (a) find an antiderivative; (b) substitute the limits; (c) state the value, and explain what a value of 0 tells you about the signed area.

Q3. Evaluate $\int_0^1 e^{3x}\,dx$, giving your answer in exact form.

Q4. Evaluate $\int_1^4 \frac{1}{\sqrt{x}}\,dx$: (a) write $\frac{1}{\sqrt{x}}$ as a power of x; (b) hence find the integral.

Q5. Evaluate $\int_1^3 \frac{2}{x}\,dx$, giving your answer in the form $a\log_e b$.

Q6. Given $\int_0^4 f(x)\,dx = 10$ and $\int_0^2 f(x)\,dx = 6$, find $\int_2^4 f(x)\,dx$.

Unscaffolded

Evaluate each definite integral, giving an exact value.

Q7. $\int_{-1}^{2} (3x^2+2x)\,dx$

Q8. $\int_{0}^{1} (x^3-x)\,dx$

Q9. $\int_{1}^{2} (2x-1)^3\,dx$

Q10. $\int_{0}^{2} e^{-x}\,dx$

Q11. $\int_{2}^{5} \frac{3}{x-1}\,dx$

Q12. $\int_{1}^{9} \sqrt{x}\,dx$

Q13. $\int_{-2}^{2} x^3\,dx$, and explain your answer using the symmetry of $y=x^3$.

Q14. $\int_{0}^{1} \left(e^x+e^{-x}\right)dx$

Q15. $\int_{1}^{4} (x-2)^2\,dx$

Q16. $\int_{3}^{1} 2x\,dx$ (note the order of the limits).

Q17. Using the property $\int_{a}^{c} f=\int_{a}^{b} f+\int_{b}^{c} f$, show that $\int_{0}^{2} x^2\,dx+\int_{2}^{3} x^2\,dx=\int_{0}^{3} x^2\,dx$, and evaluate.

Q18. $\int_{-1}^{1} (4-x^2)\,dx$

Q19. Find the value of $k>0$ for which $\int_{0}^{k} 2x\,dx=9$.

Q20. Find the value of $a>1$ for which $\int_{1}^{a} \frac{1}{x}\,dx=\log_e 5$.

Q21. $\int_{0}^{1} \left(3e^{2x}-4x\right)dx$

Answers

Q1. 14 **Q2.** 0 (equal positive and negative signed areas) **Q3.** $\frac{e^3-1}{3}$ **Q4.** 2 **Q5.** $2\log_e 3$ **Q6.** 4

Q7. 12 **Q8.** $-\frac{1}{4}$ **Q9.** 10 **Q10.** $1-e^{-2}$ **Q11.** $3\log_e 4=6\log_e 2$ **Q12.** $\frac{52}{3}$ **Q13.** 0 **Q14.** $e-\frac{1}{e}$ **Q15.** 3 **Q16.** -8 **Q17.** 9 **Q18.** $\frac{22}{3}$ **Q19.** $k=3$ **Q20.** $a=5$ **Q21.** $\frac{3e^2-7}{2}$

Fully-worked solutions

Q1. $\int_1^2 6x^2\,dx=[2x^3]_1^2=2(8)-2(1)=16-2=14.$

Q2. $\int_0^3 (x^2-2x)\,dx=[x^3/3-x^2]_0^3=(9-9)-0=0.$ The value 0 means the region above the axis (on $[2,3]$) has the same area as the region below the axis (on $[0,2]$), so the signed areas cancel.

Q3. $\int_0^1 e^{3x}\,dx=[1/3e^{3x}]_0^1=1/3e^3-1/3=\frac{e^3-1}{3}.$

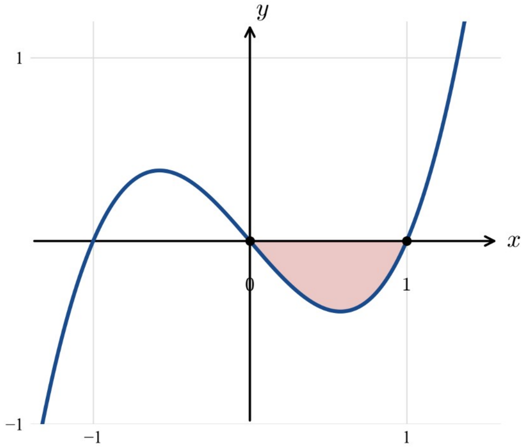
Q4. $\frac{1}{\sqrt{x}}=x^{-1/2}$, so $\int_1^4 x^{-1/2}\,dx=[2x^{1/2}]_1^4=2\sqrt{4}-2\sqrt{1}=4-2=2.$

Q5. $\int_1^3 \frac{2}{x}\,dx=[2\log_e x]_1^3=2\log_e 3-2\log_e 1=2\log_e 3.$

Q6. $\int_2^4 f=\int_0^4 f-\int_0^2 f=10-6=4.$

Q7. $\int_{-1}^2 (3x^2+2x)\,dx=[x^3+x^2]_{-1}^2=(8+4)-(-1+1)=12-0=12.$

Q8. $\int_0^1 (x^3-x)\,dx=[x^4/4-x^2/2]_0^1=(1/4-1/2)-0=-1/4.$ The result is negative because the region lies **below** the x-axis on $(0,1)$:



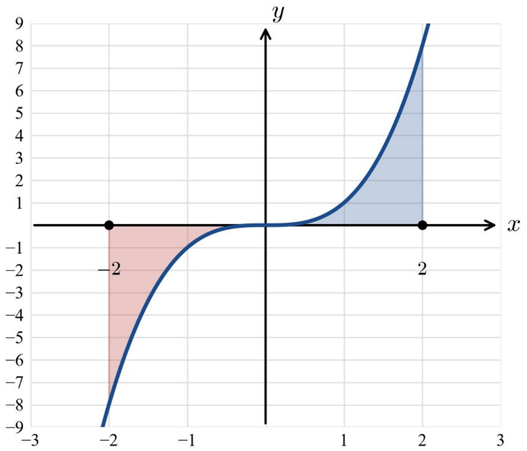
Q9. $\int_1^2 (2x-1)^3 dx = [(2x-1)^4/8]_1^2 = \frac{3^4}{8} - \frac{1^4}{8} = \frac{81-1}{8} = \frac{80}{8} = 10$.

Q10. $\int_0^2 e^{-x} dx = [-e^{-x}]_0^2 = -e^{-2} - (-1) = 1 - e^{-2}$.

Q11. $\int_2^5 \frac{3}{x-1} dx = [3\log_e(x-1)]_2^5 = 3\log_e 4 - 3\log_e 1 = 3\log_e 4 = 6\log_e 2$.

Q12. $\int_1^9 \sqrt{x}\, dx = [2/3\, x^{3/2}]_1^9 = 2/3(27) - 2/3(1) = 18 - 2/3 = \frac{52}{3}$.

Q13. $\int_{-2}^2 x^3 dx = [x^4/4]_{-2}^2 = 16/4 - 16/4 = 0$. Since $y = x^3$ is an **odd** function, the (negative) signed area on $[-2, 0]$ exactly cancels the (positive) signed area on $[0, 2]$:



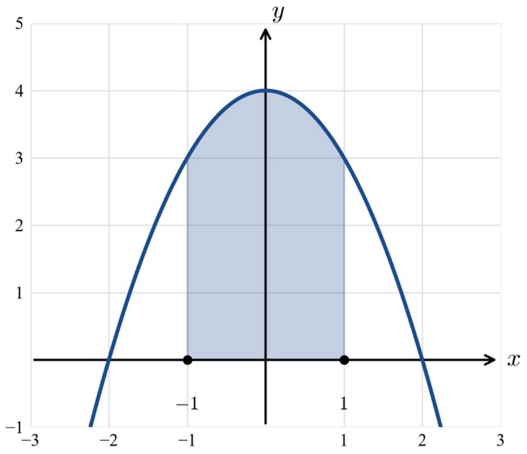
Q14. $\int_0^1 (e^x + e^{-x}) dx = [e^x - e^{-x}]_0^1 = (e - e^{-1}) - (1 - 1) = e - \frac{1}{e}$.

Q15. $\int_1^4 (x-2)^2 dx = [(x-2)^3/3]_1^4 = \frac{2^3}{3} - \frac{(-1)^3}{3} = \frac{8}{3} + \frac{1}{3} = 3$.

Q16. $\int_3^1 2x\, dx = -\int_1^3 2x\, dx = -[x^2]_1^3 = -(9-1) = -8$.

Q17. $\int_0^2 x^2 dx = [x^3/3]_0^2 = 8/3$ and $\int_2^3 x^2 dx = [x^3/3]_2^3 = 9 - 8/3 = 19/3$, so their sum is $8/3 + 19/3 = 27/3 = 9 = \int_0^3 x^2 dx$, confirming the splitting property. The value is 9.

Q18. $\int_{-1}^1 (4 - x^2) dx = [4x - x^3/3]_{-1}^1 = (4 - 1/3) - (-4 + 1/3) = 11/3 + 11/3 = \frac{22}{3}$.



Q19. $\int_0^k 2x\,dx = [x^2]_0^k = k^2$. Setting $k^2 = 9$ gives $k = 3$ (taking $k > 0$).

Q20. $\int_1^a \frac{1}{x}\,dx = [\log_e x]_1^a = \log_e a - \log_e 1 = \log_e a$. Setting $\log_e a = \log_e 5$ gives $a = 5$.

Q21. $\int_0^1 (3e^{2x} - 4x)\,dx = [3/2\,e^{2x} - 2x^2]_0^1 = (3/2\,e^2 - 2) - (3/2) = \frac{3e^2 - 7}{2}$.

Theory

The definite integral $\int_a^b f(x)\,dx$ gives the **signed** area between the curve $y=f(x)$ and the x-axis from $x=a$ to $x=b$: area above the axis counts as positive, area below as negative.

Region above the axis. If $f(x)\geq 0$ on $[a,b]$, the area is

$$A=\int_a^b f(x)\,dx.$$

Region below the axis. If $f(x)\leq 0$ on $[a,b]$, the integral is **negative**, so the area is

$$A=-\int_a^b f(x)\,dx=\left|\int_a^b f(x)\,dx\right|.$$

Region that crosses the axis. If $y=f(x)$ crosses the x-axis inside $[a,b]$, the signed areas partly cancel. To find the **total area** you must:

1. find the x-intercepts in $[a,b]$ (solve $f(x)=0$);
2. integrate over each piece separately;
3. take the **absolute value** of each piece, then add.

So the *total area* is generally **not** the same as the single integral $\int_a^b f(x)\,dx$ — that single integral is the *signed* value, in which below-axis parts subtract.

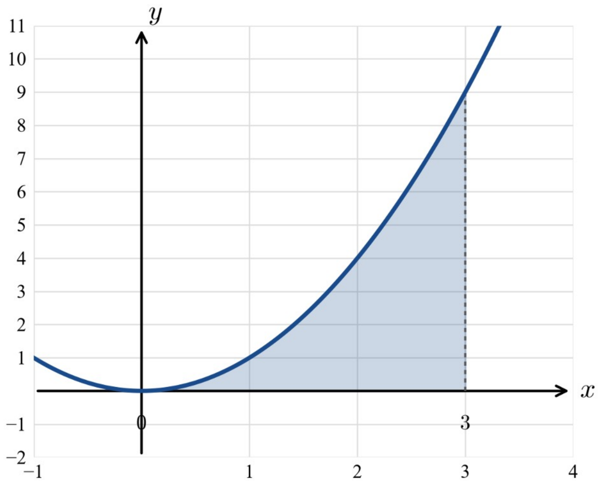
To find an area: sketch the region (shade it), find any x-intercepts between a and b, split the region there, evaluate each definite integral, and add the areas as **positive** quantities.

Worked examples

Example 1 — scaffolded

Find the area of the region bounded by $y=x^2$, the x-axis, and the lines $x=0$ and $x=3$.

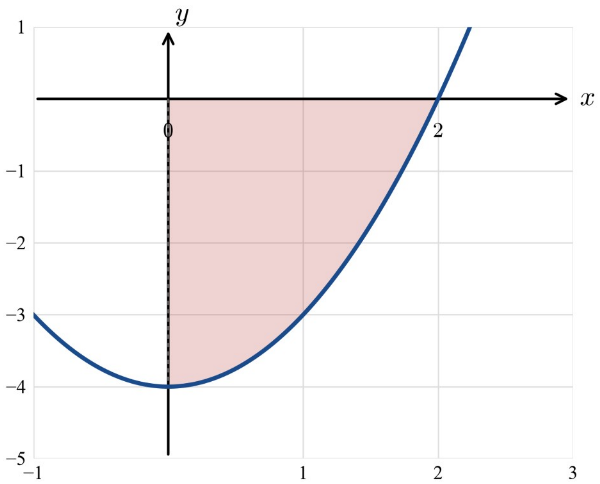
Working	Comment
On $[0,3]$, $x^2\geq 0$, so the region is **above** the axis.	Decide the sign of f before integrating.
$A=\int_0^3 x^2\,dx$	Area of an above-axis region is the integral itself.
$=\left[\frac{x^3}{3}\right]_0^3=\frac{27}{3}-0=9$	Antidifferentiate, then substitute the terminals.
∴ the area is 9 square units.	State the answer with units.



Example 2 — scaffolded

Find the area of the region bounded by $y=x^2-4$, the x-axis, and the lines $x=0$ and $x=2$.

Working	Comment
On $[0,2]$, $x^2-4\le 0$, so the region is **below** the axis.	Check the sign: $x^2\le 4$ here.
$\int_0^2 (x^2-4)\,dx=\left[\frac{x^3}{3}-4x\right]_0^2=\frac{8}{3}-8=-\frac{16}{3}$	The integral is **negative** because the region is below the axis.
$A=\left\lvert -\frac{16}{3}\right\rvert=\frac{16}{3}$	Area is the absolute value.
$\therefore$ the area is $\frac{16}{3}$ square units.	



Example 3 — unscaffolded

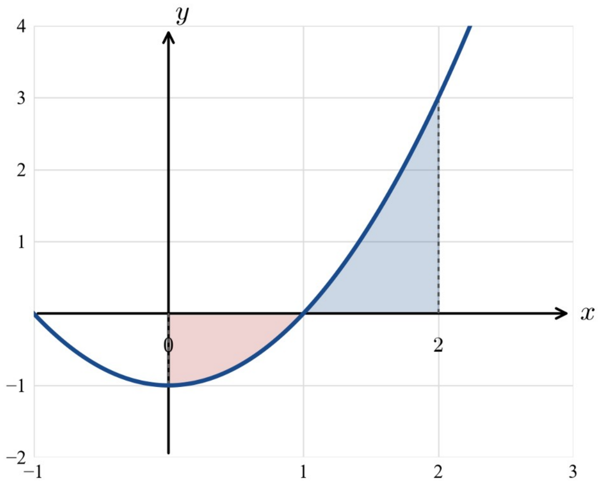
Find the total area of the region bounded by $y=x^2-1$, the x-axis, and the lines $x=0$ and $x=2$.

The curve crosses the x-axis where $x^2-1=0$, i.e. $x=1$ (in $[0,2]$), so split the region there.

On $[0,1]$ the curve is below the axis: $\int_0^1 (x^2-1)\,dx=\left[\frac{x^3}{3}-x\right]_0^1=\frac{1}{3}-1=-\frac{2}{3}$, area $\frac{2}{3}$.

On $[1,2]$ the curve is above the axis: $\int_1^2 (x^2-1)\,dx=\left[\frac{x^3}{3}-x\right]_1^2=\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)=\frac{4}{3}$.

$\therefore$ total area $=\frac{2}{3}+\frac{4}{3}=2$ square units.



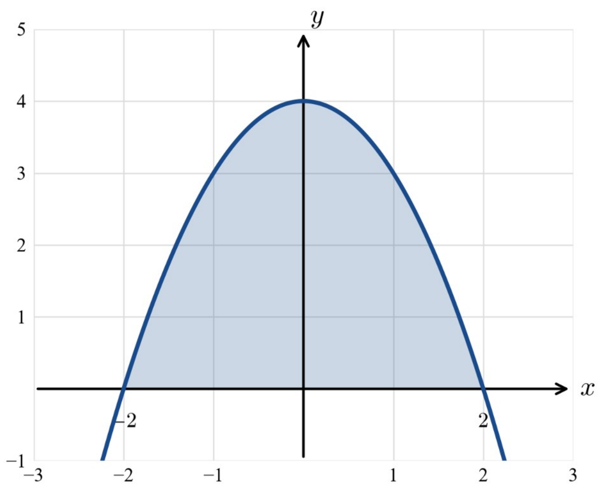
Example 4 — unscaffolded

Find the area of the region bounded by $y=4-x^2$ and the x-axis.

$4-x^2=0$ at $x=\pm 2$, and $4-x^2\geq 0$ on $[-2,2]$, so the whole region is above the axis.

$$A=\int_{-2}^{2}(4-x^2)\,dx=\left[4x-\frac{x^3}{3}\right]_{-2}^{2}=\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)=\frac{32}{3}.$$

$\therefore$ the area is $\frac{32}{3}$ square units.



Practice questions

Scaffolded

Q1. For the region under $y=x^3$ from $x=0$ to $x=2$: (a) state whether the region is above or below the x-axis; (b) write the area as a definite integral; (c) evaluate it.

Q2. Find the area bounded by $y=2x$, the x-axis, and the lines $x=1$ and $x=4$.

Q3. Find the area bounded by $y=x^2+1$, the x-axis, $x=0$ and $x=3$.

Q4. For $y=6-2x$ between $x=0$ and $x=3$: (a) show that the region is above the axis on this interval; (b) find the area.

Q5. Find the area bounded by $y=1-x^2$ and the x-axis between $x=-1$ and $x=1$.

Q6. For $y=x^2-2x$ between $x=0$ and $x=3$: (a) find the x-intercepts in this interval; (b) find the total area of the region; (c) evaluate $\int_0^3 (x^2-2x)\,dx$ and explain why it differs from the total area.

Unscaffolded

Find the exact area of the region bounded by the curve, the x-axis and the given lines (sketch and shade the region first; split at any x-intercepts).

Q7. $y=x^3$, from $x=-2$ to $x=0$.

Q8. $y=3x^2$, from $x=0$ to $x=2$.

Q9. $y=x^2-9$, from $x=0$ to $x=3$.

Q10. $y=2x-x^2$, from $x=0$ to $x=2$.

Q11. $y=e^x$, from $x=0$ to $x=2$.

Q12. $y=e^{-x}$, from $x=0$ to $x=1$.

Q13. $y=\cos x$, from $x=0$ to $x=\frac{\pi}{2}$.

Q14. $y=\sin x$, from $x=0$ to $x=\pi$.

Q15. $y=x^3-4x$, from $x=0$ to $x=2$.

Q16. $y=x^3-4x$, from $x=-2$ to $x=2$. Find the **total area**, and explain why it is not equal to $\int_{-2}^{2} (x^3-4x)\,dx$.

Q17. $y=\frac{1}{x^2}$, from $x=1$ to $x=2$.

Q18. $y=\sqrt{x}$, from $x=0$ to $x=4$.

Q19. $y=4-x$, from $x=0$ to $x=6$. Find the total area of the region(s).

Q20. $y=\frac{1}{x}$, from $x=1$ to $x=e$.

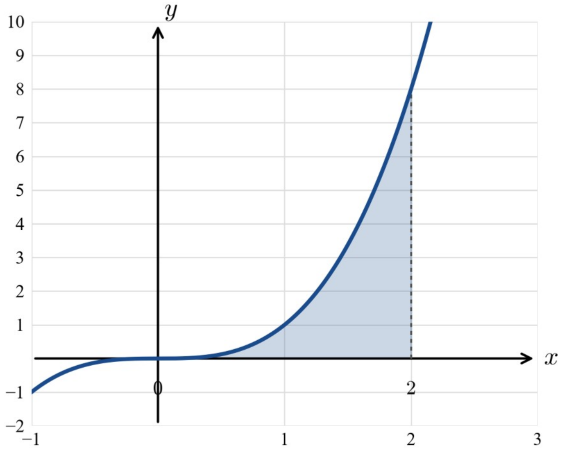
Q21. Find the exact area of the shaded region bounded by $y=(x-1)(x-3)$ and the x-axis between $x=0$ and $x=4$.

Answers

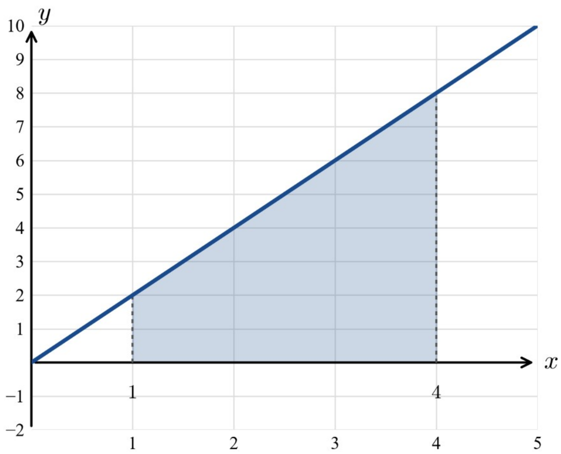
Q1. above; $\int_0^2 x^3\,dx=4$. **Q2.** 15. **Q3.** 12. **Q4.** 9. **Q5.** $\frac{4}{3}$. **Q6.** intercepts $x=0,2$; total area $\frac{8}{3}$; $\int_0^3 =0$ (below- and above-axis parts cancel). **Q7.** 4. **Q8.** 8. **Q9.** 18. **Q10.** $\frac{4}{3}$. **Q11.** e^2-1. **Q12.** $1-e^{-1}$. **Q13.** 1. **Q14.** 2. **Q15.** 4. **Q16.** total area 8 (the integral is 0). **Q17.** $\frac{1}{2}$. **Q18.** $\frac{16}{3}$. **Q19.** 10. **Q20.** 1. **Q21.** 4.

Fully-worked solutions

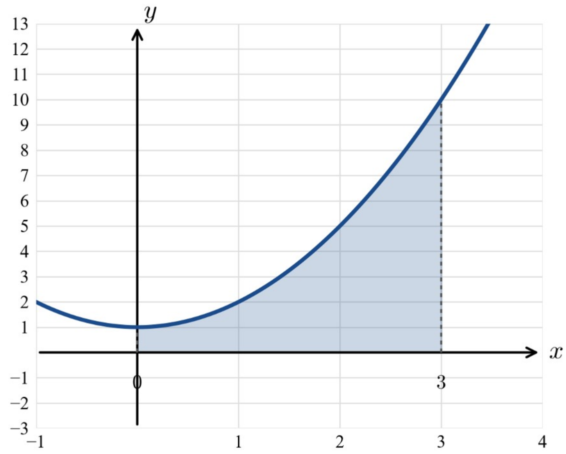
Q1. $y=x^3\geq 0$ on $[0,2]$ (above). $A=\int_0^2 x^3\,dx=\left[\frac{x^4}{4}\right]_0^2=\frac{16}{4}=4$ square units.



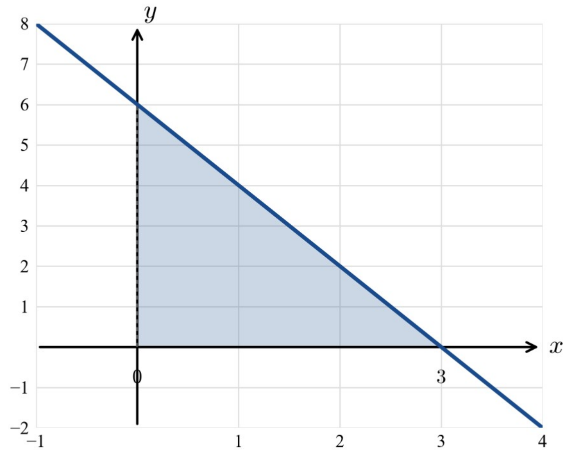
Q2. $y=2x\geq 0$ on $[1,4]$. $A=\int_1^4 2x\,dx=\left[x^2\right]_1^4=16-1=15$ square units.



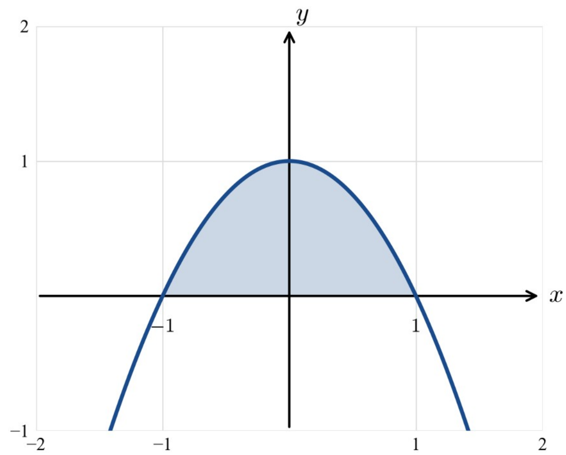
Q3. $A=\int_0^3 (x^2+1)\,dx=\left[\frac{x^3}{3}+x\right]_0^3=9+3=12$ square units.



Q4. (a) $6-2x=0$ at $x=3$, and $6-2x\geq 0$ for $x\leq 3$, so the region is above the axis on $[0,3]$. (b) $A=\int_0^3 (6-2x)\,dx=\left[6x-x^2\right]_0^3=18-9=9$ square units.

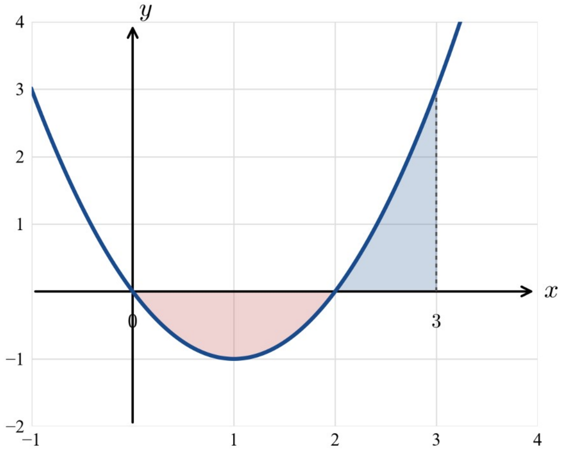


Q5. $1-x^2\geq 0$ on $[-1,1]$. $A=\int_{-1}^{1} (1-x^2)\,dx=\left[x-\dfrac{x^3}{3}\right]_{-1}^{1}=\left(1-\dfrac{1}{3}\right)-\left(-1+\dfrac{1}{3}\right)=\dfrac{4}{3}$ square units.

