Mathematical Methods Units 3 & 4

Chapter 9 — Applications of differentiation

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Significant effort has been made to ensure this content is legal. Please send any areas of concern to admin@vwx.au with appropriate document/legal references so errors can be verified and changes be made.

Contents

Section	Page
9A Tangents and normals	2
9B Rates of change	15
9C Stationary points	24
9D Classifying stationary points	37
9E Global (absolute) maxima and minima	50
9F Maximum and minimum problems	62
9G Differentiating families of functions	75
9H Newton's method	85
9I Algorithms and pseudocode	92

Chapter 9 Applications of differentiation — 9A Tangents and normals

Theory

At a point $P(a, f(a))$ on the curve $y=f(x)$, the **gradient of the tangent** is $f'(a)$.

Equation of the tangent. The tangent at $x=a$ is the straight line through $(a, f(a))$ with gradient $f'(a)$:

$$y-f(a)=f'(a)(x-a).$$

Equation of the normal. The **normal** at P is the line through P perpendicular to the tangent. Since perpendicular gradients multiply to -1, the normal has gradient $-\frac{1}{f'(a)}$ (provided $f'(a)\neq 0$):

$$y-f(a)=-\frac{1}{f'(a)}(x-a).$$

If $f'(a)=0$ the tangent is horizontal ($y=f(a)$) and the normal is the vertical line $x=a$.

Method.

1. Differentiate to find $f'(x)$.
2. Evaluate $f(a)$ (the y-coordinate of the point) and $f'(a)$ (the gradient).
3. Substitute into the tangent (or normal) formula and simplify.

The same method works for **every** family of functions — polynomial, power, exponential, logarithmic and circular — because it only needs the point and the gradient there. Keep answers **exact** (surds, e, $\log_e$, π).

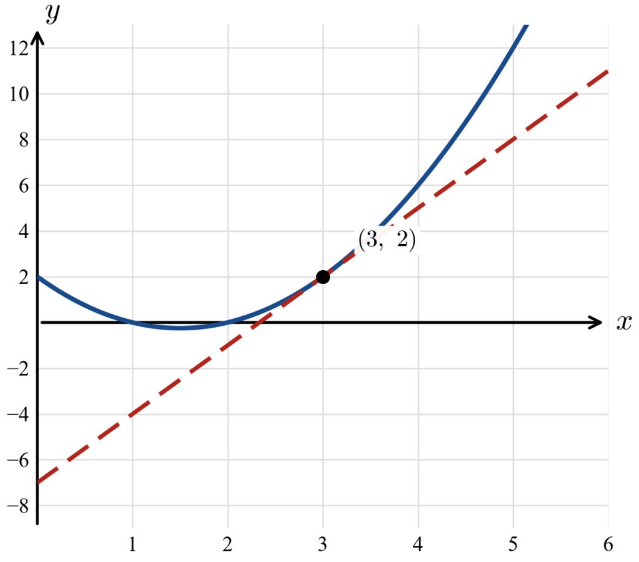
Useful follow-ups. To find where a tangent **cuts the axes**, set $y=0$ (for the x-intercept) or $x=0$ (for the y-intercept). To find where a tangent **meets the curve again**, solve $f(x)=(\text{tangent})$; the point of tangency is a repeated solution, so the other solution gives the second point. To find a tangent **with a given gradient** m, solve $f'(x)=m$ for x first.

Worked examples

Example 1 — scaffolded

Find the equation of the tangent to $y=x^2-3x+2$ at the point where $x=3$.

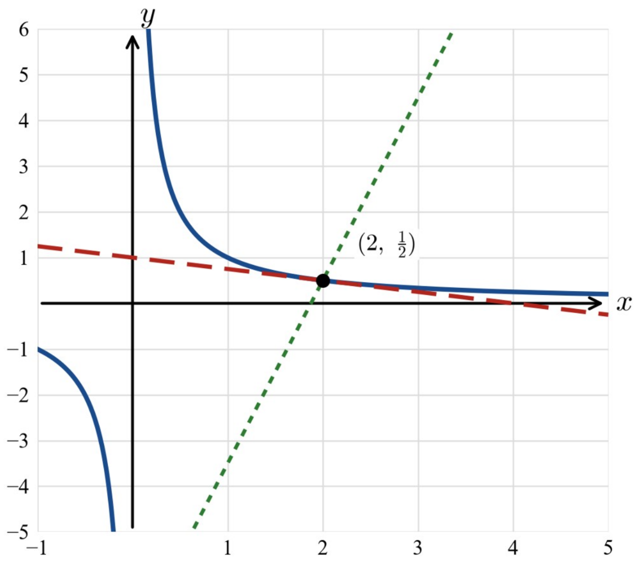
Working	Comment
$f(3)=3^2-3(3)+2=2$, so the point is $(3,2)$.	Find the y-coordinate first.
$f'(x)=2x-3$.	Differentiate.
$f'(3)=2(3)-3=3$.	The gradient of the tangent at $x=3$.
$y-2=3(x-3)$	Substitute into $y-f(a)=f'(a)(x-a)$.
$\therefore y=3x-7$	Expand and simplify.



Example 2 — scaffolded

Find the equations of the tangent and the normal to $y=\dfrac{1}{x}$ at the point where $x=2$.

Working	Comment
$f(2)=\dfrac{1}{2}$, so the point is $\left(2,\dfrac{1}{2}\right)$.	The y-coordinate.
$f(x)=x^{-1}$, so $f'(x)=-x^{-2}=-\dfrac{1}{x^2}$.	Rewrite with an index, then differentiate.
$f'(2)=-\dfrac{1}{4}$.	Tangent gradient.
Tangent: $y-\dfrac{1}{2}=-\dfrac{1}{4}(x-2)$, so $y=-\dfrac{x}{4}+1$.	Substitute and simplify.
Normal gradient $=-\dfrac{1}{f'(2)}=-\dfrac{1}{-1/4}=4$.	Perpendicular gradient.
Normal: $y-\dfrac{1}{2}=4(x-2)$, so $y=4x-\dfrac{15}{2}$.	Substitute and simplify.



Example 3 — unscaffolded

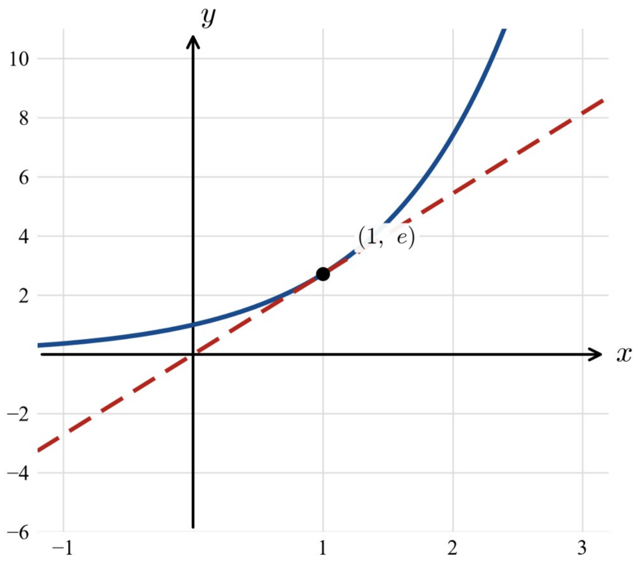
Find the equation of the tangent to $y = e^x$ at the point where $x = 1$.

$f(1) = e$, so the point is $(1, e)$.

$f'(x) = e^x$, so $f'(1) = e$.

Tangent: $y - e = e(x - 1) = ex - e$.

$\therefore y = ex$ — the tangent passes through the origin.



Example 4 — unscaffolded

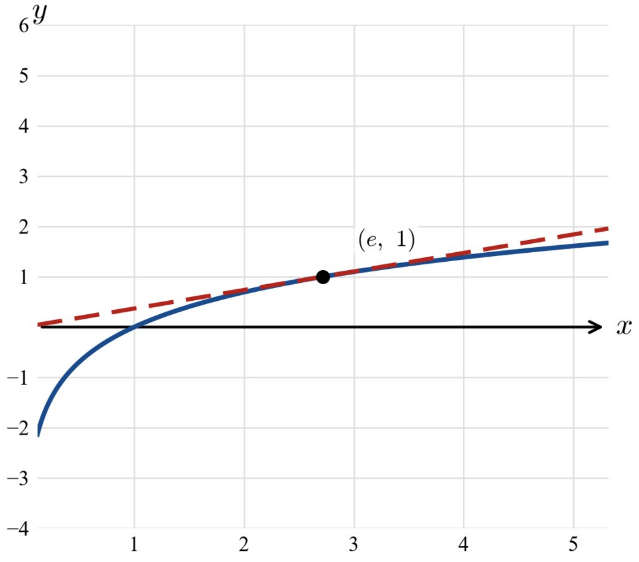
Find the equation of the tangent to $y = \log_e x$ at the point where $x = e$.

$f(e) = \log_e e = 1$, so the point is $(e, 1)$.

$f'(x) = \dfrac{1}{x}$, so $f'(e) = \dfrac{1}{e}$.

Tangent: $y - 1 = \dfrac{1}{e}(x - e) = \dfrac{x}{e} - 1$.

$\therefore y = \dfrac{x}{e}$.



Practice questions

Scaffolded

Q1. For $y = x^2 - 4x + 5$ at the point where $x = 1$: (a) find the coordinates of the point; (b) find $f'(x)$ and the gradient of the tangent; (c) find the equation of the tangent; (d) find the equation of the normal.

Q2. For $y = x^3 - 2x$ at $x = 1$: (a) find the point and the gradient; (b) find the equation of the tangent; (c) find where the tangent cuts the x-axis.

Q3. For $y = \sqrt{x}$ at $x = 4$: (a) write $\sqrt{x}$ as a power of x and differentiate; (b) find the gradient at $x = 4$; (c) find the equation of the tangent.

Q4. For $y = e^{2x}$ at $x = 0$: (a) find the point and the gradient; (b) find the equation of the tangent.

Q5. For $y = \log_e x$ at $x = 1$: (a) find the point and the gradient; (b) find the equation of the tangent; (c) find the equation of the normal.

Q6. For $y = \sin x$ at $x = \frac{\pi}{6}$: (a) find the exact coordinates of the point; (b) find the exact gradient; (c) find the equation of the tangent.

Unscaffolded

Find the equation of the tangent to each curve at the given point (Q7–Q15).

Q7. $y = x^2 - 5x + 6$ at $x = 4$.

Q8. $y = 2x^2 - 3x + 1$ at $x = 2$.

Q9. $y = x^3 - 3x^2 + 2$ at $x = 1$.

Q10. $y = \frac{1}{x}$ at $x = 1$; also find the equation of the normal.

Q11. $y = \sqrt{x}$ at $x = 9$.

Q12. $y = e^x$ at $x = 0$.

Q13. $y = e^{-x}$ at $x = 0$.

Q14. $y = \log_e x$ at $x = 2$.

Q15. $y = \cos x$ at $x = \frac{\pi}{3}$.

Q16. Find the equation of the normal to $y = x^2$ at the point $(2, 4)$.

Q17. Find the equation of the tangent to $y = x^2 - 3x$ that has gradient 1.

Q18. The tangent to $y = x^3$ at the point $(1, 1)$ meets the curve again at another point. Find the equation of the tangent and hence the coordinates of that second point.

Q19. Find the equation of the normal to $y = x^2 + 1$ at the point where $x = 1$.

Q20. Find the equations of the two tangents to $y = x^3 - 3x$ that are parallel to the x-axis.

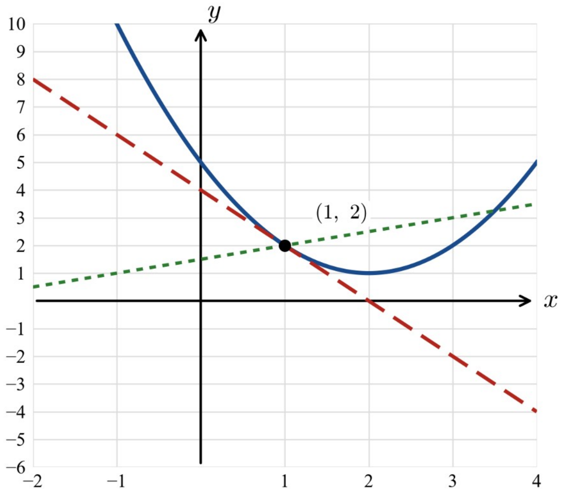
Q21. For $y = x e^x$, show that the tangent at the origin is $y = x$.

Answers

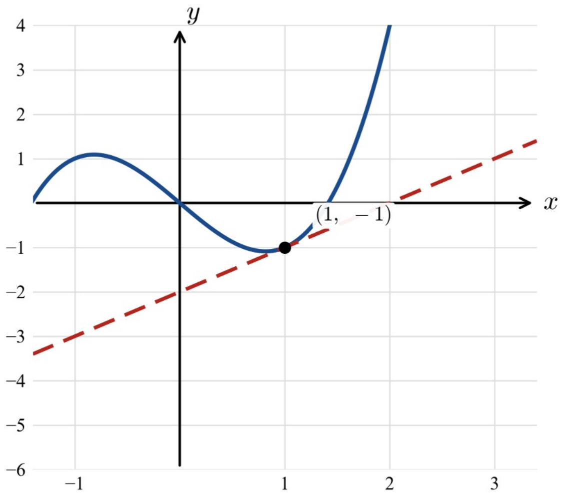
Q1. $(1,2)$; $f'(x)=2x-4$, gradient -2; tangent $y=-2x+4$; normal $y=\frac{x}{2}+\frac{3}{2}$. **Q2.** $(1,-1)$, gradient 1; tangent $y=x-2$; cuts the x-axis at $(2,0)$. **Q3.** $y=x^{1/2}$, $f'(x)=\frac{1}{2\sqrt{x}}$; gradient $\frac{1}{4}$; tangent $y=\frac{x}{4}+1$. **Q4.** $(0,1)$, gradient 2; tangent $y=2x+1$. **Q5.** $(1,0)$, gradient 1; tangent $y=x-1$; normal $y=-x+1$. **Q6.** $\left(\frac{\pi}{6},\frac{1}{2}\right)$, gradient $\frac{\sqrt{3}}{2}$; tangent $y=\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}\pi}{12}+\frac{1}{2}$. **Q7.** $y=3x-10$. **Q8.** $y=5x-7$. **Q9.** $y=-3x+3$. **Q10.** Tangent $y=-x+2$; normal $y=x$. **Q11.** $y=\frac{x}{6}+\frac{3}{2}$. **Q12.** $y=x+1$. **Q13.** $y=-x+1$. **Q14.** $y=\frac{x}{2}-1+\log_e 2$. **Q15.** $y=-\frac{\sqrt{3}}{2}x+\frac{1}{2}+\frac{\sqrt{3}\pi}{6}$. **Q16.** $y=-\frac{x}{4}+\frac{9}{2}$. **Q17.** $y=x-4$ (at the point $(2,-2)$). **Q18.** Tangent $y=3x-2$; second point $(-2,-8)$. **Q19.** $y=-\frac{x}{2}+\frac{5}{2}$. **Q20.** $y=2$ (at $(-1,2)$) and $y=-2$ (at $(1,-2)$). **Q21.** Tangent $y=x$ (shown below).

Fully-worked solutions

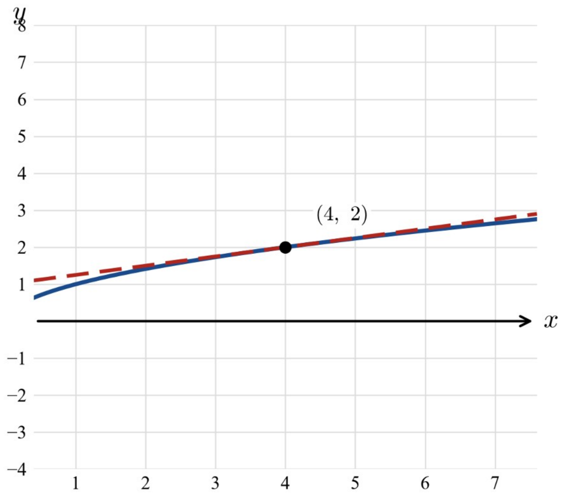
Q1. $y=x^2-4x+5$. $f(1)=1-4+5=2$, point $(1,2)$. $f'(x)=2x-4$, so $f'(1)=-2$. Tangent: $y-2=-2(x-1)\Rightarrow y=-2x+4$. Normal gradient $=-\frac{1}{-2}=\frac{1}{2}$: $y-2=\frac{1}{2}(x-1)\Rightarrow y=\frac{x}{2}+\frac{3}{2}$.



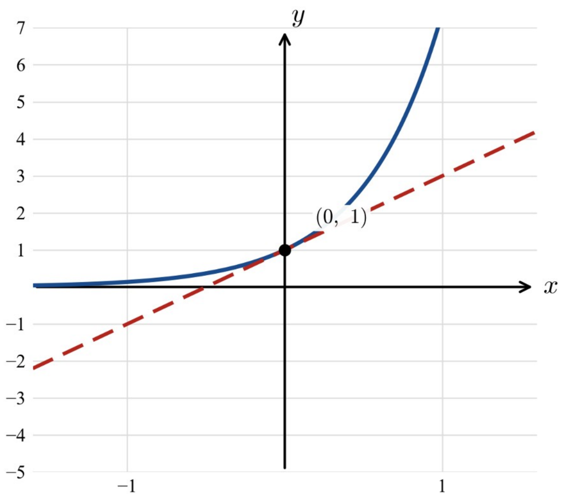
Q2. $y=x^3-2x$. $f(1)=1-2=-1$, point $(1,-1)$. $f'(x)=3x^2-2$, so $f'(1)=1$. Tangent: $y+1=1(x-1)\Rightarrow y=x-2$. It cuts the x-axis when $0=x-2$, i.e. at $(2,0)$.



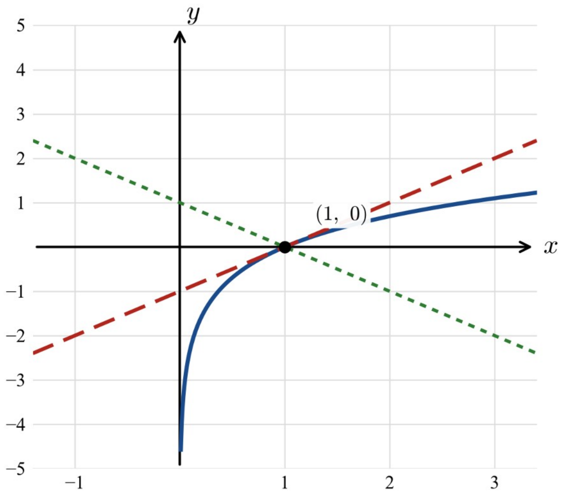
Q3. $y=\sqrt{x}=x^{1/2}$. $f(4)=2$, point $(4,2)$. $f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$, so $f'(4)=\frac{1}{4}$. Tangent: $y-2=\frac{1}{4}(x-4)\Rightarrow y=\frac{x}{4}+1$.



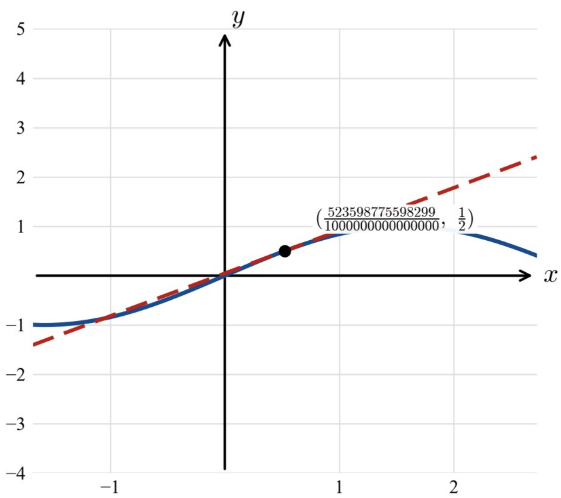
Q4. $y=e^{2x}$. $f(0)=1$, point $(0,1)$. $f'(x)=2e^{2x}$, so $f'(0)=2$. Tangent: $y-1=2(x-0)\Rightarrow y=2x+1$.



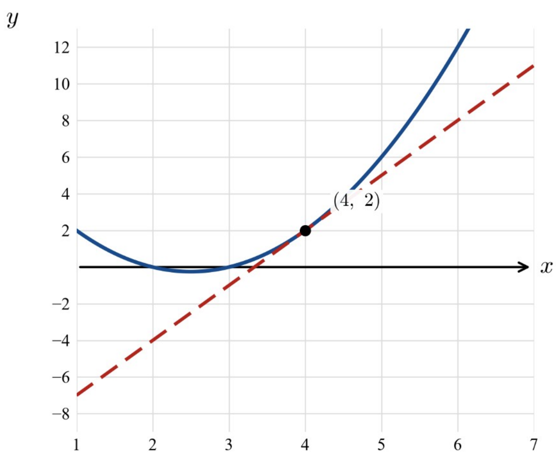
Q5. $y=\log_e x$. $f(1)=0$, point $(1,0)$. $f'(x)=\frac{1}{x}$, so $f'(1)=1$. Tangent: $y=x-1$. Normal gradient $=-1$: $y-0=-1(x-1)\Rightarrow y=-x+1$.



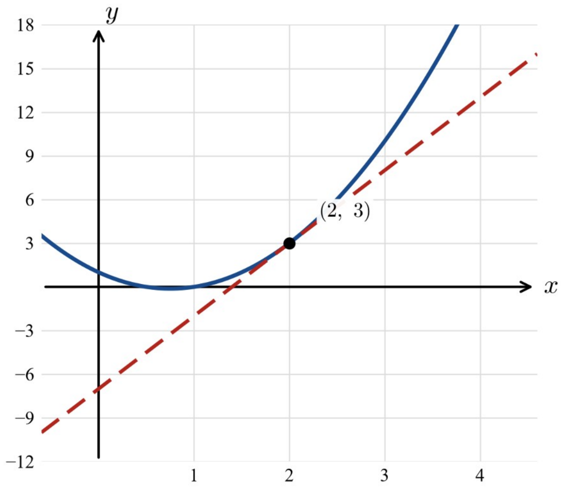
Q6. $y = \sin x$. $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$, point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$. $f'(x) = \cos x$, so $f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$. Tangent:
$y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) \Rightarrow y = \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\pi}{12} + \frac{1}{2}$.



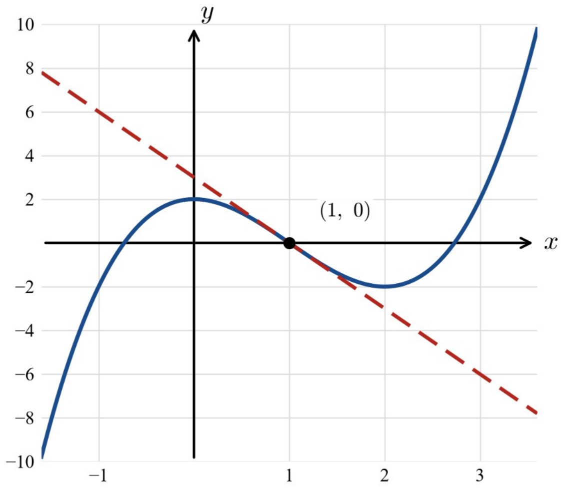
Q7. $y = x^2 - 5x + 6$. $f(4) = 16 - 20 + 6 = 2$. $f'(x) = 2x - 5$, $f'(4) = 3$. Tangent: $y - 2 = 3(x - 4) \Rightarrow y = 3x - 10$.



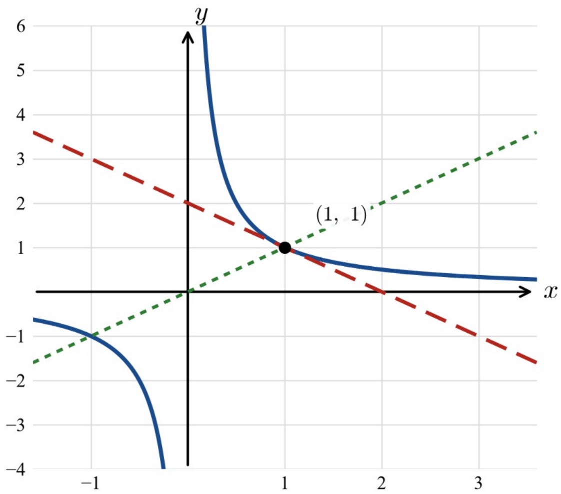
Q8. $y = 2x^2 - 3x + 1$. $f(2) = 8 - 6 + 1 = 3$. $f'(x) = 4x - 3$, $f'(2) = 5$. Tangent: $y - 3 = 5(x - 2) \Rightarrow y = 5x - 7$.



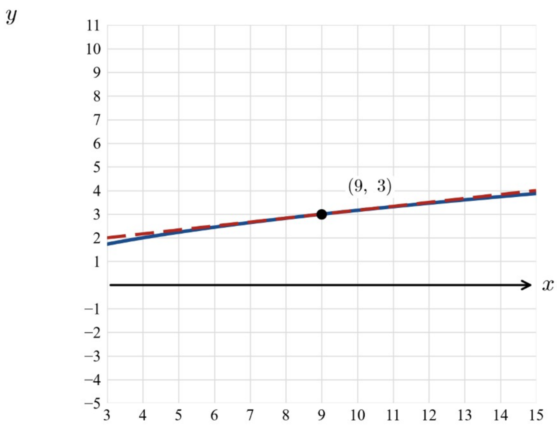
Q9. $y=x^3-3x^2+2$. $f(1)=1-3+2=0$. $f'(x)=3x^2-6x$, $f'(1)=-3$. Tangent: $y-0=-3(x-1) \Rightarrow y=-3x+3$.



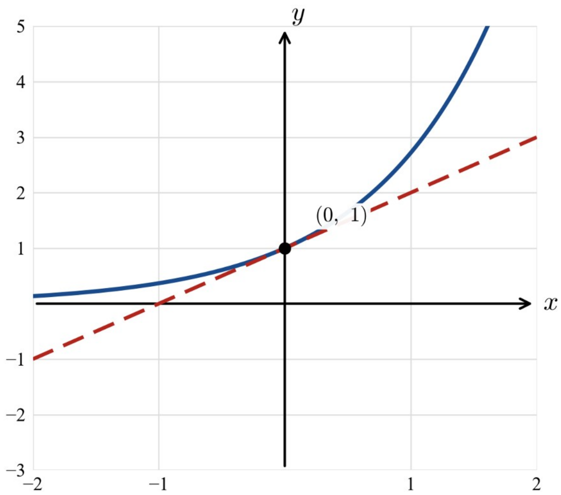
Q10. $y=\frac{1}{x}=x^{-1}$. $f(1)=1$. $f'(x)=-x^{-2}$, $f'(1)=-1$. Tangent: $y-1=-1(x-1) \Rightarrow y=-x+2$. Normal gradient $=1$: $y-1=1(x-1) \Rightarrow y=x$.



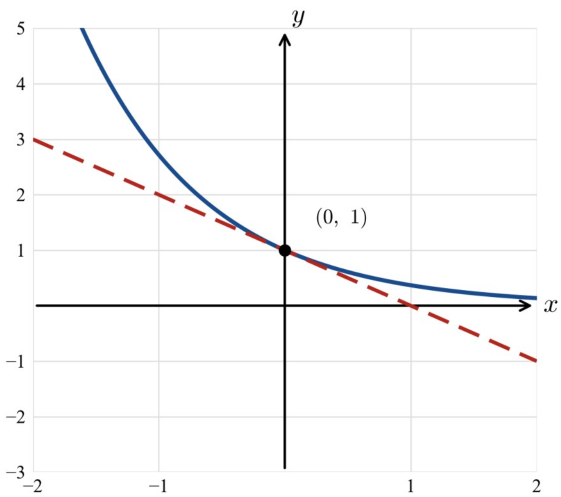
Q11. $y=\sqrt{x}$. $f(9)=3$. $f'(x)=\frac{1}{2\sqrt{x}}$, $f'(9)=\frac{1}{6}$. Tangent: $y-3=\frac{1}{6}(x-9) \Rightarrow y=\frac{x}{6}+\frac{3}{2}$.



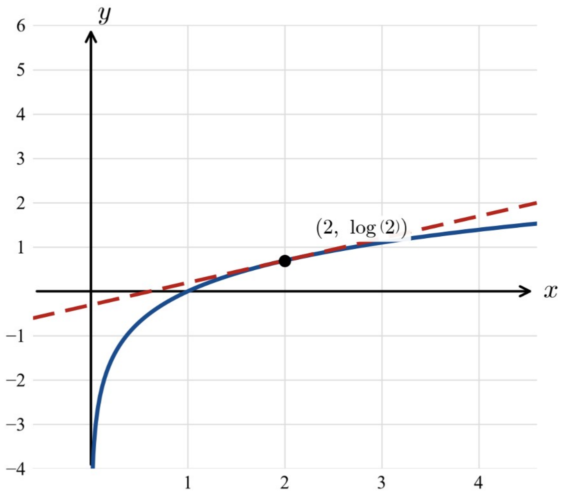
Q12. $y = e^x$. $f(0) = 1$. $f'(x) = e^x$, $f'(0) = 1$. Tangent: $y - 1 = 1(x - 0) \Rightarrow y = x + 1$.



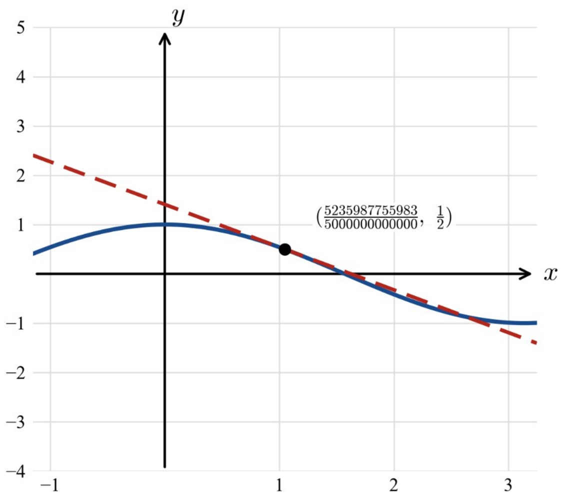
Q13. $y = e^{-x}$. $f(0) = 1$. $f'(x) = -e^{-x}$, $f'(0) = -1$. Tangent: $y - 1 = -1(x - 0) \Rightarrow y = -x + 1$.



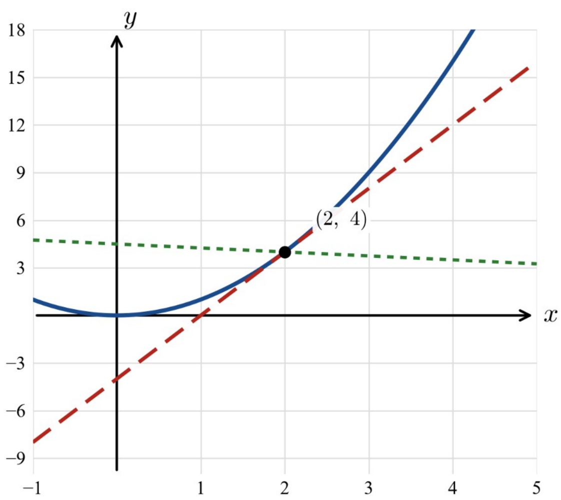
Q14. $y = \log_e x$. $f(2) = \log_e 2$. $f'(x) = \frac{1}{x}$, $f'(2) = \frac{1}{2}$. Tangent: $y - \log_e 2 = \frac{1}{2}(x - 2) \Rightarrow y = \frac{x}{2} - 1 + \log_e 2$.



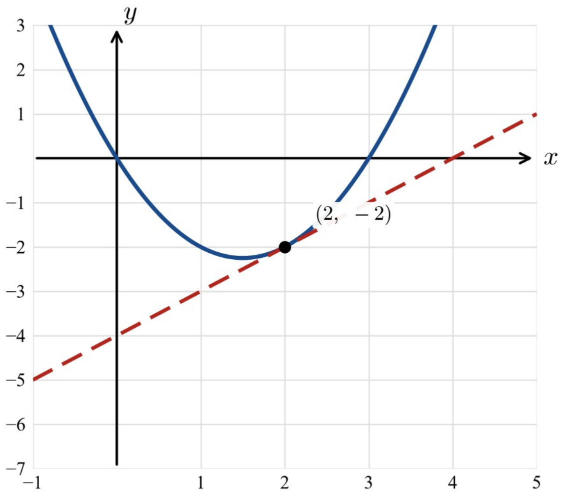
Q15. $y=\cos x$. $f\left(\frac{\pi}{3}\right)=\frac{1}{2}$. $f'(x)=-\sin x$, $f'\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$. Tangent: $y-\frac{1}{2}=-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)\Rightarrow y=-\frac{\sqrt{3}}{2}x+\frac{1}{2}+\frac{\sqrt{3}\pi}{6}$
.



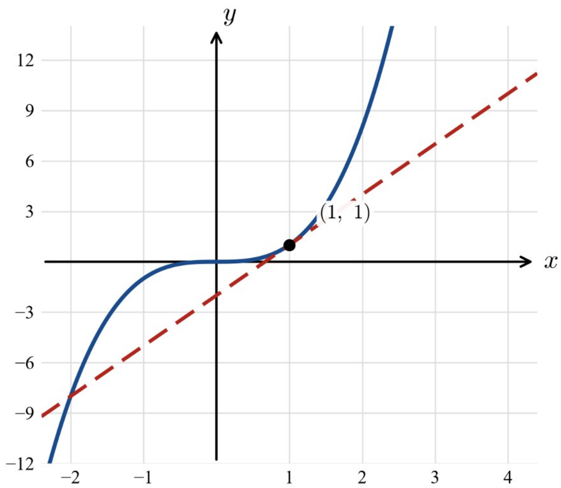
Q16. $y=x^2$. At $(2,4)$, $f'(x)=2x$ gives $f'(2)=4$. Normal gradient $=-\frac{1}{4}$: $y-4=-\frac{1}{4}(x-2)\Rightarrow y=-\frac{x}{4}+\frac{9}{2}$.



Q17. $y=x^2-3x$. $f'(x)=2x-3$. Set $f'(x)=1$: $2x-3=1\Rightarrow x=2$. Then $f(2)=4-6=-2$, so the point is $(2,-2)$. Tangent: $y+2=1(x-2)\Rightarrow y=x-4$.

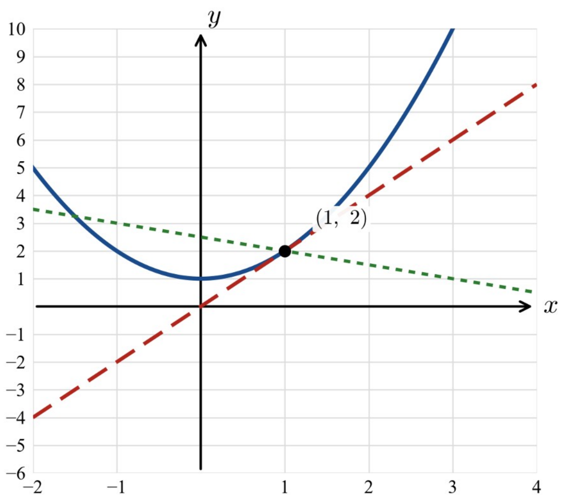


Q18. $y=x^3$. At $(1,1)$, $f'(x)=3x^2$ gives $f'(1)=3$. Tangent: $y-1=3(x-1)\Rightarrow y=3x-2$. It meets the curve where $x^3=3x-2$, i.e. $x^3-3x+2=0$. Since $x=1$ is a (repeated) root, factorise: $x^3-3x+2=(x-1)^2(x+2)$. The other root is $x=-2$, giving the second point $(-2,-8)$.

