

Mathematical Methods Units 3 & 4

Chapter 8 — Differentiation techniques

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Chapter 8 Differentiation techniques — 8A The derivative

Theory

The **gradient of a curve** at a point P is the gradient of the **tangent** to the curve at P . To find it, take a nearby point Q on the curve and let Q slide towards P : the gradient of the chord PQ approaches the gradient of the tangent.

The derivative by first principles. For $y = f(x)$, take $P(x, f(x))$ and $Q(x+h, f(x+h))$. The gradient of the chord PQ is

$$\frac{f(x+h) - f(x)}{h}.$$

As $h \rightarrow 0$, $Q \rightarrow P$ and this chord gradient approaches the gradient of the tangent. This limit is the **derivative** $f'(x)$:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative $f'(x)$ is itself a function — the **gradient function** — giving the gradient of the curve at each value of x .

Notation. For $y = f(x)$, the derivative may be written $f'(x)$, $\frac{dy}{dx}$, or $\frac{d}{dx}(f(x))$. All mean the same thing.

Finding $f'(x)$ by first principles (three steps):

1. Write down and expand $f(x+h)$.
2. Form $f(x+h) - f(x)$ and simplify; a common factor of h will appear.
3. Divide by h , then take the limit as $h \rightarrow 0$ (set the remaining h terms to 0).

Gradient at a point. The gradient of $y = f(x)$ at $x = a$ is $f'(a)$ — substitute $x = a$ into the derivative.

Equation of a tangent. The tangent to $y = f(x)$ at the point where $x = a$ passes through $(a, f(a))$ with gradient $f'(a)$, so its equation is

$$y - f(a) = f'(a)(x - a).$$

Worked examples

Example 1 — scaffolded

Find $f'(x)$ by first principles for $f(x) = x^2 + 3x$.

Working

Comment

$$f(x+h) = (x+h)^2 + 3(x+h) = x^2 + 2xh + h^2 + 3x + 3h$$

Step 1: expand $f(x+h)$.

$$f(x+h) - f(x) = 2xh + h^2 + 3h = h(2x + h + 3)$$

Step 2: subtract $f(x) = x^2 + 3x$; factor out h .

$$\frac{f(x+h) - f(x)}{h} = 2x + h + 3$$

Divide by h (valid since $h \neq 0$ in the limit).

$$f'(x) = \lim_{h \rightarrow 0} (2x + h + 3) = 2x + 3$$

Step 3: let $h \rightarrow 0$.

Example 2 — scaffolded

For $f(x) = x^2 - 4x + 1$, find $f'(x)$ by first principles, then the gradient at $x = 3$ and the equation of the tangent there.

$$f(x+h) = (x+h)^2 - 4(x+h) + 1 = x^2 + 2xh + h^2 - 4x - 4h + 1$$

Expand $f(x+h)$.

$$f(x+h) - f(x) = 2xh + h^2 - 4h = h(2x + h - 4)$$

Subtract and factor h .

$$f'(x) = \lim_{h \rightarrow 0} (2x + h - 4) = 2x - 4$$

Divide by h , then $h \rightarrow 0$.

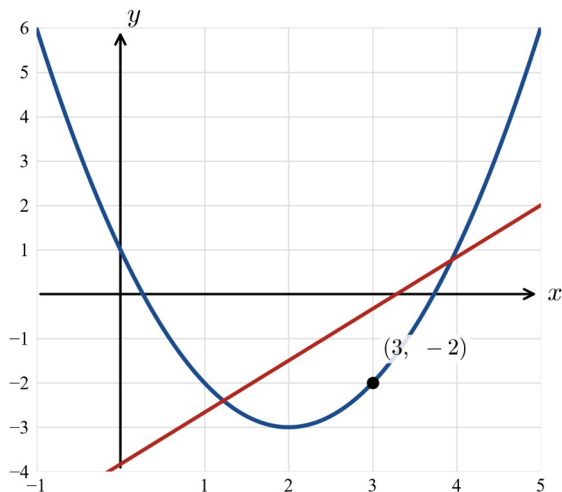
$$\text{Gradient at } x=3: f'(3) = 2(3) - 4 = 2.$$

Substitute $x=3$ into f' .

$$f(3) = 3^2 - 4(3) + 1 = -2, \text{ so the tangent passes through } (3, -2).$$

The point of tangency.

$$y - (-2) = 2(x - 3) \Rightarrow y = 2x - 8$$

Use $y - f(a) = f'(a)(x - a)$.

Example 3 — unscaffolded

Find $f'(x)$ by first principles for $f(x) = x^3$.

$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3.$$

$$f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 = h(3x^2 + 3xh + h^2).$$

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2.$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2.$$

Example 4 — unscaffolded

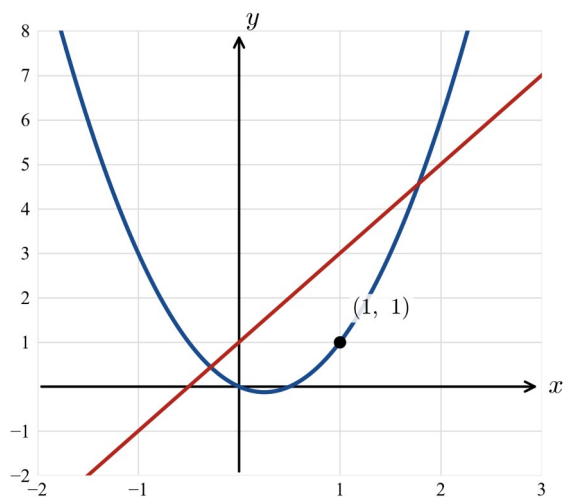
Find the equation of the tangent to $f(x) = 2x^2 - x$ at the point where $x = 1$.

By first principles, $f(x+h) - f(x) = 4xh + 2h^2 - h = h(4x + 2h - 1)$, so $f'(x) = 4x - 1$.

Gradient at $x=1$: $f'(1) = 4(1) - 1 = 3$. Point: $f(1) = 2(1)^2 - 1 = 1$, giving $(1, 1)$.

$$y - 1 = 3(x - 1) \Rightarrow y = 3x - 2.$$

$\therefore$ the tangent is $y = 3x - 2$.



Practice questions

Scaffolded

Q1. For $f(x) = x^2 - 2x$: (a) find $f'(x)$ by first principles; (b) find the gradient at $x = 3$; (c) find the equation of the tangent at $x = 3$.

Q2. For $f(x) = 3x^2$: (a) find $f'(x)$ by first principles; (b) find $f'(1)$; (c) find the tangent at $x = 1$.

Q3. For $f(x) = x^2 + x - 6$: (a) find $f'(x)$ by first principles; (b) find the gradient at $x = 2$; (c) find the tangent at $x = 2$.

Q4. For $f(x) = 5 - 2x^2$: (a) find $f'(x)$ by first principles; (b) find $f'(1)$; (c) find the tangent at $x = 1$.

Q5. For $f(x) = x^3 - x$: (a) find $f'(x)$ by first principles; (b) find the gradient at $x = 2$; (c) find the tangent at $x = 2$.

Q6. For $f(x) = 2x^2 - 3x + 1$: (a) find $f'(x)$ by first principles; (b) find $f'(0)$; (c) find the tangent at $x = 0$.

Un scaffolded

Find $f'(x)$ by first principles for each of Q7–Q11.

Q7. $f(x) = 4x^2$

Q8. $f(x) = x^2 - 7x$

Q9. $f(x) = 3 - x^2$

Q10. $f(x) = x^3 + 1$

Q11. $f(x) = 2x^3$

Q12. Find the gradient of $y = x^2 - 5x + 6$ at $x = 2$.

Q13. Find the gradient of $y = x^3$ at $x = -1$.

Q14. Find the equation of the tangent to $y = x^2$ at the point where $x = 3$.

Q15. Find the equation of the tangent to $y = x^2 - 4x$ at the point where $x = 1$.

Q16. Find the equation of the tangent to $y = x^3 - 2x$ at the point where $x = 2$.

Q17. Find the value of x at which the gradient of $y = x^2 - 6x$ is 0.

Q18. Find the coordinates of the points on $y = x^3$ at which the gradient is 12.

Q19. Show, by first principles, that the derivative of $f(x) = x^2 + bx + c$ (where b, c are constants) is $f'(x) = 2x + b$.

Q20. For $f(x) = x^2$, find the gradient of the tangent at the point where $x = a$, and hence write down the equation of that tangent in terms of a .

Q21. Find the equation of the tangent to $y = x^2$ that is parallel to the line $y = 6x - 1$.

Answers

Q1. (a) $2x - 2$; (b) 4; (c) $y = 4x - 9$. **Q2.** (a) $6x$; (b) 6; (c) $y = 6x - 3$. **Q3.** (a) $2x + 1$; (b) 5; (c) $y = 5x - 10$. **Q4.** (a) $-4x$; (b) -4 ; (c) $y = -4x + 7$. **Q5.** (a) $3x^2 - 1$; (b) 11; (c) $y = 11x - 16$. **Q6.** (a) $4x - 3$; (b) -3 ; (c) $y = -3x + 1$. **Q7.** $8x$. **Q8.** $2x - 7$. **Q9.** $-2x$. **Q10.** $3x^2$. **Q11.** $6x^2$. **Q12.** -1 . **Q13.** 3. **Q14.** $y = 6x - 9$. **Q15.** $y = -2x - 1$. **Q16.** $y = 10x - 16$. **Q17.** $x = 3$. **Q18.** $(2, 8)$ and $(-2, -8)$. **Q19.** proof. **Q20.** gradient $2a$; $y = 2ax - a^2$. **Q21.** $y = 6x - 9$.

Fully-worked solutions

Q1. $f(x+h) - f(x) = (x+h)^2 - 2(x+h) - (x^2 - 2x) = 2xh + h^2 - 2h = h(2x + h - 2)$, so $f'(x) = 2x - 2$. (b) $f'(3) = 4$. (c) $f(3) = 9 - 6 = 3$; $y - 3 = 4(x - 3) \Rightarrow y = 4x - 9$.

Q2. $f(x+h) - f(x) = 3(x+h)^2 - 3x^2 = 6xh + 3h^2 = h(6x + 3h)$, so $f'(x) = 6x$. (b) $f'(1) = 6$. (c) $f(1) = 3$; $y - 3 = 6(x - 1) \Rightarrow y = 6x - 3$.

Q3. $f(x+h) - f(x) = 2xh + h^2 + h = h(2x + h + 1)$, so $f'(x) = 2x + 1$. (b) $f'(2) = 5$. (c) $f(2) = 4 + 2 - 6 = 0$; $y - 0 = 5(x - 2) \Rightarrow y = 5x - 10$.

Q4. $f(x+h) - f(x) = 5 - 2(x+h)^2 - (5 - 2x^2) = -4xh - 2h^2 = h(-4x - 2h)$, so $f'(x) = -4x$. (b) $f'(1) = -4$. (c) $f(1) = 3$; $y - 3 = -4(x - 1) \Rightarrow y = -4x + 7$.

Q5. $f(x+h) - f(x) = (x+h)^3 - (x+h) - (x^3 - x) = 3x^2h + 3xh^2 + h^3 - h = h(3x^2 + 3xh + h^2 - 1)$, so $f'(x) = 3x^2 - 1$. (b) $f'(2) = 11$. (c) $f(2) = 8 - 2 = 6$; $y - 6 = 11(x - 2) \Rightarrow y = 11x - 16$.

Q6. $f(x+h) - f(x) = 4xh + 2h^2 - 3h = h(4x + 2h - 3)$, so $f'(x) = 4x - 3$. (b) $f'(0) = -3$. (c) $f(0) = 1$; $y - 1 = -3(x - 0) \Rightarrow y = -3x + 1$.

Q7. $f(x+h) - f(x) = 4(x+h)^2 - 4x^2 = 8xh + 4h^2 = h(8x + 4h)$, so $f'(x) = 8x$.

Q8. $f(x+h) - f(x) = 2xh + h^2 - 7h = h(2x + h - 7)$, so $f'(x) = 2x - 7$.

Q9. $f(x+h) - f(x) = 3 - (x+h)^2 - (3 - x^2) = -2xh - h^2 = h(-2x - h)$, so $f'(x) = -2x$.

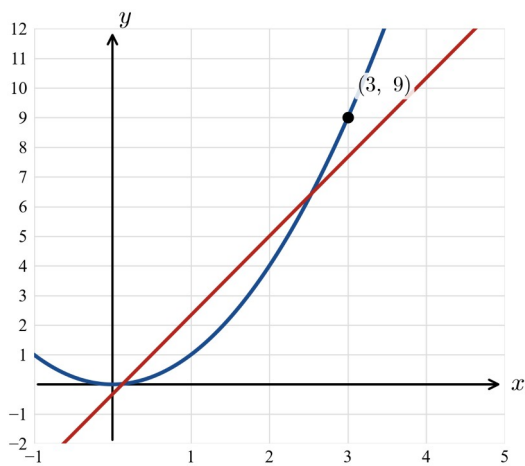
Q10. $f(x+h) - f(x) = (x+h)^3 + 1 - (x^3 + 1) = 3x^2h + 3xh^2 + h^3 = h(3x^2 + 3xh + h^2)$, so $f'(x) = 3x^2$.

Q11. $f(x+h) - f(x) = 2(x+h)^3 - 2x^3 = 6x^2h + 6xh^2 + 2h^3 = h(6x^2 + 6xh + 2h^2)$, so $f'(x) = 6x^2$.

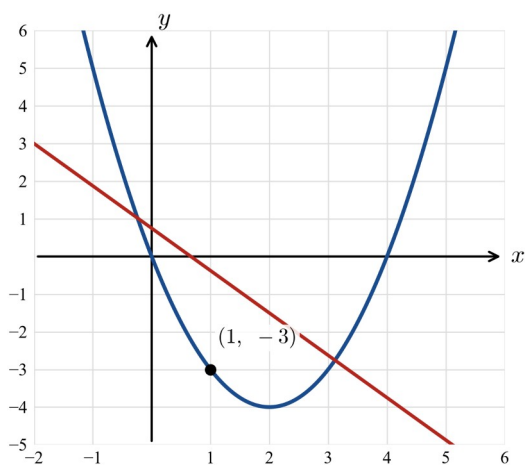
Q12. $\frac{dy}{dx} = 2x - 5$; at $x = 2$ the gradient is $2(2) - 5 = -1$.

Q13. $\frac{dy}{dx} = 3x^2$; at $x = -1$ the gradient is $3(-1)^2 = 3$.

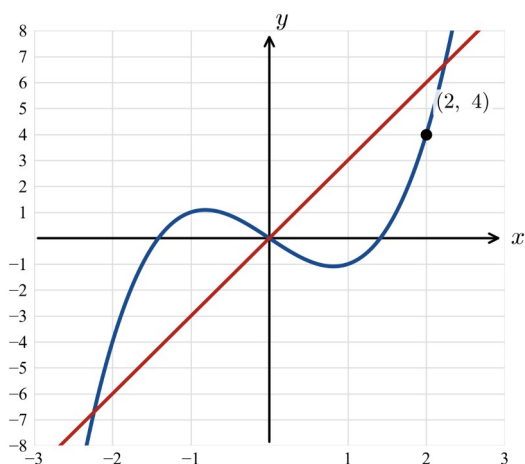
Q14. $\frac{dy}{dx} = 2x$; at $x = 3$, gradient 6 and $y = 9$. Tangent: $y - 9 = 6(x - 3)$, i.e. $y = 6x - 9$.



Q15. $\frac{dy}{dx} = 2x - 4$; at $x = 1$, gradient -2 and $y = 1 - 4 = -3$. Tangent: $y - (-3) = -2(x - 1)$, i.e. $y = -2x - 1$.



Q16. $\frac{dy}{dx} = 3x^2 - 2$; at $x = 2$, gradient 10 and $y = 8 - 4 = 4$. Tangent: $y - 4 = 10(x - 2)$, i.e. $y = 10x - 16$.



Q17. $\frac{dy}{dx} = 2x - 6 = 0 \Rightarrow x = 3$.

Q18. $\frac{dy}{dx} = 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. Points $(2, 8)$ and $(-2, -8)$.

Q19. $f(x+h) - f(x) = (x+h)^2 + b(x+h) + c - (x^2 + bx + c) = 2xh + h^2 + bh = h(2x + h + b)$. Dividing by h gives $2x + h + b$, and letting $h \rightarrow 0$ gives $f'(x) = 2x + b$, as required.

Q20. $f'(x) = 2x$, so the gradient at $x = a$ is $2a$. The point is (a, a^2) , so the tangent is $y - a^2 = 2a(x - a)$, i.e. $y = 2ax - a^2$.

Q21. A tangent parallel to $y = 6x - 1$ has gradient 6. Since $\frac{dy}{dx} = 2x$, set $2x = 6 \Rightarrow x = 3$, giving the point $(3, 9)$.

Tangent: $y - 9 = 6(x - 3)$, i.e. $y = 6x - 9$.

Chapter 8 Differentiation techniques — 8B Differentiation by rule

Theory

Finding derivatives from first principles (Section 8A) is slow. This section develops the **rules** that let you differentiate any polynomial quickly.

The derivative of a power. For any positive integer n ,

$$\text{if } f(x) = x^n \text{ then } f'(x) = n x^{n-1}.$$

The index becomes the coefficient and the new index is one less. In particular the derivative of x is 1, and the derivative of a **constant** is 0 (a constant function has zero gradient everywhere).

Constant-multiple rule. If $f(x) = k g(x)$ for a constant k , then $f'(x) = k g'(x)$. For example,

$$\frac{d}{dx}(5x^3) = 5 \times 3x^2 = 15x^2.$$

Sum and difference rules. The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x).$$

Together these let you differentiate any polynomial **term by term**. For example, if $y = 3x^4 - 2x^3 + 5x - 7$ then

$$\frac{dy}{dx} = 12x^3 - 6x^2 + 5.$$

Notation. The derivative of $y = f(x)$ is written $f'(x)$, $\frac{dy}{dx}$, or $\frac{d}{dx}(f(x))$. Always name the derivative correctly —

for $y = f(x)$ write $\frac{dy}{dx}$ or $f'(x)$, not y .

Using the derivative.

- To find the **gradient at a point** $x = a$, evaluate $f'(a)$.
- To find **where the gradient equals a value** k , solve $f'(x) = k$. (A horizontal tangent occurs where $f'(x) = 0$.)

Before differentiating, simplify. If a function is given as a product or a quotient that reduces to a polynomial,

expand or divide first, then differentiate term by term. For example, $\frac{6x^5 - 4x^3}{2x^2} = 3x^3 - 2x$, whose derivative is

$$9x^2 - 2.$$

Worked examples

Example 1 — scaffolded

Differentiate $y = 2x^3 - 5x^2 + 4x - 1$ and find the gradient at $x = 1$.

Working

$$\frac{dy}{dx} = 2(3x^2) - 5(2x) + 4(1) - 0$$

$$= 6x^2 - 10x + 4$$

$$\text{At } x = 1: \frac{dy}{dx} = 6(1)^2 - 10(1) + 4 = 0$$

Comment

Differentiate term by term: bring down the index, reduce it by one; the constant -1 differentiates to 0.

Simplify each term.

Substitute $x = 1$ into the derivative.

$\therefore$ the gradient at $x=1$ is 0.

A gradient of 0 means the tangent there is horizontal.

Example 2 — scaffolded

For $f(x) = x^3 - 4x^2 + 2x$, find $f'(x)$ and evaluate $f'(2)$.

$$f'(x) = 3x^2 - 8x + 2$$

Differentiate each term.

$$f'(2) = 3(2)^2 - 8(2) + 2$$

Substitute $x=2$.

$$= 12 - 16 + 2 = -2$$

So the curve has gradient -2 at $x=2$.

Example 3 — unscaffolded

Differentiate $y = (2x - 1)(x + 3)$.

The rule applies to sums, so **expand first**: $y = 2x^2 + 6x - x - 3 = 2x^2 + 5x - 3$.

Differentiating term by term, $\frac{dy}{dx} = 4x + 5$.

$$\therefore \frac{dy}{dx} = 4x + 5.$$

Example 4 — unscaffolded

Find the values of x at which $f(x) = x^3 - 3x^2 - 9x + 5$ has a horizontal tangent.

A horizontal tangent occurs where $f'(x) = 0$.

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1).$$

Setting $f'(x) = 0$ gives $x = 3$ or $x = -1$.

$\therefore$ the tangent is horizontal at $x = -1$ and $x = 3$.

Practice questions

Scaffolded

Q1. For $y = 2x^3 - 5x^2 + 4x - 1$: (a) find $\frac{dy}{dx}$; (b) find the gradient at $x = 1$; (c) state what the answer to (b) tells you about the tangent there.

Q2. For $y = x^4 - 2x^2$: (a) find $\frac{dy}{dx}$; (b) factorise the derivative; (c) find all values of x where the gradient is 0.

Q3. For $y = (x - 2)(x + 4)$: (a) expand the right-hand side; (b) find $\frac{dy}{dx}$; (c) find the value of x where the tangent is horizontal.

Q4. For $f(x) = x^3 + x^2 - x$: (a) find $f'(x)$; (b) factorise $f'(x)$; (c) solve $f'(x) = 0$.

Q5. For $y = \frac{1}{2}x^4 - x^3 + 3x$: (a) find $\frac{dy}{dx}$; (b) find the gradient at $x = 0$.

Q6. For $f(x) = x^3 - 6x^2 + 9x + 1$: (a) find $f'(x)$; (b) find the gradient at $x = 0$; (c) find the values of x where the gradient is 0.

Un scaffolded

Differentiate each of the following.

Q7. $y = 4x^3 - 3x^2 + 2x - 5$

Q8. $y = x^6 - 2x^4 + x^2$

Q9. $y = (3x - 2)^2$

Q10. $y = x^2(x - 3)$

Q11. $y = (x^2 - 1)(x + 2)$

Q12. For $f(x) = 2x^3 - 9x^2 + 12x$, find $f'(x)$ and solve $f'(x) = 0$.

Q13. For $y = x^4 - 8x^2$, find all values of x where the gradient is 0.

Q14. By simplifying first, differentiate $y = \frac{6x^5 - 4x^3}{2x^2}$.

Q15. Find the gradient of $y = x^3 - 2x^2 + 1$ at $x = 2$.

Q16. Find the coordinates of the points on $y = x^3 - 3x$ where the tangent is horizontal.

Q17. Find the values of x for which the curve $y = x^3 - 12x$ has gradient 15.

Q18. Show that $f(x) = x^3 + 3x$ has $f'(x) > 0$ for all real x , and explain what this means about the graph.

Q19. Differentiate $y = (2x - 3)(x^2 + 1)$.

Q20. Find the value of x for which the tangent to $y = x^2 - 5x + 6$ is horizontal.

Q21. Find the values of x for which the curve $y = x^3 - x$ has gradient 2.

Answers

Q1. (a) $6x^2 - 10x + 4$; (b) 0; (c) the tangent at $x = 1$ is horizontal. **Q2.** (a) $4x^3 - 4x$; (b) $4x(x-1)(x+1)$; (c) $x = -1, 0, 1$. **Q3.** (a) $x^2 + 2x - 8$; (b) $2x + 2$; (c) $x = -1$. **Q4.** (a) $3x^2 + 2x - 1$; (b) $(3x-1)(x+1)$; (c) $x = -1, 1/3$. **Q5.** (a) $2x^3 - 3x^2 + 3$; (b) 3. **Q6.** (a) $3x^2 - 12x + 9$; (b) 9; (c) $x = 1, 3$. **Q7.** $12x^2 - 6x + 2$. **Q8.** $6x^5 - 8x^3 + 2x$. **Q9.** $18x - 12$. **Q10.** $3x^2 - 6x$. **Q11.** $3x^2 + 4x - 1$. **Q12.** $f'(x) = 6x^2 - 18x + 12$; $x = 1, 2$. **Q13.** $x = -2, 0, 2$. **Q14.** $9x^2 - 2$. **Q15.** 4. **Q16.** $(1, -2)$ and $(-1, 2)$. **Q17.** $x = -3, 3$. **Q18.** $f'(x) = 3x^2 + 3 > 0$ for all x ; the graph is strictly increasing everywhere. **Q19.** $6x^2 - 6x + 2$. **Q20.** $x = \frac{5}{2}$. **Q21.** $x = -1, 1$.

Fully-worked solutions

Q1. $y = 2x^3 - 5x^2 + 4x - 1$. (a) $\frac{dy}{dx} = 6x^2 - 10x + 4$. (b) At $x = 1$: $6(1) - 10(1) + 4 = 0$. (c) A gradient of 0 means the tangent at $x = 1$ is **horizontal**.

Q2. $y = x^4 - 2x^2$. (a) $\frac{dy}{dx} = 4x^3 - 4x$. (b) $4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$. (c) $4x(x-1)(x+1) = 0 \Rightarrow x = 0, 1, -1$.

Q3. $y = (x-2)(x+4)$. (a) $y = x^2 + 2x - 8$. (b) $\frac{dy}{dx} = 2x + 2$. (c) $2x + 2 = 0 \Rightarrow x = -1$.

Q4. $f(x) = x^3 + x^2 - x$. (a) $f'(x) = 3x^2 + 2x - 1$. (b) $3x^2 + 2x - 1 = (3x-1)(x+1)$. (c) $(3x-1)(x+1) = 0 \Rightarrow x = \frac{1}{3}$ or $x = -1$.

Q5. $y = \frac{1}{2}x^4 - x^3 + 3x$. (a) $\frac{dy}{dx} = \frac{1}{2}(4x^3) - 3x^2 + 3 = 2x^3 - 3x^2 + 3$. (b) At $x = 0$: $2(0) - 3(0) + 3 = 3$.

Q6. $f(x) = x^3 - 6x^2 + 9x + 1$. (a) $f'(x) = 3x^2 - 12x + 9$. (b) $f'(0) = 9$. (c) $3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0 \Rightarrow x = 1$ or $x = 3$.

Q7. $\frac{dy}{dx} = 12x^2 - 6x + 2$.

Q8. $\frac{dy}{dx} = 6x^5 - 8x^3 + 2x$.

Q9. Expand: $y = 9x^2 - 12x + 4$, so $\frac{dy}{dx} = 18x - 12$.

Q10. Expand: $y = x^3 - 3x^2$, so $\frac{dy}{dx} = 3x^2 - 6x$.

Q11. Expand: $y = x^3 + 2x^2 - x - 2$, so $\frac{dy}{dx} = 3x^2 + 4x - 1$.

Q12. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$, so $f'(x) = 0$ at $x = 1$ and $x = 2$.

Q13. $\frac{dy}{dx} = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2) = 0 \Rightarrow x = -2, 0, 2$.

Q14. Simplify: $\frac{6x^5 - 4x^3}{2x^2} = 3x^3 - 2x$, so $\frac{dy}{dx} = 9x^2 - 2$.

Q15. $\frac{dy}{dx} = 3x^2 - 4x$; at $x=2$: $3(4) - 4(2) = 12 - 8 = 4$.

Q16. $\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1) = 0 \Rightarrow x=1$ or $x=-1$. Then $y(1) = 1 - 3 = -2$ and $y(-1) = -1 + 3 = 2$, giving $(1, -2)$ and $(-1, 2)$.

Q17. $\frac{dy}{dx} = 3x^2 - 12$. Set $3x^2 - 12 = 15 \Rightarrow 3x^2 = 27 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.

Q18. $f'(x) = 3x^2 + 3 = 3(x^2 + 1)$. Since $x^2 \geq 0$, we have $x^2 + 1 \geq 1 > 0$, so $f'(x) > 0$ for all real x . Therefore the graph of f is **strictly increasing** for all x (it never has a horizontal tangent and never decreases).

Q19. Expand: $y = 2x^3 - 3x^2 + 2x - 3$, so $\frac{dy}{dx} = 6x^2 - 6x + 2$.

Q20. $\frac{dy}{dx} = 2x - 5$. Horizontal tangent where $2x - 5 = 0 \Rightarrow x = \frac{5}{2}$.

Q21. $\frac{dy}{dx} = 3x^2 - 1$. Set $3x^2 - 1 = 2 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

Theory

The power rule $\frac{d}{dx}(x^n) = nx^{n-1}$ holds not only for positive integers but also for **negative integer** indices. The key skill is to **rewrite** a reciprocal power as a negative index *before* differentiating.

Rewriting. For a positive integer k ,

$$\frac{1}{x^k} = x^{-k}, \text{ so } \frac{a}{x^k} = ax^{-k}.$$

Differentiating. Apply the power rule with the negative index, then (usually) rewrite the answer with a positive index:

$$\frac{d}{dx}(x^{-k}) = -kx^{-k-1} = \frac{-k}{x^{k+1}}.$$

For example $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}.$

Sums. Differentiate term by term. A quotient such as $\frac{x^2+3}{x}$ should first be split into separate powers,

$$\frac{x^2+3}{x} = x + 3x^{-1}, \text{ and then differentiated.}$$

Domain. A negative power x^{-k} is not defined at $x=0$, so these functions and their derivatives have domain $\mathbb{R} \setminus \{0\}$.