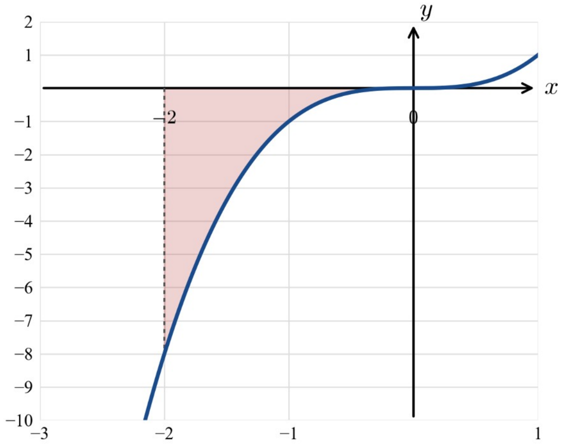


Q6. (a) $x^2-2x=x(x-2)=0$ at $x=0$ and $x=2$. (b) On $[0,2]$ the curve is below the axis:
$\int_0^2 (x^2-2x)\,dx=\left[\dfrac{x^3}{3}-x^2\right]_0^2=\dfrac{8}{3}-4=-\dfrac{4}{3}$, area $\dfrac{4}{3}$. On $[2,3]$ it is above:
$\int_2^3 (x^2-2x)\,dx=\left[\dfrac{x^3}{3}-x^2\right]_2^3=(9-9)-\left(\dfrac{8}{3}-4\right)=\dfrac{4}{3}$. Total area $\dfrac{4}{3}+\dfrac{4}{3}=\dfrac{8}{3}$. (c)

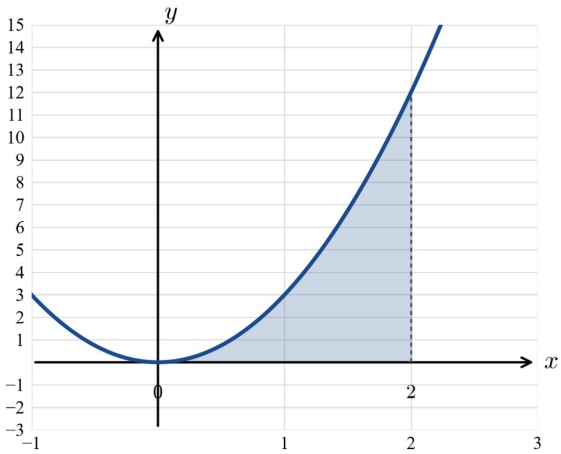
$\int_0^3 (x^2-2x)\,dx=\left[\frac{x^3}{3}-x^2\right]_0^3=9-9=0$: the equal below- and above-axis parts cancel, so the signed integral is 0, not the total area.



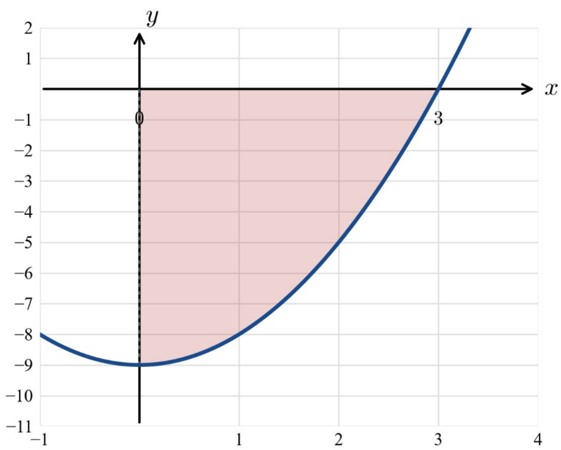
Q7. $y=x^3\le 0$ on $[-2,0]$ (below). $\int_{-2}^{0} x^3\,dx=\left[\frac{x^4}{4}\right]_{-2}^{0}=0-4=-4$, so area $=4$ square units.



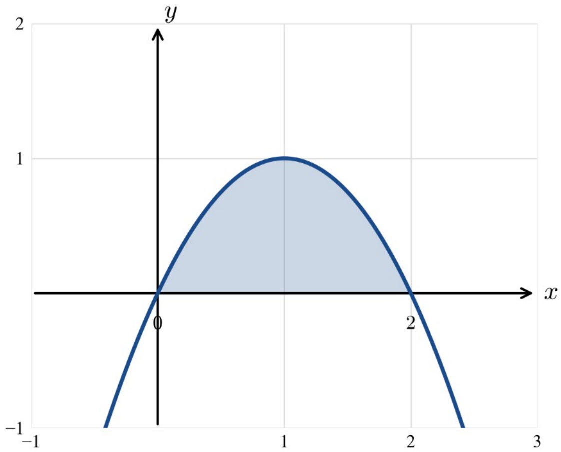
Q8. $A=\int_0^2 3x^2\,dx=\left[x^3\right]_0^2=8$ square units.



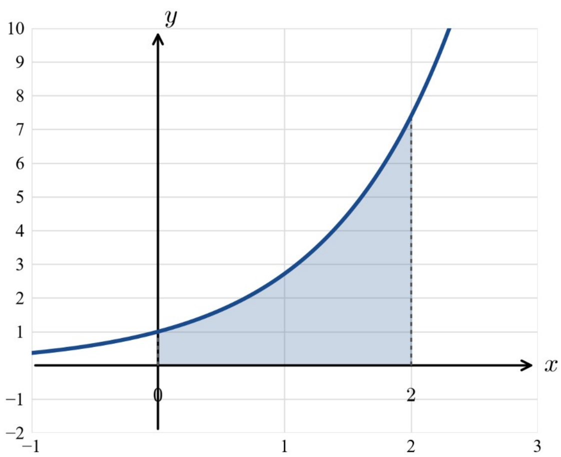
Q9. $y=x^2-9\le 0$ on $[0,3]$. $\int_0^3 (x^2-9)\,dx=\left[\frac{x^3}{3}-9x\right]_0^3=9-27=-18$, area $=18$ square units.



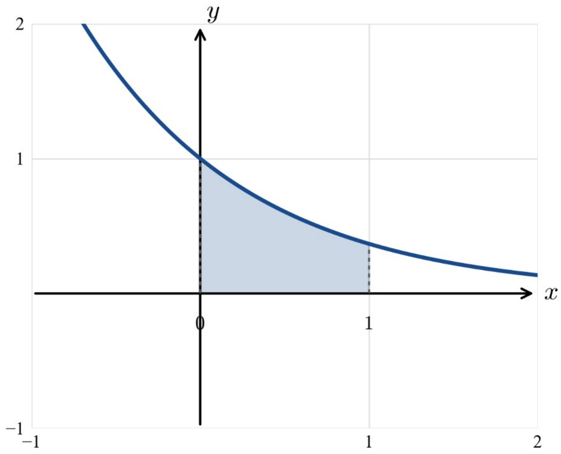
Q10. $2x - x^2 = x(2-x) \geq 0$ on $[0,2]$. $A = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$ square units.



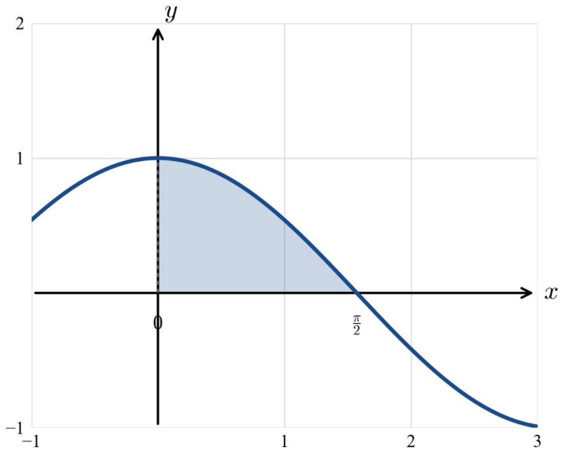
Q11. $A = \int_0^2 e^x\,dx = \left[e^x\right]_0^2 = e^2 - 1$ square units.



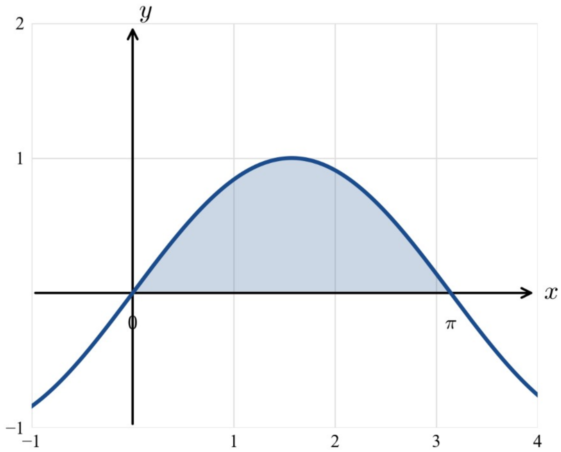
Q12. $A = \int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = -e^{-1} + 1 = 1 - e^{-1}$ square units.



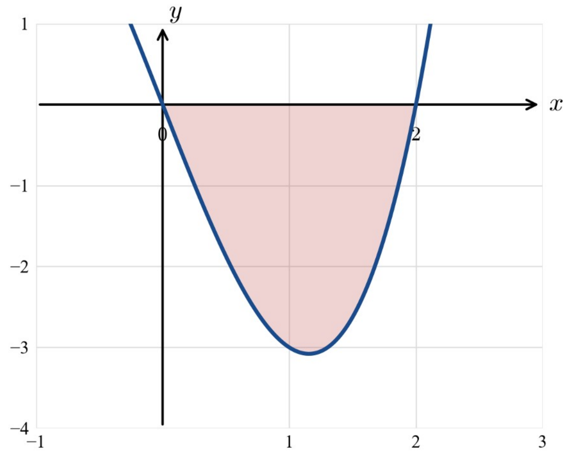
Q13. $\cos x \geq 0$ on $\left[0, \frac{\pi}{2}\right]$. $A = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$ square unit.



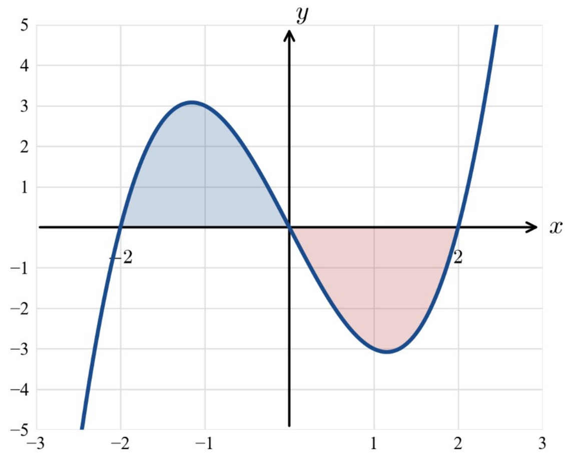
Q14. $\sin x \geq 0$ on $[0, \pi]$. $A = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = -(-1) - (-1) = 2$ square units.



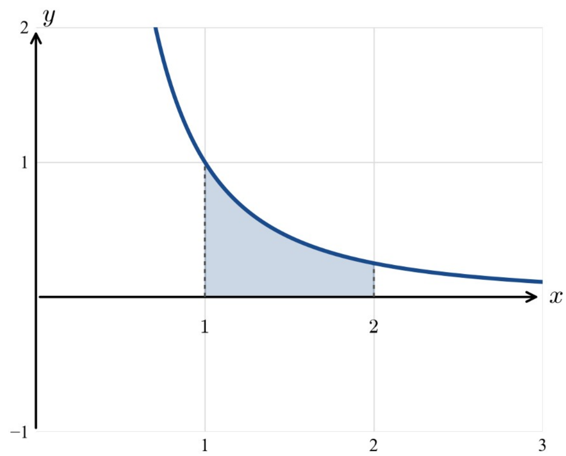
Q15. $x^3 - 4x = x(x^2 - 4)$; on $[0, 2]$ it is below the axis. $\int_0^2 (x^3 - 4x) \, dx = \left[\frac{x^4}{4} - 2x^2\right]_0^2 = 4 - 8 = -4$, area $= 4$ square units.



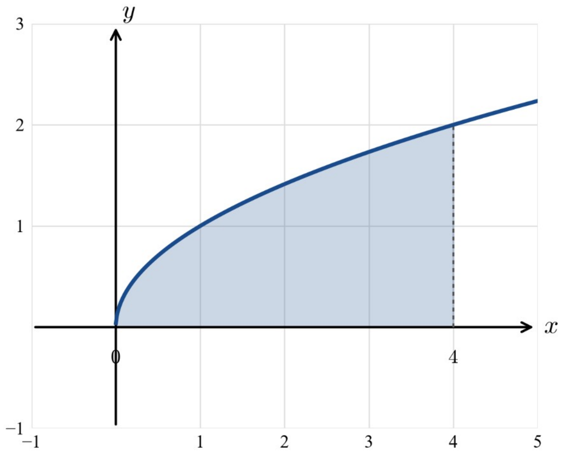
Q16. $x^3-4x=0$ at $x=-2,0,2$. On $[-2,0]$ the curve is above the axis and on $[0,2]$ below, and by symmetry each part has area 4, so the **total area is** 8. The single integral $\int_{-2}^{2}(x^3-4x)\,dx=0$ because x^3-4x is an odd function — the above- and below-axis parts cancel — so it is not the total area.



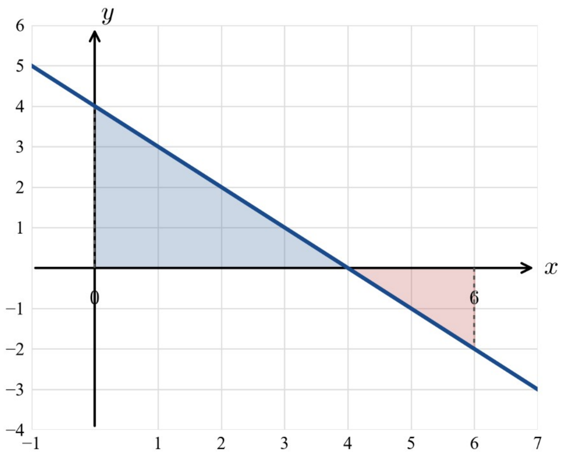
Q17. $A=\int_{1}^{2}\frac{1}{x^2}\,dx=\int_{1}^{2}x^{-2}\,dx=\left[-\frac{1}{x}\right]_{1}^{2}=-\frac{1}{2}+1=\frac{1}{2}$ square unit.



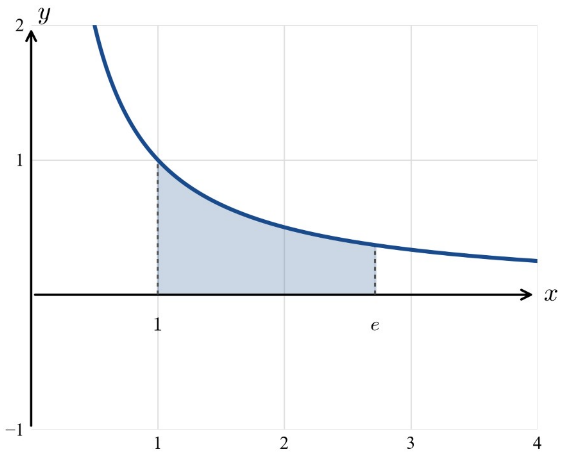
Q18. $A=\int_{0}^{4}\sqrt{x}\,dx=\int_{0}^{4}x^{1/2}\,dx=\left[\frac{2}{3}x^{3/2}\right]_{0}^{4}=\frac{2}{3}(8)=\frac{16}{3}$ square units.



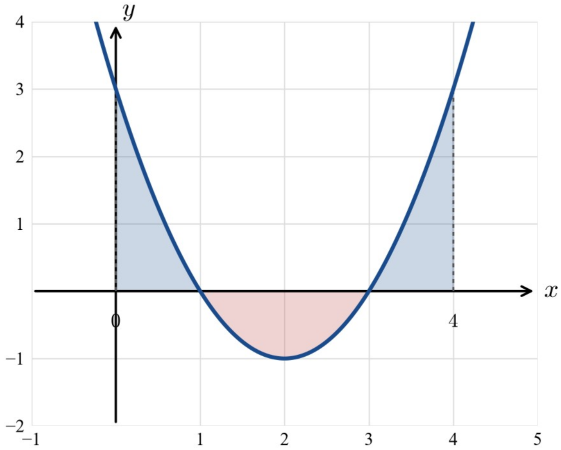
Q19. $4-x=0$ at $x=4$. On $[0,4]$ the region is above the axis: $\int_0^4 (4-x)\,dx=\left[4x-\frac{x^2}{2}\right]_0^4=16-8=8$. On $[4,6]$ it is below: $\int_4^6 (4-x)\,dx=\left[4x-\frac{x^2}{2}\right]_4^6=(24-18)-(16-8)=-2$, area 2. Total area $=8+2=10$ square units.



Q20. $\frac{1}{x}>0$ on $[1,e]$. $A=\int_1^e \frac{1}{x}\,dx=[\log_e x]_1^e=\log_e e-\log_e 1=1-0=1$ square unit.



Q21. $(x-1)(x-3)=0$ at $x=1,3$. On $[0,1]$ above, $[1,3]$ below, $[3,4]$ above. With $F(x)=\frac{x^3}{3}-2x^2+3x$:
$\int_0^1 = F(1)-F(0)=\frac{4}{3}$ (area $\frac{4}{3}$); $\int_1^3 = F(3)-F(1)=0-\frac{4}{3}=-\frac{4}{3}$ (area $\frac{4}{3}$); $\int_3^4 = F(4)-F(3)=\frac{4}{3}-0=\frac{4}{3}$. Total area $=\frac{4}{3}+\frac{4}{3}+\frac{4}{3}=4$ square units.



Theory

Because differentiation and antidifferentiation are reverse processes, each derivative of a circular function gives an antiderivative. Since $\frac{d}{dx}(\sin(kx))=k\cos(kx)$ and $\frac{d}{dx}(\cos(kx))=-k\sin(kx)$, dividing by k gives the standard antiderivatives:

$$\int\cos(kx)\,dx=1/k\sin(kx)+c \text{ and } \int\sin(kx)\,dx=-1/k\cos(kx)+c.$$

Watch the sign and the factor. Integrating $\cos$ gives $+\sin$; integrating $\sin$ gives $-\cos$. In each case divide by the coefficient k of x. More generally, for a linear inside function,

$$\int a\sin(n(x-\varepsilon))\,dx=-\frac{a}{n}\cos(n(x-\varepsilon))+c.$$

Definite integrals. As for any function, $\int_a^b f(x)\,dx=[F(x)]_a^b=F(b)-F(a)$, where F is an antiderivative. Use **exact values** of the circular functions at the endpoints (for example $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$).

Area. The area between the curve $y=f(x)$ and the x-axis from $x=a$ to $x=b$, where $f(x)\ge 0$ on $[a,b]$, equals $\int_a^b f(x)\,dx$. Where a circular-function graph dips **below** the axis the definite integral counts that part as **negative**, so a signed integral may differ from the total area — split the interval at the x-intercepts and integrate each piece.

Finding a function from a boundary condition. Antidifferentiate to get the family $F(x)+c$, then substitute the given point to find c.

Worked examples

Example 1 — scaffolded

Find $\int(2\sin(2x)-3\cos x)\,dx$.

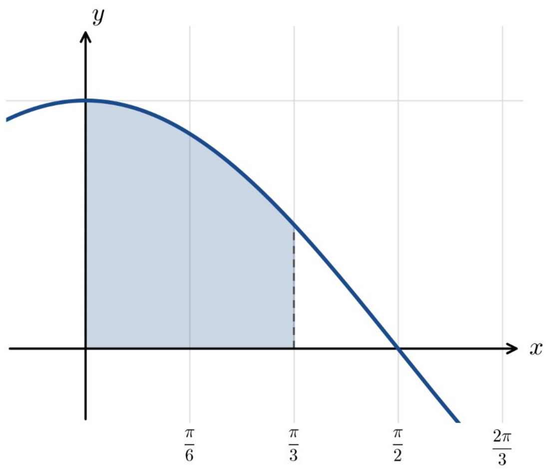
Working	Comment
$\int 2\sin(2x)\,dx=2\cdot(-1/2\cos(2x))=-\cos(2x)$	$\int\sin(kx)\,dx=-1/k\cos(kx)$; here $k=2$.
$\int 3\cos x\,dx=3\sin x$	$\int\cos x\,dx=\sin x$.
$\int(2\sin(2x)-3\cos x)\,dx=-\cos(2x)-3\sin x+c$	Combine and add the constant of integration.
Check: $\frac{d}{dx}(-\cos(2x)-3\sin x)=2\sin(2x)-3\cos x$ ✓	Differentiate back to confirm.

Example 2 — scaffolded

Evaluate $\int_0^{\pi/3}\cos x\,dx$, giving the exact value.

Working	Comment
$\int_0^{\pi/3}\cos x\,dx=[\sin x]_0^{\pi/3}$	An antiderivative of $\cos x$ is $\sin x$.

Working	Comment
$=\sin\frac{\pi}{3}-\sin 0=\frac{\sqrt{3}}{2}-0$	Substitute the terminals; use exact values.
$=\frac{\sqrt{3}}{2}$	The exact value of the integral.



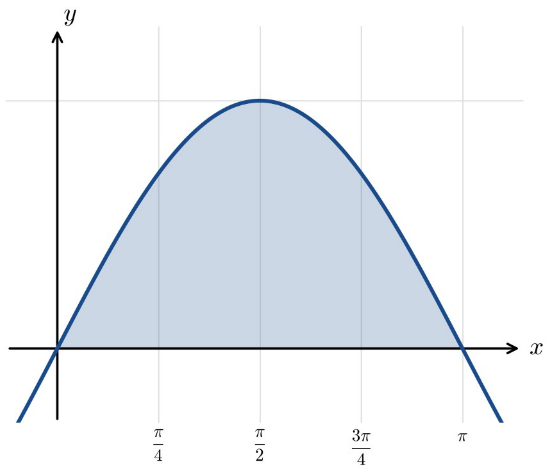
Example 3 — unscaffolded

Find the exact area bounded by $y=\sin x$, the x-axis, and the lines $x=0$ and $x=\pi$.

On $[0,\pi]$, $\sin x\geq 0$, so the area equals the definite integral.

$$\text{Area}=\int_0^{\pi}\sin x\,dx=[-\cos x]_0^{\pi}=-\cos\pi-(-\cos 0)=-(-1)+1=2.$$

∴ the area is 2 square units.



Example 4 — unscaffolded

Given $f'(x)=4\cos(2x)$ and $f(0)=3$, find $f(x)$.

$f(x)=\int 4\cos(2x)\,dx=4\cdot 1/2\sin(2x)+c=2\sin(2x)+c$.

Using $f(0)=3$: $2\sin 0+c=3\Rightarrow c=3$.

$\therefore f(x)=2\sin(2x)+3$.

Practice questions

Scaffolded

Q1. Find $\int \sin(3x)\,dx$.

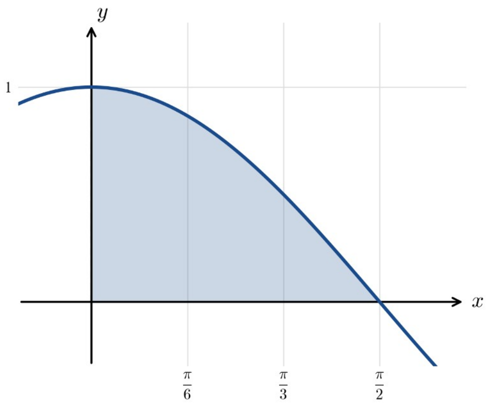
Q2. Find $\int \cos\left(\frac{x}{2}\right)dx$.

Q3. Find $\int(\sin x+\cos x)\,dx$.

Q4. Evaluate $\int_0^{\pi/2} \sin x\,dx$.

Q5. Evaluate $\int_0^{\pi/4} \cos(2x)\,dx$.

Q6. Find the exact area bounded by $y=\cos x$, the x-axis, $x=0$ and $x=\frac{\pi}{2}$.



Unscaffolded

Find each of the following (add the constant of integration for indefinite integrals; give exact values for definite integrals).

Q7. $\int 3\cos(3x)\,dx$

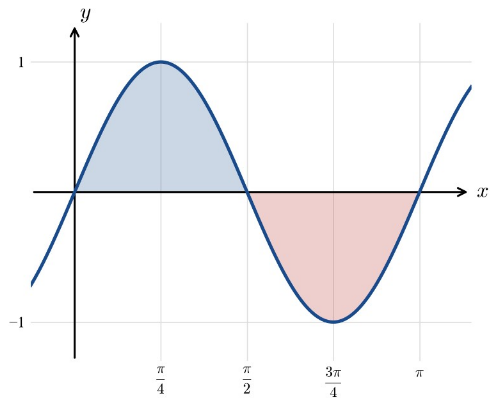
Q8. $\int -2\sin(4x)\,dx$

Q9. $\int(\cos(2x)-\sin(3x))\,dx$

Q10. $\int 2\sin\left(2x+\frac{\pi}{6}\right)dx$

Q11. $\int 4\cos(2x)\,dx$

Q12. Evaluate the definite integral $\int_0^{\pi} \sin(2x)\,dx$, and explain what its value tells you about the signed area.

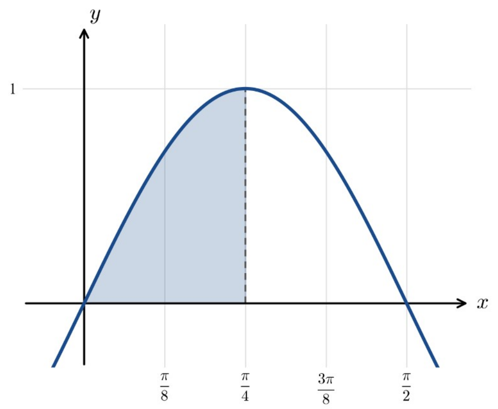


Q13. $\int_{\pi/6}^{\pi/2} \cos x \, dx$

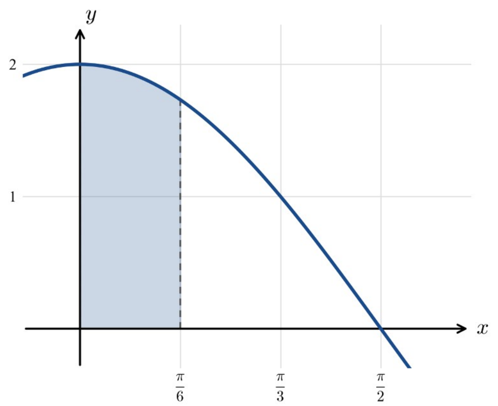
Q14. $\int_{0}^{\pi/3} \sin x \, dx$

Q15. $\int_{0}^{\pi/4} (\sin x + \cos x) \, dx$

Q16. Find the exact area bounded by $y = \sin(2x)$, the x-axis, $x = 0$ and $x = \frac{\pi}{4}$.