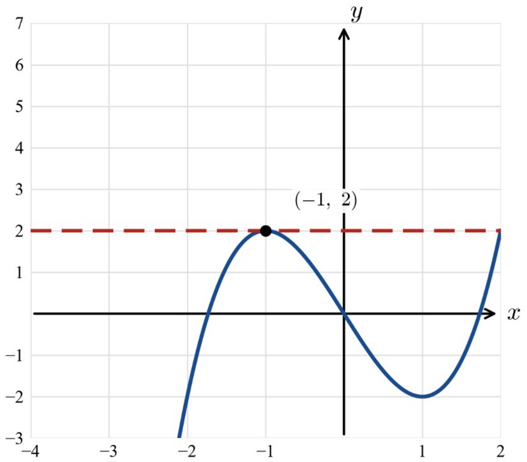


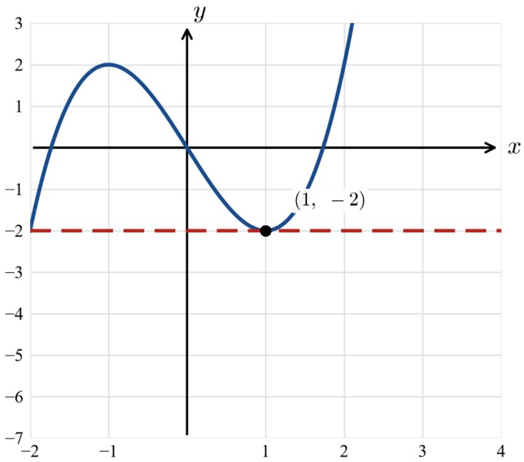
Q19. $y=x^2+1$. $f(1)=2$, point $(1,2)$. $f'(x)=2x$, $f'(1)=2$. Normal gradient $=-\dfrac{1}{2}$:

$y-2=-\dfrac{1}{2}(x-1)\Rightarrow y=-\dfrac{x}{2}+\dfrac{5}{2}$.

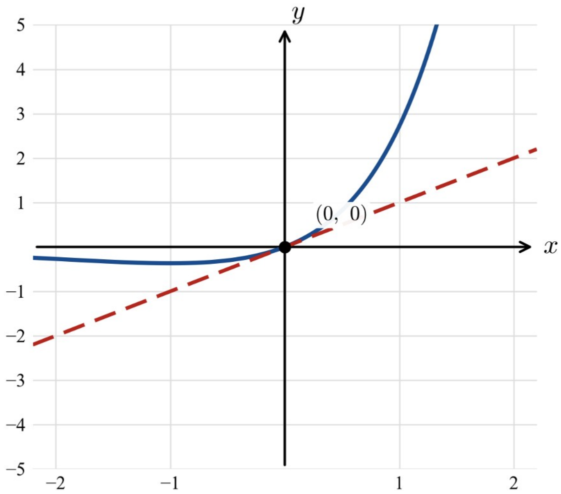


Q20. $y=x^3-3x$. A tangent is parallel to the x-axis where $f'(x)=0$: $3x^2-3=0\Rightarrow x=\pm1$. At $x=-1$: $f(-1)=-1+3=2$, so the tangent is $y=2$. At $x=1$: $f(1)=1-3=-2$, so the tangent is $y=-2$.





Q21. $y = x e^x$. At $x = 0$: $f(0) = 0$, so the point is the origin. By the product rule $f'(x) = e^x + x e^x = (1 + x) e^x$, so $f'(0) = 1$. Tangent: $y - 0 = 1(x - 0)$, i.e. $\therefore y = x$.



Theory

A derivative measures a **rate of change**. This section interprets $f'(x)$ as the rate at which a quantity changes, and distinguishes it from the *average* rate over an interval.

Average rate of change. The average rate of change of $y=f(x)$ over the interval $[a,b]$ is

$$\text{average rate}=\frac{f(b)-f(a)}{b-a}.$$

This is the **gradient of the chord (secant)** joining $(a,f(a))$ and $(b,f(b))$.

Instantaneous rate of change. The instantaneous rate of change of $y=f(x)$ at $x=a$ is the derivative $f'(a)$ — the **gradient of the tangent** to the curve at that point.

Interpreting the rate. In an applied context the derivative carries units of $\frac{\text{units of } y}{\text{units of } x}$. For example, if $h(t)$ is a height in metres and t is time in seconds, then $h'(t)$ is a **velocity** in metres per second (m/s). A **positive** rate means the quantity is increasing; a **negative** rate means it is decreasing; a rate of **zero** means the quantity is momentarily stationary (a maximum, minimum or stationary point of inflection).

Kinematics. For a particle moving in a straight line with position $x(t)$, the **velocity** is the rate of change of position, $v(t)=\frac{dx}{dt}=x'(t)$. The particle is *momentarily at rest* when $v(t)=0$.

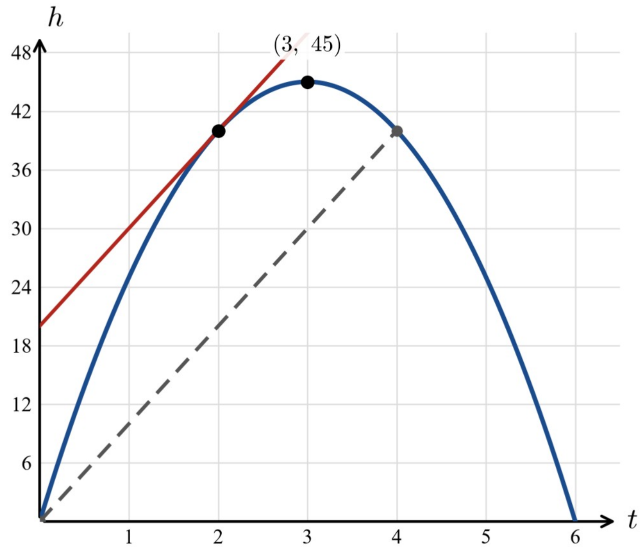
Watch the wording. *"Average rate of change over $[a,b]$"* uses the chord formula above; *"(instantaneous) rate of change at $t=a$"* uses the derivative $f'(a)$. Confusing the two is a common exam error.

Worked examples

Example 1 — scaffolded

A ball is thrown vertically upwards. Its height (metres) after t seconds is $h(t)=30t-5t^2$, for $0\le t\le 6$. (a) Find the average velocity over the first 4 seconds. (b) Find the velocity at $t=2$. (c) Find when the ball is momentarily at rest and its maximum height.

Working	Comment
(a) $\frac{h(4)-h(0)}{4-0}=\frac{(120-80)-0}{4}=\frac{40}{4}=10$ m/s.	Average rate = gradient of the chord over $[0,4]$.
(b) $h'(t)=30-10t$, so $h'(2)=30-20=10$ m/s.	Instantaneous rate = derivative at $t=2$.
(c) At rest when $h'(t)=0$: $30-10t=0\Rightarrow t=3$ s.	Velocity zero at the turning point.
Maximum height $h(3)=90-45=45$ m.	Substitute $t=3$ into h.



Here the tangent at $t = 2$ (red) and the chord over $[0, 4]$ (dashed) happen to have the same gradient, 10 m/s.

Example 2 — scaffolded

A circular oil slick has area $A(r) = \pi r^2$ square metres when its radius is r metres. (a) Find the average rate of change of area as the radius increases from 2 m to 4 m. (b) Find the rate of change of area with respect to the radius when $r = 4$.

Working	Comment
(a) $\dfrac{A(4) - A(2)}{4 - 2} = \dfrac{16\pi - 4\pi}{2} = \dfrac{12\pi}{2} = 6\pi$ m²/m.	Average rate over $[2, 4]$.
(b) $\dfrac{dA}{dr} = 2\pi r$, so at $r = 4$: $\dfrac{dA}{dr} = 8\pi$ m²/m.	Instantaneous rate = derivative at $r = 4$.

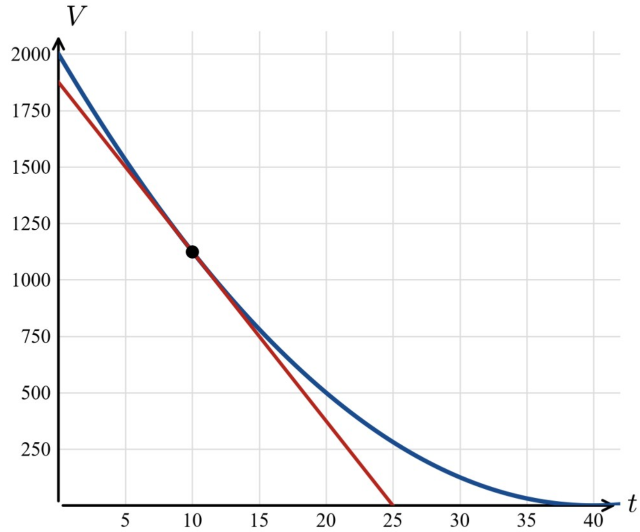
Example 3 — unscaffolded

A tank drains so that the volume of water (litres) after t minutes is $V(t) = 2000\left(1 - \frac{t}{40}\right)^2$, for $0 \le t \le 40$. Find the rate at which water is leaving the tank at $t = 10$ minutes.

$$V'(t) = 2000 \cdot 2\left(1 - \frac{t}{40}\right) \cdot \left(-\frac{1}{40}\right) = -100\left(1 - \frac{t}{40}\right).$$

At $t = 10$: $V'(10) = -100\left(1 - \frac{10}{40}\right) = -100 \cdot \frac{3}{4} = -75$ L/min.

∴ water is leaving at a rate of 75 litres per minute (the negative sign shows the volume is decreasing).



Example 4 — unscaffolded

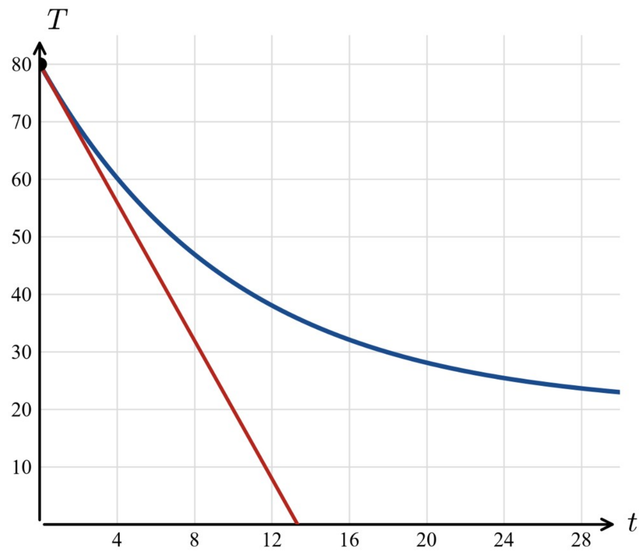
A hot drink cools so that its temperature (°C) after t minutes is $T(t)=20+60e^{-t/10}$. Find the rate of change of temperature at $t=0$ and at $t=10$, giving exact values.

$T'(t)=60\cdot\left(-\frac{1}{10}\right)e^{-t/10}=-6e^{-t/10}$.

At $t=0$: $T'(0)=-6e^{0}=-6$ °C/min.

At $t=10$: $T'(10)=-6e^{-1}=-\frac{6}{e}$ °C/min ≈ -2.21 °C/min.

∴ the drink is cooling fastest at the start (6 °C/min) and more slowly ($\frac{6}{e}$ °C/min) after 10 minutes.



Practice questions

Scaffolded

Q1. A particle moves in a straight line with position $x(t)=t^3-6t^2+9t$ metres after t seconds. (a) Find the average velocity over $[0,3]$. (b) Find the velocity function $v(t)$. (c) Find the times when the particle is momentarily at rest. (d) Sketch the position–time graph for $0\le t\le 4$.

Q2. A ball has height $h(t)=40t-5t^2$ metres after t seconds. (a) Find the average velocity over $[1,3]$. (b) Find $h'(t)$ and the velocity at $t=2$. (c) Find when the ball reaches its maximum height, and that height.

Q3. A square metal plate has side length x cm and area $A(x)=x^2$ cm². (a) Find the average rate of change of area as x increases from 3 to 5. (b) Find $\frac{dA}{dx}$. (c) Find the rate of change of area when $x=5$.

Q4. A population is modelled by $P(t)=1000e^{t/20}$, t in years. (a) Find $P'(t)$. (b) Find the rate of growth at $t=0$. (c) Find the exact rate of growth at $t=20$.

Q5. A spherical balloon has volume $V(r)=\frac{4}{3}\pi r^3$ cm³ when its radius is r cm. (a) Find $\frac{dV}{dr}$. (b) Find the rate of change of volume with respect to the radius when $r=3$.

Q6. A company's daily profit is $P(x)=-x^2+50x-200$ dollars when x items are sold. (a) Find the marginal profit $P'(x)$. (b) Find $P'(10)$ and interpret it. (c) Find the number of items that maximises the profit.

Unscaffolded

Q7. For $x(t)=2t^2-8t+3$ m, find the average velocity over $[0,4]$, the velocity at $t=1$, and when the particle is at rest.

Q8. For $h(t)=25t-5t^2$ m, find the velocity at $t=0$ and $t=3$, and the maximum height.

Q9. A quantity is $V(t)=100-4t^2$ for $0\le t\le 5$. Find the average rate of change over $[0,5]$ and the instantaneous rate at $t=2$.

Q10. The cost is $C(x)=x^2-10x+30$. Find the marginal cost $C'(3)$, and the value of x at which the cost is a minimum.

Q11. For $P(t)=5000e^{-t/10}$, find the rate of change at $t=0$ and the exact rate at $t=10$.

Q12. A circle's area is $A(r)=\pi r^2$. Find the rate of change of area with respect to r when $r=6$.

Q13. A particle has position $s(t)=t^3-3t^2$ m. Find the velocity at $t=1$ and $t=2$, and the times when it is at rest.

Q14. A projectile has height $h(t)=-4.9t^2+20t$ m. Find the velocity at $t=1$ and the exact time at which it reaches its maximum height.

Q15. For $T(t)=15+50e^{-t/5}$ °C, find the rate of change of temperature at $t=0$ and the exact rate at $t=5$.

Q16. A quantity is $N(t)=-t^2+40t+200$. Find the average rate of change over $[0,10]$, the instantaneous rate at $t=5$, and the maximum value of N.

Q17. For a sphere $V(r)=\frac{4}{3}\pi r^3$, find the rate of change of volume with respect to r when $r=2$.

Q18. A particle has position $x(t)=t^3-9t^2+24t$ m. Find the velocity at $t=0$ and the times when it is momentarily at rest.

Q19. The amount of a chemical is $W(t)=\frac{60}{t+1}$ grams, $t\ge 0$. Find $W'(t)$, and the rate of change at $t=0$ and $t=2$.

Q20. A drone's height is $h(t)=t^2-8t+20$ m for $0\le t\le 8$. Show that the average rate of change over $[2,6]$ is 0, find the instantaneous rate at $t=3$, and hence explain the difference.

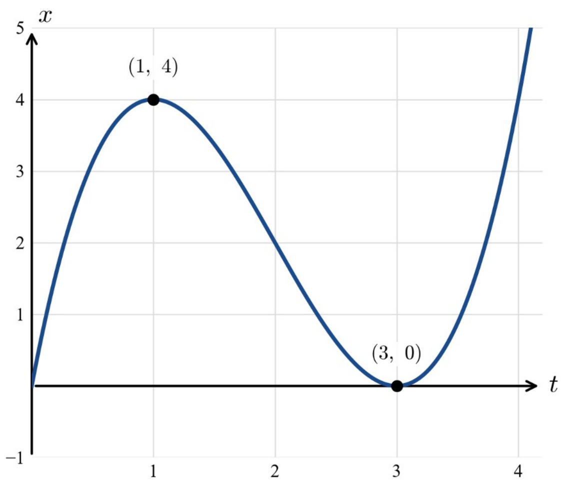
Q21. A drug's concentration is $D(t)=100e^{-t/8}$ mg/L. Find the rate of change at $t=0$ and the exact rate at $t=8$.

Answers

Q1. (a) 0 m/s; (b) $v(t)=3t^2-12t+9$; (c) $t=1$ and $t=3$ s. **Q2.** (a) 20 m/s; (b) $h'(t)=40-10t$, $h'(2)=20$ m/s; (c) $t=4$ s, 80 m. **Q3.** (a) 8 cm²/cm; (b) $2x$; (c) 10 cm²/cm. **Q4.** (a) $P'(t)=50e^{t/20}$; (b) 50 per year; (c) $50e$ per year. **Q5.** (a) $4\pi r^2$; (b) 36π cm³/cm. **Q6.** (a) $P'(x)=50-2x$; (b) $P'(10)=30$ (profit rising by \$30 per extra item); (c) 25 items. **Q7.** avg 0 m/s; $v(1)=-4$ m/s; at rest at $t=2$ s. **Q8.** $v(0)=25$ m/s, $v(3)=-5$ m/s; max height $\frac{125}{4}=31.25$ m. **Q9.** average -20; instantaneous -16. **Q10.** $C'(3)=-4$; minimum at $x=5$. **Q11.** -500; exact $-\frac{500}{e}$. **Q12.** 12π. **Q13.** $v(1)=-3$ m/s, $v(2)=0$; at rest at $t=0$ and $t=2$ s. **Q14.** $v(1)=\frac{51}{5}=10.2$ m/s; maximum at $t=\frac{100}{49}$ s. **Q15.** -10 °C/min; exact $-\frac{10}{e}$ °C/min. **Q16.** average 30; instantaneous 30; maximum $N=600$. **Q17.** 16π cm³/cm. **Q18.** $v(0)=24$ m/s; at rest at $t=2$ and $t=4$ s. **Q19.** $W'(t)=-\frac{60}{(t+1)^2}$; $W'(0)=-60$, $W'(2)=-\frac{20}{3}$ g/min. **Q20.** average 0; instantaneous -2 m/s; the height falls then rises back to the same value, so the *average* is 0 even though it is moving at $t=3$. **Q21.** $-\frac{25}{2}$ mg/L per hour; exact $-\frac{25}{2e}$.

Fully-worked solutions

Q1. $x(t)=t^3-6t^2+9t$. (a) $\frac{x(3)-x(0)}{3-0}=\frac{0-0}{3}=0$ m/s (since $x(3)=27-54+27=0$). (b) $v(t)=x'(t)=3t^2-12t+9$. (c) $v(t)=0\Rightarrow 3(t^2-4t+3)=3(t-1)(t-3)=0$, so $t=1$ or $t=3$ s. (d) Turning points $(1,4)$ (max) and $(3,0)$ (min):



Q2. $h(t)=40t-5t^2$. (a) $\frac{h(3)-h(1)}{2}=\frac{75-35}{2}=20$ m/s. (b) $h'(t)=40-10t$, $h'(2)=20$ m/s. (c) $h'(t)=0\Rightarrow t=4$ s; $h(4)=160-80=80$ m.

Q3. $A(x)=x^2$. (a) $\frac{A(5)-A(3)}{2}=\frac{25-9}{2}=8$ cm²/cm. (b) $\frac{dA}{dx}=2x$. (c) at $x=5$: $2(5)=10$ cm²/cm.

Q4. $P(t)=1000e^{t/20}$. (a) $P'(t)=1000\cdot\frac{1}{20}e^{t/20}=50e^{t/20}$. (b) $P'(0)=50$ per year. (c) $P'(20)=50e^1=50e$ per year.

Q5. $V(r)=\frac{4}{3}\pi r^3$. (a) $\frac{dV}{dr}=4\pi r^2$. (b) at $r=3$: $4\pi(9)=36\pi$ cm³/cm.

Q6. $P(x) = -x^2 + 50x - 200$. (a) $P'(x) = 50 - 2x$. (b) $P'(10) = 30$: near $x = 10$ each extra item sold increases profit by about \$30. (c) $P'(x) = 0 \Rightarrow x = 25$ items.

Q7. $x(t) = 2t^2 - 8t + 3$. Average over $[0, 4]$: $\dfrac{x(4) - x(0)}{4} = \dfrac{3 - 3}{4} = 0$ m/s. $v(t) = 4t - 8$, so $v(1) = -4$ m/s. At rest: $4t - 8 = 0 \Rightarrow t = 2$ s.

Q8. $h(t) = 25t - 5t^2$. $h'(t) = 25 - 10t$, so $v(0) = 25$ m/s and $v(3) = -5$ m/s. Maximum when $h'(t) = 0 \Rightarrow t = \dfrac{5}{2}$; $h\left(\dfrac{5}{2}\right) = 25 \cdot \dfrac{5}{2} - 5 \cdot \dfrac{25}{4} = \dfrac{125}{4} = 31.25$ m.

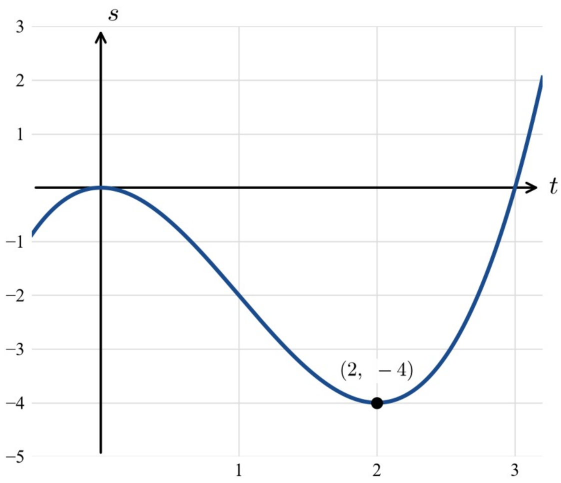
Q9. $V(t) = 100 - 4t^2$. Average over $[0, 5]$: $\dfrac{V(5) - V(0)}{5} = \dfrac{0 - 100}{5} = -20$. $V'(t) = -8t$, so $V'(2) = -16$.

Q10. $C(x) = x^2 - 10x + 30$. $C'(x) = 2x - 10$, so $C'(3) = -4$. Minimum when $C'(x) = 0 \Rightarrow x = 5$.

Q11. $P(t) = 5000e^{-t/10}$. $P'(t) = 5000 \cdot \left(-\dfrac{1}{10}\right)e^{-t/10} = -500e^{-t/10}$. $P'(0) = -500$; $P'(10) = -500e^{-1} = -\dfrac{500}{e}$.

Q12. $A(r) = \pi r^2 \Rightarrow \dfrac{dA}{dr} = 2\pi r$; at $r = 6$: 12π.

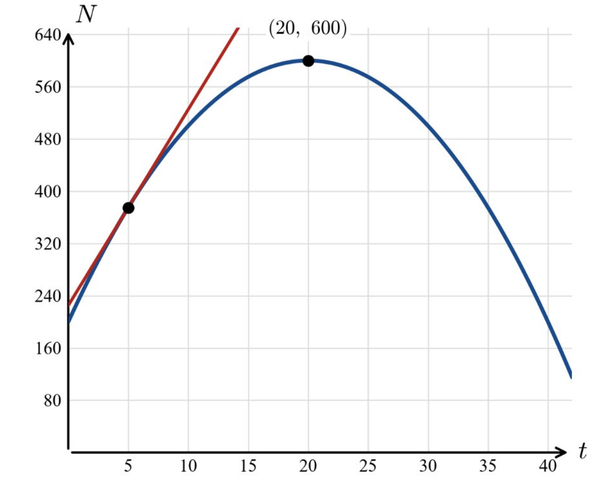
Q13. $s(t) = t^3 - 3t^2$. $v(t) = 3t^2 - 6t = 3t(t - 2)$, so $v(1) = -3$ m/s and $v(2) = 0$. At rest: $t = 0$ and $t = 2$ s.



Q14. $h(t) = -4.9t^2 + 20t = -\dfrac{49}{10}t^2 + 20t$. $h'(t) = -\dfrac{49}{5}t + 20$, so $h'(1) = 20 - \dfrac{49}{5} = \dfrac{51}{5} = 10.2$ m/s. Maximum when $h'(t) = 0 \Rightarrow t = \dfrac{100}{49}$ s.

Q15. $T(t) = 15 + 50e^{-t/5}$. $T'(t) = -10e^{-t/5}$, so $T'(0) = -10$ °C/min and $T'(5) = -10e^{-1} = -\dfrac{10}{e}$ °C/min.

Q16. $N(t) = -t^2 + 40t + 200$. Average over $[0, 10]$: $\dfrac{N(10) - N(0)}{10} = \dfrac{500 - 200}{10} = 30$. $N'(t) = 40 - 2t$, so $N'(5) = 30$. Maximum when $N'(t) = 0 \Rightarrow t = 20$; $N(20) = -400 + 800 + 200 = 600$.

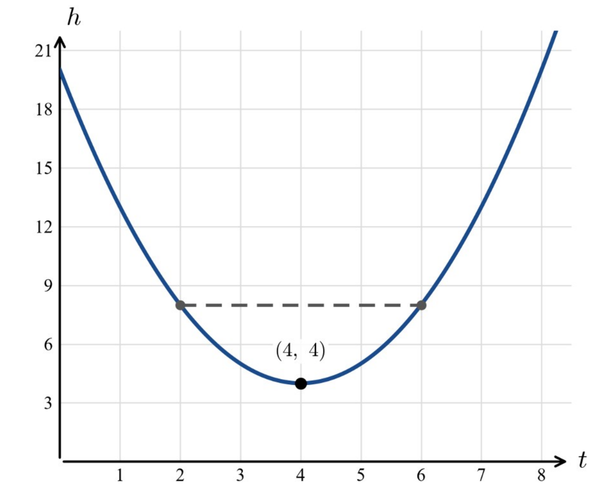


Q17. $V(r)=\frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr}=4\pi r^2$; at $r=2$: 16π cm³/cm.

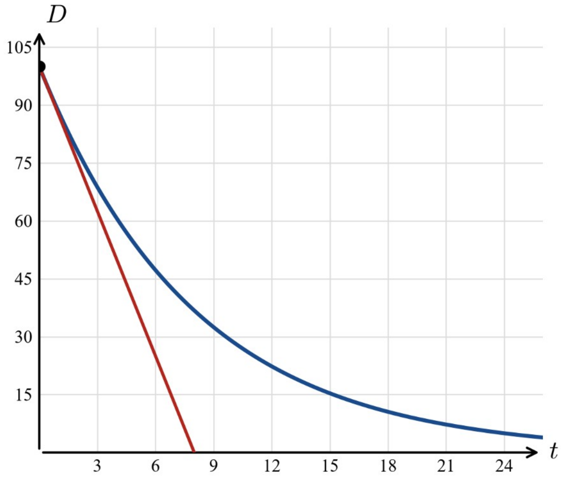
Q18. $x(t)=t^3-9t^2+24t$. $v(t)=3t^2-18t+24=3(t^2-6t+8)=3(t-2)(t-4)$, so $v(0)=24$ m/s. At rest: $t=2$ and $t=4$ s.

Q19. $W(t)=\frac{60}{t+1}=60(t+1)^{-1}$. $W'(t)=-60(t+1)^{-2}=-\frac{60}{(t+1)^2}$. $W'(0)=-60$ g/min; $W'(2)=-\frac{60}{9}=-\frac{20}{3}$ g/min.

Q20. $h(t)=t^2-8t+20$. Average over $[2,6]$: $\frac{h(6)-h(2)}{4}=\frac{8-8}{4}=0$. $h'(t)=2t-8$, so $h'(3)=-2$ m/s. The drone descends to its lowest point at $t=4$ then climbs back to the same height, so although it is moving at every instant (e.g. -2 m/s at $t=3$), its **average** rate of change over $[2,6]$ is 0.



Q21. $D(t)=100e^{-t/8}$. $D'(t)=100\cdot\left(-\frac{1}{8}\right)e^{-t/8}=-\frac{25}{2}e^{-t/8}$. $D'(0)=-\frac{25}{2}$ mg/L per hour; $D'(8)=-\frac{25}{2}e^{-1}=-\frac{25}{2e}$.


D
105
90
75
60
45
30
15
3
6
9
12
15
18
21
24
t

Theory

A **stationary point** of a function f is a point on its graph where the gradient is zero — that is, where $f'(x)=0$. At a stationary point the tangent to the curve is **horizontal**.

Finding stationary points.

1. Differentiate to obtain $f'(x)$.
2. Solve $f'(x)=0$ for x (the x-coordinates of the stationary points).
3. Substitute each solution back into $f(x)$ (not f') to find the corresponding y-coordinate.

Increasing and decreasing. On an interval, f is **strictly increasing** where $f'(x)>0$ and **strictly decreasing** where $f'(x)<0$. The stationary points divide the domain into intervals on which f' keeps a constant sign, so to describe where f is increasing or decreasing we test the sign of f' on each interval — usually set out in a **gradient (sign) table**:

x	$\cdots$	x_1	$\cdots$	x_2	$\cdots$
$f'(x)$	+	0	−	0	+
slope	↗	—	↘	—	↗

A note on inflection. Most stationary points are turning points (a maximum or a minimum), where f' **changes sign**. Occasionally $f'(x)=0$ but f' does **not** change sign — then the point is a **stationary point of inflection** (the curve flattens but keeps going the same way). Deciding which type a stationary point is comes in Section 9D; here we locate the stationary points and describe the increasing/decreasing behaviour.

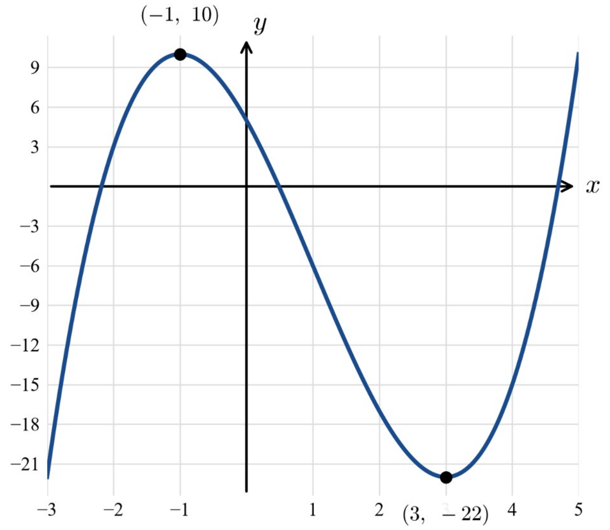
To sketch using stationary points: find the stationary points and the axis intercepts, mark them, note where the curve is increasing or decreasing, then draw a smooth curve.

Worked examples

Example 1 — scaffolded

Find the coordinates of the stationary points of $f(x)=x^3-3x^2-9x+5$ and state where f is increasing.

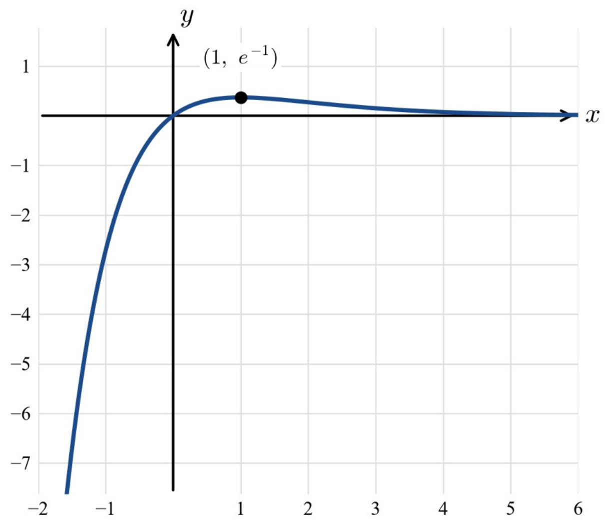
Working	Comment
$f'(x)=3x^2-6x-9$	Differentiate term by term.
$3x^2-6x-9=3(x^2-2x-3)=3(x-3)(x+1)$	Factor out 3, then factorise.
$f'(x)=0 \Rightarrow x=3$ or $x=-1$	Null factor law gives the x-coordinates.
$f(-1)=(-1)^3-3(-1)^2-9(-1)+5=10$; $f(3)=27-27-27+5=-22$	Substitute into f (not f') for the y-coordinates.
Stationary points $(-1,10)$ and $(3,-22)$.	State the coordinates.
$f'(x)>0$ for $x<-1$ and for $x>3$.	Test the sign of $f'=3(x-3)(x+1)$ on each interval: f is increasing on $(-\infty,-1)$ and $(3,\infty)$, decreasing on $(-1,3)$.



Example 2 — scaffolded

For $f(x)=xe^{-x}$, find the stationary point and state where f is increasing and decreasing.

Working	Comment
$f'(x)=1\cdot e^{-x}+x\cdot(-e^{-x})=(1-x)e^{-x}$	Product rule; factor out e^{-x}.
$e^{-x}>0$ always, so $f'(x)=0\Rightarrow 1-x=0\Rightarrow x=1$	Only the bracket can be zero.
$f(1)=1\cdot e^{-1}=e^{-1}$	Substitute to get the y-coordinate.
Stationary point $(1,e^{-1})$.	State it.
Since $e^{-x}>0$, the sign of f' is the sign of $1-x$: $f'>0$ for $x<1$, $f'<0$ for $x>1$.	f is increasing on $(-\infty,1)$ and decreasing on $(1,\infty)$.



Example 3 — unscaffolded

Find the stationary points of $f(x)=x^4-4x^3$ and describe the behaviour of f.

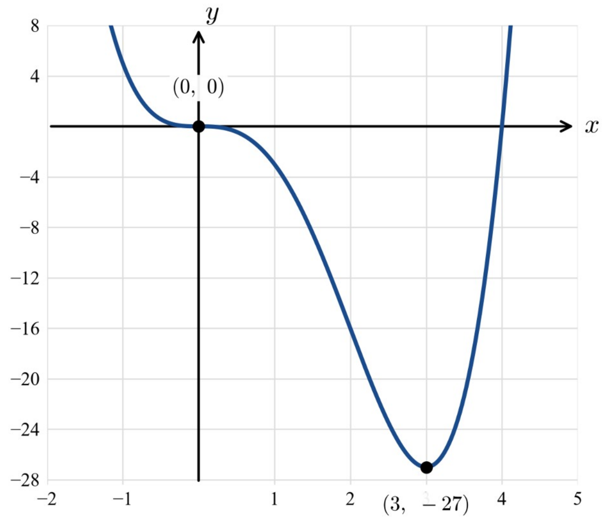
$f'(x)=4x^3-12x^2=4x^2(x-3)$.

$f'(x)=0\Rightarrow x=0$ or $x=3$.

$f(0)=0$ and $f(3)=81-108=-27$, giving stationary points $(0,0)$ and $(3,-27)$.

The factor $4x^2 \geq 0$, so the sign of f' is the sign of $x-3$: $f'<0$ for $x<3$ (with $x \neq 0$) and $f'>0$ for $x>3$. Thus f is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

$\therefore$ at $x=0$ the gradient is zero but does not change sign — a **stationary point of inflection** — while $(3, -27)$ is a minimum.



Example 4 — unscaffolded

Find the stationary point of $f(x) = x^2 - 8\log_e x$ and state its nature by testing the gradient.

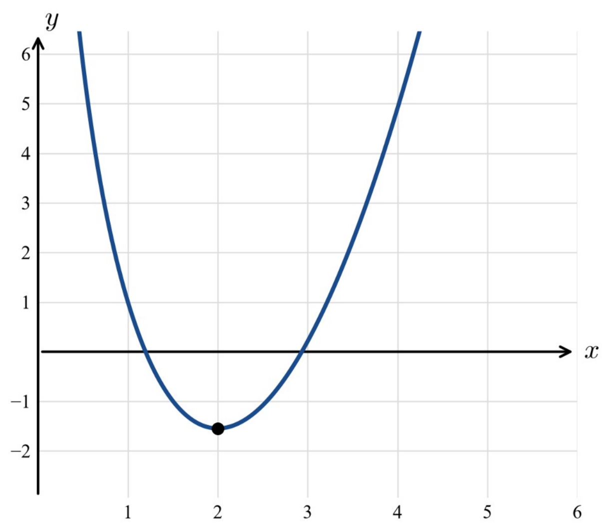
Domain: $x>0$. $f'(x) = 2x - \frac{8}{x} = \frac{2x^2-8}{x} = \frac{2(x^2-4)}{x}$.

For $x>0$, $f'(x)=0 \Rightarrow x^2=4 \Rightarrow x=2$ (rejecting $x=-2$, outside the domain).

$f(2) = 4 - 8\log_e 2$, giving the stationary point $(2, 4-8\log_e 2)$.

For $0<x<2$, $f'(x)<0$; for $x>2$, $f'(x)>0$. The gradient changes from negative to positive.

$\therefore$ the stationary point is a **minimum**; f is decreasing on $(0, 2)$ and increasing on $(2, \infty)$.



Practice questions

Scaffolded

Q1. For $f(x)=x^3-6x^2+9x-2$: (a) find $f'(x)$; (b) solve $f'(x)=0$; (c) find the coordinates of the stationary points; (d) sketch the graph.