Gradients and tangents. The derivative gives the gradient of the tangent at any point. The tangent to $y=f(x)$ at $x=a$ has gradient $f'(a)$ and passes through $(a, f(a))$, so its equation is $y=f'(a)(x-a)+f(a)$.

To differentiate a negative-power expression:

1. Rewrite every term in the form ax^n (split quotients; write reciprocals as negative powers).
2. Apply the power rule to each term.
3. Rewrite with positive indices if required, and state the domain.

Worked examples

Example 1 — scaffolded

For $y = \frac{1}{x^2}$, find $\frac{dy}{dx}$, the gradient at $x=1$, and the equation of the tangent there.

Working

Comment

$$y = \frac{1}{x^2} = x^{-2}$$

Rewrite as a negative power before differentiating.

$$\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$$

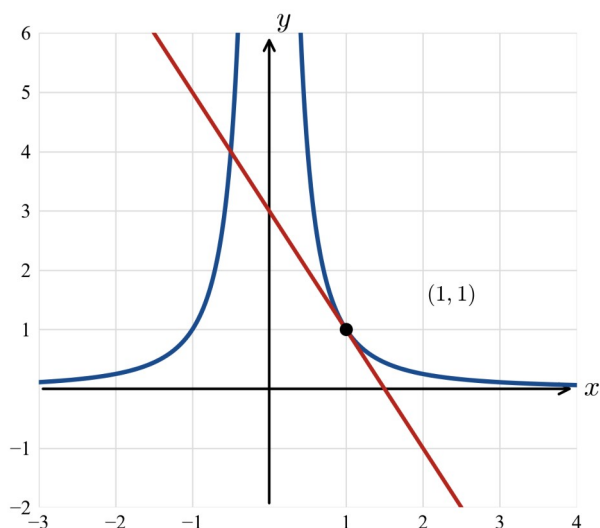
Power rule: $n = -2$, so $-2x^{-2-1}$.

$$\text{At } x=1: \frac{dy}{dx} = -\frac{2}{1^3} = -2$$

Substitute $x=1$ into the derivative.

Point $(1, 1)$, gradient -2 : $y = -2(x-1) + 1 = -2x + 3$ Tangent $y = f'(a)(x-a) + f(a)$.

$\therefore$ the tangent is $y = 3 - 2x$.



Example 2 — scaffolded

Differentiate $y = \frac{3}{x} + x$ and find the gradient at $x = 3$.

Working

$$y = 3x^{-1} + x$$

$$\frac{dy}{dx} = -3x^{-2} + 1 = 1 - \frac{3}{x^2}$$

$$\text{At } x = 3: 1 - \frac{3}{9} = 1 - \frac{1}{3} = \frac{2}{3}$$

Comment

Write the reciprocal term as a negative power.

Differentiate term by term.

Substitute $x = 3$.

Example 3 — unscaffolded

Differentiate $y = \frac{2x^2 - 1}{x^2}$.

Splitting the quotient: $y = \frac{2x^2}{x^2} - \frac{1}{x^2} = 2 - x^{-2}$.

$$\frac{dy}{dx} = -(-2)x^{-3} = 2x^{-3} = \frac{2}{x^3}$$

$$\therefore \frac{dy}{dx} = \frac{2}{x^3}, \text{ with domain } \mathbb{R} \setminus \{0\}.$$

Example 4 — unscaffolded

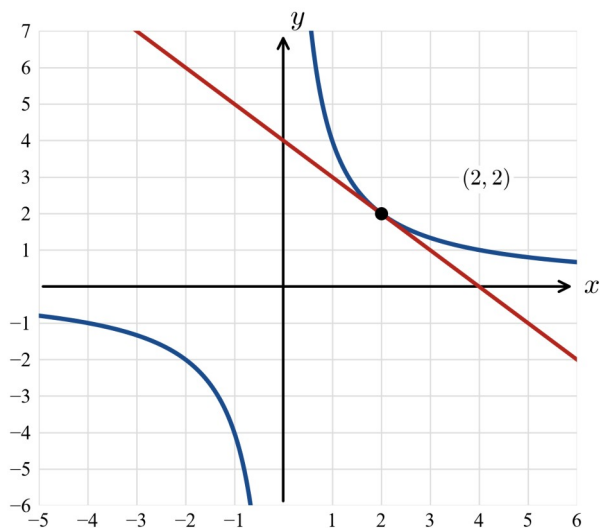
Find the equation of the tangent to $y = \frac{4}{x}$ at the point where $x = 2$.

$$y = 4x^{-1}, \text{ so } \frac{dy}{dx} = -4x^{-2} = -\frac{4}{x^2}.$$

At $x = 2$: gradient $= -\frac{4}{4} = -1$ and $y = \frac{4}{2} = 2$, giving the point $(2, 2)$.

$$\text{Tangent: } y = -1(x - 2) + 2 = -x + 4.$$

$\therefore$ the tangent is $y = 4 - x$.



Practice questions

Scaffolded

Q1. For $y = \frac{1}{x^3}$: (a) write y as a power of x ; (b) find $\frac{dy}{dx}$; (c) find the gradient at $x = 1$.

Q2. For $y = \frac{5}{x}$: (a) find $\frac{dy}{dx}$; (b) find the gradient at $x = 1$; (c) state the domain of y and of $\frac{dy}{dx}$.

Q3. Differentiate $y = x^2 + \frac{2}{x}$.

Q4. Differentiate $y = \frac{1}{x} - \frac{1}{x^2}$.

Q5. For $y = \frac{x+1}{x}$: (a) simplify y into two terms; (b) hence find $\frac{dy}{dx}$.

Q6. Find the equation of the tangent to $y = \frac{2}{x^2}$ at $x = 1$.

Un scaffolded

Differentiate each of the following (Q7–Q14), giving your answer with positive indices.

Q7. $y = \frac{6}{x}$ * **Q8.**** $y = \frac{1}{x^4}$ * **Q9.**** $y = \frac{3}{x^2}$

Q10. $y = x^3 + \frac{1}{x}$ * **Q11.**** $y = x - \frac{4}{x}$ * **Q12.**** $y = \frac{x^2+3}{x}$

Q13. $y = \frac{x^3-2}{x^2}$ * **Q14.**** $y = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

Q15. For $y = \frac{2}{x^3}$, find $\frac{dy}{dx}$ and the gradient at $x = 1$.

Q16. Find the equation of the tangent to $y = \frac{1}{x}$ at $x = 2$.

Q17. Find the values of x at which the gradient of $y = \frac{1}{x}$ is -1 .

Q18. Find the value of x at which the gradient of $y = \frac{4}{x^2}$ is -1 .

Q19. Differentiate $y = \frac{2x-1}{x^2}$.

Q20. Find the equation of the tangent to $y = \frac{3}{x}$ at $x = -1$.

Q21. Show that the tangent to $y = \frac{1}{x}$ at the point where $x = a$ (with $a \neq 0$) has gradient $-\frac{1}{a^2}$.

Answers

Q1. (a) x^{-3} ; (b) $-\frac{3}{x^4}$; (c) -3 . **Q2.** (a) $-\frac{5}{x^2}$; (b) -5 ; (c) domain $\mathbb{R} \setminus \{0\}$ for both. **Q3.** $2x - \frac{2}{x^2}$. **Q4.** $-\frac{1}{x^2} + \frac{2}{x^3}$. **Q5.** (a) $y = 1 + \frac{1}{x}$; (b) $-\frac{1}{x^2}$. **Q6.** $y = 6 - 4x$. **Q7.** $-\frac{6}{x^2}$. **Q8.** $-\frac{4}{x^5}$. **Q9.** $-\frac{6}{x^3}$. **Q10.** $3x^2 - \frac{1}{x^2}$. **Q11.** $1 + \frac{4}{x^2}$. **Q12.** $1 - \frac{3}{x^2}$. **Q13.** $1 + \frac{4}{x^3}$. **Q14.** $-\frac{1}{x^2} - \frac{2}{x^3} - \frac{3}{x^4}$. **Q15.** $\frac{dy}{dx} = -\frac{6}{x^4}$; gradient at $x = 1$ is -6 . **Q16.** $y = 1 - \frac{x}{4}$. **Q17.** $x = 1$ or $x = -1$. **Q18.** $x = 2$. **Q19.** $-\frac{2}{x^2} + \frac{2}{x^3}$. **Q20.** $y = -3x - 6$. **Q21.** Shown — gradient $-\frac{1}{d^2}$.

Fully-worked solutions

Q1. $y = x^{-3}$. (b) $\frac{dy}{dx} = -3x^{-4} = -\frac{3}{x^4}$. (c) At $x = 1$: $-\frac{3}{1} = -3$.

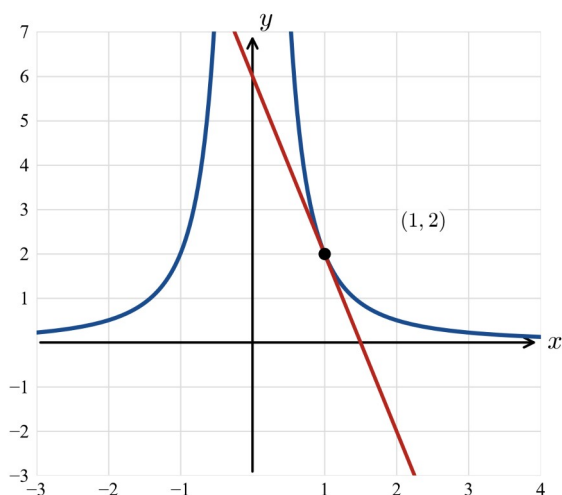
Q2. $y = 5x^{-1}$. (a) $\frac{dy}{dx} = -5x^{-2} = -\frac{5}{x^2}$. (b) At $x = 1$: -5 . (c) Since $x = 0$ is excluded, the domain of both y and $\frac{dy}{dx}$ is $\mathbb{R} \setminus \{0\}$.

Q3. $y = x^2 + 2x^{-1}$, so $\frac{dy}{dx} = 2x - 2x^{-2} = 2x - \frac{2}{x^2}$.

Q4. $y = x^{-1} - x^{-2}$, so $\frac{dy}{dx} = -x^{-2} + 2x^{-3} = -\frac{1}{x^2} + \frac{2}{x^3}$.

Q5. (a) $y = \frac{x}{x} + \frac{1}{x} = 1 + x^{-1}$. (b) $\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$.

Q6. $y = 2x^{-2}$, $\frac{dy}{dx} = -4x^{-3} = -\frac{4}{x^3}$. At $x = 1$: gradient -4 , point $(1, 2)$. Tangent $y = -4(x - 1) + 2 = -4x + 6$.



Q7. $y = 6x^{-1} \Rightarrow \frac{dy}{dx} = -6x^{-2} = -\frac{6}{x^2}$.

Q8. $y = x^{-4} \Rightarrow \frac{dy}{dx} = -4x^{-5} = -\frac{4}{x^5}$.

$$\text{Q9. } y = 3x^{-2} \Rightarrow \frac{dy}{dx} = -6x^{-3} = -\frac{6}{x^3}.$$

$$\text{Q10. } y = x^3 + x^{-1} \Rightarrow \frac{dy}{dx} = 3x^2 - x^{-2} = 3x^2 - \frac{1}{x^2}.$$

$$\text{Q11. } y = x - 4x^{-1} \Rightarrow \frac{dy}{dx} = 1 + 4x^{-2} = 1 + \frac{4}{x^2}.$$

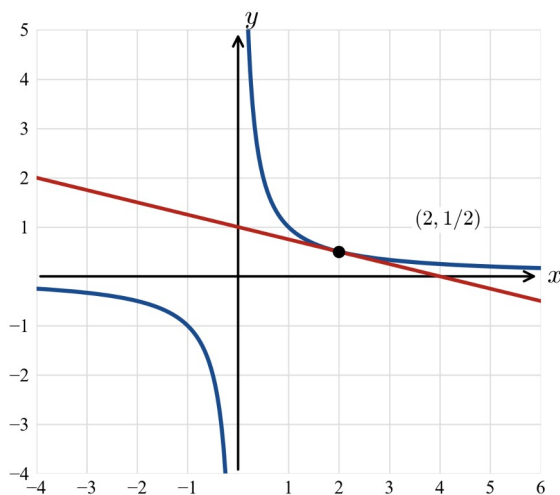
$$\text{Q12. } y = \frac{x^2}{x} + \frac{3}{x} = x + 3x^{-1} \Rightarrow \frac{dy}{dx} = 1 - 3x^{-2} = 1 - \frac{3}{x^2}.$$

$$\text{Q13. } y = \frac{x^3}{x^2} - \frac{2}{x^2} = x - 2x^{-2} \Rightarrow \frac{dy}{dx} = 1 + 4x^{-3} = 1 + \frac{4}{x^3}.$$

$$\text{Q14. } y = x^{-1} + x^{-2} + x^{-3} \Rightarrow \frac{dy}{dx} = -x^{-2} - 2x^{-3} - 3x^{-4} = -\frac{1}{x^2} - \frac{2}{x^3} - \frac{3}{x^4}.$$

$$\text{Q15. } y = 2x^{-3} \Rightarrow \frac{dy}{dx} = -6x^{-4} = -\frac{6}{x^4}. \text{ At } x = 1: -6.$$

$$\text{Q16. } y = x^{-1}, \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}. \text{ At } x = 2: \text{ gradient } -\frac{1}{4}, \text{ point } \left(2, \frac{1}{2}\right). \text{ Tangent } y = -\frac{1}{4}(x - 2) + \frac{1}{2} = -\frac{x}{4} + 1.$$

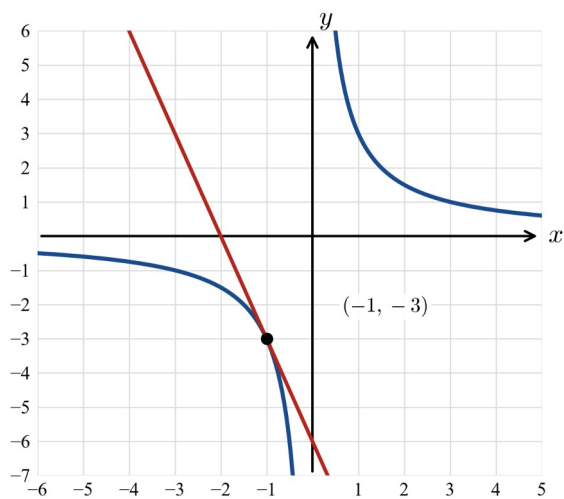


$$\text{Q17. } \frac{dy}{dx} = -\frac{1}{x^2}. \text{ Set } -\frac{1}{x^2} = -1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

$$\text{Q18. } y = 4x^{-2}, \frac{dy}{dx} = -8x^{-3} = -\frac{8}{x^3}. \text{ Set } -\frac{8}{x^3} = -1 \Rightarrow x^3 = 8 \Rightarrow x = 2.$$

$$\text{Q19. } y = \frac{2x}{x^2} - \frac{1}{x^2} = 2x^{-1} - x^{-2} \Rightarrow \frac{dy}{dx} = -2x^{-2} + 2x^{-3} = -\frac{2}{x^2} + \frac{2}{x^3}.$$

$$\text{Q20. } y = 3x^{-1}, \frac{dy}{dx} = -3x^{-2} = -\frac{3}{x^2}. \text{ At } x = -1: \text{ gradient } -3, \text{ point } (-1, -3). \text{ Tangent } y = -3(x + 1) - 3 = -3x - 6.$$



Q21. $y = x^{-1}$, so $\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$. At $x = a$ the gradient is $-\frac{1}{a^2}$, as required. ■

Chapter 8 Differentiation techniques — 8D The gradient function and its graph

Theory

The derivative $f'(x)$ gives the **gradient** of $y=f(x)$ at each x . So the graph of f' can be read directly from the shape of the graph of f :

- Where f is **increasing**, the gradient is positive, so $f'(x) > 0$ (the graph of f' is **above** the x -axis).
- Where f is **decreasing**, $f'(x) < 0$ (the graph of f' is **below** the x -axis).
- At a **stationary point** of f (a turning point or a stationary point of inflection) the gradient is zero, so $f'(x) = 0$: each stationary point of f gives an x -intercept of f' .

Reading the nature of a stationary point from f' :

- at a **local maximum** of f , the gradient changes from positive to negative, so f' **crosses the x -axis from above to below**;
- at a **local minimum**, f' crosses from below to above;
- at a **stationary point of inflection**, f' **touches** the x -axis (it has the same sign on both sides).

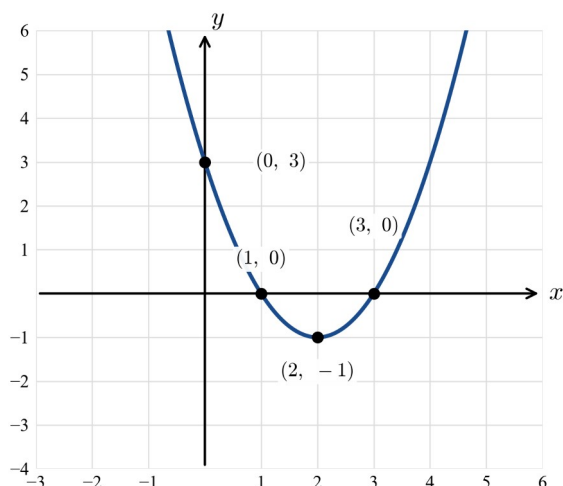
Degree. Differentiating lowers the degree by one: a quadratic f has a **linear** f' ; a cubic f has a **quadratic** f' ; a quartic f has a **cubic** f' . (A linear f has a **constant** f' .)

To sketch f' from f : mark the x -values of the stationary points of f as the x -intercepts of f' ; put f' above the axis where f is increasing and below where f is decreasing; then draw the appropriate lower-degree curve through those intercepts.

Worked examples

Example 1 — scaffolded

The graph of $f(x) = x^2 - 4x + 3$ is shown. Sketch the graph of f' .



Working

$$f'(x) = 2x - 4.$$

f has a minimum turning point at $x = 2$, so $f'(2) = 0$: the graph of f' cuts the x -axis at $(2, 0)$.

For $x < 2$, f is decreasing so $f' < 0$; for $x > 2$, f is

Comment

Differentiate: a quadratic gives a **linear** derivative.

The stationary point of f is an x -intercept of f' .

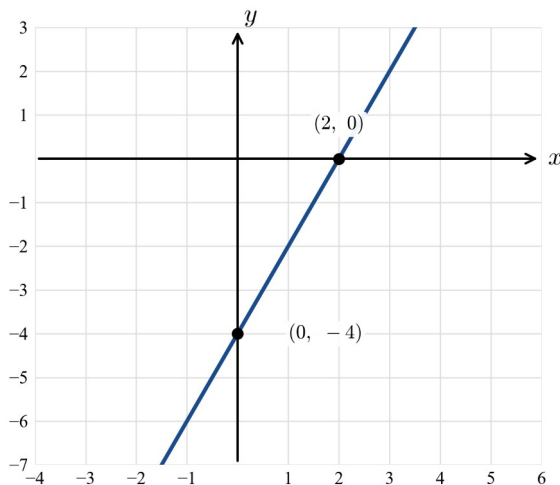
f' crosses from below to above — consistent with a

increasing so $f' > 0$.

f' is the line through $(2, 0)$ with gradient 2; its y -intercept is $(0, -4)$.

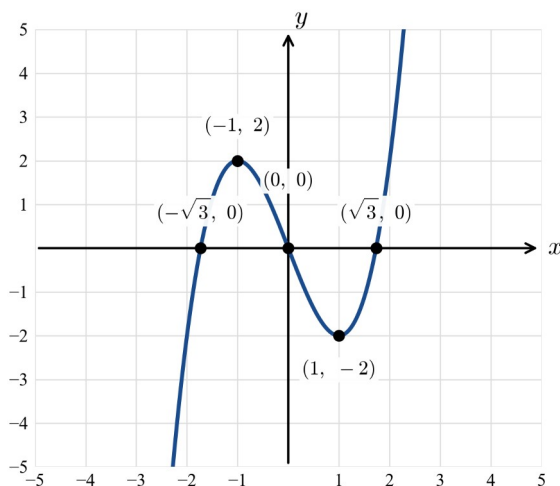
minimum.

Draw the straight line.



Example 2 — scaffolded

The graph of $f(x) = x^3 - 3x$ is shown. Sketch the graph of f' .



$$f'(x) = 3x^2 - 3.$$

A cubic gives a **quadratic** derivative.

f has a local maximum at $x = -1$ and a local minimum at $x = 1$, so $f'(-1) = f'(1) = 0$: intercepts $(-1, 0)$ and $(1, 0)$.

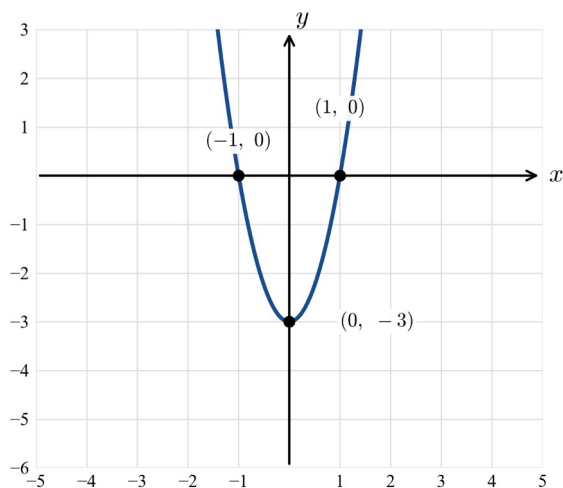
The two stationary points of f give the two x -intercepts of f' .

Between $x = -1$ and $x = 1$, f is decreasing so $f' < 0$ (the parabola dips below the axis); outside, $f' > 0$.

The parabola opens **upwards**.

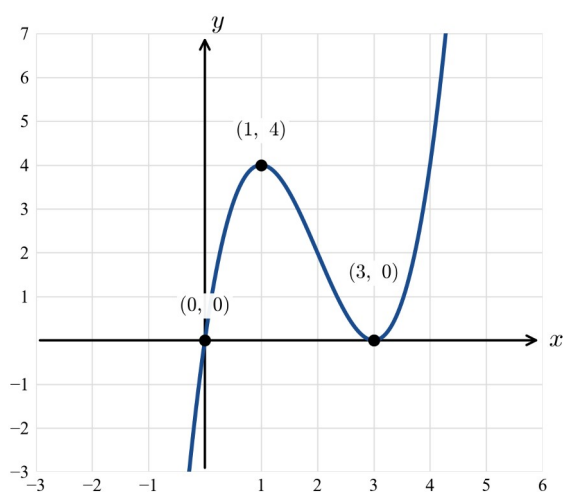
$f'(0) = -3$, giving the minimum of f' at $(0, -3)$.

Locate the vertex of f' .



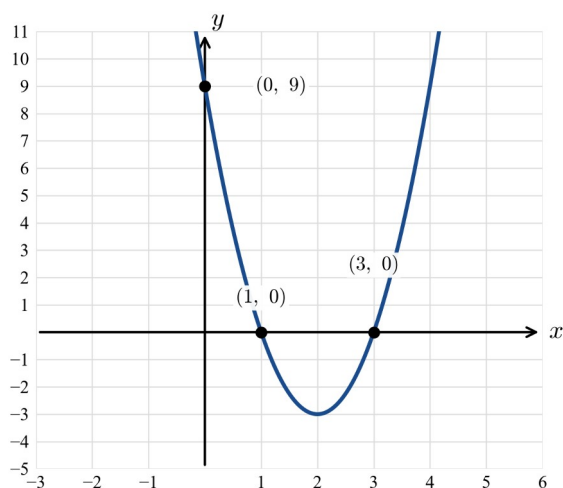
Example 3 — unscaffolded

The graph of $f(x) = x^3 - 6x^2 + 9x$ is shown. Sketch the graph of f' .



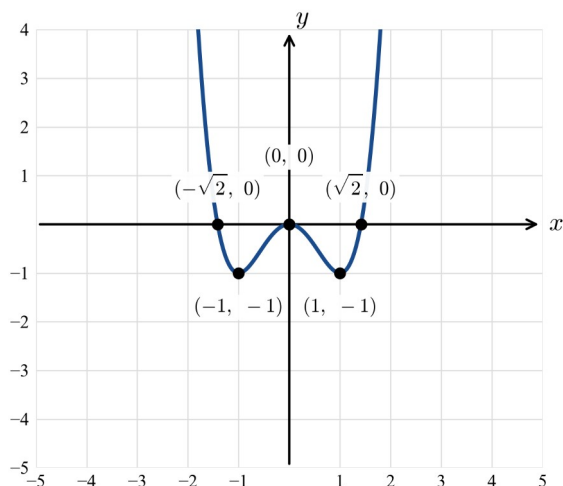
$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. The stationary points of f are at $x=1$ (a local maximum) and $x=3$ (a local minimum), so f' has x -intercepts $(1, 0)$ and $(3, 0)$. Since f is increasing for $x < 1$, decreasing for $1 < x < 3$ and increasing for $x > 3$, the graph of f' is a **positive** parabola: above the axis, below between the intercepts, then above again.

$\therefore f'$ is the upward parabola through $(1, 0)$ and $(3, 0)$ with y -intercept $(0, 9)$.



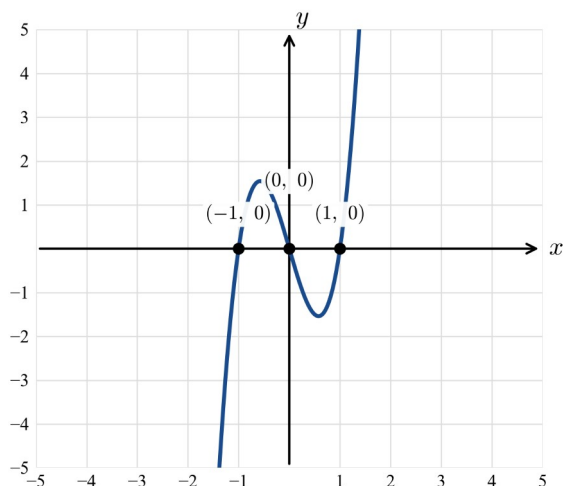
Example 4 — unscaffolded

The graph of $f(x) = x^4 - 2x^2$ is shown. Sketch the graph of f' .



$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$. The stationary points of f are at $x = -1$ (minimum), $x = 0$ (maximum) and $x = 1$ (minimum), so f' has x -intercepts $(-1, 0)$, $(0, 0)$ and $(1, 0)$. A quartic gives a **cubic** f' ; the sign pattern (down, up, down, up as x increases through the three stationary points) matches a positive cubic.

$\therefore f'$ is the cubic through $(-1, 0)$, $(0, 0)$ and $(1, 0)$.

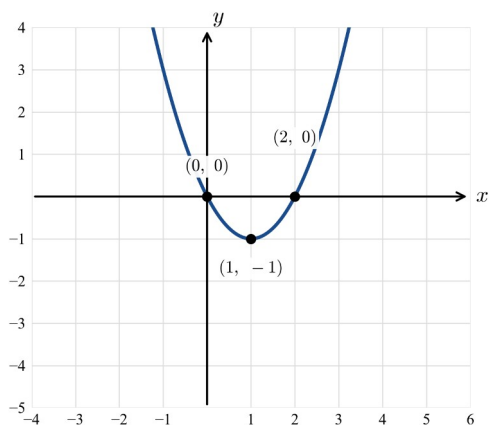


Practice questions

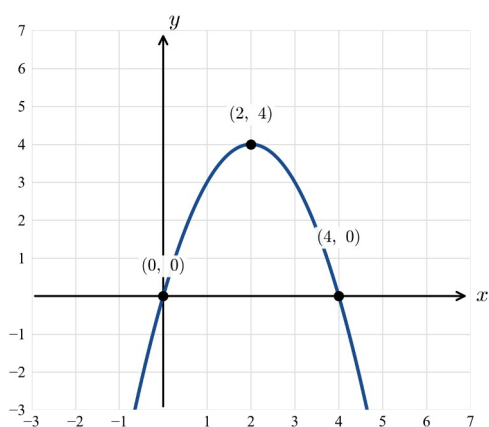
Scaffolded

For each graph of f shown: (a) find $f'(x)$; (b) state the interval(s) on which f is increasing; (c) write the coordinates of the x -intercept(s) of the graph of f' ; (d) sketch the graph of f' .

