Mathematical Methods Units 3 & 4

Chapter 7 — Combinations of functions

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Chapter 7 Combinations of functions — 7A Power functions with negative and rational indices

Theory

A **power function** has the rule $y=x^n$ for a fixed rational number $n=\frac{p}{q}$ (in lowest terms). You have already met $y=x^2$, $y=x^3$, $y=\frac{1}{x}=x^{-1}$, $y=\frac{1}{x^2}=x^{-2}$ and $y=\sqrt{x}=x^{1/2}$. This section looks at the full family of power functions with rational index and how the index controls the shape.

Reading the index $n=\frac{p}{q}$.

- Write $x^{p/q}=\left(\sqrt[q]{x}\right)^p$ — a qth root followed by a pth power.
- **Domain** depends on the denominator q: if q is **even** a real qth root needs $x\geq 0$ (and $x>0$ when $p<0$); if q is **odd** the qth root is defined for all real x.
- **Symmetry** depends on the parities of p and q (for odd q): $x^{p/q}$ is an **odd** function when p is odd and an **even** function when p is even.
- The point $(1,1)$ lies on **every** power function $y=x^n$, and (when it is in the domain) so does $(0,0)$ for $n>0$.

The standard power functions.

Rule	Domain	Range	Shape / key features
$y=x^{1/2}=\sqrt{x}$	$\{x:x\geq 0\}$	$\{y:y\geq 0\}$	increasing from endpoint $(0,0)$
$y=x^{1/3}=\sqrt[3]{x}$	$\mathbb{R}$	$\mathbb{R}$	**odd**, increasing, through $(0,0)$
$y=x^{2/3}$	$\mathbb{R}$	$\{y:y\geq 0\}$	**even**, a sharp **cusp** at $(0,0)$
$y=x^{3/2}$	$\{x:x\geq 0\}$	$\{y:y\geq 0\}$	increasing, steeper than $\sqrt{x}$
$y=x^{-1}=\frac{1}{x}$	$\mathbb{R}\setminus\{0\}$	$\mathbb{R}\setminus\{0\}$	**odd**; asymptotes $x=0$, $y=0$
$y=x^{-2}=\frac{1}{x^2}$	$\mathbb{R}\setminus\{0\}$	$\{y:y>0\}$	**even**; asymptotes $x=0$, $y=0$
$y=x^{-1/2}=\frac{1}{\sqrt{x}}$	$\{x:x>0\}$	$\{y:y>0\}$	decreasing; asymptotes $x=0$, $y=0$

Transformations. $y=a\,x^n+k$ dilates the basic graph by factor a from the x-axis (reflecting it when $a<0$) and translates it k units vertically — the endpoint, cusp or horizontal asymptote moves to height k.

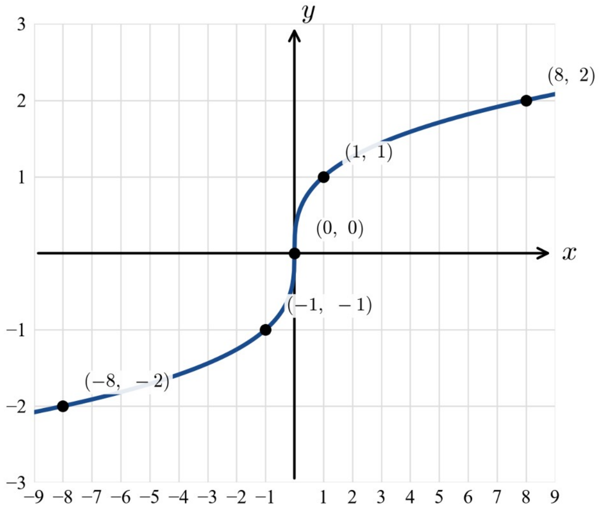
Solving $x^{p/q}=c$. Undo the index by raising both sides to the power $\frac{q}{p}$. Watch the parity of p: if p is **even** the function is even, so there are **two** solutions $x=\pm c^{q/p}$ (for $c>0$); if p is odd there is a single solution.

Worked examples

Example 1 — scaffolded

Sketch $y=x^{1/3}$, stating its domain and range, and mark the points where $x=-8,-1,0,1,8$.

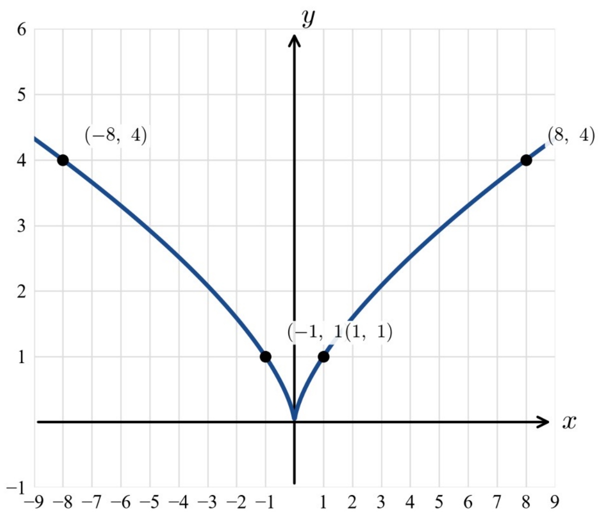
Working	Comment
$x^{1/3} = \sqrt[3]{x}$; the denominator $q = 3$ is odd, so the domain is $\mathbb{R}$.	An odd-order root is defined for negative numbers.
$p = 1$ is odd, so $y = x^{1/3}$ is an **odd** function (symmetric about the origin).	$(-x)^{1/3} = -x^{1/3}$.
$\sqrt[3]{-8} = -2$, $\sqrt[3]{-1} = -1$, $\sqrt[3]{0} = 0$, $\sqrt[3]{1} = 1$, $\sqrt[3]{8} = 2$.	Cube roots of the marked x-values.
The graph increases through $(-8,-2),(-1,-1),(0,0),(1,1),(8,2)$. Range $\mathbb{R}$.	Every real y is achieved.



Example 2 — scaffolded

Sketch $y = x^{2/3}$, stating its domain and range, and describe the behaviour at the origin.

Working	Comment
$x^{2/3} = \left(\sqrt[3]{x}\right)^2$; $q = 3$ is odd, so the domain is $\mathbb{R}$.	Cube root first, then square.
$p = 2$ is even, so $y = x^{2/3}$ is an **even** function and $y \geq 0$ for all x. Range $\{y : y \geq 0\}$.	Squaring makes every value non-negative; symmetric about the y-axis.
$(\pm 1)^{2/3} = 1$ and $(\pm 8)^{2/3} = 4$; the graph passes through $(-8,4),(-1,1),(1,1),(8,4)$.	Values agree in pairs by the even symmetry.
At the origin the two halves meet in a sharp **cusp** at $(0,0)$ (the graph is not smooth there).	Characteristic of $x^{2/3}$.



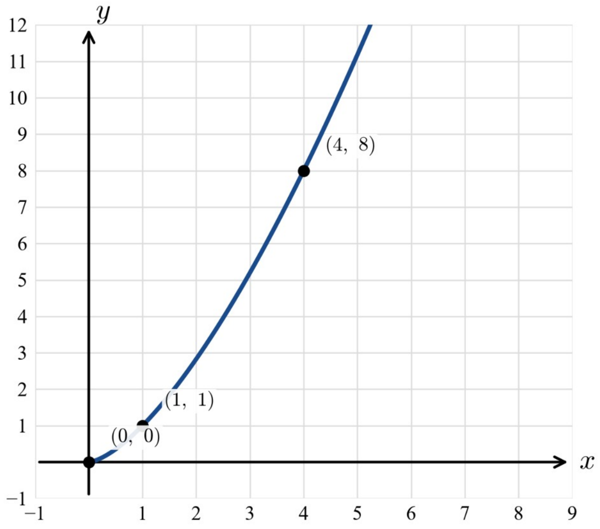
Example 3 — unscaffolded

Sketch $y=x^{3/2}$, stating its domain and range.

$x^{3/2}=\left(\sqrt{x}\right)^3$; the denominator $q=2$ is even, so the domain is $\{x : x \geq 0\}$.

Since $y \geq 0$ throughout, the range is $\{y : y \geq 0\}$. Key points: $0^{3/2}=0$, $1^{3/2}=1$, $4^{3/2}=8$.

$\therefore$ the graph increases from the endpoint $(0,0)$ through $(1,1)$ and $(4,8)$, rising more steeply than $y=\sqrt{x}$.



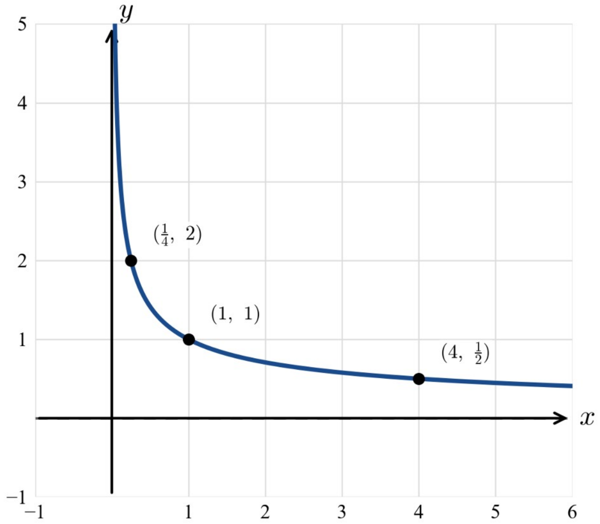
Example 4 — unscaffolded

Sketch $y=x^{-1/2}$, stating its domain, range and asymptotes.

$x^{-1/2}=\dfrac{1}{\sqrt{x}}$. The even denominator needs $x \geq 0$, and $x=0$ makes the expression undefined, so the domain is $\{x : x>0\}$ and the range is $\{y : y>0\}$.

As $x \to 0^+$, $y \to \infty$ and as $x \to \infty$, $y \to 0^+$: asymptotes $x=0$ and $y=0$. Key points: $\left(\dfrac{1}{4}, 2\right)$, $(1,1)$, $\left(4, \dfrac{1}{2}\right)$.

$\therefore$ the graph is a decreasing curve in the first quadrant.



Practice questions

Scaffolded

Q1. For $y=x^{1/2}$: (a) state the domain and range; (b) find y when $x=0,1,4,9$; (c) state whether the function is odd, even or neither; (d) sketch the graph.

Q2. For $y=x^{1/3}$: (a) state the domain and range; (b) explain why the graph exists for negative x; (c) find the points where $x=-8,0,8$; (d) sketch the graph.

Q3. For $y=x^{2/3}$: (a) state the domain and range; (b) show that the function is even; (c) find the points where $x=\pm 8$; (d) sketch the graph, showing the cusp.

Q4. For $y=x^{-1/2}$: (a) state the domain, range and asymptotes; (b) find y when $x=1$ and $x=4$; (c) sketch the graph.

Q5. For $y=x^{-2}$: (a) state the domain, range and asymptotes; (b) explain why the function is even; (c) find the points where $x=-1,1,2$; (d) sketch the graph.

Q6. For $y=2x^{1/3}$: (a) describe how the graph is obtained from $y=x^{1/3}$; (b) find the points where $x=-8,1,8$; (c) sketch the graph.

Unscaffolded

Sketch each of the following, stating the domain, range and any asymptotes, and marking the key points.

Q7. $y=x^{3/2}$

Q8. $y=x^{2/3}$

Q9. $y=x^{-1/2}$

Q10. $y=x^{1/3}+2$

Q11. $y=\sqrt{x}-1$

Q12. On the same axes, sketch $y=x^{1/2}$ and $y=x^{1/3}$, and state for which x the cube-root graph lies above the square-root graph.

Q13. $y=x^{-2}$

Q14. $y=x^{3/2}$ for $0\le x\le 4$

Q15. $y=x^{2/3}-1$

Q16. $y=-x^{1/3}$

Q17. Solve $x^{2/3}=9$ for x, giving all real solutions exactly.

Q18. Solve $x^{3/2}=8$ for x.

Q19. State the domain and range of $y=x^{-1/2}$ and explain why $x=0$ is excluded.

Q20. Show that the point $(8,4)$ lies on $y=x^{2/3}$, and find the other point on the graph with the same y-coordinate.

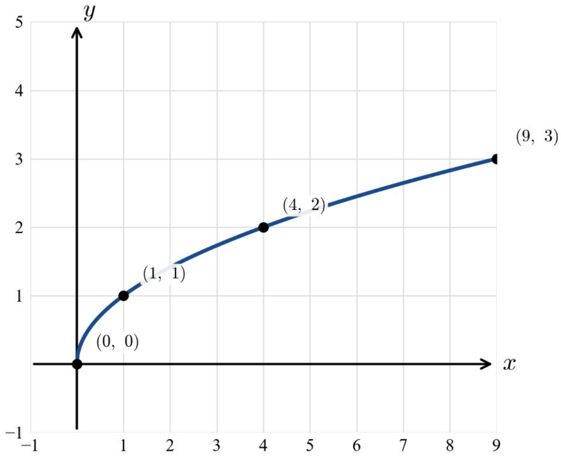
Q21. Solve $x^{-2}=\frac{1}{9}$ for x, giving all real solutions.