Q2. For $f(x)=x^3-3x$: (a) find the stationary points; (b) state where f is increasing and where it is decreasing; (c) sketch the graph.

Q3. For $f(x)=-x^2+4x+1$: (a) find the stationary point; (b) explain, using f', why it is a maximum; (c) sketch the graph.

Q4. For $f(x)=x^3-12x+3$: (a) find $f'(x)$ and solve $f'(x)=0$; (b) find the coordinates of the stationary points; (c) state the intervals where f is decreasing.

Q5. For $f(x)=x^4-2x^2$: (a) show that $f'(x)=4x(x-1)(x+1)$; (b) find the three stationary points; (c) sketch the graph.

Q6. For $f(x)=2x^3-9x^2+12x$: (a) find the stationary points; (b) construct a gradient table; (c) sketch the graph.

Unscaffolded

For each function, find the coordinates of all stationary points and state the intervals on which the function is increasing and decreasing.

Q7. $f(x)=x^3-3x^2+4$

Q8. $f(x)=x^3+3x^2-9x$

Q9. $f(x)=-x^3+3x^2$

Q10. $f(x)=x^4-8x^2$

Q11. $f(x)=3x^4-4x^3$

Q12. $f(x)=x^3-6x^2+12x-5$

Q13. $f(x)=xe^x$

Q14. $f(x)=x^2e^{-x}$

Q15. $f(x)=e^x-x$

Q16. $f(x)=x-\log_e x$

Q17. $f(x)=\dfrac{\log_e x}{x}$

Q18. $f(x)=\dfrac{x}{x^2+1}$

Q19. $f(x)=(x-1)(x-4)^2$

Q20. $f(x)=2x^3-3x^2-36x$. Hence describe the behaviour of f as x increases from $-\infty$ to ∞.

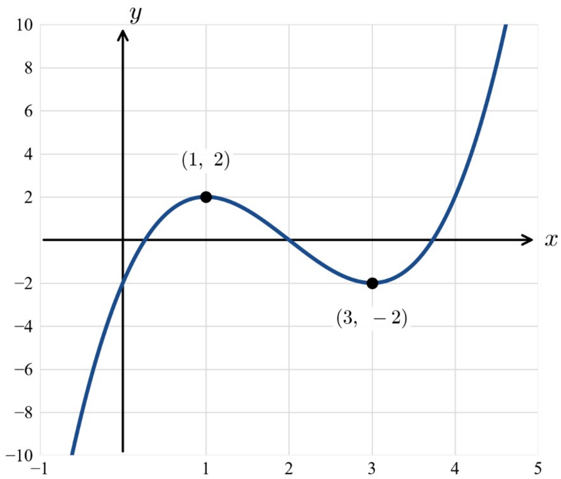
Q21. For $f(x)=x^3-3kx$, where $k>0$, show that the stationary points are $(\sqrt{k},-2k\sqrt{k})$ and $(-\sqrt{k},2k\sqrt{k})$.

Answers

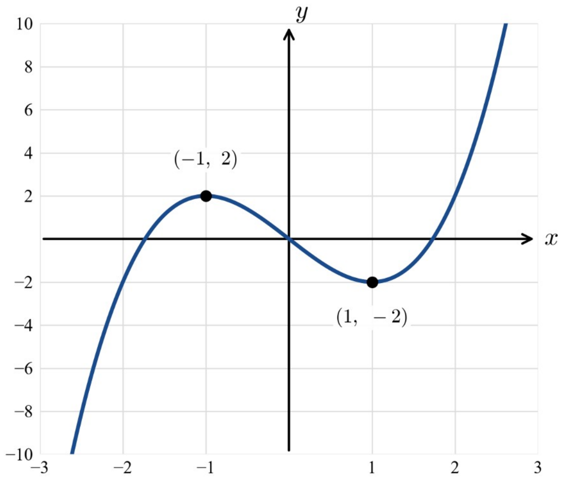
Q1. SPs $(1,2)$, $(3,-2)$. **Q2.** SPs $(-1,2)$, $(1,-2)$; increasing on $(-\infty,-1)$ and $(1,\infty)$, decreasing on $(-1,1)$. **Q3.** SP $(2,5)$, a maximum. **Q4.** SPs $(-2,19)$, $(2,-13)$; decreasing on $(-2,2)$. **Q5.** SPs $(-1,-1)$, $(0,0)$, $(1,-1)$. **Q6.** SPs $(1,5)$, $(2,4)$. **Q7.** $(0,4)$, $(2,0)$; increasing on $(-\infty,0)$ and $(2,\infty)$, decreasing on $(0,2)$. **Q8.** $(-3,27)$, $(1,-5)$; increasing on $(-\infty,-3)$ and $(1,\infty)$, decreasing on $(-3,1)$. **Q9.** $(0,0)$, $(2,4)$; decreasing on $(-\infty,0)$ and $(2,\infty)$, increasing on $(0,2)$. **Q10.** $(-2,-16)$, $(0,0)$, $(2,-16)$; decreasing on $(-\infty,-2)$ and $(0,2)$, increasing on $(-2,0)$ and $(2,\infty)$. **Q11.** $(0,0)$ (stationary point of inflection), $(1,-1)$; decreasing on $(-\infty,1)$, increasing on $(1,\infty)$. **Q12.** $(2,3)$, a stationary point of inflection; f is increasing for all x (except momentarily stationary at $x=2$). **Q13.** $(-1,-e^{-1})$; decreasing on $(-\infty,-1)$, increasing on $(-1,\infty)$. **Q14.** $(0,0)$, $(2,4e^{-2})$; decreasing on $(-\infty,0)$ and $(2,\infty)$, increasing on $(0,2)$. **Q15.** $(0,1)$; decreasing on $(-\infty,0)$, increasing on $(0,\infty)$. **Q16.** $(1,1)$; decreasing on $(0,1)$, increasing on $(1,\infty)$ (domain $x>0$). **Q17.** (e,e^{-1}); increasing on $(0,e)$, decreasing on (e,∞) (domain $x>0$). **Q18.** $\left(-1,-\frac{1}{2}\right)$, $\left(1,\frac{1}{2}\right)$; decreasing on $(-\infty,-1)$ and $(1,\infty)$, increasing on $(-1,1)$. **Q19.** $(2,4)$, $(4,0)$; increasing on $(-\infty,2)$ and $(4,\infty)$, decreasing on $(2,4)$. **Q20.** $(-2,44)$, $(3,-81)$; increasing, then decreasing on $(-2,3)$, then increasing. **Q21.** SPs $(\sqrt{k},-2k\sqrt{k})$ and $(-\sqrt{k},2k\sqrt{k})$.

Fully-worked solutions

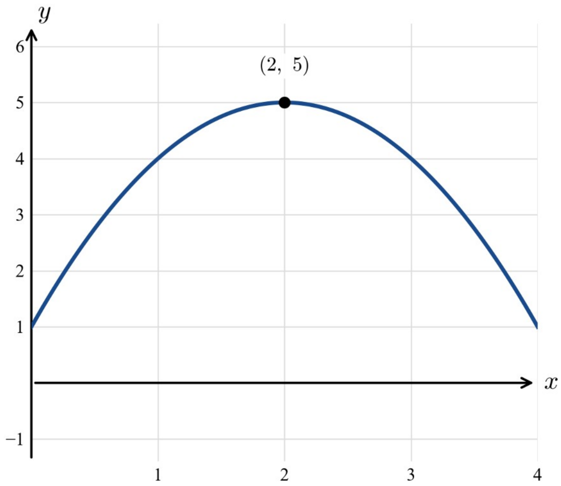
Q1. $f'(x)=3x^2-12x+9=3(x-1)(x-3)$, so $x=1$ or $x=3$. $f(1)=1-6+9-2=2$, $f(3)=27-54+27-2=-2$. SPs $(1,2)$, $(3,-2)$.



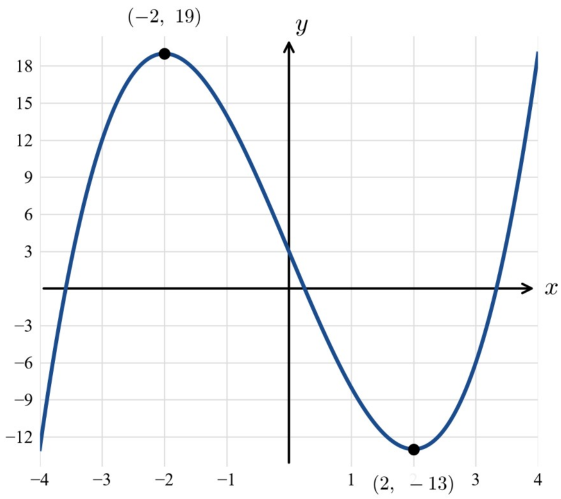
Q2. $f'(x)=3x^2-3=3(x-1)(x+1)$, so $x=\pm1$. $f(-1)=-1+3=2$, $f(1)=1-3=-2$. SPs $(-1,2)$, $(1,-2)$. Since $f'=3(x-1)(x+1)$, $f'>0$ for $x<-1$ and $x>1$ (increasing) and $f'<0$ for $-1<x<1$ (decreasing).



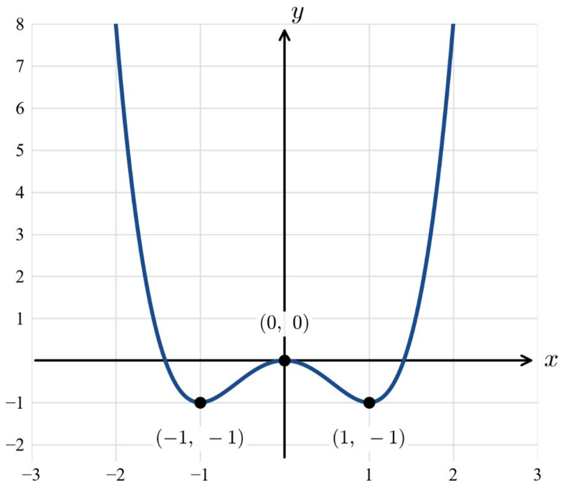
Q3. $f'(x)=-2x+4$, so $x=2$. $f(2)=-4+8+1=5$. SP $(2,5)$. $f'>0$ for $x<2$ and $f'<0$ for $x>2$, so the gradient changes $+\rightarrow-$: a **maximum**.



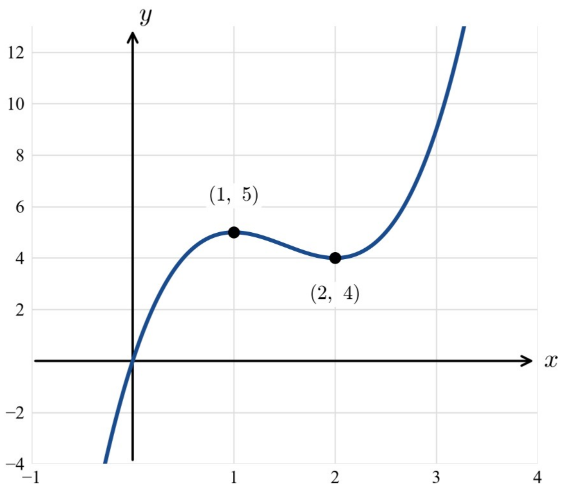
Q4. $f'(x)=3x^2-12=3(x-2)(x+2)$, so $x=\pm 2$. $f(-2)=-8+24+3=19$, $f(2)=8-24+3=-13$. SPs $(-2,19)$, $(2,-13)$. $f'<0$ for $-2<x<2$, so f is decreasing on $(-2,2)$.



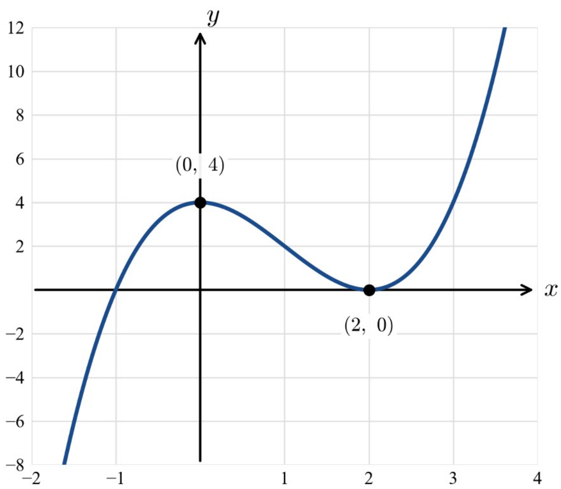
Q5. $f'(x)=4x^3-4x=4x(x^2-1)=4x(x-1)(x+1)$, so $x=0,\pm 1$. $f(0)=0$, $f(\pm 1)=1-2=-1$. SPs $(-1,-1)$, $(0,0)$, $(1,-1)$.



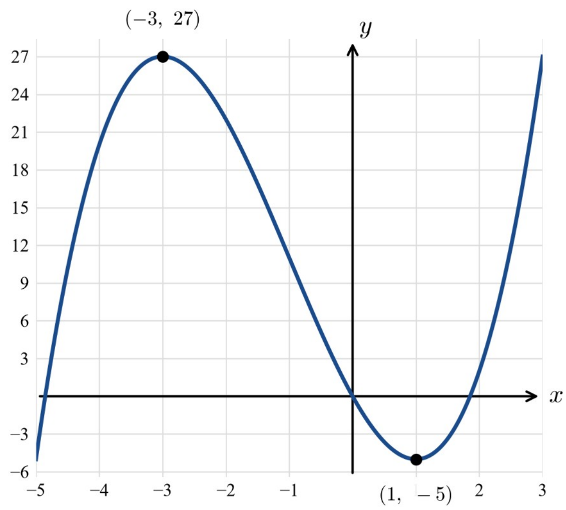
Q6. $f'(x)=6x^2-18x+12=6(x-1)(x-2)$, so $x=1,2$. $f(1)=2-9+12=5$, $f(2)=16-36+24=4$. SPs $(1,5)$, $(2,4)$. Gradient table: $f'>0$ for $x<1$, $f'<0$ for $1<x<2$, $f'>0$ for $x>2$.



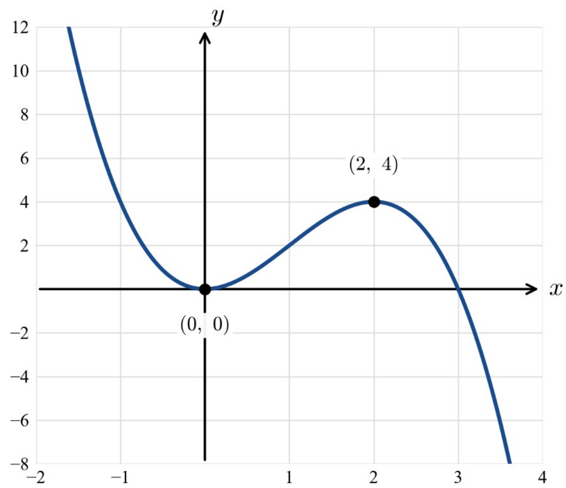
Q7. $f'(x)=3x^2-6x=3x(x-2)$, so $x=0,2$. $f(0)=4$, $f(2)=8-12+4=0$. SPs $(0,4)$, $(2,0)$; increasing on $(-\infty,0)$ and $(2,\infty)$, decreasing on $(0,2)$.



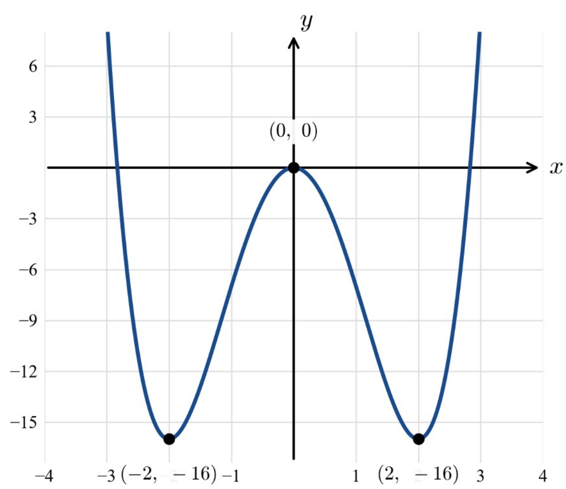
Q8. $f'(x)=3x^2+6x-9=3(x+3)(x-1)$, so $x=-3,1$. $f(-3)=-27+27+27=27$, $f(1)=1+3-9=-5$. SPs $(-3,27)$, $(1,-5)$; increasing on $(-\infty,-3)$ and $(1,\infty)$, decreasing on $(-3,1)$.



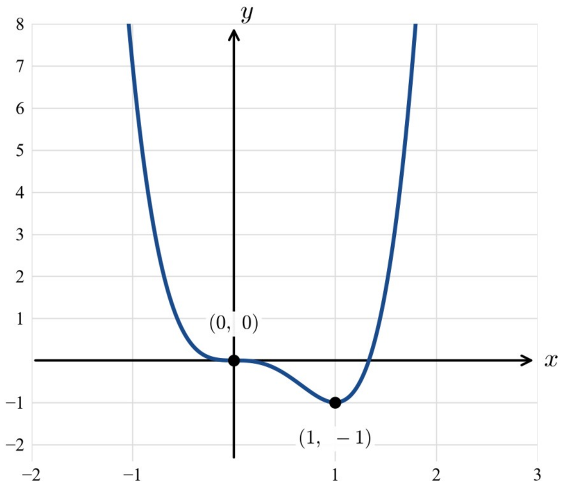
Q9. $f'(x) = -3x^2 + 6x = 3x(2-x)$, so $x = 0, 2$. $f(0) = 0$, $f(2) = -8 + 12 = 4$. SPs $(0,0)$, $(2,4)$; decreasing on $(-\infty, 0)$ and $(2, \infty)$, increasing on $(0, 2)$.



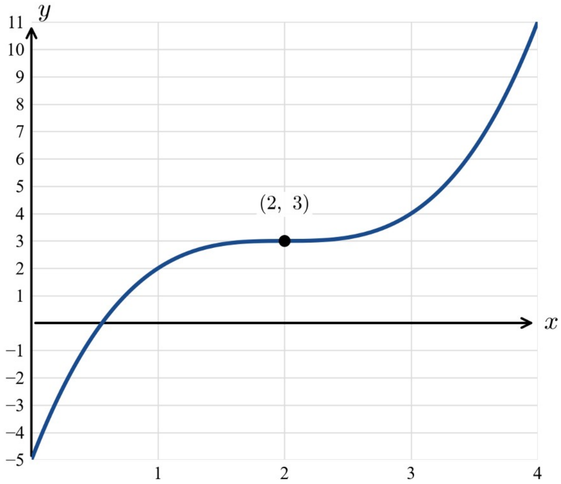
Q10. $f'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$, so $x = 0, \pm 2$. $f(0) = 0$, $f(\pm 2) = 16 - 32 = -16$. SPs $(-2, -16)$, $(0,0)$, $(2, -16)$; decreasing on $(-\infty, -2)$ and $(0, 2)$, increasing on $(-2, 0)$ and $(2, \infty)$.



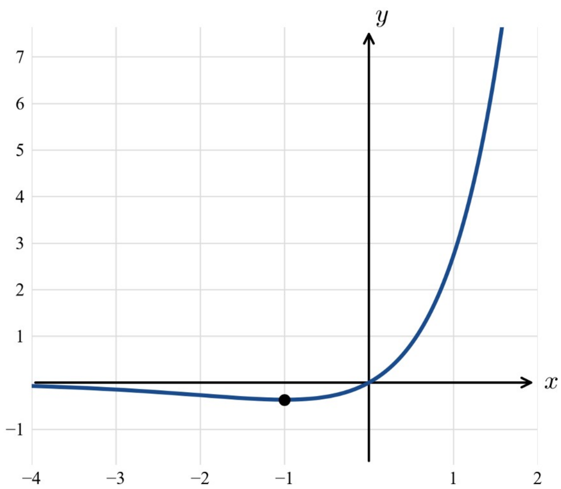
Q11. $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$, so $x = 0, 1$. $f(0) = 0$, $f(1) = 3 - 4 = -1$. Since $12x^2 \geq 0$, the sign of f' is that of $x - 1$: decreasing on $(-\infty, 1)$ (with $x \neq 0$), increasing on $(1, \infty)$. $(0,0)$ is a stationary point of inflection; $(1, -1)$ a minimum.



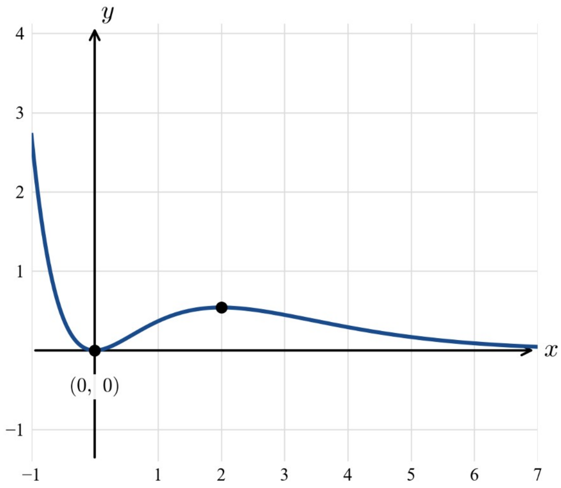
Q12. $f'(x)=3x^2-12x+12=3(x-2)^2\geq 0$, so $x=2$ is the only stationary point and f' never changes sign. $f(2)=8-24+24-5=3$. SP $(2,3)$, a **stationary point of inflection**; f is increasing for all x (momentarily stationary at $x=2$).



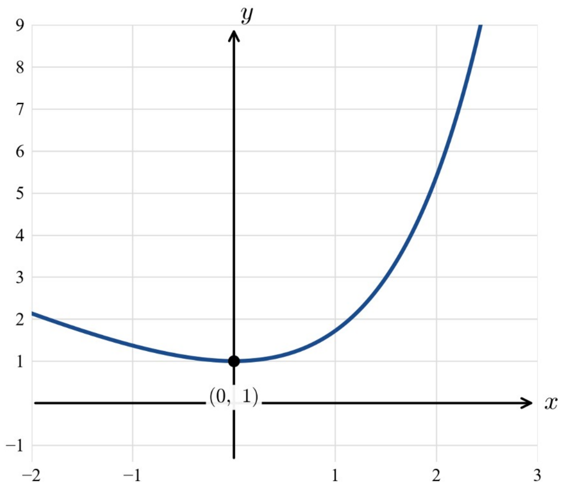
Q13. $f'(x)=e^x+xe^x=(x+1)e^x$. Since $e^x>0$, $f'=0 \Rightarrow x=-1$. $f(-1)=-e^{-1}$. SP $(-1,-e^{-1})$; decreasing on $(-\infty,-1)$, increasing on $(-1,\infty)$.



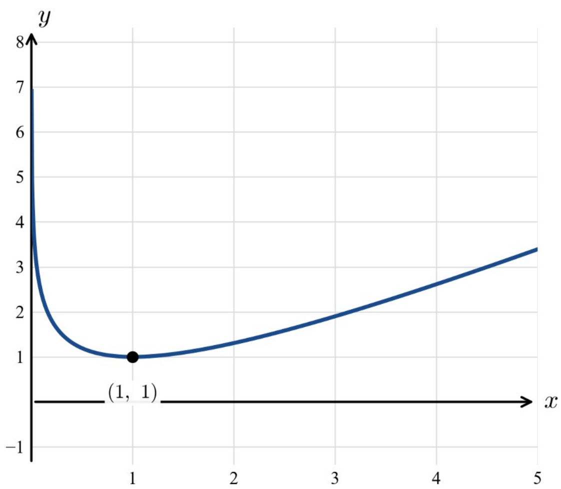
Q14. $f'(x)=2xe^{-x}-x^2e^{-x}=x(2-x)e^{-x}$, so $x=0$ or $x=2$. $f(0)=0$, $f(2)=4e^{-2}$. SPs $(0,0)$, $(2,4e^{-2})$; decreasing on $(-\infty,0)$ and $(2,\infty)$, increasing on $(0,2)$.



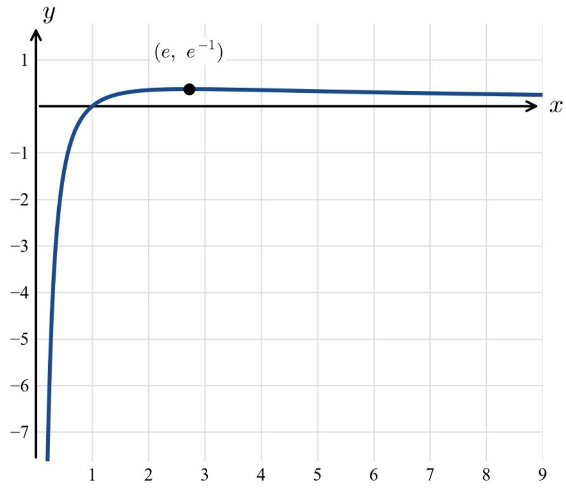
Q15. $f'(x)=e^x-1$, so $e^x=1\Rightarrow x=0$. $f(0)=1$. SP $(0,1)$; $f'<0$ for $x<0$, $f'>0$ for $x>0$: decreasing on $(-\infty,0)$, increasing on $(0,\infty)$.



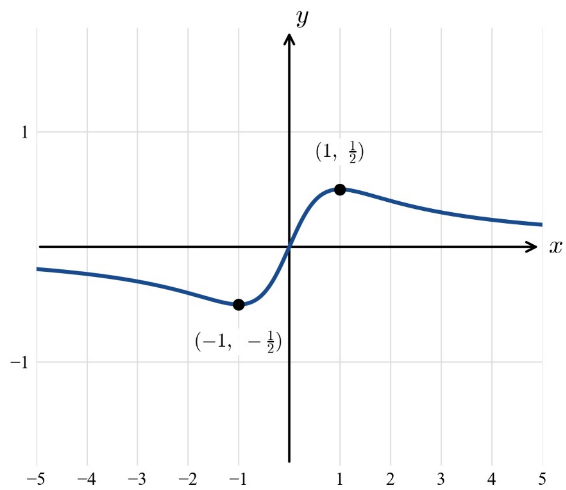
Q16. Domain $x>0$. $f'(x)=1-\dfrac{1}{x}=\dfrac{x-1}{x}$, so $x=1$. $f(1)=1$. SP $(1,1)$; for $0<x<1$, $f'<0$; for $x>1$, $f'>0$: decreasing on $(0,1)$, increasing on $(1,\infty)$.



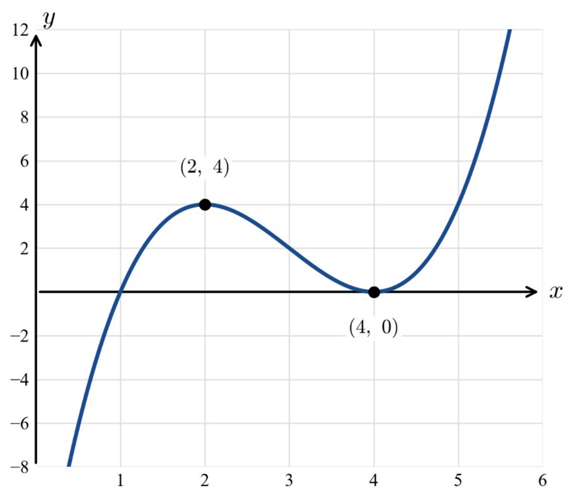
Q17. Domain $x>0$. $f'(x)=\dfrac{1-\log_e x}{x^2}$, so $\log_e x=1\Rightarrow x=e$. $f(e)=\dfrac{1}{e}=e^{-1}$. SP (e,e^{-1}); for $0<x<e$, $f'>0$; for $x>e$, $f'<0$: increasing then decreasing.



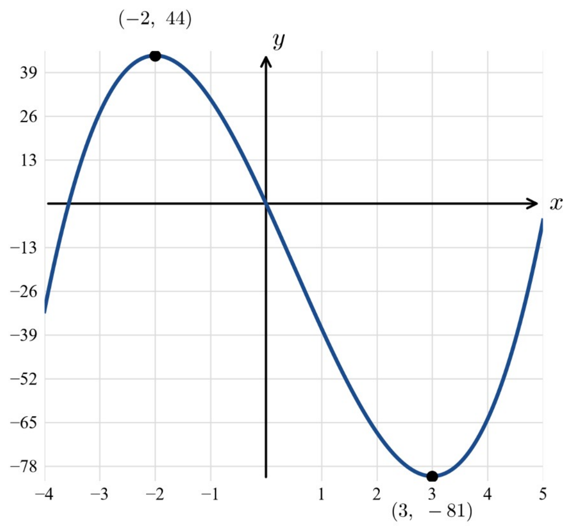
Q18. $f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$, so $x = \pm 1$. $f(-1) = -\dfrac{1}{2}$, $f(1) = \dfrac{1}{2}$. SPs $\left(-1, -\dfrac{1}{2}\right)$, $\left(1, \dfrac{1}{2}\right)$; decreasing on $(-\infty, -1)$ and $(1, \infty)$, increasing on $(-1, 1)$.



Q19. $f'(x) = (x-4)^2 + (x-1) \cdot 2(x-4) = (x-4)\left[(x-4) + 2(x-1)\right] = (x-4)(3x-6) = 3(x-4)(x-2)$, so $x = 2, 4$. $f(2) = (1)(-2)^2 = 4$, $f(4) = (3)(0)^2 = 0$. SPs $(2, 4)$, $(4, 0)$; increasing on $(-\infty, 2)$ and $(4, \infty)$, decreasing on $(2, 4)$.



Q20. $f'(x) = 6x^2 - 6x - 36 = 6(x-3)(x+2)$, so $x = -2, 3$. $f(-2) = -16 - 12 + 72 = 44$, $f(3) = 54 - 27 - 108 = -81$. SPs $(-2, 44)$, $(3, -81)$. As x increases: f increases up to $(-2, 44)$, decreases to $(3, -81)$, then increases without bound.



Q21. $f'(x)=3x^2-3k$. Setting $f'(x)=0$: $3x^2=3k \Rightarrow x^2=k \Rightarrow x=\pm\sqrt{k}$ (real since $k>0$). $f(\sqrt{k})=(\sqrt{k})^3-3k\sqrt{k}=k\sqrt{k}-3k\sqrt{k}=-2k\sqrt{k}$, and similarly $f(-\sqrt{k})=-k\sqrt{k}+3k\sqrt{k}=2k\sqrt{k}$. $\therefore$ the stationary points are $(\sqrt{k},-2k\sqrt{k})$ and $(-\sqrt{k},2k\sqrt{k})$.

Theory

A **stationary point** of f is a point where $f'(x)=0$ — the tangent is horizontal. There are three types:

- **local maximum** — the curve stops rising and starts falling; f' changes from **positive to negative**;
- **local minimum** — the curve stops falling and starts rising; f' changes from **negative to positive**;
- **stationary point of inflection** — the curve flattens but continues in the same direction; f' is zero but **does not change sign** (it has the same sign on both sides).

The gradient (sign-diagram) test. Find the stationary points by solving $f'(x)=0$. Then check the sign of f' just to the left and just to the right of each one and read off the nature from the pattern above. A sign diagram of f' makes this systematic.

x	left of a	a	right of a	nature at $x=a$
f'	+	0	−	local maximum
f'	−	0	+	local minimum
f'	+	0	+	stationary point of inflection
f'	−	0	−	stationary point of inflection

The second-derivative test. At a stationary point $x=a$:

- if $f''(a)<0$, the point is a **local maximum**;
- if $f''(a)>0$, the point is a **local minimum**;
- if $f''(a)=0$, the test is **inconclusive** — fall back on the sign diagram of f' (the point may be a maximum, a minimum, or a stationary point of inflection).

To locate and classify: (1) find $f'(x)$ and solve $f'(x)=0$ for the x-coordinates; (2) find the y-coordinate of each by substituting into f; (3) classify each using the sign diagram of f' (or the second-derivative test); (4) mark the classified stationary points, together with the axis intercepts, and draw a smooth curve.

Worked examples

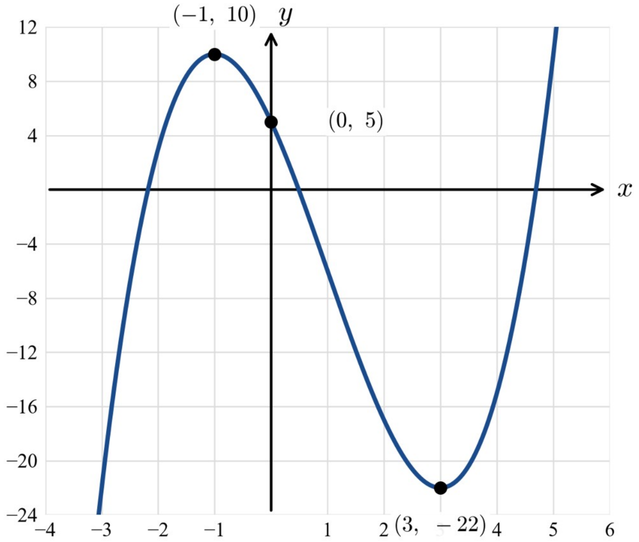
Example 1 — scaffolded

Find and classify the stationary points of $f(x)=x^3-3x^2-9x+5$, and sketch the graph.

Working	Comment
$f'(x)=3x^2-6x-9=3(x-3)(x+1)$	Differentiate, then factorise.
$f'(x)=0 \Rightarrow x=-1$ or $x=3$	The x-coordinates of the stationary points.
$f(-1)=-1-3+9+5=10$; $f(3)=27-27-27+5=-22$	Substitute into f for the y-coordinates.
Sign of $f'=3(x-3)(x+1)$: for $x<-1$, +; for $-1<x<3$, −; for $x>3$, +.	Test each factor's sign in each interval.

Working	Comment
At $x=-1$: f' goes $+\rightarrow-$, so $(-1,10)$ is a **local maximum**. At $x=3$: f' goes $-\rightarrow+$, so $(3,-22)$ is a **local minimum**.	Read the nature from the sign change.

y-intercept $(0,5)$.

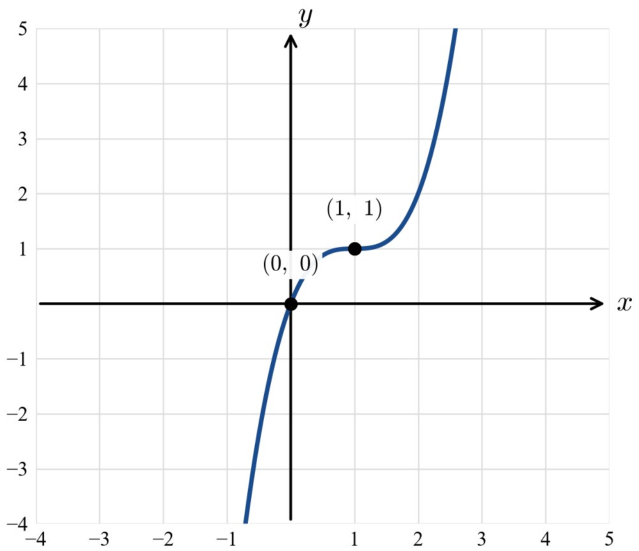


Example 2 — scaffolded

Find and classify the stationary point of $f(x)=x^3-3x^2+3x$.

Working	Comment
$f'(x)=3x^2-6x+3=3(x-1)^2$	Differentiate and factorise — a perfect square.
$f'(x)=0 \Rightarrow x=1$ (only)	The single stationary point.
$f(1)=1-3+3=1$	The y-coordinate.
$f'(x)=3(x-1)^2\geq 0$ for all x, and is 0 only at $x=1$	The gradient never becomes negative.
Since f' does **not** change sign (it is + on both sides of 1), $(1,1)$ is a **stationary point of inflection**.	Same sign either side $\Rightarrow$ inflection.

(The second-derivative test is inconclusive here: $f''(x)=6(x-1)$ gives $f''(1)=0$, so the sign diagram is needed.)



Example 3 — unscaffolded

Find and classify the stationary points of $f(x)=x^4-4x^3$, and sketch the graph.

$f'(x)=4x^3-12x^2=4x^2(x-3)$, so $f'(x)=0$ at $x=0$ and $x=3$.