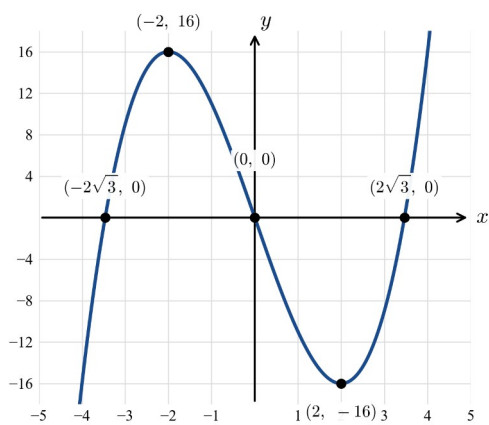
Q1. $f(x) = x^2 - 2x$



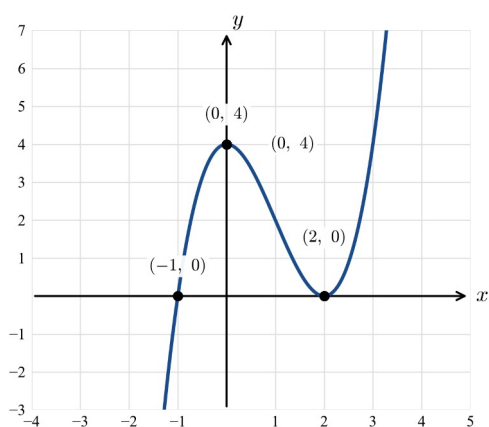
Q2. $f(x) = -x^2 + 4x$



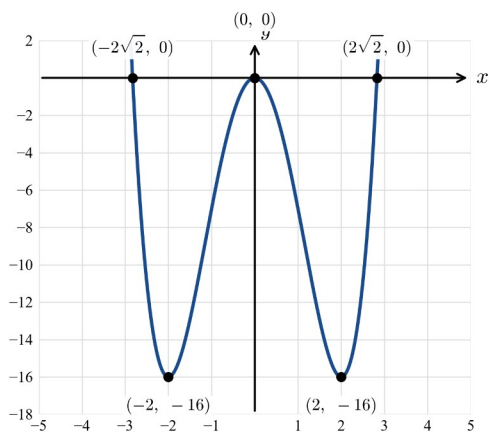
Q3. $f(x) = x^3 - 12x$



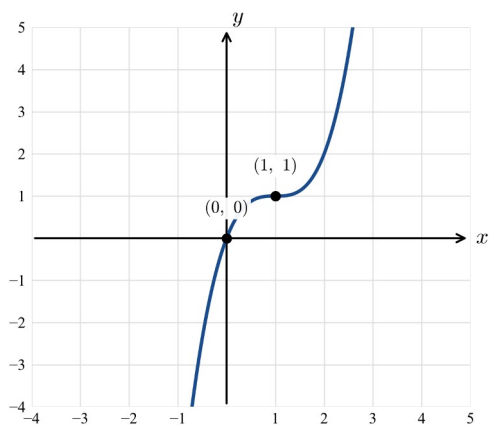
Q4. $f(x) = x^3 - 3x^2 + 4$



Q5. $f(x) = x^4 - 8x^2$



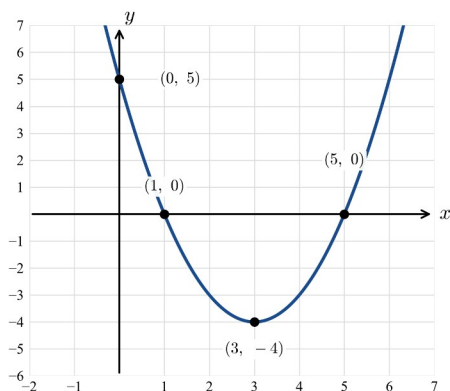
Q6. $f(x) = x^3 - 3x^2 + 3x$ (note the stationary point of inflection)



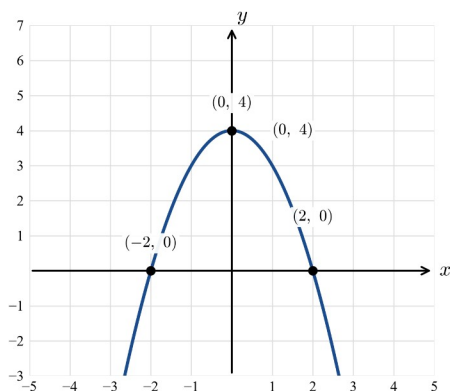
Unscaffolded

For **Q7–Q16** and **Q21**, the graph of f is shown; sketch the graph of f' , labelling its x -intercepts.

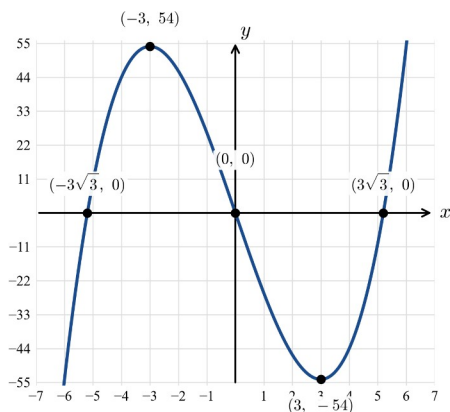
Q7. $f(x) = x^2 - 6x + 5$



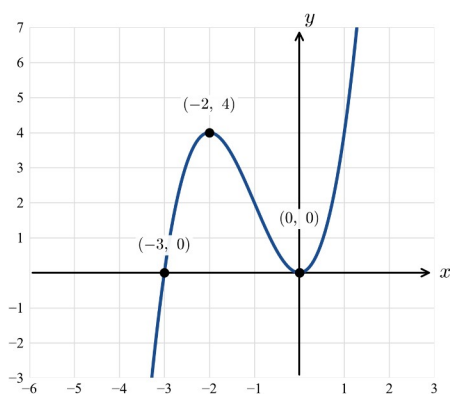
Q8. $f(x) = -x^2 + 4$



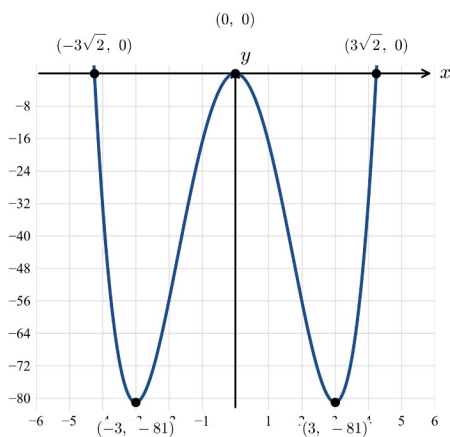
Q9. $f(x) = x^3 - 27x$



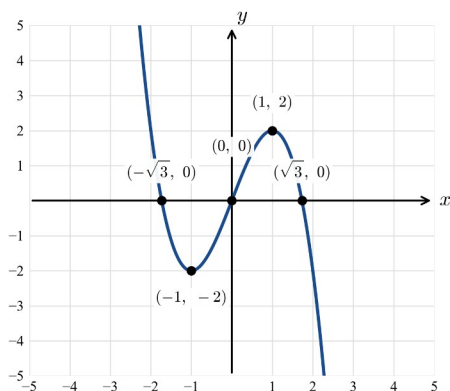
Q10. $f(x) = x^3 + 3x^2$



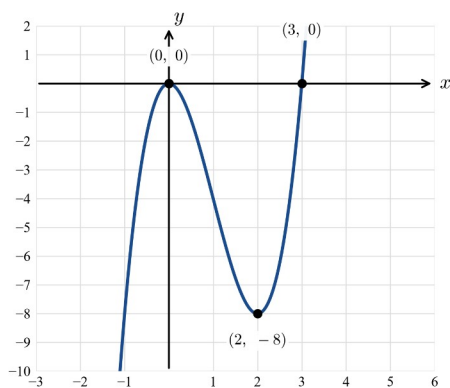
Q11. $f(x) = x^4 - 18x^2$



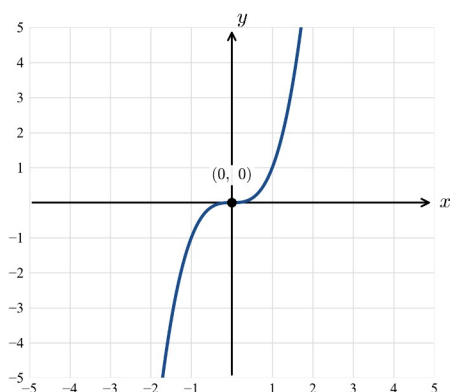
Q12. $f(x) = -x^3 + 3x$



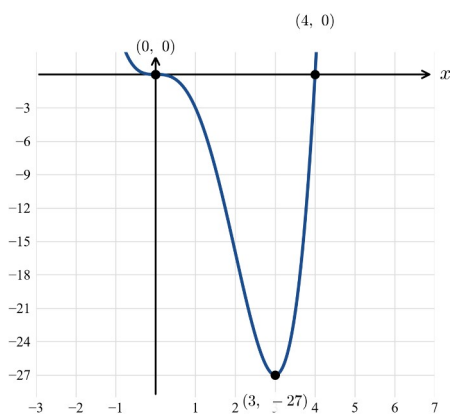
Q13. $f(x) = 2x^3 - 6x^2$



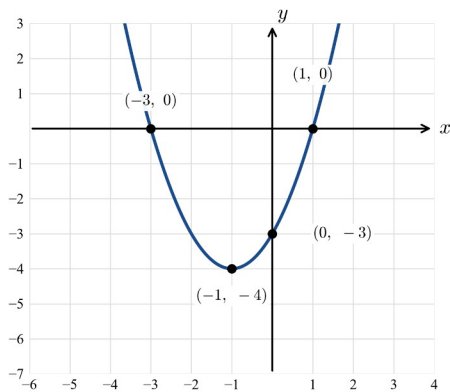
Q14. $f(x) = x^3$



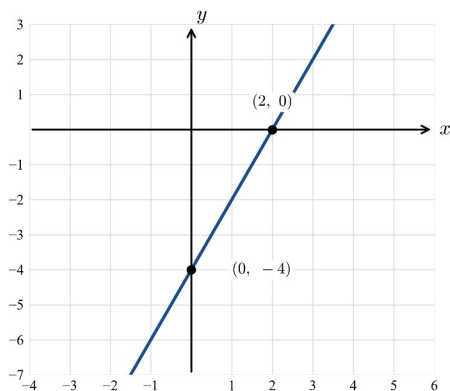
Q15. $f(x) = x^4 - 4x^3$



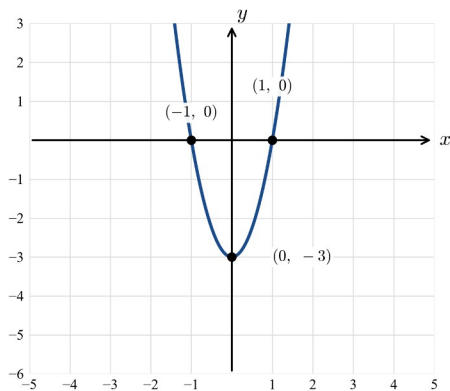
Q16. $f(x) = x^2 + 2x - 3$



Q17. The graph of $f'(x) = 2x - 4$ is shown. State the x -value of the stationary point of f and its nature, and describe where f is increasing.



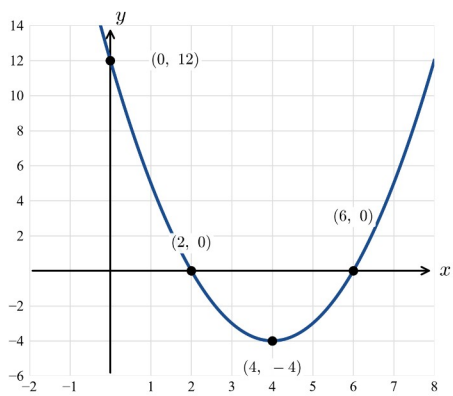
Q18. The graph of $f'(x) = 3x^2 - 3$ is shown. State the x -values of the stationary points of f and the nature of each.



Q19. A function f is increasing for $x < 1$, has a stationary point at $x = 1$, and is decreasing for $x > 1$. Sketch a possible graph of f' .

Q20. A cubic function f has a local maximum at $x = -2$ and a local minimum at $x = 2$. Sketch a possible graph of f' .

Q21. $f(x) = x^2 - 8x + 12$

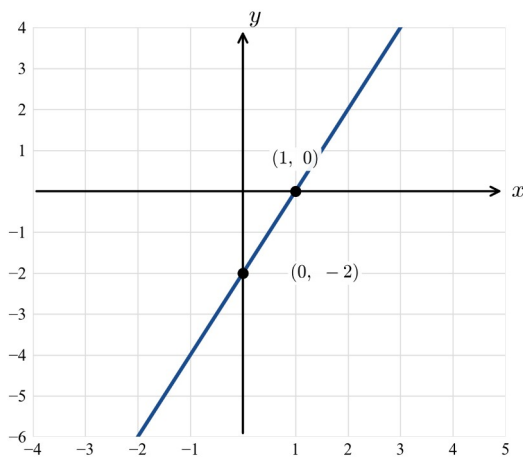


Answers

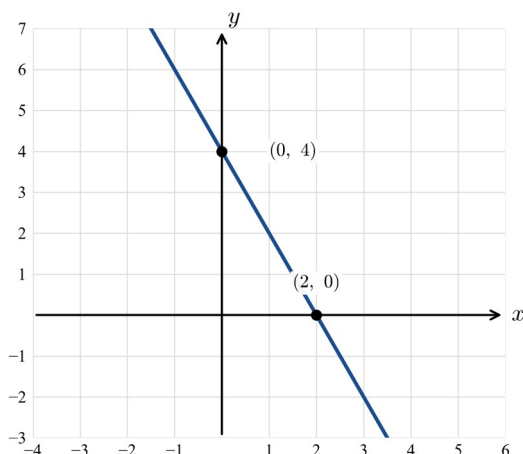
Q1. (a) $2x - 2$; (b) $x > 1$; (c) $(1, 0)$. **Q2.** (a) $4 - 2x$; (b) $x < 2$; (c) $(2, 0)$. **Q3.** (a) $3x^2 - 12$; (b) $x < -2$ or $x > 2$; (c) $(-2, 0), (2, 0)$. **Q4.** (a) $3x^2 - 6x$; (b) $x < 0$ or $x > 2$; (c) $(0, 0), (2, 0)$. **Q5.** (a) $4x^3 - 16x$; (b) $-2 < x < 0$ or $x > 2$; (c) $(-2, 0), (0, 0), (2, 0)$. **Q6.** (a) $3x^2 - 6x + 3 = 3(x - 1)^2$; (b) increasing for all x (except the stationary point $x = 1$); (c) $(1, 0)$ — f' touches the axis (stationary point of inflection). **Q7.** $f' = 2x - 6$; intercept $(3, 0)$; line, gradient 2. **Q8.** $f' = -2x$; intercept $(0, 0)$; line, gradient -2. **Q9.** $f' = 3x^2 - 27$; intercepts $(-3, 0), (3, 0)$; up-parabola, y-int $(0, -27)$. **Q10.** $f' = 3x^2 + 6x$; intercepts $(-2, 0), (0, 0)$; up-parabola. **Q11.** $f' = 4x^3 - 36x$; intercepts $(-3, 0), (0, 0), (3, 0)$; positive cubic. **Q12.** $f' = 3 - 3x^2$; intercepts $(-1, 0), (1, 0)$; down-parabola (y-int $(0, 3)$). **Q13.** $f' = 6x^2 - 12x$; intercepts $(0, 0), (2, 0)$; up-parabola. **Q14.** $f' = 3x^2$; intercept $(0, 0)$ — f' touches the axis (stationary point of inflection). **Q15.** $f' = 4x^3 - 12x^2$; intercepts $(0, 0)$ (touch) and $(3, 0)$ (cross). **Q16.** $f' = 2x + 2$; intercept $(-1, 0)$; line, gradient 2. **Q17.** Stationary point at $x = 2$, a **minimum** (since f' crosses from below to above); f increasing for $x > 2$. **Q18.** $x = -1$ (**maximum**) and $x = 1$ (**minimum**). **Q19.** $f' > 0$ for $x < 1$, $f'(1) = 0$, $f' < 0$ for $x > 1$: a graph crossing the axis from above to below at $x = 1$ (e.g. the line $y = 2 - 2x$). **Q20.** f' has x-intercepts at $x = -2$ (crossing $+$ $\rightarrow$ $-$) and $x = 2$ (crossing $-$ $\rightarrow$ $+$): an **upward** parabola with roots -2 and 2 (e.g. $y = x^2 - 4$). **Q21.** $f' = 2x - 8$; intercept $(4, 0)$; line, gradient 2.

Fully-worked solutions

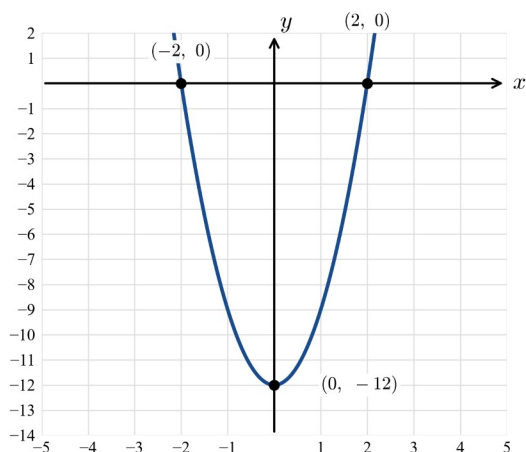
Q1. $f'(x) = 2x - 2$. f decreasing for $x < 1$, increasing for $x > 1$, minimum at $x = 1 \rightarrow$ intercept $(1, 0)$; the line has gradient 2.



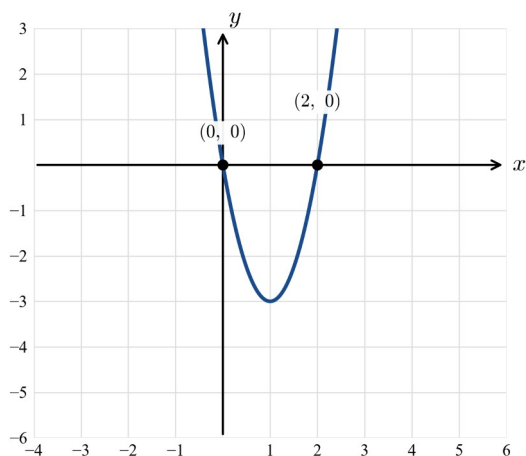
Q2. $f'(x) = 4 - 2x$. Maximum at $x = 2 \rightarrow$ intercept $(2, 0)$; f increasing for $x < 2$, so $f' > 0$ there; the line has gradient -2.



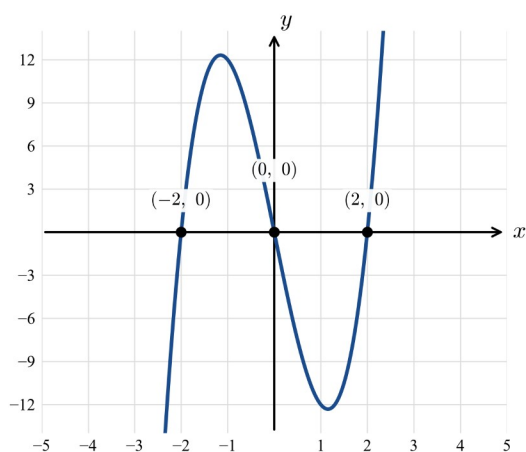
Q3. $f'(x) = 3x^2 - 12$. Maximum at $x = -2$, minimum at $x = 2 \rightarrow$ intercepts $(-2, 0), (2, 0)$; up-parabola, minimum $(0, -12)$.



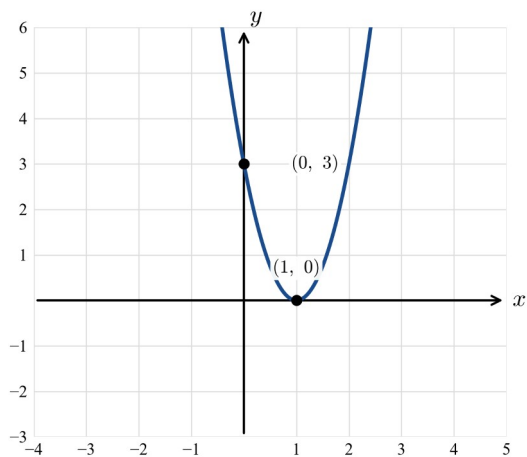
Q4. $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Maximum at $x = 0$, minimum at $x = 2 \rightarrow$ intercepts $(0, 0), (2, 0)$; up-parabola.



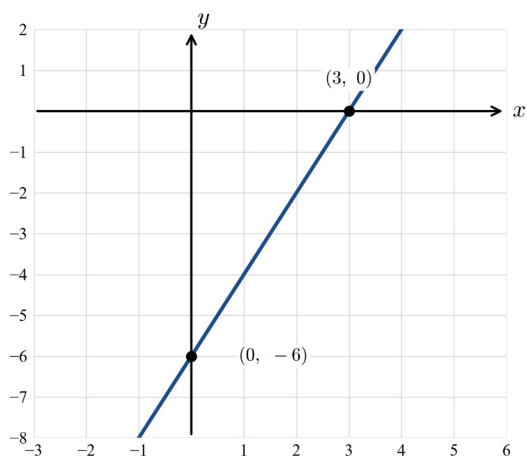
Q5. $f'(x) = 4x^3 - 16x = 4x(x - 2)(x + 2)$. Stationary points of f at $x = -2, 0, 2 \rightarrow$ intercepts $(-2, 0), (0, 0), (2, 0)$; positive cubic.



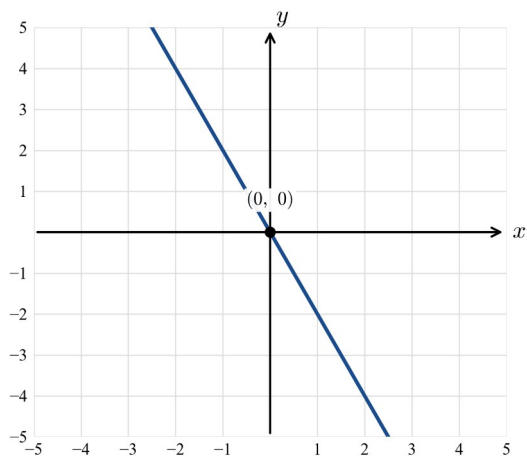
Q6. $f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2$. The only stationary point is the **stationary point of inflection** at $x = 1$; $f' = 3(x - 1)^2 \geq 0$, so f' **touches** the axis at $(1, 0)$ and is otherwise positive (f is increasing everywhere else).



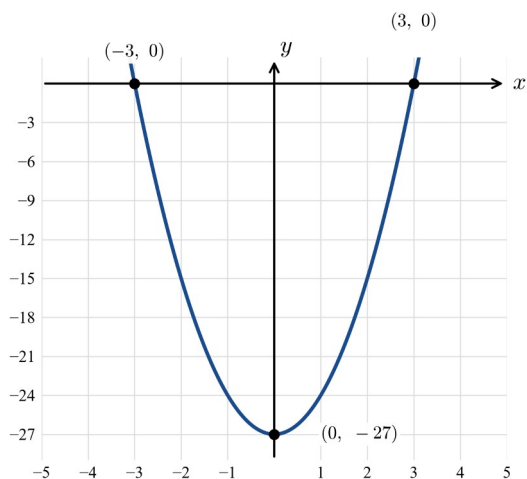
Q7. $f'(x) = 2x - 6$; minimum of f at $x = 3 \rightarrow$ intercept $(3, 0)$; line, gradient 2.



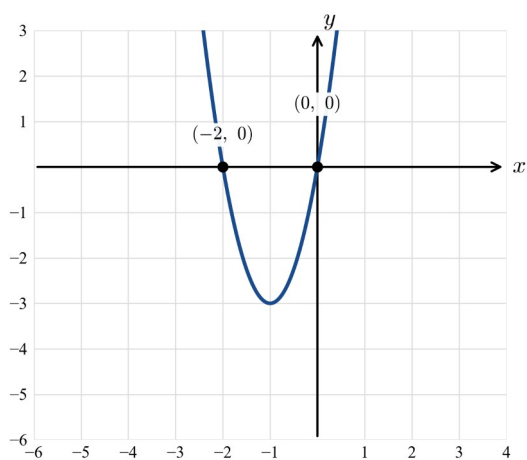
Q8. $f'(x) = -2x$; maximum of f at $x = 0 \rightarrow$ intercept $(0, 0)$; line, gradient -2 .



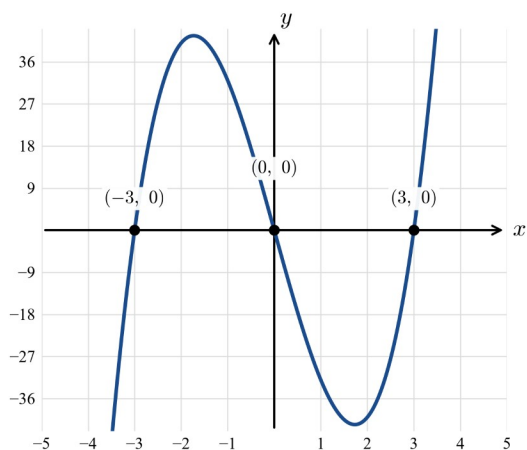
Q9. $f'(x) = 3x^2 - 27$; maximum at $x = -3$, minimum at $x = 3 \rightarrow$ intercepts $(-3, 0), (3, 0)$; up-parabola, y-intercept $(0, -27)$.



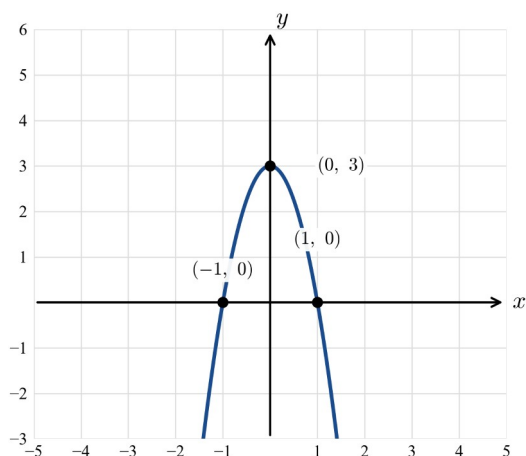
Q10. $f'(x) = 3x^2 + 6x = 3x(x+2)$; maximum at $x = -2$, minimum at $x = 0 \rightarrow$ intercepts $(-2, 0), (0, 0)$; up-parabola.



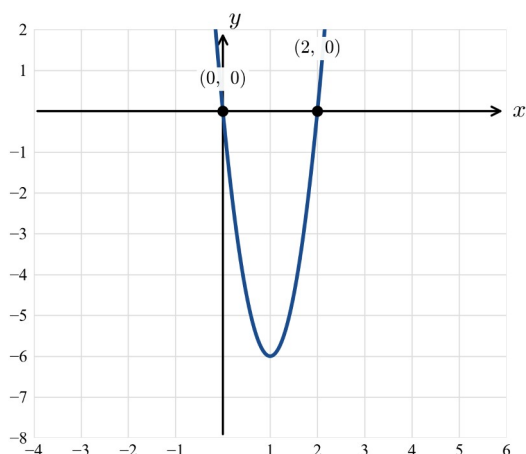
Q11. $f'(x) = 4x^3 - 36x = 4x(x-3)(x+3)$; stationary points at $x = -3, 0, 3 \rightarrow$ intercepts $(-3, 0), (0, 0), (3, 0)$; positive cubic.



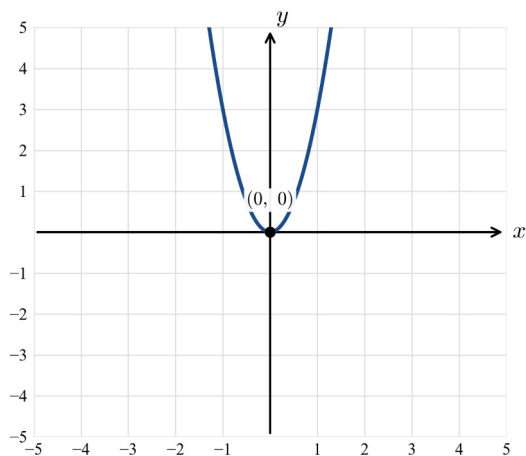
Q12. $f'(x) = 3 - 3x^2$; minimum of f at $x = -1$, maximum at $x = 1 \rightarrow$ intercepts $(-1, 0), (1, 0)$; **down**-parabola, y -intercept $(0, 3)$.