Answers

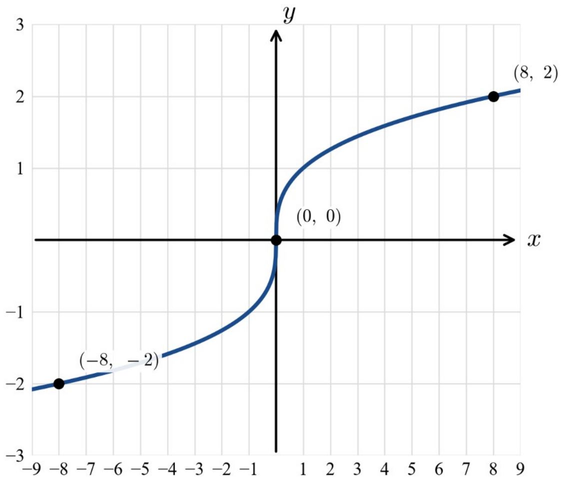
Q1. Domain $\{x : x \geq 0\}$, range $\{y : y \geq 0\}$; $(0,0),(1,1),(4,2),(9,3)$; neither. **Q2.** Domain $\mathbb{R}$, range $\mathbb{R}$; defined for $x<0$ because a cube root of a negative number is real; $(-8,-2),(0,0),(8,2)$. **Q3.** Domain $\mathbb{R}$, range $\{y : y \geq 0\}$; even; $(-8,4)$ and $(8,4)$. **Q4.** Domain $\{x : x>0\}$, range $\{y : y>0\}$; asymptotes $x=0,\ y=0$; $(1,1)$, $\left(4,\frac{1}{2}\right)$. **Q5.** Domain $\mathbb{R}\setminus\{0\}$, range $\{y : y>0\}$; asymptotes $x=0,\ y=0$; even; $(-1,1),(1,1),\left(2,\frac{1}{4}\right)$. **Q6.** Dilation of factor 2 from the x-axis; $(-8,-4),(1,2),(8,4)$. **Q7.** Domain $\{x : x \geq 0\}$, range $\{y : y \geq 0\}$; $(0,0),(1,1),(4,8),(9,27)$. **Q8.** Domain $\mathbb{R}$, range $\{y : y \geq 0\}$; cusp at $(0,0)$; $(\pm 1,1),(\pm 8,4)$. **Q9.** Domain $\{x : x>0\}$, range $\{y : y>0\}$; asymptotes $x=0,\ y=0$; $(1,1),(4,1/2),(9,1/3)$. **Q10.** Domain $\mathbb{R}$, range $\mathbb{R}$; $y=x^{1/3}$ translated up 2; through $(-8,0),(0,2),(8,4)$. **Q11.** Domain $\{x : x \geq 0\}$, range $\{y : y \geq -1\}$; endpoint $(0,-1)$; x-intercept $(1,0)$; $(4,1),(9,2)$. **Q12.** Both pass through $(0,0)$ and $(1,1)$; for $0<x<1$ the cube-root graph is above; the square-root graph is not defined for $x<0$. **Q13.** Domain $\mathbb{R}\setminus\{0\}$, range $\{y : y>0\}$; asymptotes $x=0,\ y=0$; $(-1,1),(1,1),(2,1/4)$. **Q14.** Domain $[0,4]$, range $[0,8]$; endpoints $(0,0)$ and $(4,8)$; through $(1,1)$. **Q15.** Domain $\mathbb{R}$, range $\{y : y \geq -1\}$; cusp at $(0,-1)$; through $(\pm 8,3)$. **Q16.** Domain $\mathbb{R}$, range $\mathbb{R}$; $y=x^{1/3}$ reflected in the x-axis; $(-8,2),(0,0),(8,-2)$. **Q17.** $x=\pm 27$. **Q18.** $x=4$. **Q19.** Domain $\{x : x>0\}$, range $\{y : y>0\}$; $x=0$ gives $\frac{1}{\sqrt{0}}$, which is undefined (division by zero). **Q20.** $8^{2/3}=4$, so $(8,4)$ is on the graph; by even symmetry the other point is $(-8,4)$. **Q21.** $x=\pm 3$.

Fully-worked solutions

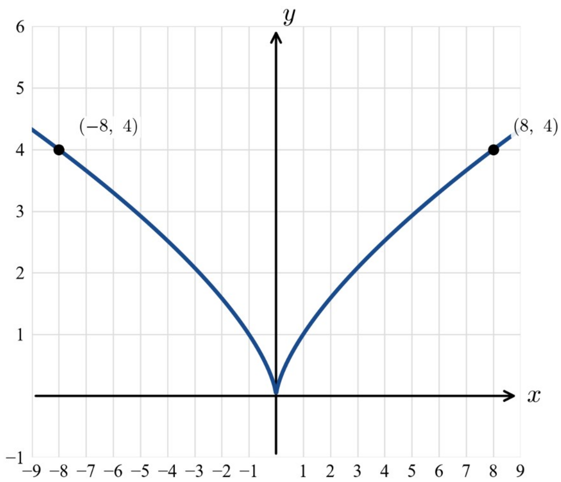
Q1. $y=x^{1/2}=\sqrt{x}$. Even denominator $\Rightarrow$ domain $\{x : x \geq 0\}$; $y \geq 0$ so range $\{y : y \geq 0\}$. $\sqrt{0}=0$, $\sqrt{1}=1$, $\sqrt{4}=2$, $\sqrt{9}=3$. Neither odd nor even (domain not symmetric).



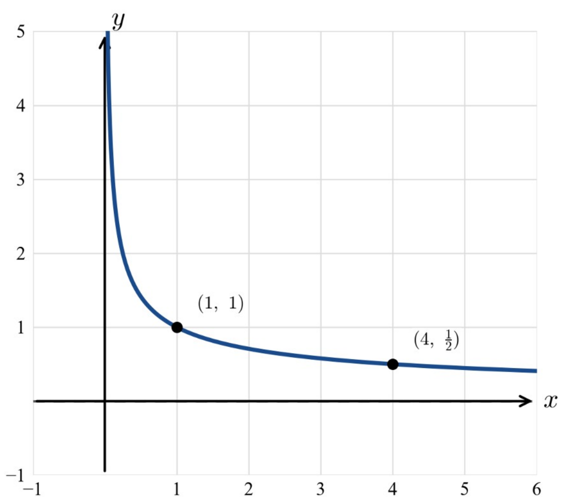
Q2. $y=x^{1/3}$. Odd denominator $\Rightarrow$ domain $\mathbb{R}$; range $\mathbb{R}$. A cube root of a negative number is a real negative number, e.g. $\sqrt[3]{-8}=-2$, so the graph exists for $x<0$. Points $(-8,-2),(0,0),(8,2)$.



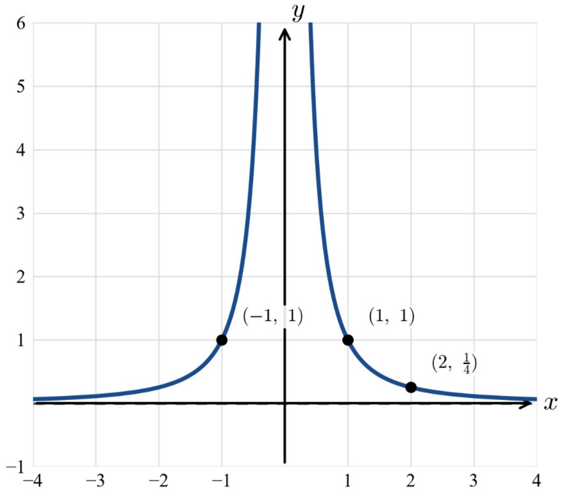
Q3. $y=x^{2/3}=\left(\sqrt[3]{x}\right)^2$. Domain $\mathbb{R}$; since a square is never negative, range $\{y: y\ge 0\}$. Even: $(-x)^{2/3}=\left(\sqrt[3]{-x}\right)^2=\left(-\sqrt[3]{x}\right)^2=\left(\sqrt[3]{x}\right)^2=x^{2/3}$. $(\pm 8)^{2/3}=4$.



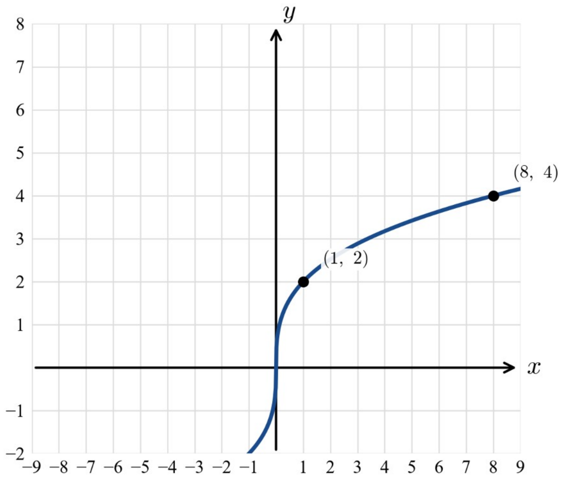
Q4. $y=x^{-1/2}=\dfrac{1}{\sqrt{x}}$. Domain $\{x: x>0\}$, range $\{y: y>0\}$; asymptotes $x=0$ and $y=0$. $1^{-1/2}=1$, $4^{-1/2}=\dfrac{1}{2}$.



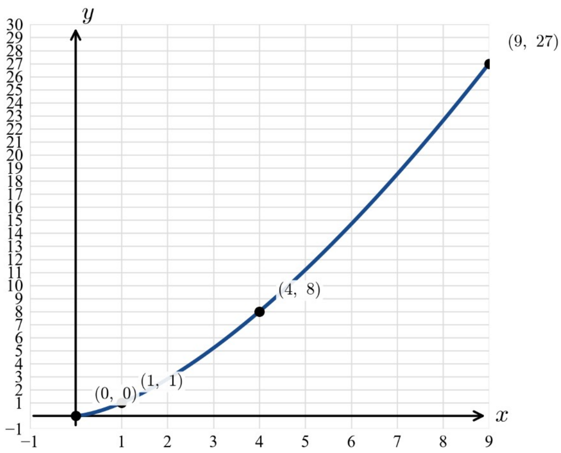
Q5. $y=x^{-2}=\dfrac{1}{x^2}$. Domain $\mathbb{R}\setminus\{0\}$; $x^2>0$ so $y>0$, range $\{y: y>0\}$; asymptotes $x=0$, $y=0$. Even: $(-x)^{-2}=\dfrac{1}{(-x)^2}=\dfrac{1}{x^2}$. Points $(-1,1),(1,1),\left(2,\dfrac{1}{4}\right)$.



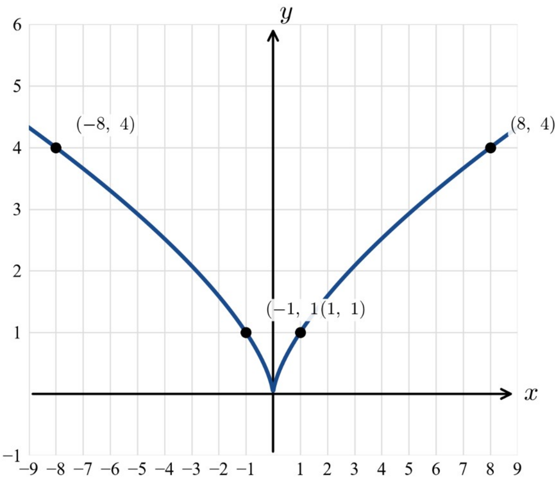
Q6. $y = 2x^{1/3}$ is $y = x^{1/3}$ dilated by factor 2 from the x-axis. $2\sqrt[3]{-8} = -4$, $2\sqrt[3]{1} = 2$, $2\sqrt[3]{8} = 4$: points $(-8, -4), (1, 2), (8, 4)$.



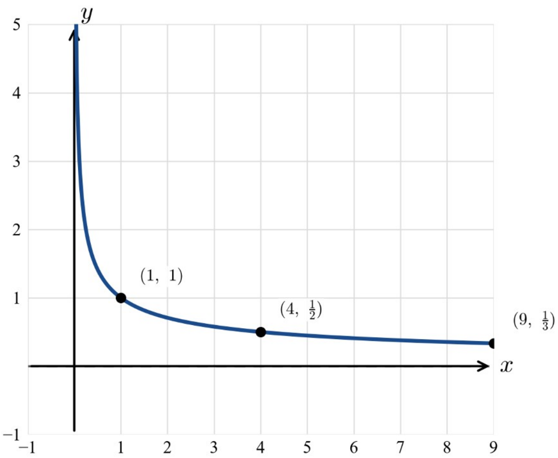
Q7. $y = x^{3/2} = \left(\sqrt{x}\right)^3$. Domain $\{x : x \geq 0\}$, range $\{y : y \geq 0\}$. $0^{3/2} = 0$, $1^{3/2} = 1$, $4^{3/2} = 8$, $9^{3/2} = 27$.



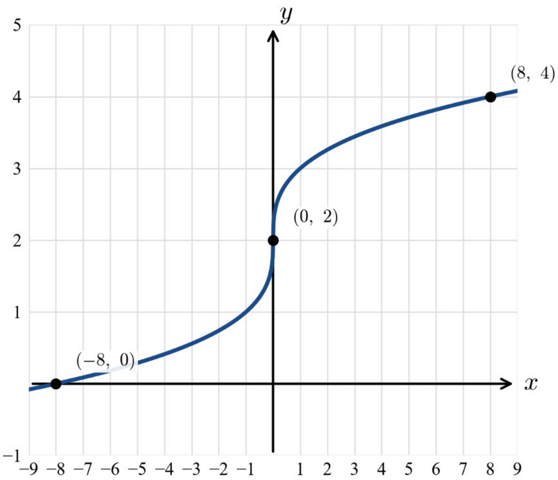
Q8. $y = x^{2/3}$. Domain $\mathbb{R}$, range $\{y : y \geq 0\}$, cusp at $(0, 0)$; $(\pm 1, 1), (\pm 8, 4)$.



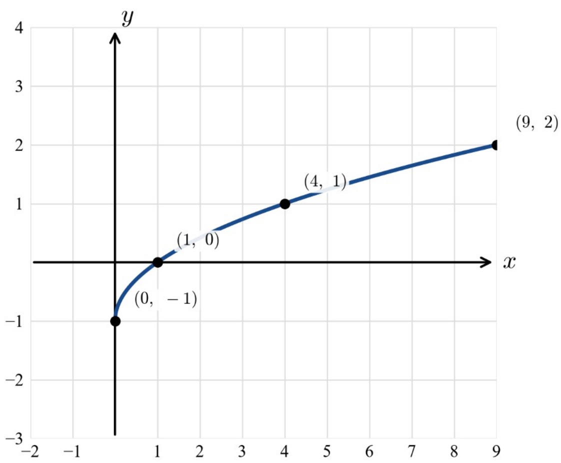
Q9. $y=x^{-1/2}$. Domain $\{x : x>0\}$, range $\{y : y>0\}$; asymptotes $x=0,\ y=0$; $(1,1),\left(4,\frac{1}{2}\right),\left(9,\frac{1}{3}\right)$.



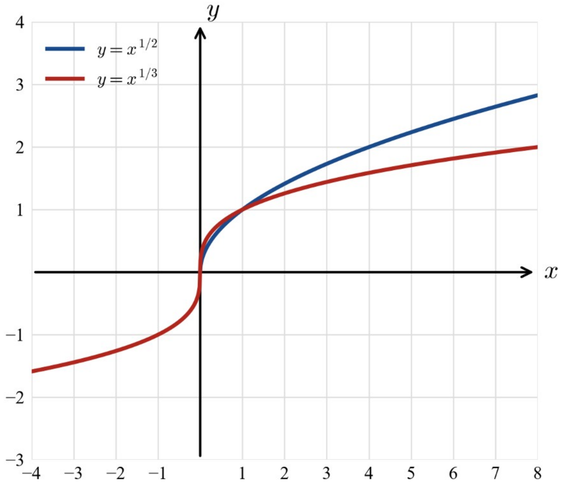
Q10. $y=x^{1/3}+2$: the cube-root graph translated up 2. Domain $\mathbb{R}$, range $\mathbb{R}$. x-intercept where $x^{1/3}=-2 \Rightarrow x=-8$, giving $(-8,0)$; y-intercept $(0,2)$; also $(8,4)$.