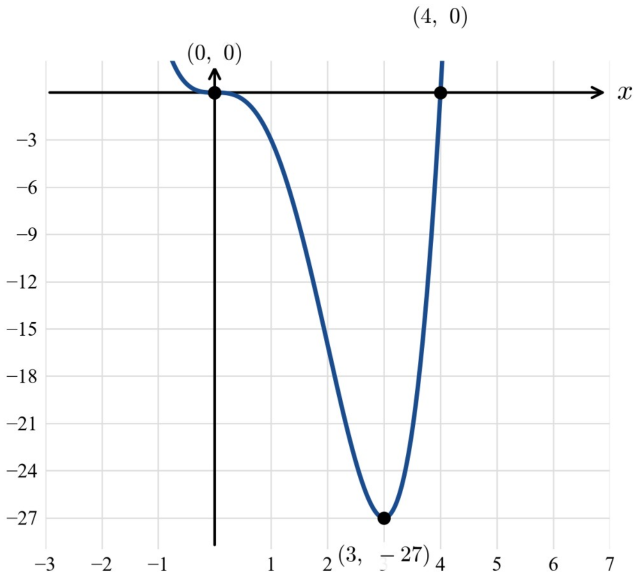
$f(0)=0$ and $f(3)=81-108=-27$.

Sign of $f'=4x^2(x-3)$: since $4x^2\geq 0$, the sign follows $(x-3)$ except that $f'=0$ at $x=0$. For $x<0$: $-$; for $0<x<3$: $-$; for $x>3$: $+$.

At $x=0$: f' is $-$ on both sides, so $(0,0)$ is a **stationary point of inflection**. At $x=3$: f' goes $-\rightarrow+$, so $(3,-27)$ is a **local minimum**.

x-intercepts: $x^4-4x^3=x^3(x-4)=0$, i.e. $(0,0)$ and $(4,0)$.

$\therefore$ the curve has a stationary point of inflection at the origin and a minimum at $(3,-27)$.



Example 4 — unscaffolded

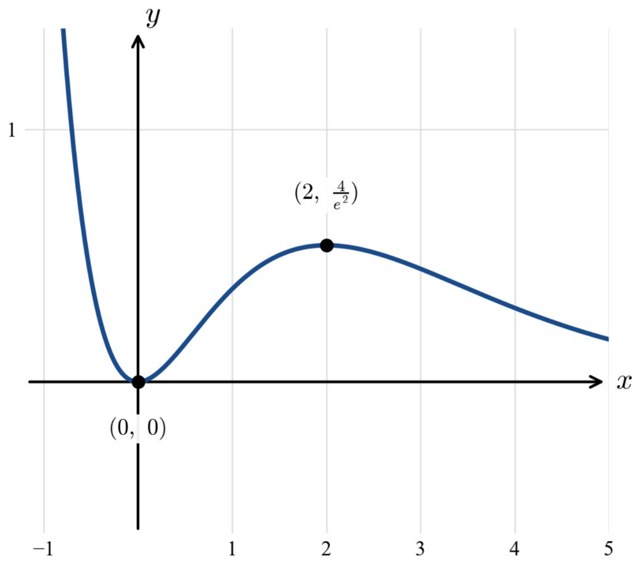
Find and classify the stationary points of $f(x)=x^2e^{-x}$.

$f'(x)=2xe^{-x}-x^2e^{-x}=e^{-x}x(2-x)$. Since $e^{-x}>0$, $f'(x)=0$ at $x=0$ and $x=2$.

$f(0)=0$ and $f(2)=4e^{-2}=\dfrac{4}{e^2}$.

Using the second-derivative test with $f''(x)=(x^2-4x+2)e^{-x}$: $f''(0)=2>0$, so $(0,0)$ is a **local minimum**; $f''(2)=-2e^{-2}<0$, so $\left(2,\dfrac{4}{e^2}\right)$ is a **local maximum**.

$\therefore$ a minimum at the origin and a maximum at $\left(2,\dfrac{4}{e^2}\right)$; note $f(x)=x^2e^{-x}\geq 0$ for all x.



Practice questions

Scaffolded

Q1. For $f(x)=x^3-6x^2+9x$: (a) find $f'(x)$ and solve $f'(x)=0$; (b) find the coordinates of the stationary points; (c) classify each using a sign diagram of f'; (d) sketch the graph.

Q2. For $f(x)=x^3-12x+5$: (a) find the stationary points; (b) classify each using the second-derivative test; (c) sketch the graph.

Q3. For $f(x)=2x^3-3x^2-12x$: (a) find and classify the stationary points; (b) state the y-intercept; (c) sketch the graph.

Q4. For $f(x)=x^3+3x^2+3x+2$: (a) show that $f'(x)=3(x+1)^2$; (b) hence find the stationary point and explain why it is a stationary point of inflection; (c) sketch the graph.

Q5. For $f(x)=x^4-2x^2$: (a) find the three stationary points; (b) classify each; (c) sketch the graph.

Q6. For $f(x)=xe^x$: (a) show that $f'(x)=(x+1)e^x$; (b) find and classify the stationary point; (c) sketch the graph.

Unscaffolded

For each function, find the coordinates of the stationary point(s), determine the nature of each, and sketch the graph.

Q7. $f(x)=x^3-3x$

Q8. $f(x)=-x^3+3x^2$

Q9. $f(x)=x^3-3x^2+2$

Q10. $f(x)=x^4-8x^2$

Q11. $f(x)=x^4-4x$

Q12. $f(x)=(x-2)^3+1$

Q13. $f(x)=x^3-3x^2+3x-2$

Q14. $f(x)=3x^4-4x^3$

Q15. $f(x)=x^2 e^x$

Q16. $f(x)=e^x-x$

Q17. $f(x)=x-e^x$

Q18. $f(x)=x\log_e x$

Q19. $f(x)=\dfrac{\log_e x}{x}$

Q20. $f(x)=x^3-6x^2+12x-5$

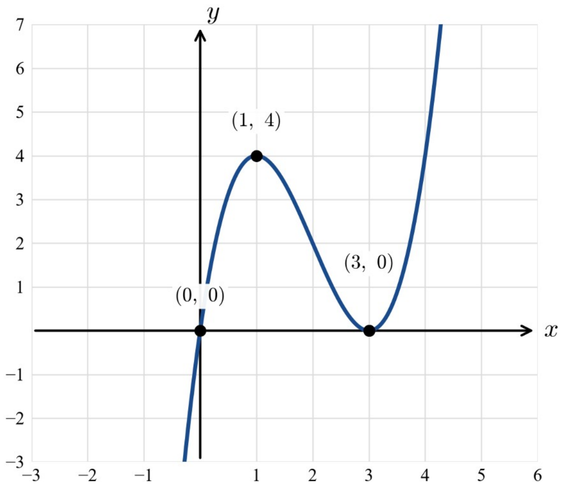
Q21. $f(x)=2x^3+3x^2-12x+1$

Answers

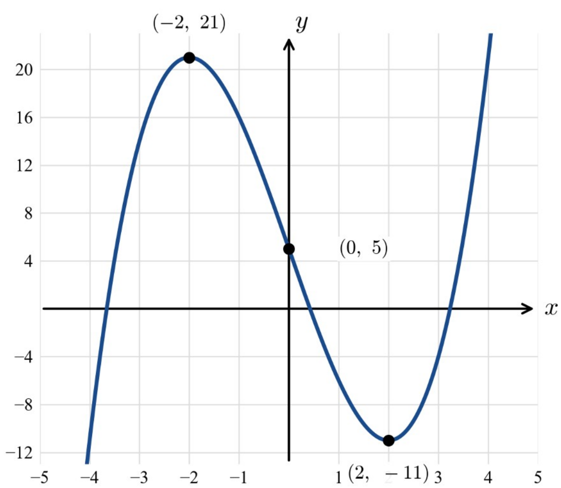
Q1. Max $(1,4)$, min $(3,0)$. **Q2.** Max $(-2,21)$, min $(2,-11)$. **Q3.** Max $(-1,7)$, min $(2,-20)$; y-intercept $(0,0)$. **Q4.** Stationary point of inflection $(-1,1)$. **Q5.** Max $(0,0)$, min $(-1,-1)$, min $(1,-1)$. **Q6.** Min $\left(-1,-\frac{1}{e}\right)$. **Q7.** Max $(-1,2)$, min $(1,-2)$. **Q8.** Min $(0,0)$, max $(2,4)$. **Q9.** Max $(0,2)$, min $(2,-2)$. **Q10.** Max $(0,0)$, min $(-2,-16)$, min $(2,-16)$. **Q11.** Min $(1,-3)$. **Q12.** Stationary point of inflection $(2,1)$. **Q13.** Stationary point of inflection $(1,-1)$. **Q14.** Stationary point of inflection $(0,0)$, min $(1,-1)$. **Q15.** Max $\left(-2,\frac{4}{e^2}\right)$, min $(0,0)$. **Q16.** Min $(0,1)$. **Q17.** Max $(0,-1)$. **Q18.** Min $\left(e^{-1},-\frac{1}{e}\right)$ (domain $x>0$). **Q19.** Max $\left(e,\frac{1}{e}\right)$ (domain $x>0$). **Q20.** Stationary point of inflection $(2,3)$. **Q21.** Max $(-2,21)$, min $(1,-6)$.

Fully-worked solutions

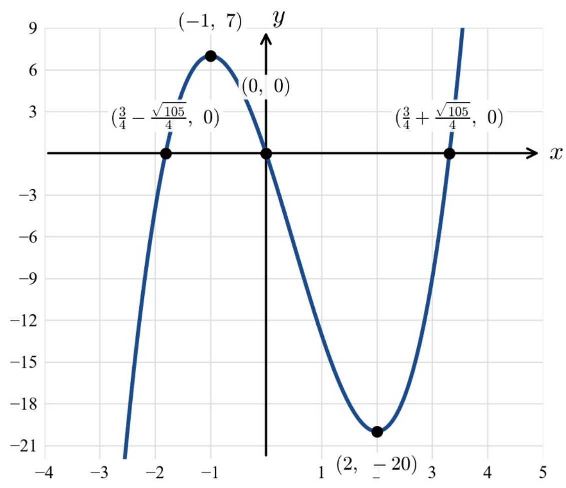
Q1. $f'(x)=3x^2-12x+9=3(x-1)(x-3)$, zero at $x=1,3$. $f(1)=4$, $f(3)=0$. f' goes $+0-$ at $x=1$ (max) and $-0+$ at $x=3$ (min): max $(1,4)$, min $(3,0)$.



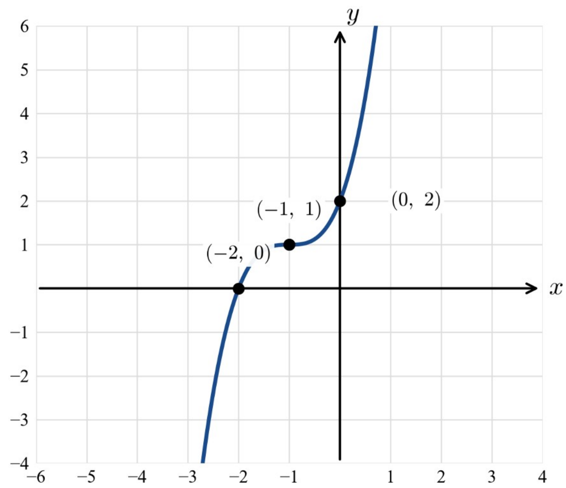
Q2. $f'(x)=3x^2-12=3(x-2)(x+2)$, zero at $x=\pm2$. $f(-2)=21$, $f(2)=-11$. $f''(x)=6x$: $f''(-2)=-12<0$ (max), $f''(2)=12>0$ (min): max $(-2,21)$, min $(2,-11)$.



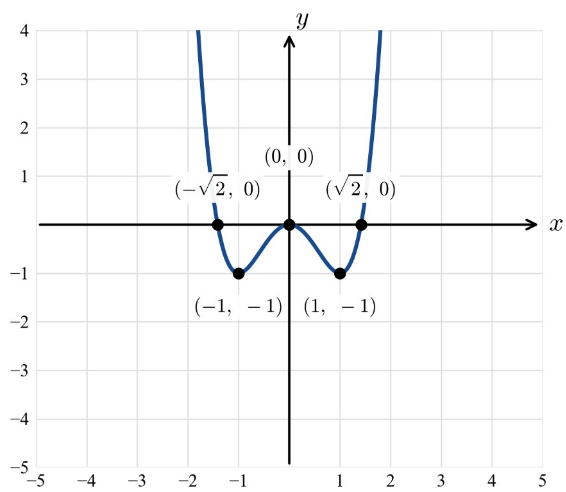
Q3. $f'(x)=6x^2-6x-12=6(x-2)(x+1)$, zero at $x=-1,2$. $f(-1)=7$, $f(2)=-20$. Sign of f': $+0-$ at $x=-1$ (max), $-0+$ at $x=2$ (min). y-intercept $(0,0)$.



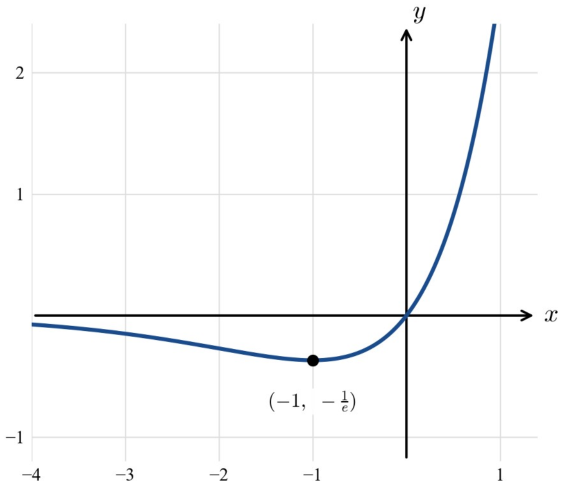
Q4. (a) $f'(x)=3x^2+6x+3=3(x+1)^2$. (b) $f'(x)=0 \Rightarrow x=-1$; $f(-1)=-1+3-3+2=1$. Since $f'(x)=3(x+1)^2 \geq 0$ and is 0 only at $x=-1$, f' does not change sign, so $(-1,1)$ is a stationary point of inflection.



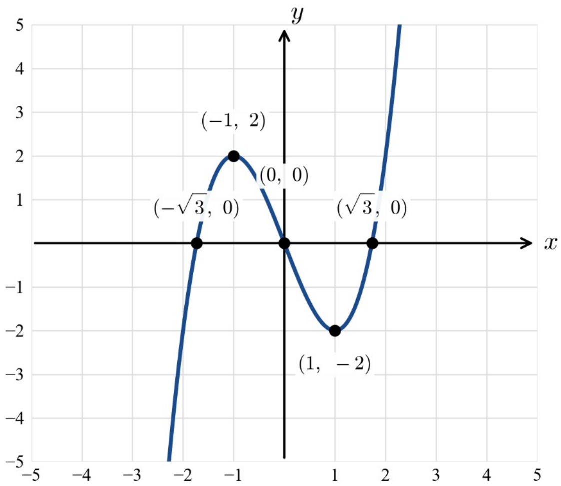
Q5. $f'(x)=4x^3-4x=4x(x-1)(x+1)$, zero at $x=-1,0,1$. $f(0)=0$, $f(\pm 1)=-1$. $f''(x)=12x^2-4$: $f''(0)=-4<0$ (max), $f''(\pm 1)=8>0$ (min). Max $(0,0)$, min $(-1,-1)$, min $(1,-1)$.



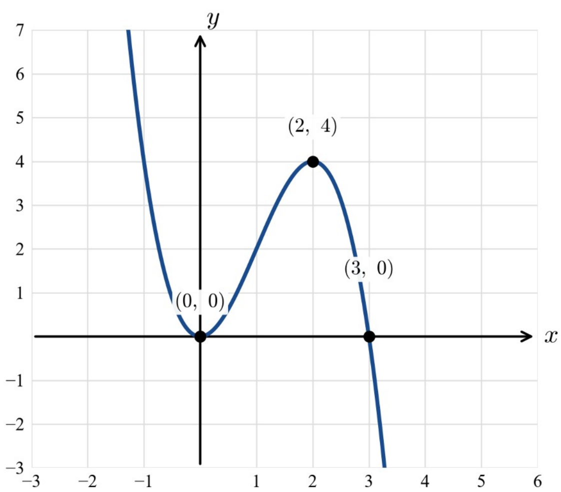
Q6. (a) $f'(x)=e^x+xe^x=(x+1)e^x$. (b) Since $e^x>0$, $f'(x)=0 \Rightarrow x=-1$; $f(-1)=-e^{-1}$. f' goes $-\,0\,+$ (as $x+1$ does), so $\left(-1,-\frac{1}{e}\right)$ is a local minimum.



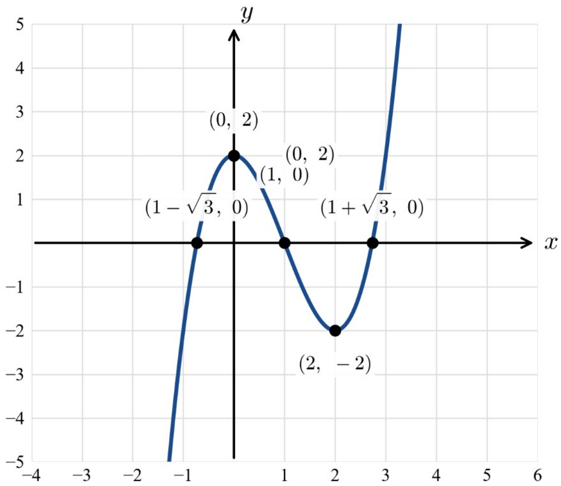
Q7. $f'(x)=3x^2-3=3(x-1)(x+1)$, zero at $x=\pm1$. $f(-1)=2$, $f(1)=-2$. $f''(x)=6x$: max $(-1,2)$, min $(1,-2)$.



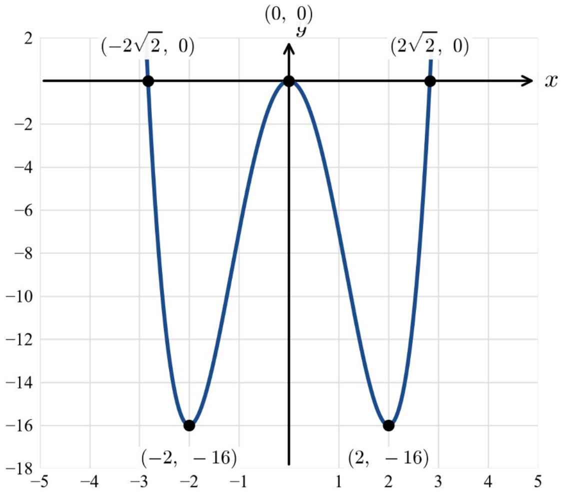
Q8. $f'(x)=-3x^2+6x=-3x(x-2)$, zero at $x=0,2$. $f(0)=0$, $f(2)=4$. $f''(x)=-6x+6$: $f''(0)=6>0$ (min), $f''(2)=-6<0$ (max). Min $(0,0)$, max $(2,4)$.



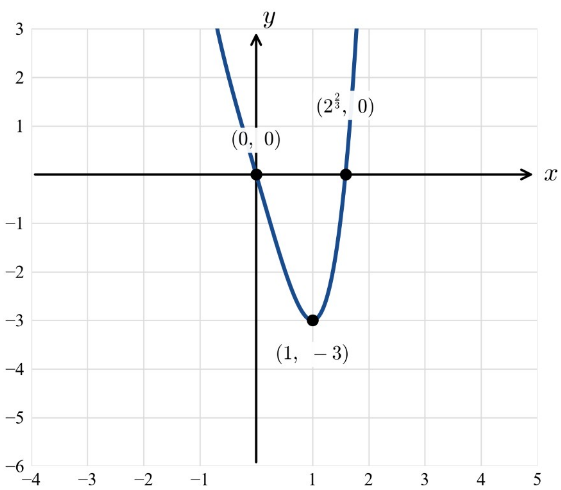
Q9. $f'(x)=3x^2-6x=3x(x-2)$, zero at $x=0,2$. $f(0)=2$, $f(2)=-2$. $f''(x)=6x-6$: $f''(0)=-6<0$ (max), $f''(2)=6>0$ (min). Max $(0,2)$, min $(2,-2)$.



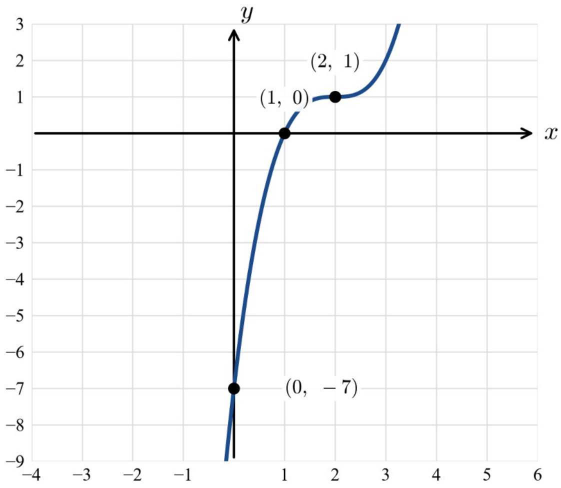
Q10. $f'(x)=4x^3-16x=4x(x-2)(x+2)$, zero at $x=-2,0,2$. $f(0)=0$, $f(\pm 2)=-16$. $f''(x)=12x^2-16$: $f''(0)=-16<0$ (max), $f''(\pm 2)=32>0$ (min). Max $(0,0)$, min $(-2,-16)$, min $(2,-16)$.



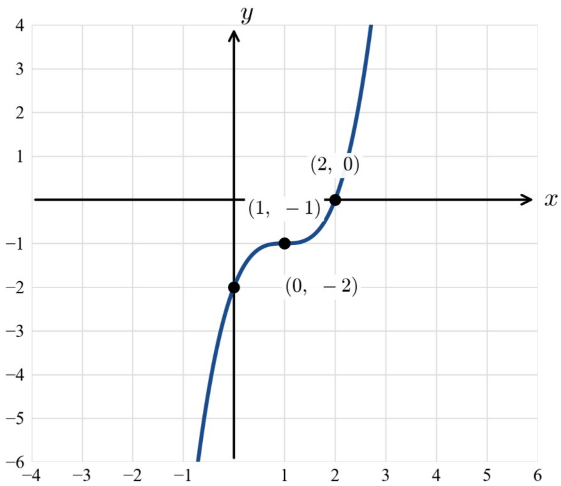
Q11. $f'(x)=4x^3-4=4(x-1)(x^2+x+1)$. The quadratic factor has no real zeros ($\Delta=-3<0$), so the only stationary point is at $x=1$; $f(1)=-3$. $f''(x)=12x^2$, $f''(1)=12>0$: minimum $(1,-3)$.



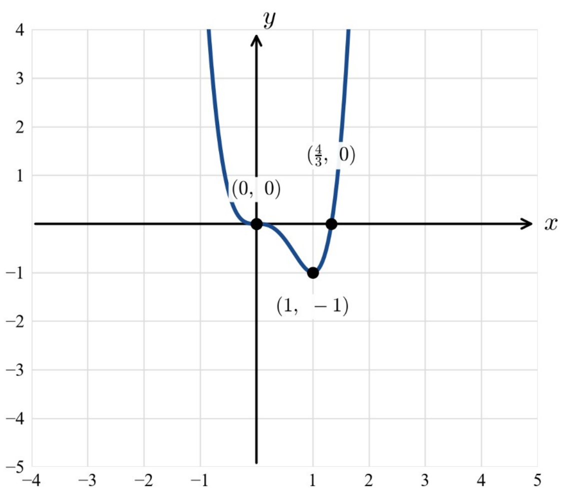
Q12. $f'(x)=3(x-2)^2\geq 0$, zero only at $x=2$; $f(2)=1$. f' does not change sign, so $(2,1)$ is a stationary point of inflection.



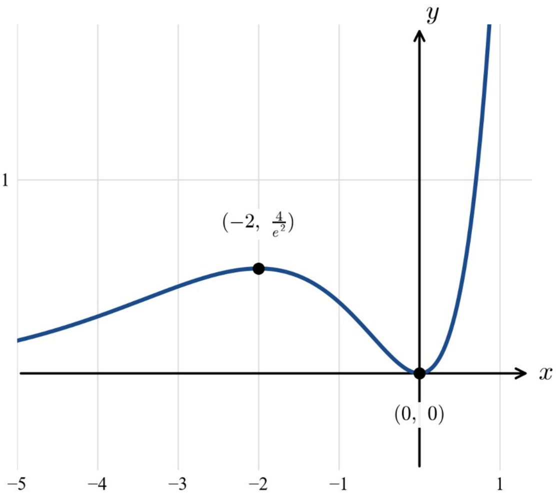
Q13. $f'(x)=3x^2-6x+3=3(x-1)^2\ge 0$, zero only at $x=1$; $f(1)=1-3+3-2=-1$. No sign change, so $(1,-1)$ is a stationary point of inflection.



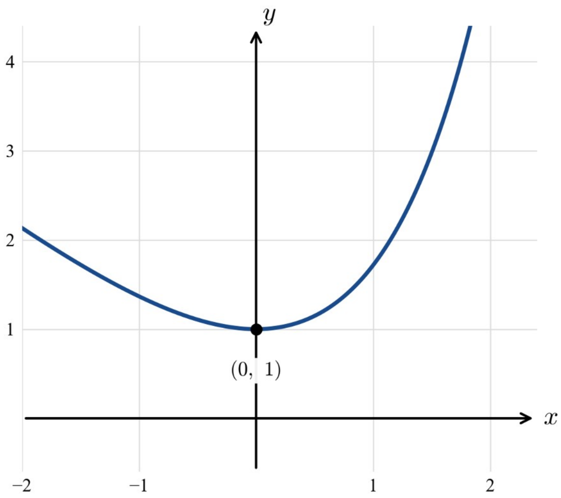
Q14. $f'(x)=12x^3-12x^2=12x^2(x-1)$, zero at $x=0,1$. $f(0)=0$, $f(1)=-1$. At $x=0$: $f'=12x^2(x-1)$ is $-$ on both sides (since $12x^2\ge 0$ and $x-1<0$ near 0), so $(0,0)$ is a stationary point of inflection. At $x=1$: f' goes $-0+$, so $(1,-1)$ is a minimum.



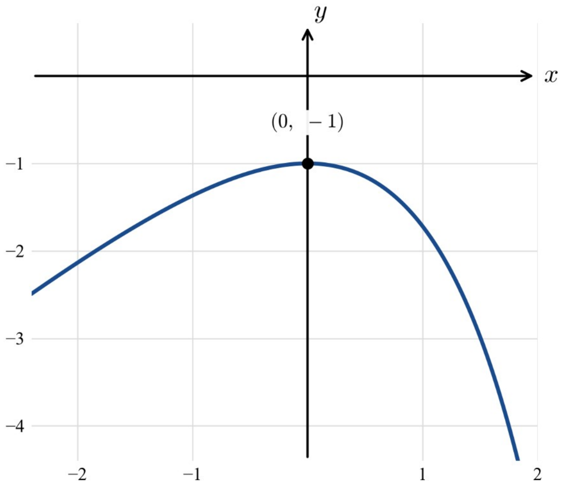
Q15. $f'(x)=2xe^x+x^2e^x=x(x+2)e^x$, zero at $x=-2,0$. $f(-2)=4e^{-2}$, $f(0)=0$. $f''(x)=(x^2+4x+2)e^x$: $f''(-2)=-2e^{-2}<0$ (max), $f''(0)=2>0$ (min). Max $\left(-2,\frac{4}{e^2}\right)$, min $(0,0)$.



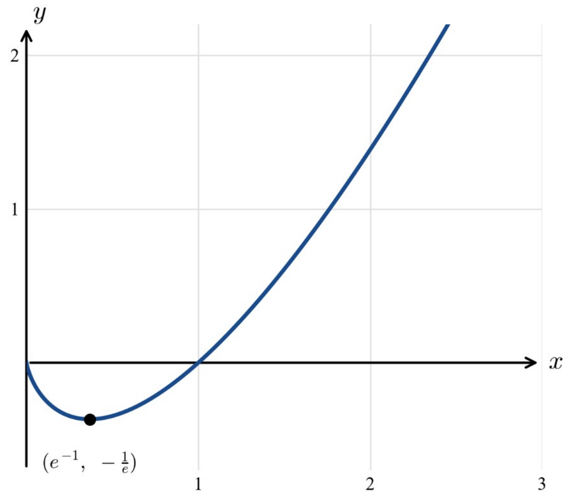
Q16. $f'(x)=e^x-1$, zero at $x=0$; $f(0)=1$. $f''(x)=e^x$, $f''(0)=1>0$: minimum $(0,1)$. (This is the least value of e^x-x.)



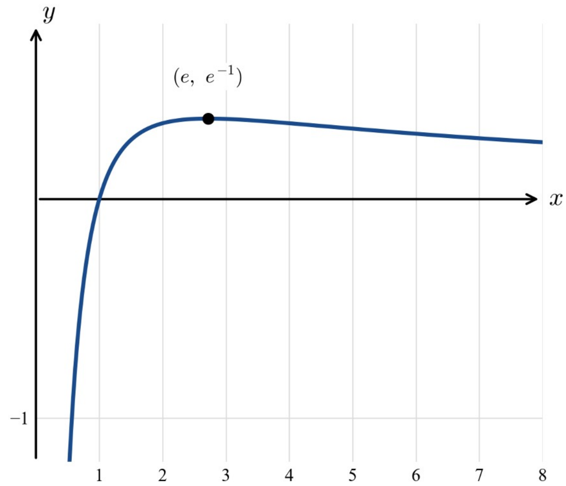
Q17. $f'(x)=1-e^x$, zero at $x=0$; $f(0)=-1$. $f''(x)=-e^x$, $f''(0)=-1<0$: maximum $(0,-1)$.



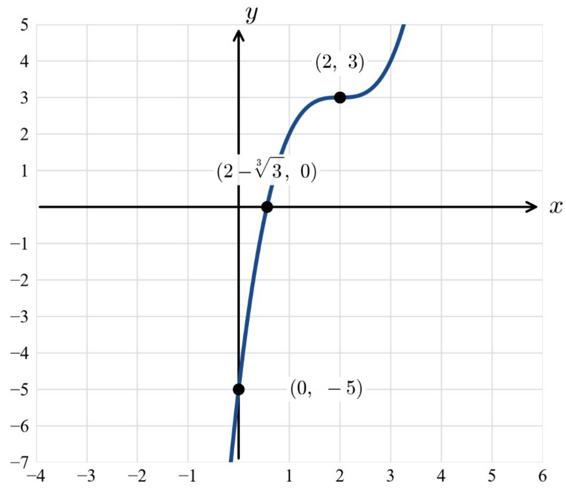
Q18. Domain $x>0$. $f'(x)=\log_e x+1$, zero when $\log_e x=-1$, i.e. $x=e^{-1}$; $f(e^{-1})=e^{-1}(-1)=-e^{-1}$. $f''(x)=\dfrac{1}{x}>0$: minimum $\left(e^{-1},-\dfrac{1}{e}\right)$.



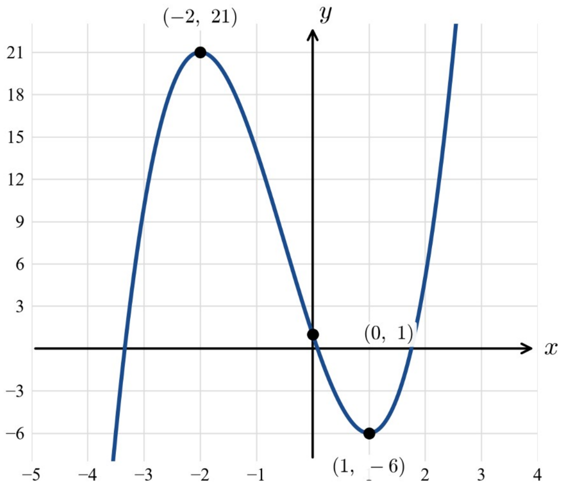
Q19. Domain $x>0$. $f'(x)=\dfrac{1-\log_e x}{x^2}$, zero when $\log_e x=1$, i.e. $x=e$; $f(e)=\dfrac{1}{e}$. For $x<e$, $f'>0$; for $x>e$, $f'<0$: maximum $\left(e,\dfrac{1}{e}\right)$.



Q20. $f'(x)=3x^2-12x+12=3(x-2)^2\geq 0$, zero only at $x=2$; $f(2)=8-24+24-5=3$. No sign change, so $(2,3)$ is a stationary point of inflection.



Q21. $f'(x)=6x^2+6x-12=6(x-1)(x+2)$, zero at $x=-2,1$. $f(-2)=21$, $f(1)=-6$. $f''(x)=12x+6$: $f''(-2)=-18<0$ (max), $f''(1)=18>0$ (min). Max $(-2,21)$, min $(1,-6)$.


(−2, 21)
y
x
(0, 1)
(1, − 6)
21
18
15
12
9
6
3
−3
−6
−5
−4
−3
−2
−1
2
3
4

Theory

A continuous function on a **closed interval** $[a,b]$ always attains an **absolute (global) maximum** value and an **absolute minimum** value somewhere on the interval. These occur either at a **stationary point** inside the interval (where $f'(x)=0$) or at an **endpoint** $x=a$ or $x=b$.

This gives a simple procedure — the **closed-interval method**:

1. Find $f'(x)$ and solve $f'(x)=0$ to locate the stationary points that lie **inside** (a,b).
2. Evaluate f at each of those stationary points **and** at the two endpoints $x=a$ and $x=b$.
3. The **largest** of these values is the absolute maximum; the **smallest** is the absolute minimum.

Local vs absolute. A *local* maximum or minimum describes the behaviour near a point; an *absolute* maximum or minimum is the greatest or least value over the **whole** interval. A local extremum need not be the absolute one — and the absolute extremum is often at an **endpoint**, not at a stationary point. Testing the endpoints is the step most often forgotten.

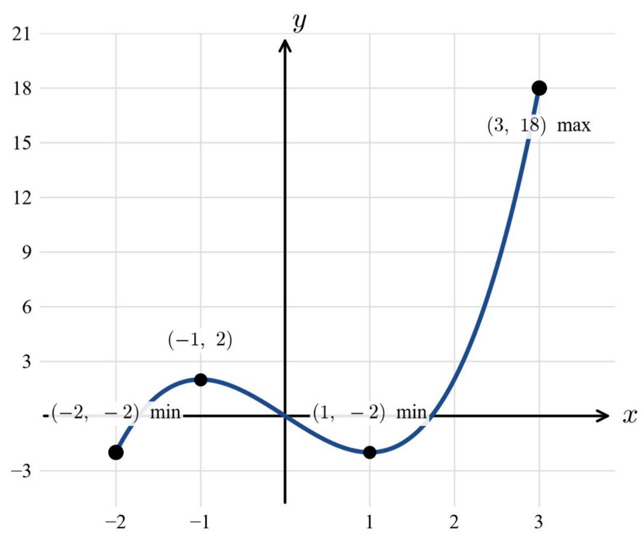
An absolute maximum or minimum **value** is a y-value; state *where* it occurs (the x-value) as well.

Worked examples

Example 1 — scaffolded

Find the absolute maximum and minimum values of $f(x)=x^3-3x$ on $[-2,3]$.

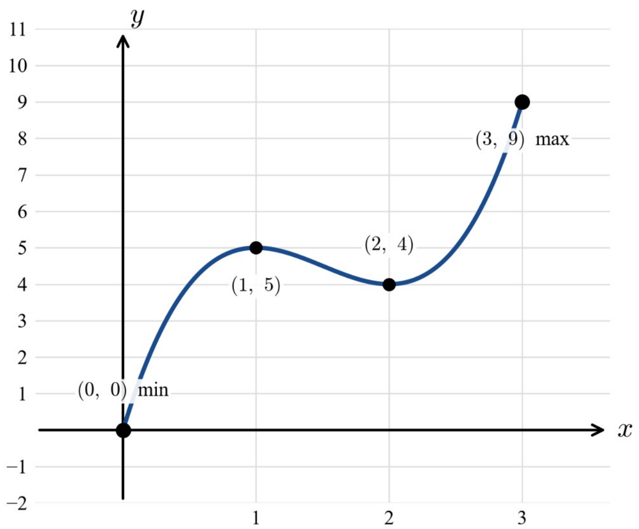
Working	Comment
$f'(x)=3x^2-3$	Differentiate.
$f'(x)=0: 3x^2-3=0 \Rightarrow x^2=1 \Rightarrow x=-1$ or $x=1$	Stationary points; both lie inside $(-2,3)$.
$f(-1)=(-1)^3-3(-1)=-1+3=2$; $f(1)=1-3=-2$	Evaluate f at the stationary points.
Endpoints: $f(-2)=-8+6=-2$; $f(3)=27-9=18$	Always test the endpoints of a closed interval.
Candidates: $-2, 2, -2, 18$.	List every candidate value.
$\therefore$ absolute **maximum** 18 at $x=3$; absolute **minimum** -2 at $x=-2$ and $x=1$.	Largest is the abs. max; smallest is the abs. min.