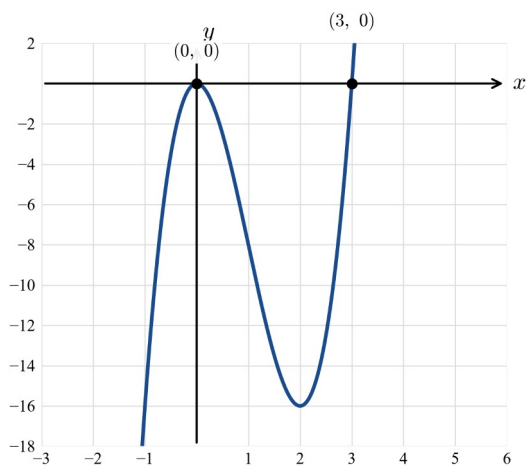
Q13. $f'(x) = 6x^2 - 12x = 6x(x - 2)$; maximum at $x=0$, minimum at $x=2 \rightarrow$ intercepts $(0, 0), (2, 0)$; up-parabola.



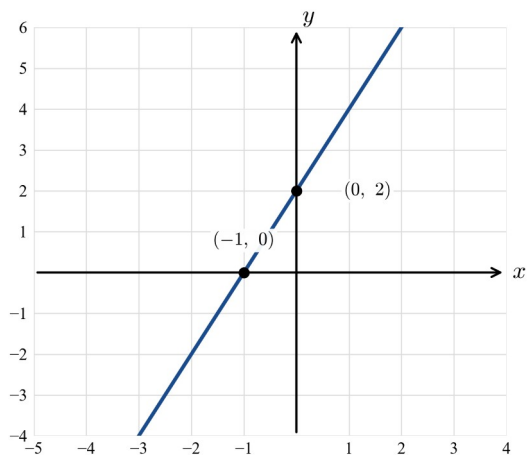
Q14. $f'(x) = 3x^2$; the stationary point of inflection at $x=0$ gives a **touch** at $(0, 0)$; $f' \geq 0$ everywhere (f is increasing except at $x=0$).



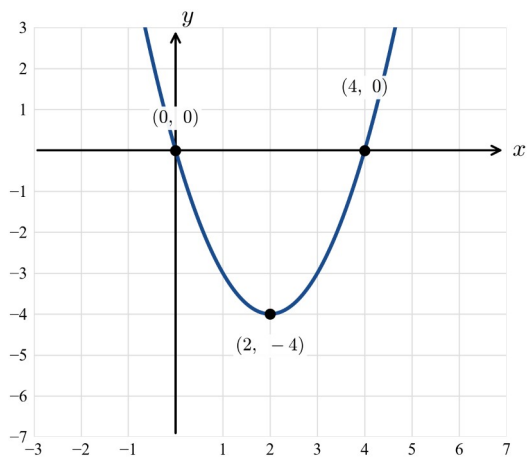
Q15. $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$; at $x=0$ a stationary point of inflection (f' touches at $(0, 0)$), at $x=3$ a minimum (f' crosses at $(3, 0)$).



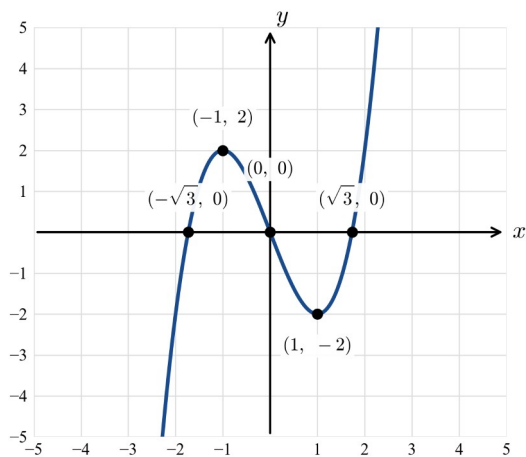
Q16. $f'(x) = 2x + 2$; minimum of f at $x = -1 \rightarrow$ intercept $(-1, 0)$; line, gradient 2.



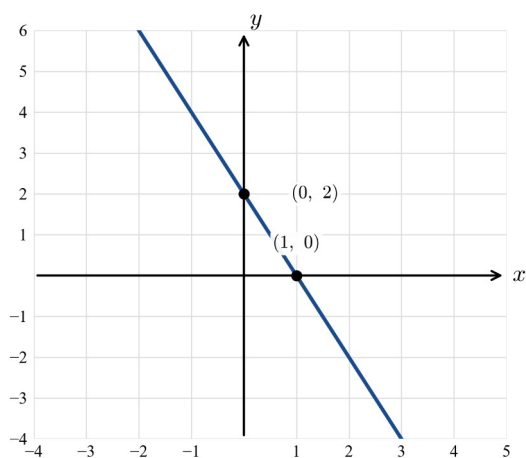
Q17. $f'(x) = 2x - 4 = 0$ at $x = 2$; $f' < 0$ for $x < 2$ and $f' > 0$ for $x > 2$, so f has a **minimum** at $x = 2$ and is increasing for $x > 2$. A possible f is shown.



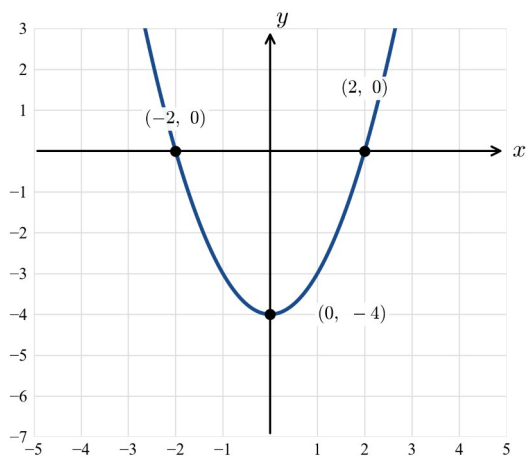
Q18. $f'(x) = 3x^2 - 3 = 0$ at $x = \pm 1$. Near $x = -1$, f' changes $+$ $\rightarrow$ $-$ (a **maximum**); near $x = 1$, f' changes $-$ $\rightarrow$ $+$ (a **minimum**). A possible f is shown.



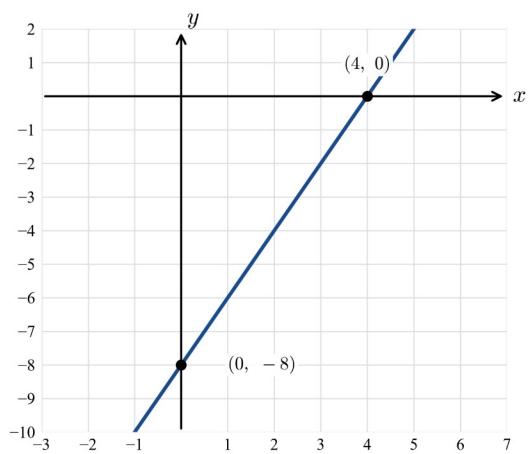
Q19. f increasing then decreasing with a stationary point at $x=1$ means $f' > 0$ for $x < 1$, $f'(1)=0$, $f' < 0$ for $x > 1$ — the graph of f' crosses the axis from above to below at $x=1$:



Q20. A maximum at $x=-2$ (f' crosses $+$ $\rightarrow$ $-$) and a minimum at $x=2$ (f' crosses $-$ $\rightarrow$ $+$) mean f' is an **upward** parabola with x-intercepts -2 and 2 :



Q21. $f'(x) = 2x - 8$; minimum of f at $x=4 \rightarrow$ intercept $(4, 0)$; line, gradient 2.



Theory

Many functions are built by **composition** — an *inner* function nested inside an *outer* one. For example $y = (3x^2 - 1)^4$ is the outer function u^4 applied to the inner function $u = 3x^2 - 1$. The **chain rule** differentiates such functions.

The chain rule. If y is a function of u and u is a function of x , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The power form (used most often). If $y = (g(x))^n$, then, taking $u = g(x)$ so that $y = u^n$,

$$\frac{dy}{dx} = n(g(x))^{n-1} \cdot g'(x).$$

In words: **bring down the power, reduce the index by one, then multiply by the derivative of the inside.** This holds for any rational n (positive, negative or fractional) — rewrite roots and reciprocals as powers first,

e.g. $\sqrt{g(x)} = (g(x))^{1/2}$ and $\frac{1}{(g(x))^k} = (g(x))^{-k}$.

To differentiate a composite function:

1. Identify the **inner** function $u = g(x)$ and the **outer** function $y = f(u)$.
2. Find $\frac{dy}{du}$ and $\frac{du}{dx}$.
3. Multiply them and write the answer **in terms of x** .
4. Simplify, factorising where possible.

Worked examples

Example 1 — scaffolded

Differentiate $y = (3x^2 - 1)^4$.

Working

Comment

$$\text{Let } u = 3x^2 - 1, \text{ so } y = u^4.$$

Identify the inner function; the outer is u^4 .

$$\frac{du}{dx} = 6x \text{ and } \frac{dy}{du} = 4u^3.$$

Differentiate each part separately.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot 6x$$

Apply the chain rule.

$$= 4(3x^2 - 1)^3 \cdot 6x = 24x(3x^2 - 1)^3$$

Substitute $u = 3x^2 - 1$ and simplify.

Example 2 — scaffolded

Differentiate $y = \sqrt{4x - 3}$.

Working

Comment

$$y = (4x - 3)^{1/2}.$$

Rewrite the root as a power before differentiating.

$$\text{Let } u = 4x - 3: \frac{du}{dx} = 4, \frac{dy}{du} = \frac{1}{2}u^{-1/2}.$$

Outer function $u^{1/2}$.

$$\frac{dy}{dx} = \frac{1}{2}(4x-3)^{-1/2} \cdot 4 = \frac{2}{(4x-3)^{1/2}}$$

Chain rule, then substitute back.

$$= \frac{2}{\sqrt{4x-3}}$$

Write the negative power as a surd.

Example 3 — unscaffolded

Differentiate $y = \frac{1}{(x^2+3)^2}$.

Rewrite as $y = (x^2+3)^{-2}$. With inside x^2+3 (derivative $2x$):

$$\frac{dy}{dx} = -2(x^2+3)^{-3} \cdot 2x = -\frac{4x}{(x^2+3)^3}$$

$$\therefore \frac{dy}{dx} = -\frac{4x}{(x^2+3)^3}$$

Example 4 — unscaffolded

For $y = (2x-1)^4$, find the gradient of the curve at $x = 1$.

$$\frac{dy}{dx} = 4(2x-1)^3 \cdot 2 = 8(2x-1)^3$$

At $x = 1$: $\frac{dy}{dx} = 8(2(1)-1)^3 = 8(1)^3 = 8$.

$\therefore$ the gradient at $x = 1$ is 8.

Practice questions**Scaffolded**

Q1. For $y = (2x+5)^3$: (a) state the inner function u ; (b) find $\frac{dy}{du}$ and $\frac{du}{dx}$; (c) hence find $\frac{dy}{dx}$.

Q2. For $y = (x^2-4x)^5$: (a) identify the inner function; (b) find $\frac{dy}{dx}$ using the chain rule; (c) factorise your answer.

Q3. For $y = (3-2x)^6$: (a) find $\frac{dy}{dx}$; (b) explain why the factor from the inside is -2 .

Q4. For $y = \sqrt{x^2+9}$: (a) rewrite y as a power; (b) find $\frac{dy}{dx}$; (c) write the answer as a single fraction.

Q5. For $y = \frac{1}{(x^2+1)^2}$: (a) rewrite y with a negative index; (b) find $\frac{dy}{dx}$.

Q6. For $y = (x^3+1)^4$: (a) find $\frac{dy}{dx}$; (b) hence find the gradient of the curve at $x = 1$.

Unscaffolded

Differentiate each of the following (simplify, factorising where possible).

Q7. $y = (4x - 7)^3$

Q8. $y = (x^2 + 2x)^4$

Q9. $y = (5 - 3x)^5$

Q10. $y = \sqrt{2x + 7}$

Q11. $y = (x^2 - 1)^{3/2}$

Q12. $y = \frac{1}{3x^2 + 2}$

Q13. $y = (x^3 - 3x)^3$

Q14. $y = \sqrt{1 - x^2}$

Q15. For $y = (2x + 3)^5$, find the gradient at $x = -1$.

Q16. For $y = (x^2 - 4)^3$, find the values of x where the gradient is 0.

Q17. $y = \frac{1}{(x - 2)^3}$

Q18. $y = (x^2 + x + 1)^4$

Q19. $y = \sqrt{x^3 + 1}$

Q20. For $y = (x^2 - 2x + 2)^3$, show that $\frac{dy}{dx} = 6(x - 1)(x^2 - 2x + 2)^2$.

Q21. Find the equation of the tangent to $y = (2x - 3)^4$ at the point where $x = 2$.

Answers

Q1. $6(2x+5)^2$ **Q2.** $10(x-2)(x^2-4x)^4$ **Q3.** $-12(3-2x)^5$ **Q4.** $\frac{x}{\sqrt{x^2+9}}$ **Q5.** $-\frac{4x}{(x^2+1)^3}$ **Q6.** $12x^2(x^3+1)^3$;
 gradient 96 **Q7.** $12(4x-7)^2$ **Q8.** $(8x+8)(x^2+2x)^3=8(x+1)(x^2+2x)^3$ **Q9.** $-15(5-3x)^4$ **Q10.** $\frac{1}{\sqrt{2x+7}}$ **Q11.**
 $3x\sqrt{x^2-1}$ **Q12.** $-\frac{6x}{(3x^2+2)^2}$ **Q13.** $9(x^2-1)(x^3-3x)^2$ **Q14.** $-\frac{x}{\sqrt{1-x^2}}$ **Q15.** 10 **Q16.** $x=-2, 0, 2$ **Q17.**
 $-\frac{3}{(x-2)^4}$ **Q18.** $4(2x+1)(x^2+x+1)^3$ **Q19.** $\frac{3x^2}{2\sqrt{x^3+1}}$ **Q20.** (proof — see solutions) **Q21.** $y=8x-15$

Fully-worked solutions

Q1. $y=(2x+5)^3$. (a) $u=2x+5$. (b) $\frac{dy}{du}=3u^2$, $\frac{du}{dx}=2$. (c) $\frac{dy}{dx}=3(2x+5)^2 \cdot 2=6(2x+5)^2$.

Q2. $y=(x^2-4x)^5$. (a) $u=x^2-4x$. (b) $\frac{dy}{dx}=5(x^2-4x)^4 \cdot (2x-4)$. (c)
 $=5 \cdot 2(x-2)(x^2-4x)^4=10(x-2)(x^2-4x)^4$.

Q3. $y=(3-2x)^6$. (a) $\frac{dy}{dx}=6(3-2x)^5 \cdot (-2)=-12(3-2x)^5$. (b) The inside is $3-2x$, whose derivative is -2 ;
 the chain rule multiplies by this factor.

Q4. $y=\sqrt{x^2+9}=(x^2+9)^{1/2}$. (b) $\frac{dy}{dx}=\frac{1}{2}(x^2+9)^{-1/2} \cdot 2x=x(x^2+9)^{-1/2}$. (c) $=\frac{x}{\sqrt{x^2+9}}$.

Q5. $y=(x^2+1)^{-2}$. $\frac{dy}{dx}=-2(x^2+1)^{-3} \cdot 2x=-\frac{4x}{(x^2+1)^3}$.

Q6. $y=(x^3+1)^4$. (a) $\frac{dy}{dx}=4(x^3+1)^3 \cdot 3x^2=12x^2(x^3+1)^3$. (b) At $x=1$: $12(1)^2(1+1)^3=12 \cdot 8=96$.

Q7. $\frac{dy}{dx}=3(4x-7)^2 \cdot 4=12(4x-7)^2$.

Q8. $\frac{dy}{dx}=4(x^2+2x)^3 \cdot (2x+2)=(8x+8)(x^2+2x)^3=8(x+1)(x^2+2x)^3$.

Q9. $\frac{dy}{dx}=5(5-3x)^4 \cdot (-3)=-15(5-3x)^4$.

Q10. $y=(2x+7)^{1/2}$; $\frac{dy}{dx}=\frac{1}{2}(2x+7)^{-1/2} \cdot 2=\frac{1}{\sqrt{2x+7}}$.

Q11. $y=(x^2-1)^{3/2}$; $\frac{dy}{dx}=\frac{3}{2}(x^2-1)^{1/2} \cdot 2x=3x\sqrt{x^2-1}$ (for $|x| \geq 1$).

Q12. $y=(3x^2+2)^{-1}$; $\frac{dy}{dx}=-(3x^2+2)^{-2} \cdot 6x=-\frac{6x}{(3x^2+2)^2}$.

Q13. $\frac{dy}{dx}=3(x^3-3x)^2 \cdot (3x^2-3)=9(x^2-1)(x^3-3x)^2$.

Q14. $y = (1 - x^2)^{1/2}$; $\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1 - x^2}}$ (for $-1 < x < 1$).

Q15. $\frac{dy}{dx} = 5(2x + 3)^4 \cdot 2 = 10(2x + 3)^4$. At $x = -1$: $10(2(-1) + 3)^4 = 10(1)^4 = 10$.

Q16. $\frac{dy}{dx} = 3(x^2 - 4)^2 \cdot 2x = 6x(x^2 - 4)^2$. Setting $6x(x^2 - 4)^2 = 0$ gives $x = 0$ or $x^2 = 4$, so $x = -2, 0, 2$.

Q17. $y = (x - 2)^{-3}$; $\frac{dy}{dx} = -3(x - 2)^{-4} \cdot 1 = -\frac{3}{(x - 2)^4}$.

Q18. $\frac{dy}{dx} = 4(x^2 + x + 1)^3 \cdot (2x + 1) = 4(2x + 1)(x^2 + x + 1)^3$.

Q19. $y = (x^3 + 1)^{1/2}$; $\frac{dy}{dx} = \frac{1}{2}(x^3 + 1)^{-1/2} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3 + 1}}$.

Q20. $y = (x^2 - 2x + 2)^3$. Using the chain rule with inside $x^2 - 2x + 2$ (derivative $2x - 2$):

$\frac{dy}{dx} = 3(x^2 - 2x + 2)^2 \cdot (2x - 2) = 3 \cdot 2(x - 1)(x^2 - 2x + 2)^2 = 6(x - 1)(x^2 - 2x + 2)^2$, as required.

Q21. $y = (2x - 3)^4$. At $x = 2$: $y = (2(2) - 3)^4 = 1^4 = 1$, so the point is $(2, 1)$. $\frac{dy}{dx} = 4(2x - 3)^3 \cdot 2 = 8(2x - 3)^3$; at $x = 2$: $8(1)^3 = 8$, the gradient. Tangent: $y - 1 = 8(x - 2) \Rightarrow y = 8x - 15$.

Chapter 8 Differentiation techniques — 8F Derivatives of rational powers

Theory

The power rule extends to **rational** indices. If $y = x^n$ where n is any rational number, then

$$\frac{dy}{dx} = n x^{n-1}.$$

For a non-integer index this holds for $x > 0$ (where x^n and x^{n-1} are defined).

Rewrite surds as fractional indices before differentiating. Recall:

$$\sqrt{x} = x^{1/2}, \sqrt[3]{x} = x^{1/3}, \sqrt[n]{x^m} = x^{m/n}, \frac{1}{\sqrt{x}} = x^{-1/2}.$$

Then apply the rule term by term, together with the **constant-multiple** and **sum/difference** rules:

$$\frac{d}{dx}(a x^{p/q}) = a \cdot \frac{p}{q} x^{p/q-1}.$$

Linear inside a power. For $y = (a x + b)^n$ the chain rule (Section 8E) gives

$$\frac{dy}{dx} = n a (a x + b)^{n-1}.$$

For example $\frac{d}{dx} \sqrt{a x + b} = \frac{a}{2\sqrt{a x + b}}.$

Domains. The derivative is often defined on a **smaller** domain than the function itself — for instance $y = \sqrt{x}$ has domain $x \geq 0$, but $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ is defined only for $x > 0$ (the curve has a vertical tangent at the origin).

To differentiate a rational-power expression:

1. Write every term as $a x^n$ (or $(a x + b)^n$) using index laws.
2. Differentiate each term with the power rule.
3. Where asked, rewrite the answer back in surd form and simplify.

Worked examples

Example 1 — scaffolded

For $y = \sqrt{x}$, find $\frac{dy}{dx}$ and hence the gradient of the curve at $x = 4$.

Working	Comment
$y = \sqrt{x} = x^{1/2}$	Write the surd as a fractional index.
$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$	Power rule: bring down 1/2, subtract 1 from the index.
At $x = 4$: $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$	Substitute $x = 4$; $\sqrt{4} = 2$.
$\therefore$ the gradient at $x = 4$ is $\frac{1}{4}$.	State the answer exactly.

Example 2 — scaffolded

Differentiate $y = x^{2/3}$.

Working

$$y = x^{2/3}$$

$$\frac{dy}{dx} = \frac{2}{3} x^{2/3-1} = \frac{2}{3} x^{-1/3}$$

$$= \frac{2}{3\sqrt[3]{x}}$$

Comment

Already in index form.

Power rule with $n = 2/3$.

Rewrite the negative fractional index as a surd.

Example 3 — unscaffolded

Differentiate $y = \sqrt{x} + \frac{1}{\sqrt{x}}$.

Write in index form: $y = x^{1/2} + x^{-1/2}$.

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} - \frac{1}{2} x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{x-1}{2x\sqrt{x}} \text{ (combining over a common denominator).}$$

Example 4 — unscaffolded

Differentiate $y = \sqrt{2x+1}$.

Write as $y = (2x+1)^{1/2}$. Using the chain rule with the linear inside,

$$\frac{dy}{dx} = \frac{1}{2} (2x+1)^{-1/2} \times 2 = (2x+1)^{-1/2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{2x+1}}$$

Practice questions

Scaffolded

Q1. For $y = x^{1/3} = \sqrt[3]{x}$: (a) write the index form; (b) find $\frac{dy}{dx}$; (c) express the answer in surd form.

Q2. For $y = x^{3/2}$: (a) find $\frac{dy}{dx}$; (b) express the answer in surd form.

Q3. For $y = 4\sqrt{x}$: (a) find $\frac{dy}{dx}$; (b) hence find the gradient of the curve at $x = 4$.

Q4. For $y = x^{5/2}$: find $\frac{dy}{dx}$.

Q5. For $y = 2\sqrt{x} - 3\sqrt[3]{x}$: (a) write each term in index form; (b) find $\frac{dy}{dx}$ and express it in surd form.

Q6. For $y = (3x-2)^{3/2}$: find $\frac{dy}{dx}$ (leave your answer in surd form).

Un scaffolded

Differentiate each of the following with respect to x , giving your answer in surd form where appropriate.

Q7. $y = \sqrt{x} + x$

Q8. $y = x\sqrt{x}$

Q9. $y = \sqrt{x}(x - 1)$

Q10. $y = \frac{\sqrt{x} + 1}{x}$

Q11. $y = \frac{1}{\sqrt{x}}$

Q12. $y = x^{2/3} + x^{1/3}$

Q13. $y = \sqrt{x^2 + 1}$

Q14. $y = \sqrt{4 - x}$

Q15. $y = 5x^{4/5}$

Q16. $y = \sqrt{2x}$; hence find the gradient at $x = 8$.

Q17. $y = 3\sqrt{x} - \frac{2}{\sqrt{x}}$

Q18. $y = x\sqrt{x+1}$

Q19. Find the equation of the tangent to $y = \sqrt{x}$ at the point where $x = 9$.

Q20. Find the equation of the tangent to $y = x^{2/3}$ at the point where $x = 8$.

Q21. For $y = x^{3/2}$, find the value of x at which the gradient equals 3.

Answers

Q1. $\frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$ **Q2.** $\frac{3}{2}\sqrt{x}$ **Q3.** $\frac{2}{\sqrt{x}}$; gradient = 1 **Q4.** $\frac{5}{2}x^{3/2}$ **Q5.** $\frac{1}{\sqrt{x}} - \frac{1}{3\sqrt[3]{x^2}}$ **Q6.** $\frac{9}{2}\sqrt{3x-2}$ **Q7.** $1 + \frac{1}{2\sqrt{x}}$ **Q8.** $\frac{3}{2}\sqrt{x}$ **Q9.** $\frac{3x-1}{2\sqrt{x}}$ **Q10.** $-\frac{1}{2x\sqrt{x}} - \frac{1}{x^2}$ **Q11.** $-\frac{1}{2x\sqrt{x}}$ **Q12.** $\frac{2}{3\sqrt[3]{x}} + \frac{1}{3\sqrt[3]{x^2}}$ **Q13.** $\frac{x}{\sqrt{x^2+1}}$ **Q14.** $-\frac{1}{2\sqrt{4-x}}$ **Q15.** $\frac{4}{\sqrt[5]{x}}$
Q16. $\frac{1}{\sqrt{2x}}$; gradient = $\frac{1}{4}$ **Q17.** $\frac{3}{2\sqrt{x}} + \frac{1}{x\sqrt{x}}$ **Q18.** $\frac{3x+2}{2\sqrt{x+1}}$ **Q19.** $y = \frac{1}{6}x + \frac{3}{2}$ **Q20.** $y = \frac{1}{3}x + \frac{4}{3}$ **Q21.** $x = 4$

Fully-worked solutions

Q1. $y = x^{1/3}$. $\frac{dy}{dx} = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$.

Q2. $y = x^{3/2}$. $\frac{dy}{dx} = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$.

Q3. $y = 4\sqrt{x} = 4x^{1/2}$. $\frac{dy}{dx} = 4 \cdot \frac{1}{2}x^{-1/2} = 2x^{-1/2} = \frac{2}{\sqrt{x}}$. At $x = 4$: $\frac{2}{\sqrt{4}} = \frac{2}{2} = 1$.

Q4. $y = x^{5/2}$. $\frac{dy}{dx} = \frac{5}{2}x^{5/2-1} = \frac{5}{2}x^{3/2}$.

Q5. $y = 2x^{1/2} - 3x^{1/3}$. $\frac{dy}{dx} = 2 \cdot \frac{1}{2}x^{-1/2} - 3 \cdot \frac{1}{3}x^{-2/3} = x^{-1/2} - x^{-2/3} = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt[3]{x^2}}$.

Q6. $y = (3x-2)^{3/2}$. Chain rule: $\frac{dy}{dx} = \frac{3}{2}(3x-2)^{1/2} \times 3 = \frac{9}{2}(3x-2)^{1/2} = \frac{9}{2}\sqrt{3x-2}$.

Q7. $y = x^{1/2} + x$. $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + 1 = 1 + \frac{1}{2\sqrt{x}}$.

Q8. $y = x\sqrt{x} = x^{3/2}$. $\frac{dy}{dx} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$.

Q9. $y = \sqrt{x}(x-1) = x^{3/2} - x^{1/2}$. $\frac{dy}{dx} = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = \frac{3\sqrt{x}}{2} - \frac{1}{2\sqrt{x}} = \frac{3x-1}{2\sqrt{x}}$.

Q10. $y = \frac{\sqrt{x}+1}{x} = x^{-1/2} + x^{-1}$. $\frac{dy}{dx} = -\frac{1}{2}x^{-3/2} - x^{-2} = -\frac{1}{2x\sqrt{x}} - \frac{1}{x^2}$.

Q11. $y = x^{-1/2}$. $\frac{dy}{dx} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x\sqrt{x}}$.

Q12. $y = x^{2/3} + x^{1/3}$. $\frac{dy}{dx} = \frac{2}{3}x^{-1/3} + \frac{1}{3}x^{-2/3} = \frac{2}{3\sqrt[3]{x}} + \frac{1}{3\sqrt[3]{x^2}}$.

Q13. $y = (x^2+1)^{1/2}$. Chain rule: $\frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \times 2x = \frac{x}{\sqrt{x^2+1}}$.

Q14. $y = (4-x)^{1/2}$. Chain rule: $\frac{dy}{dx} = \frac{1}{2}(4-x)^{-1/2} \times (-1) = -\frac{1}{2\sqrt{4-x}}$.

Q15. $y = 5x^{4/5}$. $\frac{dy}{dx} = 5 \cdot \frac{4}{5}x^{-1/5} = 4x^{-1/5} = \frac{4}{\sqrt[5]{x}}$.

Q16. $y = \sqrt{2x} = (2x)^{1/2}$. $\frac{dy}{dx} = \frac{1}{2}(2x)^{-1/2} \times 2 = (2x)^{-1/2} = \frac{1}{\sqrt{2x}}$. At $x=8$: $\frac{1}{\sqrt{16}} = \frac{1}{4}$.

Q17. $y = 3x^{1/2} - 2x^{-1/2}$. $\frac{dy}{dx} = \frac{3}{2}x^{-1/2} + x^{-3/2} = \frac{3}{2\sqrt{x}} + \frac{1}{x\sqrt{x}}$.