Q17. Find the exact area bounded by $y = 2\cos x$, the x-axis, $x = 0$ and $x = \frac{\pi}{6}$.

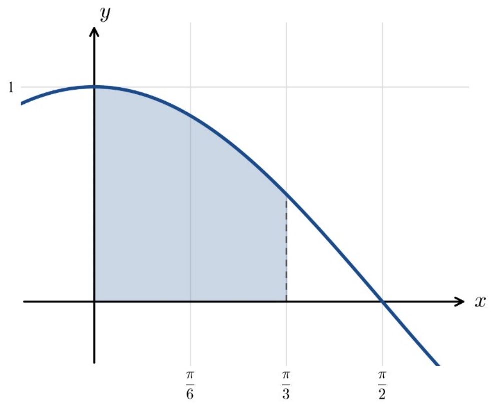


Q18. Given $f'(x) = 6\cos(3x)$ and $f(0) = 2$, find $f(x)$.

Q19. Given $f'(x) = -\sin x$ and $f(0) = 1$, find $f(x)$.

Q20. Show that $\int_0^{\pi} \sin x \, dx = 2$, and hence write down the area between $y = \sin x$ and the x-axis over one full period $[0, 2\pi]$.

Q21. Find the exact area bounded by $y = \cos x$, the x-axis, $x = 0$ and $x = \frac{\pi}{3}$.



Answers

Q1. $-\frac{1}{3}\cos(3x)+c$ **Q2.** $2\sin\left(\frac{x}{2}\right)+c$ **Q3.** $-\cos x+\sin x+c$ **Q4.** 1 **Q5.** $\frac{1}{2}$ **Q6.** 1 **Q7.** $\sin(3x)+c$ **Q8.** $\frac{1}{2}\cos(4x)+c$ **Q9.** $\frac{1}{2}\sin(2x)+\frac{1}{3}\cos(3x)+c$ **Q10.** $-\cos\left(2x+\frac{\pi}{6}\right)+c$ **Q11.** $2\sin(2x)+c$ **Q12.** 0 (the area below the axis on $[\pi/2,\pi]$ cancels the equal area above it on $[0,\pi/2]$) **Q13.** $\frac{1}{2}$ **Q14.** $\frac{1}{2}$ **Q15.** 1 **Q16.** $\frac{1}{2}$ **Q17.** 1 **Q18.** $f(x)=2\sin(3x)+2$ **Q19.** $f(x)=\cos x$ **Q20.** 2; area over $[0,2\pi]$ is 4 **Q21.** $\frac{\sqrt{3}}{2}$

Fully-worked solutions

Q1. $\int\sin(3x)\,dx=-\frac{1}{3}\cos(3x)+c$. Check: $\frac{d}{dx}\left(-\frac{1}{3}\cos 3x\right)=\sin 3x$. ✓

Q2. $\int\cos\left(\frac{x}{2}\right)dx=\frac{1}{1/2}\sin\left(\frac{x}{2}\right)+c=2\sin\left(\frac{x}{2}\right)+c$.

Q3. $\int(\sin x+\cos x)\,dx=-\cos x+\sin x+c$.

Q4. $\int_0^{\pi/2}\sin x\,dx=[-\cos x]_0^{\pi/2}=-\cos\frac{\pi}{2}+\cos 0=0+1=1$.

Q5. $\int_0^{\pi/4}\cos(2x)\,dx=\left[\frac{1}{2}\sin(2x)\right]_0^{\pi/4}=\frac{1}{2}\sin\frac{\pi}{2}-0=\frac{1}{2}$.

Q6. $\int_0^{\pi/2}\cos x\,dx=[\sin x]_0^{\pi/2}=\sin\frac{\pi}{2}-0=1$. Since $\cos x\geq 0$ on $\left[0,\frac{\pi}{2}\right]$, the area is 1 square unit.

Q7. $\int 3\cos(3x)\,dx=3\cdot\frac{1}{3}\sin(3x)+c=\sin(3x)+c$.

Q8. $\int -2\sin(4x)\,dx=-2\cdot\left(-\frac{1}{4}\cos(4x)\right)+c=\frac{1}{2}\cos(4x)+c$.

Q9. $\int(\cos(2x)-\sin(3x))\,dx=\frac{1}{2}\sin(2x)-\left(-\frac{1}{3}\cos(3x)\right)+c=\frac{1}{2}\sin(2x)+\frac{1}{3}\cos(3x)+c$.

Q10. $\int 2\sin\left(2x+\frac{\pi}{6}\right)dx=2\cdot\left(-\frac{1}{2}\right)\cos\left(2x+\frac{\pi}{6}\right)+c=-\cos\left(2x+\frac{\pi}{6}\right)+c$.

Q11. $\int 4\cos(2x)\,dx=4\cdot\frac{1}{2}\sin(2x)+c=2\sin(2x)+c$.

Q12. $\int_0^{\pi}\sin(2x)\,dx=\left[-\frac{1}{2}\cos(2x)\right]_0^{\pi}=-\frac{1}{2}\cos 2\pi+\frac{1}{2}\cos 0=-\frac{1}{2}+\frac{1}{2}=0$. On $\left[0,\frac{\pi}{2}\right]$ the graph is above the axis and on $\left[\frac{\pi}{2},\pi\right]$ it is below by an equal amount, so the signed area is 0 (the actual total area is 1).

Q13. $\int_{\pi/6}^{\pi/2}\cos x\,dx=[\sin x]_{\pi/6}^{\pi/2}=\sin\frac{\pi}{2}-\sin\frac{\pi}{6}=1-\frac{1}{2}=\frac{1}{2}$.

Q14. $\int_0^{\pi/3} \sin x\, dx = [-\cos x]_0^{\pi/3} = -\cos\frac{\pi}{3} + \cos 0 = -\frac{1}{2} + 1 = \frac{1}{2}$.

Q15. $\int_0^{\pi/4} (\sin x + \cos x)\, dx = [-\cos x + \sin x]_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (-1+0) = 1$.

Q16. $\int_0^{\pi/4} \sin(2x)\, dx = \left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/4} = -\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos 0 = 0 + \frac{1}{2} = \frac{1}{2}$. As $\sin(2x) \geq 0$ on $\left[0, \frac{\pi}{4}\right]$, the area is $\frac{1}{2}$ square unit.

Q17. $\int_0^{\pi/6} 2\cos x\, dx = [2\sin x]_0^{\pi/6} = 2\sin\frac{\pi}{6} - 0 = 2 \cdot \frac{1}{2} = 1$. The area is 1 square unit.

Q18. $f(x) = \int 6\cos(3x)\, dx = 6 \cdot \frac{1}{3}\sin(3x) + c = 2\sin(3x) + c$. Then $f(0) = 2\sin 0 + c = c = 2$, so $f(x) = 2\sin(3x) + 2$.

Q19. $f(x) = \int -\sin x\, dx = \cos x + c$. Then $f(0) = \cos 0 + c = 1 + c = 1$, so $c = 0$ and $f(x) = \cos x$.

Q20. $\int_0^{\pi} \sin x\, dx = [-\cos x]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2$, as required. Over $[0, 2\pi]$ the graph encloses one region above the axis on $[0, \pi]$ (area 2) and one equal region below on $[\pi, 2\pi]$ (area 2), so the **total area** is $2 + 2 = 4$ square units.

Q21. $\int_0^{\pi/3} \cos x\, dx = [\sin x]_0^{\pi/3} = \sin\frac{\pi}{3} - 0 = \frac{\sqrt{3}}{2}$. The area is $\frac{\sqrt{3}}{2}$ square units.

Theory

This section brings together every antidifferentiation rule met so far. The key skill is **recognising which rule applies** to a given integrand — and, where the integrand is a product or a quotient, **simplifying first**.

The standard antiderivatives (each includes an arbitrary constant $+c$):

Integrand	Antiderivative
x^n (with $n \neq -1$)	$\dfrac{x^{n+1}}{n+1}+c$
$(ax+b)^n$ (with $n \neq -1$)	$\dfrac{(ax+b)^{n+1}}{a(n+1)}+c$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\log_e\lvert ax+b\rvert+c$
e^{kx}	$\dfrac{1}{k}e^{kx}+c$
$\sin(kx)$	$-\dfrac{1}{k}\cos(kx)+c$
$\cos(kx)$	$\dfrac{1}{k}\sin(kx)+c$

Antidifferentiation is **linear**: $\int(af(x)+bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$.

Simplify before integrating. There is no product or quotient rule for integration. If the integrand is a product, **expand** it; if it is a quotient over a single power, **split** it into separate terms:

$$\frac{x^4-2x}{x^2}=x^2-\frac{2}{x},\quad (x-1)(x+3)=x^2+2x-3.$$

Definite integrals. If F is an antiderivative of f, then $\int_a^b f(x)\,dx=[F(x)]_a^b=F(b)-F(a)$. The constant c cancels, so it is omitted in a definite integral.

A decision guide. (1) Is it a **sum**? Integrate term by term. (2) A **power of** x? Add one to the index and divide. (3) A **power of a linear expression** $(ax+b)^n$? Same, then divide by a as well. (4) $\dfrac{1}{ax+b}$? A **logarithm**. (5) e^{kx}, $\sin(kx)$, $\cos(kx)$? Divide by k. (6) A **product or quotient**? Expand or split first. Always include $+c$ for an indefinite integral.

Worked examples

Example 1 — scaffolded

Find $\int(x^3-4x+5)\,dx$.

Working	Comment
$\int(x^3-4x+5)\,dx$	A sum of powers of x — integrate term by term.
$=\dfrac{x^4}{4}-4\cdot\dfrac{x^2}{2}+5x+c$	Add one to each index and divide by the new index; $\int 5\,dx=5x$.

Working	Comment
$=\frac{x^4}{4}-2x^2+5x+c$	Simplify. Don't forget $+c$.

Example 2 — scaffolded

Find $\int(3x-2)^4\,dx$.

Working	Comment
Integrand is $(ax+b)^n$ with $a=3$, $b=-2$, $n=4$.	Recognise a power of a **linear** expression.
$\int(3x-2)^4\,dx=\frac{(3x-2)^5}{3\times5}+c$	Add one to the index, then divide by the index **and** by $a=3$.
$=\frac{(3x-2)^5}{15}+c$	Simplify.

Example 3 — unscaffolded

Find $\int\left(2e^{-x}+3\cos(3x)\right)dx$.

$\int 2e^{-x}\,dx=2\cdot\frac{1}{-1}e^{-x}=-2e^{-x}$, and $\int 3\cos(3x)\,dx=3\cdot\frac{1}{3}\sin(3x)=\sin(3x)$.

$\therefore\int\left(2e^{-x}+3\cos(3x)\right)dx=-2e^{-x}+\sin(3x)+c$.

Example 4 — unscaffolded

Evaluate $\int_1^4\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$.

Write in index form: $\sqrt{x}=x^{1/2}$, $\frac{1}{\sqrt{x}}=x^{-1/2}$. An antiderivative is $\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}=\frac{2}{3}x^{3/2}+2x^{1/2}$.

$$\int_1^4\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=\left[\frac{2}{3}x^{3/2}+2\sqrt{x}\right]_1^4=\left(\frac{2}{3}\cdot8+2\cdot2\right)-\left(\frac{2}{3}+2\right)=\frac{28}{3}-\frac{8}{3}.$$

$$\therefore\int_1^4\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=\frac{20}{3}.$$

Practice questions

Scaffolded

Q1. Find (a) $\int\left(4x^3-6x^2+2\right)dx$; (b) $\int\left(x^2-\frac{3}{x^2}\right)dx$.

Q2. Find $\int(5x-1)^3\,dx$.

Q3. Find (a) $\int e^{4x}\,dx$; (b) $\int 3e^{-2x}\,dx$.

Q4. Find $\int\left(2\sin(2x)-\cos x\right)dx$.

Q5. Find $\int\frac{1}{3x+2}\,dx$.

Q6. A function has $f'(x)=4x-3$ and $f(2)=5$. Find $f(x)$.

Unscaffolded

Find each of the following. For Q16–Q18, evaluate the definite integral exactly.

Q7. $\int(6x^2+4x-3)\,dx$

Q8. $\int \frac{x^4-2x}{x^2}\,dx$

Q9. $\int(2x+3)^5\,dx$

Q10. $\int\sqrt{4x-3}\,dx$

Q11. $\int \frac{1}{(2x+5)^2}\,dx$

Q12. $\int(e^{2x}-e^{-x})\,dx$

Q13. $\int(3\cos(2x)+4\sin x)\,dx$

Q14. $\int \frac{5}{2-x}\,dx$

Q15. $\int(x-1)(x+3)\,dx$

Q16. $\int_0^2(3x^2-2x)\,dx$

Q17. $\int_1^e \frac{1}{x}\,dx$

Q18. $\int_0^{\pi/2} 2\cos x\,dx$

Q19. $\int(e^x+1)^2\,dx$

Q20. A function has $f'(x)=6x^2-4x+1$ and $f(1)=3$. Find $f(x)$.

Q21. The volume V litres of water in a tank changes at the rate $\frac{dV}{dt}=3t^2-2$ (litres per minute), and $V=5$ when $t=0$. Find V in terms of t.

Answers

Q1. (a) x^4-2x^3+2x+c; (b) $\frac{x^3}{3}+\frac{3}{x}+c$. **Q2.** $\frac{(5x-1)^4}{20}+c$. **Q3.** (a) $\frac{1}{4}e^{4x}+c$; (b) $-\frac{3}{2}e^{-2x}+c$. **Q4.** $-\cos(2x)-\sin x+c$. **Q5.** $\frac{1}{3}\log_e|3x+2|+c$. **Q6.** $f(x)=2x^2-3x+3$. **Q7.** $2x^3+2x^2-3x+c$. **Q8.** $\frac{x^3}{3}-2\log_e|x|+c$. **Q9.** $\frac{(2x+3)^6}{12}+c$. **Q10.** $\frac{(4x-3)^{3/2}}{6}+c$. **Q11.** $-\frac{1}{2(2x+5)}+c$. **Q12.** $\frac{1}{2}e^{2x}+e^{-x}+c$. **Q13.** $\frac{3}{2}\sin(2x)-4\cos x+c$. **Q14.** $-5\log_e|2-x|+c$. **Q15.** $\frac{x^3}{3}+x^2-3x+c$. **Q16.** 4. **Q17.** 1. **Q18.** 2. **Q19.** $\frac{1}{2}e^{2x}+2e^x+x+c$. **Q20.** $f(x)=2x^3-2x^2+x+2$. **Q21.** $V=t^3-2t+5$.

Fully-worked solutions

Q1. (a) $\int(4x^3-6x^2+2)\,dx=4\cdot\frac{x^4}{4}-6\cdot\frac{x^3}{3}+2x+c=x^4-2x^3+2x+c$. (b) $\frac{3}{x^2}=3x^{-2}$, so $\int(x^2-3x^{-2})\,dx=\frac{x^3}{3}-3\cdot\frac{x^{-1}}{-1}+c=\frac{x^3}{3}+\frac{3}{x}+c$.

Q2. $(5x-1)^3$ is $(ax+b)^n$ with $a=5$, $n=3$: $\int(5x-1)^3\,dx=\frac{(5x-1)^4}{5\times4}+c=\frac{(5x-1)^4}{20}+c$.

Q3. (a) $\int e^{4x}\,dx=\frac{1}{4}e^{4x}+c$. (b) $\int 3e^{-2x}\,dx=3\cdot\frac{1}{-2}e^{-2x}+c=-\frac{3}{2}e^{-2x}+c$.

Q4. $\int(2\sin(2x)-\cos x)\,dx=2\cdot\frac{1}{-2}\cos(2x)-\sin x+c=-\cos(2x)-\sin x+c$.

Q5. $\int\frac{1}{3x+2}\,dx=\frac{1}{3}\log_e|3x+2|+c$ (the $\frac{1}{ax+b}$ rule with $a=3$).

Q6. $f(x)=\int(4x-3)\,dx=2x^2-3x+c$. Using $f(2)=5$: $2(4)-3(2)+c=8-6+c=2+c=5$, so $c=3$. $\therefore f(x)=2x^2-3x+3$.

Q7. $\int(6x^2+4x-3)\,dx=6\cdot\frac{x^3}{3}+4\cdot\frac{x^2}{2}-3x+c=2x^3+2x^2-3x+c$.

Q8. Split first: $\frac{x^4-2x}{x^2}=x^2-\frac{2}{x}$. $\int\left(x^2-\frac{2}{x}\right)dx=\frac{x^3}{3}-2\log_e|x|+c$.

Q9. $\int(2x+3)^5\,dx=\frac{(2x+3)^6}{2\times6}+c=\frac{(2x+3)^6}{12}+c$.

Q10. $\sqrt{4x-3}=(4x-3)^{1/2}$. $\int(4x-3)^{1/2}\,dx=\frac{(4x-3)^{3/2}}{4\times\frac{3}{2}}+c=\frac{(4x-3)^{3/2}}{6}+c$.

Q11. $\frac{1}{(2x+5)^2}=(2x+5)^{-2}$. $\int(2x+5)^{-2}\,dx=\frac{(2x+5)^{-1}}{2\times(-1)}+c=-\frac{1}{2(2x+5)}+c$.

Q12. $\int(e^{2x}-e^{-x})\,dx=\frac{1}{2}e^{2x}-\frac{1}{-1}e^{-x}+c=\frac{1}{2}e^{2x}+e^{-x}+c$.

Q13. $\int(3\cos(2x)+4\sin x)\,dx=3\cdot\frac{1}{2}\sin(2x)+4\cdot(-\cos x)+c=\frac{3}{2}\sin(2x)-4\cos x+c$.

Q14. $\int\frac{5}{2-x}\,dx$. Here $ax+b=2-x$ has $a=-1$: $\int\frac{5}{2-x}\,dx=5\cdot\frac{1}{-1}\log_e|2-x|+c=-5\log_e|2-x|+c$.

Q15. Expand first: $(x-1)(x+3)=x^2+2x-3$. $\int(x^2+2x-3)\,dx=\frac{x^3}{3}+x^2-3x+c$.

Q16. $\int_0^2(3x^2-2x)\,dx=[x^3-x^2]_0^2=(8-4)-(0)=4$.

Q17. $\int_1^e\frac{1}{x}\,dx=[\log_e|x|]_1^e=\log_e e-\log_e 1=1-0=1$.

Q18. $\int_0^{\pi/2}2\cos x\,dx=[2\sin x]_0^{\pi/2}=2\sin\frac{\pi}{2}-2\sin 0=2-0=2$.

Q19. Expand first: $(e^x+1)^2=e^{2x}+2e^x+1$. $\int(e^{2x}+2e^x+1)\,dx=\frac{1}{2}e^{2x}+2e^x+x+c$.

Q20. $f(x)=\int(6x^2-4x+1)\,dx=2x^3-2x^2+x+c$. Using $f(1)=3$: $2-2+1+c=1+c=3$, so $c=2$.
$\therefore f(x)=2x^3-2x^2+x+2$.

Q21. $V=\int(3t^2-2)\,dt=t^3-2t+c$. Using $V=5$ when $t=0$: $0-0+c=5$, so $c=5$. $\therefore V=t^3-2t+5$.

Theory

If $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region **between** the curves $y = f(x)$ (the **upper** curve) and $y = g(x)$ (the **lower** curve) from $x = a$ to $x = b$ is

$$A = \int_a^b (f(x) - g(x))\, dx = \int_a^b (\text{upper} - \text{lower})\, dx.$$

Finding the limits. When a region is *enclosed* by two curves, the limits a and b are the x-coordinates of the **points of intersection**, found by solving $f(x) = g(x)$.

Deciding which curve is on top. Between two adjacent intersection points, test a value (or read the graph) to see which curve is larger — that one is the upper curve.

When the curves swap over. If the upper and lower curves **change** at an interior intersection, split the region there and integrate each piece separately, taking the area of each as a positive quantity:

$$A = \int_a^c (\text{upper}_1 - \text{lower}_1)\, dx + \int_c^b (\text{upper}_2 - \text{lower}_2)\, dx.$$

(Equivalently, add the absolute values of the signed integrals on each piece.)

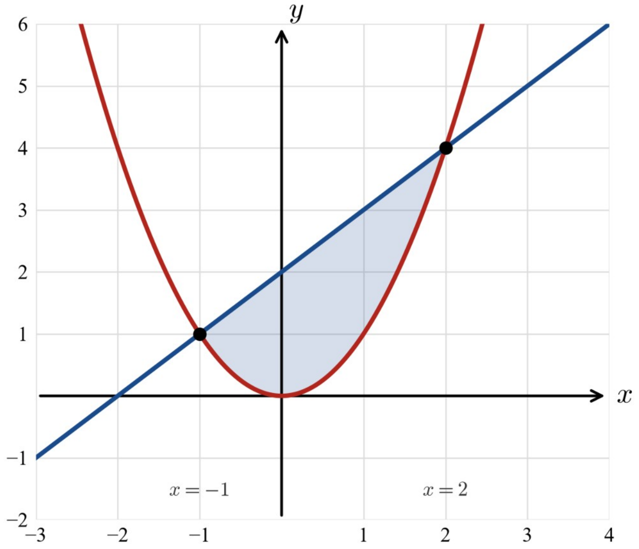
Method. (1) Find the intersection points by solving $f(x) = g(x)$. (2) On each interval decide the upper curve. (3) Integrate $(\text{upper} - \text{lower})$ over each interval. (4) Add the pieces; give an exact value.

Worked examples

Example 1 — scaffolded

Find the area of the region enclosed by the line $y = x + 2$ and the parabola $y = x^2$.

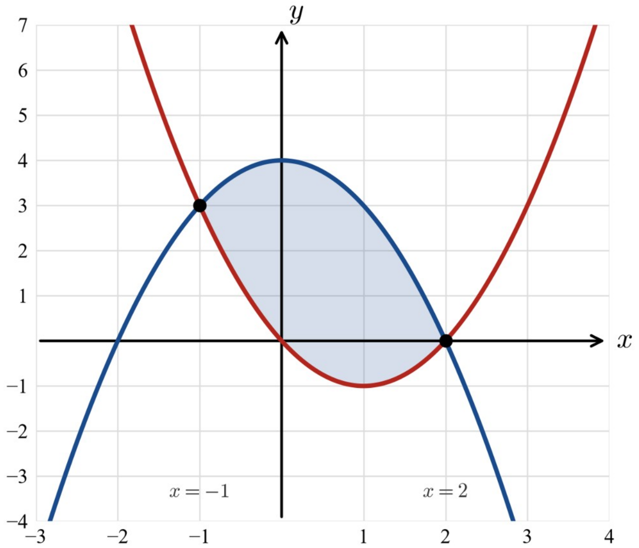
Working	Comment
$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$, so $x = -1$ or $x = 2$.	Intersections give the limits.
On $[-1, 2]$ the line is above the parabola, so upper $= x + 2$, lower $= x^2$.	Test $x = 0$: line $= 2$, parabola $= 0$.
$A = \int_{-1}^{2} ((x+2) - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$	Set up (upper − lower).
$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{9}{2}$	$\therefore$ area $= \frac{9}{2}$.



Example 2 — scaffolded

Find the area enclosed between $y=4-x^2$ and $y=x^2-2x$.

Working	Comment
$4-x^2=x^2-2x \Rightarrow 2x^2-2x-4=0 \Rightarrow x^2-x-2=0$, so $x=-1$ or $x=2$.	Solve for the limits.
On $[-1,2]$, $y=4-x^2$ is the upper curve.	Test $x=0$: $4>0$.
$A=\int_{-1}^{2}\left((4-x^2)-(x^2-2x)\right)dx=\int_{-1}^{2}(4+2x-2x^2)\,dx$	Simplify the integrand.
$=\left[4x+x^2-\frac{2x^3}{3}\right]_{-1}^{2}=\frac{20}{3}-\left(-\frac{7}{3}\right)=9$	$\therefore$ area $=9$.



Example 3 — unscaffolded

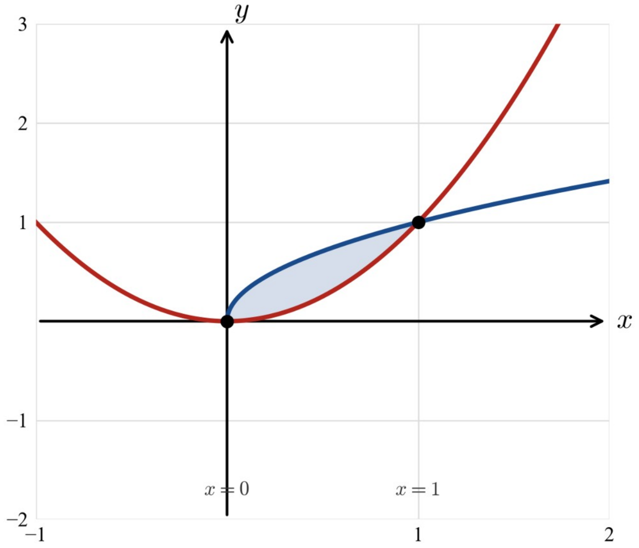
Find the exact area of the region enclosed by $y=\sqrt{x}$ and $y=x^2$.

Intersections: $\sqrt{x}=x^2 \Rightarrow x=x^4 \Rightarrow x(x^3-1)=0$, so $x=0$ or $x=1$.

On $[0,1]$, $\sqrt{x}\geq x^2$ (test $x=1/4$: $1/2>1/16$), so upper $=\sqrt{x}$.

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \left[\frac{2x^{3/2}}{3} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

$\therefore$ the area is $\frac{1}{3}$.



Example 4 — unscaffolded

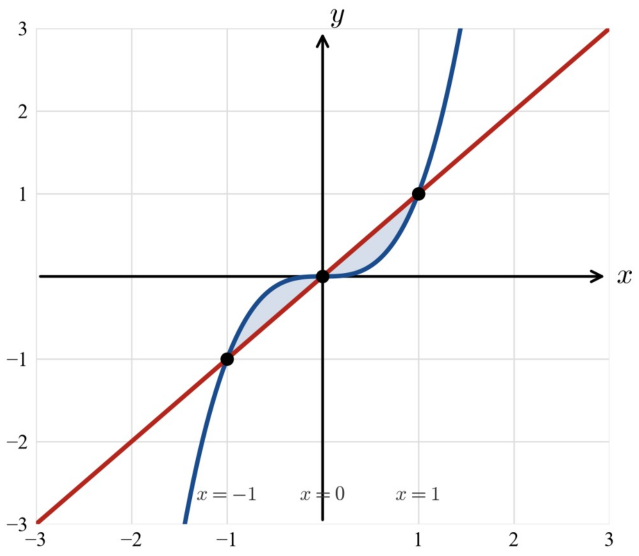
Find the total area of the region enclosed by $y = x^3$ and $y = x$.