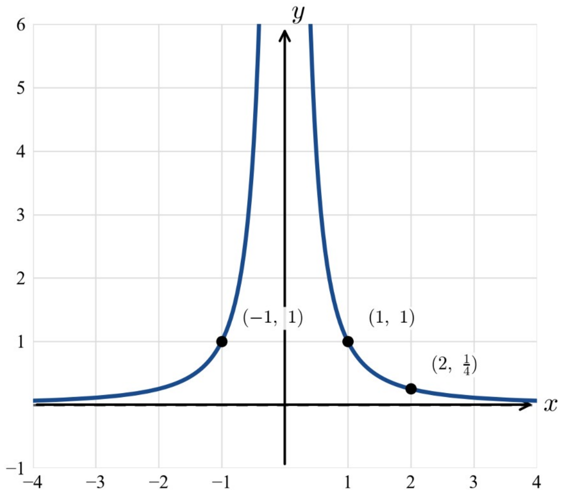
Q11. $y=\sqrt{x}-1$: the square-root graph translated down 1. Domain $\{x : x\ge 0\}$, range $\{y : y\ge -1\}$; endpoint $(0,-1)$; x-intercept $\sqrt{x}=1 \Rightarrow x=1$, giving $(1,0)$; also $(4,1),(9,2)$.



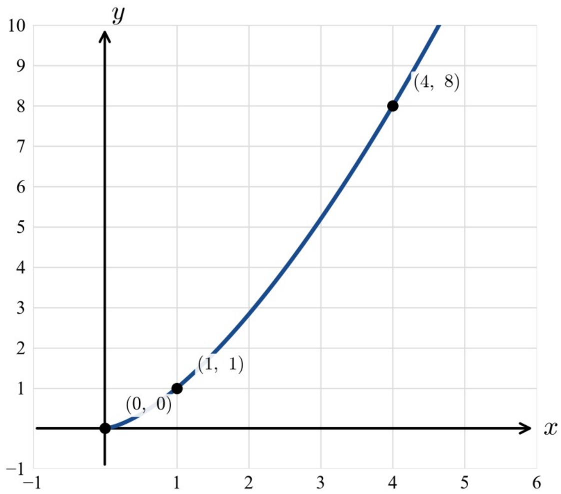
Q12. Both $y=x^{1/2}$ and $y=x^{1/3}$ pass through $(0,0)$ and $(1,1)$. For $0<x<1$ the cube-root graph lies **above** the square-root graph; for $x>1$ it lies below. The square-root graph is undefined for $x<0$, while the cube-root graph continues into the third quadrant.



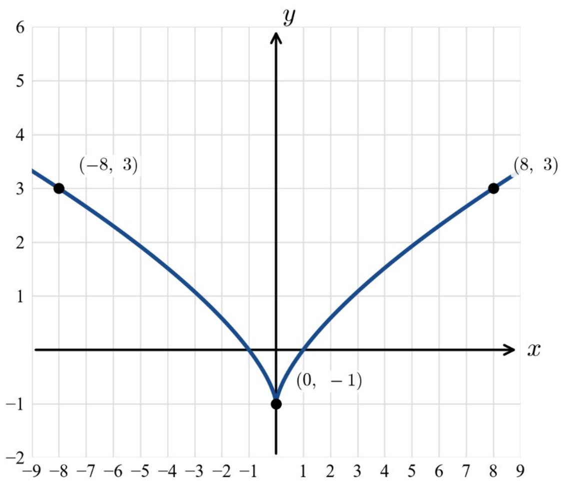
Q13. $y=x^{-2}=\frac{1}{x^2}$. Domain $\mathbb{R}\setminus\{0\}$, range $\{y:y>0\}$; asymptotes $x=0,\ y=0$; even, through $(-1,1),(1,1),\left(2,\frac{1}{4}\right)$.



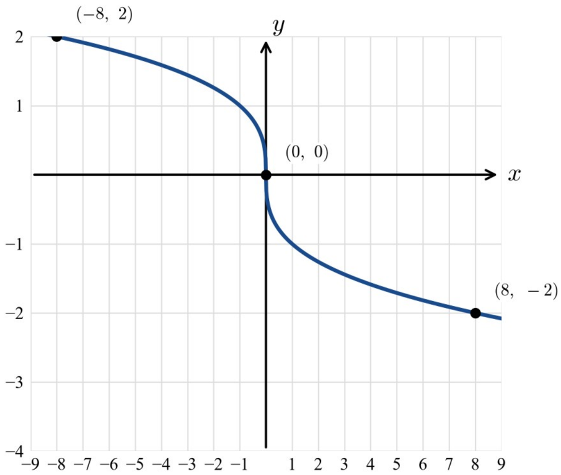
Q14. $y=x^{3/2}$ restricted to $0\le x\le 4$. Domain $[0,4]$, range $[0,8]$; endpoints $(0,0)$ and $(4,8)$, through $(1,1)$.



Q15. $y=x^{2/3}-1$: the $x^{2/3}$ graph translated down 1. Domain $\mathbb{R}$, range $\{y: y\geq -1\}$; cusp at $(0,-1)$; x-intercepts where $x^{2/3}=1 \Rightarrow x=\pm 1$, giving $(\pm 1,0)$; also $(\pm 8,3)$.



Q16. $y=-x^{1/3}$: the cube-root graph reflected in the x-axis. Domain $\mathbb{R}$, range $\mathbb{R}$; decreasing, through $(-8,2),(0,0),(8,-2)$.



Q17. $x^{2/3}=9$. Raise both sides to the power $\dfrac{3}{2}$: $x=\pm 9^{3/2}=\pm 27$. Because $p=2$ is even the function is even, so **both** signs are valid: $x=\pm 27$.

Q18. $x^{3/2}=8$. Raise both sides to the power $\dfrac{2}{3}$: $x=8^{2/3}=4$. (The domain is $x\geq 0$, so there is one solution.) $x=4$.

Q19. $y = x^{-1/2} = \frac{1}{\sqrt{x}}$. The even root requires $x \geq 0$, and $x = 0$ gives $\frac{1}{\sqrt{0}} = \frac{1}{0}$, which is undefined. Hence domain $\{x : x > 0\}$ and range $\{y : y > 0\}$.

Q20. $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$, so $(8, 4)$ lies on $y = x^{2/3}$. Since $x^{2/3}$ is even, $(-8)^{2/3} = 4$ as well, so the other point is $(-8, 4)$.

Q21. $x^{-2} = \frac{1}{9} \Rightarrow \frac{1}{x^2} = \frac{1}{9} \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.

Theory

This section draws together two ways of building new functions from old ones: **composing** them and **inverting** them.

Composite functions. The composite of f *with* g is

$$(f\circ g)(x)=f(g(x)),$$

found by applying g first, then f. For $f(g(x))$ to be defined, every output of g must be an allowable input of f:

$$\operatorname{ran}(g)\subseteq\operatorname{dom}(f).$$

- The **domain** of $f\circ g$ is $\{x\in\operatorname{dom}(g): g(x)\in\operatorname{dom}(f)\}$.
- In general $f\circ g\neq g\circ f$ — **order matters**.

Inverse functions. A function f has an **inverse function** f^{-1} **if and only if** f **is one-to-one**. The inverse *undoes* f:

$$f\left(f^{-1}(x)\right)=x \text{ and } f^{-1}\left(f(x)\right)=x.$$

To find the rule of f^{-1}: write $y=f(x)$, **swap** x **and** y, then solve for y.

- $\operatorname{dom}(f^{-1})=\operatorname{ran}(f)$ and $\operatorname{ran}(f^{-1})=\operatorname{dom}(f)$.
- The graph of $y=f^{-1}(x)$ is the **reflection of the graph of** $y=f(x)$ **in the line** $y=x$.
- If f is **many-to-one**, first **restrict its domain** to a piece on which it is one-to-one; the chosen restriction fixes which branch of the inverse you obtain (for example, $y=x^2$ needs $x\geq 0$ or $x\leq 0$).

To find an inverse, then, you (1) confirm f is one-to-one (restricting if needed), (2) swap and solve, and (3) state the domain of f^{-1} as the range of f.

Worked examples

Example 1 — scaffolded

For $f(x)=2x-3$ and $g(x)=x^2+1$, find $f(g(x))$ and $g(f(x))$, and state the domain of each.

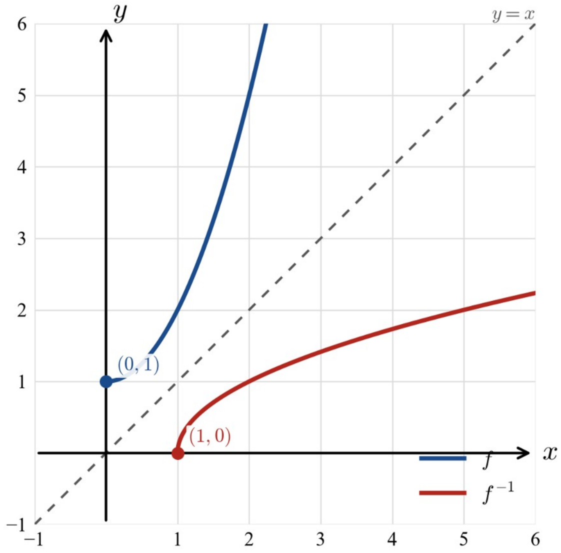
Working	Comment
$f(g(x))=2(g(x))-3=2(x^2+1)-3$	Substitute $g(x)$ into f: replace the input of f by x^2+1.
$=2x^2-1$	Expand and simplify. Domain $\mathbb{R}$.
$g(f(x))=(f(x))^2+1=(2x-3)^2+1$	Now substitute $f(x)$ into g.
$=4x^2-12x+10$	Expand. Domain $\mathbb{R}$.
$\therefore f(g(x))\neq g(f(x))$	The two composites differ — order matters.

Example 2 — scaffolded

The function $f:[0,\infty)\to\mathbb{R}$, $f(x)=x^2+1$ is one-to-one. Find f^{-1}, stating its domain and range, and sketch f and f^{-1} on one set of axes.

Working	Comment
f is one-to-one on $[0,\infty)$, so f^{-1} exists.	A restricted-domain parabola passes the horizontal-line test.
Let $y=x^2+1$. Swap: $x=y^2+1$.	Interchange x and y.

Working	Comment
$y^2=x-1 \Rightarrow y=\pm\sqrt{x-1}$; take $y=\sqrt{x-1}$.	Choose the + branch, since $\operatorname{ran}(f^{-1})=\operatorname{dom}(f)=[0,\infty)$.
$f^{-1}(x)=\sqrt{x-1}$, $\operatorname{dom}(f^{-1})=[1,\infty)$, $\operatorname{ran}(f^{-1})=[0,\infty)$.	$\operatorname{dom}(f^{-1})=\operatorname{ran}(f)=[1,\infty)$.



Example 3 — unscaffolded

For $f(x)=\sqrt{x}$ (domain $[0,\infty)$) and $g(x)=x-5$, find $f(g(x))$ and $g(f(x))$, and state the domain of each.

$f(g(x))=\sqrt{x-5}$. This requires $x-5\geq 0$, so the domain is $\{x : x\geq 5\}$.

$g(f(x))=\sqrt{x}-5$. This requires $x\geq 0$, so the domain is $\{x : x\geq 0\}$.

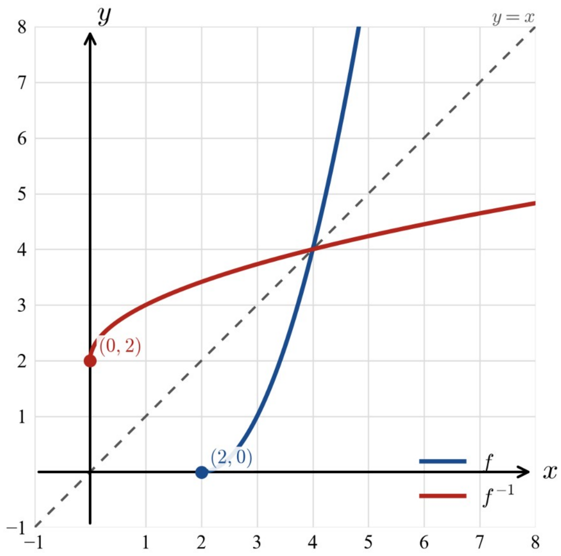
$\therefore$ the two composites have different rules **and** different domains.

Example 4 — unscaffolded

The function $f:[2,\infty)\to\mathbb{R}$, $f(x)=(x-2)^2$. Find f^{-1} and sketch f and f^{-1}.

On $[2,\infty)$, f is one-to-one. Let $y=(x-2)^2$; swapping, $x=(y-2)^2$, so $y-2=\pm\sqrt{x}$. Since $\operatorname{ran}(f^{-1})=\operatorname{dom}(f)=[2,\infty)$, take the + branch: $y=2+\sqrt{x}$.

$\therefore f^{-1}(x)=2+\sqrt{x}$, with $\operatorname{dom}(f^{-1})=[0,\infty)$ and $\operatorname{ran}(f^{-1})=[2,\infty)$.



Practice questions

Scaffolded

Q1. For $f(x)=3x+1$ and $g(x)=x^2$: (a) find $f(g(x))$; (b) find $g(f(x))$; (c) state whether $f\circ g=g\circ f$.

Q2. For $f(x)=\dfrac{x+3}{2}$: (a) explain why f^{-1} exists; (b) find the rule of f^{-1}; (c) verify that $f\left(f^{-1}(x)\right)=x$.

Q3. For $f(x)=\dfrac{1}{x}$ and $g(x)=x-1$: (a) find $f(g(x))$ and its domain; (b) find $g(f(x))$ and its domain.

Q4. For $f(x)=x^3-1$: (a) find $f^{-1}(x)$; (b) state its domain and range.

Q5. For $f(x)=\sqrt{x}$ and $g(x)=2x+1$: (a) find $f(g(x))$ and its domain; (b) find $g(f(x))$ and its domain.

Q6. The function $f:[-1,\infty)\to\mathbb{R}$, $f(x)=(x+1)^2$: (a) state why f is one-to-one; (b) find $f^{-1}(x)$ and its domain; (c) sketch f and f^{-1}.

Unscaffolded

Q7. For $f(x)=4-2x$ and $g(x)=x^2+3$, find $f(g(x))$ and $g(f(x))$.

Q8. Find the inverse of $f(x)=\dfrac{1}{x+2}$, and state its domain.

Q9. For $f(x)=\sqrt{x-3}$, find $f^{-1}(x)$, stating its domain, and sketch f and f^{-1}.

Q10. Find the inverse of $f(x)=2x^3+1$.

Q11. The function $f:(-\infty,0]\to\mathbb{R}$, $f(x)=x^2-4$. Find $f^{-1}(x)$, stating its domain and range, and sketch f and f^{-1}.

Q12. For $f(x)=x^2$ and $g(x)=\sqrt{x}$, find $f(g(x))$ and $g(f(x))$, being careful to state each domain. Explain why $g(f(x))\neq x$ for all real x.

Q13. Find the inverse of $f(x)=\dfrac{2x-1}{x+3}$, and state the domain and range of f^{-1}.