Q18. $y = x(x+1)^{1/2}$. Product rule with a chain-rule factor:

$$\frac{dy}{dx} = 1 \cdot (x+1)^{1/2} + x \cdot \frac{1}{2}(x+1)^{-1/2} = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} = \frac{2(x+1) + x}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}.$$

Q19. $y = \sqrt{x}$, so $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. At $x=9$: point $(9, 3)$ and gradient $m = \frac{1}{2\sqrt{9}} = \frac{1}{6}$. Tangent:

$$y - 3 = \frac{1}{6}(x - 9) \Rightarrow y = \frac{1}{6}x + \frac{3}{2}.$$

Q20. $y = x^{2/3}$, so $\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$. At $x=8$: point $(8, 8^{2/3}) = (8, 4)$ and gradient $m = \frac{2}{3\sqrt[3]{8}} = \frac{2}{3 \times 2} = \frac{1}{3}$. Tangent:

$$y - 4 = \frac{1}{3}(x - 8) \Rightarrow y = \frac{1}{3}x + \frac{4}{3}.$$

Q21. $y = x^{3/2}$, so $\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$. Set $\frac{3}{2}\sqrt{x} = 3 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$.

Theory

The natural exponential function $y = e^x$ has the remarkable property that it is **its own derivative**:

$$\frac{d}{dx}(e^x) = e^x.$$

Constant multiple in the index. For a constant k ,

$$\frac{d}{dx}(e^{kx}) = k e^{kx}.$$

The chain rule for exponentials. If the index is a function $f(x)$, then

$$\frac{d}{dx}(e^{f(x)}) = f'(x) e^{f(x)}.$$

That is, differentiate the index and multiply by the original exponential. For example $\frac{d}{dx}(e^{x^2}) = 2x e^{x^2}$.

Constant multiples and sums. The constant-multiple, sum and difference rules apply as usual:

$$\frac{d}{dx}(A e^{kx}) = A k e^{kx}, \quad \frac{d}{dx}(e^{2x} + e^{-2x}) = 2e^{2x} - 2e^{-2x}.$$

Gradients and tangents. Because $e^{f(x)} > 0$ for all x , the sign of the derivative $f'(x) e^{f(x)}$ is the sign of $f'(x)$. The gradient at $x = a$ is the value of the derivative there, and the tangent at $(a, f(a))$ has equation $y = f'(a)(x - a) + f(a)$.

Worked examples

Example 1 — scaffolded

Differentiate $y = e^{3x}$.

Working

Comment

$y = e^{3x}$, of the form e^{kx} with $k = 3$.

Identify the constant multiplying x in the index.

$$\frac{dy}{dx} = 3e^{3x}.$$

Multiply the exponential by $k = 3$.

Example 2 — scaffolded

Differentiate $y = 2e^{x^2-1}$.

Working

Comment

The index is $f(x) = x^2 - 1$, so $f'(x) = 2x$.

Differentiate the index for the chain rule.

$$\frac{dy}{dx} = 2 \cdot f'(x) e^{f(x)} = 2(2x) e^{x^2-1}.$$

Apply $\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$, keeping the factor of 2.

$$\therefore \frac{dy}{dx} = 4x e^{x^2-1}.$$

Simplify.

Example 3 — unscaffolded

For $y = e^{2x} + 3e^{-x}$, find $\frac{dy}{dx}$ and the gradient at $x = 0$.

$$\frac{dy}{dx} = 2e^{2x} + 3(-1)e^{-x} = 2e^{2x} - 3e^{-x}.$$

$$\text{At } x=0: \frac{dy}{dx} = 2e^0 - 3e^0 = 2 - 3 = -1.$$

$\therefore$ the gradient at $x=0$ is -1 .

Example 4 — unscaffolded

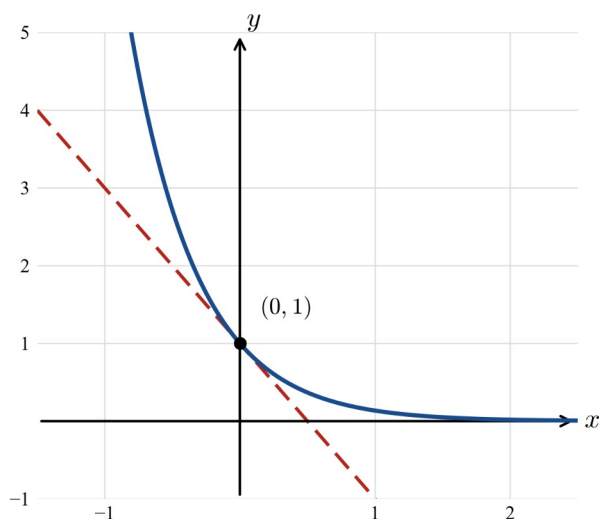
Find the equation of the tangent to $y = e^{-2x}$ at the point where $x = 0$.

At $x=0$: $y = e^0 = 1$, so the point is $(0, 1)$.

$$\frac{dy}{dx} = -2e^{-2x}, \text{ so at } x=0 \text{ the gradient is } -2e^0 = -2.$$

Tangent: $y - 1 = -2(x - 0)$, so $y = 1 - 2x$.

$\therefore$ the tangent is $y = 1 - 2x$.



Practice questions

Scaffolded

Q1. Differentiate $y = e^{4x}$.

Q2. Differentiate $y = e^x + e^{-x}$.

Q3. For $y = 5e^{2x}$: (a) find $\frac{dy}{dx}$; (b) hence find the gradient at $x = 0$.

Q4. Differentiate $y = e^{x^2+1}$ using the chain rule.

Q5. Differentiate $y = 3e^{-2x} - e^x$.

Q6. For $y = e^x$: (a) state $\frac{dy}{dx}$; (b) find the gradient at $x = 0$; (c) find the equation of the tangent at the point $(0, 1)$.

Un scaffolded

Differentiate each of the following with respect to x .

Q7. $y = e^{5x}$

Q8. $y = e^{-3x}$

Q9. $y = e^{x/2}$

Q10. $y = 4e^{2x}$

Q11. $y = e^{x^2}$

Q12. $y = e^{3x^2-1}$

Q13. $y = e^{2x} + e^{-2x}$

Q14. $y = e^{x^3}$

Q15. $y = 2e^{-x^2}$

Q16. For $y = e^{x^2-4x}$, find $\frac{dy}{dx}$ and hence the value of x for which the gradient is zero.

Q17. Find the gradient of $y = e^{3x}$ at the point where $x = 0$.

Q18. Find the equation of the tangent to $y = e^{2x}$ at the point where $x = 0$.

Q19. Find the equation of the tangent to $y = e^{-x}$ at the point where $x = 0$.

Q20. Show that $\frac{d}{dx}(e^{2x} + e^{-2x}) = 2(e^{2x} - e^{-2x})$.

Q21. For $y = e^{x^2-2x}$, show that the gradient is zero at $x = 1$, and find the coordinates of the corresponding point.

Answers

Q1. $4e^{4x}$. **Q2.** $e^x - e^{-x}$. **Q3.** (a) $10e^{2x}$; (b) 10. **Q4.** $2xe^{x^2+1}$. **Q5.** $-6e^{-2x} - e^x$. **Q6.** (a) e^x ; (b) 1; (c) $y = x + 1$. **Q7.** $5e^{5x}$. **Q8.** $-3e^{-3x}$. **Q9.** $\frac{1}{2}e^{x/2}$. **Q10.** $8e^{2x}$. **Q11.** $2xe^{x^2}$. **Q12.** $6xe^{3x^2-1}$. **Q13.** $2e^{2x} - 2e^{-2x}$. **Q14.** $3x^2e^{x^3}$. **Q15.** $-4xe^{-x^2}$. **Q16.** $\frac{dy}{dx} = (2x - 4)e^{x^2-4x}$; gradient zero at $x = 2$. **Q17.** 3. **Q18.** $y = 2x + 1$. **Q19.** $y = 1 - x$. **Q20.** (proof — see solutions). **Q21.** Gradient zero at $x = 1$; point $(1, e^{-1})$.

Fully-worked solutions

Q1. $\frac{dy}{dx} = 4e^{4x}$ (multiply by $k = 4$).

Q2. $\frac{dy}{dx} = e^x + (-1)e^{-x} = e^x - e^{-x}$.

Q3. (a) $\frac{dy}{dx} = 5 \cdot 2e^{2x} = 10e^{2x}$. (b) At $x = 0$: $10e^0 = 10$.

Q4. Index $f(x) = x^2 + 1$, $f'(x) = 2x$, so $\frac{dy}{dx} = 2xe^{x^2+1}$.

Q5. $\frac{dy}{dx} = 3(-2)e^{-2x} - e^x = -6e^{-2x} - e^x$.

Q6. (a) $\frac{dy}{dx} = e^x$. (b) At $x = 0$: $e^0 = 1$. (c) Point $(0, 1)$, gradient 1: $y - 1 = 1(x - 0)$, so $y = x + 1$.

Q7. $5e^{5x}$. **Q8.** $-3e^{-3x}$. **Q9.** $\frac{1}{2}e^{x/2}$. **Q10.** $4 \cdot 2e^{2x} = 8e^{2x}$.

Q11. Index x^2 , derivative $2x$: $\frac{dy}{dx} = 2xe^{x^2}$.

Q12. Index $3x^2 - 1$, derivative $6x$: $\frac{dy}{dx} = 6xe^{3x^2-1}$.

Q13. $\frac{dy}{dx} = 2e^{2x} + (-2)e^{-2x} = 2e^{2x} - 2e^{-2x}$.

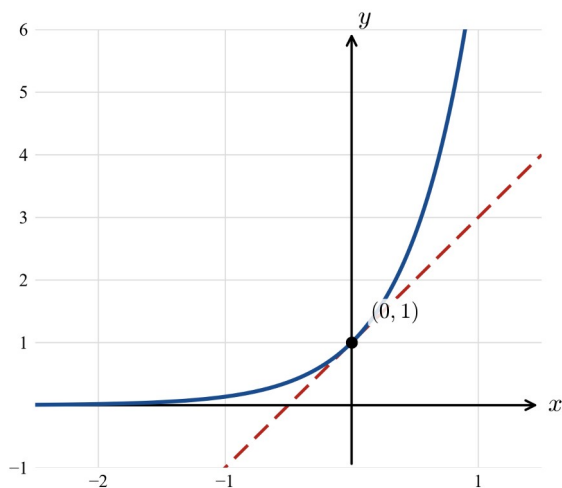
Q14. Index x^3 , derivative $3x^2$: $\frac{dy}{dx} = 3x^2e^{x^3}$.

Q15. $\frac{dy}{dx} = 2(-2x)e^{-x^2} = -4xe^{-x^2}$.

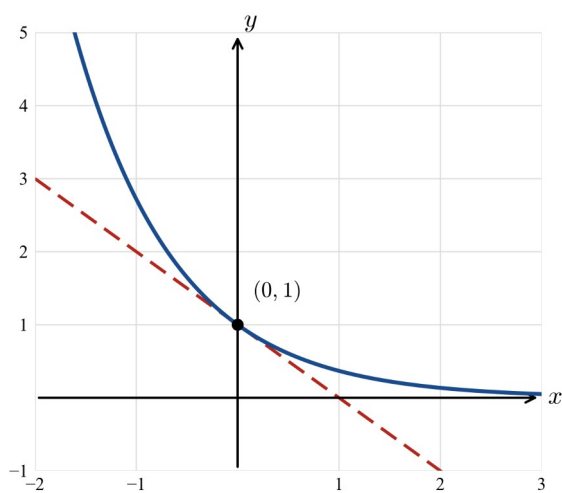
Q16. Index $x^2 - 4x$, derivative $2x - 4$: $\frac{dy}{dx} = (2x - 4)e^{x^2-4x}$. Since $e^{x^2-4x} > 0$ always, the gradient is zero when $2x - 4 = 0$, i.e. $x = 2$.

Q17. $\frac{dy}{dx} = 3e^{3x}$; at $x = 0$, gradient $= 3e^0 = 3$.

Q18. Point $(0, 1)$; $\frac{dy}{dx} = 2e^{2x}$, gradient $2e^0 = 2$; tangent $y - 1 = 2(x - 0)$, so $y = 2x + 1$.

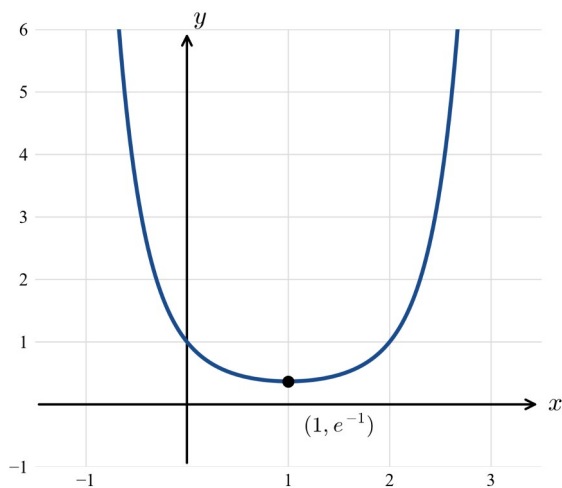


Q19. Point $(0, 1)$; $\frac{dy}{dx} = -e^{-x}$, gradient $-e^0 = -1$; tangent $y - 1 = -1(x - 0)$, so $y = 1 - x$.



Q20. $\frac{d}{dx}(e^{2x} + e^{-2x}) = 2e^{2x} + (-2)e^{-2x} = 2e^{2x} - 2e^{-2x} = 2(e^{2x} - e^{-2x})$, as required.

Q21. $\frac{dy}{dx} = (2x - 2)e^{x^2 - 2x}$. Since $e^{x^2 - 2x} > 0$, the gradient is zero when $2x - 2 = 0$, i.e. $x = 1$. Then $y = e^{1-2} = e^{-1}$, giving the point $(1, e^{-1})$.



Theory

The natural logarithm function $y = \log_e x$ has derivative

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0.$$

Using the chain rule. If $y = \log_e(f(x))$ then, writing $u = f(x)$ so that $y = \log_e u$,

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{f'(x)}{f(x)}.$$

So the derivative of the log of a function is **its own derivative over itself**. In particular

$$\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{ax+b}.$$

Simplify first with the logarithm laws. Before differentiating, use $\log_e(mn) = \log_e m + \log_e n$, $\log_e(m/n) = \log_e m - \log_e n$ and $\log_e(m^n) = n \log_e m$ to break a complicated logarithm into a sum of simpler ones — each is then easy to differentiate.

Domain. $\log_e(f(x))$ is only defined where $f(x) > 0$, so always state the domain from the condition $f(x) > 0$.

To differentiate $y = \log_e(f(x))$: either apply $\frac{f'(x)}{f(x)}$ directly, or first expand with the log laws and differentiate term by term.

Worked examples

Example 1 — scaffolded

Differentiate $y = \log_e(3x^2 + 1)$.

Working

Comment

Let $f(x) = 3x^2 + 1$, so $y = \log_e(f(x))$.

Identify the inside function.

$f'(x) = 6x$.

Differentiate the inside.

$\frac{dy}{dx} = \frac{f'(x)}{f(x)} = \frac{6x}{3x^2 + 1}$.

Apply $\frac{f'(x)}{f(x)}$.

Domain: $3x^2 + 1 > 0$ for all x , so the domain is $\mathbb{R}$.

The inside is always positive here.

$\therefore \frac{dy}{dx} = \frac{6x}{3x^2 + 1}$.

Example 2 — scaffolded

Differentiate $y = \log_e\left(\frac{2x+1}{x-3}\right)$.

Working

Comment

$y = \log_e(2x+1) - \log_e(x-3)$.

Use the quotient law **first** — it avoids the quotient rule.

$$\frac{dy}{dx} = \frac{2}{2x+1} - \frac{1}{x-3}.$$

Differentiate each logarithm using $\frac{f'}{f}$.

$$\text{Domain: } \frac{2x+1}{x-3} > 0, \text{ i.e. } x < -\frac{1}{2} \text{ or } x > 3.$$

The original quotient must be positive.

$$\therefore \frac{dy}{dx} = \frac{2}{2x+1} - \frac{1}{x-3}.$$

Example 3 — unscaffolded

For $y = \log_e(x^2 - 4)$, find $\frac{dy}{dx}$ and the gradient at $x = 3$.

$$\frac{dy}{dx} = \frac{2x}{x^2 - 4}, \text{ with domain } x^2 - 4 > 0, \text{ i.e. } x < -2 \text{ or } x > 2.$$

$$\text{At } x = 3: \text{ gradient} = \frac{2(3)}{3^2 - 4} = \frac{6}{5}.$$

$$\therefore \frac{dy}{dx} = \frac{2x}{x^2 - 4} \text{ and the gradient at } x = 3 \text{ is } \frac{6}{5}.$$

Example 4 — unscaffolded

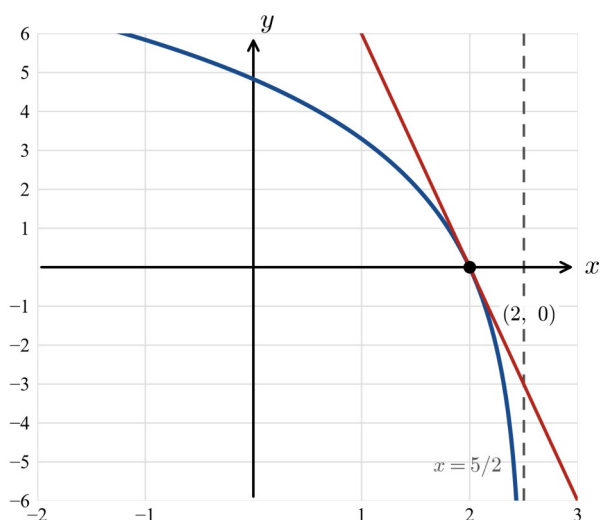
For $y = 3 \log_e(5 - 2x)$, find the equation of the tangent at $x = 2$.

$$\frac{dy}{dx} = 3 \cdot \frac{-2}{5 - 2x} = \frac{-6}{5 - 2x}, \text{ with domain } 5 - 2x > 0, \text{ i.e. } x < \frac{5}{2}.$$

$$\text{At } x = 2: y = 3 \log_e(5 - 4) = 3 \log_e 1 = 0, \text{ so the point is } (2, 0); \text{ gradient} = \frac{-6}{5 - 4} = -6.$$

$$\text{Tangent: } y - 0 = -6(x - 2), \text{ so } y = -6x + 12.$$

$$\therefore \text{ the tangent at } x = 2 \text{ is } y = -6x + 12.$$



Practice questions

Scaffolded

Q1. For $y = \log_e(4x - 1)$: (a) state the domain; (b) find $\frac{dy}{dx}$; (c) find the gradient at $x = \frac{1}{2}$.

Q2. For $y = \log_e(x^2 + 2x + 2)$: (a) show that $x^2 + 2x + 2 > 0$ for all x (hence the domain is $\mathbb{R}$); (b) find $\frac{dy}{dx}$.

Q3. For $y = \log_e(x^3)$: (a) use a logarithm law to write y in the form $k \log_e x$; (b) hence find $\frac{dy}{dx}$; (c) state the domain.

Q4. For $y = \log_e((x+1)^2)$: (a) state the domain; (b) using $y = 2 \log_e(x+1)$ on $x > -1$, find $\frac{dy}{dx}$.

Q5. For $y = \log_e(\sqrt{2x+3})$: (a) write y as $\frac{1}{2} \log_e(2x+3)$; (b) hence find $\frac{dy}{dx}$; (c) state the domain.

Q6. For $y = 5 \log_e(1-x)$: (a) find $\frac{dy}{dx}$; (b) state the domain; (c) find the equation of the tangent at $x = 0$.

Un scaffolded

Differentiate each of the following, and state the maximal domain.

Q7. $y = \log_e(6x)$

Q8. $y = \log_e(2x+5)$

Q9. $y = \log_e(x^2+1)$

Q10. $y = \log_e(3-x)$

Q11. $y = \log_e(x^2-9)$

Q12. $y = 4 \log_e(2x)$

Q13. $y = \log_e((x-1)(x+2))$

Q14. $y = \log_e\left(\frac{x}{x+4}\right)$

Q15. $y = \log_e(x^3-2x)$

Q16. $y = 2 \log_e(5x-4)$

Q17. $y = \log_e((2x+1)^3)$

Q18. $y = \log_e(\sqrt{x^2+4})$

Q19. Find the gradient of $y = \log_e(x^2+3)$ at $x = 1$.

Q20. Find the equation of the tangent to $y = \log_e(2x-3)$ at $x = 2$.

Q21. Find the coordinates of the point on $y = \log_e(x^2-4x+5)$ where the gradient is zero.

Answers

Q1. (a) $x > \frac{1}{4}$; (b) $\frac{4}{4x-1}$; (c) 4. **Q2.** (b) $\frac{2(x+1)}{x^2+2x+2}$. **Q3.** (a) $3 \log_e x$; (b) $\frac{3}{x}$; (c) $x > 0$. **Q4.** (a) $x \neq -1$; (b) $\frac{2}{x+1}$. **Q5.** (b) $\frac{1}{2x+3}$; (c) $x > -\frac{3}{2}$. **Q6.** (a) $\frac{-5}{1-x}$ (i.e. $\frac{5}{x-1}$); (b) $x < 1$; (c) $y = -5x$. **Q7.** $\frac{1}{x}$, $x > 0$. **Q8.** $\frac{2}{2x+5}$, $x > -\frac{5}{2}$. **Q9.** $\frac{2x}{x^2+1}$, $\mathbb{R}$. **Q10.** $\frac{-1}{3-x}$ (i.e. $\frac{1}{x-3}$), $x < 3$. **Q11.** $\frac{2x}{x^2-9}$, $x < -3$ or $x > 3$. **Q12.** $\frac{4}{x}$, $x > 0$. **Q13.** $\frac{1}{x-1} + \frac{1}{x+2}$, $x < -2$ or $x > 1$. **Q14.** $\frac{1}{x} - \frac{1}{x+4}$, $x < -4$ or $x > 0$. **Q15.** $\frac{3x^2-2}{x^3-2x}$, $-\sqrt{2} < x < 0$ or $x > \sqrt{2}$. **Q16.** $\frac{10}{5x-4}$, $x > \frac{4}{5}$. **Q17.** $\frac{6}{2x+1}$, $x > -\frac{1}{2}$. **Q18.** $\frac{x}{x^2+4}$, $\mathbb{R}$. **Q19.** $\frac{1}{2}$. **Q20.** $y = 2x - 4$. **Q21.** $(2, 0)$.

Fully-worked solutions

Q1. $y = \log_e(4x-1)$. (a) Need $4x-1 > 0$, so $x > \frac{1}{4}$. (b) $\frac{dy}{dx} = \frac{4}{4x-1}$. (c) At $x = \frac{1}{2}$: $\frac{4}{4(\frac{1}{2})-1} = \frac{4}{1} = 4$.

Q2. $y = \log_e(x^2+2x+2)$. (a) $x^2+2x+2 = (x+1)^2+1 \geq 1 > 0$ for all x , so the domain is $\mathbb{R}$. (b) $\frac{dy}{dx} = \frac{2x+2}{x^2+2x+2} = \frac{2(x+1)}{x^2+2x+2}$.

Q3. $y = \log_e(x^3)$. (a) For $x > 0$, $\log_e(x^3) = 3 \log_e x$. (b) $\frac{dy}{dx} = \frac{3}{x}$. (c) Need $x^3 > 0$, so $x > 0$.

Q4. $y = \log_e((x+1)^2)$. (a) $(x+1)^2 > 0$ requires $x \neq -1$. (b) On $x > -1$, $y = 2 \log_e(x+1)$, so $\frac{dy}{dx} = \frac{2}{x+1}$ (this expression holds for all $x \neq -1$).

Q5. $y = \log_e(\sqrt{2x+3}) = \frac{1}{2} \log_e(2x+3)$. $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2}{2x+3} = \frac{1}{2x+3}$. Domain: $2x+3 > 0$, so $x > -\frac{3}{2}$.

Q6. $y = 5 \log_e(1-x)$. $\frac{dy}{dx} = 5 \cdot \frac{-1}{1-x} = \frac{-5}{1-x}$. Domain: $1-x > 0$, so $x < 1$. At $x = 0$: $y = 5 \log_e 1 = 0$ and gradient $= \frac{-5}{1} = -5$, so the tangent is $y = -5x$.

Q7. $y = \log_e(6x) = \log_e 6 + \log_e x$, so $\frac{dy}{dx} = \frac{1}{x}$; domain $x > 0$.

Q8. $\frac{dy}{dx} = \frac{2}{2x+5}$; domain $2x+5 > 0$, i.e. $x > -\frac{5}{2}$.

Q9. $\frac{dy}{dx} = \frac{2x}{x^2+1}$; $x^2+1 > 0$ always, domain $\mathbb{R}$.

Q10. $\frac{dy}{dx} = \frac{-1}{3-x} = \frac{1}{x-3}$; domain $3-x > 0$, i.e. $x < 3$.

Q11. $\frac{dy}{dx} = \frac{2x}{x^2-9}$; $x^2-9 > 0$, i.e. $x < -3$ or $x > 3$.

Q12. $y = 4 \log_e(2x) = 4 \log_e 2 + 4 \log_e x$, so $\frac{dy}{dx} = \frac{4}{x}$; domain $x > 0$.

Q13. $y = \log_e((x-1)(x+2)) = \log_e(x-1) + \log_e(x+2)$, so $\frac{dy}{dx} = \frac{1}{x-1} + \frac{1}{x+2}$. Domain: $(x-1)(x+2) > 0$, i.e. $x < -2$ or $x > 1$.

Q14. $y = \log_e\left(\frac{x}{x+4}\right) = \log_e x - \log_e(x+4)$, so $\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+4}$. Domain: $\frac{x}{x+4} > 0$, i.e. $x < -4$ or $x > 0$.

Q15. $\frac{dy}{dx} = \frac{3x^2 - 2}{x^3 - 2x}$. Domain: $x^3 - 2x = x(x^2 - 2) > 0$, i.e. $-\sqrt{2} < x < 0$ or $x > \sqrt{2}$.

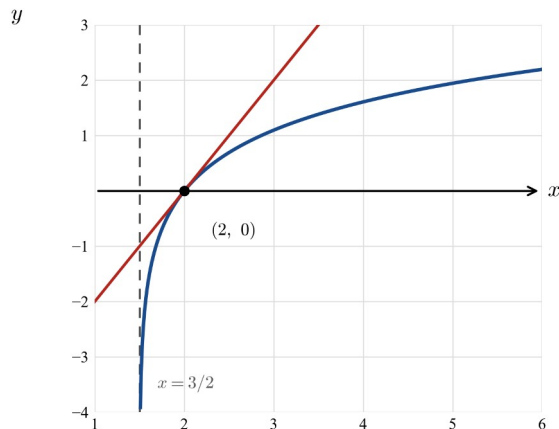
Q16. $\frac{dy}{dx} = 2 \cdot \frac{5}{5x-4} = \frac{10}{5x-4}$; domain $5x-4 > 0$, i.e. $x > \frac{4}{5}$.

Q17. $y = \log_e((2x+1)^3) = 3 \log_e(2x+1)$, so $\frac{dy}{dx} = 3 \cdot \frac{2}{2x+1} = \frac{6}{2x+1}$. Domain: $2x+1 > 0$, i.e. $x > -\frac{1}{2}$.

Q18. $y = \log_e(\sqrt{x^2+4}) = \frac{1}{2} \log_e(x^2+4)$, so $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{x^2+4} = \frac{x}{x^2+4}$; domain $\mathbb{R}$.

Q19. $\frac{dy}{dx} = \frac{2x}{x^2+3}$. At $x=1$: $\frac{2}{1+3} = \frac{1}{2}$.

Q20. $\frac{dy}{dx} = \frac{2}{2x-3}$. At $x=2$: $y = \log_e(4-3) = \log_e 1 = 0$ and gradient $= \frac{2}{1} = 2$. Tangent: $y - 0 = 2(x - 2)$, i.e. $y = 2x - 4$.



Q21. $\frac{dy}{dx} = \frac{2x-4}{x^2-4x+5}$. Since $x^2 - 4x + 5 = (x-2)^2 + 1 > 0$ for all x , the domain is $\mathbb{R}$ and the gradient is zero when $2x - 4 = 0$, i.e. $x = 2$. Then $y = \log_e((2)^2 - 8 + 5) = \log_e 1 = 0$, so the point is $(2, 0)$.

Chapter 8 Differentiation techniques — 8I Derivatives of circular functions

Theory

For x measured in **radians**, the circular functions have the derivatives

$$\frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}.$$

(These rules hold only in radian measure — never differentiate a circular function of an angle in degrees.)