Example 2 — scaffolded

Find the absolute maximum and minimum values of $f(x)=2x^3-9x^2+12x$ on $[0,3]$.

Working	Comment
$f'(x)=6x^2-18x+12=6(x-1)(x-2)$	Differentiate and factorise.
$f'(x)=0 \Rightarrow x=1$ or $x=2$	Both stationary points lie inside $(0,3)$.
$f(1)=2-9+12=5$; $f(2)=16-36+24=4$	Evaluate at the stationary points.
Endpoints: $f(0)=0$; $f(3)=54-81+36=9$	Test the endpoints.
Candidates: $0,5,4,9$.	Here the maximum is at an **endpoint**.
$\therefore$ absolute **maximum** 9 at $x=3$; absolute **minimum** 0 at $x=0$.	Both extremes are at endpoints, not the stationary points.



Example 3 — unscaffolded

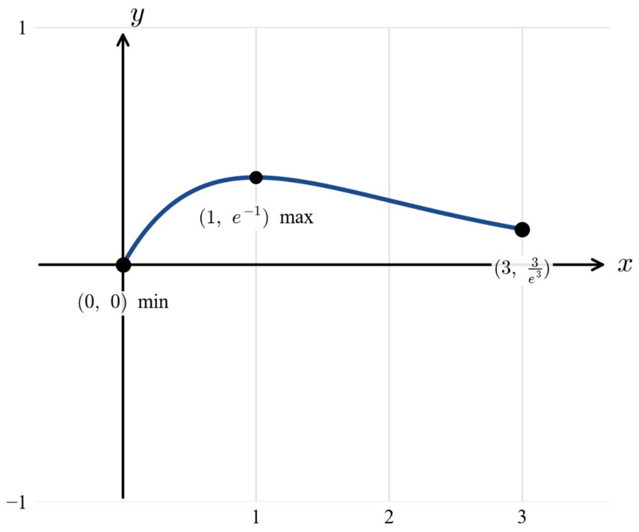
Find the absolute maximum and minimum values of $f(x)=xe^{-x}$ on $[0,3]$.

$f'(x)=e^{-x}-xe^{-x}=(1-x)e^{-x}$. Since $e^{-x}>0$, $f'(x)=0$ when $x=1$ (inside $[0,3]$).

$f(1)=e^{-1}$. Endpoints: $f(0)=0$ and $f(3)=3e^{-3}\approx 0.149$.

Comparing $0, e^{-1}\approx 0.368, 3e^{-3}$:

$\therefore$ absolute maximum e^{-1} at $x=1$; absolute minimum 0 at $x=0$.



Example 4 — unscaffolded

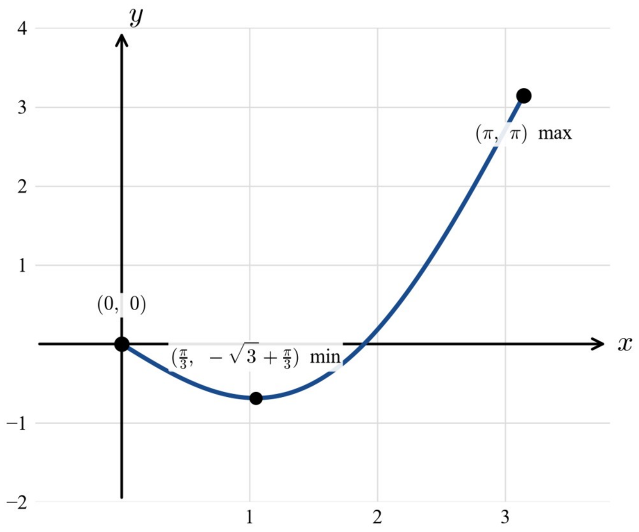
Find the absolute maximum and minimum values of $f(x)=x-2\sin x$ on $[0,\pi]$.

$f'(x)=1-2\cos x$. $f'(x)=0 \Rightarrow \cos x=\frac{1}{2} \Rightarrow x=\frac{\pi}{3}$ (inside $[0,\pi]$).

$f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}-2\sin\frac{\pi}{3}=\frac{\pi}{3}-\sqrt{3}$. Endpoints: $f(0)=0$ and $f(\pi)=\pi-2\sin\pi=\pi$.

Comparing $0, \frac{\pi}{3}-\sqrt{3}\approx-0.68, \pi$:

$\therefore$ absolute maximum π at $x=\pi$; absolute minimum $\frac{\pi}{3}-\sqrt{3}$ at $x=\frac{\pi}{3}$.



Practice questions

Scaffolded

For each function on the given closed interval: (a) find the stationary point(s) inside the interval; (b) evaluate the function at the stationary point(s) and at both endpoints; (c) state the absolute maximum and minimum values and where each occurs; (d) sketch the graph over the interval, labelling these points.

Q1. $f(x)=x^2-4x+1$ on $[0,3]$.

Q2. $f(x)=x^3-12x$ on $[-3,3]$.

Q3. $f(x)=4-x^2$ on $[-1,3]$.

Q4. $f(x)=x^3-3x^2$ on $[-1,4]$.

Q5. $f(x)=e^x-x$ on $[-1,2]$.

Q6. $f(x)=x+\frac{1}{x}$ on $\left[\frac{1}{2},3\right]$.

Unscaffolded

For each function, find the absolute maximum and minimum values on the given closed interval, stating where each occurs, and sketch the graph over the interval.

Q7. $f(x)=x^3-6x^2+9x$ on $[0,4]$.

Q8. $f(x)=-x^3+3x$ on $[-2,2]$.

Q9. $f(x)=x^4-2x^2$ on $[-2,2]$.

Q10. $f(x)=x^3-3x^2+2$ on $[-1,3]$.

Q11. $f(x)=x-\log_e x$ on $\left[\frac{1}{2},3\right]$.

Q12. $f(x)=2\cos x+x$ on $[0,2\pi]$.

Q13. $f(x)=x^3-3x^2-9x+5$ on $[-2,4]$.

Q14. $f(x)=12x-x^3$ on $[-3,3]$.

Q15. $f(x)=x^4-8x^2$ on $[-3,1]$.

Q16. $f(x)=(x-2)e^x$ on $[0,3]$.

Q17. $f(x)=x\log_e x$ on $[1,e]$.

Q18. $f(x)=\sin x+\cos x$ on $[0,\pi]$.

Q19. $f(x)=2x^3-3x^2$ on $[-1,2]$.

Q20. A ball's height (metres) after x seconds is $h(x)=4x-x^2$, for $0\le x\le 5$. Find the absolute maximum and minimum height over this interval.

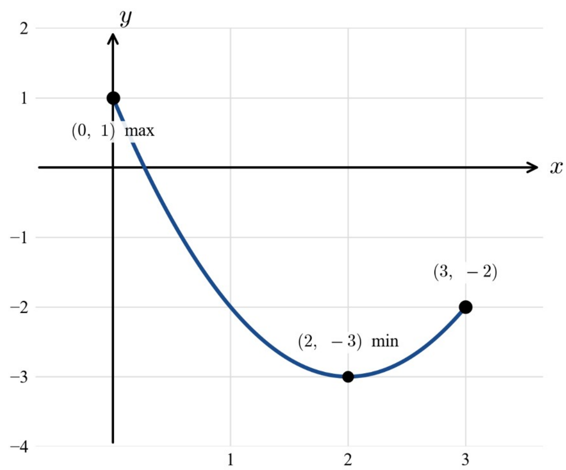
Q21. $f(x)=x+\frac{4}{x}$ on $[1,5]$.

Answers

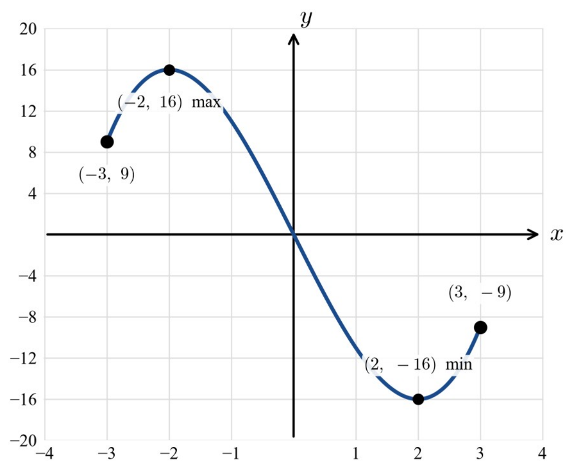
Q1. Max 1 at $x=0$; min -3 at $x=2$. **Q2.** Max 16 at $x=-2$; min -16 at $x=2$. **Q3.** Max 4 at $x=0$; min -5 at $x=3$. **Q4.** Max 16 at $x=4$; min -4 at $x=-1$ and $x=2$. **Q5.** Max e^2-2 at $x=2$; min 1 at $x=0$. **Q6.** Max $\frac{10}{3}$ at $x=3$; min 2 at $x=1$. **Q7.** Max 4 at $x=1$ and $x=4$; min 0 at $x=0$ and $x=3$. **Q8.** Max 2 at $x=-2$ and $x=1$; min -2 at $x=-1$ and $x=2$. **Q9.** Max 8 at $x=-2$ and $x=2$; min -1 at $x=-1$ and $x=1$. **Q10.** Max 2 at $x=0$ and $x=3$; min -2 at $x=-1$ and $x=2$. **Q11.** Max $3-\log_e 3$ at $x=3$; min 1 at $x=1$. **Q12.** Max $2+2\pi$ at $x=2\pi$; min $\frac{5\pi}{6}-\sqrt{3}$ at $x=\frac{5\pi}{6}$. **Q13.** Max 10 at $x=-1$; min -22 at $x=3$. **Q14.** Max 16 at $x=2$; min -16 at $x=-2$. **Q15.** Max 9 at $x=-3$; min -16 at $x=-2$. **Q16.** Max e^3 at $x=3$; min $-e$ at $x=1$. **Q17.** Max e at $x=e$; min 0 at $x=1$. **Q18.** Max $\sqrt{2}$ at $x=\frac{\pi}{4}$; min -1 at $x=\pi$. **Q19.** Max 4 at $x=2$; min -5 at $x=-1$. **Q20.** Max height 4 m at $x=2$; min height -5… (see solution — restricted to the physical model). **Q21.** Max $\frac{29}{5}$ at $x=5$; min 4 at $x=2$.

Fully-worked solutions

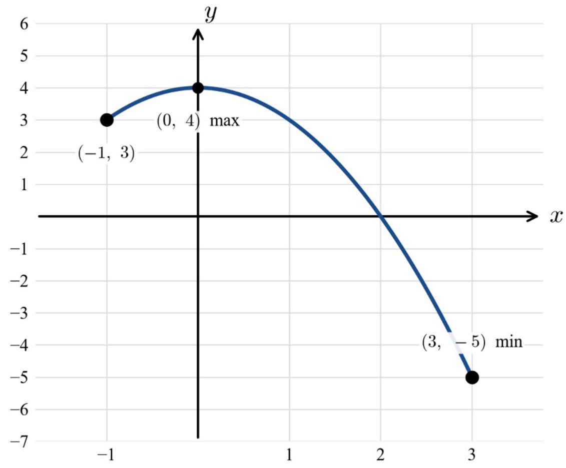
Q1. $f'(x)=2x-4=0 \Rightarrow x=2$. $f(2)=4-8+1=-3$. Endpoints $f(0)=1$, $f(3)=9-12+1=-2$. Candidates $1,-3,-2$: max 1 at $x=0$, min -3 at $x=2$.



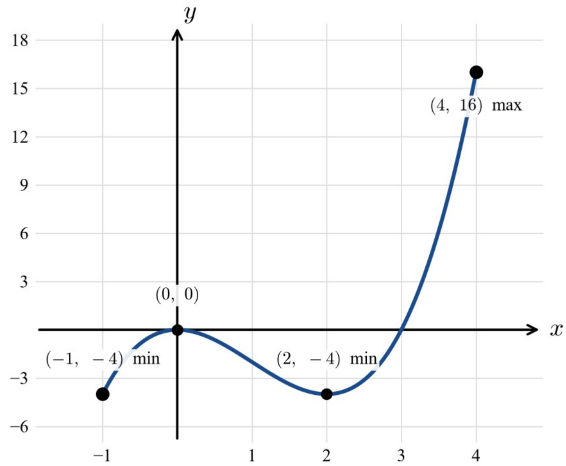
Q2. $f'(x)=3x^2-12=0 \Rightarrow x=\pm 2$. $f(-2)=-8+24=16$, $f(2)=8-24=-16$. Endpoints $f(-3)=-27+36=9$, $f(3)=27-36=-9$. Max 16 at $x=-2$, min -16 at $x=2$.



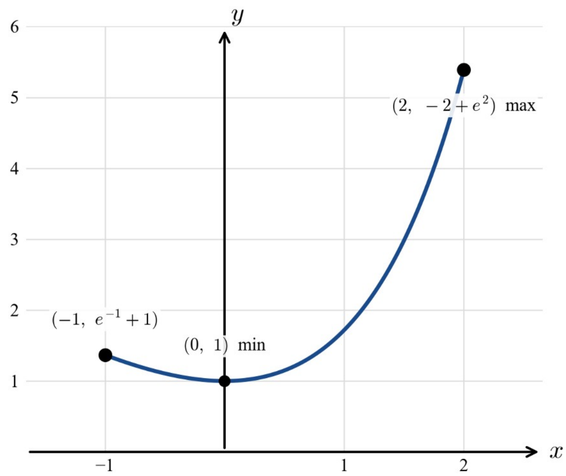
Q3. $f'(x)=-2x=0 \Rightarrow x=0$, $f(0)=4$. Endpoints $f(-1)=3$, $f(3)=-5$. Max 4 at $x=0$, min -5 at $x=3$.



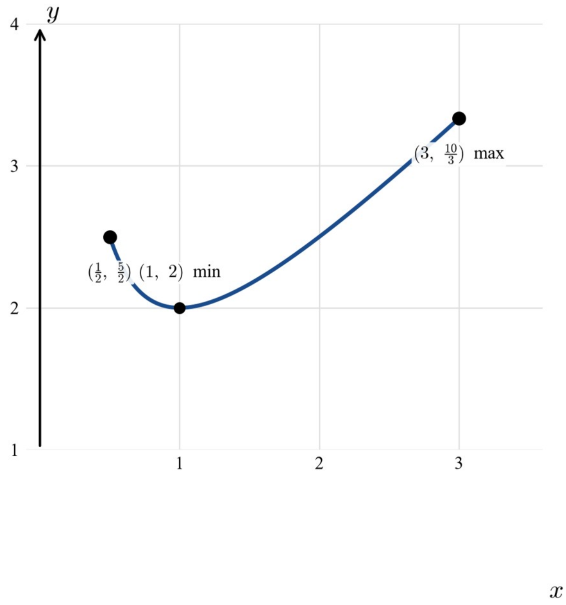
Q4. $f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. $f(0) = 0$, $f(2) = 8 - 12 = -4$. Endpoints $f(-1) = -1 - 3 = -4$, $f(4) = 64 - 48 = 16$. Max 16 at $x = 4$; min -4 at $x = -1$ and $x = 2$.



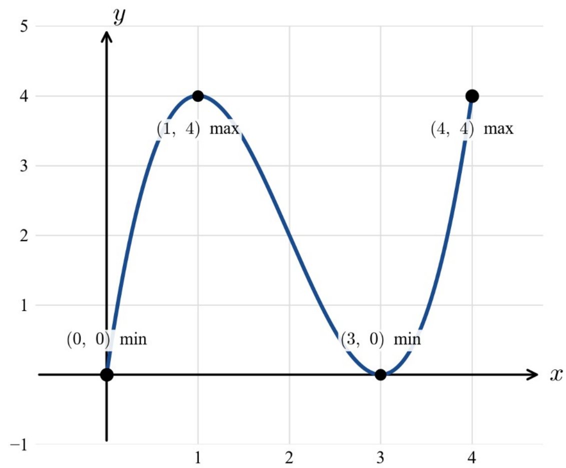
Q5. $f'(x) = e^x - 1 = 0 \Rightarrow x = 0$, $f(0) = 1$. Endpoints $f(-1) = e^{-1} + 1 \approx 1.37$, $f(2) = e^2 - 2 \approx 5.39$. Max $e^2 - 2$ at $x = 2$; min 1 at $x = 0$.



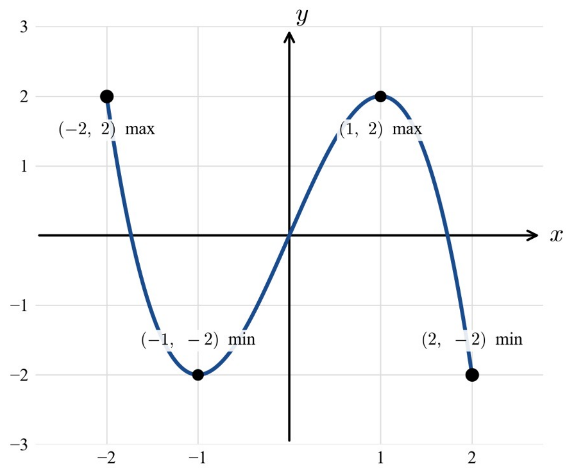
Q6. $f'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = 1$ (in the interval), $f(1) = 2$. Endpoints $f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$, $f(3) = 3 + \frac{1}{3} = \frac{10}{3}$. Max $\frac{10}{3}$ at $x = 3$; min 2 at $x = 1$.



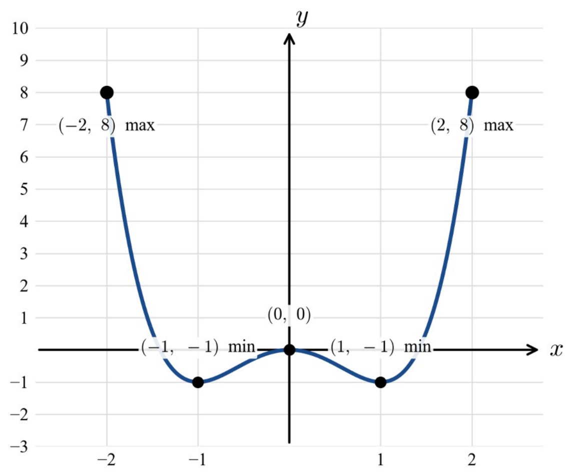
Q7. $f'(x)=3x^2-12x+9=3(x-1)(x-3)=0 \Rightarrow x=1$ or $x=3$. $f(1)=4$, $f(3)=0$. Endpoints $f(0)=0$, $f(4)=64-96+36=4$. Max 4 at $x=1$ and $x=4$; min 0 at $x=0$ and $x=3$.



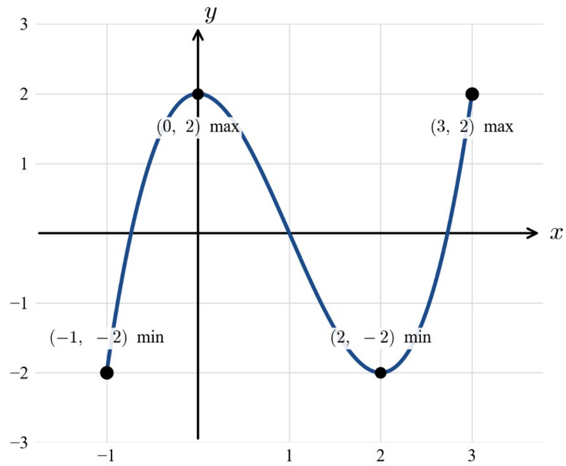
Q8. $f'(x)=-3x^2+3=0 \Rightarrow x=\pm1$. $f(-1)=1-3=-2$, $f(1)=-1+3=2$. Endpoints $f(-2)=8-6=2$, $f(2)=-8+6=-2$. Max 2 at $x=-2$ and $x=1$; min -2 at $x=-1$ and $x=2$.



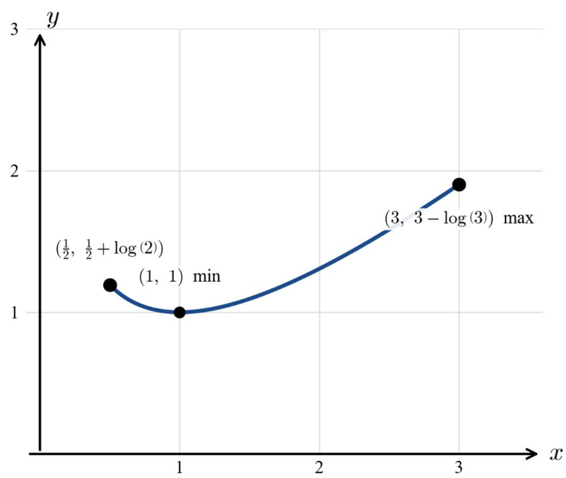
Q9. $f'(x)=4x^3-4x=4x(x-1)(x+1)=0 \Rightarrow x=0,\pm1$. $f(0)=0$, $f(\pm1)=1-2=-1$. Endpoints $f(\pm2)=16-8=8$. Max 8 at $x=\pm2$; min -1 at $x=\pm1$.



Q10. $f'(x)=3x^2-6x=3x(x-2)=0 \Rightarrow x=0$ or $x=2$. $f(0)=2$, $f(2)=8-12+2=-2$. Endpoints $f(-1)=-1-3+2=-2$, $f(3)=27-27+2=2$. Max 2 at $x=0$ and $x=3$; min -2 at $x=-1$ and $x=2$.

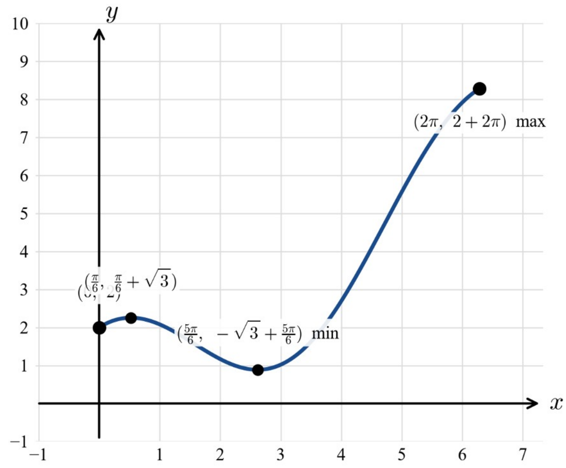


Q11. $f'(x)=1-\dfrac{1}{x}=0 \Rightarrow x=1$, $f(1)=1-0=1$. Endpoints $f\left(\dfrac{1}{2}\right)=\dfrac{1}{2}+\log_e 2\approx 1.19$, $f(3)=3-\log_e 3\approx 1.90$. Max $3-\log_e 3$ at $x=3$; min 1 at $x=1$.

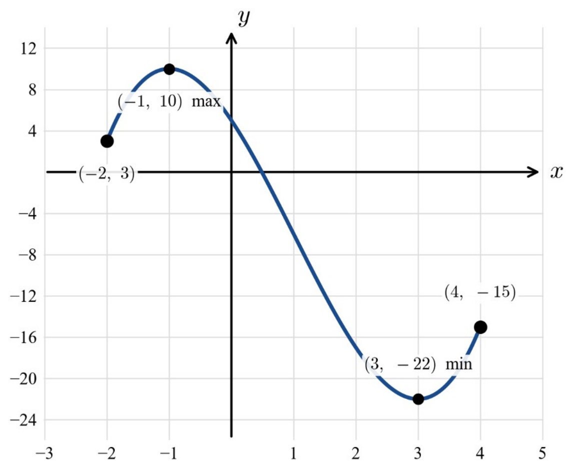


Q12. $f'(x)=1-2\sin x=0 \Rightarrow \sin x=\dfrac{1}{2} \Rightarrow x=\dfrac{\pi}{6}$ or $x=\dfrac{5\pi}{6}$. $f\left(\dfrac{\pi}{6}\right)=\dfrac{\pi}{6}+\sqrt{3}\approx 2.26$, $f\left(\dfrac{5\pi}{6}\right)=\dfrac{5\pi}{6}-\sqrt{3}\approx 0.89$.

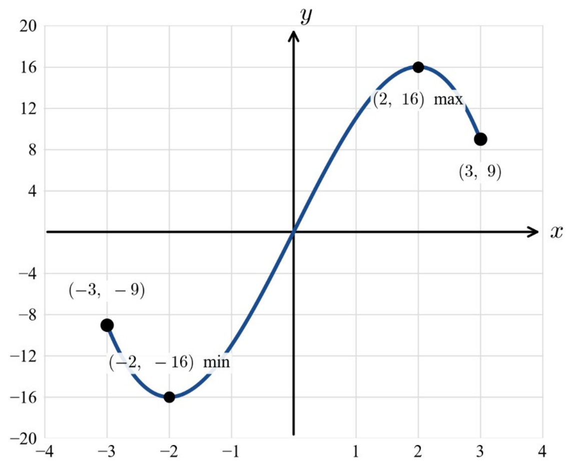
Endpoints $f(0)=2$, $f(2\pi)=2+2\pi\approx 8.28$. Max $2+2\pi$ at $x=2\pi$; min $\dfrac{5\pi}{6}-\sqrt{3}$ at $x=\dfrac{5\pi}{6}$.



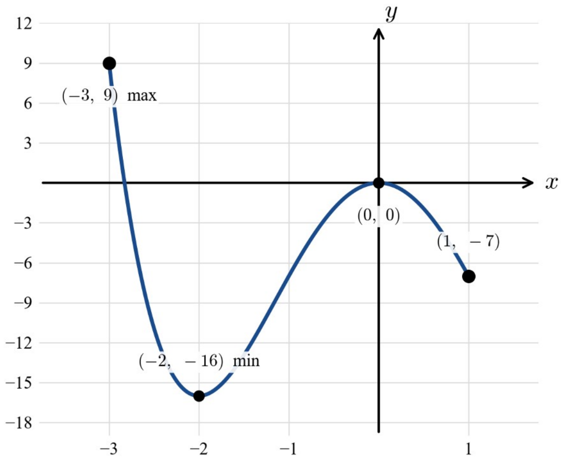
Q13. $f'(x)=3x^2-6x-9=3(x-3)(x+1)=0 \Rightarrow x=-1$ or $x=3$. $f(-1)=-1-3+9+5=10$, $f(3)=27-27-27+5=-22$. Endpoints $f(-2)=-8-12+18+5=3$, $f(4)=64-48-36+5=-15$. Max 10 at $x=-1$; min -22 at $x=3$.



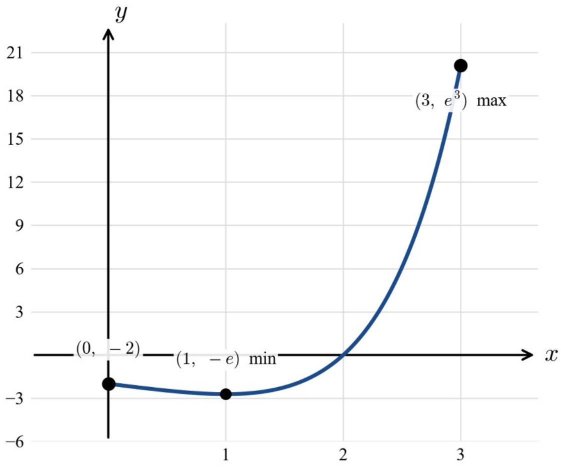
Q14. $f'(x)=12-3x^2=0 \Rightarrow x=\pm 2$. $f(-2)=-24+8=-16$, $f(2)=24-8=16$. Endpoints $f(-3)=-36+27=-9$, $f(3)=36-27=9$. Max 16 at $x=2$; min -16 at $x=-2$.



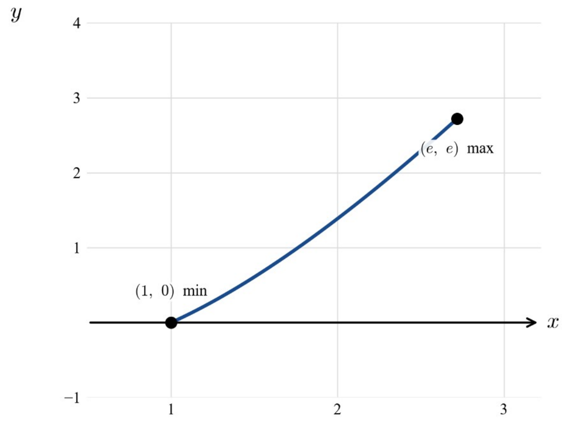
Q15. $f'(x)=4x^3-16x=4x(x-2)(x+2)=0$; inside $[-3,1]$: $x=-2$ and $x=0$. $f(-2)=16-32=-16$, $f(0)=0$. Endpoints $f(-3)=81-72=9$, $f(1)=1-8=-7$. Max 9 at $x=-3$; min -16 at $x=-2$.



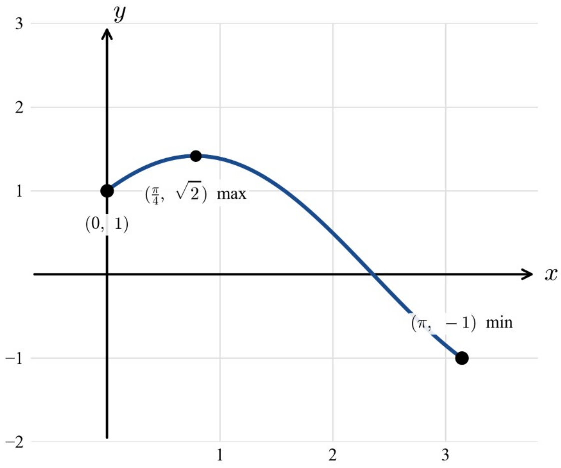
Q16. $f'(x)=e^x+(x-2)e^x=(x-1)e^x=0 \Rightarrow x=1$, $f(1)=-e$. Endpoints $f(0)=-2$, $f(3)=e^3\approx 20.1$. Max e^3 at $x=3$; min $-e$ at $x=1$.



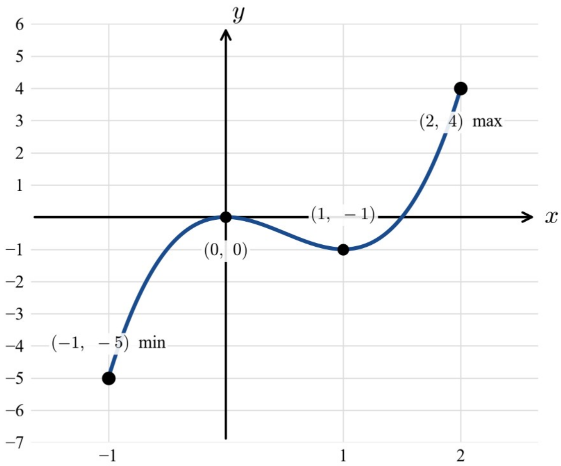
Q17. $f'(x)=\log_e x+1=0 \Rightarrow x=e^{-1}\approx 0.37$, which is **outside** $[1, e]$, so there is no interior stationary point and f is increasing on $[1, e]$. Endpoints $f(1)=0$, $f(e)=e$. Max e at $x=e$; min 0 at $x=1$.



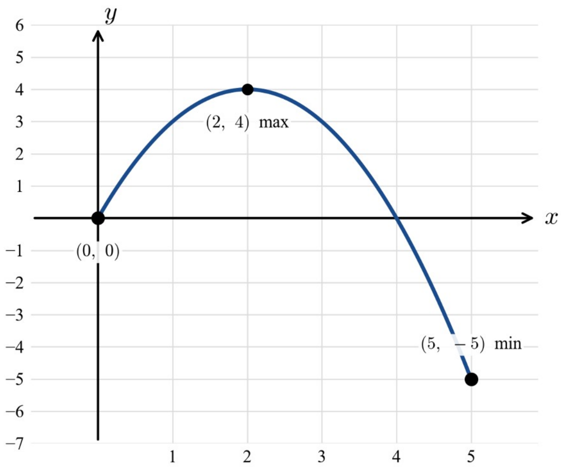
Q18. $f'(x)=\cos x-\sin x=0 \Rightarrow \tan x=1 \Rightarrow x=\dfrac{\pi}{4}$, $f\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}=\sqrt{2}$. Endpoints $f(0)=1$, $f(\pi)=-1$. Max $\sqrt{2}$ at $x=\dfrac{\pi}{4}$; min -1 at $x=\pi$.



Q19. $f'(x)=6x^2-6x=6x(x-1)=0 \Rightarrow x=0$ or $x=1$. $f(0)=0$, $f(1)=2-3=-1$. Endpoints $f(-1)=-2-3=-5$, $f(2)=16-12=4$. Max 4 at $x=2$; min -5 at $x=-1$.

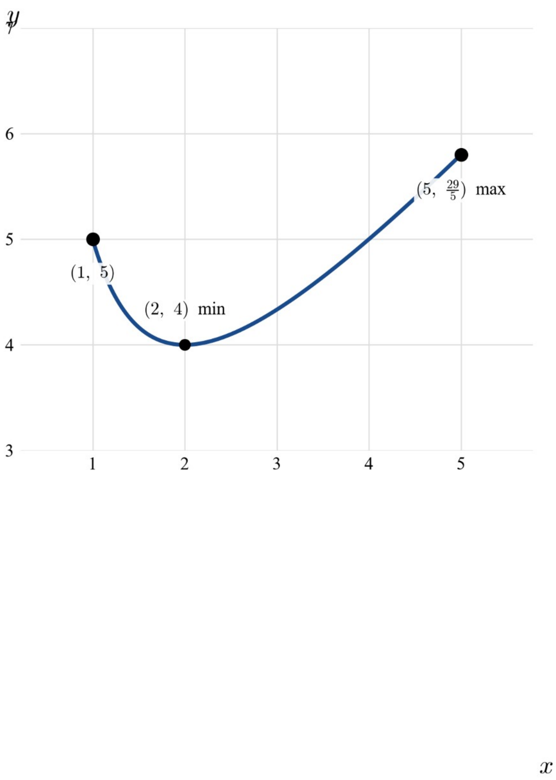


Q20. $h'(x)=4-2x=0 \Rightarrow x=2$, $h(2)=8-4=4$. Endpoints $h(0)=0$ and $h(5)=20-25=-5$. Over $0 \le x \le 5$ the greatest value is 4 m at $x=2$ (the maximum height) and the least value is -5 (at $x=5$). In the physical model the ball reaches the ground $h=0$ at $x=4$; the maximum height is 4 m at $t=2$ s.



Q21. $f'(x)=1-\dfrac{4}{x^2}=0 \Rightarrow x^2=4 \Rightarrow x=2$ (in $[1,5]$), $f(2)=2+2=4$. Endpoints $f(1)=1+4=5$, $f(5)=5+\dfrac{4}{5}=\dfrac{29}{5}$.

Max $\dfrac{29}{5}$ at $x=5$; min 4 at $x=2$.


y
x
(1, 5)
(2, 4) min
(5, 29/5) max

Theory

Many practical problems ask for the **largest** or **smallest** possible value of some quantity — the maximum area that can be enclosed, the minimum cost of materials, the greatest volume of a box. Calculus solves these by finding the stationary point of a suitable function.

The optimisation method (five steps).

1. **Define a variable** and draw a diagram if it helps. Let x be the quantity you can control.
2. **Write the quantity to be optimised** (area, volume, cost, …) as an expression. If it involves more than one variable, **use the constraint** given in the problem to eliminate all but one, so the quantity is a function of x alone. **State the domain** of x (the physically possible values).
3. **Differentiate** and solve $\frac{dV}{dx}=0$ (or $\frac{dA}{dx}=0$, …) for the stationary value of x.
4. **Justify** that this gives a maximum (or minimum) — using a sign diagram of the derivative, or the sign of the second derivative ($f''<0 \Rightarrow$ maximum, $f''>0 \Rightarrow$ minimum).
5. **State the answer** in context, including units, and check it lies in the domain.