Intersections: $x^3 = x \Rightarrow x\left(x^2 - 1\right) = 0$, so $x = -1, 0, 1$. The curves **swap** at $x = 0$, so split there.

On $[-1, 0]$, $x^3 \geq x$ (test $x = -1/2$: $-1/8 > -1/2$); on $[0, 1]$, $x \geq x^3$.

$$A = \int_{-1}^{0} \left(x^3 - x\right) dx + \int_0^1 \left(x - x^3\right) dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$\therefore$ the total area is $\frac{1}{2}$.



Practice questions

Scaffolded

Q1. For the region enclosed by $y=2x$ and $y=x^2$: (a) find the points of intersection; (b) state which curve is the upper curve; (c) find the exact area.

Q2. For the region enclosed by $y=8-x^2$ and $y=x^2$: (a) find the intersections; (b) find the exact area; (c) sketch the region.

Q3. For the region enclosed by $y=\sqrt{x}$ and $y=x$: (a) find the intersections; (b) find the exact area.

Q4. For the region enclosed by $y=4x$ and $y=x^2$: (a) find the intersections; (b) find the exact area.

Q5. For the region enclosed by $y=1-x^2$ and $y=x^2-1$: (a) find the intersections; (b) find the exact area.

Q6. For the region enclosed by $y=x+6$ and $y=x^2$: (a) find the intersections; (b) find the exact area.

Unscaffolded

Find the exact area of the region enclosed between each pair of curves.

Q7. $y=x$ and $y=\frac{x^2}{2}$

Q8. $y=3\sqrt{x}$ and $y=x$

Q9. $y=x^3$ and $y=4x$

Q10. $y=16-x^2$ and $y=x^2-16$

Q11. $y=9-x^2$ and $y=x^2-9$

Q12. $y=x^3$ and $y=9x$

Q13. $y=3-x$ and $y=x^2-2x-1$

Q14. $y=x^2$ and $y=x^3$

Q15. $y=2x+8$ and $y=x^2$

Q16. $y=5-x^2$ and $y=x^2+1$

Q17. $y=12-x^2$ and $y=2x^2$

Q18. $y=\sin x$ and $y=\cos x$ between their intersections at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$

Q19. $y=x^2$ and $y=x^4$

Q20. $y=8-2x^2$ and $y=x^2-1$

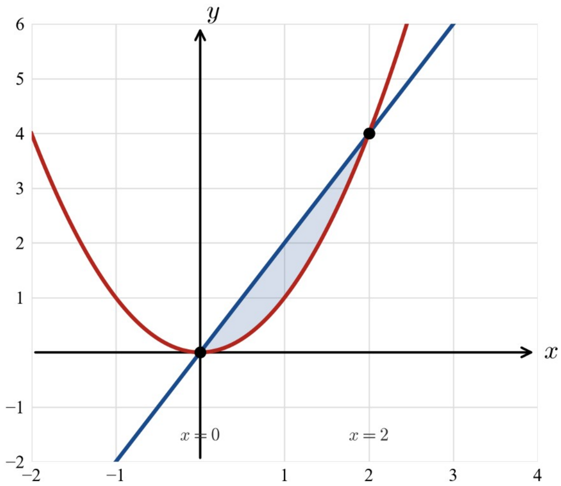
Q21. $y=x^3-4x$ and $y=x$

Answers

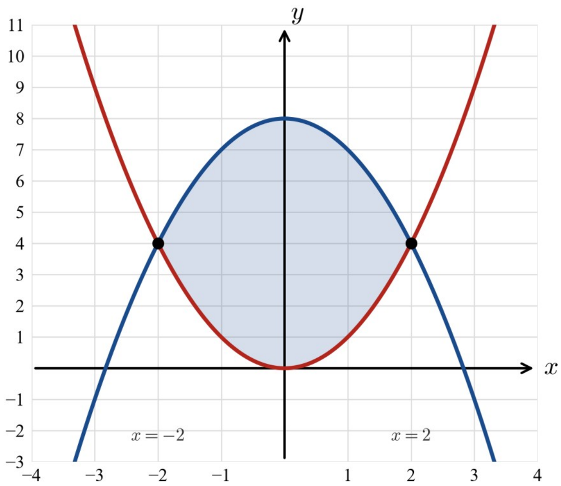
Q1. $(0,0),(2,4)$; upper is $y=2x$; area $\frac{4}{3}$. **Q2.** $x=\pm 2$; area $\frac{64}{3}$. **Q3.** $(0,0),(1,1)$; area $\frac{1}{6}$. **Q4.** $x=0,4$; area $\frac{32}{3}$. **Q5.** $x=\pm 1$; area $\frac{8}{3}$. **Q6.** $x=-2,3$; area $\frac{125}{6}$. **Q7.** $\frac{2}{3}$ **Q8.** $\frac{27}{2}$ **Q9.** 8 **Q10.** $\frac{512}{3}$ **Q11.** 72 **Q12.** $\frac{81}{2}$ **Q13.** $\frac{17\sqrt{17}}{6}$ **Q14.** $\frac{1}{12}$ **Q15.** 36 **Q16.** $\frac{16\sqrt{2}}{3}$ **Q17.** 32 **Q18.** $2\sqrt{2}$ **Q19.** $\frac{4}{15}$ **Q20.** $12\sqrt{3}$ **Q21.** $\frac{25}{2}$

Fully-worked solutions

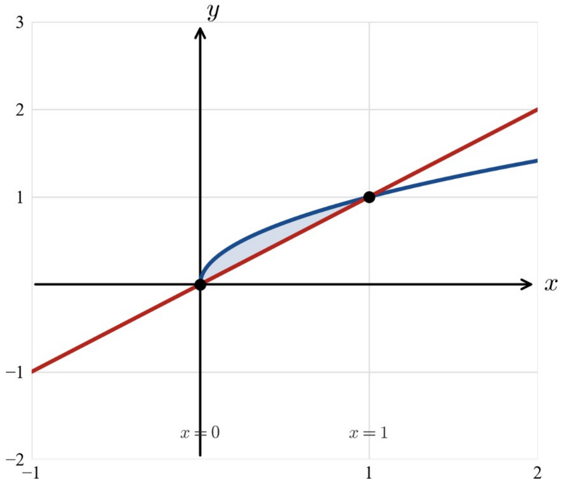
Q1. $2x=x^2 \Rightarrow x^2-2x=0 \Rightarrow x=0,2$. Upper $=2x$. $A=\int_0^2 (2x-x^2)\,dx=\left[x^2-\frac{x^3}{3}\right]_0^2=4-\frac{8}{3}=\frac{4}{3}$.



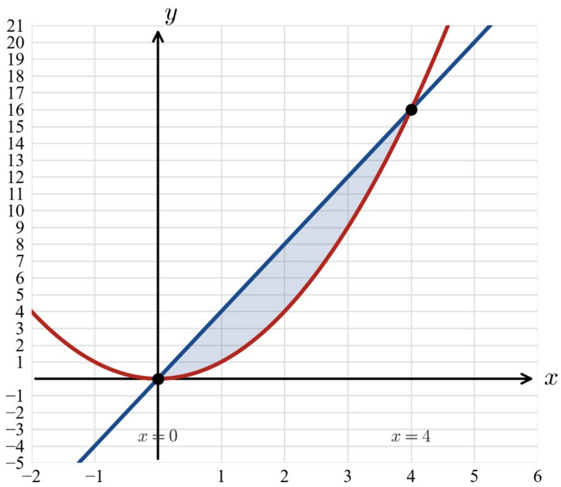
Q2. $8-x^2=x^2 \Rightarrow x^2=4 \Rightarrow x=\pm 2$. Upper $=8-x^2$. $A=\int_{-2}^{2} (8-2x^2)\,dx=\left[8x-\frac{2x^3}{3}\right]_{-2}^{2}=\frac{32}{3}-\left(-\frac{32}{3}\right)=\frac{64}{3}$.



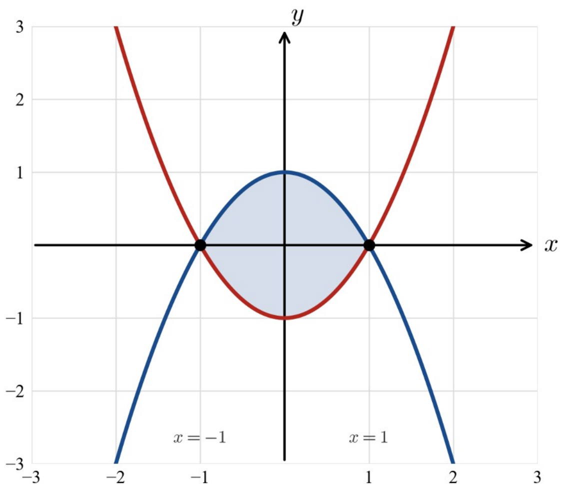
Q3. $\sqrt{x}=x \Rightarrow x=x^2 \Rightarrow x=0,1$. Upper $=\sqrt{x}$. $A=\int_0^1 (\sqrt{x}-x)\,dx=\left[\frac{2x^{3/2}}{3}-\frac{x^2}{2}\right]_0^1=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$.



Q4. $4x = x^2 \Rightarrow x = 0, 4$. Upper $= 4x$. $A = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$.

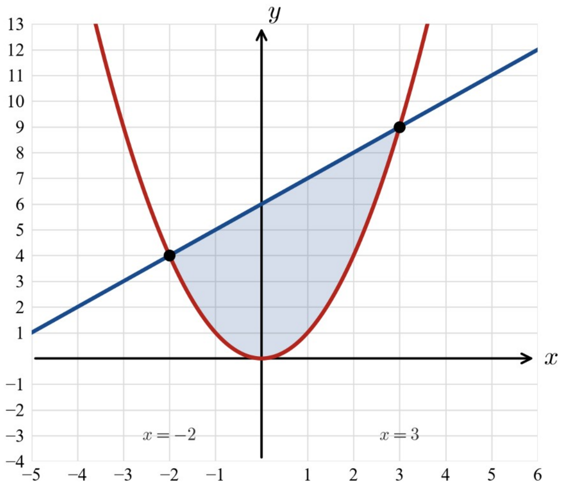


Q5. $1 - x^2 = x^2 - 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$. Upper $= 1 - x^2$. $A = \int_{-1}^{1} (2 - 2x^2)\,dx = \left[2x - \frac{2x^3}{3}\right]_{-1}^{1} = \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}$.

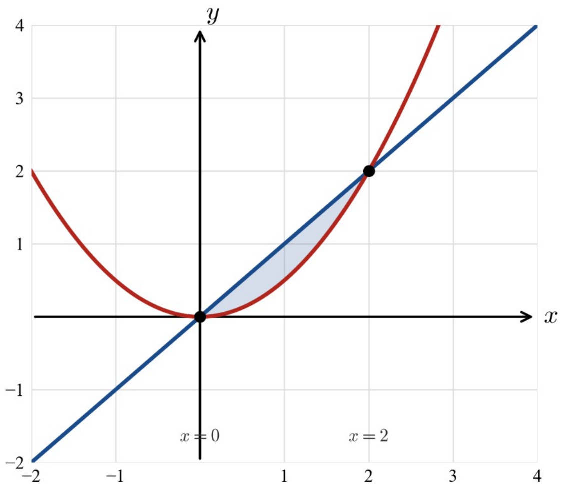


Q6. $x + 6 = x^2 \Rightarrow x^2 - x - 6 = 0 \Rightarrow x = -2, 3$. Upper $= x + 6$.

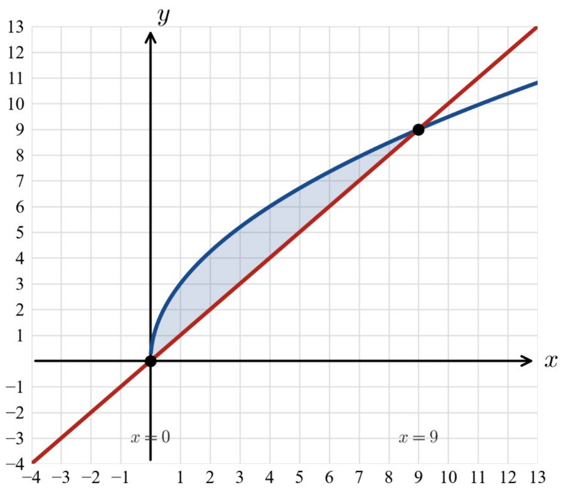
$A = \int_{-2}^{3} (x + 6 - x^2)\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}$.



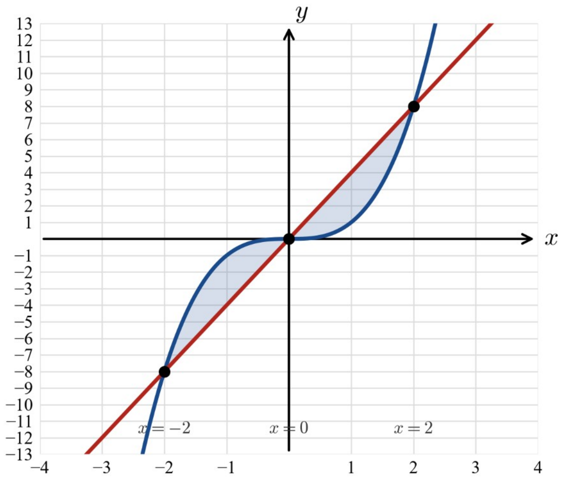
Q7. $x=\dfrac{x^2}{2}\Rightarrow 2x=x^2\Rightarrow x=0,2$. Upper $=x$. $A=\displaystyle\int_0^2\left(x-\frac{x^2}{2}\right)dx=\left[\frac{x^2}{2}-\frac{x^3}{6}\right]_0^2=2-\frac{4}{3}=\frac{2}{3}$.



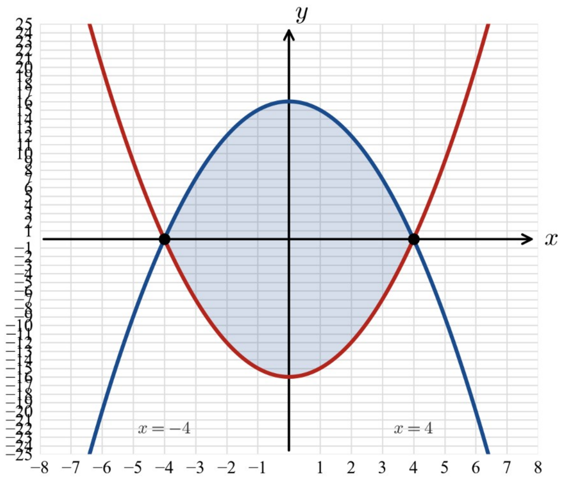
Q8. $3\sqrt{x}=x\Rightarrow 9x=x^2\Rightarrow x=0,9$. Upper $=3\sqrt{x}$. $A=\displaystyle\int_0^9(3\sqrt{x}-x)\,dx=\left[2x^{3/2}-\frac{x^2}{2}\right]_0^9=54-\frac{81}{2}=\frac{27}{2}$.



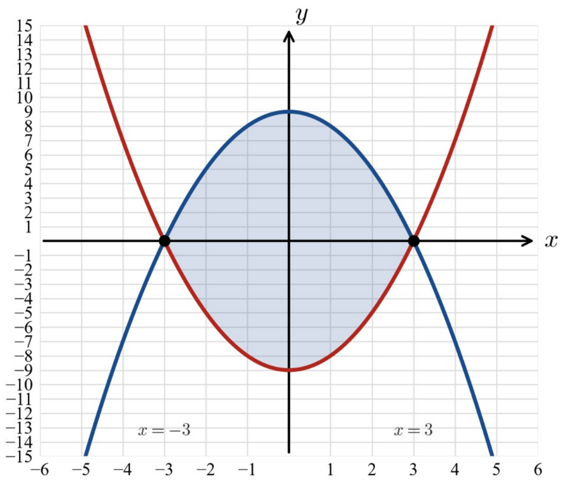
Q9. $x^3=4x\Rightarrow x(x^2-4)=0\Rightarrow x=-2,0,2$ (swap at 0). $A=\displaystyle\int_{-2}^{0}(x^3-4x)\,dx+\int_0^2(4x-x^3)\,dx=4+4=8$.



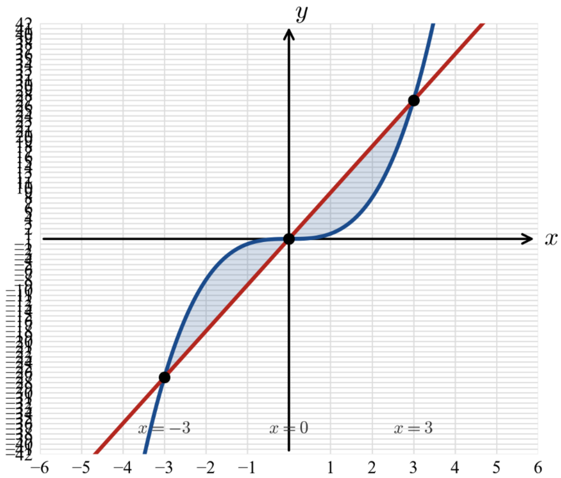
Q10. $16-x^2=x^2-16 \Rightarrow x^2=16 \Rightarrow x=\pm 4$. Upper $=16-x^2$. $A=\int_{-4}^{4}\left(32-2x^2\right)dx=\left[32x-\frac{2x^3}{3}\right]_{-4}^{4}=\frac{512}{3}$.



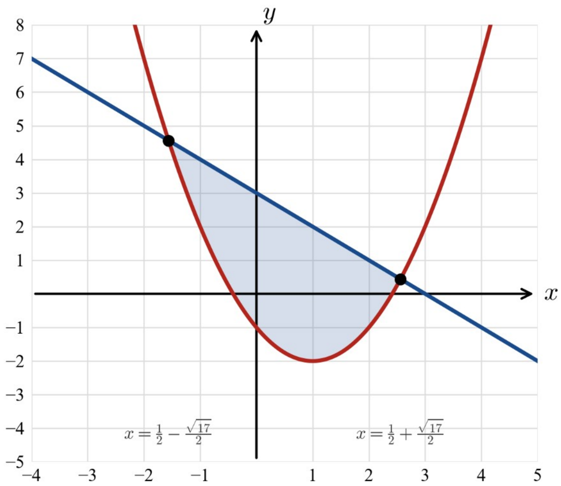
Q11. $9-x^2=x^2-9 \Rightarrow x^2=9 \Rightarrow x=\pm 3$. Upper $=9-x^2$. $A=\int_{-3}^{3}\left(18-2x^2\right)dx=\left[18x-\frac{2x^3}{3}\right]_{-3}^{3}=72$.



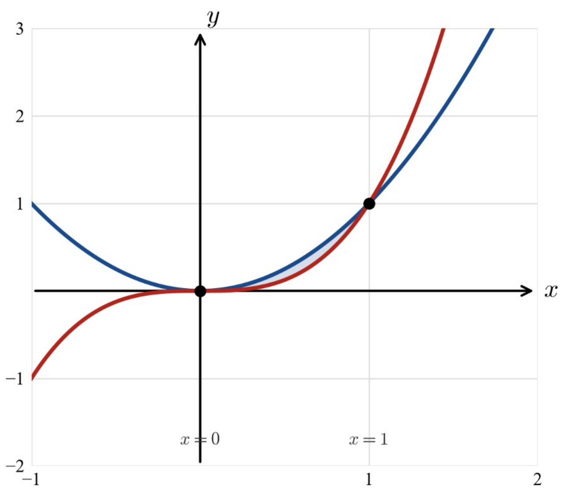
Q12. $x^3=9x \Rightarrow x\left(x^2-9\right)=0 \Rightarrow x=-3,0,3$ (swap at 0). $A=\int_{-3}^{0}\left(x^3-9x\right)dx+\int_{0}^{3}\left(9x-x^3\right)dx=\frac{81}{4}+\frac{81}{4}=\frac{81}{2}$.



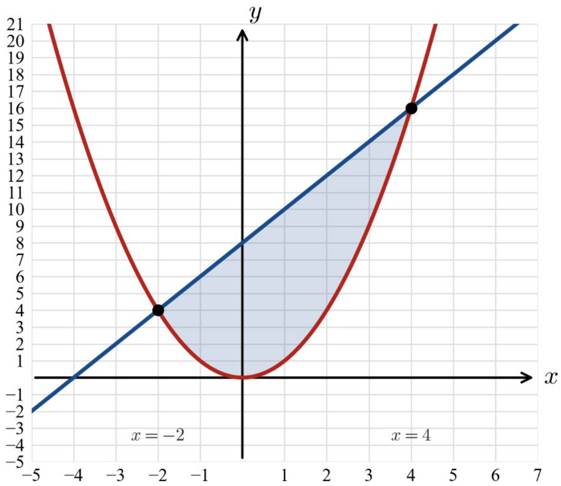
Q13. $3-x=x^2-2x-1 \Rightarrow x^2-x-4=0 \Rightarrow x=\dfrac{1\pm\sqrt{17}}{2}$. Upper $=3-x$. Letting the roots be $\alpha<\beta$ with $\beta-\alpha=\sqrt{17}$,
$A=\displaystyle\int_{\alpha}^{\beta}\big((3-x)-(x^2-2x-1)\big)\,dx=\int_{\alpha}^{\beta}(4+x-x^2)\,dx=\frac{17\sqrt{17}}{6}$.



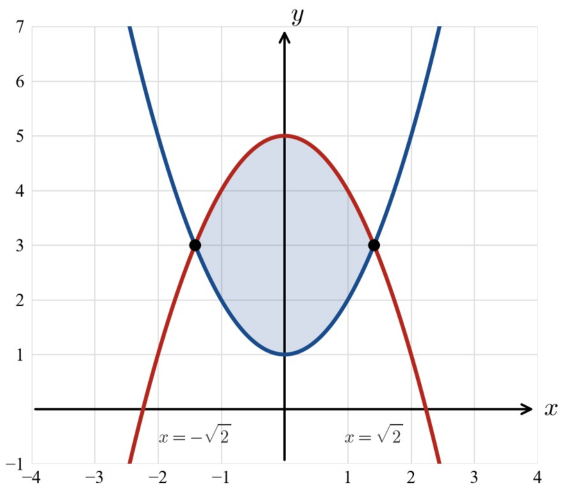
Q14. $x^2=x^3 \Rightarrow x^2(1-x)=0 \Rightarrow x=0,1$. On $[0,1]$, $x^2\ge x^3$, so upper $=x^2$.
$A=\displaystyle\int_0^1(x^2-x^3)\,dx=\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$.



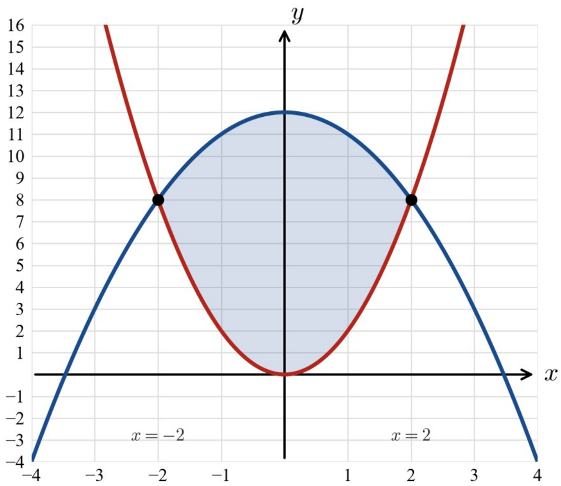
Q15. $2x+8=x^2 \Rightarrow x^2-2x-8=0 \Rightarrow x=-2,4$. Upper $=2x+8$. $A=\displaystyle\int_{-2}^{4}(2x+8-x^2)\,dx=\left[x^2+8x-\frac{x^3}{3}\right]_{-2}^{4}=36$.



Q16. $5-x^2=x^2+1 \Rightarrow 2x^2=4 \Rightarrow x=\pm\sqrt{2}$. Upper $=5-x^2$. $A=\int_{-\sqrt{2}}^{\sqrt{2}}(4-2x^2)\,dx=\left[4x-\dfrac{2x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}=\dfrac{16\sqrt{2}}{3}$.

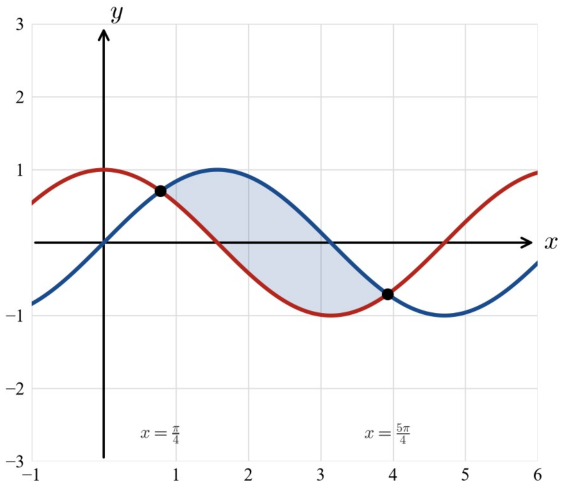


Q17. $12-x^2=2x^2 \Rightarrow 3x^2=12 \Rightarrow x=\pm 2$. Upper $=12-x^2$. $A=\int_{-2}^{2}(12-3x^2)\,dx=\left[12x-x^3\right]_{-2}^{2}=32$.



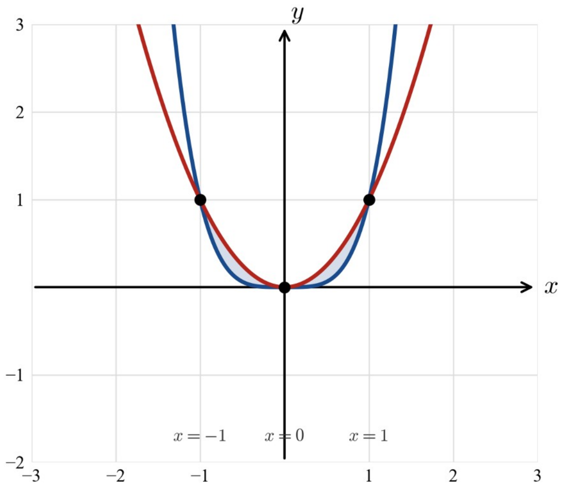
Q18. On $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]$, $\sin x \geq \cos x$, so upper $=\sin x$.