The chain rule. Combining these with the chain rule gives, for a differentiable function f ,

$$\frac{d}{dx}(\sin f(x)) = f'(x) \cos f(x), \frac{d}{dx}(\cos f(x)) = -f'(x) \sin f(x), \frac{d}{dx}(\tan f(x)) = \frac{f'(x)}{\cos^2 f(x)}.$$

In particular, for constants a , n and b ,

$$\frac{d}{dx}(a \sin(nx+b)) = an \cos(nx+b), \frac{d}{dx}(a \cos(nx+b)) = -an \sin(nx+b).$$

Gradients and tangents. The gradient of $y = f(x)$ at $x = a$ is $f'(a)$; the tangent there passes through $(a, f(a))$ with this gradient. Evaluate at exact-value angles ($\pi/6, \pi/4, \pi/3, \dots$) so the answer stays exact.

Worked examples

Example 1 — scaffolded

Differentiate $y = \sin(3x)$.

Working

$y = \sin(3x)$, a function of the form $\sin f(x)$ with $f(x) = 3x$.

$$f'(x) = 3.$$

$$\frac{dy}{dx} = f'(x) \cos f(x) = 3 \cos(3x).$$

Comment

Identify the inner function.

Differentiate the inside.

Apply $\frac{d}{dx} \sin f(x) = f'(x) \cos f(x)$.

Example 2 — scaffolded

Differentiate $y = 4 \sin\left(\frac{\pi x}{2}\right)$.

Working

Here $a = 4$, and the inside is $\frac{\pi}{2}x$, so $n = \frac{\pi}{2}$.

$$\frac{dy}{dx} = 4 \cdot \frac{\pi}{2} \cos\left(\frac{\pi x}{2}\right)$$

$$= 2\pi \cos\left(\frac{\pi x}{2}\right).$$

Comment

Compare with $a \sin(nx)$.

Use $\frac{d}{dx} a \sin(nx) = an \cos(nx)$.

Simplify the constant.

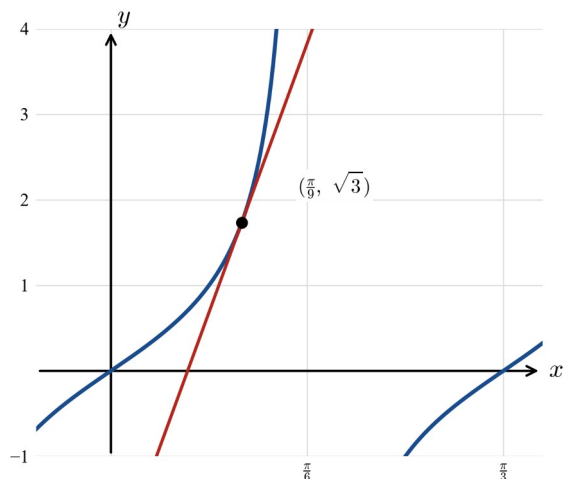
Example 3 — unscaffolded

For $y = \tan(3x)$, find $\frac{dy}{dx}$ and hence the gradient of the curve at $x = \frac{\pi}{9}$.

$$\frac{dy}{dx} = \frac{3}{\cos^2(3x)}.$$

At $x = \frac{\pi}{9}$: $3x = \frac{\pi}{3}$, and $\cos \frac{\pi}{3} = \frac{1}{2}$, so $\cos^2 \frac{\pi}{3} = \frac{1}{4}$.

$$\therefore \frac{dy}{dx} = \frac{3}{1/4} = 12.$$



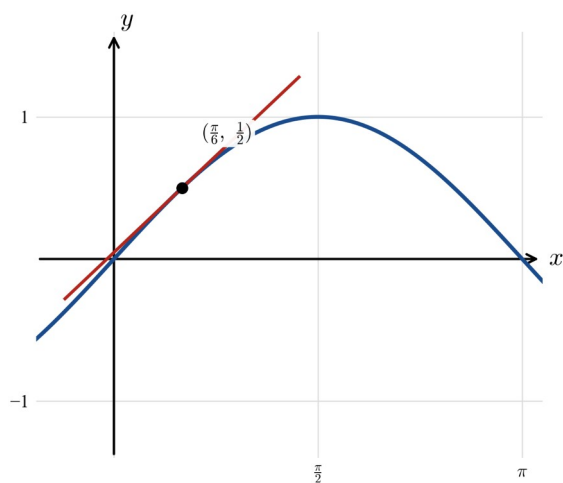
Example 4 — unscaffolded

Find the equation of the tangent to $y = \sin x$ at the point where $x = \frac{\pi}{6}$.

$$\frac{dy}{dx} = \cos x, \text{ so the gradient at } x = \frac{\pi}{6} \text{ is } \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

The point of contact is $(\frac{\pi}{6}, \sin \frac{\pi}{6}) = (\frac{\pi}{6}, \frac{1}{2})$.

$$\therefore y - \frac{1}{2} = \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6} \right), \text{ i.e. } y = \frac{\sqrt{3}}{2} x + \frac{1}{2} - \frac{\sqrt{3}\pi}{12}.$$



Practice questions

Scaffolded

Q1. For $y = \cos(4x)$: (a) identify the inner function; (b) differentiate the inside; (c) write $\frac{dy}{dx}$.

Q2. Differentiate $y = \sin\left(\frac{x}{2}\right)$, showing the chain-rule step.

Q3. For $y = 3 \sin(2x)$: (a) find $\frac{dy}{dx}$; (b) hence find the gradient at $x = 0$.

Q4. Differentiate $y = \tan(2x)$, stating the rule you use.

Q5. Differentiate $y = 2 \cos(3x) - \sin x$.

Q6. Differentiate $y = \sin\left(2x + \frac{\pi}{4}\right)$.

Un scaffolded

Differentiate each of the following with respect to x (Q18–Q21 have an extra part).

Q7. $y = \sin(5x)$

Q8. $y = \cos\left(\frac{x}{3}\right)$

Q9. $y = \tan\left(\frac{x}{2}\right)$

Q10. $y = 4 \sin\left(2x - \frac{\pi}{3}\right)$

Q11. $y = -3 \cos(2x)$

Q12. $y = \sin(x^2)$

Q13. $y = \cos\left(3x + \frac{\pi}{6}\right)$

Q14. $y = 2 \tan(4x)$

Q15. $y = \sin(2x) + \cos(3x)$

Q16. $y = 5 \cos(\pi x)$

Q17. $y = \tan\left(2x - \frac{\pi}{4}\right)$

Q18. For $y = 3 \sin x - 2 \cos x$, find $\frac{dy}{dx}$ and hence the gradient at $x = \frac{\pi}{3}$.

Q19. Find the equation of the tangent to $y = \cos(2x)$ at $x = \frac{\pi}{4}$.

Q20. $y = 2 \sin\left(\frac{x}{2} + \frac{\pi}{6}\right)$

Q21. Show that $\frac{d}{dx} \sin(2x) = 2 \cos(2x)$, and hence find the values of x in $[0, \pi]$ at which the gradient of $y = \sin(2x)$ is zero.

Answers

Q1. $-4 \sin(4x)$ **Q2.** $\frac{1}{2} \cos\left(\frac{x}{2}\right)$ **Q3.** (a) $6 \cos(2x)$; (b) 6 **Q4.** $\frac{2}{\cos^2(2x)}$ **Q5.** $-6 \sin(3x) - \cos x$ **Q6.**
 $2 \cos\left(2x + \frac{\pi}{4}\right)$ **Q7.** $5 \cos(5x)$ **Q8.** $-\frac{1}{3} \sin\left(\frac{x}{3}\right)$ **Q9.** $\frac{1}{2 \cos^2(x/2)}$ **Q10.** $8 \cos\left(2x - \frac{\pi}{3}\right)$ **Q11.** $6 \sin(2x)$ **Q12.**
 $2x \cos(x^2)$ **Q13.** $-3 \sin\left(3x + \frac{\pi}{6}\right)$ **Q14.** $\frac{8}{\cos^2(4x)}$ **Q15.** $2 \cos(2x) - 3 \sin(3x)$ **Q16.** $-5\pi \sin(\pi x)$ **Q17.**
 $\frac{2}{\cos^2(2x - \pi/4)}$ **Q18.** $3 \cos x + 2 \sin x$; at $x = \frac{\pi}{3}$, gradient $\frac{3}{2} + \sqrt{3}$ **Q19.** $y = -2\left(x - \frac{\pi}{4}\right)$ **Q20.** $\cos\left(\frac{x}{2} + \frac{\pi}{6}\right)$ **Q21.**
 $x = \frac{\pi}{4}, \frac{3\pi}{4}$

Fully-worked solutions

Q1. Inner $f(x) = 4x$, $f'(x) = 4$; $\frac{d}{dx} \cos f(x) = -f'(x) \sin f(x) = -4 \sin(4x)$.

Q2. Inner $\frac{x}{2}$, derivative $\frac{1}{2}$; $\frac{dy}{dx} = \frac{1}{2} \cos\left(\frac{x}{2}\right)$.

Q3. (a) $\frac{dy}{dx} = 3 \cdot 2 \cos(2x) = 6 \cos(2x)$. (b) At $x=0$, $6 \cos 0 = 6$.

Q4. $\frac{d}{dx} \tan(2x) = \frac{2}{\cos^2(2x)}$ (inner $2x$, derivative 2).

Q5. $\frac{dy}{dx} = 2 \cdot (-3 \sin(3x)) - \cos x = -6 \sin(3x) - \cos x$.

Q6. $\frac{dy}{dx} = 2 \cos\left(2x + \frac{\pi}{4}\right)$ (inner $2x + \pi/4$, derivative 2).

Q7. $5 \cos(5x)$.

Q8. $-\frac{1}{3} \sin\left(\frac{x}{3}\right)$.

Q9. $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\cos^2(x/2)} = \frac{1}{2 \cos^2(x/2)}$.

Q10. $\frac{dy}{dx} = 4 \cdot 2 \cos\left(2x - \frac{\pi}{3}\right) = 8 \cos\left(2x - \frac{\pi}{3}\right)$.

Q11. $\frac{dy}{dx} = -3 \cdot (-2 \sin(2x)) = 6 \sin(2x)$.

Q12. Inner x^2 , derivative $2x$; $\frac{dy}{dx} = 2x \cos(x^2)$.

Q13. $\frac{dy}{dx} = -3 \sin\left(3x + \frac{\pi}{6}\right)$.

$$\text{Q14. } \frac{dy}{dx} = 2 \cdot \frac{4}{\cos^2(4x)} = \frac{8}{\cos^2(4x)}.$$

$$\text{Q15. } \frac{dy}{dx} = 2 \cos(2x) - 3 \sin(3x).$$

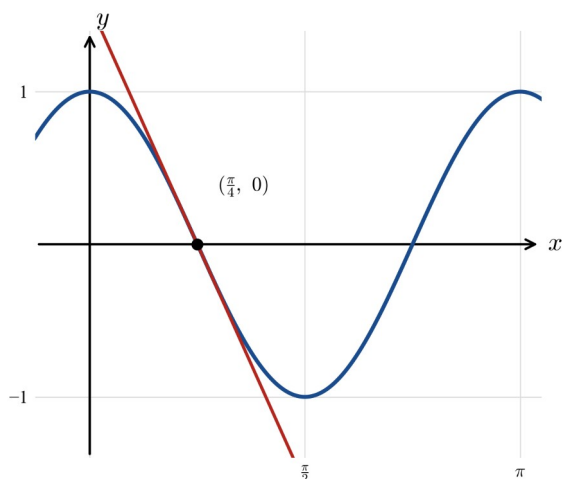
$$\text{Q16. } \frac{dy}{dx} = -5\pi \sin(\pi x) \text{ (inner } \pi x, \text{ derivative } \pi).$$

$$\text{Q17. } \frac{dy}{dx} = \frac{2}{\cos^2(2x - \pi/4)}.$$

$$\text{Q18. } \frac{dy}{dx} = 3 \cos x + 2 \sin x. \text{ At } x = \frac{\pi}{3}: 3 \cos \frac{\pi}{3} + 2 \sin \frac{\pi}{3} = 3 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{3}}{2} = \frac{3}{2} + \sqrt{3}.$$

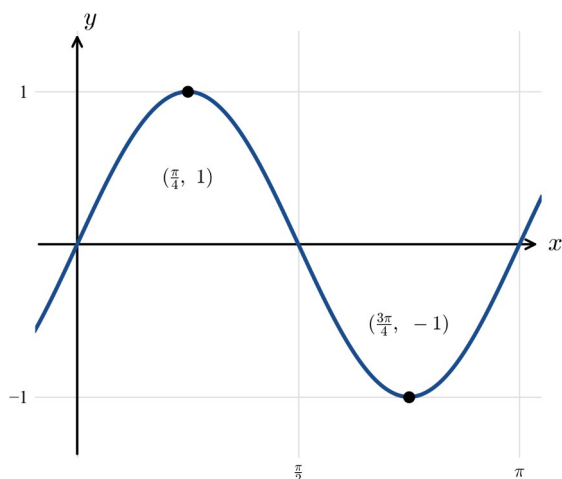
$$\text{Q19. } \frac{dy}{dx} = -2 \sin(2x); \text{ at } x = \frac{\pi}{4}, \text{ gradient} = -2 \sin \frac{\pi}{2} = -2. \text{ Point of contact } \left(\frac{\pi}{4}, \cos \frac{\pi}{2} \right) = \left(\frac{\pi}{4}, 0 \right).$$

$$\therefore y - 0 = -2 \left(x - \frac{\pi}{4} \right), \text{ i.e. } y = -2x + \frac{\pi}{2}.$$



$$\text{Q20. } \frac{dy}{dx} = 2 \cdot \frac{1}{2} \cos\left(\frac{x}{2} + \frac{\pi}{6}\right) = \cos\left(\frac{x}{2} + \frac{\pi}{6}\right).$$

Q21. Inner $2x$, derivative 2: $\frac{d}{dx} \sin(2x) = 2 \cos(2x)$. Gradient zero when $2 \cos(2x) = 0 \Rightarrow \cos(2x) = 0 \Rightarrow 2x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ (for $2x \in [0, 2\pi]$), so $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$.



Chapter 8 Differentiation techniques — 8J The product rule

Theory

Many functions are **products** of two simpler functions — for example $y = x^2 e^x$ is the product of x^2 and e^x . The derivative of a product is **not** the product of the derivatives. Instead we use the **product rule**.

The product rule. If $y = u(x)v(x)$, where u and v are differentiable, then

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}, \text{ often written } (uv)' = u'v + uv'.$$

In words: *differentiate the first factor times the second, plus the first factor times the derivative of the second.*

Method.

1. Identify the two factors u and v .
2. Differentiate each: find u' and v' (use the chain rule for a factor such as e^{2x} or $\sin 2x$).
3. Substitute into $u'v + uv'$.
4. **Simplify** — usually by taking out a common factor (a power of x , an exponential, or a surd). Examiners reward a fully factorised derivative, and it makes solving $\frac{dy}{dx} = 0$ much easier.

Combining with the chain rule. When a factor is itself a composite, differentiate it with the chain rule inside step 2. For example, for $y = (3x - 2)e^{2x}$ we have $v = e^{2x}$ so $v' = 2e^{2x}$.

A common exam error is to forget one of the two terms, or to miss a bracket when the derivative of a factor is a difference. Keep $u'v$ and uv' clearly separated until you factorise.

Worked examples

Example 1 — scaffolded

Differentiate $y = x^2 e^x$.

Working

Comment

Let $u = x^2$ and $v = e^x$.

Identify the two factors.

$u' = 2x$ and $v' = e^x$.

Differentiate each factor.

$\frac{dy}{dx} = u'v + uv' = 2xe^x + x^2e^x$.

Apply the product rule.

$= xe^x(x + 2)$.

Take out the common factor xe^x .

$\therefore \frac{dy}{dx} = x(x + 2)e^x$.

Example 2 — scaffolded

Differentiate $y = x \log_e x$.

Working

Comment

Let $u = x$ and $v = \log_e x$.

Identify the factors; $x > 0$.

$u' = 1$ and $v' = \frac{1}{x}$.

$\frac{d}{dx}(\log_e x) = \frac{1}{x}$.

Working	Comment
$\frac{dy}{dx} = 1 \cdot \log_e x + x \cdot \frac{1}{x}$	Apply the product rule.
$= \log_e x + 1.$	Simplify.
$\therefore \frac{dy}{dx} = \log_e x + 1.$	

Example 3 — unscaffolded

Differentiate $y = e^{2x} \sin x$.

Let $u = e^{2x}$, $v = \sin x$, so $u' = 2e^{2x}$ (chain rule) and $v' = \cos x$.

$$\frac{dy}{dx} = 2e^{2x} \sin x + e^{2x} \cos x = e^{2x}(2 \sin x + \cos x).$$

$$\therefore \frac{dy}{dx} = e^{2x}(2 \sin x + \cos x).$$

Example 4 — unscaffolded

Differentiate $y = x\sqrt{x+1}$.

Write $v = (x+1)^{1/2}$, so $v' = 1/2(x+1)^{-1/2} = \frac{1}{2\sqrt{x+1}}$, and $u = x$, $u' = 1$.

$$\frac{dy}{dx} = 1 \cdot \sqrt{x+1} + x \cdot \frac{1}{2\sqrt{x+1}} = \frac{2(x+1) + x}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}.$$

$$\therefore \frac{dy}{dx} = \frac{3x+2}{2\sqrt{x+1}}.$$

Practice questions

Scaffolded

Q1. For $y = x^3 e^x$: (a) state u and v ; (b) find u' and v' ; (c) apply the product rule; (d) hence write $\frac{dy}{dx}$ in fully factorised form.

Q2. For $y = (2x+1)e^{3x}$: (a) state u and v ; (b) find u' and v' (use the chain rule for v); (c) apply the product rule; (d) simplify to a single factorised expression.

Q3. For $y = x^2 \log_e x$: (a) state u and v ; (b) find u' and v' ; (c) apply the product rule; (d) take out the common factor.

Q4. For $y = x \cos x$: (a) state u and v ; (b) find u' and v' ; (c) find $\frac{dy}{dx}$; (d) evaluate the gradient at $x = 0$.

Q5. For $y = (x-1)\sqrt{x}$: (a) write $\sqrt{x}$ as a power and state u and v ; (b) find u' and v' ; (c) apply the product rule; (d) express $\frac{dy}{dx}$ as a single fraction.

Q6. For $y = x^2 e^{-x}$: (a) state u and v ; (b) find u' and v' ; (c) apply the product rule; (d) hence find the values of x where the tangent is horizontal.

Un scaffolded

Find $\frac{dy}{dx}$ for each of the following, giving your answer in a fully factorised (or simplest) form.

Q7. $y = x e^x$

Q8. $y = x^2 \sin x$

Q9. $y = (3x - 2) e^{2x}$

Q10. $y = x^3 \log_e x$

Q11. $y = e^x \cos x$

Q12. $y = x^2 \sqrt{x - 1}$

Q13. $y = (x^2 + 1) e^x$

Q14. $y = x \sin 2x$

Q15. $y = e^{-x} \sin x$

Q16. $y = (2x + 3) \log_e x$

Q17. $y = x^2 e^{3x}$

Q18. $y = \sqrt{x} e^x$

Q19. For $y = (x - 1)^2 e^x$, find $\frac{dy}{dx}$ and hence the coordinates of the points where the tangent is horizontal.

Q20. Show that if $y = x \log_e (2x)$ then $\frac{dy}{dx} = \log_e (2x) + 1$.

Q21. $y = e^{2x} \cos 3x$

Answers

Q1. $x^2(x+3)e^x$ **Q2.** $(6x+5)e^{3x}$ **Q3.** $x(2\log_e x + 1)$ **Q4.** $\cos x - x \sin x$; at $x=0$, gradient = 1 **Q5.** $\frac{3x-1}{2\sqrt{x}}$ **Q6.** $x(2-x)e^{-x}$; horizontal tangents at $x=0$ and $x=2$ **Q7.** $(x+1)e^x$ **Q8.** $2x \sin x + x^2 \cos x = x(2 \sin x + x \cos x)$ **Q9.** $(6x-1)e^{2x}$ **Q10.** $x^2(3\log_e x + 1)$ **Q11.** $e^x(\cos x - \sin x)$ **Q12.** $\frac{x(5x-4)}{2\sqrt{x-1}}$ **Q13.** $(x+1)^2 e^x$ **Q14.** $\sin 2x + 2x \cos 2x$ **Q15.** $e^{-x}(\cos x - \sin x)$ **Q16.** $2\log_e x + 2 + \frac{3}{x}$ **Q17.** $x(3x+2)e^{3x}$ **Q18.** $\frac{(2x+1)e^x}{2\sqrt{x}}$ **Q19.** $(x-1)(x+1)e^x$; horizontal tangents at $(-1, 4e^{-1})$ and $(1, 0)$ **Q20.** $\log_e(2x) + 1$ (shown) **Q21.** $e^{2x}(2\cos 3x - 3\sin 3x)$

Fully-worked solutions

Q1. $u = x^3, v = e^x; u' = 3x^2, v' = e^x. \frac{dy}{dx} = 3x^2 e^x + x^3 e^x = x^2 e^x(3+x) = x^2(x+3)e^x.$

Q2. $u = 2x+1, v = e^{3x}; u' = 2, v' = 3e^{3x}. \frac{dy}{dx} = 2e^{3x} + (2x+1) \cdot 3e^{3x} = e^{3x}(2+6x+3) = (6x+5)e^{3x}.$

Q3. $u = x^2, v = \log_e x; u' = 2x, v' = \frac{1}{x}. \frac{dy}{dx} = 2x \log_e x + x^2 \cdot \frac{1}{x} = 2x \log_e x + x = x(2 \log_e x + 1).$

Q4. $u = x, v = \cos x; u' = 1, v' = -\sin x. \frac{dy}{dx} = \cos x - x \sin x. \text{ At } x=0: \cos 0 - 0 = 1.$

Q5. $y = (x-1)x^{1/2}; u = x-1, v = x^{1/2}; u' = 1, v' = \frac{1}{2\sqrt{x}}. \frac{dy}{dx} = \sqrt{x} + \frac{x-1}{2\sqrt{x}} = \frac{2x + (x-1)}{2\sqrt{x}} = \frac{3x-1}{2\sqrt{x}}.$

Q6. $u = x^2, v = e^{-x}; u' = 2x, v' = -e^{-x}. \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = x(2-x)e^{-x}. \text{ As } e^{-x} \neq 0, \frac{dy}{dx} = 0 \text{ when } x(2-x) = 0,$
i.e. $x=0$ or $x=2$.

Q7. $u = x, v = e^x; \frac{dy}{dx} = e^x + x e^x = (x+1)e^x.$

Q8. $u = x^2, v = \sin x; \frac{dy}{dx} = 2x \sin x + x^2 \cos x = x(2 \sin x + x \cos x).$

Q9. $u = 3x-2, v = e^{2x}; v' = 2e^{2x}. \frac{dy}{dx} = 3e^{2x} + (3x-2) \cdot 2e^{2x} = e^{2x}(3+6x-4) = (6x-1)e^{2x}.$

Q10. $u = x^3, v = \log_e x; \frac{dy}{dx} = 3x^2 \log_e x + x^3 \cdot \frac{1}{x} = x^2(3 \log_e x + 1).$

Q11. $u = e^x, v = \cos x; \frac{dy}{dx} = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x).$

Q12. $u = x^2, v = (x-1)^{1/2}; v' = \frac{1}{2\sqrt{x-1}}. \frac{dy}{dx} = 2x\sqrt{x-1} + \frac{x^2}{2\sqrt{x-1}} = \frac{4x(x-1) + x^2}{2\sqrt{x-1}} = \frac{5x^2 - 4x}{2\sqrt{x-1}} = \frac{x(5x-4)}{2\sqrt{x-1}}.$

Q13. $u = x^2+1, v = e^x; \frac{dy}{dx} = 2xe^x + (x^2+1)e^x = e^x(x^2+2x+1) = (x+1)^2 e^x.$

Q14. $u = x$, $v = \sin 2x$; $v' = 2 \cos 2x$: $\frac{dy}{dx} = \sin 2x + 2x \cos 2x$.

Q15. $u = e^{-x}$, $v = \sin x$; $u' = -e^{-x}$. $\frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x = e^{-x}(\cos x - \sin x)$.

Q16. $u = 2x + 3$, $v = \log_e x$: $\frac{dy}{dx} = 2 \log_e x + (2x + 3) \cdot \frac{1}{x} = 2 \log_e x + 2 + \frac{3}{x}$.

Q17. $u = x^2$, $v = e^{3x}$; $v' = 3e^{3x}$. $\frac{dy}{dx} = 2xe^{3x} + 3x^2 e^{3x} = x(2 + 3x)e^{3x} = x(3x + 2)e^{3x}$.

Q18. $u = x^{1/2}$, $v = e^x$; $u' = \frac{1}{2\sqrt{x}}$. $\frac{dy}{dx} = \frac{e^x}{2\sqrt{x}} + \sqrt{x}e^x = \frac{e^x + 2xe^x}{2\sqrt{x}} = \frac{(2x + 1)e^x}{2\sqrt{x}}$.

Q19. $u = (x - 1)^2$, $v = e^x$; $u' = 2(x - 1)$. $\frac{dy}{dx} = 2(x - 1)e^x + (x - 1)^2 e^x = (x - 1)e^x[2 + (x - 1)] = (x - 1)(x + 1)e^x$.

$\frac{dy}{dx} = 0$ when $x = 1$ or $x = -1$. At $x = 1$: $y = (0)^2 e^1 = 0$, point $(1, 0)$. At $x = -1$: $y = (-2)^2 e^{-1} = 4e^{-1}$, point $(-1, 4e^{-1})$.

Q20. $u = x$, $v = \log_e(2x)$; $u' = 1$, and by the chain rule $v' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$. $\frac{dy}{dx} = 1 \cdot \log_e(2x) + x \cdot \frac{1}{x} = \log_e(2x) + 1$, as required.

Q21. $u = e^{2x}$, $v = \cos 3x$; $u' = 2e^{2x}$, $v' = -3 \sin 3x$. $\frac{dy}{dx} = 2e^{2x} \cos 3x - 3e^{2x} \sin 3x = e^{2x}(2 \cos 3x - 3 \sin 3x)$.

Chapter 8 Differentiation techniques — 8K The quotient rule

Theory

The **quotient rule** differentiates a function written as one expression divided by another. If $y = \frac{u}{v}$, where u and v are both functions of x (and $v \neq 0$), then

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2},$$

where $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$. In function notation, if $f(x) = \frac{g(x)}{h(x)}$ then

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}.$$

Where it comes from. Writing $\frac{u}{v} = uv^{-1}$ and applying the product and chain rules gives

$$u'v^{-1} + u(-v^{-2})v' = \frac{u'}{v} - \frac{uv'}{v^2} = \frac{u'v - uv'}{v^2}.$$

Using the rule.

1. Identify u (the numerator) and v (the denominator), and differentiate each to get u' and v' .
2. Substitute into $\frac{u'v - uv'}{v^2}$ — **the numerator has the $u'v$ term first**; the order matters because of the minus sign.
3. Expand the numerator carefully, **keeping every bracket**, then simplify. Leave the denominator as v^2 (factored) rather than expanding it.

Bracket care. The commonest error is dropping a bracket in $-uv'$, which flips a sign. Always write $u'v - uv'$ with the second product fully bracketed before expanding.

Zero and undefined gradients. $f'(x) = 0$ where the **numerator** is zero (and $v \neq 0$); $f'(x)$ is undefined where $v = 0$. This is how stationary points and vertical tangents of a quotient are found.

Sometimes a quotient can be simplified to a form that is quicker to differentiate (e.g. $\frac{x+2}{x^2-4} = \frac{1}{x-2}$, $x \neq \pm 2$) — always check first.

Worked examples

Example 1 — scaffolded

Differentiate $y = \frac{x}{x+2}$.

Working

$$u = x, v = x + 2.$$

$$u' = 1, v' = 1.$$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{(1)(x+2) - (x)(1)}{(x+2)^2}$$

Comment

Identify numerator and denominator.

Differentiate each.

Substitute — $u'v$ term first, then $-uv'$.

$$= \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

Expand the numerator and simplify; leave v^2 factored.

Example 2 — scaffolded

Differentiate $y = \frac{2x-1}{x^2+1}$.

Working

Comment

$$u = 2x - 1, v = x^2 + 1.$$

Identify u and v .

$$u' = 2, v' = 2x.$$

Differentiate each.

$$\frac{dy}{dx} = \frac{2(x^2+1) - (2x-1)(2x)}{(x^2+1)^2}$$

Substitute; bracket the whole uv' product.

$$= \frac{2x^2+2 - (4x^2-2x)}{(x^2+1)^2} = \frac{-2x^2+2x+2}{(x^2+1)^2}$$

Expand carefully — the bracket flips both signs.

$$= \frac{-2(x^2-x-1)}{(x^2+1)^2}$$

Factor the numerator.