Q14. For $f(x)=\dfrac{1}{\sqrt{x}}$ and $g(x)=x-2$, find $f(g(x))$ and its domain.

Q15. For $f(x)=3-\sqrt{x+1}$, find $f^{-1}(x)$, stating its domain, and sketch f and f^{-1}.

Q16. Show that $f(x)=\dfrac{x}{x-1}$ (with $x\neq 1$) is its own inverse, i.e. $f^{-1}=f$.

Q17. For $f(x)=(x-1)^2$ and $g(x)=x+1$, find $f(g(x))$ and $g(f(x))$.

Q18. Find the inverse of $f(x)=\dfrac{2}{x-3}+1$, and state the equations of the asymptotes of f^{-1}.

Q19. The function $f:(-\infty,4]\to\mathbb{R}$, $f(x)=\sqrt{4-x}$. Find $f^{-1}(x)$, stating its domain, and sketch f and f^{-1}.

Q20. The function $f(x)=\dfrac{x}{x+1}$ has domain $(-1,\infty)$. **Show that** $f^{-1}(x)=\dfrac{x}{1-x}$, and state its domain.

Q21. The functions f and g satisfy $f(x)=2x-1$ and $f(g(x))=x^2+3$. Find $g(x)$.

Answers

Q1. (a) $3x^2+1$; (b) $9x^2+6x+1$; (c) no. **Q2.** (b) $f^{-1}(x)=2x-3$; domain $\mathbb{R}$. **Q3.** (a) $\frac{1}{x-1}$, domain $\mathbb{R}\setminus\{1\}$; (b) $\frac{1}{x}-1$, domain $\mathbb{R}\setminus\{0\}$. **Q4.** (a) $f^{-1}(x)=\sqrt[3]{x+1}$; (b) domain $\mathbb{R}$, range $\mathbb{R}$. **Q5.** (a) $\sqrt{2x+1}$, domain $\left\{x: x\geq -\frac{1}{2}\right\}$; (b) $2\sqrt{x}+1$, domain $\{x: x\geq 0\}$. **Q6.** (b) $f^{-1}(x)=\sqrt{x}-1$, domain $[0,\infty)$. **Q7.** $f(g(x))=-2x^2-2$; $g(f(x))=4x^2-16x+19$. **Q8.** $f^{-1}(x)=\frac{1}{x}-2$, domain $\mathbb{R}\setminus\{0\}$. **Q9.** $f^{-1}(x)=x^2+3$, domain $[0,\infty)$. **Q10.** $f^{-1}(x)=\sqrt[3]{\frac{x-1}{2}}$. **Q11.** $f^{-1}(x)=-\sqrt{x+4}$, domain $[-4,\infty)$, range $(-\infty,0]$. **Q12.** $f(g(x))=x$ (domain $[0,\infty)$); $g(f(x))=|x|$ (domain $\mathbb{R}$); they differ because $g(f(x))=|x|=x$ only for $x\geq 0$. **Q13.** $f^{-1}(x)=\frac{3x+1}{2-x}$; domain $\mathbb{R}\setminus\{2\}$, range $\mathbb{R}\setminus\{-3\}$. **Q14.** $f(g(x))=\frac{1}{\sqrt{x-2}}$, domain $\{x: x>2\}$. **Q15.** $f^{-1}(x)=(x-3)^2-1$, domain $\{x: x\leq 3\}$. **Q16.** Self-inverse (see solution). **Q17.** $f(g(x))=x^2$; $g(f(x))=x^2-2x+2$. **Q18.** $f^{-1}(x)=\frac{2}{x-1}+3$; asymptotes $x=1$, $y=3$. **Q19.** $f^{-1}(x)=4-x^2$, domain $[0,\infty)$. **Q20.** $f^{-1}(x)=\frac{x}{1-x}$, domain $(-\infty,1)$. **Q21.** $g(x)=\frac{x^2+4}{2}$.

Fully-worked solutions

Q1. (a) $f(g(x))=3(x^2)+1=3x^2+1$. (b) $g(f(x))=(3x+1)^2=9x^2+6x+1$. (c) Since $3x^2+1\neq 9x^2+6x+1$, $f\circ g\neq g\circ f$.

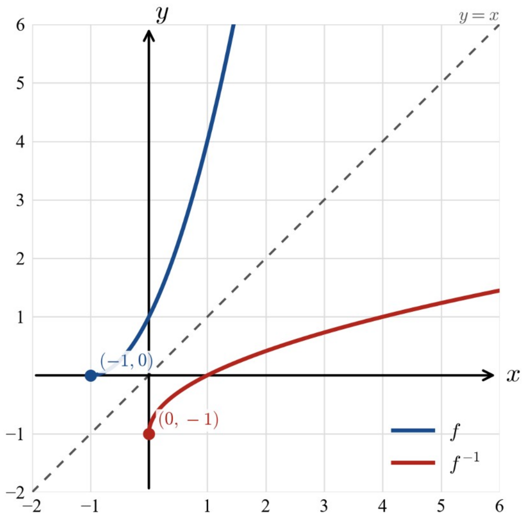
Q2. (a) f is linear (non-constant), hence one-to-one, so f^{-1} exists. (b) $y=\frac{x+3}{2}$; swap: $x=\frac{y+3}{2}\Rightarrow 2x=y+3\Rightarrow y=2x-3$, so $f^{-1}(x)=2x-3$. (c) $f(2x-3)=\frac{(2x-3)+3}{2}=\frac{2x}{2}=x$. ✓

Q3. (a) $f(g(x))=\frac{1}{x-1}$, defined for $x\neq 1$: domain $\mathbb{R}\setminus\{1\}$. (b) $g(f(x))=\frac{1}{x}-1$, defined for $x\neq 0$: domain $\mathbb{R}\setminus\{0\}$.

Q4. (a) $y=x^3-1$; swap: $x=y^3-1\Rightarrow y^3=x+1\Rightarrow y=\sqrt[3]{x+1}$, so $f^{-1}(x)=\sqrt[3]{x+1}$. (b) A cubic is one-to-one over $\mathbb{R}$, so domain and range of f^{-1} are both $\mathbb{R}$.

Q5. (a) $f(g(x))=\sqrt{2x+1}$, requiring $2x+1\geq 0$, i.e. $x\geq -\frac{1}{2}$. (b) $g(f(x))=2\sqrt{x}+1$, requiring $x\geq 0$.

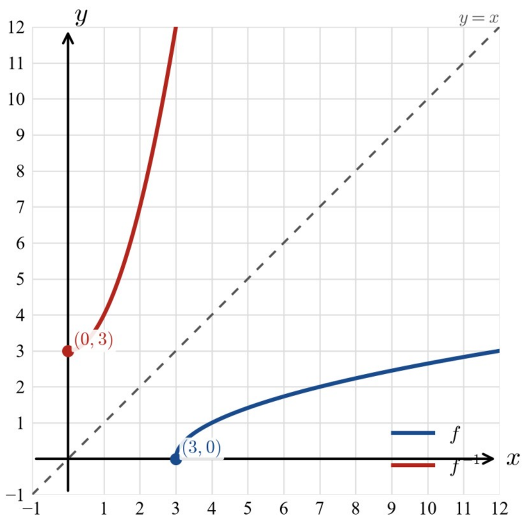
Q6. (a) On $[-1,\infty)$ the parabola $y=(x+1)^2$ is increasing, so it is one-to-one. (b) $y=(x+1)^2$; swap: $x=(y+1)^2\Rightarrow y+1=\pm\sqrt{x}$; since $\operatorname{ran}(f^{-1})=[-1,\infty)$, take $y=\sqrt{x}-1$. So $f^{-1}(x)=\sqrt{x}-1$, domain $[0,\infty)$. (c)



Q7. $f(g(x)) = 4 - 2(x^2+3) = 4 - 2x^2 - 6 = -2x^2 - 2$.
$g(f(x)) = (4-2x)^2 + 3 = 16 - 16x + 4x^2 + 3 = 4x^2 - 16x + 19$.

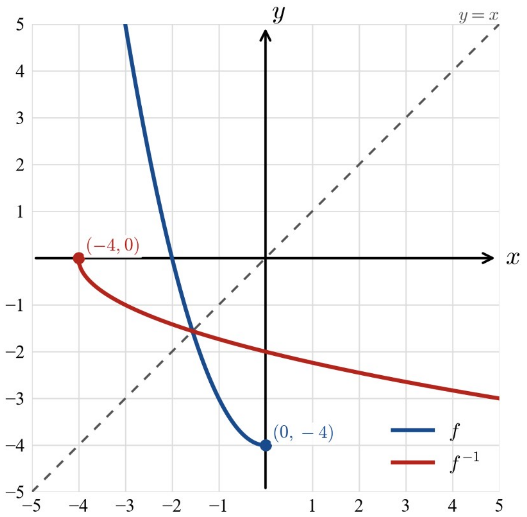
Q8. $y = \dfrac{1}{x+2}$; swap: $x = \dfrac{1}{y+2} \Rightarrow y+2 = \dfrac{1}{x} \Rightarrow y = \dfrac{1}{x} - 2$. So $f^{-1}(x) = \dfrac{1}{x} - 2$, domain $\mathbb{R} \setminus \{0\}$.

Q9. $y = \sqrt{x-3}$; swap: $x = \sqrt{y-3}$, and since $x \geq 0$, $x^2 = y - 3 \Rightarrow y = x^2 + 3$. So $f^{-1}(x) = x^2 + 3$ with domain $[0, \infty)$ (the range of f).



Q10. $y = 2x^3 + 1$; swap: $x = 2y^3 + 1 \Rightarrow y^3 = \dfrac{x-1}{2} \Rightarrow y = \sqrt[3]{\dfrac{x-1}{2}}$.

Q11. On $(-\infty, 0]$, $f(x) = x^2 - 4$ is decreasing, so one-to-one; its range is $[-4, \infty)$. Swap $x = y^2 - 4 \Rightarrow y^2 = x + 4 \Rightarrow y = \pm\sqrt{x+4}$; since $\operatorname{ran}(f^{-1}) = (-\infty, 0]$, take $y = -\sqrt{x+4}$. So $f^{-1}(x) = -\sqrt{x+4}$, domain $[-4, \infty)$, range $(-\infty, 0]$.

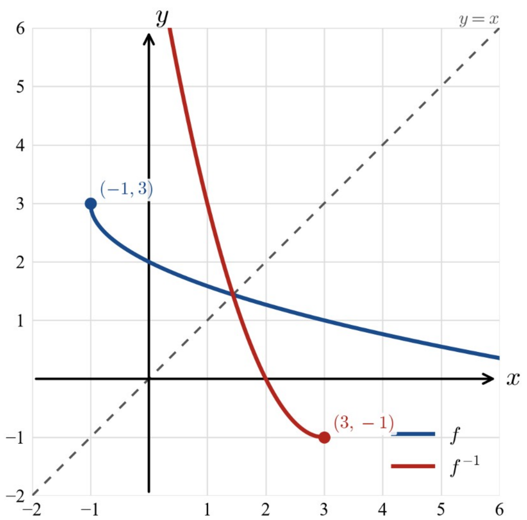


Q12. $f(g(x))=(\sqrt{x})^2=x$, with domain $[0,\infty)$ (the domain of g). $g(f(x))=\sqrt{x^2}=|x|$, with domain $\mathbb{R}$. For $x<0$, $|x|=-x\neq x$, so $g(f(x))\neq x$ there — the square root returns the non-negative root.

Q13. $y=\dfrac{2x-1}{x+3}$; swap: $x=\dfrac{2y-1}{y+3}\Rightarrow x(y+3)=2y-1\Rightarrow xy+3x=2y-1\Rightarrow y(x-2)=-1-3x\Rightarrow y=\dfrac{3x+1}{2-x}$. So $f^{-1}(x)=\dfrac{3x+1}{2-x}$; domain $\mathbb{R}\setminus\{2\}$ (the range of f), range $\mathbb{R}\setminus\{-3\}$.

Q14. $f(g(x))=\dfrac{1}{\sqrt{x-2}}$. This needs $x-2>0$ (strictly, since the denominator cannot be 0), so the domain is $\{x:x>2\}$.

Q15. $y=3-\sqrt{x+1}$; swap: $x=3-\sqrt{y+1}\Rightarrow\sqrt{y+1}=3-x\Rightarrow y+1=(3-x)^2\Rightarrow y=(x-3)^2-1$. Since $\operatorname{ran}(f)=(-\infty,3]$, $\operatorname{dom}(f^{-1})=\{x:x\leq 3\}$.

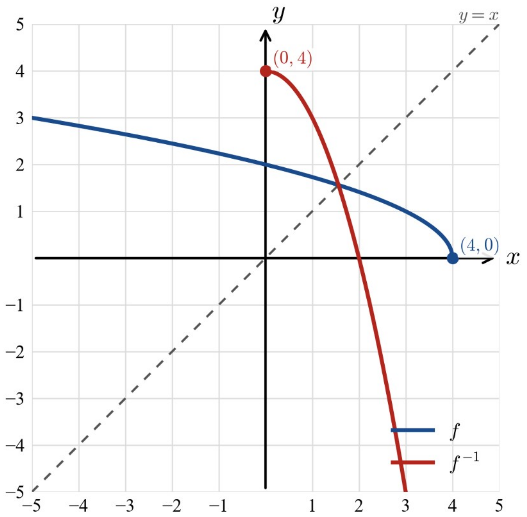


Q16. $y=\dfrac{x}{x-1}$; swap: $x=\dfrac{y}{y-1}\Rightarrow x(y-1)=y\Rightarrow xy-x=y\Rightarrow y(x-1)=x\Rightarrow y=\dfrac{x}{x-1}$. The inverse rule equals the original, so $f^{-1}=f$: f is **self-inverse**.

Q17. $f(g(x))=(g(x)-1)^2=((x+1)-1)^2=x^2$. $g(f(x))=f(x)+1=(x-1)^2+1=x^2-2x+2$.

Q18. $y=\dfrac{2}{x-3}+1$; swap: $x=\dfrac{2}{y-3}+1\Rightarrow x-1=\dfrac{2}{y-3}\Rightarrow y-3=\dfrac{2}{x-1}\Rightarrow y=\dfrac{2}{x-1}+3$. So $f^{-1}(x)=\dfrac{2}{x-1}+3$, with asymptotes $x=1$ and $y=3$ (the reflections of f's asymptotes $x=3$, $y=1$).

Q19. $y=\sqrt{4-x}$; swap: $x=\sqrt{4-y}$, and since $x\geq 0$, $x^2=4-y\Rightarrow y=4-x^2$. So $f^{-1}(x)=4-x^2$ with domain $[0,\infty)$ (the range of f).