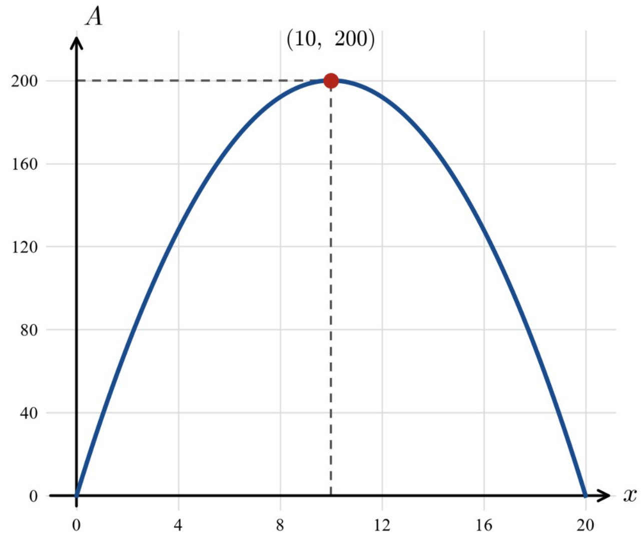
Always keep the domain in mind: sometimes the optimum occurs at an **endpoint** of the domain rather than at a stationary point.

Worked examples

Example 1 — scaffolded

A farmer uses 40 m of fencing to enclose a rectangular paddock, using an existing straight wall as one side (so only **three** sides are fenced). Find the dimensions giving the maximum area.

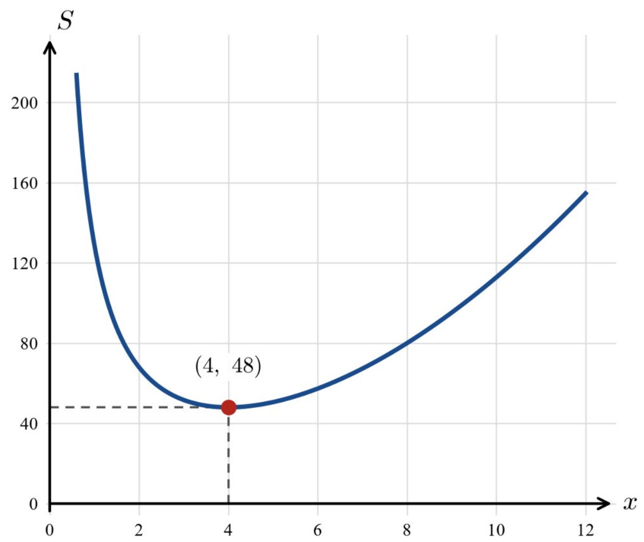
Working	Comment
Let the two sides perpendicular to the wall be x m each. The side parallel to the wall is $40-2x$ m.	Only three sides are fenced, so $2x+(\text{length})=40$.
Area $A=x(40-2x)=40x-2x^2$, with domain $0<x<20$.	Express A in one variable; $40-2x>0$ gives the domain.
$\frac{dA}{dx}=40-4x$. Setting $40-4x=0$ gives $x=10$.	Differentiate and solve $A'=0$.
$\frac{d^2A}{dx^2}=-4<0$, so $x=10$ gives a **maximum**.	Second-derivative test confirms a maximum.
Length $=40-2(10)=20$ m, so $A=10\times 20=200$. $\therefore$ maximum area 200 m² (10 m by 20 m).	State the answer in context.



Example 2 — scaffolded

An open-top box with a square base is to have a volume of 32 cm³. Find the dimensions that use the least material (minimum surface area).

Working	Comment
Let the base be x cm by x cm and the height h cm. Volume: $x^2h=32$, so $h=\frac{32}{x^2}$.	Use the volume constraint to express h in terms of x.
Surface area (base + four sides): $S=x^2+4xh=x^2+4x\cdot\frac{32}{x^2}=x^2+\frac{128}{x}$, $x>0$.	Open top, so no lid. Substitute for h.
$\frac{dS}{dx}=2x-\frac{128}{x^2}$. Setting this to 0: $2x^3=128\Rightarrow x^3=64\Rightarrow x=4$.	Solve $S'=0$.
$\frac{d^2S}{dx^2}=2+\frac{256}{x^3}>0$, so $x=4$ gives a **minimum**.	Second-derivative test confirms a minimum.
$h=\frac{32}{16}=2$, so $S=16+\frac{128}{4}=48$. ∴ minimum surface area 48 cm² (base 4 cm, height 2 cm).	State the answer with dimensions.



Example 3 — unscaffolded

An open box is made from a square sheet of card of side 12 cm by cutting a square of side x cm from each corner and folding up the sides. Find the value of x that maximises the volume.

The base is $(12-2x)$ by $(12-2x)$ and the height is x, so

$$V=x(12-2x)^2, 0<x<6.$$

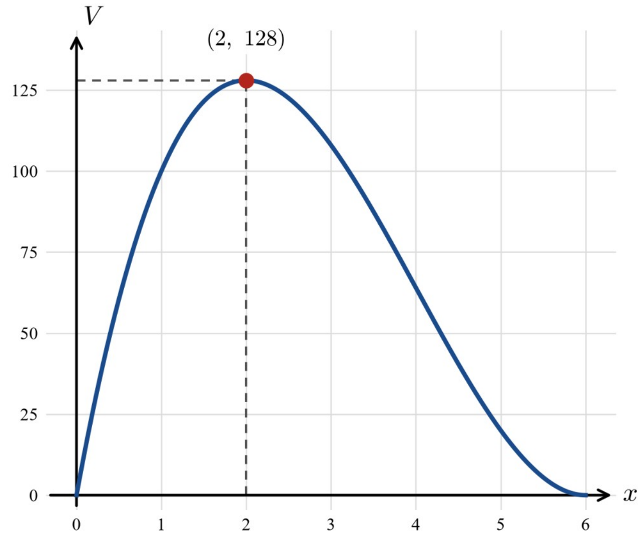
$\frac{dV}{dx}=(12-2x)^2+x\cdot 2(12-2x)(-2)=(12-2x)[(12-2x)-4x]=(12-2x)(12-6x)$.

Setting $V'=0$: $12-2x=0$ (i.e. $x=6$, rejected) or $12-6x=0$, giving $x=2$.

For $0<x<2$, $V'>0$; for $2<x<6$, $V'<0$, so $x=2$ gives a maximum.

$V(2)=2(12-4)^2=2\times 64=128$.

∴ the maximum volume is 128 cm³, when $x=2$ cm.



Example 4 — unscaffolded

A rectangle has its base on the x-axis and its two upper vertices on the parabola $y=12-x^2$. Find the maximum possible area of the rectangle.

By symmetry the upper vertices are $(\pm x, 12-x^2)$, so the rectangle has width $2x$ and height $12-x^2$:

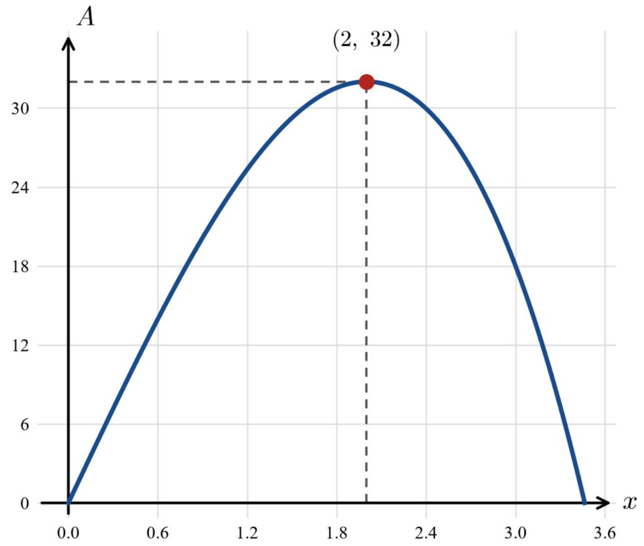
$$A=2x(12-x^2)=24x-2x^3, 0<x<2\sqrt{3}.$$

$\frac{dA}{dx}=24-6x^2=0 \Rightarrow x^2=4 \Rightarrow x=2$ (taking the positive value).

$\frac{d^2A}{dx^2}=-12x<0$ at $x=2$, so this is a maximum.

$A=2(2)(12-4)=4\times 8=32$.

∴ the maximum area is 32 square units.



Practice questions

Scaffolded

Q1. A rectangle has a perimeter of 60 cm. Let one side be x cm. (a) Write the area A in terms of x and state the domain; (b) find $\frac{dA}{dx}$ and solve $\frac{dA}{dx}=0$; (c) justify that this gives a maximum; (d) state the maximum area.

Q2. A rectangular enclosure is fenced on three sides (100 m of fencing), the fourth side being a wall. Let each of the two equal sides be x m. (a) Write the area A in terms of x and state the domain; (b) find the value of x for which the area is greatest; (c) state the maximum area.

Q3. Two positive numbers have a sum of 12. Let one of them be x. (a) Write the sum of their squares, S, in terms of x ; (b) find the value of x that minimises S; (c) state the minimum value of S.

Q4. An open box is made from an 18 cm square sheet by cutting squares of side x from the corners. (a) Write the volume V in terms of x and state the domain; (b) find the value of x that maximises V; (c) state the maximum volume.

Q5. A rectangle in the first quadrant has one side on the x-axis, one side on the y-axis, and its top-right vertex on the line $y=6-x$. Let the vertex be $(x, 6-x)$. (a) Write the area A in terms of x; (b) find the maximum area.

Q6. An open-top box with a square base is to hold 108 cm³. (a) Show that the surface area is $S=x^2+\frac{432}{x}$, where x is the base length; (b) find the dimensions that minimise S; (c) state the minimum surface area.

Unscaffolded

In each optimisation problem, define your variable, use calculus to find the optimum, justify its nature, and state the answer with units where appropriate.

Q7. A rectangle has a perimeter of 80 m. Find the maximum possible area.

Q8. A rectangular field is fenced on three sides with 60 m of fencing (the fourth side is a river). Find the maximum area.

Q9. Two positive numbers add to 20. Find the maximum value of their product.

Q10. An open box is made from a 24 cm square sheet by cutting squares of side x from the corners. Find the maximum volume.

Q11. A **closed** box with a square base is to have a volume of 27 cm³. Find the dimensions that minimise the total surface area.

Q12. A rectangle has its base on the x-axis and its upper vertices on $y=27-x^2$. Find the maximum area.

Q13. Find the point on the curve $y^2=x$ that is closest to the point $(4,0)$, and state the minimum distance.

Q14. A company's profit, in dollars, from selling x items is $P=-x^2+120x-1100$. Find the number of items that maximises the profit, and the maximum profit.

Q15. For $x>0$, find the minimum value of $y=x+\frac{16}{x}$.

Q16. Two positive numbers have a sum of 40. Find the minimum value of the sum of their squares.

Q17. A rectangular pen is fenced on three sides with 24 m of fencing (the fourth side is a wall). Find the maximum area.

Q18. An open-top box with a square base has a total surface area of 300 cm². Find the base length that maximises the volume, and state the maximum volume.

Q19. For $x>0$, find the minimum value of $y=x^2+\frac{250}{x}$.

Q20. A piece of wire 100 cm long is bent to form a rectangle. Find the dimensions that give the maximum area, and state that area.

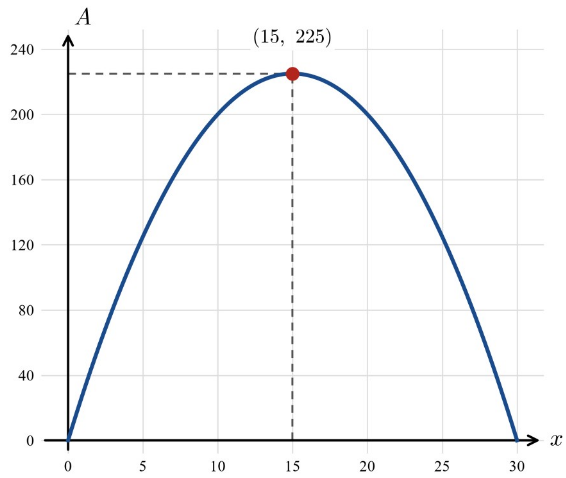
Q21. An open box is made from a 30 cm square sheet by cutting squares of side x from the corners. Show that $V=4x(15-x)^2$, and hence find the maximum volume.

Answers

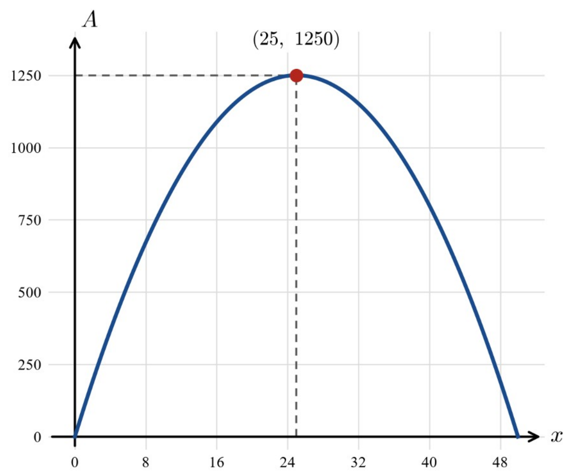
Q1. $A=x(30-x)$, $0<x<30$; max area 225 cm² (a 15 cm by 15 cm square). **Q2.** $A=x(100-2x)$, $0<x<50$; max area 1250 m² ($x=25$). **Q3.** $S=x^2+(12-x)^2$; minimum 72 at $x=6$ (numbers 6 and 6). **Q4.** $V=x(18-2x)^2$, $0<x<9$; max volume 432 cm³ at $x=3$. **Q5.** $A=x(6-x)$; max area 9 (at $x=3$). **Q6.** base 6 cm, height 3 cm; minimum surface area 108 cm². **Q7.** Max area 400 m² (a 20 m square). **Q8.** Max area 450 m² (15 m by 30 m). **Q9.** Max product 100 (10 and 10). **Q10.** Max volume 1024 cm³ at $x=4$. **Q11.** A cube 3 cm by 3 cm by 3 cm; minimum surface area 54 cm². **Q12.** Max area 108 (at $x=3$). **Q13.** Closest point $\left(\frac{7}{2},\sqrt{\frac{7}{2}}\right)$; minimum distance $\frac{\sqrt{15}}{2}$. **Q14.** 60 items; maximum profit $\$2500$. **Q15.** Minimum 8 (at $x=4$). **Q16.** Minimum 800 (20 and 20). **Q17.** Max area 72 m² ($x=6$). **Q18.** Base 10 cm; maximum volume 500 cm³. **Q19.** Minimum 75 (at $x=5$). **Q20.** A 25 cm by 25 cm square; maximum area 625 cm². **Q21.** Maximum volume 2000 cm³ at $x=5$.

Fully-worked solutions

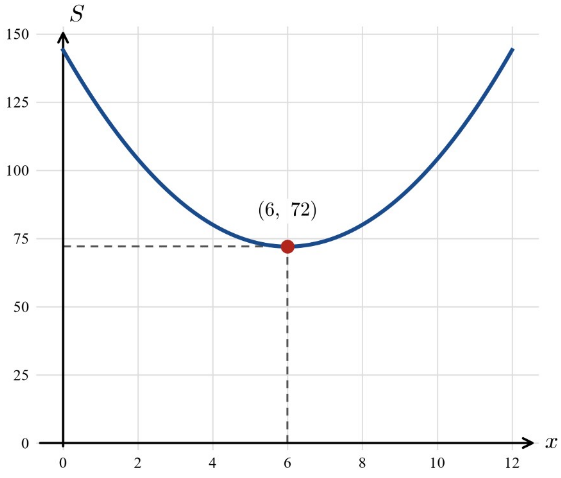
Q1. $A=x(30-x)=30x-x^2$, $0<x<30$. $A'=30-2x=0\Rightarrow x=15$; $A''=-2<0$, a maximum. $A=15\times 15=225$ cm².



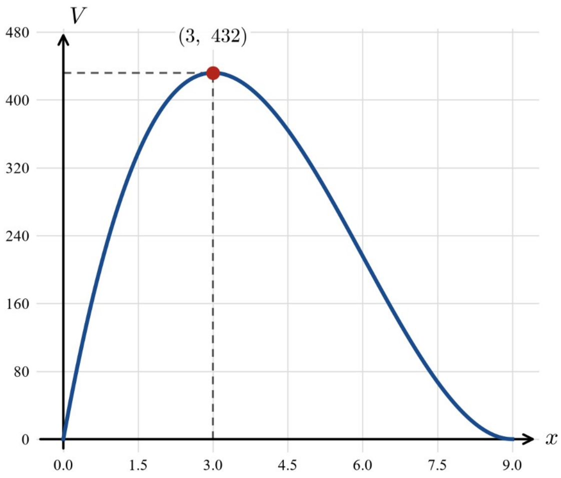
Q2. Each equal side x, the third side $100-2x$. $A=x(100-2x)$, $0<x<50$. $A'=100-4x=0\Rightarrow x=25$; $A''=-4<0$, a maximum. Third side $100-50=50$, $A=25\times 50=1250$ m².



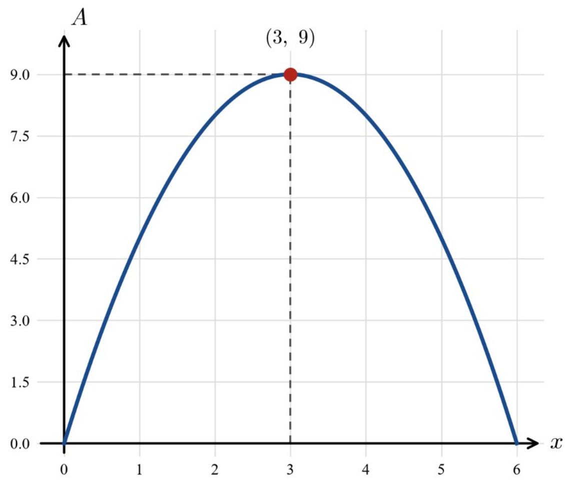
Q3. Numbers x and $12-x$. $S=x^2+(12-x)^2=2x^2-24x+144$. $S'=4x-24=0\Rightarrow x=6$; $S''=4>0$, a minimum. $S=36+36=72$.



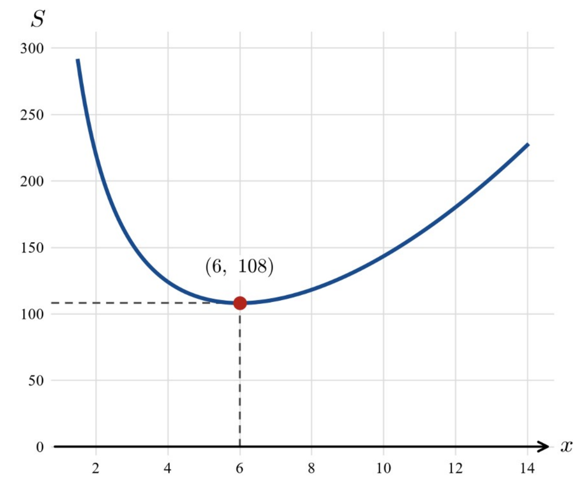
Q4. $V = x(18-2x)^2$, $0<x<9$. $V' = (18-2x)^2 + x \cdot 2(18-2x)(-2) = (18-2x)(18-6x)$. $V'=0$ at $x=9$ (rejected) or $x=3$. Sign of V' changes + to − at $x=3$, a maximum. $V(3) = 3(12)^2 = 432$ cm³.



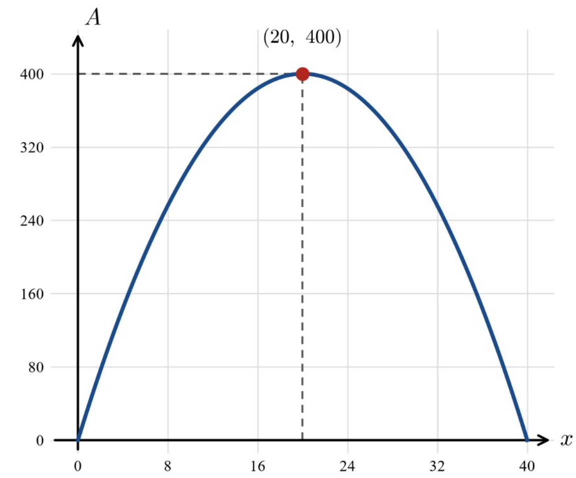
Q5. $A = x(6-x) = 6x - x^2$, $0<x<6$. $A' = 6-2x = 0 \Rightarrow x=3$; $A'' = -2 < 0$, a maximum. $A = 3\times3 = 9$.



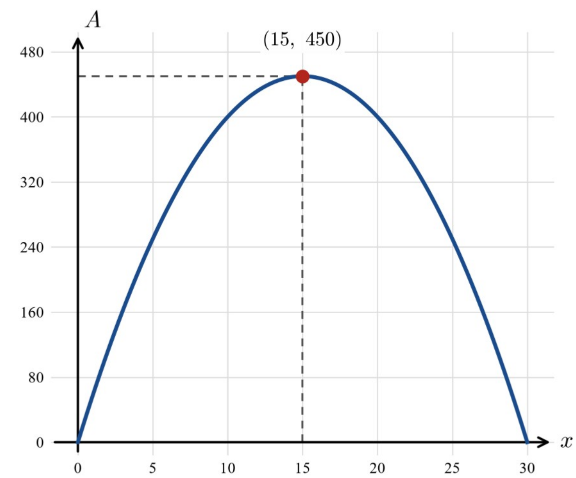
Q6. Base x, height h, with $x^2 h = 108$ so $h = \dfrac{108}{x^2}$. $S = x^2 + 4xh = x^2 + \dfrac{432}{x}$. $S' = 2x - \dfrac{432}{x^2} = 0 \Rightarrow x^3 = 216 \Rightarrow x=6$; $S'' = 2 + \dfrac{864}{x^3} > 0$, a minimum. $h = \dfrac{108}{36} = 3$, $S = 36 + \dfrac{432}{6} = 108$ cm².



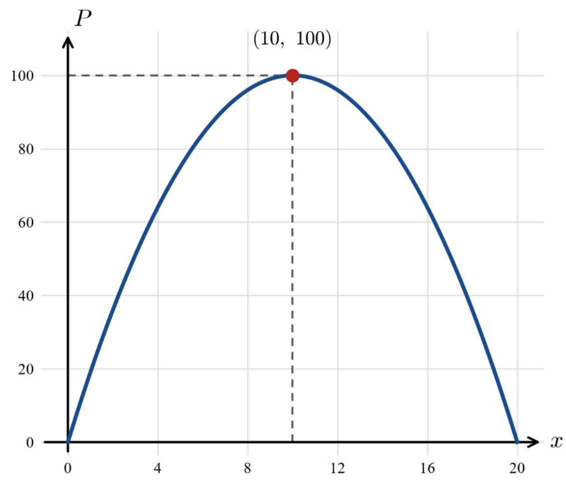
Q7. Sides x and $40-x$ (half the perimeter is 40). $A=x(40-x)$, $0<x<40$. $A'=40-2x=0 \Rightarrow x=20$; $A''=-2<0$, a maximum. $A=20\times 20=400$ m².



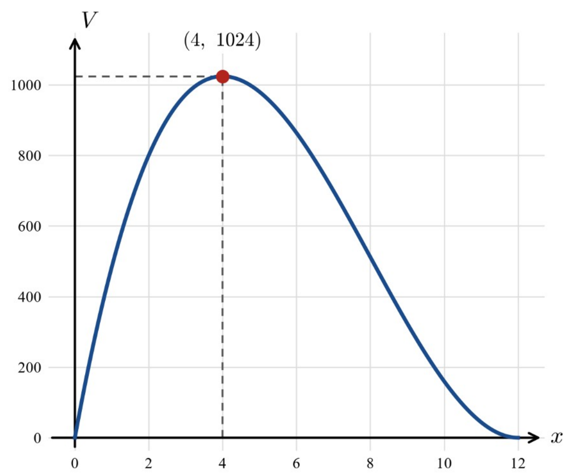
Q8. Two equal sides x, third side $60-2x$. $A=x(60-2x)$, $0<x<30$. $A'=60-4x=0 \Rightarrow x=15$; $A''=-4<0$, a maximum. Third side 30, $A=15\times 30=450$ m².



Q9. Numbers x and $20-x$. $P=x(20-x)=20x-x^2$. $P'=20-2x=0 \Rightarrow x=10$; $P''=-2<0$, a maximum. $P=10\times 10=100$.

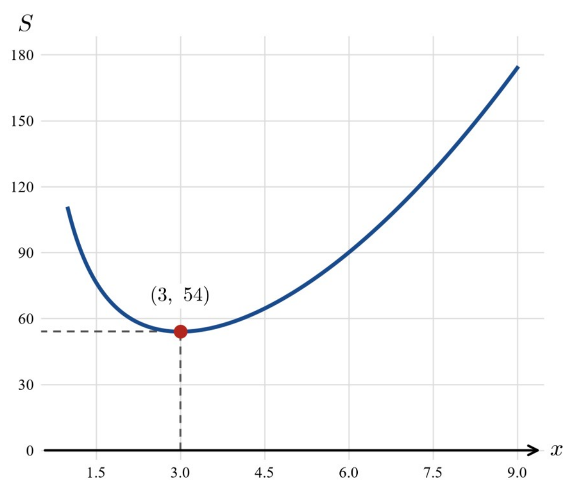


Q10. $V = x(24-2x)^2$, $0 < x < 12$. $V' = (24-2x)(24-6x)$; $V' = 0$ at $x = 12$ (rejected) or $x = 4$. Sign change + to − at $x = 4$, a maximum. $V(4) = 4(16)^2 = 1024$ cm³.

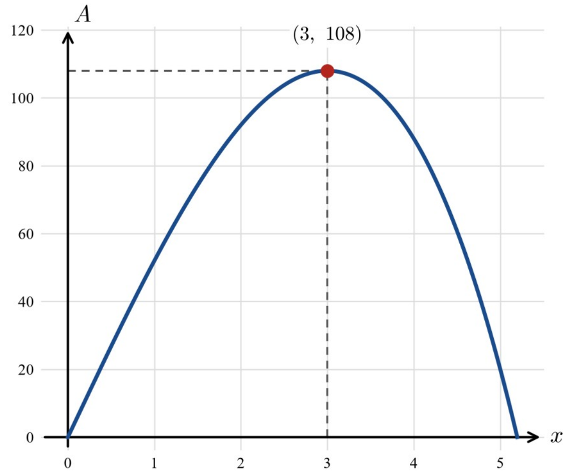


Q11. Base x, height h, $x^2h = 27$ so $h = \dfrac{27}{x^2}$. Closed box: $S = 2x^2 + 4xh = 2x^2 + \dfrac{108}{x}$.

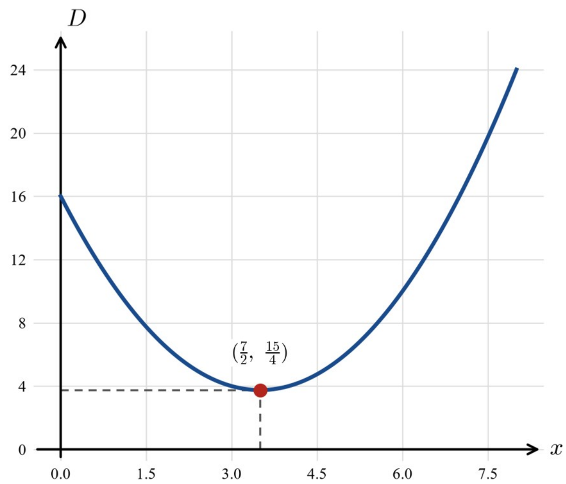
$S' = 4x - \dfrac{108}{x^2} = 0 \Rightarrow x^3 = 27 \Rightarrow x = 3$; $S'' > 0$, a minimum. $h = \dfrac{27}{9} = 3$ (a cube), $S = 18 + 36 = 54$ cm².



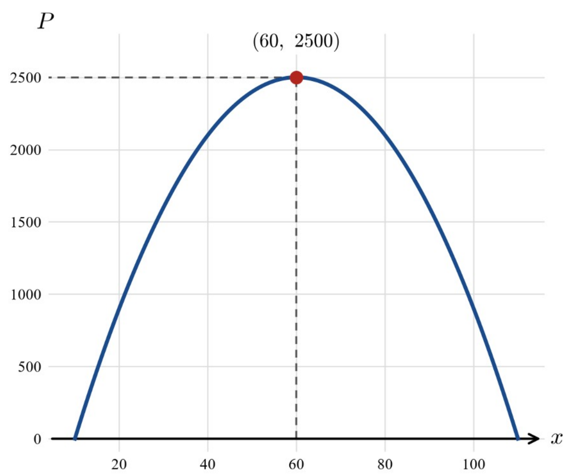
Q12. Upper vertices $(\pm x, 27 - x^2)$. $A = 2x(27 - x^2) = 54x - 2x^3$, $0 < x < 3\sqrt{3}$. $A' = 54 - 6x^2 = 0 \Rightarrow x^2 = 9 \Rightarrow x = 3$; $A'' = -12x < 0$, a maximum. $A = 2(3)(27-9) = 108$.



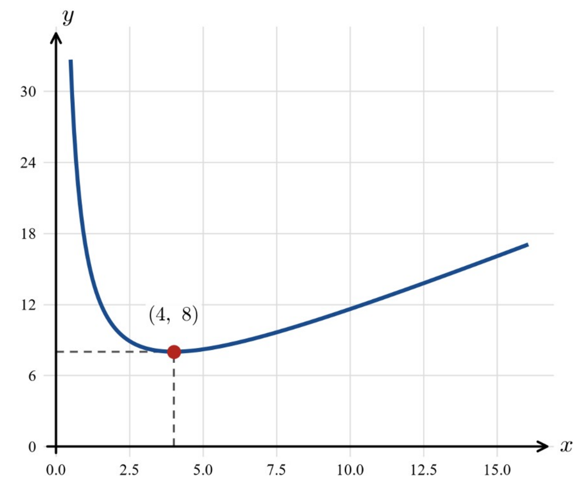
Q13. A point on $y^2=x$ is $(x,\sqrt{x})$. Its squared distance to $(4,0)$ is $D=(x-4)^2+x=x^2-7x+16$.
$D'=2x-7=0\Rightarrow x=\dfrac{7}{2}$; $D''=2>0$, a minimum. $D=\dfrac{49}{4}-\dfrac{49}{2}+16=\dfrac{15}{4}$, so the distance is $\sqrt{\dfrac{15}{4}}=\dfrac{\sqrt{15}}{2}$, at the point $\left(\dfrac{7}{2},\sqrt{\dfrac{7}{2}}\right)$.



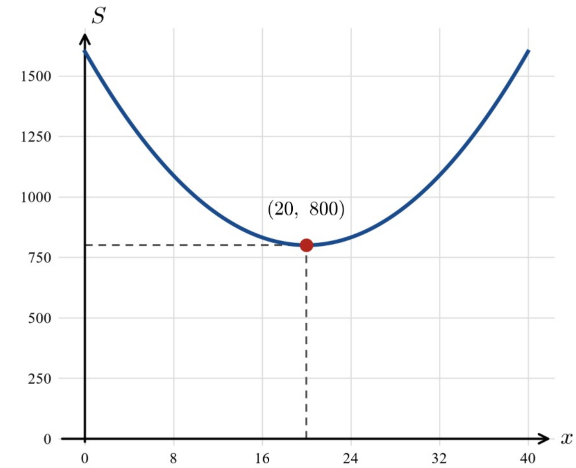
Q14. $P=-x^2+120x-1100$. $P'=-2x+120=0\Rightarrow x=60$; $P''=-2<0$, a maximum.
$P=-3600+7200-1100=2500$. $\therefore$ selling 60 items gives the maximum profit of $\$2500$.



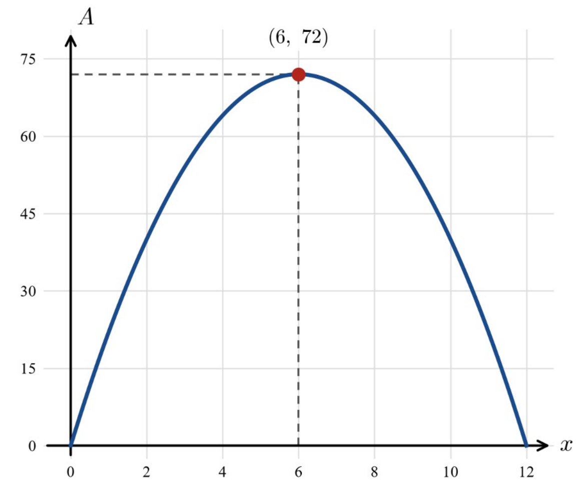
Q15. $y=x+\dfrac{16}{x}$, $x>0$. $y'=1-\dfrac{16}{x^2}=0\Rightarrow x^2=16\Rightarrow x=4$; $y''=\dfrac{32}{x^3}>0$, a minimum. $y=4+\dfrac{16}{4}=8$.



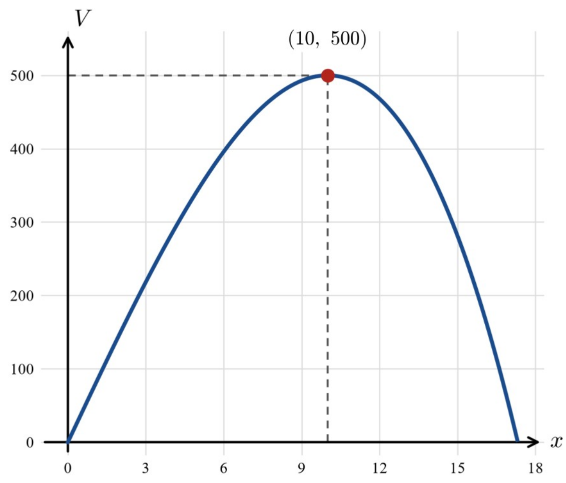
Q16. Numbers x and $40 - x$. $S = x^2 + (40 - x)^2 = 2x^2 - 80x + 1600$. $S' = 4x - 80 = 0 \Rightarrow x = 20$; $S'' = 4 > 0$, a minimum. $S = 400 + 400 = 800$.



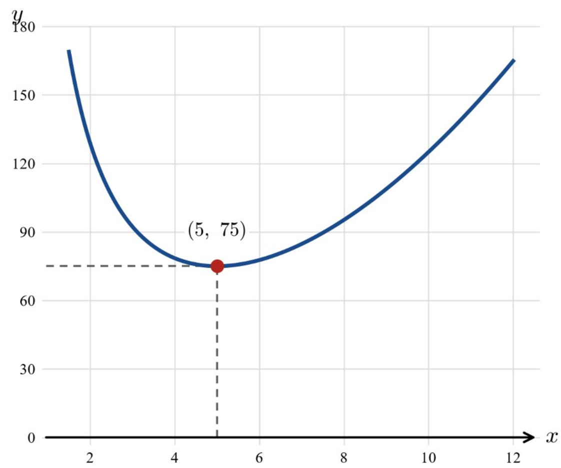
Q17. Two equal sides x, third side $24 - 2x$. $A = x(24 - 2x)$, $0 < x < 12$. $A' = 24 - 4x = 0 \Rightarrow x = 6$; $A'' = -4 < 0$, a maximum. Third side 12, $A = 6 \times 12 = 72$ m².