Example 3 — unscaffolded

Differentiate $y = \frac{e^x}{x}$.

$$u = e^x, v = x, \text{ so } u' = e^x \text{ and } v' = 1.$$

$$\frac{dy}{dx} = \frac{e^x \cdot x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}.$$

$$\therefore \frac{dy}{dx} = \frac{e^x(x-1)}{x^2}.$$

Example 4 — unscaffolded

Differentiate $y = \frac{\log_e x}{x}$.

$$u = \log_e x, v = x, \text{ so } u' = \frac{1}{x} \text{ and } v' = 1.$$

$$\frac{dy}{dx} = \frac{\frac{1}{x} \cdot x - \log_e x \cdot 1}{x^2} = \frac{1 - \log_e x}{x^2}.$$

$$\therefore \frac{dy}{dx} = \frac{1 - \log_e x}{x^2}.$$

Practice questions

Scaffolded

Q1. For $y = \frac{x}{x-3}$: (a) state u , v , u' and v' ; (b) apply the quotient rule; (c) simplify $\frac{dy}{dx}$.

Q2. For $y = \frac{x+1}{x-1}$: (a) find u' and v' ; (b) hence find $\frac{dy}{dx}$ in simplest form.

Q3. For $y = \frac{3x}{x^2+2}$: (a) find $\frac{dy}{dx}$; (b) factorise the numerator.

Q4. For $y = \frac{x-1}{x^2}$: (a) find $\frac{dy}{dx}$ using the quotient rule; (b) check your answer by first writing $y = x^{-1} - x^{-2}$ and differentiating.

Q5. For $y = \frac{e^x}{x+1}$, find $\frac{dy}{dx}$ in simplest form.

Q6. For $y = \frac{2x+1}{3x-2}$: (a) find $\frac{dy}{dx}$; (b) explain why the gradient is never zero.

Un scaffolded

Find $\frac{dy}{dx}$ for each of the following, giving your answer in simplest form.

Q7. $y = \frac{x+4}{x-4}$

Q8. $y = \frac{x^2}{x+1}$

Q9. $y = \frac{x^2-1}{x^2+1}$

Q10. $y = \frac{1-x}{1+x}$

Q11. $y = \frac{e^x}{x^2}$

Q12. $y = \frac{e^{2x}}{x}$

Q13. $y = \frac{\sin x}{\cos x}$ (and state what this confirms).

Q14. $y = \frac{\cos x}{x}$

Q15. $y = \frac{x}{e^x}$

Q16. $y = \frac{\log_e x}{x^2}$

Q17. $y = \frac{2x-3}{x^2+1}$

Q18. $y = \frac{x+2}{x^2-4}$ (simplify first, then differentiate; state the domain).

Q19. $y = \frac{\sqrt{x}}{x+1}$

Q20. Find $\frac{dy}{dx}$ for $y = \frac{x^2+1}{x-1}$, and **hence find the coordinates of the points where the tangent is horizontal.**

Q21. Show that the derivative of $y = \frac{x}{x^2+4}$ **is** $\frac{4-x^2}{(x^2+4)^2}$, **and hence find the values of x where the gradient is zero.**

Answers

Q1. $\frac{-3}{(x-3)^2}$ **Q2.** $\frac{-2}{(x-1)^2}$ **Q3.** $\frac{3(2-x^2)}{(x^2+2)^2}$ **Q4.** $\frac{2-x}{x^3}$ **Q5.** $\frac{xe^x}{(x+1)^2}$ **Q6.** $\frac{-7}{(3x-2)^2}$; the numerator is the constant $-7 \neq 0$, so the gradient is never 0. **Q7.** $\frac{-8}{(x-4)^2}$ **Q8.** $\frac{x(x+2)}{(x+1)^2}$ **Q9.** $\frac{4x}{(x^2+1)^2}$ **Q10.** $\frac{-2}{(1+x)^2}$ **Q11.** $\frac{e^x(x-2)}{x^3}$ **Q12.** $\frac{e^{2x}(2x-1)}{x^2}$ **Q13.** $\frac{1}{\cos^2 x}$; this confirms $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$. **Q14.** $\frac{-(x \sin x + \cos x)}{x^2}$ **Q15.** $\frac{1-x}{e^x}$ **Q16.** $\frac{1-2 \log_e x}{x^3}$ **Q17.** $\frac{-2(x^2-3x-1)}{(x^2+1)^2}$ **Q18.** $\frac{-1}{(x-2)^2}$, $x \neq \pm 2$ **Q19.** $\frac{1-x}{2\sqrt{x}(x+1)^2}$ **Q20.** $\frac{x^2-2x-1}{(x-1)^2}$; horizontal tangents at $(1+\sqrt{2}, 2+2\sqrt{2})$ and $(1-\sqrt{2}, 2-2\sqrt{2})$. **Q21.** gradient zero at $x = -2$ and $x = 2$.

Fully-worked solutions

$$\text{Q1. } u = x, v = x - 3, u' = 1, v' = 1. \frac{dy}{dx} = \frac{(1)(x-3) - (x)(1)}{(x-3)^2} = \frac{x-3-x}{(x-3)^2} = \frac{-3}{(x-3)^2}.$$

$$\text{Q2. } u = x + 1, v = x - 1, u' = 1, v' = 1. \frac{dy}{dx} = \frac{(1)(x-1) - (x+1)(1)}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}.$$

$$\text{Q3. } u = 3x, v = x^2 + 2, u' = 3, v' = 2x. \frac{dy}{dx} = \frac{3(x^2+2) - 3x(2x)}{(x^2+2)^2} = \frac{3x^2+6-6x^2}{(x^2+2)^2} = \frac{6-3x^2}{(x^2+2)^2} = \frac{3(2-x^2)}{(x^2+2)^2}.$$

$$\text{Q4. } u = x - 1, v = x^2, u' = 1, v' = 2x. \frac{dy}{dx} = \frac{(1)(x^2) - (x-1)(2x)}{x^4} = \frac{x^2-2x^2+2x}{x^4} = \frac{-x^2+2x}{x^4} = \frac{2-x}{x^3}. \text{ Check: } y = x^{-1} - x^{-2} \Rightarrow \frac{dy}{dx} = -x^{-2} + 2x^{-3} = \frac{-x+2}{x^3} = \frac{2-x}{x^3}. \checkmark$$

$$\text{Q5. } u = e^x, v = x + 1, u' = e^x, v' = 1. \frac{dy}{dx} = \frac{e^x(x+1) - e^x(1)}{(x+1)^2} = \frac{e^x(x+1-1)}{(x+1)^2} = \frac{xe^x}{(x+1)^2}.$$

$$\text{Q6. } u = 2x + 1, v = 3x - 2, u' = 2, v' = 3. \frac{dy}{dx} = \frac{2(3x-2) - (2x+1)(3)}{(3x-2)^2} = \frac{6x-4-6x-3}{(3x-2)^2} = \frac{-7}{(3x-2)^2}. \text{ The numerator is the constant } -7 \neq 0, \text{ so } \frac{dy}{dx} \neq 0 \text{ for any } x: \text{ the gradient is never zero.}$$

$$\text{Q7. } u = x + 4, v = x - 4. \frac{dy}{dx} = \frac{(1)(x-4) - (x+4)(1)}{(x-4)^2} = \frac{x-4-x-4}{(x-4)^2} = \frac{-8}{(x-4)^2}.$$

$$\text{Q8. } u = x^2, v = x + 1, u' = 2x, v' = 1. \frac{dy}{dx} = \frac{2x(x+1) - x^2(1)}{(x+1)^2} = \frac{2x^2+2x-x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}.$$

$$\text{Q9. } u = x^2 - 1, v = x^2 + 1, u' = 2x, v' = 2x. \frac{dy}{dx} = \frac{2x(x^2+1) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{2x[(x^2+1) - (x^2-1)]}{(x^2+1)^2} = \frac{2x(2)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

$$\text{Q10. } u = 1 - x, v = 1 + x, u' = -1, v' = 1. \frac{dy}{dx} = \frac{(-1)(1+x) - (1-x)(1)}{(1+x)^2} = \frac{-1-x-1+x}{(1+x)^2} = \frac{-2}{(1+x)^2}.$$

$$\text{Q11. } u = e^x, v = x^2, u' = e^x, v' = 2x. \frac{dy}{dx} = \frac{e^x(x^2) - e^x(2x)}{x^4} = \frac{e^x x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}.$$

$$\text{Q12. } u = e^{2x}, v = x, u' = 2e^{2x}, v' = 1. \frac{dy}{dx} = \frac{2e^{2x}(x) - e^{2x}(1)}{x^2} = \frac{e^{2x}(2x-1)}{x^2}.$$

$$\text{Q13. } u = \sin x, v = \cos x, u' = \cos x, v' = -\sin x. \frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}. \text{ Since } \frac{\sin x}{\cos x} = \tan x, \text{ this confirms } \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}.$$

$$\text{Q14. } u = \cos x, v = x, u' = -\sin x, v' = 1. \frac{dy}{dx} = \frac{-\sin x(x) - \cos x(1)}{x^2} = \frac{-(x \sin x + \cos x)}{x^2}.$$

$$\text{Q15. } u = x, v = e^x, u' = 1, v' = e^x. \frac{dy}{dx} = \frac{(1)e^x - x e^x}{(e^x)^2} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}.$$

$$\text{Q16. } u = \log_e x, v = x^2, u' = \frac{1}{x}, v' = 2x. \frac{dy}{dx} = \frac{\frac{1}{x}x^2 - \log_e x(2x)}{x^4} = \frac{x - 2x \log_e x}{x^4} = \frac{1 - 2 \log_e x}{x^3}.$$

$$\text{Q17. } u = 2x - 3, v = x^2 + 1, u' = 2, v' = 2x. \frac{dy}{dx} = \frac{2(x^2 + 1) - (2x - 3)(2x)}{(x^2 + 1)^2} = \frac{2x^2 + 2 - (4x^2 - 6x)}{(x^2 + 1)^2} = \frac{-2x^2 + 6x + 2}{(x^2 + 1)^2} = \frac{-2(x^2 - 3x - 1)}{(x^2 + 1)^2}.$$

$$\text{Q18. First simplify: } y = \frac{x+2}{x^2-4} = \frac{x+2}{(x-2)(x+2)} = \frac{1}{x-2}, \text{ valid for } x \neq \pm 2. \text{ Then}$$

$$y = (x-2)^{-1} \Rightarrow \frac{dy}{dx} = -(x-2)^{-2} = \frac{-1}{(x-2)^2}, x \neq \pm 2. \text{ (The quotient rule on the original gives the same result.)}$$

$$\text{Q19. } u = \sqrt{x} = x^{1/2}, v = x+1, u' = \frac{1}{2\sqrt{x}}, v' = 1. \frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x}}(x+1) - \sqrt{x}(1)}{(x+1)^2}. \text{ Multiply numerator and denominator by}$$

$$2\sqrt{x}: \frac{(x+1) - 2x}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2}.$$

$$\text{Q20. } u = x^2 + 1, v = x - 1, u' = 2x, v' = 1. \frac{dy}{dx} = \frac{2x(x-1) - (x^2+1)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

$$\text{Horizontal tangents where the numerator is zero: } x^2 - 2x - 1 = 0 \Rightarrow x = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}. \text{ At } x = 1 + \sqrt{2}:$$

$$y = \frac{(1+\sqrt{2})^2 + 1}{(1+\sqrt{2}) - 1} = \frac{4 + 2\sqrt{2}}{\sqrt{2}} = 2\sqrt{2} + 2. \text{ At } x = 1 - \sqrt{2}: y = \frac{(1-\sqrt{2})^2 + 1}{(1-\sqrt{2}) - 1} = \frac{4 - 2\sqrt{2}}{-\sqrt{2}} = 2 - 2\sqrt{2}. \therefore \text{ horizontal tangents at } (1+\sqrt{2}, 2+2\sqrt{2}) \text{ and } (1-\sqrt{2}, 2-2\sqrt{2}).$$

$$\text{Q21. } u = x, v = x^2 + 4, u' = 1, v' = 2x. \frac{dy}{dx} = \frac{(1)(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}, \text{ as required. The}$$

$$\text{gradient is zero where the numerator is zero: } 4 - x^2 = 0 \Rightarrow x = \pm 2.$$

Theory

Limits. The statement $\lim_{x \rightarrow a} f(x) = L$ means that the values of $f(x)$ can be made as close as we like to L by taking x sufficiently close to a (but not equal to a). The limit describes the behaviour of f near $x = a$, which need not equal $f(a)$.

Evaluating a limit.

- If f is a polynomial (or any function continuous at a), then $\lim_{x \rightarrow a} f(x) = f(a)$ — just substitute.
- If substitution gives the **indeterminate form** $\frac{0}{0}$, factorise the numerator and denominator and cancel the common factor, then substitute. For example

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 6.$$

One-sided limits. $\lim_{x \rightarrow a^-} f(x)$ is the limit as x approaches a from the **left** ($x < a$); $\lim_{x \rightarrow a^+} f(x)$ is the limit from the **right** ($x > a$). The two-sided limit exists **iff** both one-sided limits exist and are equal:

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L.$$

Continuity at a point. A function f is **continuous at** $x = a$ if all three conditions hold:

- $f(a)$ is defined;
- $\lim_{x \rightarrow a} f(x)$ exists (the two one-sided limits agree);
- $\lim_{x \rightarrow a} f(x) = f(a)$.

If any condition fails, f is **discontinuous** at a . A rational function is discontinuous wherever its denominator is zero.

Hybrid (piecewise) functions. A hybrid function is continuous at a join $x = a$ precisely when the two pieces **meet** there — the left piece's value at a equals the right piece's value at a . To find a parameter that makes a hybrid continuous, set the two pieces equal at the join and solve.

Worked examples

Example 1 — scaffolded

Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.

Working

Comment

Substituting $x = 3$ gives $\frac{0}{0}$.

The indeterminate form signals a common factor.

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3 \text{ for } x \neq 3.$$

Factorise and cancel $(x - 3)$.

$$\lim_{x \rightarrow 3} (x + 3) = 3 + 3 = 6.$$

Substitute into the simplified expression.

Working

Comment

$$\lim_{x \rightarrow 3} x^2 - 9$$

$$\therefore \frac{x^2 - 9}{x - 3} = 6.$$

State the limit.

Example 2 — scaffolded

For $f(x) = \begin{cases} x+3, & x \leq 2 \\ x^2, & x > 2 \end{cases}$, find $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$, and state whether $\lim_{x \rightarrow 2} f(x)$ exists.

Working

Comment

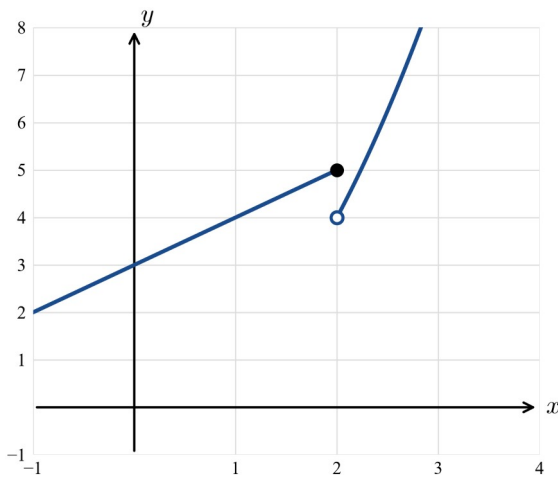
$$\lim_{x \rightarrow 2^-} f(x) = 2 + 3 = 5.$$

From the left, f is the piece $x + 3$.

$$\lim_{x \rightarrow 2^+} f(x) = 2^2 = 4.$$

From the right, f is the piece x^2 .

The one-sided limits differ ($5 \neq 4$), so $\lim_{x \rightarrow 2} f(x)$ **does not exist**.

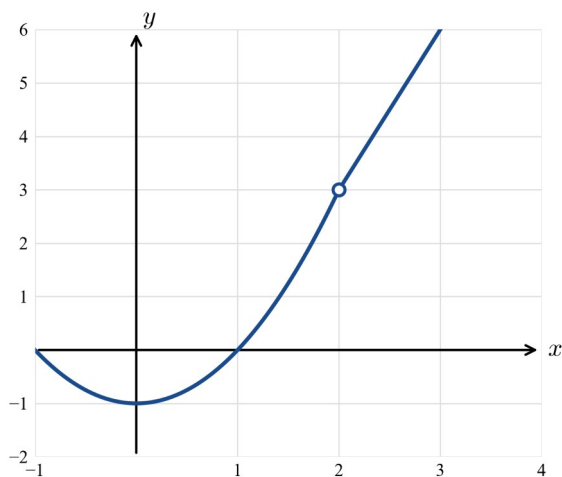
The graph jumps at $x = 2$.**Example 3 — unscaffolded**

Determine whether $f(x) = \begin{cases} x^2 - 1, & x < 2 \\ 3x - 3, & x \geq 2 \end{cases}$ is continuous at $x = 2$.

$$\lim_{x \rightarrow 2^-} f(x) = 2^2 - 1 = 3 \text{ and } \lim_{x \rightarrow 2^+} f(x) = 3(2) - 3 = 3, \text{ so } \lim_{x \rightarrow 2} f(x) = 3.$$

$$\text{Also } f(2) = 3(2) - 3 = 3, \text{ so } \lim_{x \rightarrow 2} f(x) = f(2).$$

$\therefore f$ is continuous at $x = 2$.



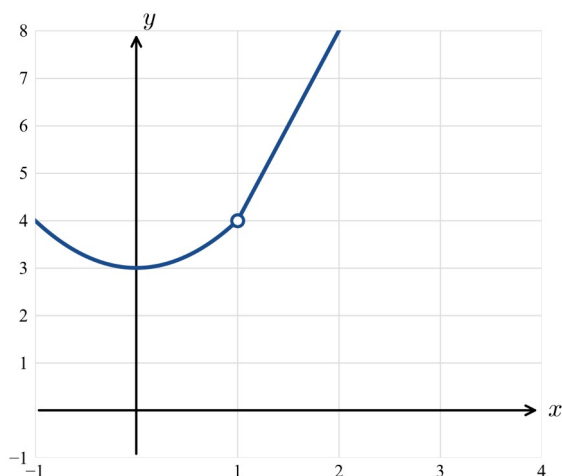
Example 4 — unscaffolded

Find the value of a for which $f(x) = \begin{cases} x^2 + a, & x \leq 1 \\ 4x, & x > 1 \end{cases}$ is continuous at $x = 1$.

For continuity the pieces must meet at $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = 1^2 + a = 1 + a$ and $\lim_{x \rightarrow 1^+} f(x) = 4(1) = 4$.

Setting them equal: $1 + a = 4$, so $a = 3$.

$\therefore a = 3$ (and then $f(1) = 1 + 3 = 4$, matching the right piece).



Practice questions

Scaffolded

Q1. For $\frac{\lim_{x \rightarrow 2} x^2 - 4}{x - 2}$: (a) explain why direct substitution fails; (b) factorise and cancel; (c) hence evaluate the limit.

Q2. Evaluate $\frac{\lim_{x \rightarrow -3} x^2 + x - 6}{x + 3}$ by factorising and cancelling.

Q3. For $f(x) = \begin{cases} x - 1, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$: (a) find $\lim_{x \rightarrow 2^-} f(x)$; (b) find $\lim_{x \rightarrow 2^+} f(x)$; (c) does $\lim_{x \rightarrow 2} f(x)$ exist? Is f continuous at $x = 2$?

Q4. Determine whether $f(x) = \begin{cases} x+1, & x \leq 2 \\ 3x-3, & x > 2 \end{cases}$ is continuous at $x=2$, justifying your answer with the three continuity conditions.

Q5. Find the value of a for which $f(x) = \begin{cases} ax+1, & x < 2 \\ x^2-1, & x \geq 2 \end{cases}$ is continuous at $x=2$.

Q6. State the value(s) of x at which $f(x) = \frac{x-1}{x^2-4}$ is discontinuous, and give a reason.

Unscaffolded

Q7. Evaluate $\lim_{x \rightarrow 4} \frac{x^2-16}{x-4}$.

Q8. Evaluate $\lim_{x \rightarrow 1} \frac{x^3-1}{x-1}$.

Q9. Evaluate $\lim_{x \rightarrow 2} \frac{x^2-5x+6}{x-2}$.

Q10. Evaluate $\lim_{x \rightarrow 0} \frac{x^2+3x}{x}$.

Q11. Evaluate $\lim_{x \rightarrow 3} \frac{x^2-9}{x^2-2x-3}$.

Q12. Evaluate $\lim_{x \rightarrow -2} \frac{x^3+8}{x+2}$.

Q13. Evaluate $\lim_{x \rightarrow 2} (3x^2-x+1)$.

Q14. For $f(x) = \begin{cases} 2x, & x < 1 \\ x+3, & x \geq 1 \end{cases}$, find the two one-sided limits at $x=1$ and state whether $\lim_{x \rightarrow 1} f(x)$ exists.

Q15. Show that $f(x) = \begin{cases} x+1, & x \leq 2 \\ 3x-3, & x > 2 \end{cases}$ is continuous at $x=2$.

Q16. Show that $f(x) = \begin{cases} x^2, & x < 1 \\ 2x, & x \geq 1 \end{cases}$ is **not** continuous at $x=1$, and describe the discontinuity.

Q17. Find the value of a for which $f(x) = \begin{cases} ax+2, & x \leq 3 \\ x^2-4, & x > 3 \end{cases}$ is continuous at $x=3$.

Q18. Find the value of a for which $f(x) = \begin{cases} x^2+1, & x < 2 \\ a-x, & x \geq 2 \end{cases}$ is continuous at $x=2$.

Q19. State all value(s) of x at which $f(x) = \frac{x+1}{(x-2)(x+3)}$ is discontinuous.

Q20. A function is defined by $f(x) = \begin{cases} x^2, & x \leq 1 \\ ax + b, & x > 1 \end{cases}$. Given that f is continuous at $x = 1$ and that $f(3) = 7$, find the values of a and b .

Q21. For $f(x) = \begin{cases} x + 2, & x \leq 3 \\ x^2 - 4, & x > 3 \end{cases}$, show that f is continuous at $x = 3$, and hence evaluate $\lim_{x \rightarrow 3} f(x)$.

Answers

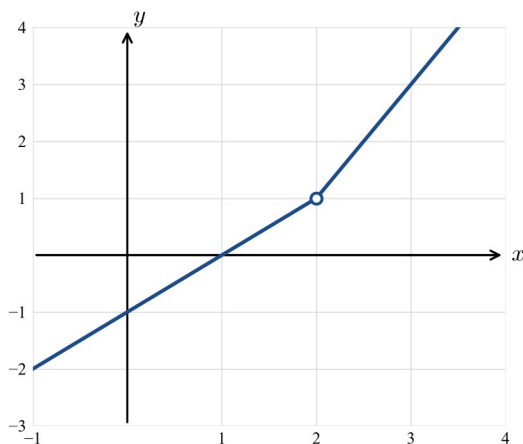
Q1. 4 **Q2.** -5 **Q3.** left 1, right 1; limit = 1 exists; $f(2)=1$ so continuous **Q4.** continuous ($\lim_{x \rightarrow 2} f(x)=3=f(2)$)
Q5. $a=1$ **Q6.** $x=-2$ and $x=2$ (denominator zero) **Q7.** 8 **Q8.** 3 **Q9.** -1 **Q10.** 3 **Q11.** $\frac{3}{2}$ **Q12.** 12 **Q13.** 11
Q14. left 2, right 4; limit does not exist **Q15.** continuous ($\lim_{x \rightarrow 2} f(x)=3=f(2)$) **Q16.** discontinuous — jump from 1 to 2 at $x=1$ **Q17.** $a=1$ **Q18.** $a=7$ **Q19.** $x=2$ and $x=-3$ **Q20.** $a=3, b=-2$ **Q21.** continuous; $\lim_{x \rightarrow 3} f(x)=5$

Fully-worked solutions

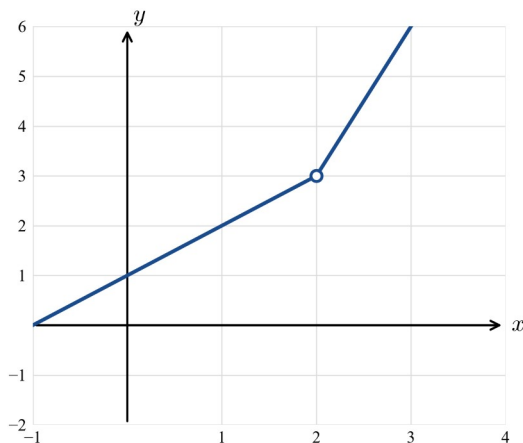
Q1. (a) At $x=2$, $\frac{x^2-4}{x-2}=\frac{0}{0}$, undefined. (b) $\frac{(x-2)(x+2)}{x-2}=x+2$ for $x \neq 2$. (c) $\lim_{x \rightarrow 2} (x+2)=4$.

Q2. $\frac{x^2+x-6}{x+3}=\frac{(x+3)(x-2)}{x+3}=x-2$ for $x \neq -3$; $\lim_{x \rightarrow -3} (x-2)=-5$.

Q3. (a) From the left, $f=x-1$: $\lim_{x \rightarrow 2^-} f(x)=1$. (b) From the right, $f=2x-3$: $\lim_{x \rightarrow 2^+} f(x)=1$. (c) The one-sided limits agree, so $\lim_{x \rightarrow 2} f(x)=1$; and $f(2)=2(2)-3=1$, so f is continuous at $x=2$.



Q4. $\lim_{x \rightarrow 2^-} f(x)=2+1=3$, $\lim_{x \rightarrow 2^+} f(x)=3(2)-3=3$, so $\lim_{x \rightarrow 2} f(x)=3$; $f(2)=2+1=3$. All three conditions hold, so f is continuous at $x=2$.



Q5. Meet at $x=2$: $a(2)+1=2^2-1 \Rightarrow 2a+1=3 \Rightarrow a=1$.

Q6. $f(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x-2)(x+2)}$; the denominator is zero at $x=2$ and $x=-2$, so f is discontinuous there (undefined).

Q7. $\frac{(x-4)(x+4)}{x-4} = x+4 \rightarrow 8$.

Q8. $\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1 \rightarrow 1+1+1=3$.

Q9. $\frac{(x-2)(x-3)}{x-2} = x-3 \rightarrow -1$.

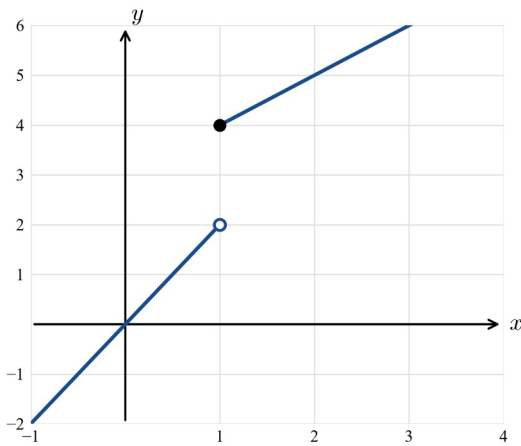
Q10. $\frac{x(x+3)}{x} = x+3 \rightarrow 3$.

Q11. $\frac{(x-3)(x+3)}{(x-3)(x+1)} = \frac{x+3}{x+1} \rightarrow \frac{6}{4} = \frac{3}{2}$.

Q12. $\frac{x^3+8}{x+2} = \frac{(x+2)(x^2-2x+4)}{x+2} = x^2-2x+4 \rightarrow 4+4+4=12$.

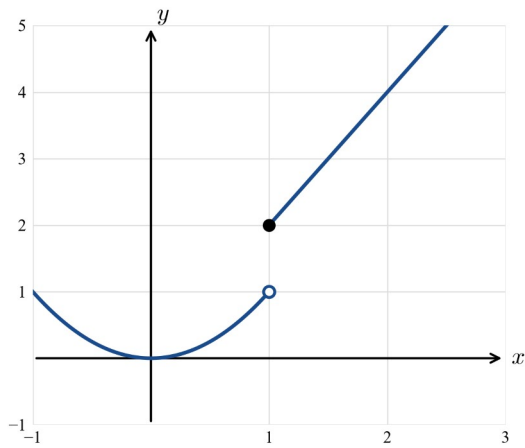
Q13. $3x^2 - x + 1$ is a polynomial, so substitute: $3(4) - 2 + 1 = 11$.

Q14. $\lim_{x \rightarrow 1^-} f(x) = 2(1) = 2$, $\lim_{x \rightarrow 1^+} f(x) = 1 + 3 = 4$. The one-sided limits differ, so $\lim_{x \rightarrow 1} f(x)$ does not exist (a jump discontinuity).



Q15. $\lim_{x \rightarrow 2^-} f(x) = 2 + 1 = 3$, $\lim_{x \rightarrow 2^+} f(x) = 3(2) - 3 = 3$, so $\lim_{x \rightarrow 2} f(x) = 3$; $f(2) = 2 + 1 = 3$. Since $\lim_{x \rightarrow 2} f(x) = f(2)$, f is continuous at $x=2$.