Q20. $y=\dfrac{x}{x+1}$; swap: $x=\dfrac{y}{y+1}\Rightarrow x(y+1)=y\Rightarrow xy+x=y\Rightarrow x=y-xy=y(1-x)\Rightarrow y=\dfrac{x}{1-x}$. So $f^{-1}(x)=\dfrac{x}{1-x}$, as required. The domain of f^{-1} is the range of f: writing $f(x)=1-\dfrac{1}{x+1}$, on $(-1,\infty)$ the range is $(-\infty,1)$, so $\operatorname{dom}(f^{-1})=(-\infty,1)$.

Q21. $f(g(x))=2g(x)-1$. Setting this equal to x^2+3: $2g(x)-1=x^2+3\Rightarrow 2g(x)=x^2+4\Rightarrow g(x)=\dfrac{x^2+4}{2}$.

Theory

Given two functions f and g, we can build new functions by adding, subtracting or multiplying their outputs.

Definitions. For each x in a common domain,

$$(f+g)(x)=f(x)+g(x),(f-g)(x)=f(x)-g(x),(fg)(x)=f(x)g(x).$$

Domain of a combination. Each of these is only defined where **both** f and g are defined, so

$$dom(f+g)=dom(f-g)=dom(fg)=dom f\cap dom g.$$

For example, if $f(x)=\sqrt{x-1}$ (domain $[1,\infty)$) and $g(x)=\sqrt{5-x}$ (domain $(-\infty,5]$), then $f+g$ and fg both have domain $[1,\infty)\cap(-\infty,5]=[1,5]$.

Addition of ordinates. The graph of $y=(f+g)(x)$ can be sketched by adding the y-coordinates (**ordinates**) of f and g at each value of x. Useful landmarks:

- where $g(x)=0$, the sum equals $f(x)$ (so $y=f+g$ meets the graph of f);
- where $f(x)=0$, the sum equals $g(x)$;
- where $f(x)=-g(x)$, the sum is 0 (an x-intercept of $f+g$);
- where one function is much larger in size than the other, the sum follows the larger one.

Products. For the graph of $y=(fg)(x)$:

- $(fg)(x)=0$ wherever $f(x)=0$ **or** $g(x)=0$ — the zeros of a product are the combined zeros of its factors;
- the **sign** of the product comes from the signs of the factors: it is positive where f and g have the same sign and negative where they have opposite signs.

Worked examples

Example 1 — scaffolded

For $f(x)=x^2-1$ and $g(x)=x+1$, find the rules and domains of $f+g$, $f-g$ and fg, and evaluate $(f+g)(2)$.

Working	Comment
Both f and g are polynomials, so $dom f=dom g=\mathbb{R}$.	Polynomials are defined for all real x.
$(f+g)(x)=(x^2-1)+(x+1)=x^2+x$.	Add the rules and simplify.
$(f-g)(x)=(x^2-1)-(x+1)=x^2-x-2$.	Subtract; watch the signs.
$(fg)(x)=(x^2-1)(x+1)=x^3+x^2-x-1$.	Multiply and expand.
Each combination has domain $\mathbb{R}$ (the intersection of the two domains).	$\mathbb{R}\cap\mathbb{R}=\mathbb{R}$.
$(f+g)(2)=2^2+2=6$.	Substitute $x=2$ into the sum.

Example 2 — scaffolded

For $f(x)=\sqrt{x-1}$ and $g(x)=\sqrt{5-x}$, find the rule and domain of $f+g$, and evaluate $(f+g)(2)$.

Working	Comment
$dom f$: need $x-1\geq 0$, so $x\geq 1$, i.e. $[1,\infty)$.	A square root needs a non-negative radicand.
$dom g$: need $5-x\geq 0$, so $x\leq 5$, i.e. $(-\infty,5]$.	Likewise for g.

Working	Comment
$(f+g)(x)=\sqrt{x-1}+\sqrt{5-x}$, domain $[1,\infty)\cap(-\infty,5]=[1,5]$.	The combined domain is the intersection.
$(f+g)(2)=\sqrt{1}+\sqrt{3}=1+\sqrt{3}$.	Substitute $x=2$; keep the surd exact.

Example 3 — unscaffolded

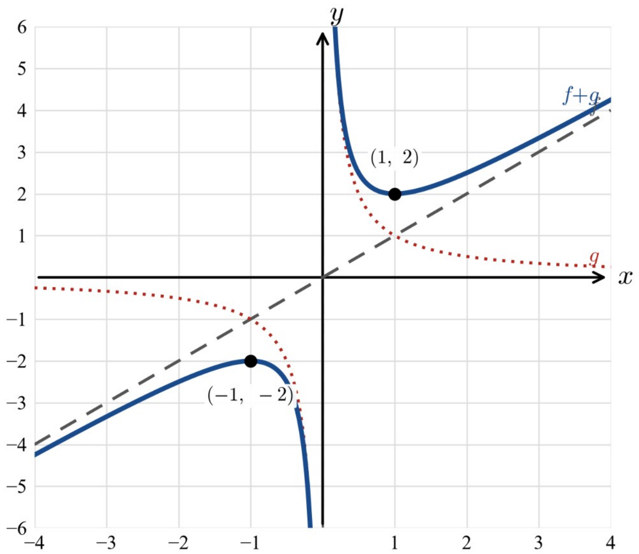
Using addition of ordinates, sketch $y=x+\frac{1}{x}$, where $f(x)=x$ and $g(x)=\frac{1}{x}$.

The domain is $\mathbb{R}\setminus\{0\}$ (the reciprocal is undefined at $x=0$).

For large $|x|$ the term $\frac{1}{x}$ is small, so the sum is close to the line $y=x$; near $x=0$ the term $\frac{1}{x}$ dominates, so the graph follows the reciprocal and has a vertical asymptote at $x=0$.

Turning points: $\frac{d}{dx}\left(x+\frac{1}{x}\right)=1-\frac{1}{x^2}=0\Rightarrow x^2=1\Rightarrow x=\pm1$, giving a local minimum $(1,2)$ and a local maximum $(-1,-2)$.

∴ the graph consists of two branches, each approaching $y=x$ away from the origin and the asymptote $x=0$ near it.



Example 4 — unscaffolded

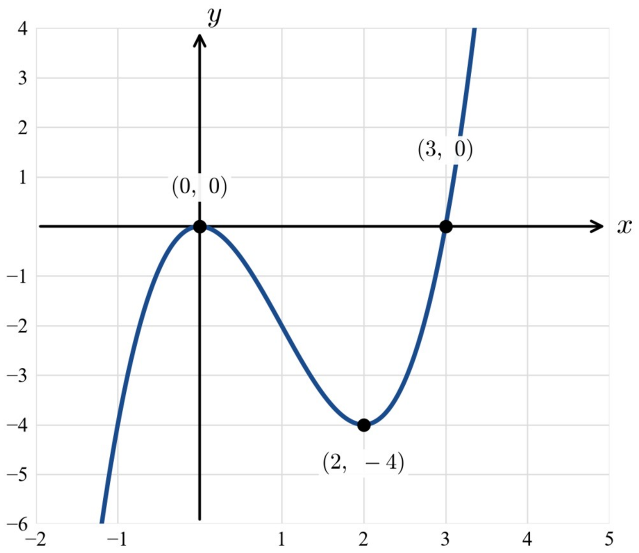
Sketch $y=(fg)(x)$ where $f(x)=x^2$ and $g(x)=x-3$.

$(fg)(x)=x^2(x-3)=x^3-3x^2$, a cubic with domain $\mathbb{R}$.

Zeros: $x^2(x-3)=0\Rightarrow x=0$ (from f, a repeated factor) or $x=3$ (from g). At $x=0$ the double factor x^2 makes the graph **touch** the axis; at $x=3$ it crosses.

Sign: for $x<0$, $x^2>0$ and $x-3<0$, so the product is negative; for $0<x<3$ still negative; for $x>3$ positive. The turning points are $(0,0)$ and $(2,-4)$.

∴ the graph rises from below, touches the axis at the origin, dips to a minimum $(2,-4)$, then rises through $(3,0)$.



Practice questions

Scaffolded

Q1. For $f(x)=x^2$ and $g(x)=3x-2$: (a) find $(f+g)(x)$; (b) find $(f-g)(x)$; (c) find $(fg)(x)$; (d) state the domain of each and evaluate $(f+g)(1)$.

Q2. For $f(x)=\sqrt{x}$ and $g(x)=\sqrt{6-x}$: (a) write the domain of f and of g; (b) find the rule and domain of $f+g$; (c) find the rule and domain of fg; (d) evaluate $(f+g)(2)$.

Q3. For $y=x-\frac{1}{x}$ (with $f(x)=x$, $g(x)=-\frac{1}{x}$): (a) state the domain; (b) find the exact x-intercepts; (c) sketch the graph by addition of ordinates; (d) describe the behaviour near $x=0$.

Q4. For $f(x)=x$ and $g(x)=\sqrt{x}$: (a) find the rule and domain of $f+g$; (b) find the rule of fg; (c) evaluate $(f+g)(4)$.

Q5. For $y=x(x-2)^2$ (the product of $f(x)=x$ and $g(x)=(x-2)^2$): (a) state the zeros and the behaviour of the graph at each; (b) state the y-intercept; (c) sketch the graph.

Q6. For $f(x)=\sqrt{x+3}$ and $g(x)=\frac{1}{x-1}$: (a) write the domain of f and of g; (b) state the domain of $f+g$; (c) state the domain of fg.

Unscaffolded

Q7. For $f(x)=x^2+1$ and $g(x)=2-x$, find $(f+g)(x)$, $(f-g)(x)$ and $(fg)(x)$, and state the domain of each.

Q8. For $f(x)=\sqrt{x-2}$ and $g(x)=\sqrt{8-x}$, find the rule and domain of $f+g$, and evaluate $(f+g)(3)$.

Q9. Sketch $y=\frac{x}{2}+\frac{2}{x}$ by addition of ordinates, labelling the turning points and stating the domain.

Q10. For $f(x)=2x+1$ and $g(x)=x^2-x$, find $(f+g)(x)$, $(f-g)(x)$ and $(fg)(x)$, and evaluate $(fg)(2)$.

Q11. Sketch $y=x(x^2-4)$, labelling all axis intercepts.

Q12. For $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{1}{x-3}$, state the domain of $f+g$.

Q13. For $f(x)=\sqrt{x}$ and $g(x)=x-6$, state the domain of $f+g$ and solve $(f+g)(x)=0$.

Q14. Sketch $y=(x-1)^2(x+3)$, labelling the axis intercepts.

Q15. For $f(x)=\sqrt{4-x}$ and $g(x)=\sqrt{x+1}$, state the domain of $f\,g$ and sketch $y=(f\,g)(x)$.

Q16. Sketch $y=-x+\frac{4}{x}$ by addition of ordinates, labelling the exact x-intercepts and stating the domain.

Q17. Sketch $y=x^3-x$ (as $f-g$ with $f(x)=x^3$, $g(x)=x$), labelling all axis intercepts.

Q18. Sketch $y=x^2(x+1)$, labelling all axis intercepts.

Q19. For $f(x)=\sqrt{x}$ and $g(x)=\sqrt{x-2}$, state the domain of $f+g$.

Q20. Let $f(x)=x^2-4$ and $g(x)=x+2$. **Show that** $(f\,g)(x)=(x-2)(x+2)^2$, and **hence** state the values of x for which $(f\,g)(x)\geq 0$.

Q21. For $f(x)=\sqrt{x}$ and $g(x)=\sqrt{x-3}$, state, with reasons, the domain of $f+g$, and explain why $f\,g$ has the same domain.

Answers

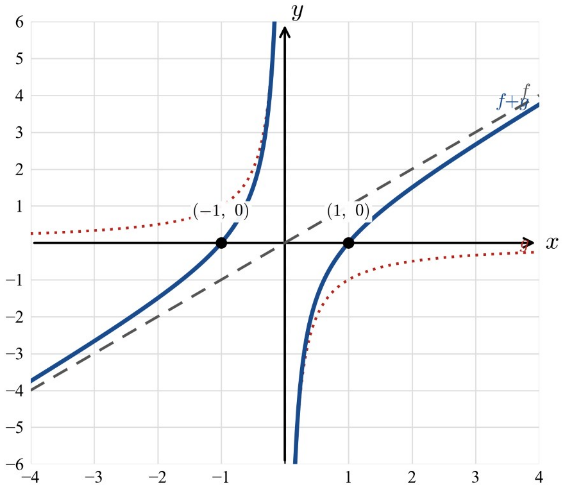
Q1. (a) x^2+3x-2; (b) x^2-3x+2; (c) $3x^3-2x^2$; (d) domain $\mathbb{R}$ for each, $(f+g)(1)=2$. **Q2.** (a) $[0,\infty)$ and $(-\infty,6]$; (b) $\sqrt{x}+\sqrt{6-x}$, domain $[0,6]$; (c) $\sqrt{x(6-x)}$, domain $[0,6]$; (d) $2+\sqrt{2}$. **Q3.** (a) $\mathbb{R}\setminus\{0\}$; (b) $(-1,0)$ and $(1,0)$; (d) as $x\to 0$ the graph follows $-1/x$ (asymptote $x=0$). **Q4.** (a) $x+\sqrt{x}$, domain $[0,\infty)$; (b) $x\sqrt{x}$; (c) 6. **Q5.** (a) $x=0$ (crosses) and $x=2$ (touches, repeated factor); (b) $(0,0)$. **Q6.** (a) $[-3,\infty)$ and $\mathbb{R}\setminus\{1\}$; (b) $[-3,1)\cup(1,\infty)$; (c) $[-3,1)\cup(1,\infty)$. **Q7.** $(f+g)(x)=x^2-x+3$; $(f-g)(x)=x^2+x-1$; $(fg)(x)=-x^3+2x^2-x+2$; domain $\mathbb{R}$ for each. **Q8.** $\sqrt{x-2}+\sqrt{8-x}$, domain $[2,8]$; $(f+g)(3)=1+\sqrt{5}$. **Q9.** Domain $\mathbb{R}\setminus\{0\}$; turning points $(2,2)$ and $(-2,-2)$. **Q10.** $(f+g)(x)=x^2+x+1$; $(f-g)(x)=-x^2+3x+1$; $(fg)(x)=2x^3-x^2-x$; $(fg)(2)=10$. **Q11.** Intercepts $(-2,0)$, $(0,0)$, $(2,0)$. **Q12.** $\mathbb{R}\setminus\{-2,3\}$. **Q13.** Domain $[0,\infty)$; $(f+g)(x)=0$ at $x=4$. **Q14.** Intercepts $(-3,0)$, $(1,0)$ (touch), $(0,3)$. **Q15.** Domain $[-1,4]$; a semicircular-shaped arch from $(-1,0)$ to $(4,0)$. **Q16.** Domain $\mathbb{R}\setminus\{0\}$; x-intercepts $(-2,0)$ and $(2,0)$. **Q17.** Intercepts $(-1,0)$, $(0,0)$, $(1,0)$. **Q18.** Intercepts $(-1,0)$ and $(0,0)$ (touch). **Q19.** $[2,\infty)$. **Q20.** $(fg)(x)=(x-2)(x+2)^2$; $(fg)(x)\ge 0$ for $x=-2$ or $x\ge 2$. **Q21.** Domain $[3,\infty)$.