$A=\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)\,dx=\left[-\cos x-\sin x\right]_{\pi/4}^{5\pi/4}=\sqrt{2}-(-\sqrt{2})=2\sqrt{2}$.

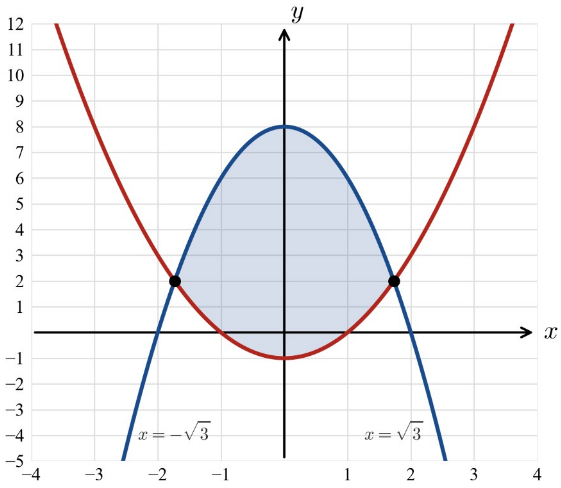


Q19. $x^2=x^4 \Rightarrow x^2(x^2-1)=0 \Rightarrow x=-1,0,1$. On $[-1,1]$, $x^2 \geq x^4$, so upper $=x^2$. By symmetry

$$A=2\int_0^1 (x^2-x^4)\,dx=2\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1=2\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{4}{15}.$$

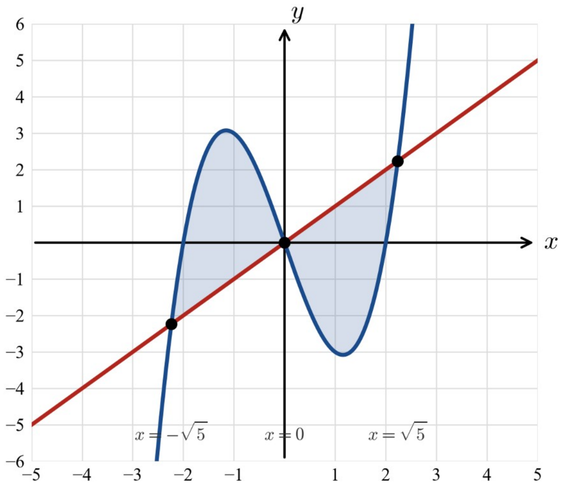


Q20. $8-2x^2=x^2-1 \Rightarrow 3x^2=9 \Rightarrow x=\pm\sqrt{3}$. Upper $=8-2x^2$. $A=\int_{-\sqrt{3}}^{\sqrt{3}} (9-3x^2)\,dx=\left[9x-x^3\right]_{-\sqrt{3}}^{\sqrt{3}}=12\sqrt{3}$.



Q21. $x^3-4x=x \Rightarrow x^3-5x=0 \Rightarrow x(x^2-5)=0 \Rightarrow x=-\sqrt{5},0,\sqrt{5}$ (swap at 0).

$$A=\int_{-\sqrt{5}}^{0} \left((x^3-4x)-x\right)dx+\int_0^{\sqrt{5}} \left(x-(x^3-4x)\right)dx=\frac{25}{4}+\frac{25}{4}=\frac{25}{2}.$$



y
x
$x = -\sqrt{5}$
$x = 0$
$x = \sqrt{5}$

Theory

Integration lets us recover a quantity from its **rate of change** and measure the **average** and **total** effect of a changing quantity.

A function from its rate of change. If $\frac{dy}{dx}=f(x)$ and one value of y is known (a *boundary condition*), then

$$y=\int f(x)\,dx=F(x)+c,$$

and the boundary condition fixes the constant c. Always find c — a missing constant is the most common error here.

Average value. The **average value** of a continuous function f over $[a,b]$ is

$$\acute{f}=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Geometrically, $\acute{f}$ is the height of the rectangle on base $[a,b]$ whose area equals the area under $y=f(x)$ (shown as the dashed line in the diagrams below).

Total change from a rate. If a quantity changes at rate $r(t)$, the **total change** between $t=a$ and $t=b$ is

$$\int_a^b r(t)\,dt.$$

For example, if water flows into a tank at $r(t)$ litres/min, then $\int_a^b r(t)\,dt$ is the total volume added between times a and b.

Kinematics. For motion in a straight line, velocity is the rate of change of position and acceleration is the rate of change of velocity:

$$x(t)=\int v(t)\,dt,\quad v(t)=\int a(t)\,dt.$$

- **Displacement** over $[a,b]$ is $\int_a^b v\,dt$ (a signed quantity).
- **Distance travelled** is $\int_a^b |v|\,dt$; if v changes sign, split the integral at the time when $v=0$ and add the magnitudes.

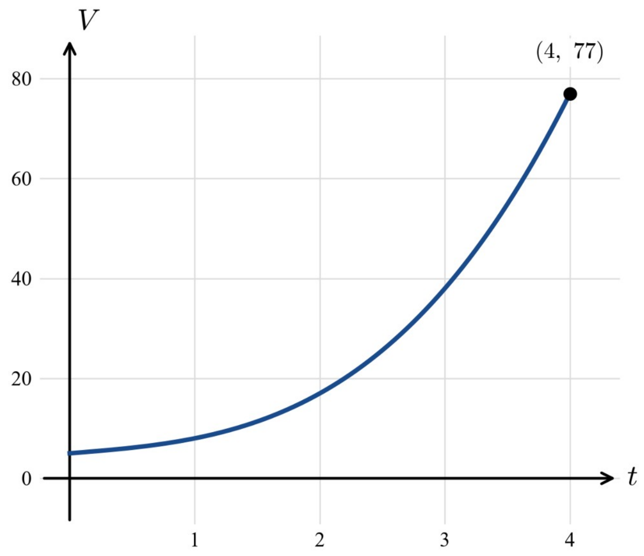
Worked examples

Example 1 — scaffolded

Water flows into a tank at a rate $\frac{dV}{dt}=3t^2+2$ litres per minute. Initially the tank holds 5 litres. Find $V(t)$, and the volume after 4 minutes.

Working	Comment
$V=\int(3t^2+2)\,dt=t^3+2t+c$	Antidifferentiate the rate.
$V(0)=5: 0+0+c=5$, so $c=5$.	Apply the boundary condition to find c.
$\therefore V(t)=t^3+2t+5$	The rule for the volume.

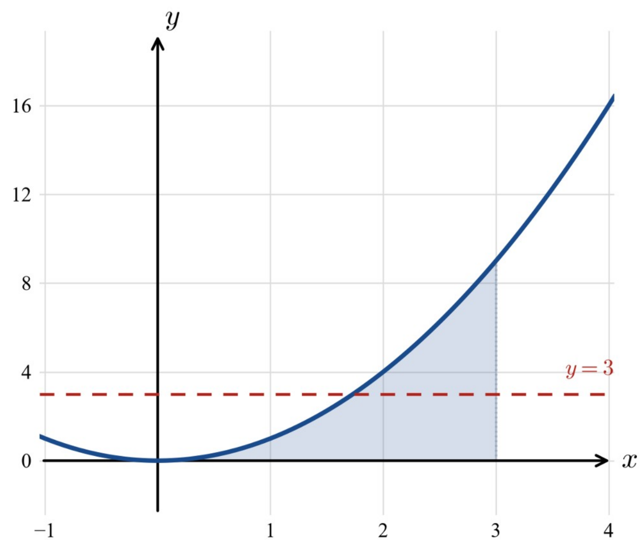
Working	Comment
$V(4)=4^3+2(4)+5=64+8+5=77$ litres.	Substitute $t=4$.



Example 2 — scaffolded

Find the average value of $f(x)=x^2$ over the interval $[0,3]$.

Working	Comment
$\acute{f}=\dfrac{1}{3-0}\displaystyle\int_0^3 x^2\,dx$	Use $\acute{f}=\dfrac{1}{b-a}\displaystyle\int_a^b f\,dx$.
$=\dfrac{1}{3}\left[\dfrac{x^3}{3}\right]_0^3=\dfrac{1}{3}\cdot 9=3$	Evaluate the definite integral, then divide by the width.
$\therefore$ the average value is 3.	The dashed line $y=3$ cuts off the same area as the curve.



Example 3 — unscaffolded

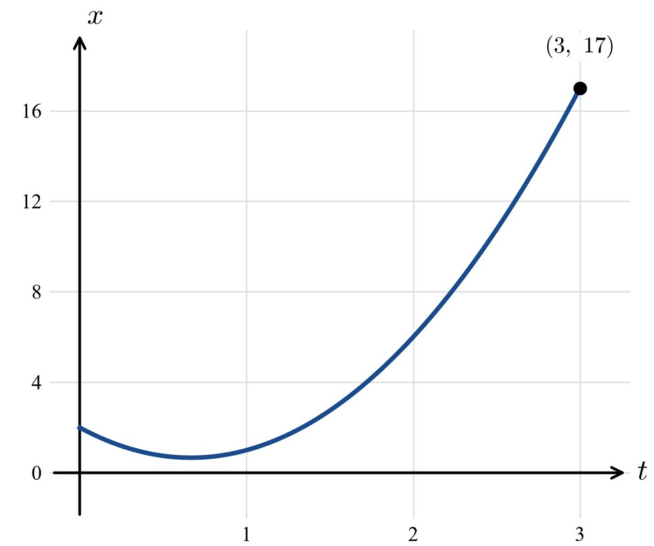
A particle moves in a straight line with velocity $v(t)=6t-4$ (m/s). Its initial position is $x(0)=2$ m. Find $x(t)$, its position after 3 s, and its displacement over the first 3 seconds.

$x(t)=\int(6t-4)\,dt=3t^2-4t+c$. Since $x(0)=2$, $c=2$, so $x(t)=3t^2-4t+2$.

$x(3)=3(9)-4(3)+2=27-12+2=17$ m.

Displacement over $[0,3]$ is $x(3)-x(0)=17-2=15$ m.

$\therefore$ the particle is at 17 m after 3 s, having been displaced 15 m.

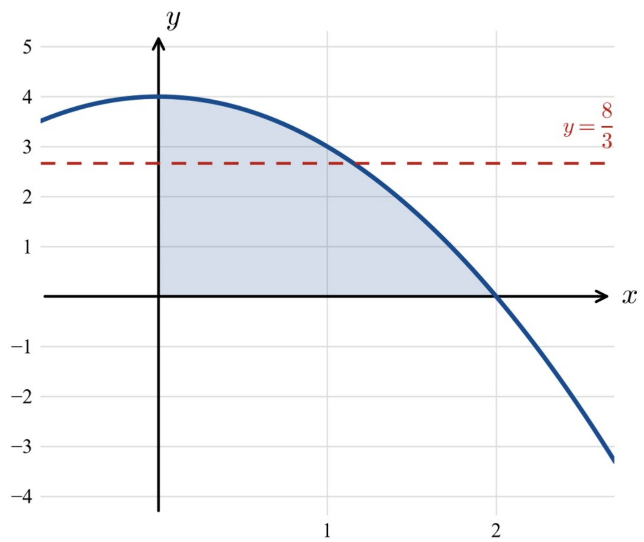


Example 4 — unscaffolded

Find the average value of $f(x)=4-x^2$ over $[0,2]$.

$$\acute{f}=\frac{1}{2-0}\int_0^2 (4-x^2)\,dx=\frac{1}{2}\left[4x-\frac{x^3}{3}\right]_0^2=\frac{1}{2}\left(8-\frac{8}{3}\right)=\frac{1}{2}\cdot\frac{16}{3}=\frac{8}{3}.$$

$\therefore$ the average value is $\frac{8}{3}$.



Practice questions

Scaffolded

Q1. For $f(x)=x^3$ on $[0,2]$: (a) evaluate $\int_0^2 x^3\,dx$; (b) hence find the average value of f over $[0,2]$.

Q2. For $f(x)=\sqrt{x}$ on $[0,4]$: (a) evaluate $\int_0^4 \sqrt{x}\,dx$; (b) hence find the average value of f.

Q3. For $f(x)=e^x$ on $[0,2]$: (a) evaluate $\int_0^2 e^x\,dx$; (b) hence find the average value, in exact form.

Q4. For $f(x)=\sin x$ on $[0,\pi]$: (a) evaluate $\int_0^{\pi} \sin x\,dx$; (b) hence find the average value.

Q5. Water flows into a tank at a rate $r(t)=4t+3$ litres/min. (a) Write an integral for the total volume added in the first 5 minutes; (b) evaluate it.

Q6. A population grows at a rate $r(t)=20e^{t/10}$ per year. (a) Write an integral for the total growth over the first 10 years; (b) evaluate it in exact form.

Unscaffolded

Q7. Find y given $\frac{dy}{dx}=3x^2-4x$ and $y(1)=2$.

Q8. Find P given $\frac{dP}{dt}=4e^{t/2}$ and $P(0)=10$.

Q9. Find y given $\frac{dy}{dx}=6x^2-2$ and $y(2)=20$.

Q10. Find y given $\frac{dy}{dx}=\cos x$ and $y(0)=3$.

Q11. Find y given $\frac{dy}{dx}=\frac{1}{2x+1}$ (for $x>-1/2$) and $y(0)=0$.

Q12. Find the average value of $f(x)=\frac{1}{x}$ over $[1,e]$.

Q13. Find the average value of $f(x)=3x^2+1$ over $[1,3]$.

Q14. Find the average value, in exact form, of $f(x)=e^{2x}$ over $[0,2]$.

Q15. Find the average value of $f(x)=2\sin x$ over $[0,\pi]$.

Q16. A particle has acceleration $a(t)=10-2t$ (m/s²) and initial velocity $v(0)=3$ m/s. Find $v(t)$ and $v(2)$.

Q17. A particle has velocity $v(t)=3t^2-12$ (m/s) and starts at the origin, $x(0)=0$. Find $x(t)$ and its position after 4 s.

Q18. A particle has velocity $v(t)=2t-6$ (m/s) for $t\in[0,5]$. Find its displacement and the total distance travelled over the first 5 seconds.

Q19. A colony of bacteria grows so that $\frac{dP}{dt}=100e^{t/5}$, and $P(0)=500$. Find $P(t)$.

Q20. Find the average value of $f(x)=\frac{6}{x^2}$ over $[2,3]$.

Q21. Water drains from a tank at a rate $r(t)=60-6t$ litres/min for $t\in[0,10]$. Find the total volume drained over these 10 minutes.

Answers

Q1. (a) 4; (b) 2. **Q2.** (a) $\frac{16}{3}$; (b) $\frac{4}{3}$. **Q3.** (a) e^2-1; (b) $\frac{e^2-1}{2}$. **Q4.** (a) 2; (b) $\frac{2}{\pi}$. **Q5.** (a) $\int_0^5 (4t+3)\,dt$; (b) 65 litres.

Q6. (a) $\int_0^{10} 20e^{t/10}\,dt$; (b) $200(e-1)$. **Q7.** $y=x^3-2x^2+3$. **Q8.** $P=8e^{t/2}+2$. **Q9.** $y=2x^3-2x+8$. **Q10.** $y=\sin x+3$. **Q11.** $y=\frac{1}{2}\log_e(2x+1)$. **Q12.** $\frac{1}{e-1}$. **Q13.** 14. **Q14.** $\frac{e^4-1}{4}$. **Q15.** $\frac{4}{\pi}$. **Q16.** $v(t)=-t^2+10t+3$; $v(2)=19$ m/s. **Q17.** $x(t)=t^3-12t$; $x(4)=16$ m. **Q18.** Displacement -5 m; distance 13 m. **Q19.** $P=500e^{t/5}$. **Q20.** 1. **Q21.** 300 litres.

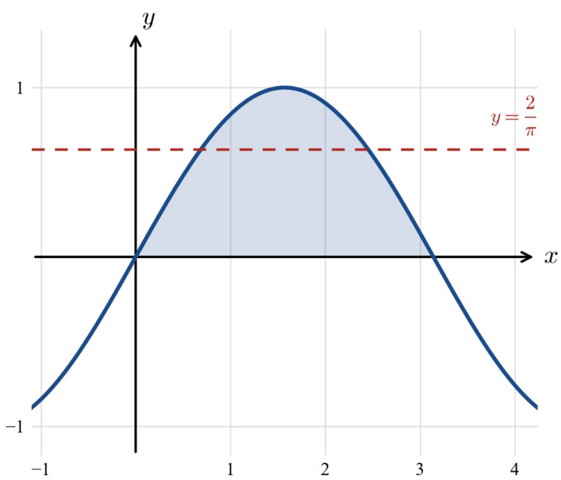
Fully-worked solutions

Q1. $\int_0^2 x^3\,dx=\left[\frac{x^4}{4}\right]_0^2=\frac{16}{4}=4$; average $=\frac{1}{2}\cdot 4=2$.

Q2. $\int_0^4 x^{1/2}\,dx=\left[\frac{2}{3}x^{3/2}\right]_0^4=\frac{2}{3}(8)=\frac{16}{3}$; average $=\frac{1}{4}\cdot\frac{16}{3}=\frac{4}{3}$.

Q3. $\int_0^2 e^x\,dx=\left[e^x\right]_0^2=e^2-1$; average $=\frac{e^2-1}{2}$.

Q4. $\int_0^\pi \sin x\,dx=\left[-\cos x\right]_0^\pi=-(-1)-(-1)=2$; average $=\frac{2}{\pi}$.



Q5. $\int_0^5 (4t+3)\,dt=\left[2t^2+3t\right]_0^5=50+15=65$ litres.

Q6. $\int_0^{10} 20e^{t/10}\,dt=\left[200e^{t/10}\right]_0^{10}=200e-200=200(e-1)$.

Q7. $y=\int(3x^2-4x)\,dx=x^3-2x^2+c$. $y(1)=2: 1-2+c=2\Rightarrow c=3$. $\therefore y=x^3-2x^2+3$.

Q8. $P=\int 4e^{t/2}\,dt=8e^{t/2}+c$. $P(0)=10: 8+c=10\Rightarrow c=2$. $\therefore P=8e^{t/2}+2$.

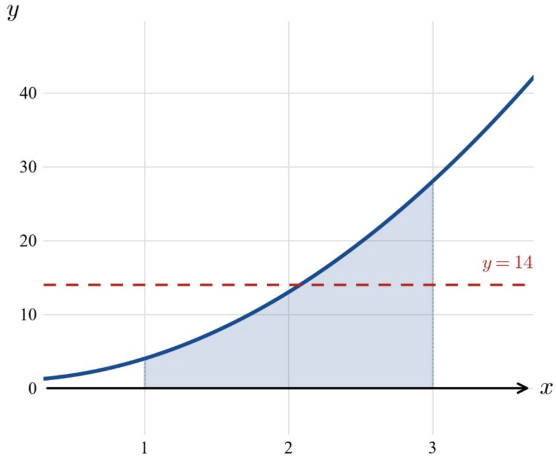
Q9. $y=\int(6x^2-2)\,dx=2x^3-2x+c$. $y(2)=20: 16-4+c=20\Rightarrow c=8$. $\therefore y=2x^3-2x+8$.

Q10. $y=\int\cos x\,dx=\sin x+c$. $y(0)=3: 0+c=3\Rightarrow c=3$. $\therefore y=\sin x+3$.

Q11. $y=\int \frac{1}{2x+1}dx=\frac{1}{2}\log_e(2x+1)+c$. $y(0)=0: \frac{1}{2}\log_e 1+c=0 \Rightarrow c=0$. $\therefore y=\frac{1}{2}\log_e(2x+1)$.

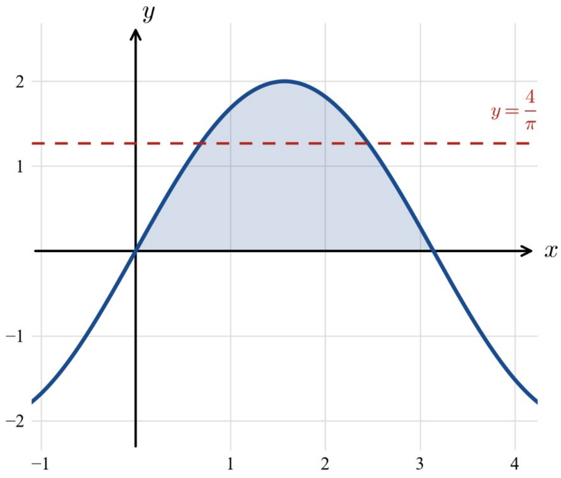
Q12. $\frac{1}{e-1}\int_1^e \frac{1}{x}dx=\frac{1}{e-1}[\log_e x]_1^e=\frac{1}{e-1}(1-0)=\frac{1}{e-1}$.

Q13. $\frac{1}{2}\int_1^3 (3x^2+1)dx=\frac{1}{2}[x^3+x]_1^3=\frac{1}{2}((27+3)-(1+1))=\frac{1}{2}(28)=14$.



Q14. $\frac{1}{2}\int_0^2 e^{2x}dx=\frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_0^2=\frac{1}{4}(e^4-1)=\frac{e^4-1}{4}$.

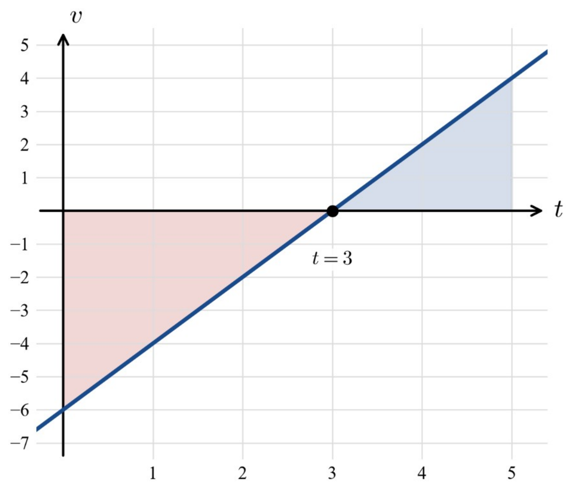
Q15. $\frac{1}{\pi}\int_0^{\pi} 2\sin x\,dx=\frac{1}{\pi}[-2\cos x]_0^{\pi}=\frac{1}{\pi}(2+2)=\frac{4}{\pi}$.



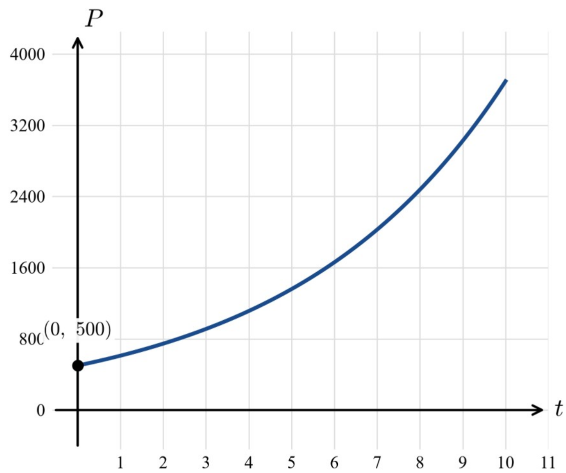
Q16. $v=\int(10-2t)dt=10t-t^2+c$. $v(0)=3 \Rightarrow c=3$, so $v(t)=-t^2+10t+3$. $v(2)=-4+20+3=19$ m/s.

Q17. $x=\int(3t^2-12)dt=t^3-12t+c$. $x(0)=0 \Rightarrow c=0$, so $x(t)=t^3-12t$. $x(4)=64-48=16$ m.

Q18. $v=2t-6=0$ at $t=3$ (velocity is negative on $[0,3)$, positive on $(3,5]$). Displacement $=\int_0^5 (2t-6)dt=[t^2-6t]_0^5=25-30=-5$ m. Distance $=\int_0^3 -(2t-6)dt+\int_3^5 (2t-6)dt=|-9|+4=9+4=13$ m.



Q19. $P=\int 100e^{t/5}dt=500e^{t/5}+c$. $P(0)=500: 500+c=500 \Rightarrow c=0$. $\therefore P=500e^{t/5}$.



Q20. $\frac{1}{1}\int_2^3 6x^{-2}dx=\left[-\frac{6}{x}\right]_2^3=-2-(-3)=1$.

Q21. $\int_0^{10}(60-6t)dt=\left[60t-3t^2\right]_0^{10}=600-300=300$ litres.

Chapter 10 Anti-differentiation and integration — 10K The fundamental theorem of calculus

Theory

Section 10E used the fundamental theorem of calculus to *evaluate* a definite integral as $\int_a^b f(x)\,dx = F(b) - F(a)$.

This section looks at the theorem from the other direction — how a definite integral with a **variable** upper limit defines a function whose derivative is the original integrand.

The signed-area function. For a continuous function f and a fixed number a, define

$$A(x) = \int_a^x f(t)\,dt.$$

For each value of x this is the signed area between the graph of f and the t-axis from $t = a$ to $t = x$. Note that $A(a) = \int_a^a f(t)\,dt = 0$. (A dummy variable t is used inside the integral so it is not confused with the limit x.)

The fundamental theorem of calculus (part 1). If f is continuous, then A is differentiable and

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

In words: *differentiating the integral of f gives back f*. This is because a small increase h in x adds a thin strip of area $\approx f(x)h$, so $A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$. The lower limit a only shifts A by a constant, so it does **not** affect the derivative.

Variable upper limit — the chain rule. If the upper limit is itself a function $g(x)$, then, writing $A(u) = \int_a^u f(t)\,dt$ with $u = g(x)$,

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\,g'(x).$$

Variable lower limit. Reversing the limits changes the sign, so

$$\frac{d}{dx}\int_{g(x)}^{b} f(t)\,dt = -f(g(x))\,g'(x).$$

Connection with antiderivatives. Because $A'(x) = f(x)$, the area function A is an antiderivative of f — the particular one with $A(a) = 0$. If F is any antiderivative of f then $A(x) = F(x) - F(a)$, which is exactly the rule used to evaluate definite integrals.