Q16. $\lim_{x \rightarrow 1^-} f(x) = 1^2 = 1$, $\lim_{x \rightarrow 1^+} f(x) = 2(1) = 2$. The one-sided limits differ, so $\lim_{x \rightarrow 1} f(x)$ does not exist and f is not continuous at $x=1$; the graph jumps from 1 to 2.



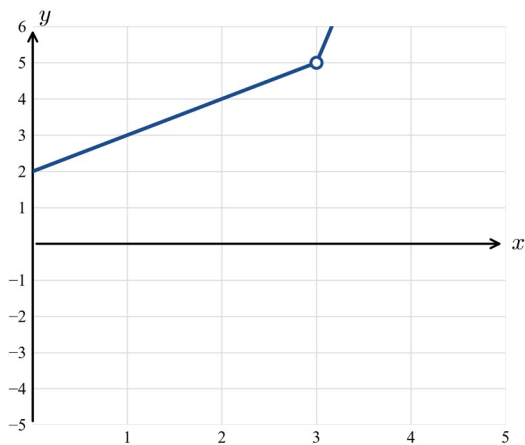
Q17. Meet at $x=3$: $a(3)+2=3^2-4 \Rightarrow 3a+2=5 \Rightarrow a=1$.

Q18. Meet at $x=2$: $2^2+1=a-2 \Rightarrow 5=a-2 \Rightarrow a=7$.

Q19. Denominator $(x-2)(x+3)=0$ at $x=2$ and $x=-3$; f is discontinuous there.

Q20. Continuity at $x=1$: $1^2=a(1)+b$, so $a+b=1$. Also $f(3)=3a+b=7$. Subtracting: $2a=6$, so $a=3$ and $b=1-3=-2$. $\therefore a=3, b=-2$.

Q21. $\lim_{x \rightarrow 3^-} f(x)=3+2=5$, $\lim_{x \rightarrow 3^+} f(x)=3^2-4=5$, so $\lim_{x \rightarrow 3} f(x)=5$; and $f(3)=3+2=5$, so f is continuous at $x=3$.
 $\therefore \lim_{x \rightarrow 3} f(x)=5$.



Theory

A function f is **differentiable** at $x=a$ if the derivative $f'(a)$ exists — that is, the gradient of the curve has a single well-defined value there. Formally this requires the **left-hand** and **right-hand** gradients at $x=a$ to be *equal* (and finite).

Where differentiability fails. A function is *not* differentiable at a point in any of these four cases:

1. **Discontinuity** — the graph has a jump or a hole at $x=a$; there is no single gradient.
2. **Sharp corner** — the graph is continuous but the gradient changes abruptly, so the left and right gradients differ (for example $y=|x|$ at $x=0$).
3. **Cusp** — the two sides approach a point with gradients running to $+\infty$ and $-\infty$ (for example $y=x^{2/3}$ at $x=0$).
4. **Vertical tangent** — the gradient tends to $\pm\infty$ (for example $y=x^{1/3}$ at $x=0$).

Differentiable $\Rightarrow$ continuous. If f is differentiable at $x=a$ then f is continuous there. The converse is **false**: $y=|x|$ is continuous everywhere but not differentiable at $x=0$ (a corner). So continuity is *necessary* but not *sufficient* for differentiability.

Smoothness of a hybrid (piecewise) function. Suppose

$$f(x) = \begin{cases} g(x) & x \leq a \\ h(x) & x > a \end{cases}$$

For f to be **differentiable at the join** $x=a$, **two** conditions must hold:

- **Continuous:** the pieces meet — $g(a)=h(a)$.
- **Equal gradient:** the pieces have the same slope — $g'(a)=h'(a)$.

Two conditions determine two unknown parameters. (If only the first holds, the graph has a corner: continuous but not differentiable.)

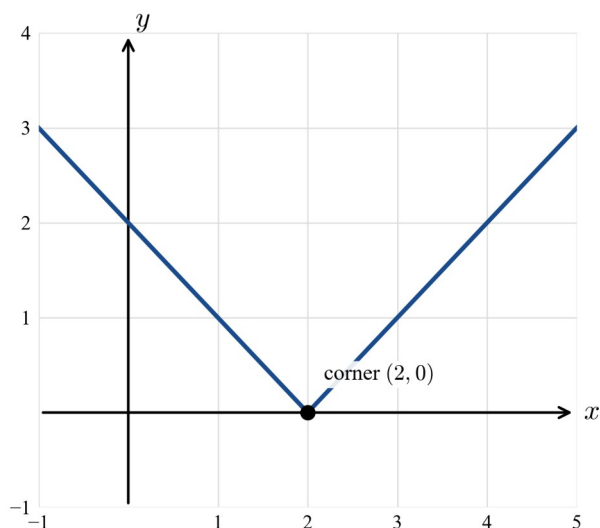
Stating where f is differentiable. Give the set of x -values, e.g. $\mathbb{R}$, or $\mathbb{R} \setminus \{a\}$ when the only failure is at $x=a$.

Worked examples

Example 1 — scaffolded

State where $f(x)=|x-2|$ is not differentiable, giving a reason, and sketch the graph.

Working	Comment
For $x > 2$, $f(x)=x-2$ so $f'(x)=1$; for $x < 2$, $f(x)=-(x-2)$ so $f'(x)=-1$.	Split the modulus at $x=2$ where $x-2=0$.
At $x=2$ the left gradient is -1 and the right gradient is $+1$: these are unequal.	The one-sided gradients disagree.
$\therefore f$ is not differentiable at $x=2$ — the graph has a sharp corner there. f is differentiable for $x \in \mathbb{R} \setminus \{2\}$.	A corner, not a discontinuity: f is still continuous at $x=2$.



Example 2 — scaffolded

Determine whether $f(x) = \begin{cases} x^2 & x \leq 1 \\ 2x - 1 & x > 1 \end{cases}$ is differentiable at $x = 1$.

Working

Comment

Continuity: $g(1) = 1^2 = 1$ and $h(1) = 2(1) - 1 = 1$, so $g(1) = h(1)$.

Both pieces give the same value at $x = 1$.

$\therefore f$ is continuous at $x = 1$.

The first condition holds.

Gradient: $g'(x) = 2x \Rightarrow g'(1) = 2$;

Differentiate each piece and evaluate at $x = 1$.

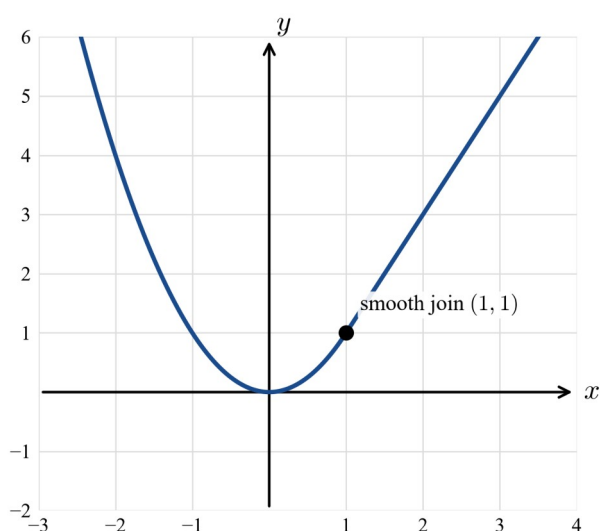
$h'(x) = 2 \Rightarrow h'(1) = 2$.

Since $g'(1) = h'(1) = 2$, the gradients match.

The second condition holds.

$\therefore f$ is **differentiable** at $x = 1$, with $f'(1) = 2$.

Both conditions met — a smooth join.



Example 3 — unscaffolded

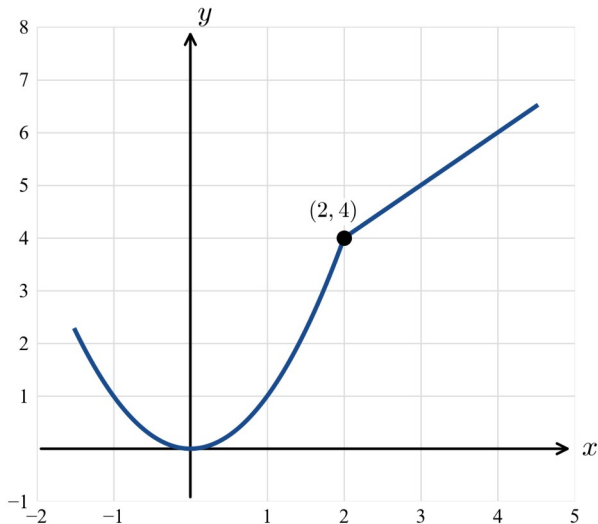
The function $f(x) = \begin{cases} ax^2 + b & x \leq 2 \\ x + 2 & x > 2 \end{cases}$ is differentiable at $x = 2$. Find the values of a and b .

Continuity at $x = 2$: $a(2)^2 + b = 2 + 2$, i.e. $4a + b = 4$. (1)

Equal gradient at $x=2$: $\frac{d}{dx}(ax^2+b)=2ax \Rightarrow 4a$; $\frac{d}{dx}(x+2)=1$. So $4a=1$. (2)

From (2), $a=\frac{1}{4}$. Substituting into (1): $1+b=4$, so $b=3$.

$$\therefore a=\frac{1}{4}, b=3.$$



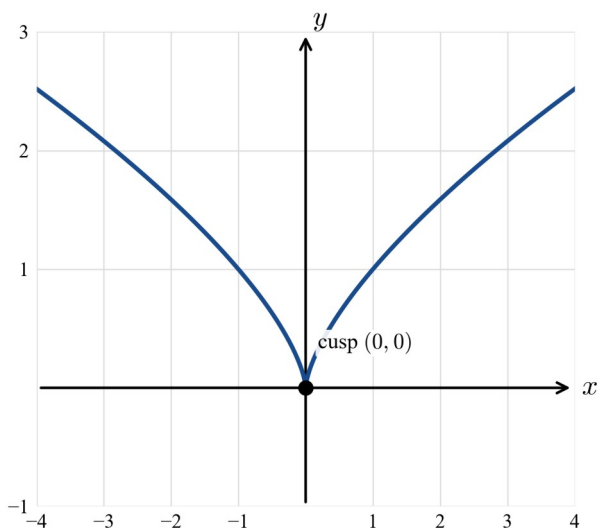
Example 4 — unscaffolded

Explain why $f(x)=x^{2/3}$ is not differentiable at $x=0$.

$f(x)=x^{2/3}$ is defined and continuous for all x , with $f(0)=0$. Differentiating, $f'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3x^{1/3}}$.

As $x \rightarrow 0^+$, $f'(x) \rightarrow +\infty$; as $x \rightarrow 0^-$, $f'(x) \rightarrow -\infty$. The one-sided gradients are infinite and of opposite sign.

$\therefore f$ is not differentiable at $x=0$ — the graph has a **cusp** there (though it is continuous).



Practice questions

Scaffolded

Q1. For $f(x) = |x + 3|$: (a) find $f'(x)$ for $x > -3$ and for $x < -3$; (b) state where f is not differentiable and why; (c) sketch the graph.

Q2. For $f(x) = \begin{cases} x^2 & x \leq 1 \\ x+1 & x > 1 \end{cases}$: (a) test whether f is continuous at $x = 1$; (b) hence state whether f is differentiable at $x = 1$.

Q3. For $f(x) = x^{1/3}$: (a) find $f'(x)$; (b) describe the behaviour of $f'(x)$ as $x \rightarrow 0$; (c) state, with a reason, why f is not differentiable at $x = 0$.

Q4. The function $f(x) = \begin{cases} x^2 + 3 & x \leq 2 \\ ax + b & x > 2 \end{cases}$ is differentiable at $x = 2$. (a) write the continuity condition; (b) write the equal-gradient condition; (c) hence find a and b .

Q5. For $f(x) = |x^2 - 9|$: (a) state the x -values where the graph meets the x -axis; (b) explain why f is not differentiable at those points; (c) sketch the graph.

Q6. The function $f(x) = \begin{cases} ax^2 & x \leq 1 \\ 2x + b & x > 1 \end{cases}$ is smooth (differentiable) at $x = 1$. (a) write the two conditions at $x = 1$; (b) solve them for a and b .

Un scaffolded

Q7. State where $f(x) = |x + 3|$ is not differentiable, give the reason, and sketch the graph.

Q8. State where $f(x) = |x^2 - 9|$ is not differentiable, give the reason, and sketch the graph.

Q9. Explain why $f(x) = x^{2/3}$ is not differentiable at $x = 0$, and name the feature.

Q10. Explain why $f(x) = x^{1/3}$ is not differentiable at $x = 0$, and name the feature.

Q11. Determine whether $f(x) = \begin{cases} x^2 & x \leq 1 \\ 2x - 1 & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q12. Determine whether $f(x) = \begin{cases} x^2 & x \leq 1 \\ x + 1 & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q13. Determine whether $f(x) = \begin{cases} x^2 & x \leq 1 \\ 3x - 2 & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q14. Find a and b so that $f(x) = \begin{cases} x^3 & x \leq 1 \\ ax + b & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q15. Find a and b so that $f(x) = \begin{cases} x^2 & x \leq 1 \\ ax + b & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q16. State where $f(x) = |2x + 1|$ is not differentiable and find the left and right gradients there.

Q17. Explain why $f(x) = (x - 1)^{2/3}$ is not differentiable at $x = 1$, and name the feature.

Q18. Find the value of k for which $f(x) = \begin{cases} x^2 & x \leq 1 \\ kx - 1 & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q19. Sketch the graph of the derivative of $f(x) = |x|$, stating its rule as a hybrid function and where it is undefined.

Q20. Determine whether $f(x) = \begin{cases} x+1 & x \leq 1 \\ x-1 & x > 1 \end{cases}$ is differentiable at $x = 1$, giving a reason.

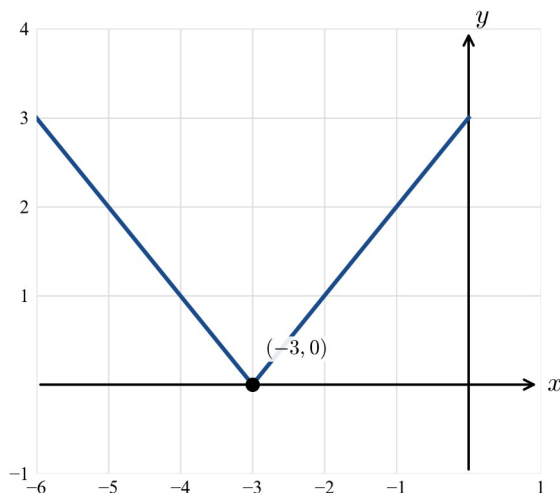
Q21. Explain why every differentiable function is continuous, and give an example of a function that is continuous everywhere but not differentiable at one point.

Answers

Q1. $f' = 1$ ($x > -3$), $f' = -1$ ($x < -3$); not differentiable at $x = -3$ (corner). **Q2.** Continuous ($1 = 1$? no: $g(1) = 1$, $h(1) = 2$) — **not continuous**, so not differentiable. **Q3.** $f'(x) = \frac{1}{3x^{2/3}} \rightarrow +\infty$ as $x \rightarrow 0$; not differentiable at $x = 0$ (vertical tangent). **Q4.** $4 = 2a + b$ and $4 = a$; $a = 4$, $b = -1$. **Q5.** Meets axis at $x = -3, 3$; corners there (gradient jumps $\neq 6$); not differentiable at $x = \pm 3$. **Q6.** $a = b + 2$ and $2a = 2$; $a = 1$, $b = -1$. **Q7.** Not differentiable at $x = -3$ (corner). **Q8.** Not differentiable at $x = -3$ and $x = 3$ (corners). **Q9.** Cusp at $x = 0$: $f' = \frac{2}{3x^{1/3}} \rightarrow \pm\infty$. **Q10.** Vertical tangent at $x = 0$: $f' = \frac{1}{3x^{2/3}} \rightarrow +\infty$. **Q11.** Differentiable ($1 = 1$ and gradients $2 = 2$). **Q12.** Not differentiable (not continuous: $1 \neq 2$). **Q13.** Continuous ($1 = 1$) but not differentiable (gradients $2 \neq 3$) — corner. **Q14.** $a = 3$, $b = -2$. **Q15.** $a = 2$, $b = -1$. **Q16.** Not differentiable at $x = -\frac{1}{2}$; left gradient -2 , right gradient $+2$. **Q17.** Cusp at $x = 1$: $f' = \frac{2}{3(x-1)^{1/3}} \rightarrow \pm\infty$. **Q18.** $k = 2$. **Q19.** $f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$, undefined at $x = 0$. **Q20.** Not differentiable (jump discontinuity at $x = 1$: $2 \neq 0$). **Q21.** Differentiability requires the difference quotient to have a finite limit, which forces $f(x) \rightarrow f(a)$ (continuity); e.g. $y = |x|$ is continuous but not differentiable at $x = 0$.

Fully-worked solutions

Q1. For $x > -3$, $f(x) = x + 3$ so $f'(x) = 1$; for $x < -3$, $f(x) = -(x + 3)$ so $f'(x) = -1$. At $x = -3$ the left gradient $-1 \neq$ right gradient 1 , so f is not differentiable at $x = -3$ — a **corner**. Elsewhere f is differentiable.

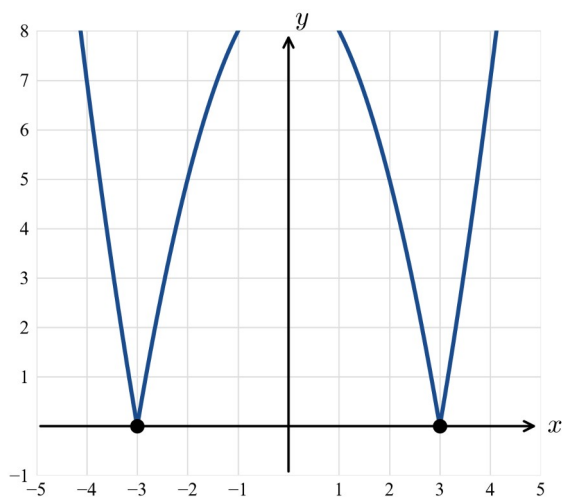


Q2. $g(1) = 1^2 = 1$ and $h(1) = 1 + 1 = 2$. Since $1 \neq 2$, f is **not continuous** at $x = 1$; a discontinuous function cannot be differentiable there. $\therefore$ not differentiable at $x = 1$.

Q3. $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$. As $x \rightarrow 0$ (either side), $x^{2/3} \rightarrow 0^+$ so $f'(x) \rightarrow +\infty$. The tangent becomes vertical, so f is not differentiable at $x = 0$ — a **vertical tangent**.

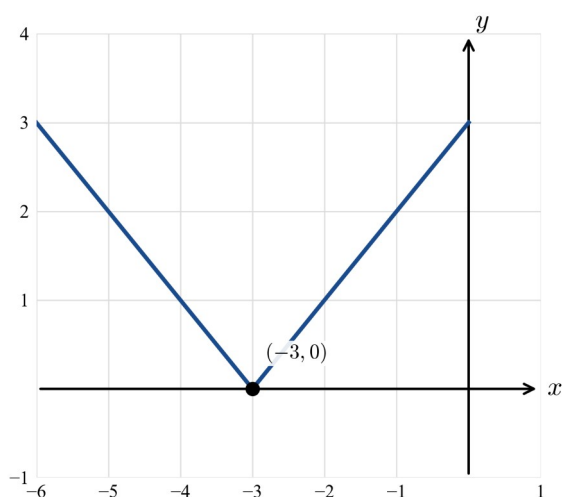
Q4. Continuity: $2^2 + 3 = 2a + b$, i.e. $2a + b = 7$. Gradient: $\frac{d}{dx}(x^2 + 3) = 2x \Rightarrow 4$ and $\frac{d}{dx}(ax + b) = a$, so $a = 4$. Then $2(4) + b = 7 \Rightarrow b = -1$. $\therefore a = 4, b = -1$.

Q5. $x^2 - 9 = 0$ at $x = -3, 3$, so the graph touches the axis there. Near a simple root the parabola $x^2 - 9$ crosses with nonzero gradient $2x$ ($\neq 6$ at $x = \mp 3$); taking the modulus reflects the below-axis part upward, creating a corner at each root. $\therefore$ not differentiable at $x = -3$ and $x = 3$.

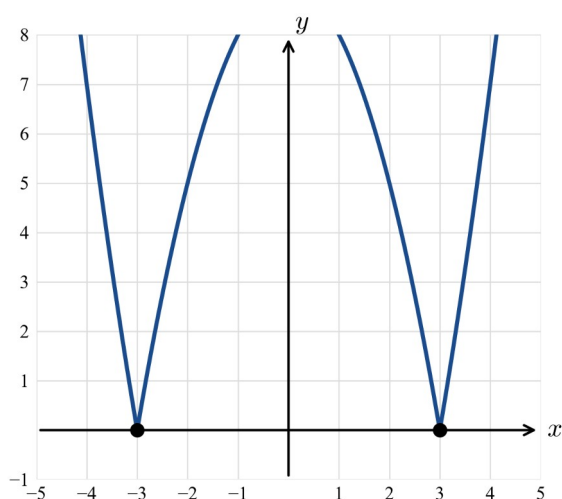


Q6. Continuity: $a(1)^2 = 2(1) + b$, i.e. $a = b + 2$. Gradient: $\frac{d}{dx}(ax^2) = 2ax \Rightarrow 2a$ and $\frac{d}{dx}(2x + b) = 2$, so $2a = 2 \Rightarrow a = 1$. Then $1 = b + 2 \Rightarrow b = -1$. $\therefore a = 1, b = -1$.

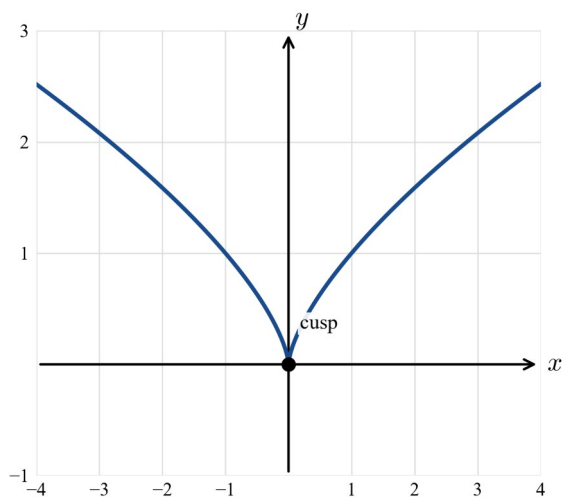
Q7. As in Q1: corner at $x = -3$, so not differentiable there (continuous elsewhere).



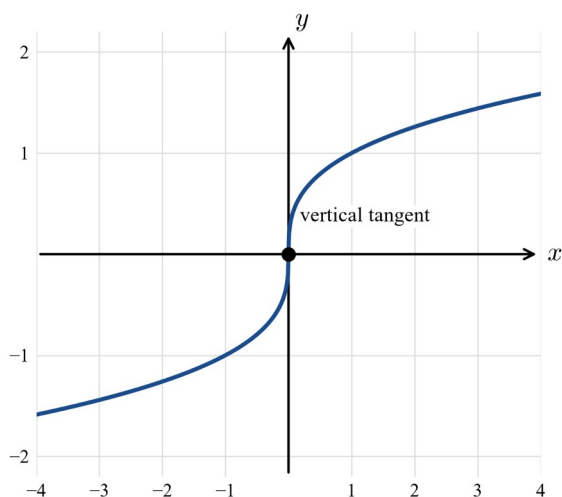
Q8. $|x^2 - 9|$ has corners where $x^2 - 9 = 0$, i.e. $x = -3$ and $x = 3$: the graph is continuous but the gradient jumps ($-6 \rightarrow 6$ at $x = 3$, $6 \rightarrow -6$ at $x = -3$). $\therefore$ not differentiable at $x = \pm 3$.



Q9. $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$. As $x \rightarrow 0^+$, $f' \rightarrow +\infty$; as $x \rightarrow 0^-$, $f' \rightarrow -\infty$. Opposite-signed infinite gradients give a **cusp**, so f is not differentiable at $x = 0$.



Q10. $f'(x) = \frac{1}{3x^{2/3}} \rightarrow +\infty$ as $x \rightarrow 0$ from either side. The tangent is vertical, so f is not differentiable at $x=0$ — a **vertical tangent**.



Q11. Continuity: $1^2=1$ and $2(1)-1=1$ — equal. Gradient: $2x \Rightarrow 2$ and 2 — equal. Both conditions hold, so f is **differentiable** at $x=1$ (with $f'(1)=2$).

Q12. $g(1)=1$, $h(1)=2$: not continuous, so **not differentiable** at $x=1$.

Q13. $g(1)=1$, $h(1)=1$: continuous. Gradients $g'(1)=2$, $h'(1)=3$: unequal. So f is continuous but **not differentiable** at $x=1$ — a corner.

Q14. Continuity: $1^3=a+b \Rightarrow a+b=1$. Gradient: $3x^2 \Rightarrow 3$ and a , so $a=3$. Then $b=1-3=-2$. $\therefore a=3, b=-2$.

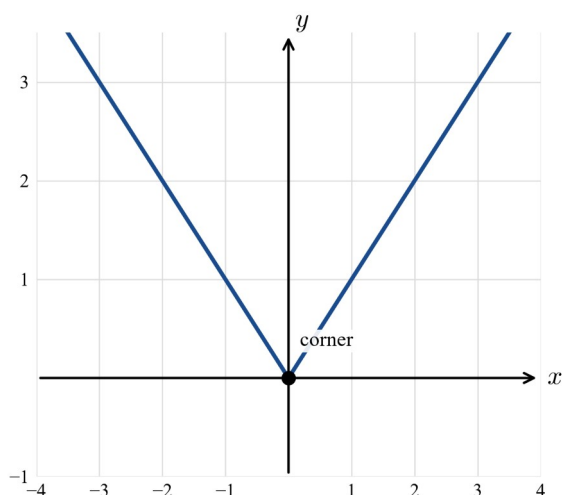
Q15. Continuity: $1=a+b$. Gradient: $2x \Rightarrow 2$ and a , so $a=2$. Then $b=1-2=-1$. $\therefore a=2, b=-1$.

Q16. For $x > -1/2$, $f(x)=2x+1$ so $f'=2$; for $x < -1/2$, $f(x)=-(2x+1)$ so $f'=-2$. At $x=-1/2$ the left gradient is -2 and the right gradient is $+2$; unequal, so not differentiable there (corner).

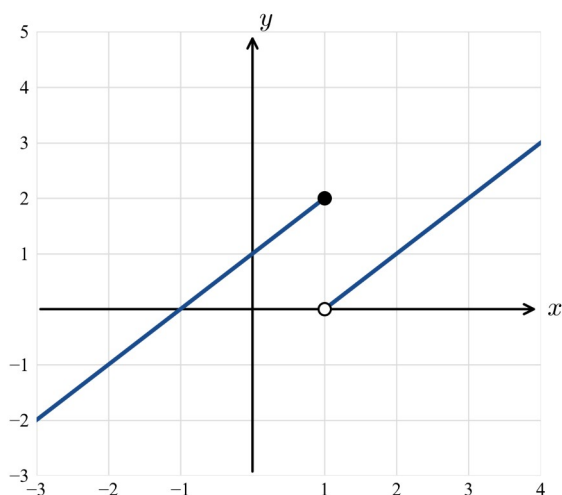
Q17. $f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$. As $x \rightarrow 1^+$, $f' \rightarrow +\infty$; as $x \rightarrow 1^-$, $f' \rightarrow -\infty$. A **cusp** at $x=1$, so not differentiable there.

Q18. Continuity: $1^2=k(1)-1 \Rightarrow k-1=1 \Rightarrow k=2$. Gradient: $2x \Rightarrow 2$ and k , so $k=2$. Both give $k=2$, so f is differentiable at $x=1$ when $k=2$.

Q19. For $x > 0$, $f(x) = x$ so $f'(x) = 1$; for $x < 0$, $f(x) = -x$ so $f'(x) = -1$; at $x = 0$ the derivative is undefined (corner). Thus $f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$ — a step, with an open circle at each end near $x = 0$.



Q20. $g(1) = 1 + 1 = 2$ and $h(1) = 1 - 1 = 0$. Since $2 \neq 0$ the graph jumps at $x = 1$ (a discontinuity), so f is **not differentiable** at $x = 1$.



Q21. If $f'(a)$ exists then $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ is finite; multiplying by $(x - a) \rightarrow 0$ gives $f(x) - f(a) \rightarrow 0$,

i.e. $f(x) \rightarrow f(a)$, which is continuity at a . The converse fails: $y = |x|$ is continuous everywhere but has a corner at $x = 0$, so it is not differentiable there.