Fully-worked solutions

Q1. $f(x)=x^2$, $g(x)=3x-2$, both with domain $\mathbb{R}$. (a) $(f+g)(x)=x^2+3x-2$. (b) $(f-g)(x)=x^2-(3x-2)=x^2-3x+2$. (c) $(fg)(x)=x^2(3x-2)=3x^3-2x^2$. (d) each has domain $\mathbb{R}$; $(f+g)(1)=1+3-2=2$.

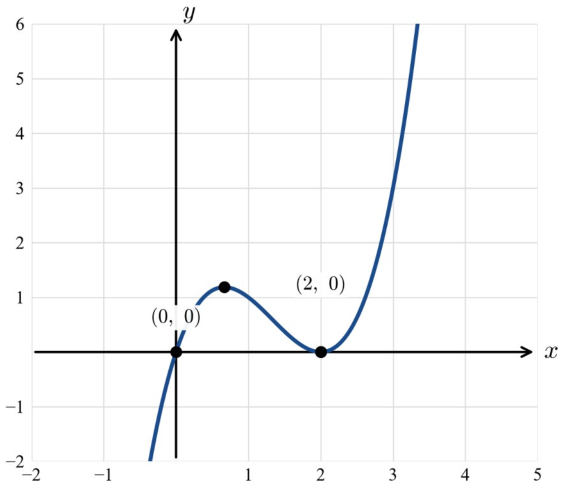
Q2. (a) $dom\, f=[0,\infty)$, $dom\, g=(-\infty,6]$. (b) $(f+g)(x)=\sqrt{x}+\sqrt{6-x}$, domain $[0,\infty)\cap(-\infty,6]=[0,6]$. (c) $(fg)(x)=\sqrt{x}\sqrt{6-x}=\sqrt{x(6-x)}$, domain $[0,6]$. (d) $(f+g)(2)=\sqrt{2}+\sqrt{4}=2+\sqrt{2}$.

Q3. $y=x-\dfrac{1}{x}$. (a) Domain $\mathbb{R}\setminus\{0\}$. (b) $x-\dfrac{1}{x}=0\Rightarrow x^2=1\Rightarrow x=\pm 1$, giving $(-1,0)$ and $(1,0)$. (c) See graph. (d) Near $x=0$ the $-\dfrac{1}{x}$ term dominates, so there is a vertical asymptote $x=0$; for large $|x|$ the graph approaches the line $y=x$.



Q4. (a) $(f+g)(x)=x+\sqrt{x}$; domain $\mathbb{R}\cap[0,\infty)=[0,\infty)$. (b) $(fg)(x)=x\sqrt{x}=x^{3/2}$. (c) $(f+g)(4)=4+\sqrt{4}=4+2=6$.

Q5. $y=x(x-2)^2$. (a) Zeros $x=0$ (single factor — the graph crosses) and $x=2$ (repeated factor $(x-2)^2$ — the graph touches). (b) y-intercept $(0,0)$. (c) See graph; the local turning point at $(2,0)$ sits on the axis.

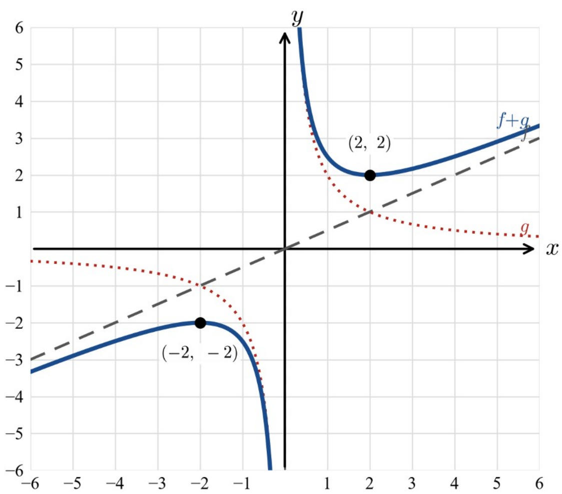


Q6. (a) $dom\, f=[-3,\infty)$ (need $x+3\geq 0$); $dom\, g=\mathbb{R}\setminus\{1\}$ (need $x\neq 1$). (b) $dom(f+g)=[-3,\infty)\cap(\mathbb{R}\setminus\{1\})=[-3,1)\cup(1,\infty)$. (c) $dom(f\,g)$ is the same intersection, $[-3,1)\cup(1,\infty)$.

Q7. $(f+g)(x)=(x^2+1)+(2-x)=x^2-x+3$; $(f-g)(x)=(x^2+1)-(2-x)=x^2+x-1$; $(f\,g)(x)=(x^2+1)(2-x)=-x^3+2x^2-x+2$. Both functions are defined on $\mathbb{R}$, so every combination has domain $\mathbb{R}$.

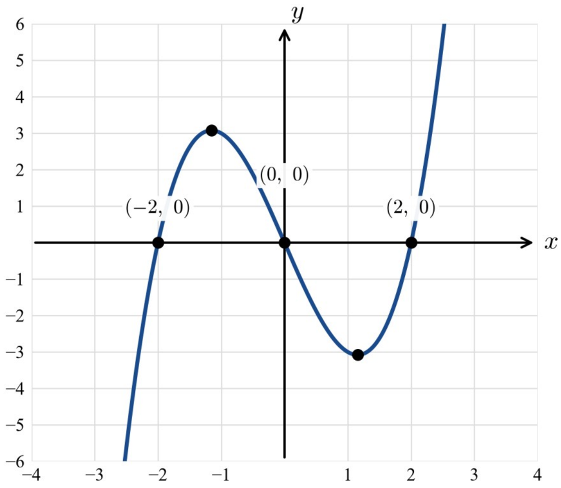
Q8. $dom\, f=[2,\infty)$, $dom\, g=(-\infty,8]$, so $(f+g)(x)=\sqrt{x-2}+\sqrt{8-x}$ has domain $[2,8]$. $(f+g)(3)=\sqrt{1}+\sqrt{5}=1+\sqrt{5}$.

Q9. $y=\frac{x}{2}+\frac{2}{x}$, domain $\mathbb{R}\setminus\{0\}$. Turning points: $\frac{1}{2}-\frac{2}{x^2}=0\Rightarrow x^2=4\Rightarrow x=\pm 2$, giving a local minimum $(2,2)$ and a local maximum $(-2,-2)$. Away from the origin the graph approaches the line $y=\frac{x}{2}$; near $x=0$ it follows $\frac{2}{x}$ with asymptote $x=0$.



Q10. $(f+g)(x)=(2x+1)+(x^2-x)=x^2+x+1$; $(f-g)(x)=(2x+1)-(x^2-x)=-x^2+3x+1$; $(f\,g)(x)=(2x+1)(x^2-x)=2x^3-x^2-x$; $(f\,g)(2)=2(8)-4-2=10$.

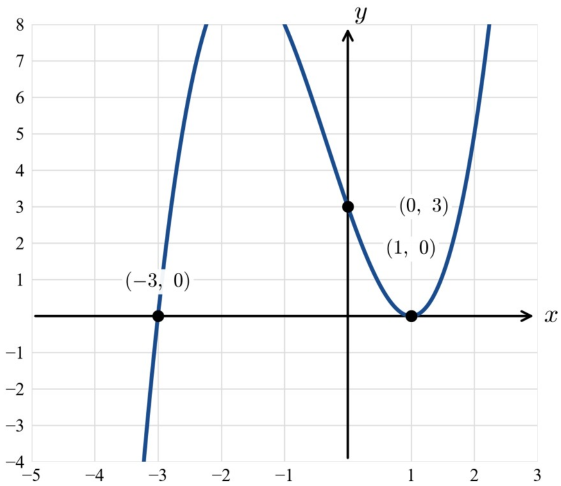
Q11. $y=x(x^2-4)=x(x-2)(x+2)$. Zeros $x=-2,0,2$, giving $(-2,0)$, $(0,0)$, $(2,0)$; the leading term x^3 gives the usual positive-cubic end behaviour.



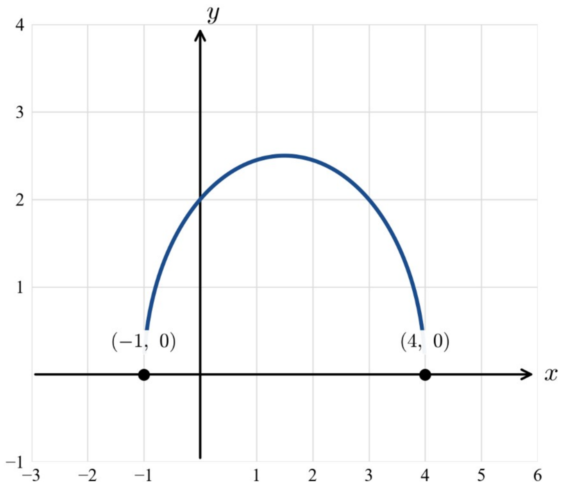
Q12. $dom\, f=\mathbb{R}\setminus\{-2\}$, $dom\, g=\mathbb{R}\setminus\{3\}$, so $dom(f+g)=\mathbb{R}\setminus\{-2,3\}$.

Q13. $dom\, f=[0,\infty)$, $dom\, g=\mathbb{R}$, so $dom(f+g)=[0,\infty)$. Solving $\sqrt{x}+x-6=0\Rightarrow\sqrt{x}=6-x$; squaring gives $x=(6-x)^2=x^2-12x+36$, i.e. $x^2-13x+36=0=(x-4)(x-9)$. Testing in the original equation, $x=4$ works ($2+4-6=0$) but $x=9$ does not ($3+9-6=6\neq0$). $\therefore x=4$.

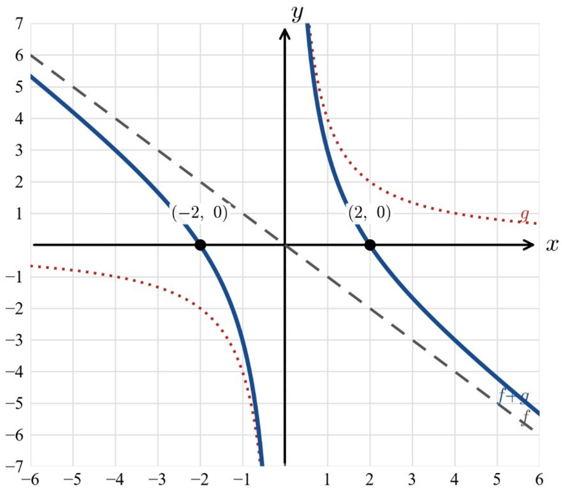
Q14. $y=(x-1)^2(x+3)$. Zeros $x=1$ (repeated — touch) and $x=-3$ (cross); y-intercept $y=(0-1)^2(0+3)=3$, giving $(0,3)$.



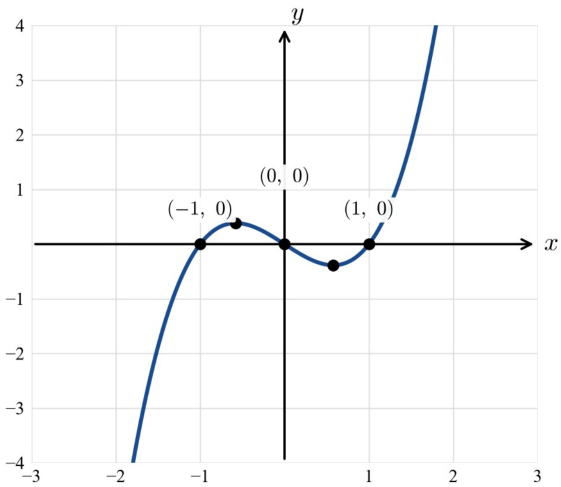
Q15. $dom\, f=(-\infty,4]$, $dom\, g=[-1,\infty)$, so $(fg)(x)=\sqrt{4-x}\sqrt{x+1}=\sqrt{(4-x)(x+1)}$ has domain $[-1,4]$. The graph is an arch from $(-1,0)$ up to a maximum and back down to $(4,0)$.



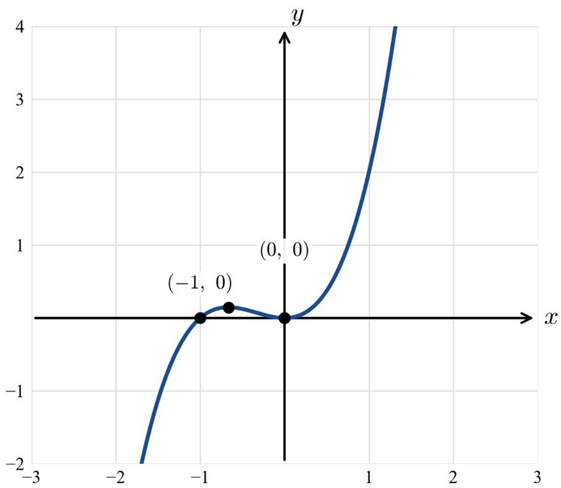
Q16. $y=-x+\dfrac{4}{x}$, domain $\mathbb{R}\setminus\{0\}$. x-intercepts: $-x+\dfrac{4}{x}=0\Rightarrow x^2=4\Rightarrow x=\pm 2$, giving $(-2,0)$ and $(2,0)$. Away from 0 the graph approaches $y=-x$; near 0 it follows $\dfrac{4}{x}$ with asymptote $x=0$.



Q17. $y=x^3-x=x(x-1)(x+1)$. Zeros $x=-1,0,1$, giving $(-1,0)$, $(0,0)$, $(1,0)$.



Q18. $y=x^2(x+1)$. Zeros $x=0$ (repeated — touch) and $x=-1$ (cross), giving $(0,0)$ and $(-1,0)$.



Q19. $dom\,f=[0,\infty)$, $dom\,g=[2,\infty)$, so $dom(f+g)=[0,\infty)\cap[2,\infty)=[2,\infty)$.

Q20. $(fg)(x)=(x^2-4)(x+2)=(x-2)(x+2)(x+2)=(x-2)(x+2)^2$, as required. Since $(x+2)^2\geq 0$ for all x, the sign of the product is the sign of $(x-2)$, except at $x=-2$ where the product is 0. Hence $(fg)(x)\geq 0$ when $x-2\geq 0$, i.e. $x\geq 2$, together with the isolated point $x=-2$. $\therefore (fg)(x)\geq 0$ for $x=-2$ or $x\geq 2$.

Q21. $f(x)=\sqrt{x}$ needs $x\geq 0$ (domain $[0,\infty)$) and $g(x)=\sqrt{x-3}$ needs $x\geq 3$ (domain $[3,\infty)$). The sum is defined only where both are, so $dom(f+g)=[0,\infty)\cap[3,\infty)=[3,\infty)$. The product fg is defined on exactly the same intersection of domains, so it too has domain $[3,\infty)$.