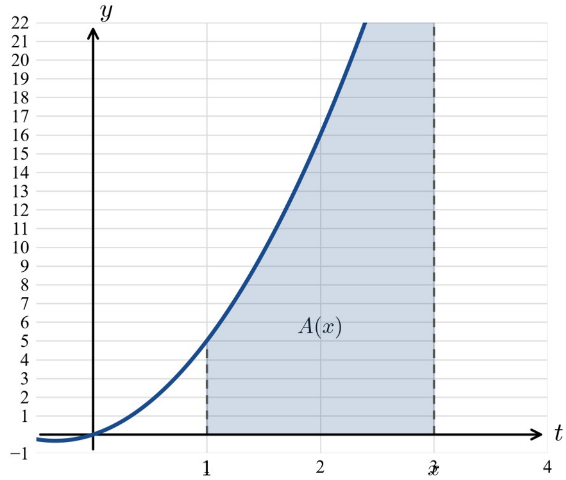
To differentiate an integral with a variable limit: (1) identify the integrand f and the variable limit; (2) substitute the limit into f; (3) multiply by the derivative of that limit (and attach a minus sign if it is the lower limit).

Worked examples

Example 1 — scaffolded

Let $A(x)=\int_1^x (3t^2+2t)\,dt$. Find $A(x)$ by integrating, verify that $A'(x)=3x^2+2x$, and state $A(1)$.

Working	Comment
$A(x)=\left[t^3+t^2\right]_1^x=(x^3+x^2)-(1+1)=x^3+x^2-2$	Antidifferentiate, then substitute the limits.
$A'(x)=3x^2+2x$	Differentiate A; this is the integrand with t replaced by x.
$A(1)=1+1-2=0$	As expected, $A(a)=\int_a^a f=0$.
$\therefore \frac{d}{dx}\int_1^x (3t^2+2t)\,dt=3x^2+2x=f(x)$	The derivative of the integral returns the integrand.



Example 2 — scaffolded

Find $\frac{d}{dx}\int_0^{x^2} \cos t\,dt$.

Working	Comment
The upper limit is $g(x)=x^2$, so $g'(x)=2x$.	The limit is a function of x — use the chain-rule form.
$\frac{d}{dx}\int_0^{x^2} \cos t\,dt=\cos(x^2)\cdot 2x$	Substitute the limit into the integrand, then multiply by $g'(x)$.
$=2x\cos(x^2)$	Tidy up.

Example 3 — unscaffolded

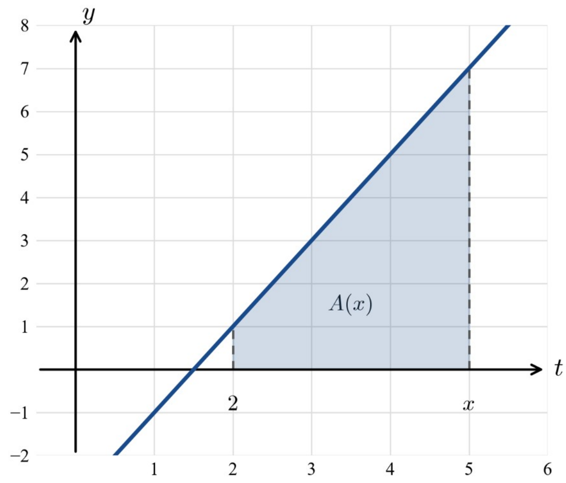
The area function $A(x)=\int_2^x (2t-3)\,dt$. Find $A(x)$, the value of $A(5)$, and $A'(x)$.

$A(x)=\left[t^2-3t\right]_2^x=(x^2-3x)-(4-6)=x^2-3x+2$.

$A(5)=25-15+2=12$.

$A'(x)=2x-3$, which is the integrand with t replaced by x.

$\therefore A(5)=12$ and $A'(x)=2x-3$.



Example 4 — unscaffolded

Find $\frac{d}{dx}\int_{x}^{4}(t^2+1)\,dt$.

Here x is the **lower** limit, so the derivative carries a minus sign:

$$\frac{d}{dx}\int_{x}^{4}(t^2+1)\,dt=-(x^2+1).$$

$\therefore \frac{d}{dx}\int_{x}^{4}(t^2+1)\,dt=-x^2-1.$

Practice questions

Scaffolded

Q1. For $A(x)=\int_{0}^{x}(t^2+1)\,dt$: (a) find $A(x)$ by integrating; (b) hence find $A'(x)$; (c) state $\frac{d}{dx}\int_{0}^{x}(t^2+1)\,dt$ directly.

Q2. For $A(x)=\int_{1}^{x}e^{2t}\,dt$: (a) find $A(x)$; (b) find $A'(x)$; (c) verify that $A'(x)$ equals the integrand with $t=x$.

Q3. For $A(x)=\int_{0}^{x}\sin t\,dt$: (a) find $A(x)$; (b) hence find $A'(x)$; (c) state $A(0)$.

Q4. Find $\frac{d}{dx}\int_{0}^{x^2}3t\,dt$: (a) name the upper limit $g(x)$ and its derivative; (b) apply the chain-rule form; (c) simplify.

Q5. For $A(x)=\int_{1}^{x}4t^3\,dt$: (a) find $A(x)$; (b) find $A(2)$; (c) find $A'(x)$.

Q6. Find $\frac{d}{dx}\int_{x}^{2}(t+1)\,dt$: (a) explain why the answer is negative; (b) write down the derivative.

Unscaffolded

Find the derivative with respect to x (Q7–Q12, Q14–Q17, Q19, Q21), or the stated value (Q13, Q18, Q20).

Q7. $\frac{d}{dx}\int_{2}^{x}(t^2-4t)\,dt$

Q8. $\frac{d}{dx}\int_{0}^{x}\cos 2t\,dt$

Q9. $\frac{d}{dx}\int_{0}^{x^2}e^t\,dt$

Q10. $\frac{d}{dx}\int_{1}^{x^3}2t\,dt$

Q11. $\frac{d}{dx}\int_{0}^{2x}\sin t\,dt$

Q12. $\frac{d}{dx}\int_{x}^{5}t^2\,dt$

Q13. For $A(x)=\int_{0}^{x}6t^2\,dt$, find $A(x)$, $A(2)$ and $A'(x)$.

Q14. $\frac{d}{dx}\int_{1}^{x}(3t^2-2t)\,dt$

Q15. $\frac{d}{dx}\int_{0}^{x}e^{-t}\,dt$

Q16. $\frac{d}{dx}\int_{1}^{\sqrt{x}}t\,dt$

Q17. $\frac{d}{dx}\int_{2}^{x^2}t^3\,dt$

Q18. For $A(x)=\int_{0}^{x}(t^2+2t)\,dt$, find $A(x)$, $A(3)$ and $A'(x)$.

Q19. $\frac{d}{dx}\int_{x}^{2x}t\,dt$ (both limits depend on x — differentiate $A(x)$ after integrating).

Q20. Let $A(x)=\int_{2}^{x}(3t^2-4)\,dt$. Find $A'(x)$ and $A(2)$, and hence find the value of $x>0$ for which $A'(x)=8$.

Q21. $\frac{d}{dx}\int_{0}^{x^2}\sqrt{t+1}\,dt$

Answers

Q1. (a) $A(x)=\frac{x^3}{3}+x$; (b) x^2+1; (c) x^2+1. **Q2.** (a) $\frac{e^{2x}}{2}-\frac{e^2}{2}$; (b) e^{2x}; (c) verified. **Q3.** (a) $1-\cos x$; (b) $\sin x$; (c) 0. **Q4.** (a) $g(x)=x^2$, $g'(x)=2x$; (b) $3x^2\cdot 2x$; (c) $6x^3$. **Q5.** (a) x^4-1; (b) 15; (c) $4x^3$. **Q6.** (a) x is the lower limit; (b) $-(x+1)=-x-1$. **Q7.** x^2-4x. **Q8.** $\cos 2x$. **Q9.** $2xe^{x^2}$. **Q10.** $6x^5$. **Q11.** $2\sin 2x$. **Q12.** $-x^2$. **Q13.** $A(x)=2x^3$, $A(2)=16$, $A'(x)=6x^2$. **Q14.** $3x^2-2x$. **Q15.** e^{-x}. **Q16.** $\frac{1}{2}$. **Q17.** $2x^7$. **Q18.** $A(x)=\frac{x^3}{3}+x^2$, $A(3)=18$, $A'(x)=x^2+2x$. **Q19.** $3x$. **Q20.** $A'(x)=3x^2-4$, $A(2)=0$, $x=2$. **Q21.** $2x\sqrt{x^2+1}$.

Fully-worked solutions

Q1. (a) $A(x)=\left[\frac{t^3}{3}+t\right]_0^x=\frac{x^3}{3}+x$. (b) $A'(x)=x^2+1$. (c) By the FTC, the derivative of the integral is the integrand with $t=x$: x^2+1.

Q2. (a) $A(x)=\left[\frac{e^{2t}}{2}\right]_1^x=\frac{e^{2x}}{2}-\frac{e^2}{2}$. (b) $A'(x)=e^{2x}$. (c) This is e^{2t} with $t=x$, as the FTC predicts.

Q3. (a) $A(x)=[-\cos t]_0^x=-\cos x-(-1)=1-\cos x$. (b) $A'(x)=\sin x$. (c) $A(0)=1-\cos 0=0$.

Q4. (a) $g(x)=x^2$, $g'(x)=2x$. (b) $\frac{d}{dx}\int_0^{x^2} 3t\,dt=3(x^2)\cdot 2x$. (c) $=6x^3$.

Q5. (a) $A(x)=[t^4]_1^x=x^4-1$. (b) $A(2)=16-1=15$. (c) $A'(x)=4x^3$.

Q6. (a) x is the lower limit, and $\frac{d}{dx}\int_x^b f(t)\,dt=-f(x)$. (b) $-(x+1)=-x-1$.

Q7. Constant lower limit, so the derivative is the integrand with $t=x$: x^2-4x.

Q8. $\cos 2x$.

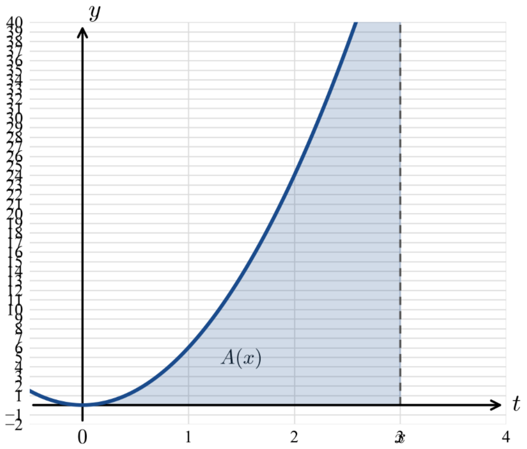
Q9. Chain rule with $g(x)=x^2$: $e^{x^2}\cdot 2x=2xe^{x^2}$.

Q10. $g(x)=x^3$, $g'(x)=3x^2$: $2(x^3)\cdot 3x^2=6x^5$.

Q11. $g(x)=2x$, $g'(x)=2$: $\sin(2x)\cdot 2=2\sin 2x$.

Q12. x is the lower limit: $-x^2$.

Q13. $A(x)=[2t^3]_0^x=2x^3$; $A(2)=2(8)=16$; $A'(x)=6x^2$.



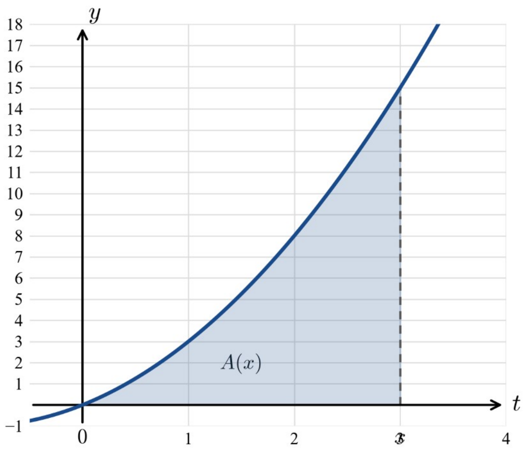
Q14. Constant lower limit: $3x^2 - 2x$.

Q15. e^{-x}.

Q16. $g(x) = \sqrt{x}$, $g'(x) = \dfrac{1}{2\sqrt{x}}$: $\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}$.

Q17. $g(x) = x^2$, $g'(x) = 2x$: $(x^2)^3 \cdot 2x = x^6 \cdot 2x = 2x^7$.

Q18. $A(x) = \left[\dfrac{t^3}{3} + t^2\right]_0^x = \dfrac{x^3}{3} + x^2$; $A(3) = 9 + 9 = 18$; $A'(x) = x^2 + 2x$.



Q19. Integrate first: $A(x) = \left[\dfrac{t^2}{2}\right]_x^{2x} = \dfrac{(2x)^2}{2} - \dfrac{x^2}{2} = 2x^2 - \dfrac{x^2}{2} = \dfrac{3x^2}{2}$, so $A'(x) = 3x$. (Check by the FTC: $f(2x) \cdot 2 - f(x) \cdot 1 = 2x \cdot 2 - x = 3x$.)

Q20. $A'(x) = 3x^2 - 4$ (constant lower limit). $A(2) = \int_2^2 (3t^2 - 4)\,dt = 0$. Setting $A'(x) = 8$: $3x^2 - 4 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$; the positive value is $x = 2$.

Q21. $g(x) = x^2$, $g'(x) = 2x$: $\sqrt{x^2+1} \cdot 2x = 2x\sqrt{x^2+1}$.

Chapter 10 Anti-differentiation and integration — 10L Sketching the graph of an antiderivative

Theory

Section 10A introduced antidifferentiation as the reverse of differentiation: an **antiderivative** F of a function f is any function with $F'(x)=f(x)$. Here we work *graphically* — given only the **graph of** f, deduce the shape of the graph of an antiderivative F. This is the reverse of the work in Section 9D, where the graph of f' was read off from the graph of f. The relationship is the same one, read in the opposite direction: f **is the gradient function of** F**.**

Slope of F comes from the height of f. Because $F'(x)=f(x)$, the *value* (height) of f at each x is the *gradient* of F at that x. So, reading the graph of f:

Where the graph of f …	… the graph of F is
is **above** the axis ($f(x)>0$)	**increasing** (positive gradient)
is **below** the axis ($f(x)<0$)	**decreasing** (negative gradient)
crosses/touches the axis ($f(x)=0$)	**stationary** (zero gradient)

The x-intercepts of f locate the stationary points of F. At each x-intercept of f, decide the nature of the stationary point of F from the way f changes sign there (the same first-derivative sign test used in Chapter 9):

- f changes $- \to +$ (below to above): F has a **local minimum**;
- f changes $+ \to -$ (above to below): F has a **local maximum**;
- f **touches** the axis without changing sign: F has a **stationary point of inflection**.

Concavity of F comes from whether f is rising or falling. Since $F''(x)=f'(x)$, the graph of F is **concave up** where f is increasing and **concave down** where f is decreasing. A **turning point of** f is therefore a **point of inflection of** F (where the concavity changes).

Degree goes up by one. Each antidifferentiation raises the degree by 1: if f is a straight line, F is a parabola; if f is a parabola, F is a cubic; and so on. (Contrast Section 9D, where differentiating *lowered* the degree.)

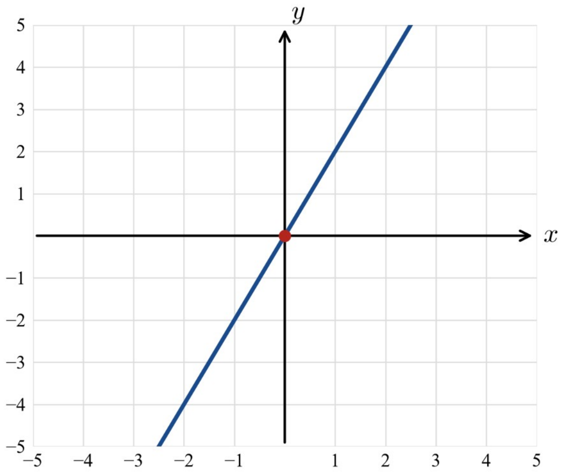
The $+c$ family. Antidifferentiation is only determined up to a constant: if F is one antiderivative, so is $F+c$ for any constant c. Graphically, **every vertical translation of F is also a valid antiderivative** — they all have the same shape, stationary x-coordinates and concavity, just shifted up or down. A single known point (for example a stated value $F(0)$) fixes which member of the family is required.

To sketch an antiderivative F from the graph of f: (1) mark the x-intercepts of f — these are the x-coordinates of the stationary points of F; (2) use the sign of f between them to decide where F increases or decreases, and hence the nature of each stationary point; (3) note the turning points of f as points of inflection of F; (4) draw a smooth curve of degree one higher, then translate it vertically through any given point.

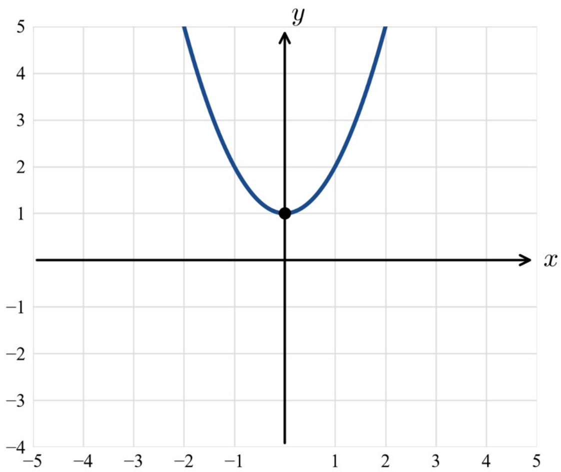
Worked examples

Example 1 — scaffolded

The graph of f is shown. Sketch a possible antiderivative F (with $F'=f$) that passes through $(0,1)$.

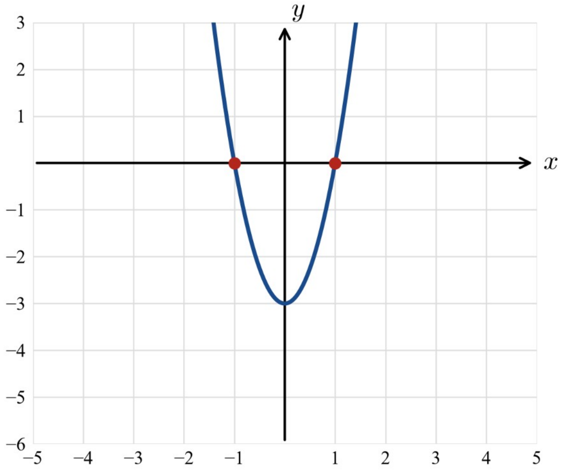


Working	Comment
$f < 0$ for $x < 0$ and $f > 0$ for $x > 0$	Read the sign of f from the graph (below/above the axis).
So F is decreasing for $x < 0$ and increasing for $x > 0$	Height of f = gradient of F.
$f = 0$ at $x = 0$, changing $- \to +$	The single x-intercept of f locates a stationary point of F…
$\therefore$ F has a **local minimum** at $x = 0$	…and the sign change $- \to +$ makes it a minimum.
f is a line (degree 1), so F is a parabola (degree 2); through $(0, 1)$	Degree rises by one; the point fixes the $+c$.

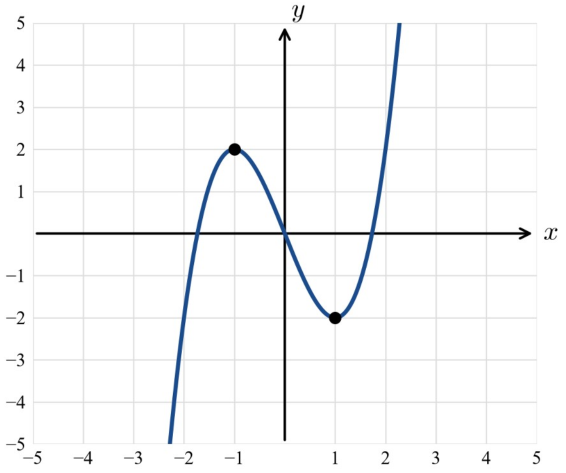


Example 2 — scaffolded

The graph of f is shown. Sketch a possible antiderivative F, showing the nature of its stationary points.

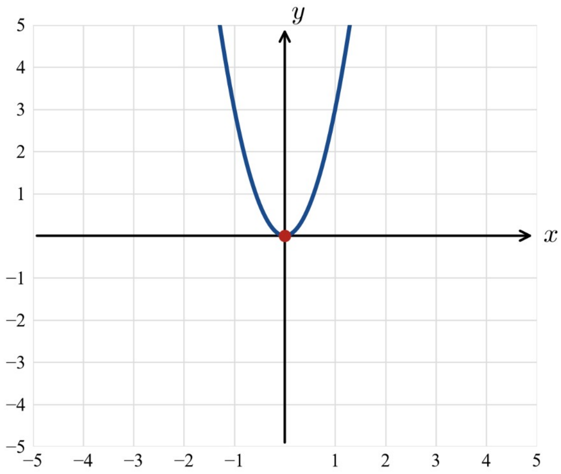


Working	Comment
x-intercepts of f at $x = -1$ and $x = 1$	These give the x-coordinates of F's stationary points.
$f > 0$ for $x < -1$, $f < 0$ for $-1 < x < 1$, $f > 0$ for $x > 1$	Sign of f between the intercepts.
So F increases, then decreases, then increases	Gradient of F follows the sign of f.
At $x = -1$: $+ \to - \Rightarrow$ **local maximum**; at $x = 1$: $- \to + \Rightarrow$ **local minimum**	First-derivative sign test.
f has a minimum at $x = 0 \Rightarrow F$ has a **point of inflection** at $x = 0$	Turning point of f = inflection of F.
$\therefore$ F is a cubic with a max at $x = -1$ and a min at $x = 1$	Degree rises from 2 to 3.

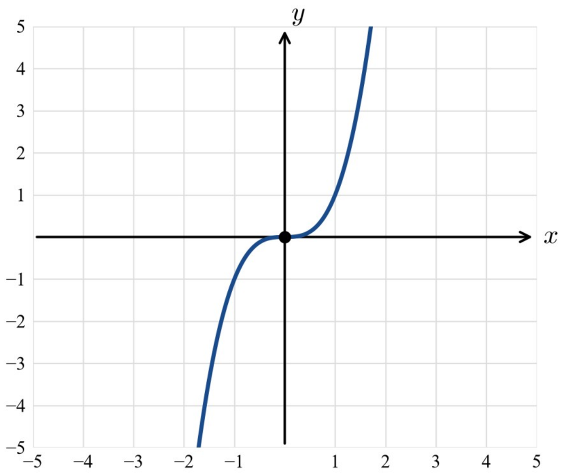


Example 3 — unscaffolded

The graph of f is shown. Sketch a possible antiderivative F through the origin.

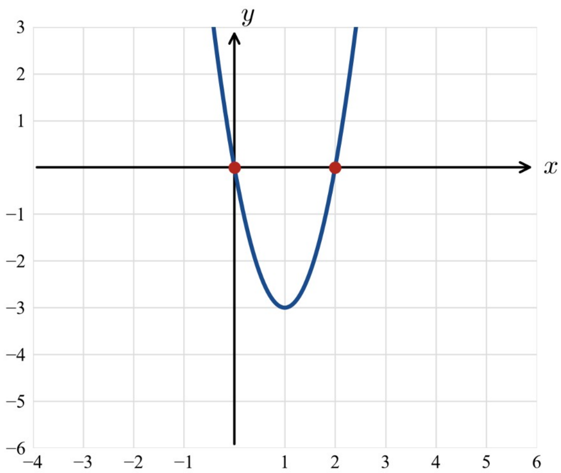


$f(x) \geq 0$ everywhere, so F is **increasing** for all x. The graph of f *touches* the axis at $x=0$ without changing sign, so F has a **stationary point of inflection** at $x=0$ (not a maximum or minimum). As f is a parabola, F is a cubic. $\therefore$ a possible F is an increasing cubic with a stationary point of inflection at the origin.

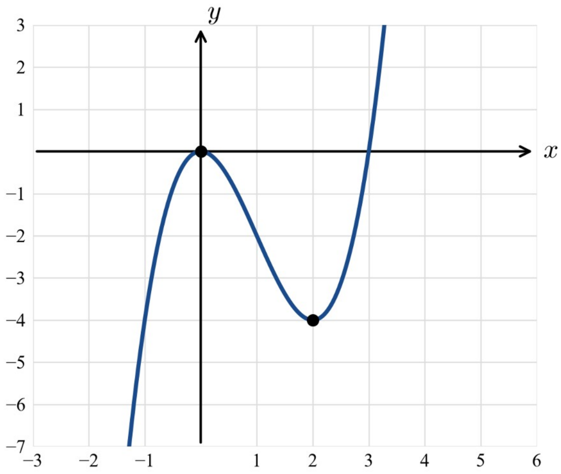


Example 4 — unscaffolded

The graph of f is shown. Sketch a possible antiderivative F through the origin, showing all key features.



The x-intercepts of f are at $x=0$ and $x=2$, so F is stationary there. From the graph, $f>0$ for $x<0$, $f<0$ for $0<x<2$ and $f>0$ for $x>2$; hence F increases, decreases, then increases. At $x=0$ the sign change is $+ \to -$ (**local maximum**) and at $x=2$ it is $- \to +$ (**local minimum**). The turning point of f at $x=1$ gives a **point of inflection** of F. $\therefore$ F is a cubic with a maximum at $(0,0)$, a minimum at $x=2$ and an inflection at $x=1$.

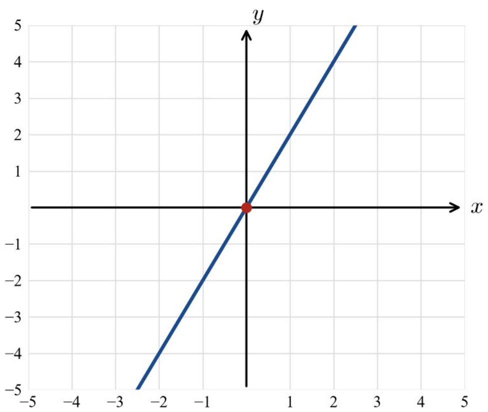


Practice questions

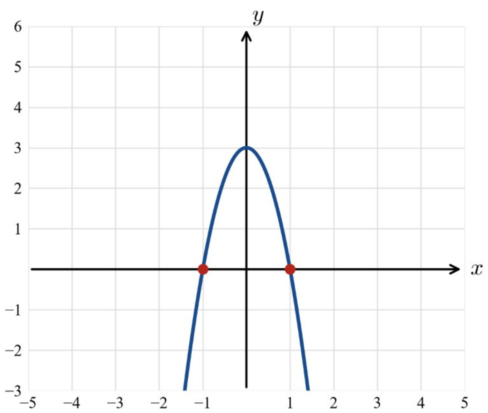
Scaffolded

For each graph of f: (a) state the interval(s) on which the antiderivative F is increasing; (b) give the x-coordinate and nature of each stationary point of F; (c) sketch a possible graph of F through the stated point.