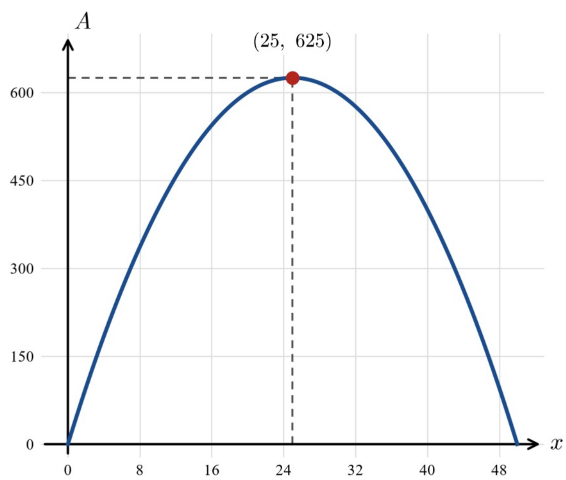
Q18. Base x, height h; open-top surface $x^2 + 4xh = 300$, so $h = \frac{300 - x^2}{4x}$. $V = x^2 h = \frac{x(300 - x^2)}{4} = \frac{300x - x^3}{4}$, $0 < x < \sqrt{300}$. $V' = \frac{300 - 3x^2}{4} = 0 \Rightarrow x^2 = 100 \Rightarrow x = 10$; $V'' = \frac{-6x}{4} < 0$, a maximum. $h = \frac{300 - 100}{40} = 5$, $V = 100 \times 5 = 500$ cm³.



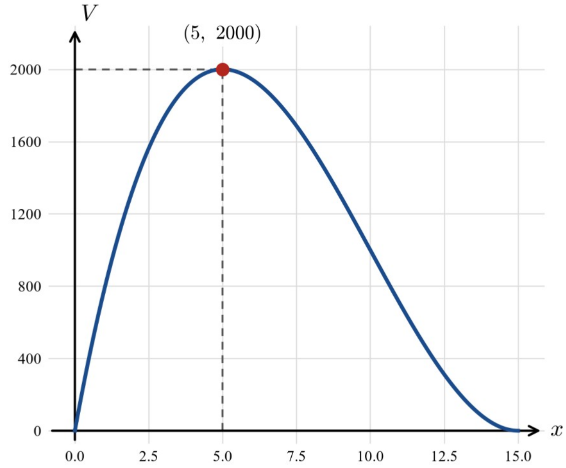
Q19. $y = x^2 + \frac{250}{x}$, $x > 0$. $y' = 2x - \frac{250}{x^2} = 0 \Rightarrow 2x^3 = 250 \Rightarrow x^3 = 125 \Rightarrow x = 5$; $y'' = 2 + \frac{500}{x^3} > 0$, a minimum. $y = 25 + \frac{250}{5} = 75$.



Q20. Sides x and $50 - x$ (half of 100). $A = x(50 - x)$, $0 < x < 50$. $A' = 50 - 2x = 0 \Rightarrow x = 25$; $A'' = -2 < 0$, a maximum. $A = 25 \times 25 = 625$ cm² (a 25 cm square).



Q21. Base $(30 - 2x)$, height x: $V = x(30 - 2x)^2 = x[2(15 - x)]^2 = 4x(15 - x)^2$, $0 < x < 15$. $V' = 4(15 - x)^2 + 4x \cdot 2(15 - x)(-1) = 4(15 - x)(15 - 3x)$. $V' = 0$ at $x = 15$ (rejected) or $x = 5$. Sign change + to − at $x = 5$, a maximum. $V(5) = 4(5)(10)^2 = 2000$ cm³.



V
(5, 2000)
2000
1600
1200
800
400
0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
x

Theory

A **family of functions** is a set of functions sharing a common rule that contains one or more **parameters** (constants such as a, k). Each value of the parameter gives one member of the family. Calculus lets us describe how the **stationary points** of the family depend on the parameter.

Differentiating with a parameter. Treat the parameter as a **constant** and differentiate with respect to x in the usual way. For example, if $y = x^3 - 3ax$ then $\frac{dy}{dx} = 3x^2 - 3a$.

Stationary points in terms of the parameter. Solve $\frac{dy}{dx} = 0$ for x; the solutions are expressions in the parameter. Substitute back to get the y-coordinates. For $y = x^3 - 3ax$ with $a > 0$, $3x^2 - 3a = 0$ gives $x = \pm\sqrt{a}$.

Classifying. Use the sign of the second derivative $\frac{d^2y}{dx^2}$ at each stationary point (a **local minimum** where it is positive, a **local maximum** where it is negative), or a sign diagram of $\frac{dy}{dx}$. A **stationary point of inflection** occurs where $\frac{dy}{dx} = 0$ but does not change sign.

Conditions on the parameter. The **number** of stationary points often depends on the parameter. When $\frac{dy}{dx} = 0$ is a quadratic, the **discriminant** decides: two stationary points when it is positive, one (a stationary point of inflection) when it is zero, none when it is negative.

Locus of the turning point. As the parameter varies, the turning point moves; eliminating the parameter between its x- and y-coordinates gives the **path** (locus) it traces.

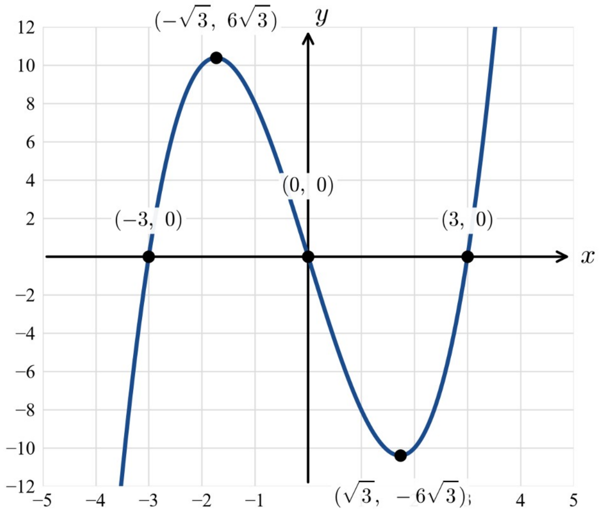
Worked examples

Example 1 — scaffolded

For the family $y = x^3 - 3ax$ where $a > 0$, find the stationary points in terms of a, classify them, and state what happens when $a \le 0$.

Working	Comment
$\frac{dy}{dx} = 3x^2 - 3a$	Differentiate, treating a as a constant.
$3x^2 - 3a = 0 \Rightarrow x^2 = a \Rightarrow x = \pm\sqrt{a}$	Solve $\frac{dy}{dx} = 0$; valid real solutions since $a > 0$.
$y(\sqrt{a}) = a\sqrt{a} - 3a\sqrt{a} = -2a\sqrt{a}$; $y(-\sqrt{a}) = 2a\sqrt{a}$	Substitute back; $a\sqrt{a} = a^{3/2}$.
$\frac{d^2y}{dx^2} = 6x$: at $x = \sqrt{a}$, $6\sqrt{a} > 0$ (min); at $x = -\sqrt{a}$, $-6\sqrt{a} < 0$ (max)	Second-derivative test.
$\therefore$ local **min** $\left(\sqrt{a}, -2a^{3/2}\right)$, local **max** $\left(-\sqrt{a}, 2a^{3/2}\right)$	Two stationary points for every $a > 0$.

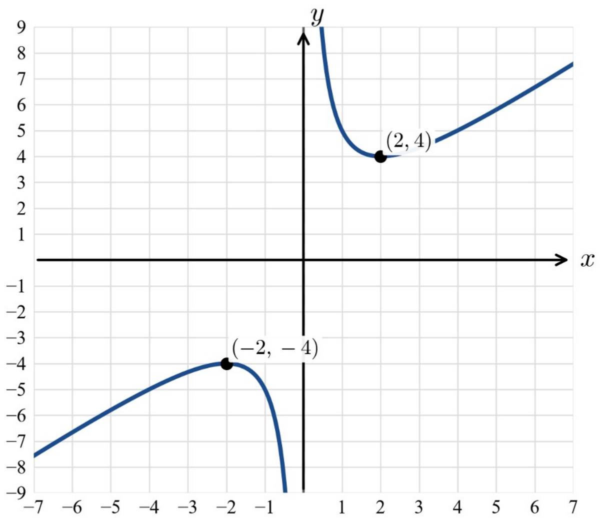
When $a = 0$, $y = x^3$ has a **stationary point of inflection** at $(0, 0)$; when $a < 0$, $3x^2 - 3a = 0$ has no real solution, so there are **no stationary points**. A representative member ($a = 3$) is shown below.



Example 2 — scaffolded

For $y = x + \dfrac{a}{x}$ $(a > 0,\ x \neq 0)$, find and classify the stationary points in terms of a.

Working	Comment
$y = x + a x^{-1} \Rightarrow \dfrac{dy}{dx} = 1 - a x^{-2} = 1 - \dfrac{a}{x^2}$	Rewrite, then differentiate.
$1 - \dfrac{a}{x^2} = 0 \Rightarrow x^2 = a \Rightarrow x = \pm\sqrt{a}$	Solve $\dfrac{dy}{dx} = 0$.
$y(\sqrt{a}) = \sqrt{a} + \dfrac{a}{\sqrt{a}} = 2\sqrt{a}$; $y(-\sqrt{a}) = -2\sqrt{a}$	Substitute back.
$\dfrac{d^2y}{dx^2} = \dfrac{2a}{x^3}$: at $x = \sqrt{a}$ positive (min); at $x = -\sqrt{a}$ negative (max) $\therefore$ min $\left(\sqrt{a}, 2\sqrt{a}\right)$, max $\left(-\sqrt{a}, -2\sqrt{a}\right)$	Classify.



Example 3 — unscaffolded

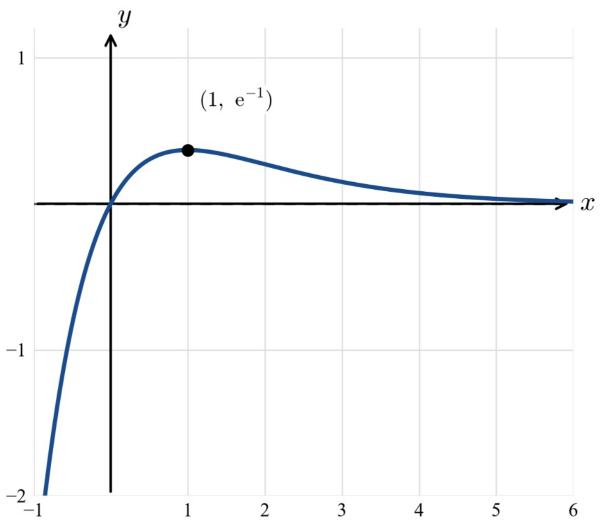
For $y = x e^{-ax}$ $(a > 0)$, find the stationary point in terms of a and classify it.

By the product rule, $\dfrac{dy}{dx} = e^{-ax} + x\left(-a e^{-ax}\right) = e^{-ax}(1 - ax)$.

Since $e^{-ax} > 0$, $\dfrac{dy}{dx} = 0 \Rightarrow 1 - ax = 0 \Rightarrow x = \dfrac{1}{a}$.

Then $y = \dfrac{1}{a} e^{-1} = \dfrac{e^{-1}}{a}$, and $\dfrac{d^2y}{dx^2} = a e^{-ax}(ax - 2)$, which at $x = \dfrac{1}{a}$ equals $-ae^{-1} < 0$.

$\therefore$ a local **maximum** at $\left(\dfrac{1}{a}, \dfrac{e^{-1}}{a}\right)$.



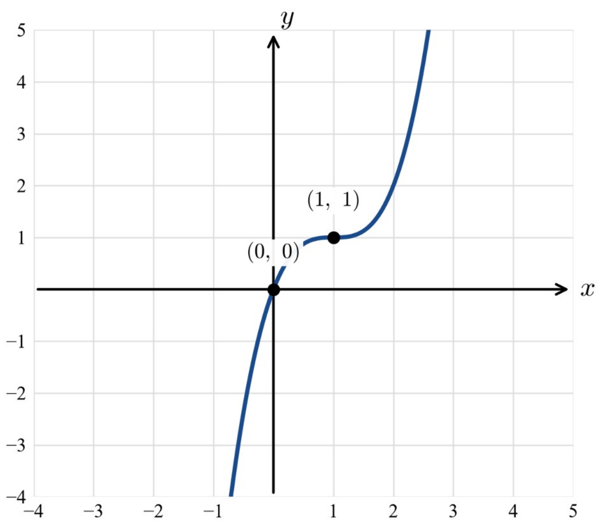
Example 4 — unscaffolded

Find the value(s) of a for which $y = x^3 - ax^2 + 3x$ has **exactly one** stationary point.

$\dfrac{dy}{dx} = 3x^2 - 2ax + 3$. This quadratic has exactly one solution (a repeated root, giving a single stationary point of inflection) when its discriminant is zero:

$$\Delta = (-2a)^2 - 4(3)(3) = 4a^2 - 36 = 4(a-3)(a+3) = 0 \Rightarrow a = 3 \text{ or } a = -3.$$

$\therefore$ exactly one stationary point when $a = 3$ or $a = -3$ (and two when $|a| > 3$, none when $|a| < 3$).



Practice questions

Take each parameter to be **positive** unless stated otherwise.

Scaffolded

Q1. For $y = x^3 - 3a^2x$: (a) find $\dfrac{dy}{dx}$; (b) find the stationary points in terms of a; (c) classify each; (d) sketch a representative member.

Q2. For $y = x + \dfrac{a^2}{x}$ $(x \neq 0)$: (a) find $\dfrac{dy}{dx}$; (b) find the stationary points in terms of a; (c) classify each.

Q3. For $y = ax^2 - x^4$: (a) find $\dfrac{dy}{dx}$ and factorise it; (b) find the three stationary points in terms of a; (c) classify each.

Q4. For $y = xe^{ax}$: (a) find $\dfrac{dy}{dx}$; (b) find the stationary point in terms of a; (c) classify it.

Q5. For $y = x^3 - ax$: (a) find the stationary points in terms of a; (b) classify each.

Q6. For $y = \dfrac{x}{x^2 + a}$: (a) show that $\dfrac{dy}{dx} = \dfrac{a - x^2}{(x^2 + a)^2}$; (b) find and classify the stationary points in terms of a.

Unscaffolded

For each family (parameter positive), find the stationary points in terms of the parameter and classify them; state any parameter conditions asked for.

Q7. $y = ax - x^3$

Q8. $y = x^2e^{-ax}$

Q9. $y = x - \dfrac{a}{x}$ — show that it has **no** stationary points.

Q10. $y = x^4 - 2a^2x^2$

Q11. $y = x^2 + \dfrac{a}{x}$

Q12. $y = (x - a)^2(x + a)$

Q13. $y = x^3 - 3ax^2$

Q14. $y = ax - x^2$

Q15. $y = a\log_e x - x$ $(x > 0)$

Q16. $y = x^3 + ax$ — explain why it has **no** stationary points.

Q17. For $y = x^3 + ax^2 + x$, find the value of a $(a > 0)$ for which the graph has exactly one stationary point.

Q18. $y = x^2(x - a)$

Q19. $y = x + \dfrac{a}{x^2}$

Q20. The turning point of $y = x^2 - 2ax + a$ moves as a varies. Show that it always lies on the curve $y = x - x^2$.

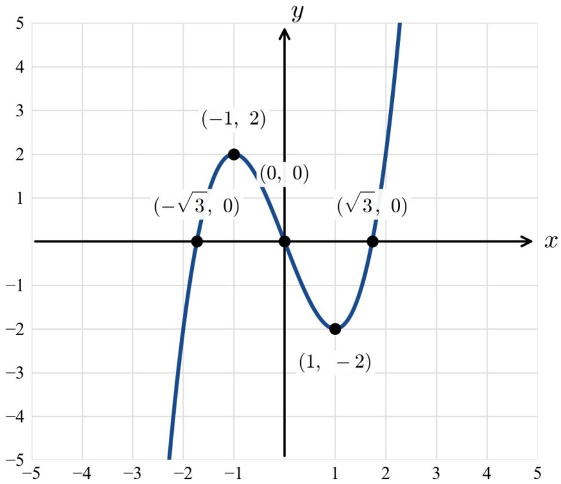
Q21. For $y = x^3 - kx$ (k real), describe how the number of stationary points depends on k.

Answers

Q1. $\frac{dy}{dx}=3x^2-3a^2$; min $(a,-2a^3)$, max $(-a,2a^3)$. **Q2.** min $(a,2a)$, max $(-a,-2a)$. **Q3.** min $(0,0)$; max $\left(\pm\frac{\sqrt{2a}}{2},\frac{a^2}{4}\right)$. **Q4.** min $\left(-\frac{1}{a},-\frac{e^{-1}}{a}\right)$. **Q5.** max $\left(-\frac{\sqrt{3a}}{3},\frac{2\sqrt{3}}{9}a^{3/2}\right)$, min $\left(\frac{\sqrt{3a}}{3},-\frac{2\sqrt{3}}{9}a^{3/2}\right)$. **Q6.** max $\left(\sqrt{a},\frac{1}{2\sqrt{a}}\right)$, min $\left(-\sqrt{a},-\frac{1}{2\sqrt{a}}\right)$. **Q7.** max $\left(\frac{\sqrt{3a}}{3},\frac{2\sqrt{3}}{9}a^{3/2}\right)$, min $\left(-\frac{\sqrt{3a}}{3},-\frac{2\sqrt{3}}{9}a^{3/2}\right)$. **Q8.** min $(0,0)$, max $\left(\frac{2}{a},\frac{4e^{-2}}{a^2}\right)$. **Q9.** $\frac{dy}{dx}=1+\frac{a}{x^2}>0$ for all $x\neq 0$, so it is strictly increasing on each branch — no stationary points. **Q10.** max $(0,0)$; min $(\pm a,-a^4)$. **Q11.** min $\left((a/2)^{1/3},\frac{3}{2}(2a^2)^{1/3}\right)$ (using $(a/2)^{1/3}=\sqrt[3]{4a}/2$). **Q12.** max $\left(-\frac{a}{3},\frac{32a^3}{27}\right)$, min $(a,0)$. **Q13.** max $(0,0)$, min $(2a,-4a^3)$. **Q14.** max $\left(\frac{a}{2},\frac{a^2}{4}\right)$. **Q15.** max $(a,a(\log_e a-1))$. **Q16.** $\frac{dy}{dx}=3x^2+a>0$ for all x (as $a>0$), so no stationary points. **Q17.** $a=\sqrt{3}$. **Q18.** max $(0,0)$, min $\left(\frac{2a}{3},-\frac{4a^3}{27}\right)$. **Q19.** min $\left(\sqrt[3]{2a},\frac{3}{2}\sqrt[3]{2a}\right)$. **Q20.** turning point $(a,a-a^2)$; eliminating a gives $y=x-x^2$. **Q21.** two stationary points when $k>0$; a stationary point of inflection at $(0,0)$ when $k=0$; none when $k<0$.

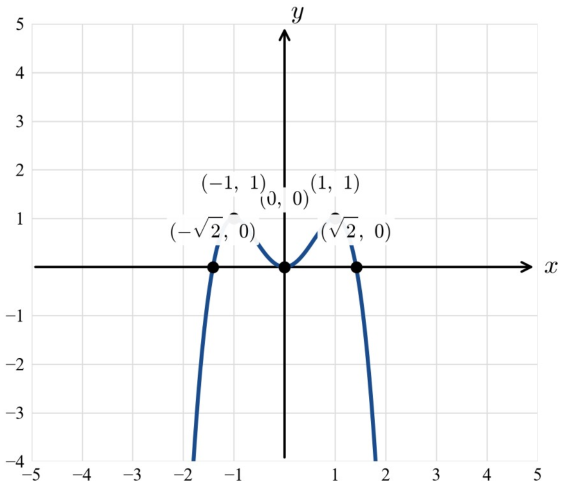
Fully-worked solutions

Q1. $y=x^3-3a^2x$, $\frac{dy}{dx}=3x^2-3a^2=3(x^2-a^2)=0\Rightarrow x=\pm a$. $y(a)=a^3-3a^3=-2a^3$, $y(-a)=2a^3$. $\frac{d^2y}{dx^2}=6x$: min $(a,-2a^3)$, max $(-a,2a^3)$.



Q2. $\frac{dy}{dx}=1-\frac{a^2}{x^2}=0\Rightarrow x^2=a^2\Rightarrow x=\pm a$. $y(a)=a+a=2a$, $y(-a)=-2a$. $\frac{d^2y}{dx^2}=\frac{2a^2}{x^3}$: min $(a,2a)$, max $(-a,-2a)$.

Q3. $\frac{dy}{dx}=2ax-4x^3=2x(a-2x^2)=0\Rightarrow x=0$ or $x^2=\frac{a}{2}$, i.e. $x=\pm\frac{\sqrt{2a}}{2}$. $y(0)=0$; $y\left(\pm\frac{\sqrt{2a}}{2}\right)=\frac{a^2}{4}$. $\frac{d^2y}{dx^2}=2a-12x^2$: at $x=0$ it is $2a>0$ (min); at $x=\pm\frac{\sqrt{2a}}{2}$ it is $-4a<0$ (max).



Q4. $\frac{dy}{dx}=e^{ax}+x(ae^{ax})=e^{ax}(1+ax)=0 \Rightarrow x=-\frac{1}{a}$. $y=-\frac{1}{a}e^{-1}=-\frac{e^{-1}}{a}$. $\frac{d^2y}{dx^2}=ae^{ax}(2+ax)$, at $x=-1/a$ equals $ae^{-1}>0$ (min).

Q5. $\frac{dy}{dx}=3x^2-a=0 \Rightarrow x=\pm\sqrt{\frac{a}{3}}=\pm\frac{\sqrt{3a}}{3}$. $y\left(\frac{\sqrt{3a}}{3}\right)=-\frac{2\sqrt{3}}{9}a^{3/2}$ (min, since $\frac{d^2y}{dx^2}=6x>0$ there), and the negative root gives the max $+\frac{2\sqrt{3}}{9}a^{3/2}$.

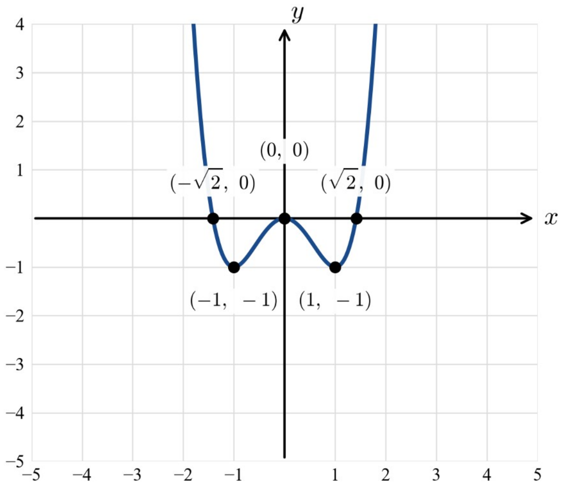
Q6. $\frac{dy}{dx}=\frac{(x^2+a)-x(2x)}{(x^2+a)^2}=\frac{a-x^2}{(x^2+a)^2}=0 \Rightarrow x=\pm\sqrt{a}$. $y(\sqrt{a})=\frac{\sqrt{a}}{2a}=\frac{1}{2\sqrt{a}}$ (max), $y(-\sqrt{a})=-\frac{1}{2\sqrt{a}}$ (min); sign of $\frac{dy}{dx}$ confirms (numerator $a-x^2$ is positive between the roots).

Q7. $\frac{dy}{dx}=a-3x^2=0 \Rightarrow x=\pm\frac{\sqrt{3a}}{3}$. $\frac{d^2y}{dx^2}=-6x$: max at $x=\frac{\sqrt{3a}}{3}$ with $y=\frac{2\sqrt{3}}{9}a^{3/2}$, min at $x=-\frac{\sqrt{3a}}{3}$.

Q8. $\frac{dy}{dx}=2xe^{-ax}-ax^2e^{-ax}=xe^{-ax}(2-ax)=0 \Rightarrow x=0$ or $x=\frac{2}{a}$. $y(0)=0$ (min, $\frac{d^2y}{dx^2}(0)=2>0$); $y\left(\frac{2}{a}\right)=\frac{4}{a^2}e^{-2}$ (max).

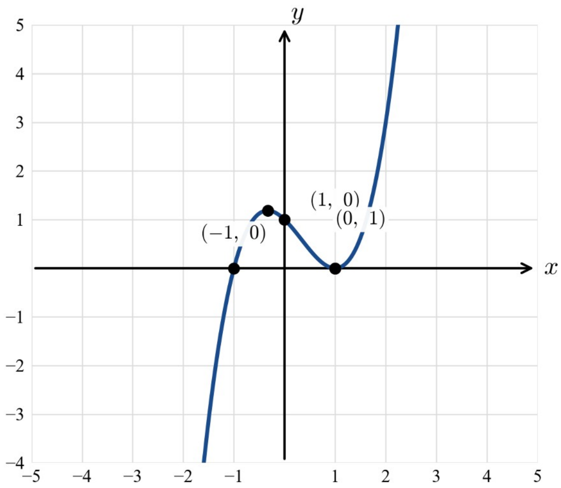
Q9. $y=x-\frac{a}{x}$, $\frac{dy}{dx}=1+\frac{a}{x^2}$. Since $a>0$ and $x^2>0$, $\frac{dy}{dx}>0$ for all $x\neq 0$; the function is strictly increasing on each branch, so it has **no stationary points**.

Q10. $\frac{dy}{dx}=4x^3-4a^2x=4x(x^2-a^2)=0 \Rightarrow x=0,\pm a$. $y(0)=0$ (max, $\frac{d^2y}{dx^2}(0)=-4a^2<0$); $y(\pm a)=a^4-2a^4=-a^4$ (min).

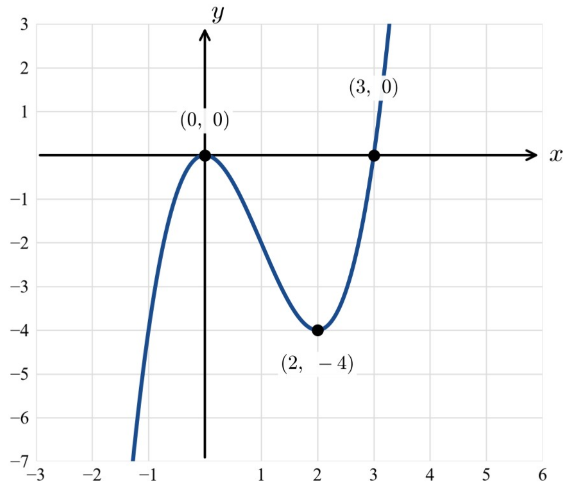


Q11. $\frac{dy}{dx}=2x-\frac{a}{x^2}=0\Rightarrow 2x^3=a\Rightarrow x=\left(\frac{a}{2}\right)^{1/3}$. $\frac{d^2y}{dx^2}=2+\frac{2a}{x^3}=2+\frac{2a}{a/2}=6>0$, so it is a **minimum**;
$y=\left(\frac{a}{2}\right)^{2/3}+a\left(\frac{2}{a}\right)^{1/3}=\frac{3}{2}(2a^2)^{1/3}$.

Q12. $y=(x-a)^2(x+a)$. $\frac{dy}{dx}=2(x-a)(x+a)+(x-a)^2=(x-a)(3x+a)=0\Rightarrow x=a$ or $x=-\frac{a}{3}$. $y(a)=0$ (min), $y\left(-\frac{a}{3}\right)=\frac{32a^3}{27}$ (max); $\frac{d^2y}{dx^2}=6x-2a$ confirms.



Q13. $\frac{dy}{dx}=3x^2-6ax=3x(x-2a)=0\Rightarrow x=0,2a$. $y(0)=0$ (max, $\frac{d^2y}{dx^2}(0)=-6a<0$);
$y(2a)=8a^3-12a^3=-4a^3$ (min).



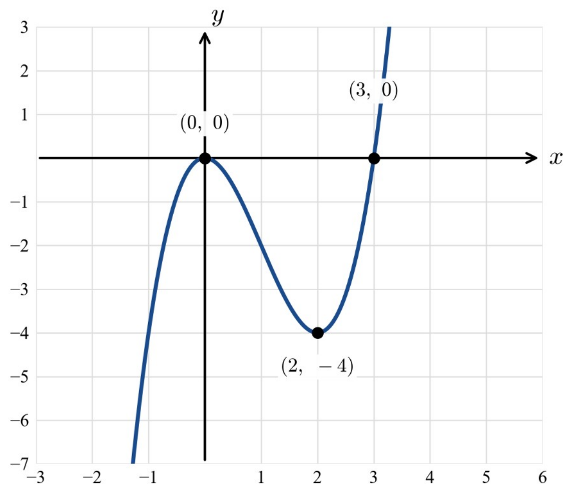
Q14. $\frac{dy}{dx}=a-2x=0 \Rightarrow x=\frac{a}{2}$, $y=\frac{a^2}{4}$; $\frac{d^2y}{dx^2}=-2<0$ (max).

Q15. $\frac{dy}{dx}=\frac{a}{x}-1=\frac{a-x}{x}=0 \Rightarrow x=a$ (in domain $x>0$). $y=a\log_e a-a=a(\log_e a-1)$; $\frac{d^2y}{dx^2}=-\frac{a}{x^2}<0$ (max).

Q16. $\frac{dy}{dx}=3x^2+a$. As $a>0$, $3x^2+a>0$ for all x, so the graph is strictly increasing and has **no stationary points**.

Q17. $\frac{dy}{dx}=3x^2+2ax+1$. Exactly one stationary point when $\Delta=(2a)^2-4(3)(1)=4a^2-12=0 \Rightarrow a^2=3$, so $a=\sqrt{3}$ (taking $a>0$).

Q18. $\frac{dy}{dx}=2x(x-a)+x^2=x(3x-2a)=0 \Rightarrow x=0,\frac{2a}{3}$. $y(0)=0$ (max); $y\left(\frac{2a}{3}\right)=-\frac{4a^3}{27}$ (min); $\frac{d^2y}{dx^2}=6x-2a$ confirms.



Q19. $y=x+\frac{a}{x^2}$, $\frac{dy}{dx}=1-\frac{2a}{x^3}=0 \Rightarrow x^3=2a \Rightarrow x=\sqrt[3]{2a}$. $\frac{d^2y}{dx^2}=\frac{6a}{x^4}>0$ (min); $y=\sqrt[3]{2a}+\frac{a}{(2a)^{2/3}}=\frac{3}{2}\sqrt[3]{2a}$.

Q20. $y=x^2-2ax+a$ is a parabola with turning point at $x=a$ (axis of symmetry), where $y=a^2-2a^2+a=a-a^2$. So the turning point is $(a,a-a^2)$. Writing $X=a$, $Y=a-a^2=X-X^2$, so the turning point always lies on $y=x-x^2$.

Q21. $\frac{dy}{dx} = 3x^2 - k$. If $k > 0$, $x = \pm\sqrt{\frac{k}{3}}$ — **two** stationary points (one max, one min). If $k = 0$, $\frac{dy}{dx} = 3x^2 \geq 0$ with a single zero at $x = 0$ that does not change sign — a **stationary point of inflection** at $(0, 0)$. If $k < 0$, $3x^2 - k > 0$ for all x — **no** stationary points.

Theory

Newton's method is a numerical procedure for finding an approximate solution of an equation $f(x)=0$. Starting from an initial estimate x_0, it produces a sequence of improving estimates using the rule

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

Geometric interpretation. At the point $(x_n, f(x_n))$ the tangent to the curve $y=f(x)$ has gradient $f'(x_n)$. This tangent meets the x-axis at $x=x_{n+1}$; that intercept is the next estimate. Repeating the process, the estimates usually move rapidly towards a point where the curve crosses the axis — a solution of $f(x)=0$.

As an algorithm (pseudocode). Using ← for assignment:

```
Inputs: f, f ′ (the derivative of f), x0 (starting estimate), n (number of
iterations)

x ← x0
for i from 1 to n
    x ← x − f(x) / f ′(x)
end for
return x
```

When Newton's method fails. The method can break down or fail to converge if:

- $f'(x_n)=0$ for some estimate — the tangent is horizontal and never meets the x-axis, so x_{n+1} is undefined (division by zero);
- the starting estimate x_0 is too far from the root, so the estimates diverge or jump to a different root;
- the estimates oscillate and do not settle.

To apply the method: (1) write $f(x)$ and differentiate to get $f'(x)$; (2) substitute the current estimate into $x-\frac{f(x)}{f'(x)}$; (3) repeat, keeping enough decimal places.

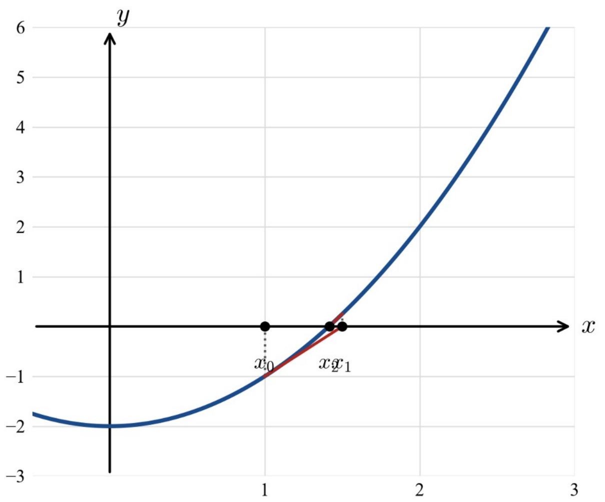
Worked examples

Example 1 — scaffolded

Use Newton's method with $f(x)=x^2-2$ and $x_0=1$ to find two further estimates of $\sqrt{2}$.

Working	Comment
$f(x)=x^2-2$, so $f'(x)=2x$.	$\sqrt{2}$ is the positive solution of $x^2-2=0$.
$x_{n+1}=x_n-\frac{x_n^2-2}{2x_n}$	Substitute f and f' into the iteration rule.
$x_1=1-\frac{1^2-2}{2(1)}=1-\frac{-1}{2}=\frac{3}{2}$	First iteration from $x_0=1$.
$x_2=\frac{3}{2}-\frac{(3/2)^2-2}{2(3/2)}=\frac{3}{2}-\frac{1/4}{3}=\frac{17}{12}$	Second iteration. $\frac{17}{12}\approx 1.41667$.

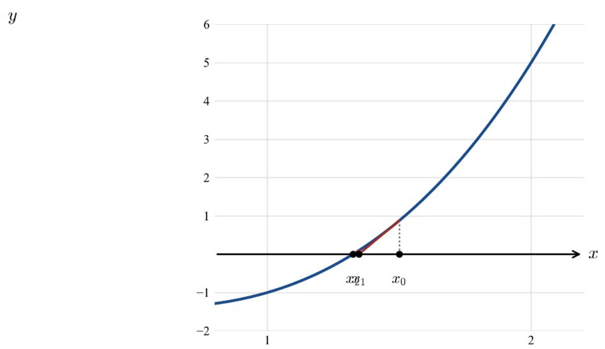
Working	Comment
$\therefore x_2 = \dfrac{17}{12} \approx 1.4167$, close to $\sqrt{2} \approx 1.4142$.	The estimate is accurate to two decimal places after two steps.



Example 2 — scaffolded

The equation $x^3 - x - 1 = 0$ has one real solution. Starting from $x_0 = 1.5$, apply two iterations of Newton's method, giving each estimate correct to five decimal places.

Working	Comment
$f(x) = x^3 - x - 1$, so $f'(x) = 3x^2 - 1$.	Differentiate the cubic.
$x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$	The iteration rule.
$x_1 = 1.5 - \dfrac{1.5^3 - 1.5 - 1}{3(1.5)^2 - 1} = 1.5 - \dfrac{0.875}{5.75} = 1.34783$	First iteration.
$x_2 = 1.34783 - \dfrac{1.34783^3 - 1.34783 - 1}{3(1.34783)^2 - 1} = 1.32520$	Second iteration.
$\therefore$ the solution is approximately 1.32520.	Further iterations give 1.32472.



Example 3 — unscaffolded

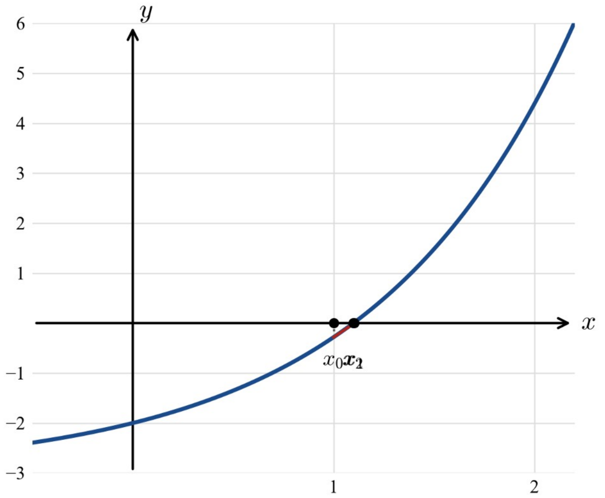
Use Newton's method with $x_0 = 1$ to find an approximate solution of $e^x - 3 = 0$, correct to four decimal places, and state the exact solution.