Theory

Function notation lets us state general properties of a function precisely. If f is a function, then $f(\text{expression})$ means **substitute that whole expression** for x in the rule. For example, if $f(x)=x^2+1$ then $f(x+2)=(x+2)^2+1$ and $f(2a)=(2a)^2+1=4a^2+1$.

Identities. An **identity** is a statement that is true for **every** value of the variable(s) in the domain — not just for some. To **prove** an identity, simplify each side (using algebra, index laws or logarithm laws) until they are visibly equal. To **disprove** a proposed identity, it is enough to give a single **counterexample** — one set of values for which the two sides differ.

Behaviour of the standard families. Each family satisfies its own identities:

- **Linear** $f(x)=mx+c$: in general $f(a+b)\neq f(a)+f(b)$ and $f(2x)\neq 2f(x)$ (the constant c is counted too many times) — unless $c=0$.
- **Power** $f(x)=x^2$: $f(xy)=(xy)^2=x^2y^2=f(x)f(y)$, but $f(x+y)\neq f(x)+f(y)$.
- **Square root** $f(x)=\sqrt{x}$ (for $x\geq 0$): $f(xy)=\sqrt{xy}=\sqrt{x}\sqrt{y}=f(x)f(y)$.
- **Exponential** $f(x)=a^x$: $f(x+y)=a^{x+y}=a^x a^y=f(x)f(y)$, and $f(x-y)=\dfrac{f(x)}{f(y)}$ (index laws).
- **Logarithm** $f(x)=\log_a x$ (for $x>0$): $f(xy)=f(x)+f(y)$, $f\left(\dfrac{x}{y}\right)=f(x)-f(y)$, and $f(x^n)=nf(x)$ (logarithm laws).

Even and odd functions. Replacing x by $-x$ tests symmetry: f is **even** if $f(-x)=f(x)$ (graph symmetric about the y-axis), and **odd** if $f(-x)=-f(x)$ (graph has rotational symmetry about the origin). A function may be neither.

Functional equations. Sometimes a function is described by a relation it satisfies (for example $f(x+1)-f(x)=3$). Substituting the general form of f and comparing coefficients determines the unknowns.

Worked examples

Example 1 — scaffolded

Let $f(x)=x^2+1$. Determine whether $f(x+2)=f(x)+f(2)$ is an identity.

Working	Comment
$f(x+2)=(x+2)^2+1=x^2+4x+5$.	Substitute $x+2$ for x and expand.
$f(x)+f(2)=(x^2+1)+(2^2+1)=x^2+6$.	Evaluate $f(2)=5$ and add.
$x^2+4x+5=x^2+6\Rightarrow 4x=1\Rightarrow x=\dfrac{1}{4}$.	The two sides are equal only at this single value.
Since it fails for (e.g.) $x=2$: $f(4)=17$ but $f(2)+f(2)=10$, the statement is **not** an identity.	One counterexample is enough to disprove it.

Example 2 — scaffolded

Let $f(x)=2^x$. Show that $f(x+y)=f(x)f(y)$ for all real x and y.

Working	Comment
$f(x+y)=2^{x+y}$.	Substitute $x+y$.

Working	Comment
$=2^x \times 2^y$	Index law $a^{m+n}=a^m a^n$.
$=f(x)f(y)$.	Rewrite each factor in function notation.
This holds for all real x, y, so it **is** an identity.	The index law is valid for all real indices.

Example 3 — unscaffolded

Let $f(x)=\dfrac{1}{x}$, $x\neq 0$. Determine whether $f(x)+f(y)=f(x+y)$, and simplify $f(x)+f(y)$.

$f(x)+f(y)=\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{y+x}{xy}=\dfrac{x+y}{xy}$, whereas $f(x+y)=\dfrac{1}{x+y}$.

Testing $x=2$, $y=1$: $f(2)+f(1)=\dfrac{3}{2}$ but $f(3)=\dfrac{1}{3}$.

$\therefore f(x)+f(y)=\dfrac{x+y}{xy}$, and $f(x)+f(y)=f(x+y)$ is **not** an identity.

Example 4 — unscaffolded

Show that $f(x)=x^3-x$ is an odd function.

$f(-x)=(-x)^3-(-x)=-x^3+x=-(x^3-x)=-f(x)$.

$\therefore f(-x)=-f(x)$ for all x, so f is odd.

Practice questions

Scaffolded

Q1. Let $f(x)=3x-2$. (a) Find $f(2x)$ and $2f(x)$. (b) Is $f(2x)=2f(x)$ an identity? Justify your answer. (c) Find $f(a+b)$ and $f(a)+f(b)$, and state whether they are equal.

Q2. Let $f(x)=x^2$. (a) Show that $f(xy)=f(x)f(y)$. (b) Determine, with a counterexample, whether $f(x+y)=f(x)+f(y)$.

Q3. Let $f(x)=\sqrt{x}$, $x\geq 0$. (a) Show that $f(xy)=f(x)f(y)$ for $x, y\geq 0$. (b) Is $f(x+y)=f(x)+f(y)$ an identity? Justify.

Q4. Let $f(x)=e^x$. (a) Show that $f(x+y)=f(x)f(y)$. (b) Show that $f(x-y)=\dfrac{f(x)}{f(y)}$.

Q5. Let $f(x)=\log_2 x$, $x>0$. (a) Show that $f(xy)=f(x)+f(y)$. (b) Show that $f(x^n)=nf(x)$.

Q6. Using $f(-x)$, classify each function as even, odd or neither: (a) $f(x)=x^4-3x^2$; (b) $g(x)=\dfrac{x}{x^2+1}$; (c) $h(x)=x^2+x$.

Unscaffolded

Q7. For $f(x)=x^2-2$, find and simplify $f(f(x))$.

Q8. For $f(x)=2x+1$, show that $f(a+b)+f(a-b)=2f(a)$.

Q9. For $f(x)=\dfrac{1}{x}$ ($x\neq 0$), simplify $f(x)+f\left(\dfrac{1}{x}\right)$.

Q10. For $f(x)=\frac{x}{x+1}$ ($x \neq 0, -1$), show that $f(x)+f\left(\frac{1}{x}\right)=1$.

Q11. Determine whether $f(x)=x^5-4x^3+x$ is even, odd or neither.

Q12. For $f(x)=\log_e x$ ($x>0$), show that $f(x^2)=2f(x)$ and $f\left(\frac{x}{y}\right)=f(x)-f(y)$.

Q13. For $f(x)=3^x$, simplify $\frac{f(x+1)}{f(x)}$.

Q14. A function $f(x)=ax+b$ satisfies $f(x+1)-f(x)=3$ for all x. Find the value of a.

Q15. Classify each as even, odd or neither: (a) $f(x)=2x^4+x^2-1$; (b) $g(x)=x^3+x^2$.

Q16. For $f(x)=x^2$, show that $f(x+h)-f(x)=2xh+h^2$.

Q17. For $f(x)=\frac{1}{x}$, show that $\frac{f(x+h)-f(x)}{h}=-\frac{1}{x(x+h)}$.

Q18. For $f(x)=\sqrt{x}$, determine whether $f(a)+f(b)=f(a+b)$, giving a counterexample if it fails.

Q19. For $f(x)=e^x$, show that $f(x)f(-x)=1$.

Q20. For $f(x)=2^x$, show that $f(x)f(-x)=1$ is an identity.

Q21. For $f(x)=e^x$: (a) show that $f(x+y)=f(x)f(y)$; (b) **hence** show that $f(2x)=(f(x))^2$.

Answers

Q1. (a) $f(2x)=6x-2$, $2f(x)=6x-4$. (b) No. (c) $f(a+b)=3(a+b)-2$, $f(a)+f(b)=3(a+b)-4$; not equal. **Q2.** (a) identity. (b) not an identity (e.g. $x=2, y=1$: $9\neq 5$). **Q3.** (a) identity. (b) not an identity. **Q4.** both identities. **Q5.** both identities. **Q6.** (a) even; (b) odd; (c) neither. **Q7.** x^4-4x^2+2. **Q8.** identity ($=4a+2=2f(a)$). **Q9.** $x+\frac{1}{x}$. **Q10.** identity ($=1$). **Q11.** odd. **Q12.** both identities. **Q13.** 3. **Q14.** $a=3$. **Q15.** (a) even; (b) neither. **Q16.** identity. **Q17.** identity. **Q18.** not an identity (e.g. $a=b=1$: $2\neq\sqrt{2}$). **Q19.** identity. **Q20.** identity. **Q21.** (b) $f(2x)=e^{2x}=(e^x)^2=(f(x))^2$.

Fully-worked solutions

Q1. $f(x)=3x-2$. (a) $f(2x)=3(2x)-2=6x-2$; $2f(x)=2(3x-2)=6x-4$. (b) $6x-2=6x-4$ would require $-2=-4$, which is false, so $f(2x)=2f(x)$ is **not** an identity. (c) $f(a+b)=3(a+b)-2=3a+3b-2$; $f(a)+f(b)=(3a-2)+(3b-2)=3a+3b-4$. These are not equal (they differ by 2).

Q2. $f(x)=x^2$. (a) $f(xy)=(xy)^2=x^2y^2=f(x)f(y)$. ✓ (b) $f(x+y)=(x+y)^2=x^2+2xy+y^2$, while $f(x)+f(y)=x^2+y^2$. Taking $x=2, y=1$: $f(3)=9$ but $f(2)+f(1)=5$. ∴ not an identity.

Q3. $f(x)=\sqrt{x}$. (a) For $x, y\geq 0$, $f(xy)=\sqrt{xy}=\sqrt{x}\sqrt{y}=f(x)f(y)$. ✓ (b) $f(x+y)=\sqrt{x+y}$, $f(x)+f(y)=\sqrt{x}+\sqrt{y}$. At $x=y=1$: $\sqrt{2}\neq 2$. ∴ not an identity.

Q4. $f(x)=e^x$. (a) $f(x+y)=e^{x+y}=e^x e^y=f(x)f(y)$. ✓ (b) $f(x-y)=e^{x-y}=\frac{e^x}{e^y}=\frac{f(x)}{f(y)}$. ✓

Q5. $f(x)=\log_2 x$. (a) $f(xy)=\log_2(xy)=\log_2 x+\log_2 y=f(x)+f(y)$. ✓ (b) $f(x^n)=\log_2(x^n)=n\log_2 x=nf(x)$. ✓

Q6. (a) $f(-x)=(-x)^4-3(-x)^2=x^4-3x^2=f(x)$: **even**. (b) $g(-x)=\frac{-x}{(-x)^2+1}=\frac{-x}{x^2+1}=-g(x)$: **odd**. (c) $h(-x)=(-x)^2+(-x)=x^2-x$, which is neither $h(x)$ nor $-h(x)$: **neither**.

Q7. $f(f(x))=f(x^2-2)=(x^2-2)^2-2=x^4-4x^2+4-2=x^4-4x^2+2$.

Q8. $f(a+b)+f(a-b)=(2(a+b)+1)+(2(a-b)+1)=2a+2b+1+2a-2b+1=4a+2=2(2a+1)=2f(a)$. ✓

Q9. $f\left(\frac{1}{x}\right)=\frac{1}{1/x}=x$, so $f(x)+f\left(\frac{1}{x}\right)=\frac{1}{x}+x$.

Q10. $f\left(\frac{1}{x}\right)=\frac{1/x}{1/x+1}=\frac{1/x}{(1+x)/x}=\frac{1}{x+1}$. Then $f(x)+f\left(\frac{1}{x}\right)=\frac{x}{x+1}+\frac{1}{x+1}=\frac{x+1}{x+1}=1$. ✓

Q11. $f(-x)=(-x)^5-4(-x)^3+(-x)=-x^5+4x^3-x=-(x^5-4x^3+x)=-f(x)$: **odd**.

Q12. $f(x^2)=\log_e(x^2)=2\log_e x=2f(x)$ (valid since $x>0$); and $f\left(\frac{x}{y}\right)=\log_e\frac{x}{y}=\log_e x-\log_e y=f(x)-f(y)$. ✓

Q13. $\frac{f(x+1)}{f(x)}=\frac{3^{x+1}}{3^x}=3^{(x+1)-x}=3^1=3$.

Q14. $f(x+1)-f(x)=(a(x+1)+b)-(ax+b)=a$. Setting $a=3$ gives the required relation, so $a=3$.

Q15. (a) $f(-x)=2(-x)^4+(-x)^2-1=2x^4+x^2-1=f(x)$: **even**. (b) $g(-x)=(-x)^3+(-x)^2=-x^3+x^2$, which is neither $g(x)$ nor $-g(x)$: **neither**.

Q16. $f(x+h)-f(x)=(x+h)^2-x^2=x^2+2xh+h^2-x^2=2xh+h^2$. ✓

Q17. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}=\dfrac{\frac{x-(x+h)}{x(x+h)}}{h}=\dfrac{-h}{x(x+h)h}=-\dfrac{1}{x(x+h)}$. ✓

Q18. $f(a)+f(b)=\sqrt{a}+\sqrt{b}$ and $f(a+b)=\sqrt{a+b}$. At $a=b=1$: $\sqrt{1}+\sqrt{1}=2$ but $\sqrt{2}\approx 1.41$. ∴ not an identity.

Q19. $f(x)f(-x)=e^x\cdot e^{-x}=e^{x-x}=e^0=1$. ✓

Q20. $f(x)f(-x)=2^x\cdot 2^{-x}=2^{x-x}=2^0=1$, true for all x: an identity. ✓

Q21. (a) $f(x+y)=e^{x+y}=e^x e^y=f(x)f(y)$. (b) Putting $y=x$: $f(2x)=f(x+x)=f(x)f(x)=(f(x))^2$. ✓

Theory

A **family of functions** is a set of functions that share the same form but differ through one or more **parameters** (constants such as a, k, m). For example, $y = a x^2$ (for $a \in \mathbb{R} \setminus \{0\}$) is the family of all parabolas with vertex at the origin, and $y = m x + 3$ is the family of all lines through $(0, 3)$.

Describing the effect of a parameter. Changing a parameter transforms the graph. In $y = \frac{k}{x} + 2$, the parameter k dilates the hyperbola (and reflects it when $k < 0$), while **every** member has the same asymptotes $x = 0$ and $y = 2$. Identifying the features that do **not** depend on the parameter is often the key to a question.

Finding a particular member. To find the member of a family passing through a given point, substitute the coordinates and solve for the parameter.

Solving literal equations. A *literal equation* contains pronumerals in place of some coefficients. Solve it exactly as an ordinary equation — collect the unknown (x) on one side and factorise:

$$a x + b = c x + d \Rightarrow x(a - c) = d - b \Rightarrow x = \frac{d - b}{a - c}, a \neq c.$$

Always state the restriction that makes the solution valid (here $a \neq c$; for fractions, no denominator may be zero).