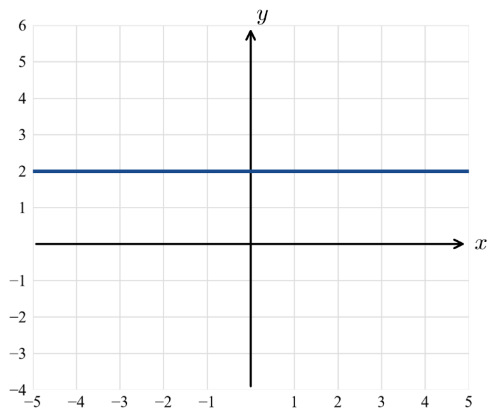
Q1. Sketch F through $(0,-4)$.



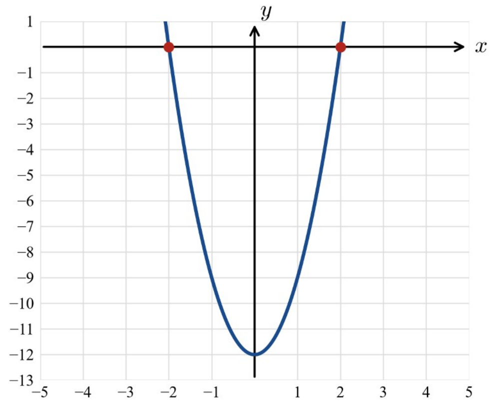
Q2. Sketch F through the origin.



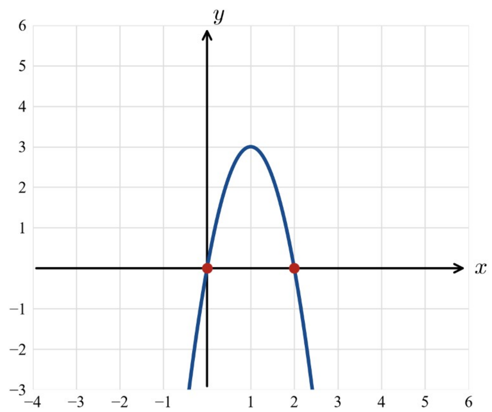
Q3. The graph of f is shown. **State** the shape of F and sketch F through $(0,1)$.



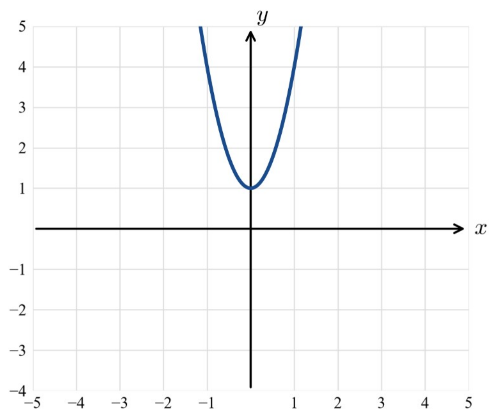
Q4. Sketch F through the origin.



Q5. Sketch F through the origin.



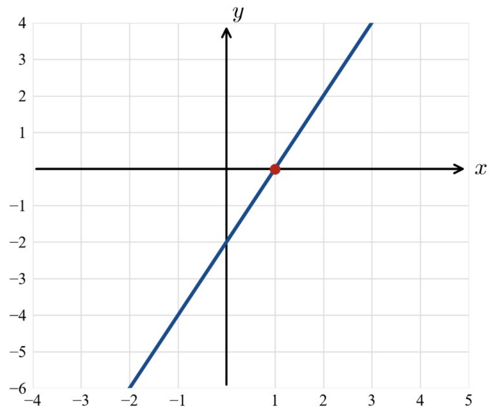
Q6. Explain why F has no stationary points, then sketch F through the origin.



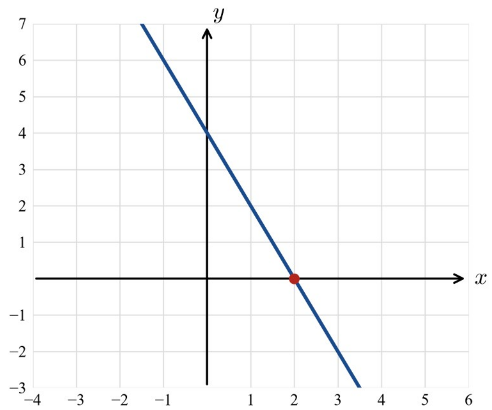
Unscaffolded

For each graph of f, **sketch** a possible antiderivative F, showing the x-coordinate and nature of every stationary point (and any point of inflection). Unless a point is given, any vertical translation is acceptable.

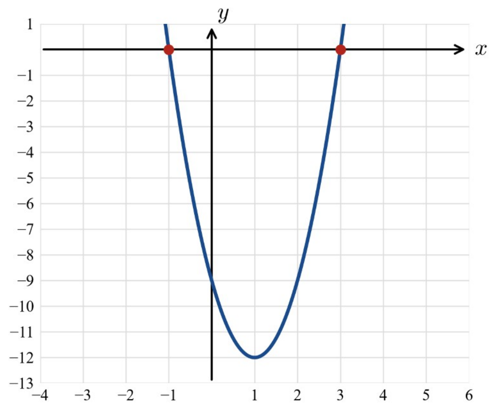
Q7. Sketch a possible F.



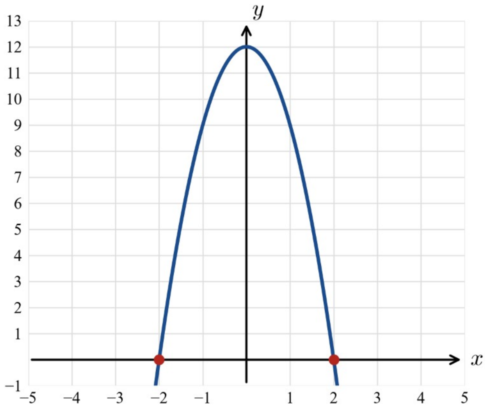
Q8. Sketch a possible F.



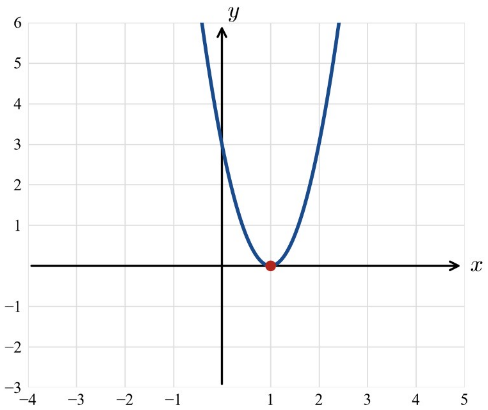
Q9. Sketch a possible F (through the origin).



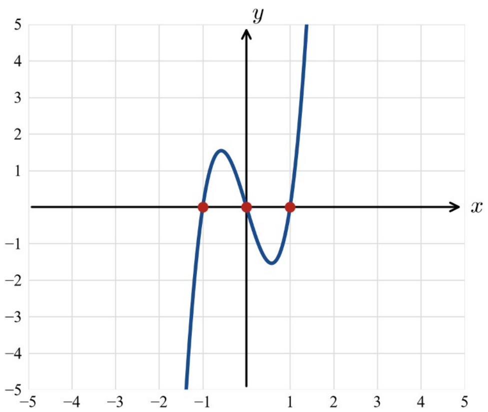
Q10. Sketch a possible F (through the origin).



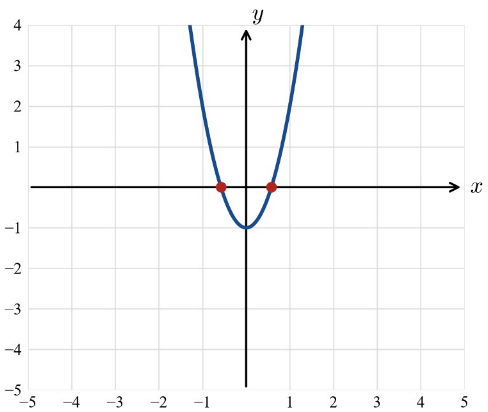
Q11. The graph of f is shown. **Explain** why F has a stationary point of inflection, and sketch a possible F through the origin.



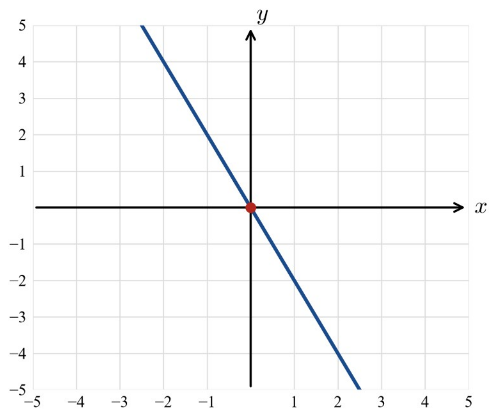
Q12. Determine the number of stationary points of F and their nature, then sketch a possible F through the origin.



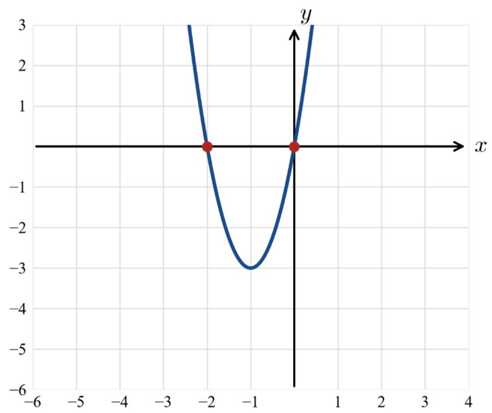
Q13. Sketch a possible F. (The x-intercepts of f are not integers — mark them as $x = \pm a$.)



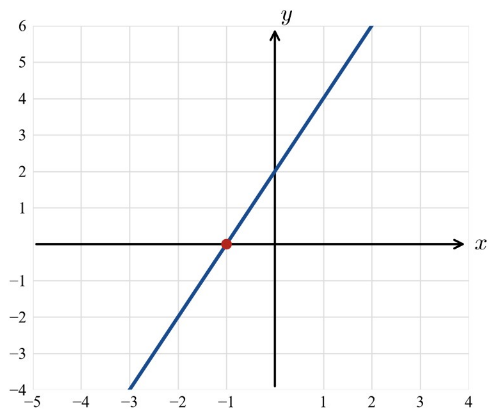
Q14. Sketch a possible F through $(0,1)$.



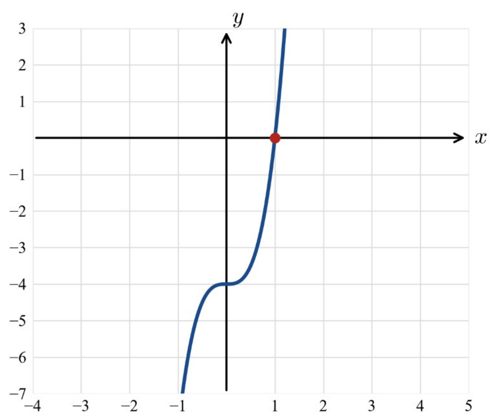
Q15. Sketch a possible F through the origin.



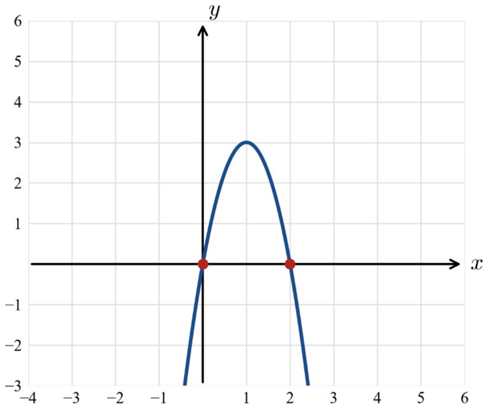
Q16. Sketch a possible F.



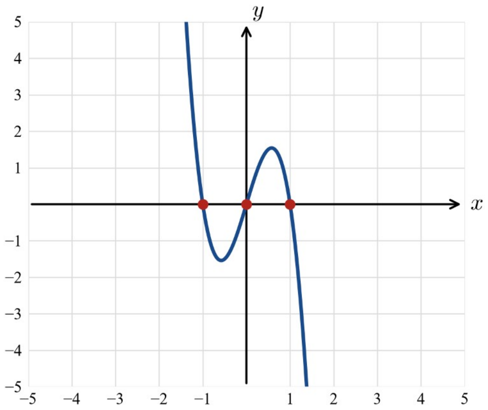
Q17. The graph of f is shown. **Show** that F has exactly one stationary point, state its nature, and sketch a possible F through the origin.



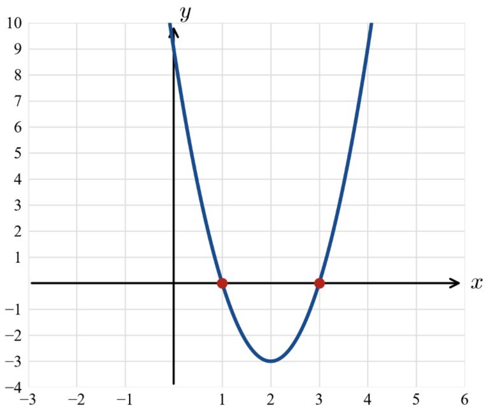
Q18. Sketch a possible F through the origin.



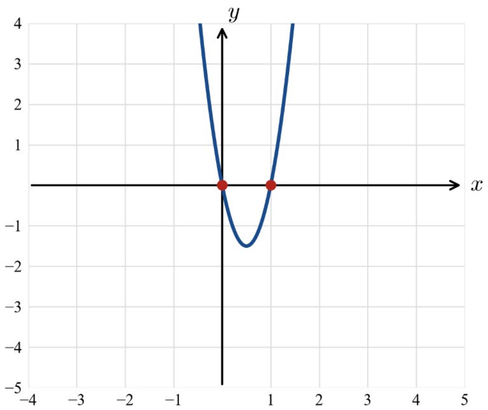
Q19. Determine the nature of each stationary point of F, then sketch a possible F through the origin.



Q20. Sketch a possible F through the origin, and **hence** state the number of x-intercepts of this particular F.



Q21. Sketch a possible F through the origin.

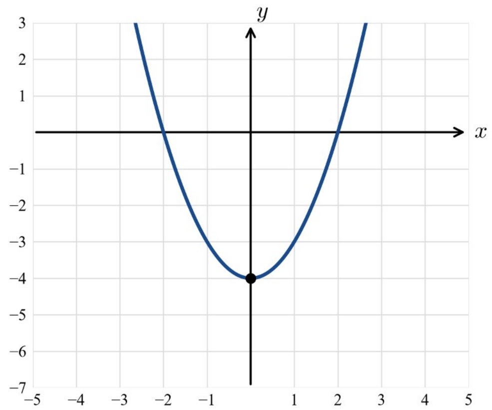


Answers

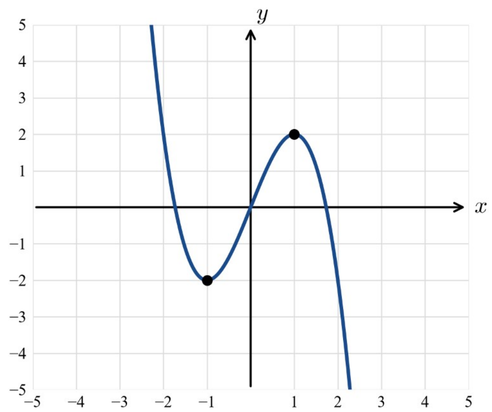
Q1. (a) increasing for $x > 0$; (b) minimum at $x = 0$; (c) parabola, min $(0, -4)$. **Q2.** (a) increasing for $-1 < x < 1$; (b) min at $x = -1$, max at $x = 1$; (c) cubic, min $(-1, -2)$, max $(1, 2)$. **Q3.** (a) increasing for all x; (b) none; (c) straight line of gradient 2 through $(0, 1)$. **Q4.** (a) increasing for $x < -2$ and $x > 2$; (b) max at $x = -2$, min at $x = 2$; (c) cubic, max $(-2, 16)$, min $(2, -16)$. **Q5.** (a) increasing for $0 < x < 2$; (b) min at $x = 0$, max at $x = 2$; (c) cubic, min $(0, 0)$, max $(2, 4)$. **Q6.** (a) increasing for all x; (b) none — $f > 0$ everywhere; (c) increasing cubic through the origin. **Q7.** Minimum at $x = 1$; parabola, min $(1, -1)$. **Q8.** Maximum at $x = 2$; parabola (∩-shaped), max $(2, 4)$. **Q9.** Maximum at $x = -1$, minimum at $x = 3$; cubic, max $(-1, 5)$, min $(3, -27)$. **Q10.** Minimum at $x = -2$, maximum at $x = 2$; cubic, min $(-2, -16)$, max $(2, 16)$. **Q11.** Stationary point of inflection at $x = 1$ ($f \geq 0$, touches the axis without a sign change); increasing cubic, SPI $(1, 0)$. **Q12.** Three stationary points: minima at $x = \pm 1$, maximum at $x = 0$ (W-shaped quartic), $(-1, -1)$, $(0, 0)$, $(1, -1)$. **Q13.** Maximum at $x = -a$, minimum at $x = a$ where $a = \dfrac{1}{\sqrt{3}}$; cubic. **Q14.** Maximum at $x = 0$; parabola (∩-shaped), max $(0, 1)$. **Q15.** Maximum at $x = -2$, minimum at $x = 0$; cubic, max $(-2, 4)$, min $(0, 0)$. **Q16.** Minimum at $x = -1$; parabola, min $(-1, -4)$. **Q17.** One stationary point, a minimum at $x = 1$ ($(1, -3)$); quartic, concave up throughout. **Q18.** Minimum at $x = 0$, maximum at $x = 2$; cubic, min $(0, -4)$, max $(2, 0)$. **Q19.** Maxima at $x = \pm 1$, minimum at $x = 0$ (M-shaped quartic), $(-1, 1)$, $(0, 0)$, $(1, 1)$. **Q20.** Maximum at $x = 1$, minimum at $x = 3$; cubic, max $(1, 4)$, min $(3, 0)$; the minimum sits on the axis, so this F has 2 x-intercepts. **Q21.** Maximum at $x = 0$, minimum at $x = 1$; cubic, max $(0, 0)$, min $(1, -1)$.

Fully-worked solutions

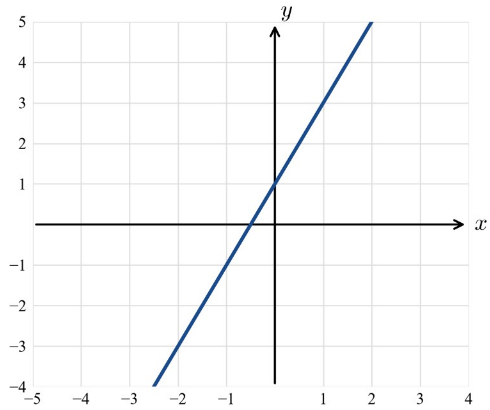
Q1. f is a line: $f < 0$ for $x < 0$, $f > 0$ for $x > 0$, zero at $x = 0$ ($- \to +$). So F decreases then increases, giving a **minimum** at $x = 0$. F is a parabola; through $(0, -4)$:



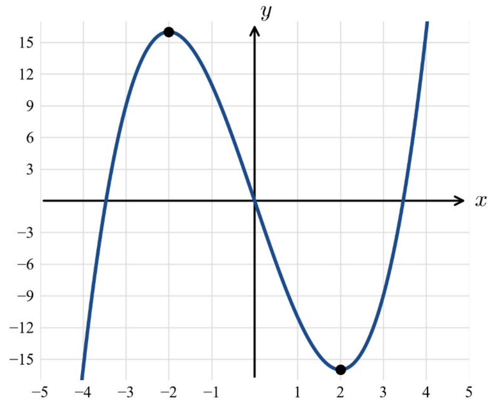
Q2. x-intercepts of f at $x = \pm 1$. $f < 0$ for $x < -1$, $f > 0$ for $-1 < x < 1$, $f < 0$ for $x > 1$. Hence F decreases–increases–decreases: **minimum** at $x = -1$ ($- \to +$), **maximum** at $x = 1$ ($+ \to -$). F is a cubic through the origin:



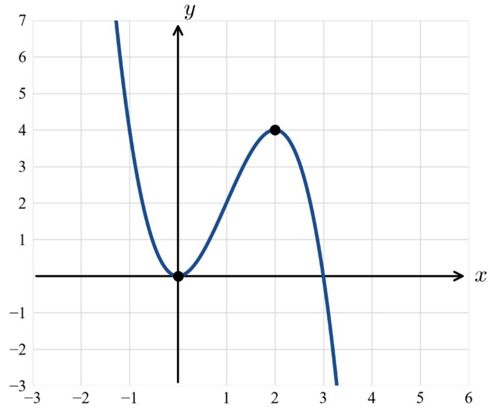
Q3. f is the constant 2, so $F' = 2$ everywhere: F has constant gradient 2 and **no stationary points** — a straight line. Through $(0,1)$, F is $y = 2x + 1$:



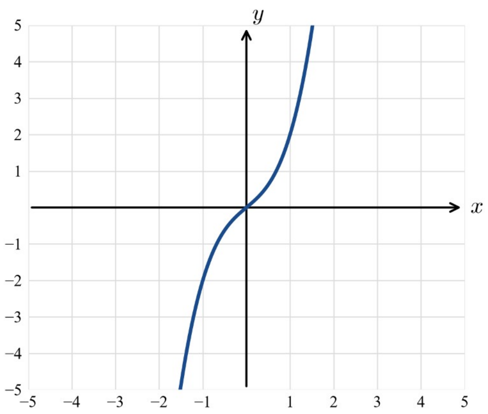
Q4. x-intercepts of f at $x = \pm 2$; $f > 0$ for $|x| > 2$ and $f < 0$ for $|x| < 2$. So F increases, decreases, increases: **maximum** at $x = -2$, **minimum** at $x = 2$. Cubic through the origin:



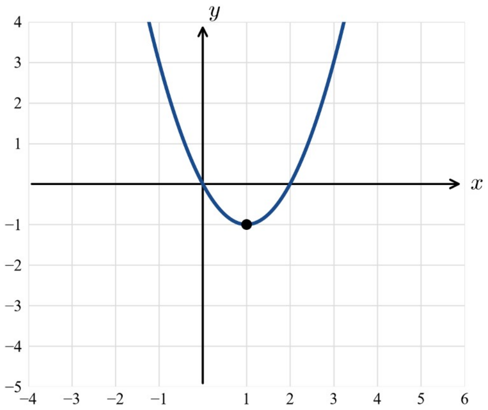
Q5. x-intercepts at $x = 0, 2$; $f < 0$ for $x < 0$, $f > 0$ for $0 < x < 2$, $f < 0$ for $x > 2$. So **minimum** at $x = 0$ and **maximum** at $x = 2$. Cubic through the origin:



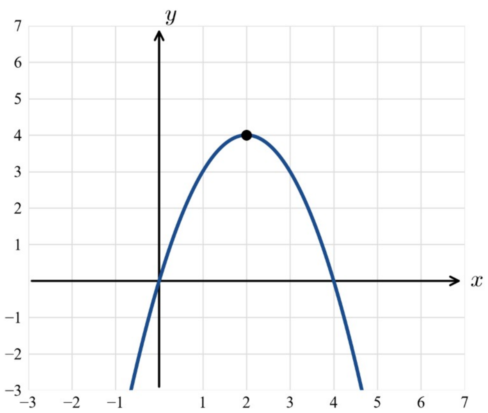
Q6. The graph of f lies entirely **above** the axis ($f > 0$ for all x), so F has positive gradient everywhere and is **strictly increasing** — it never turns, so there are no stationary points. (The minimum of f at $x = 0$ gives a non-stationary point of inflection.) An increasing cubic through the origin:



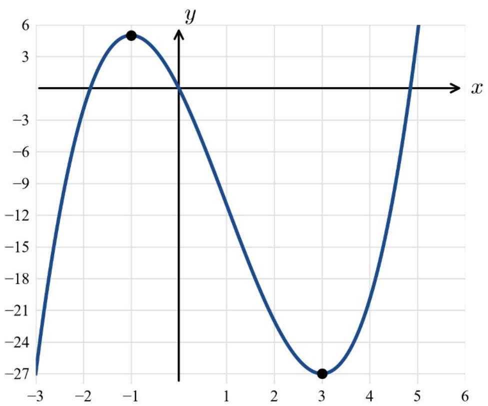
Q7. $f=0$ at $x=1$, changing $- \to +$: **minimum** at $x=1$. F is a parabola:



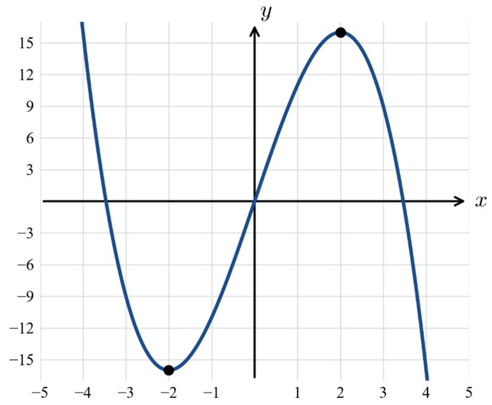
Q8. $f=0$ at $x=2$, changing $+ \to -$: **maximum** at $x=2$. F is a $\cap$-shaped parabola:



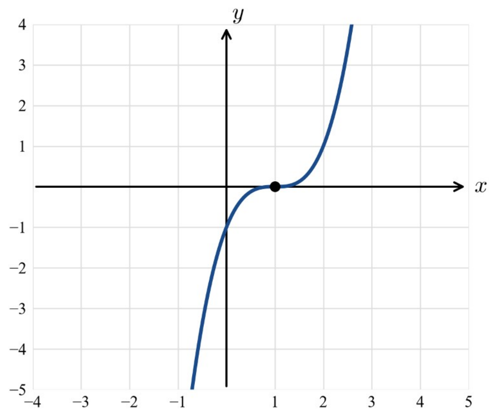
Q9. x-intercepts of f at $x=-1, 3$; $f>0$, then $f<0$, then $f>0$. **Maximum** at $x=-1$, **minimum** at $x=3$. Cubic through the origin:



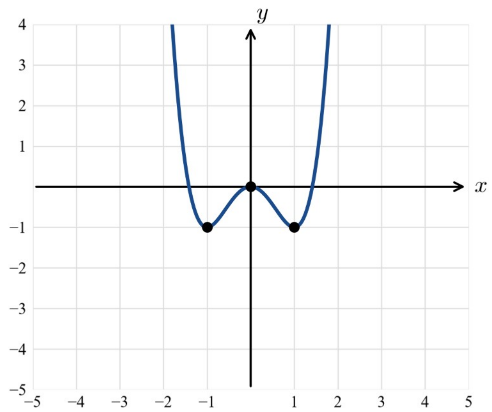
Q10. x-intercepts at $x=\pm 2$; $f<0$ for $|x|>2$, $f>0$ for $|x|<2$. **Minimum** at $x=-2$, **maximum** at $x=2$. Cubic through the origin:



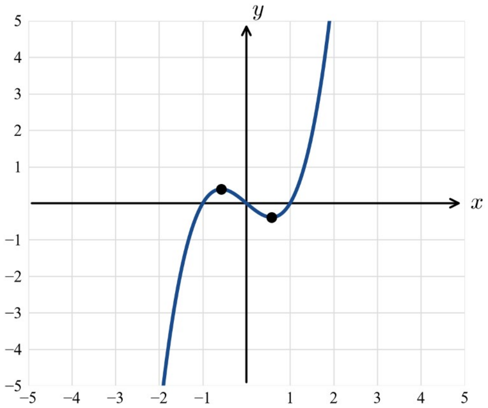
Q11. $f\geq 0$ for all x and $f=0$ at $x=1$, where the graph **touches** the axis without changing sign. Because the gradient of F is momentarily zero but stays positive on both sides, F has a **stationary point of inflection** at $x=1$, not a turning point. F is an increasing cubic through the origin:



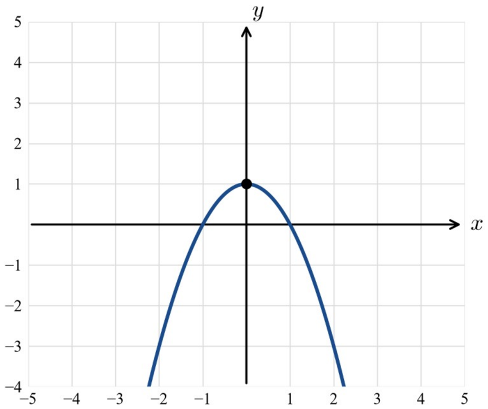
Q12. x-intercepts of f at $x=-1,0,1$; the sign of f is $-,+,-,+$ across these. So F has **three** stationary points: **minimum** at $x=-1$, **maximum** at $x=0$, **minimum** at $x=1$ — a W-shaped quartic through the origin:



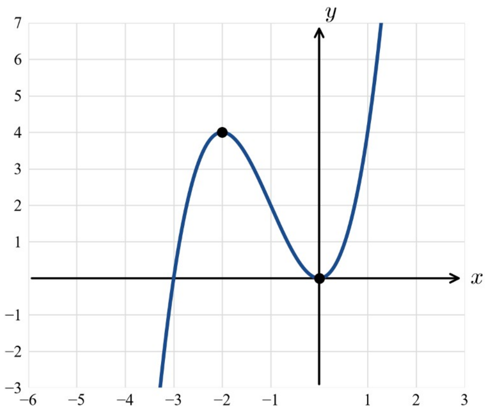
Q13. f is a $\cup$-parabola with x-intercepts at $x=\pm a$, $a=1/\sqrt{3}\approx 0.58$. $f>0$ for $|x|>a$, $f<0$ for $|x|<a$: **maximum** at $x=-a$, **minimum** at $x=a$. Cubic:



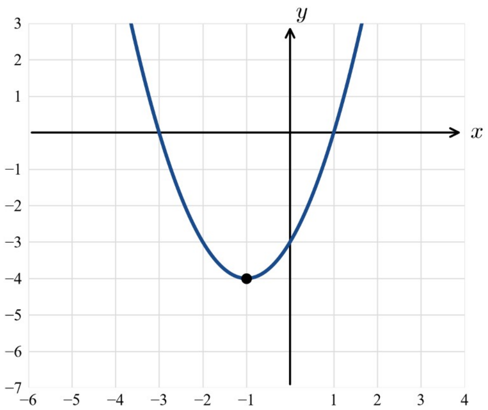
Q14. $f = 0$ at $x = 0$, changing $+ \to -$: **maximum** at $x = 0$. ∩-shaped parabola through $(0,1)$:



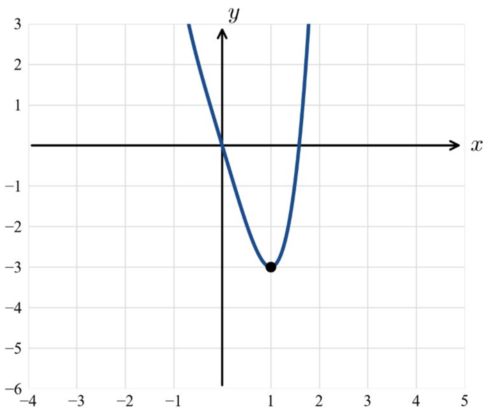
Q15. x-intercepts at $x = -2, 0$; $f > 0$, $f < 0$, $f > 0$. **Maximum** at $x = -2$, **minimum** at $x = 0$. Cubic through the origin:



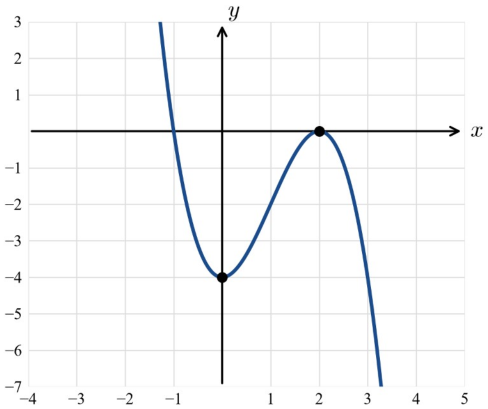
Q16. $f = 0$ at $x = -1$, changing $- \to +$: **minimum** at $x = -1$. Parabola:



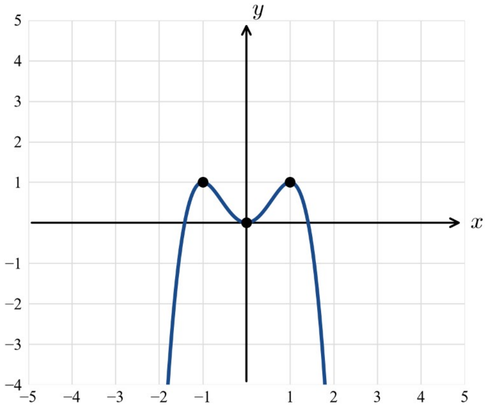
Q17. f is a cubic with a single x-intercept at $x=1$; $f<0$ for $x<1$ and $f>0$ for $x>1$, so there is exactly **one** stationary point, a **minimum**, at $x=1$. (For $x<1$, F decreases; for $x>1$, it increases.) F is a quartic through the origin:



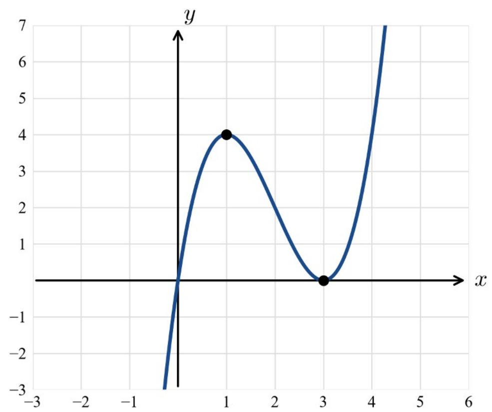
Q18. x-intercepts at $x=0,2$; $f<0$, $f>0$, $f<0$. **Minimum** at $x=0$, **maximum** at $x=2$. Cubic through the origin:



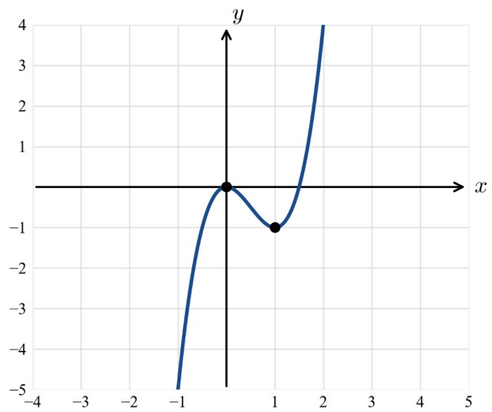
Q19. x-intercepts of f at $x=-1,0,1$; the sign of f is $+,-,+,-$ across these. So the stationary points of F are: **maximum** at $x=-1$, **minimum** at $x=0$, **maximum** at $x=1$ — an M-shaped quartic through the origin:



Q20. x-intercepts of f at $x=1,3$; $f>0$, $f<0$, $f>0$. **Maximum** at $x=1$, **minimum** at $x=3$. The cubic through the origin has its minimum *on* the x-axis at $(3,0)$, so it meets the axis at the origin and touches at $x=3$: **two** x-intercepts.



Q21. x-intercepts of f at $x = 0, 1$; $f > 0$, $f < 0$, $f > 0$. **Maximum** at $x = 0$, **minimum** at $x = 1$. Cubic through the origin:



Topic 2 Test A — Technology-active

One bound reference and an approved CAS calculator are permitted. Reading time + 50 minutes writing. Total 50 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (5 marks)

Consider $f(x)=x^3-6x^2+9x+2$.

a. Find $f'(x)$. 1 mark

b. Find the coordinates of the stationary points of the graph of f. 2 marks

c. State the nature of each stationary point. 2 marks

Question 2 (4 marks)

Consider the curve $y=x^2-4x+5$.

a. Find the equation of the tangent to the curve at the point where $x=3$. 2 marks

b. Find the equation of the normal to the curve at that point. 2 marks

Question 3 (8 marks)

A rectangular enclosure is built against a straight wall. The wall forms one side; the other three sides use 100 m of fencing. Let the width of each side perpendicular to the wall be W metres.

a. Show that the enclosed area is $A=100W-2W^2$. 2 marks

b. Find $\frac{dA}{dW}$. 1 mark

c. Find the width W that maximises the area, justifying that it gives a maximum. 3 marks

d. State the maximum area. 2 marks

Question 4 (6 marks)

The volume of water in a tank is $V(t)=t^3-12t^2+36t$ litres, for $0\le t\le 6$, where t is in minutes.

a. Find the rate of change $\frac{dV}{dt}$. 1 mark

b. Find the times when the rate of change is zero. 2 marks

c. Find the maximum volume of water in the tank over this interval. 3 marks

Question 5 (7 marks)

Consider $y=x^2-2x$.

a. Find the x-intercepts. 1 mark

b. Find the exact area of the region enclosed by the curve and the x-axis. 3 marks

c. Evaluate $\int_0^3 (x^2-2x)\,dx$, and explain why this differs from the total area between the curve and the x-axis from $x=0$ to $x=3$. 3 marks

Question 6 (8 marks)

a. Find the coordinates of the points of intersection of $y=x^2$ and $y=2x+3$. 3 marks

b. Find the exact area of the region enclosed between the two curves. 5 marks

Question 7 (6 marks)

Consider $g(x)=e^{2x}-4x$.

a. Find $g'(x)$. 1 mark

b. Find the exact coordinates of the stationary point. 3 marks

c. Determine its nature. 2 marks

Question 8 (6 marks)

a. Evaluate $\int_0^{\pi/2} 2\cos(x)\,dx$. 3 marks

b. Hence find the average value of $f(x)=2\cos(x)$ over the interval $\left[0,\frac{\pi}{2}\right]$. 3 marks

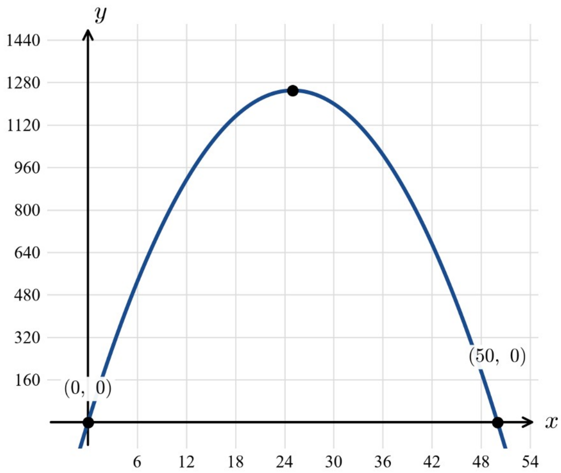
End of Topic 2 Test A

Test A — solutions

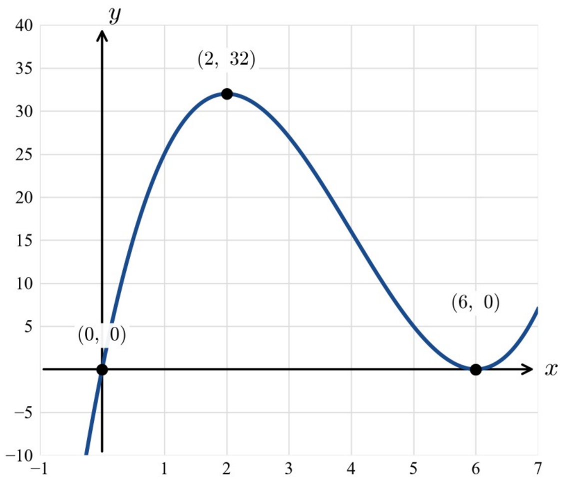
Q1. (a) $f'(x)=3x^2-12x+9=3(x-1)(x-3)$. (b) $f'(x)=0 \Rightarrow x=1$ or $x=3$; $f(1)=6$, $f(3)=2$, so the stationary points are $(1,6)$ and $(3,2)$. (c) $f''(x)=6x-12$: $f''(1)=-6<0$ so $(1,6)$ is a **local maximum**; $f''(3)=6>0$ so $(3,2)$ is a **local minimum**.

Q2. At $x=3$: $y=9-12+5=2$, and $y'=2x-4$ so the gradient is 2. (a) Tangent: $y-2=2(x-3) \Rightarrow y=2x-4$. (b) Normal gradient $=-\frac{1}{2}$: $y-2=-\frac{1}{2}(x-3) \Rightarrow y=-\frac{x}{2}+\frac{7}{2}$.

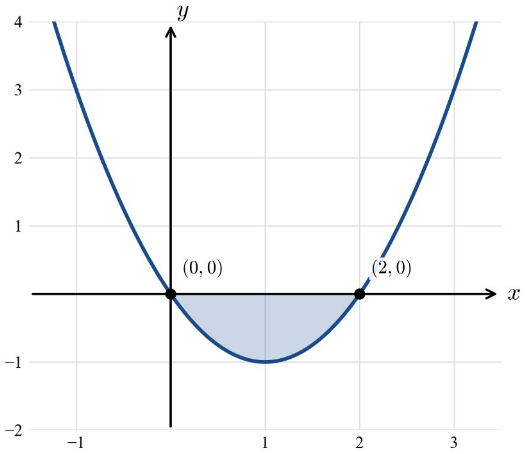
Q3. (a) The three fenced sides are the two widths and the length: $2W+L=100$, so $L=100-2W$. Then $A=LW=(100-2W)W=100W-2W^2$. (b) $\frac{dA}{dW}=100-4W$. (c) $\frac{dA}{dW}=0 \Rightarrow W=25$. Since $\frac{d^2A}{dW^2}=-4<0$, this is a maximum. (d) $A=100(25)-2(25)^2=2500-1250=1250\,\text{m}^2$.



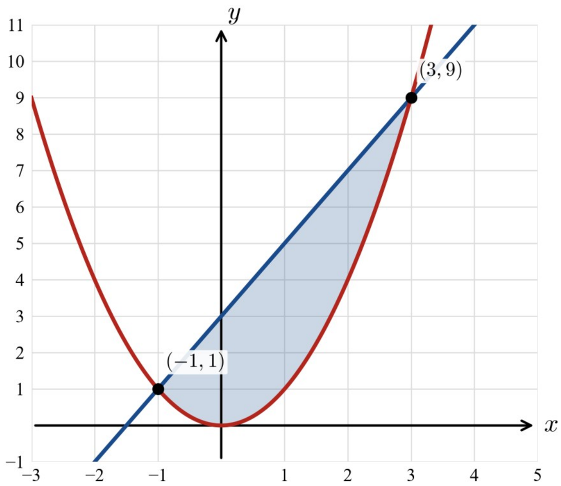
Q4. (a) $\frac{dV}{dt}=3t^2-24t+36=3(t-2)(t-6)$. (b) $\frac{dV}{dt}=0 \Rightarrow t=2$ or $t=6$. (c) $V(2)=8-48+72=32$; the endpoints give $V(0)=0$ and $V(6)=216-432+216=0$. The maximum volume is 32 litres (at $t=2$).



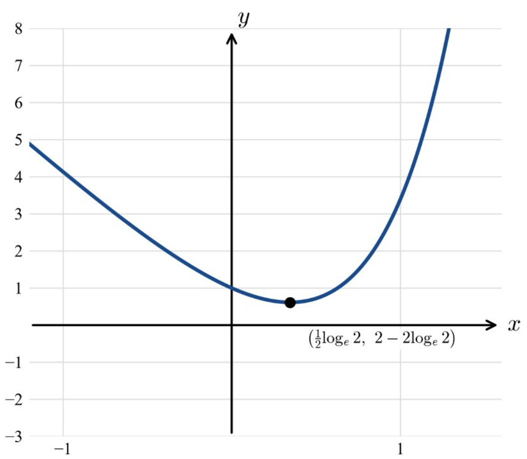
Q5. (a) $x^2-2x=x(x-2)=0 \Rightarrow x=0$ or $x=2$. (b) On $[0,2]$ the curve is below the x-axis, so the area is $\left|\int_0^2 (x^2-2x)\,dx\right|=\left|\left[\frac{x^3}{3}-x^2\right]_0^2\right|=\left|\frac{8}{3}-4\right|=\frac{4}{3}$. (c) $\int_0^3 (x^2-2x)\,dx=\left[\frac{x^3}{3}-x^2\right]_0^3=9-9=0$. From $x=0$ to $x=2$ the region lies below the axis (signed area $-\frac{4}{3}$) and from $x=2$ to $x=3$ it lies above (signed area $+\frac{4}{3}$). The definite integral adds these signed values, giving 0, whereas the total area is $\frac{4}{3}+\frac{4}{3}=\frac{8}{3}$.



Q6. (a) $x^2=2x+3 \Rightarrow x^2-2x-3=0 \Rightarrow (x-3)(x+1)=0$, so $x=-1$ or $x=3$, giving $(-1,1)$ and $(3,9)$. (b) On $[-1,3]$ the line $y=2x+3$ is above $y=x^2$, so the area is
$\int_{-1}^{3}\left((2x+3)-x^2\right)dx=\left[x^2+3x-\frac{x^3}{3}\right]_{-1}^{3}=(9+9-9)-\left(1-3+\frac{1}{3}\right)=9-\left(-\frac{5}{3}\right)=\frac{32}{3}$.



Q7. (a) $g'(x)=2e^{2x}-4$. (b) $g'(x)=0 \Rightarrow e^{2x}=2 \Rightarrow 2x=\log_e 2 \Rightarrow x=\frac{1}{2}\log_e 2$. Then
$g=e^{\log_e 2}-4\cdot\frac{1}{2}\log_e 2=2-2\log_e 2$, so the stationary point is $\left(\frac{1}{2}\log_e 2, 2-2\log_e 2\right)$. (c) $g''(x)=4e^{2x}>0$ for all x, so it is a **local minimum**.



Q8. (a) $\int_0^{\pi/2} 2\cos(x)\,dx = [2\sin(x)]_0^{\pi/2} = 2\sin\frac{\pi}{2} - 2\sin 0 = 2$. (b) Average value $= \frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} 2\cos(x)\,dx = \frac{2}{\pi/2} = \frac{4}{\pi}$.

Topic 2 Test B — Technology-free

No calculator and no notes. 45 minutes. Total 40 marks. All numerical answers must be exact. Where more than one mark is available, appropriate working must be shown.

Question 1 (6 marks)

Differentiate each of the following with respect to x.

a. $y=(2x-1)^4$ — 2 marks

b. $y=x^2e^{3x}$ — 2 marks

c. $y=\frac{x+1}{x^2+1}$ — 2 marks

Question 2 (4 marks)

Find, by first principles, the derivative of $f(x)=x^2-3x$.

Question 3 (4 marks)

Find the equation of the tangent to the curve $y=\log_e(2x)$ at the point where $x=\frac{1}{2}$.

Question 4 (6 marks)

a. Find $\int(3x^2-4x+1)\,dx$. — 2 marks

b. Find $\int 2e^{4x}\,dx$. — 2 marks

c. Given $f'(x)=6x-4$ and $f(1)=3$, find $f(x)$. — 2 marks

Question 5 (6 marks)

Evaluate each of the following exactly.

a. $\int_0^1 (4x^3-2x)\,dx$ — 2 marks

b. $\int_0^1 e^{2x}\,dx$ — 2 marks

c. $\int_{\pi/6}^{\pi/2} \sin(x)\,dx$ — 2 marks

Question 6 (6 marks)

Find the exact area of the region bounded by the curve $y=4-x^2$ and the x-axis.

Question 7 (8 marks)

For the function $f(x)=x^3-3x^2$:

a. find $f'(x)$; — 1 mark

b. find the coordinates of the stationary points and determine the nature of each; — 3 marks

c. find the coordinates of the axis intercepts; — 2 marks

d. sketch the graph of $y = f(x)$, labelling the stationary points and axis intercepts. 2 marks

End of Topic 2 Test B

Test B — solutions

Q1. (a) Chain rule: $\frac{dy}{dx}=4(2x-1)^3\cdot 2=8(2x-1)^3$. (b) Product rule: $\frac{dy}{dx}=2xe^{3x}+x^2\cdot 3e^{3x}=xe^{3x}(2+3x)$. (c) Quotient rule: $\frac{dy}{dx}=\frac{(1)(x^2+1)-(x+1)(2x)}{(x^2+1)^2}=\frac{x^2+1-2x^2-2x}{(x^2+1)^2}=\frac{-x^2-2x+1}{(x^2+1)^2}$.

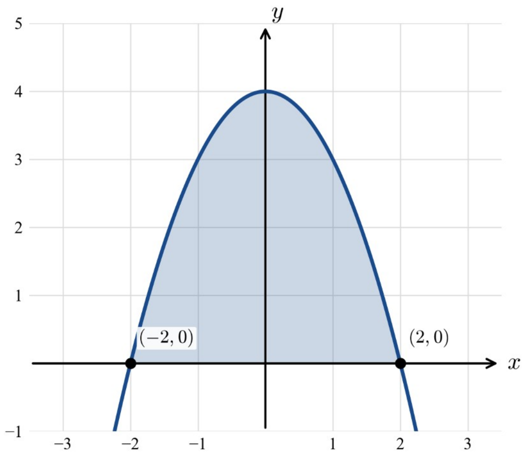
Q2. $f'(x)=\frac{\lim_{h\to 0} f(x+h)-f(x)}{h}=\frac{\lim_{h\to 0}[(x+h)^2-3(x+h)]-[x^2-3x]}{h}$
$=\frac{\lim_{h\to 0} 2xh+h^2-3h}{h}=\lim_{h\to 0}(2x+h-3)=2x-3\cdot$

Q3. At $x=\frac{1}{2}$: $y=\log_e\left(2\cdot\frac{1}{2}\right)=\log_e 1=0$. Now $y=\log_e(2x)$ so $\frac{dy}{dx}=\frac{1}{x}$, giving gradient $\frac{1}{1/2}=2$. Tangent: $y-0=2\left(x-\frac{1}{2}\right)\Rightarrow y=2x-1$.

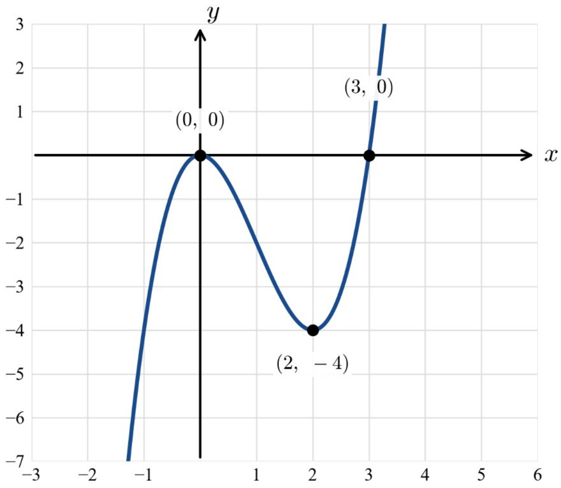
Q4. (a) $\int(3x^2-4x+1)\,dx=x^3-2x^2+x+c$. (b) $\int 2e^{4x}\,dx=\frac{2}{4}e^{4x}+c=\frac{1}{2}e^{4x}+c$. (c) $f(x)=\int(6x-4)\,dx=3x^2-4x+c$. Since $f(1)=3$: $3-4+c=3\Rightarrow c=4$, so $f(x)=3x^2-4x+4$.

Q5. (a) $\int_0^1(4x^3-2x)\,dx=[x^4-x^2]_0^1=(1-1)-0=0$. (b) $\int_0^1 e^{2x}\,dx=\left[\frac{1}{2}e^{2x}\right]_0^1=\frac{1}{2}e^2-\frac{1}{2}=\frac{e^2-1}{2}$. (c) $\int_{\pi/6}^{\pi/2}\sin(x)\,dx=[-\cos(x)]_{\pi/6}^{\pi/2}=-\cos\frac{\pi}{2}+\cos\frac{\pi}{6}=0+\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}$.

Q6. The curve meets the x-axis where $4-x^2=0$, i.e. $x=-2$ and $x=2$; the curve is above the axis between them. Area = $\int_{-2}^{2}(4-x^2)\,dx=\left[4x-\frac{x^3}{3}\right]_{-2}^{2}=\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)=16-\frac{16}{3}=\frac{32}{3}$ square units.



Q7. (a) $f'(x)=3x^2-6x=3x(x-2)$. (b) $f'(x)=0\Rightarrow x=0$ or $x=2$. $f(0)=0$ and $f(2)=8-12=-4$. $f''(x)=6x-6$: $f''(0)=-6<0$ so $(0,0)$ is a **local maximum**; $f''(2)=6>0$ so $(2,-4)$ is a **local minimum**. (c) $f(x)=x^2(x-3)=0\Rightarrow x=0$ (touch) or $x=3$; intercepts $(0,0)$ and $(3,0)$. (d)



y
x
(0, 0)
(3, 0)
(2, −4)
3
2
1
−1
−2
−3
−4
−5
−6
−7
−3
−2
−1
1
2
3
4
5
6