$f(x) = e^x - 3$, so $f'(x) = e^x$, and $x_{n+1} = x_n - \dfrac{e^{x_n} - 3}{e^{x_n}}$.

$x_1 = 1 - \frac{e-3}{e} = 1.10364.$

$x_2 = 1.10364 - \frac{e^{1.10364} - 3}{e^{1.10364}} = 1.09862.$

$\therefore$ the solution is approximately 1.0986. The exact solution is $x = \log_e 3 \approx 1.0986$.



Example 4 — unscaffolded

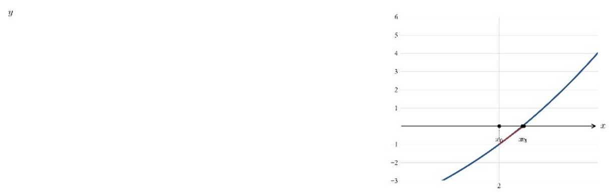
Apply two iterations of Newton's method to $f(x) = x^3 - 2x - 5$, starting from $x_0 = 2$, giving each estimate correct to four decimal places.

$f'(x) = 3x^2 - 2$, so $x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$.

$x_1 = 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2 - \frac{-1}{10} = 2.1000.$

$x_2 = 2.1 - \frac{2.1^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = 2.1 - \frac{0.061}{11.23} = 2.0946.$

$\therefore$ the solution is approximately 2.0946.



Practice questions

Scaffolded

Q1. For $f(x) = x^2 - 3$ with $x_0 = 2$ (approximating $\sqrt{3}$): (a) write the iteration formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for this f; (b) find x_1 as an exact fraction; (c) find x_2 as an exact fraction, and its decimal value; (d) compare x_2 with $\sqrt{3}$.

Q2. For $f(x) = x^2 - 7$ with $x_0 = 3$: (a) state $f'(x)$; (b) find x_1 and x_2 (exact fractions); (c) give x_2 correct to four decimal places.

Q3. For $f(x)=x^3-20$ with $x_0=3$ (approximating $\sqrt[3]{20}$): (a) write the iteration formula; (b) find x_1; (c) find x_2 correct to four decimal places.

Q4. For $f(x)=x^3+x-3$ with $x_0=1$: (a) state $f'(x)$; (b) find x_1 (exact); (c) find x_2 (exact), and its decimal value.

Q5. Consider the pseudocode:

```
x ← 2
for i from 1 to 2
    x ← x − (x^2 − 5) / (2*x)
end for
return x
```

(a) What equation is being solved, and what value is being approximated?
(b) Trace the algorithm, writing down x after each pass of the loop.

Q6. Explain, with reference to the tangent line, why x_{n+1} is the x-intercept of the tangent to $y=f(x)$ at $(x_n, f(x_n))$, and state one situation in which Newton's method fails.

Unscaffolded

For **Q7–Q14** and **Q17–Q21**, apply **two iterations** of Newton's method from the given x_0, giving each required estimate to four decimal places (or exact fractions where indicated).

Q7. $f(x)=x^2-10$, $x_0=3$ (approximate $\sqrt{10}$).

Q8. $f(x)=x^3-7$, $x_0=2$ (approximate $\sqrt[3]{7}$).

Q9. $f(x)=x^3-2x-1$, $x_0=1.5$.

Q10. $f(x)=x^2-x-3$, $x_0=2$ (exact fractions for x_1, x_2).

Q11. $f(x)=x^3+2x-1$, $x_0=0.5$.

Q12. $f(x)=x^4-5$, $x_0=1$. Comment on what happens to the estimate after the first step.

Q13. $f(x)=e^x-2$, $x_0=0.5$. State the exact solution.

Q14. $f(x)=xe^x-1$, $x_0=0.5$.

Q15. Write pseudocode (using $\leftarrow$) that applies Newton's method n times to solve $f(x)=0$ from a starting value x_0, taking f, f', x_0 and n as inputs.

Q16. For $f(x)=x^3-3x$, explain why Newton's method fails if the starting value is $x_0=1$.

Q17. $f(x)=x^3-5x-1$, $x_0=2$.

Q18. $f(x)=x^3-x-2$, $x_0=1.5$.

Q19. $f(x)=x^2-6$, $x_0=2.5$ (exact fractions).

Q20. $f(x)=x^3-3x-5$, $x_0=2$.

Q21. $f(x)=\cos x-x$, $x_0=1$ (radians). Two iterations to four decimal places.

Answers

Q1. (a) $x_{n+1}=x_n-\frac{x_n^2-3}{2x_n}$; (b) $\frac{7}{4}$; (c) $\frac{97}{56}\approx 1.7321$; (d) $\sqrt{3}\approx 1.7321$ — equal to 4 d.p. **Q2.** (a) $2x$; (b) $\frac{8}{3},\frac{127}{48}$; (c) 2.6458. **Q3.** (a) $x_{n+1}=x_n-\frac{x_n^3-20}{3x_n^2}$; (b) $\frac{74}{27}$; (c) 2.7147. **Q4.** (a) $3x^2+1$; (b) $\frac{5}{4}$; (c) $\frac{17}{14}\approx 1.2143$. **Q5.** (a) $x^2-5=0$; it approximates $\sqrt{5}$. (b) $x=2.25$ then $x=\frac{161}{72}\approx 2.2361$. **Q6.** The tangent at $(x_n,f(x_n))$ has equation $y-f(x_n)=f'(x_n)(x-x_n)$; setting $y=0$ gives $x=x_n-\frac{f(x_n)}{f'(x_n)}=x_{n+1}$. It fails when $f'(x_n)=0$ (horizontal tangent). **Q7.** $x_1=\frac{19}{6},x_2=\frac{721}{228}\approx 3.1623$. **Q8.** $x_1=\frac{23}{12},x_2\approx 1.9129$. **Q9.** $x_1=\frac{31}{19}\approx 1.6316,x_2\approx 1.6182$. **Q10.** $x_1=\frac{7}{3},x_2=\frac{76}{33}\approx 2.3030$. **Q11.** $x_1=\frac{5}{11}\approx 0.4545,x_2\approx 0.4534$. **Q12.** $x_1=2,x_2=\frac{53}{32}\approx 1.6563$. The first step overshoots to 2 (above the root $\sqrt[4]{5}\approx 1.4953$), then the estimates return towards it. **Q13.** $x_1\approx 0.7131,x_2\approx 0.6933$; exact solution $x=\log_e 2\approx 0.6931$. **Q14.** $x_1\approx 0.5710,x_2\approx 0.5672$. **Q15.** See solutions. **Q16.** $f'(x)=3x^2-3$, and $f'(1)=0$: the tangent at $x_0=1$ is horizontal, so it never meets the x-axis and x_1 is undefined. **Q17.** $x_1=\frac{17}{7}\approx 2.4286,x_2\approx 2.3356$. **Q18.** $x_1=\frac{35}{23}\approx 1.5217,x_2\approx 1.5214$. **Q19.** $x_1=\frac{49}{20},x_2=\frac{4801}{1960}\approx 2.4495$. **Q20.** $x_1=\frac{7}{3}\approx 2.3333,x_2\approx 2.2806$. **Q21.** $x_1\approx 0.7504,x_2\approx 0.7391$.

Fully-worked solutions

Q1. $f(x)=x^2-3$, $f'(x)=2x$. (a) $x_{n+1}=x_n-\frac{x_n^2-3}{2x_n}$. (b) $x_1=2-\frac{4-3}{4}=2-\frac{1}{4}=\frac{7}{4}$. (c) $x_2=\frac{7}{4}-\frac{(7/4)^2-3}{2(7/4)}=\frac{7}{4}-\frac{1/16}{7/2}=\frac{97}{56}\approx 1.7321$. (d) $\sqrt{3}\approx 1.7321$, so x_2 agrees to four decimal places.

Q2. $f'(x)=2x$. $x_1=3-\frac{9-7}{6}=\frac{8}{3}$; $x_2=\frac{8}{3}-\frac{64/9-7}{16/3}=\frac{127}{48}\approx 2.6458$.

Q3. $f'(x)=3x^2$, $x_{n+1}=x_n-\frac{x_n^3-20}{3x_n^2}$. $x_1=3-\frac{27-20}{27}=\frac{74}{27}\approx 2.7407$; $x_2\approx 2.7147$.

Q4. $f'(x)=3x^2+1$. $x_1=1-\frac{1+1-3}{3+1}=1-\frac{-1}{4}=\frac{5}{4}$; $x_2=\frac{5}{4}-\frac{(5/4)^3+5/4-3}{3(5/4)^2+1}=\frac{17}{14}\approx 1.2143$.

Q5. (a) The update uses $\frac{x^2-5}{2x}$, so it solves $x^2-5=0$ and approximates $\sqrt{5}$. (b) Start $x=2$. Pass 1: $x=2-\frac{4-5}{4}=2+\frac{1}{4}=2.25$. Pass 2: $x=2.25-\frac{2.25^2-5}{4.5}=\frac{161}{72}\approx 2.2361$.

Q6. The tangent to $y=f(x)$ at $(x_n,f(x_n))$ is $y-f(x_n)=f'(x_n)(x-x_n)$. Putting $y=0$: $-f(x_n)=f'(x_n)(x-x_n)$, so $x=x_n-\frac{f(x_n)}{f'(x_n)}=x_{n+1}$, i.e. the tangent's x-intercept is the next estimate. The method fails when $f'(x_n)=0$ (a horizontal tangent, which never meets the x-axis).

Q7. $f'(x)=2x$. $x_1=3-\frac{9-10}{6}=\frac{19}{6}$; $x_2=\frac{19}{6}-\frac{(19/6)^2-10}{19/3}=\frac{721}{228}\approx 3.1623(\approx\sqrt{10})$.

Q8. $f'(x)=3x^2$. $x_1=2-\dfrac{8-7}{12}=\dfrac{23}{12}\approx 1.9167$; $x_2=\dfrac{23}{12}-\dfrac{(23/12)^3-7}{3(23/12)^2}\approx 1.9129$.

Q9. $f'(x)=3x^2-2$. $x_1=1.5-\dfrac{1.5^3-3-1}{3(1.5)^2-2}=\dfrac{31}{19}\approx 1.6316$; $x_2\approx 1.6182$.

Q10. $f'(x)=2x-1$. $x_1=2-\dfrac{4-2-3}{3}=2+\dfrac{1}{3}=\dfrac{7}{3}$; $x_2=\dfrac{7}{3}-\dfrac{(7/3)^2-7/3-3}{2(7/3)-1}=\dfrac{76}{33}\approx 2.3030$.

Q11. $f'(x)=3x^2+2$. $x_1=0.5-\dfrac{0.125+1-1}{0.75+2}=\dfrac{5}{11}\approx 0.4545$; $x_2\approx 0.4534$.

Q12. $f'(x)=4x^3$. $x_1=1-\dfrac{1-5}{4}=1+1=2$; $x_2=2-\dfrac{16-5}{32}=\dfrac{53}{32}\approx 1.6563$. The first step overshoots to 2 (past the root $\sqrt[4]{5}\approx 1.4953$) because the gradient at $x_0=1$ is small; the estimates then move back towards the root.

Q13. $f'(x)=e^x$. $x_1=0.5-\dfrac{e^{0.5}-2}{e^{0.5}}\approx 0.7131$; $x_2\approx 0.6933$. The exact solution is $x=\log_e 2\approx 0.6931$.

Q14. $f(x)=xe^x-1$, $f'(x)=e^x(x+1)$. $x_1=0.5-\dfrac{0.5e^{0.5}-1}{e^{0.5}(1.5)}\approx 0.5710$; $x_2\approx 0.5672$.

Q15.

```
Inputs: f, f ', x0, n

x ← x0
for i from 1 to n
    x ← x − f(x) / f '(x)
end for
return x
```

Q16. $f(x)=x^3-3x$, so $f'(x)=3x^2-3$. At $x_0=1$, $f'(1)=3-3=0$. The iteration requires dividing by $f'(x_0)$, which is 0, so x_1 is undefined; geometrically the tangent at $x_0=1$ is horizontal and never crosses the x-axis.

Q17. $f'(x)=3x^2-5$. $x_1=2-\dfrac{8-10-1}{12-5}=2+\dfrac{3}{7}=\dfrac{17}{7}\approx 2.4286$; $x_2\approx 2.3356$.

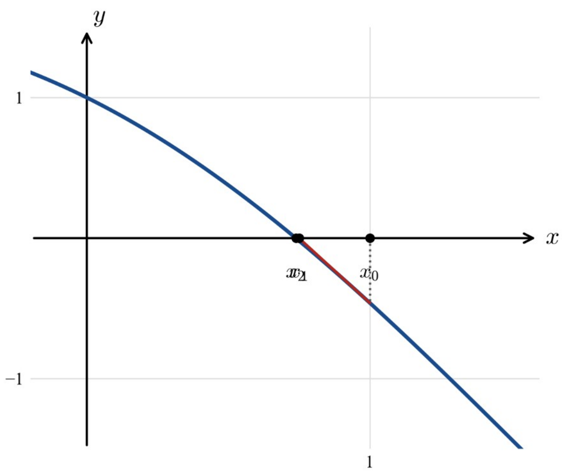
Q18. $f'(x)=3x^2-1$. $x_1=1.5-\dfrac{1.5^3-1.5-2}{3(1.5)^2-1}=\dfrac{35}{23}\approx 1.5217$; $x_2\approx 1.5214$.

Q19. $f'(x)=2x$. $x_1=2.5-\dfrac{6.25-6}{5}=\dfrac{49}{20}=2.45$; $x_2=\dfrac{49}{20}-\dfrac{(49/20)^2-6}{49/10}=\dfrac{4801}{1960}\approx 2.4495\,(\approx\sqrt{6})$.

Q20. $f'(x)=3x^2-3$. $x_1=2-\dfrac{8-6-5}{12-3}=2+\dfrac{3}{9}=\dfrac{7}{3}\approx 2.3333$; $x_2\approx 2.2806$.

Q21. $f(x)=\cos x-x$, $f'(x)=-\sin x-1$ (radians). $x_1=1-\dfrac{\cos 1-1}{-\sin 1-1}\approx 0.7504$;

$x_2=0.7504-\dfrac{\cos 0.7504-0.7504}{-\sin 0.7504-1}\approx 0.7391$.


y
1
x
x1
x0
−1
1

Theory

An **algorithm** is a finite, ordered sequence of unambiguous steps that solves a problem or performs a computation. **Computational thinking** is the approach used to design one: **abstraction** (ignore irrelevant detail), **decomposition** (break a problem into smaller parts), **pattern recognition** (spot repetition and structure), and **algorithm design** (set out the ordered steps).

Pseudocode is a structured, language-neutral way of writing an algorithm. VCAA pseudocode uses three kinds of control structure.

- **Sequencing** — steps are carried out in order, from top to bottom.
- **Decision-making (selection)** — a choice made with `if … then … else … end if`.
- **Repetition (iteration)** — a block repeated with `for … do … end for` (a set number of times) or `while … do … end while` (while a condition holds).

Assignment uses a left arrow: `x ← 3` means "store the value 3 in the variable x". The statement `x ← x + 1` reads the current value of x, adds 1, and stores the result back in x.

```
s ← 0                        -- sequencing: initialise
for i from 1 to 5 do         -- repetition: repeat 5 times
    s ← s + i                --   accumulate
end for
print s
```

Tracing an algorithm means working through it by hand, recording the value of every variable after each step in a **trace table**, to find the output. **Writing** an algorithm means designing the pseudocode for a described task.

Algorithms used in Methods.

- **Newton's method** — to approximate a solution of $f(x)=0$: repeat $x \leftarrow x - \frac{f(x)}{f'(x)}$.
- **The trapezium rule** — to approximate $\int_a^b f(x)\,dx$: add the areas of n trapezium strips of width $h=\frac{b-a}{n}$.
- **Bisection (interval-halving)** — to trap a root of f in an interval $[a,b]$ where f changes sign: repeatedly halve the interval, keeping the half in which the sign change occurs.
- **Accumulation** — a running total built up in a loop (a sum or a product).
- **Simulation** — estimating a probability or average by running many random trials and recording the results.

Worked examples

Example 1 — scaffolded

Trace the algorithm below and state its output.

```
s ← 0
for i from 1 to 5 do
    s ← s + i*i
end for
print s
```

Working	Comment
Start: $s=0$.	Initialise before the loop.
$i=1$: $s=0+1=1$	Add 1^2.

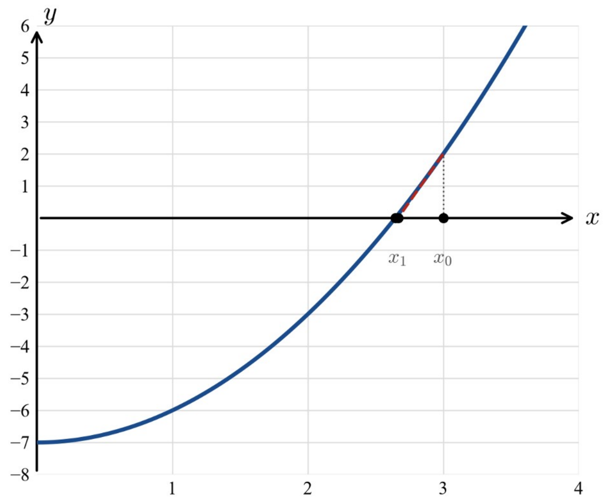
Working	Comment
$i=2$: $s=1+4=5$	Add 2^2.
$i=3$: $s=5+9=14$	Add 3^2.
$i=4$: $s=14+16=30$	Add 4^2.
$i=5$: $s=30+25=55$	Add 5^2; loop ends.
Output 55.	This algorithm computes $\sum_{i=1}^{5} i^2$.

Example 2 — scaffolded

The algorithm below is Newton's method applied to $f(x)=x^2-7$ (so $f'(x)=2x$, and $x-\frac{f(x)}{f'(x)}$ simplifies to $\frac{1}{2}\left(x+\frac{7}{x}\right)$). Trace two iterations from $x \leftarrow 3$ and give the exact values.

```
x ← 3
repeat 2 times
    x ← (x + 7/x) / 2
end repeat
print x
```

Working	Comment
Start $x=3$.	The starting estimate x_0.
Iteration 1: $x=\frac{1}{2}\left(3+\frac{7}{3}\right)=\frac{1}{2}\cdot\frac{16}{3}=\frac{8}{3}$	$x_1=\frac{8}{3}\approx 2.6667$.
Iteration 2: $x=\frac{1}{2}\left(\frac{8}{3}+\frac{7}{8/3}\right)=\frac{1}{2}\left(\frac{8}{3}+\frac{21}{8}\right)=\frac{127}{48}$	$x_2=\frac{127}{48}\approx 2.6458$.
$\therefore$ output $\frac{127}{48}\approx 2.6458$, converging to $\sqrt{7}\approx 2.6458$.	The tangents at each estimate meet the axis at the next.



Newton's method for $y=x^2-7$

Example 3 — unscaffolded

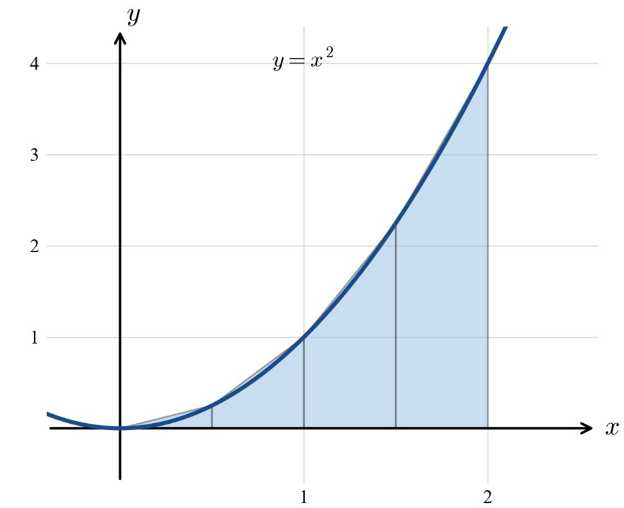
Write an algorithm, in pseudocode, that uses the trapezium rule with n strips to estimate $\int_a^b f(x)\,dx$. Then use it, by hand, to estimate $\int_0^2 x^2\,dx$ with $n=4$ strips.

```
input a, b, n
h ← (b - a) / n
total ← (f(a) + f(b)) / 2
for i from 1 to n-1 do
    total ← total + f(a + i*h)
end for
area ← h * total
print area
```

With $f(x)=x^2$, $a=0$, $b=2$, $n=4$: $h=\frac{1}{2}$, and the interior points are $x=\frac{1}{2},1,\frac{3}{2}$.

$$\text{area}=1/2\left[f(0)+f(2)/2+f(1/2)+f(1)+f(3/2)\right]=1/2\left[0+4/2+1/4+1+9/4\right]=1/2(2+14/4)=11/4.$$

$\therefore$ the estimate is $\frac{11}{4}=2.75$ (the exact value is $\frac{8}{3}\approx2.667$; the trapezium rule over-estimates here because $y=x^2$ is concave up).



Four trapezium strips under $y=x^2$

Example 4 — unscaffolded

$f(x)=x^2-2$ satisfies $f(1)=-1<0$ and $f(2)=2>0$, so a solution of $f(x)=0$ lies in $[1,2]$. Perform three iterations of the bisection algorithm and state the interval containing the root.

```
a ← 1 ;  b ← 2
repeat 3 times
    m ← (a + b) / 2
    if f(m) > 0 then b ← m else a ← m end if
end repeat
```

Iteration 1: $m=\frac{3}{2}$, $f(3/2)=9/4-2=1/4>0$, so $b\leftarrow\frac{3}{2}$. Interval $[1,3/2]$.

Iteration 2: $m=\frac{5}{4}$, $f(5/4)=25/16-2=-7/16<0$, so $a\leftarrow\frac{5}{4}$. Interval $[5/4,3/2]$.

Iteration 3: $m=\frac{11}{8}$, $f(11/8)=121/64-2=-7/64<0$, so $a\leftarrow\frac{11}{8}$. Interval $\left[\frac{11}{8},\frac{3}{2}\right]$.

$\therefore$ after three iterations the root of $x^2-2=0$ (that is, $\sqrt{2}\approx1.4142$) is trapped in $\left[\frac{11}{8},\frac{3}{2}\right]=[1.375,1.5]$.

Practice questions

Scaffolded

Q1. Consider the algorithm below.

```
p ← 1
for i from 1 to 4 do
    p ← p * i
end for
print p
```

(a) Complete a trace table showing p after each pass. (b) State the output. (c) What quantity does this algorithm compute?

Q2. For the algorithm below:

```
s ← 0
for k from 1 to 6 do
    if k is even then s ← s + k end if
end for
print s
```

(a) State the values of k for which the `if` condition is true. (b) Trace the value of s. (c) State the output.

Q3. For the algorithm below:

```
n ← 1 ;  c ← 0
while n < 100 do
    n ← 2*n ;  c ← c + 1
end while
print n, c
```

(a) Trace n and c. (b) State the output. (c) Explain in words what c counts.

Q4. For the algorithm below:

```
s ← 0 ;  n ← 0
while s < 50 do
    n ← n + 1 ;  s ← s + n
end while
print n
```

(a) Trace n and s until the loop stops. (b) State the output n. (c) Why does the loop use `while` rather than `for`?

Q5. Write pseudocode that computes and prints the sum $1^3+2^3+\cdots+m^3$ for an input value m, and state the output when $m=4$.

Q6. Newton's method for $f(x)=x^3-20$ uses $x \leftarrow x-\dfrac{x^3-20}{3x^2}$. (a) Write the pseudocode to perform two iterations from $x \leftarrow 3$. (b) Compute x_1 and x_2 as exact fractions. (c) To what value is the sequence converging?

Unscaffolded

Q7. Trace `s ← 0; for i from 1 to 5 do s ← s + (2*i - 1) end for; print s` and state the output. What does it compute?

Q8. Trace the algorithm `a ← 1; b ← 1; for i from 1 to 6 do t ← a + b; a ← b; b ← t end for; print b` (record a and b each pass) and state the output.

Q9. For `x ← 6; c ← 0; while x > 1 do (if x is even then x ← x/2 else x ← 3*x + 1 end if); c ← c + 1 end while; print c`, list the sequence of values of x and state the number of steps c.

Q10. Use the trapezium-rule algorithm of Example 3 with $f(x)=\frac{1}{x}$, $a=1$, $b=3$, $n=4$ to estimate $\int_1^3 \frac{1}{x}\,dx$. Give the exact estimate and compare it with the exact value $\log_e 3$.

Q11. Trace `n ← 1; while 2^n ≤ 1000 do n ← n + 1 end while; print n` and state the smallest n for which $2^n > 1000$.

Q12. For the nested loop `c ← 0; for i from 1 to 5 do (for j from 1 to i do c ← c + 1 end for) end for; print c`, state the output and explain why it equals $1+2+3+4+5$.

Q13. Write pseudocode that finds and prints the largest value in a list $L=[4,9,2,15,7]$ using an `if` statement, and state the output.

Q14. Newton's method for $f(x)=\cos x - x$ (radians) uses $x \leftarrow x - \frac{\cos x - x}{-\sin x - 1}$. Starting from $x \leftarrow 1$, state x_2 correct to four decimal places.

Q15. Describe, in pseudocode, a **simulation** that estimates the probability that a fair six-sided die shows a 6, using N trials. What value should the estimate approach as N becomes large?

Q16. Write pseudocode that prints $\sum_{i=1}^{n} i$ for an input n, and state the output when $n=20$. Which exact formula does the loop reproduce?

Q17. Trace Euclid's algorithm `while a ≠ b do (if a > b then a ← a - b else b ← b - a end if) end while; print a` with $a=48$, $b=36$, and state the output.

Q18. $f(x)=x^3 - x - 2$ has $f(1)=-2$ and $f(2)=4$. Perform three iterations of the bisection algorithm (as in Example 4) and state the interval containing the root.

Q19. Trace `for i from 1 to 4 do A[i] ← i*i end for` and write the resulting list A.

Q20. For `sum ← 0; k ← 0; while sum < 0.99 do k ← k + 1; sum ← sum + 1/2^k end while; print k`, state the value of k and the exact value of `sum` when the loop stops.

Q21. Write pseudocode that counts and prints how many of the integers $1, 2, \ldots, 100$ are divisible by 3, and state the output.

Answers

Q1. (a) $p=1,1,2,6,24$. (b) 24. (c) $4!=4\times3\times2\times1$. **Q2.** (a) $k=2,4,6$. (b) $s=0,2,2,6,6,12$. (c) 12. **Q3.** (a) $n=1,2,4,\ldots,128$; $c=0,1,2,\ldots,7$. (b) $n=128$, $c=7$. (c) the number of doublings of 1 needed to first exceed 100. **Q4.** (a) stops when $s=55$ at $n=10$. (b) $n=10$. (c) the number of passes is not known in advance — the loop runs until the running total first reaches 50. **Q5.** Output 100 $(1+8+27+64)$. **Q6.** (b) $x_1=\frac{74}{27}\approx2.7407$, $x_2=\frac{301027}{110889}\approx2.7147$. (c) $\sqrt[3]{20}\approx2.7144$. **Q7.** 25; it computes $1+3+5+7+9=\sum(2i-1)$, the sum of the first five odd numbers $(=5^2)$. **Q8.** $b=21$ (a Fibonacci sequence: $1,2,3,5,8,13,21$). **Q9.** $6,3,10,5,16,8,4,2,1$; $c=8$ steps. **Q10.** Estimate $\frac{67}{60}\approx1.1167$; exact $\log_e 3\approx1.0986$ (the rule over-estimates, since $y=1/x$ is concave up). **Q11.** $n=10$ (since $2^{10}=1024>1000$). **Q12.** 15; the inner loop runs i times for each i, so the total is $1+2+3+4+5=15$. **Q13.** Output 15. **Q14.** $x_2\approx0.7391$. **Q15.** Count the trials that give a 6 and divide by N; the estimate approaches $\frac{1}{6}\approx0.1667$. **Q16.** Output 210; the loop reproduces $\frac{n(n+1)}{2}=\frac{20\cdot21}{2}$. **Q17.** 12 (the gcd of 48 and 36). **Q18.** Root trapped in $\left[\frac{3}{2},\frac{13}{8}\right]=[1.5,1.625]$. **Q19.** $A=[1,4,9,16]$. **Q20.** $k=7$, sum $=\frac{127}{128}\approx0.9922$. **Q21.** Output 33.

Fully-worked solutions

Q1. (a)

pass i	p
start	1
1	1
2	2
3	6
4	24

(b) 24. (c) It multiplies $1\times2\times3\times4$, i.e. $4!$.

Q2. (a) `k is even` is true for $k=2,4,6$. (b) s: starts 0; unchanged at $k=1$; $0+2=2$ at $k=2$; unchanged at $k=3$; $2+4=6$ at $k=4$; unchanged at $k=5$; $6+6=12$ at $k=6$. (c) Output 12 (the sum of the even numbers from 1 to 6).

Q3. (a)

n	1	2	4	8	16	32	64	128
c	0	1	2	3	4	5	6	7

The loop stops once $n\geq100$, i.e. at $n=128$. (b) $n=128$, $c=7$. (c) c counts how many times 1 must be doubled to first exceed 100.

Q4. (a) $n=1,s=1$; $n=2,s=3$; $n=3,s=6$; $n=4,s=10$; $n=5,s=15$; $n=6,s=21$; $n=7,s=28$; $n=8,s=36$; $n=9,s=45$; $n=10,s=55$. Since $45<50\leq55$, the loop stops at $n=10$. (b) $n=10$. (c) A `while` loop is needed because the number of passes depends on the data (here, when the running total first reaches 50) and is not fixed in advance.

Q5.

```
input m
s ← 0
for i from 1 to m do
    s ← s + i*i*i
```

```
end for
print s
```

For $m=4$: $s=1+8+27+64=100$.

Q6. (a)

```
x ← 3
repeat 2 times
    x ← x - (x^3 - 20) / (3*x^2)
end repeat
print x
```

(b) $x_1=3-\frac{27-20}{27}=3-\frac{7}{27}=\frac{74}{27}\approx 2.7407$. Then $x_2=\frac{74}{27}-\frac{(74/27)^3-20}{3(74/27)^2}=\frac{301027}{110889}\approx 2.7147$. (c) The iteration solves $x^3-20=0$, so it converges to $\sqrt[3]{20}\approx 2.7144$.

Q7. $s: 0\to 1\to 4\to 9\to 16\to 25$. Output 25. Each pass adds the next odd number $2i-1$, so it computes $1+3+5+7+9$, the sum of the first five odd numbers, which equals $5^2=25$.

Q8.

pass	*a*	*b*
start	1	1
1	1	2
2	2	3
3	3	5
4	5	8
5	8	13
6	13	21

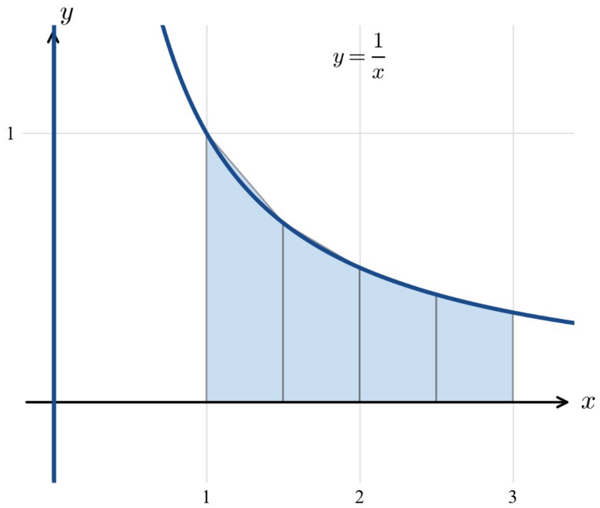
Output $b=21$ (each term is the sum of the previous two — a Fibonacci sequence).

Q9. From $x=6$: even $\to 3$; odd $\to 10$; even $\to 5$; odd $\to 16$; even $\to 8\to 4\to 2\to 1$. The sequence is $6, 3, 10, 5, 16, 8, 4, 2, 1$, reaching 1 after $c=8$ steps.

Q10. $h=\frac{3-1}{4}=\frac{1}{2}$; interior points $x=3/2, 2, 5/2$.

$$\text{area}=1/2\left[f(1)+f(3)/2+f(3/2)+f(2)+f(5/2)\right]=1/2\left[1+\frac{1}{3}/2+2/3+1/2+2/5\right]=\frac{67}{60}\approx 1.1167.$$

The exact value is $\int_1^3 1/x\,dx=\log_e 3\approx 1.0986$; the estimate is a slight over-estimate because $y=1/x$ is concave up.



Four trapezium strips under $y = 1/x$

Q11. Doubling from $n=1$: the condition $2^n \le 1000$ holds while $2^n = 2, 4, \ldots, 512$ (i.e. $n = 1, \ldots, 9$), and fails at $n = 10$ since $2^{10} = 1024 > 1000$. Output $n = 10$.

Q12. For each i the inner loop executes i times, adding 1 each time. Total $= 1+2+3+4+5 = 15$.

Q13.

```
m ← L[1]
for i from 2 to 5 do
    if L[i] > m then m ← L[i] end if
end for
print m
```

Tracing $L = [4, 9, 2, 15, 7]$: $m = 4 \to 9 \to 9 \to 15 \to 15$. Output 15.

Q14. $x_0 = 1$. $x_1 = 1 - \dfrac{\cos 1 - 1}{-\sin 1 - 1} \approx 0.7504$; $x_2 = 0.7504 - \dfrac{\cos(0.7504) - 0.7504}{-\sin(0.7504) - 1} \approx 0.7391$. So $x_2 \approx 0.7391$ (the solution of $\cos x = x$).

Q15.

```
count ← 0
for trial from 1 to N do
    r ← a random integer from 1 to 6
    if r = 6 then count ← count + 1 end if
end for
print count / N
```

As N increases, the relative frequency $\dfrac{\text{count}}{N}$ approaches the true probability $\dfrac{1}{6} \approx 0.1667$.

Q16.

```
input n
s ← 0
for i from 1 to n do
    s ← s + i
end for
print s
```

For $n = 20$: $s = 1 + 2 + \cdots + 20 = \dfrac{20 \times 21}{2} = 210$. The loop reproduces the formula $\dfrac{n(n+1)}{2}$.

Q17. $a=48, b=36$: $a>b \Rightarrow a=12$; now $a=12, b=36$: $b>a \Rightarrow b=24$; $b=24, a=12 \Rightarrow b=12$; now $a=b=12$, stop. Output 12 — the greatest common divisor of 48 and 36.

Q18. $f(x)=x^3-x-2$. Iteration 1: $m=3/2$, $f(3/2)=27/8-3/2-2=-1/8<0 \Rightarrow a=3/2$, interval $[3/2, 2]$. Iteration 2: $m=7/4$, $f(7/4)=343/64-7/4-2>0 \Rightarrow b=7/4$, interval $[3/2, 7/4]$. Iteration 3: $m=13/8$, $f(13/8)>0 \Rightarrow b=13/8$, interval $\left[\frac{3}{2}, \frac{13}{8}\right]=[1.5, 1.625]$, which contains the root (≈ 1.5214).

Q19. $A[1]=1, A[2]=4, A[3]=9, A[4]=16$, so $A=[1,4,9,16]$.

Q20. sum accumulates $1/2, 3/4, 7/8, \ldots$; after k terms $\text{sum}=1-1/2^k$. This first reaches 0.99 when $1/2^k \le 0.01$, i.e. $2^k \ge 100$, so $k=7$ ($2^7=128$). Then $\text{sum}=1-1/128=\frac{127}{128}\approx 0.9922$.

Q21.

```
count ← 0
for i from 1 to 100 do
    if i is divisible by 3 then count ← count + 1 end if
end for
print count
```

The multiples of 3 up to 100 are $3, 6, \ldots, 99$, and $99=3\times 33$, so there are 33 of them. Output 33.