Number of solutions and a parameter. For a quadratic $a x^2 + b x + c = 0$, the **discriminant** $\Delta = b^2 - 4 a c$ decides the number of real solutions: two if $\Delta > 0$, one if $\Delta = 0$, none if $\Delta < 0$. Setting up Δ as an expression in a parameter, then solving the resulting inequality, tells you which members of a family meet a curve (or a line) twice, once (a tangent), or not at all.

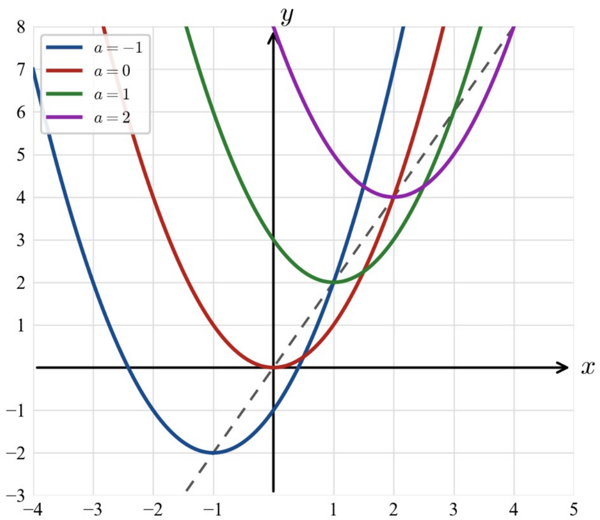
Worked examples

Example 1 — scaffolded

Consider the family of parabolas $y = (x - a)^2 + 2a$, where a is a real parameter.

Working	Comment
In turning-point form the vertex is at $x = a$, $y = 2a$: turning point $(a, 2a)$.	Compare with $y = (x - h)^2 + k$: here $h = a$, $k = 2a$.
As a varies, the point $(a, 2a)$ satisfies $y = 2x$. So **every turning point lies on the line** $y = 2x$.	The turning point traces out this line (its *locus*).
Through $(1, 5)$: $(1 - a)^2 + 2a = 5 \Rightarrow a^2 + 1 = 5 \Rightarrow a^2 = 4$, so $a = 2$ or $a = -2$.	Substitute the point and solve for the parameter.
$\therefore$ two members pass through $(1, 5)$: $y = (x - 2)^2 + 4$ and $y = (x + 2)^2 - 4$.	Each value of a gives a member of the family.

$y = 2x$



Example 2 — scaffolded

Solve each literal equation for x, stating any restriction.

Working	Comment
(a) $ax-3=bx+5 \Rightarrow ax-bx=8 \Rightarrow x(a-b)=8$, so $x=\dfrac{8}{a-b}$, $a \neq b$.	Collect x-terms; divide by the coefficient $a-b$ (nonzero).
(b) $\dfrac{x}{a}+\dfrac{x}{b}=1 \Rightarrow x\left(\dfrac{b+a}{ab}\right)=1$, so $x=\dfrac{ab}{a+b}$, $a,b \neq 0, a+b \neq 0$.	Add the fractions, then divide. State every restriction.

Example 3 — unscaffolded

For what values of k does $x^2-kx+4=0$ have (i) two solutions, (ii) one solution, (iii) no solutions?

The discriminant is $\Delta=(-k)^2-4(1)(4)=k^2-16$.

(i) Two solutions: $\Delta>0 \Rightarrow k^2>16 \Rightarrow k<-4$ or $k>4$.

(ii) One solution: $\Delta=0 \Rightarrow k^2=16 \Rightarrow k=-4$ or $k=4$.

(iii) No solutions: $\Delta<0 \Rightarrow k^2<16 \Rightarrow -4<k<4$.

$\therefore$ the number of solutions changes as k passes through ± 4.

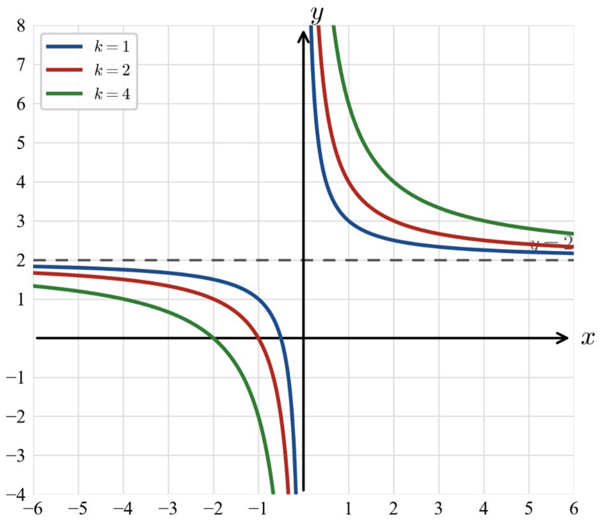
Example 4 — unscaffolded

Consider the family $y=\dfrac{k}{x}+2$, $k \neq 0$. Describe the effect of k, and find the member passing through $(2,5)$.

Every member is the hyperbola $y=\dfrac{1}{x}$ dilated by factor k from the x-axis (reflected in the x-axis when $k<0$), then translated up 2. All members share the asymptotes $x=0$ and $y=2$.

Through $(2,5)$: $\dfrac{k}{2}+2=5 \Rightarrow \dfrac{k}{2}=3 \Rightarrow k=6$.

$\therefore$ the member is $y=\dfrac{6}{x}+2$.



Practice questions

Scaffolded

Q1. For the family $y=a(x-2)^2-1$: (a) state the point that every member passes through, whatever the value of a; (b) find a if the graph passes through $(0,3)$; (c) describe the effect of a on the graph.

Q2. Solve $cx+d=e$ for x, stating a restriction.

Q3. Solve $\frac{a}{x}=b$ for x, stating restrictions.

Q4. For $x^2+bx+9=0$, find the value(s) of b for which there is exactly one solution.

Q5. For the family $y=mx+2$: (a) state the point common to every member; (b) find m if the line passes through $(3,8)$.

Q6. Solve $px-q=r(x+1)$ for x, stating a restriction.

Unscaffolded

Solve each literal equation for x (Q7–Q11, Q16, Q19), stating restrictions; for the family/parameter questions, answer as asked and give exact values.

Q7. $3x+a=b-x$

Q8. $\frac{x}{2}+a=\frac{x}{3}+b$

Q9. $ax+b=cx+d$

Q10. $\frac{1}{x}+\frac{1}{a}=\frac{1}{b}$

Q11. $\frac{k}{x-1}=3$

Q12. For $x^2-6x+c=0$, find the values of c for which the equation has (i) two solutions, (ii) one solution, (iii) no solutions.

Q13. For $kx^2+4x+1=0$, find the value of k ($k\neq 0$) for which there is exactly one solution.

Q14. The family $y=(x-h)^2-h$. Show that every turning point lies on the line $y=-x$, and find the exact value(s) of h for which the graph passes through $(2,1)$.

Q15. Find the value(s) of m for which the line $y=mx-1$ is a tangent to the parabola $y=x^2$.

Q16. $a^2x-a=x+1$

Q17. For the family $y=\frac{2}{x-a}+1$, find the value of a for which the graph passes through $(3,3)$.

Q18. Find the values of k for which $x^2=k(x-1)$ has (i) two solutions, (ii) one solution, (iii) no solutions.

Q19. $\frac{x-a}{b}+\frac{x-b}{a}=2$

Q20. The family $y=a(x+1)(x-3)$. State the two points common to every member, and find a if the graph passes through $(1,-8)$.

Q21. Find the exact values of m for which the line $y=mx+1$ meets the parabola $y=x^2+3$ at two distinct points.

Answers

Q1. (a) $(2,-1)$; (b) $a=1$; (c) a dilates it from the x-axis (reflected if $a<0$); vertex stays $(2,-1)$. **Q2.** $x=\dfrac{e-d}{c}$, $c\neq 0$. **Q3.** $x=\dfrac{a}{b}$, $b\neq 0$ (and $x\neq 0 \Rightarrow a\neq 0$). **Q4.** $b=\pm 6$. **Q5.** (a) $(0,2)$; (b) $m=2$. **Q6.** $x=\dfrac{q+r}{p-r}$, $p\neq r$. **Q7.** $x=\dfrac{b-a}{4}$. **Q8.** $x=6(b-a)$. **Q9.** $x=\dfrac{d-b}{a-c}$, $a\neq c$. **Q10.** $x=\dfrac{ab}{a-b}$, $a\neq b$ (and $a,b\neq 0$). **Q11.** $x=\dfrac{k+3}{3}$ ($=1+k/3$), $k\neq 0$. **Q12.** (i) $c<9$; (ii) $c=9$; (iii) $c>9$. **Q13.** $k=4$. **Q14.** Turning point $(h,-h)$ lies on $y=-x$; passes through $(2,1)$ when $h=\dfrac{5\pm\sqrt{13}}{2}$. **Q15.** $m=\pm 2$. **Q16.** $x=\dfrac{1}{a-1}$, $a\neq\pm 1$. **Q17.** $a=2$. **Q18.** (i) $k<0$ or $k>4$; (ii) $k=0$ or $k=4$; (iii) $0<k<4$. **Q19.** $x=a+b$ ($a+b\neq 0$). **Q20.** Common points $(-1,0)$ and $(3,0)$; $a=2$. **Q21.** $m<-2\sqrt{2}$ or $m>2\sqrt{2}$.

Fully-worked solutions

Q1. $y=a(x-2)^2-1$. (a) When $x=2$, $y=a(0)-1=-1$ for every a, so all members pass through $(2,-1)$. (b) $a(0-2)^2-1=3\Rightarrow 4a=4\Rightarrow a=1$. (c) a is a dilation of factor a from the x-axis (a reflection as well when $a<0$); the vertex $(2,-1)$ is unchanged.

Q2. $cx+d=e\Rightarrow cx=e-d\Rightarrow x=\dfrac{e-d}{c}$, provided $c\neq 0$.

Q3. $\dfrac{a}{x}=b\Rightarrow a=bx\Rightarrow x=\dfrac{a}{b}$, provided $b\neq 0$ (and $x\neq 0$, so $a\neq 0$).

Q4. One solution $\Rightarrow \Delta=0$: $b^2-4(1)(9)=0\Rightarrow b^2=36\Rightarrow b=\pm 6$.

Q5. (a) When $x=0$, $y=2$ for every m, so all members pass through $(0,2)$. (b) $3m+2=8\Rightarrow 3m=6\Rightarrow m=2$.

Q6. $px-q=r(x+1)=rx+r\Rightarrow px-rx=q+r\Rightarrow x(p-r)=q+r\Rightarrow x=\dfrac{q+r}{p-r}$, provided $p\neq r$.

Q7. $3x+a=b-x\Rightarrow 4x=b-a\Rightarrow x=\dfrac{b-a}{4}$.

Q8. $\dfrac{x}{2}+a=\dfrac{x}{3}+b\Rightarrow \dfrac{x}{2}-\dfrac{x}{3}=b-a\Rightarrow \dfrac{x}{6}=b-a\Rightarrow x=6(b-a)$.

Q9. $ax+b=cx+d\Rightarrow x(a-c)=d-b\Rightarrow x=\dfrac{d-b}{a-c}$, provided $a\neq c$.

Q10. $\dfrac{1}{x}+\dfrac{1}{a}=\dfrac{1}{b}\Rightarrow \dfrac{1}{x}=\dfrac{1}{b}-\dfrac{1}{a}=\dfrac{a-b}{ab}\Rightarrow x=\dfrac{ab}{a-b}$, provided $a\neq b$ (and $a,b\neq 0$).

Q11. $\dfrac{k}{x-1}=3\Rightarrow x-1=\dfrac{k}{3}\Rightarrow x=1+\dfrac{k}{3}=\dfrac{k+3}{3}$, provided $k\neq 0$ (else there is no solution).

Q12. $\Delta=(-6)^2-4c=36-4c$. (i) Two: $36-4c>0\Rightarrow c<9$. (ii) One: $c=9$. (iii) None: $c>9$.

Q13. For a quadratic ($k\neq 0$), one solution $\Rightarrow \Delta=0$: $4^2-4k=16-4k=0\Rightarrow k=4$.

Q14. Vertex of $y=(x-h)^2-h$ is $(h,-h)$, which satisfies $y=-x$, so every turning point lies on $y=-x$. Through $(2,1)$: $(2-h)^2-h=1\Rightarrow 4-4h+h^2-h=1\Rightarrow h^2-5h+3=0\Rightarrow h=\dfrac{5\pm\sqrt{25-12}}{2}=\dfrac{5\pm\sqrt{13}}{2}$.

Q15. Set $mx-1=x^2\Rightarrow x^2-mx+1=0$. Tangent $\Rightarrow \Delta=0$: $m^2-4=0\Rightarrow m=\pm 2$.

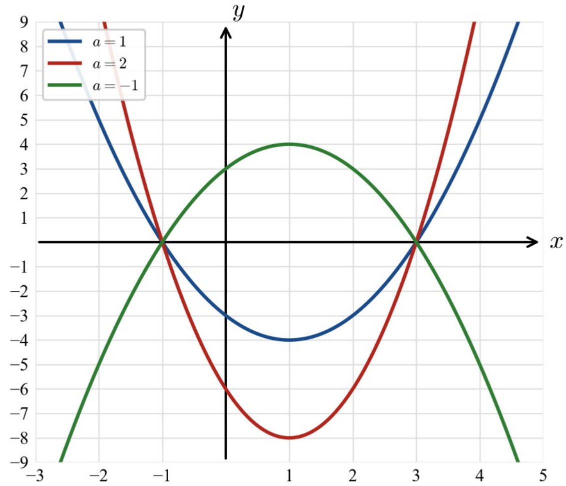
Q16. $a^2x - a = x + 1 \Rightarrow x(a^2 - 1) = 1 + a \Rightarrow x = \dfrac{1+a}{(a-1)(a+1)} = \dfrac{1}{a-1}$, provided $a \neq \pm 1$.

Q17. $\dfrac{2}{3-a} + 1 = 3 \Rightarrow \dfrac{2}{3-a} = 2 \Rightarrow 3 - a = 1 \Rightarrow a = 2$.

Q18. $x^2 = k(x-1) \Rightarrow x^2 - kx + k = 0$, $\Delta = k^2 - 4k = k(k-4)$. (i) Two: $k(k-4) > 0 \Rightarrow k < 0$ or $k > 4$. (ii) One: $k = 0$ or $k = 4$. (iii) None: $0 < k < 4$.

Q19. Multiply by ab: $a(x-a) + b(x-b) = 2ab \Rightarrow ax - a^2 + bx - b^2 = 2ab \Rightarrow x(a+b) = a^2 + 2ab + b^2 = (a+b)^2$. So $x = a + b$, provided $a + b \neq 0$.

Q20. $y = a(x+1)(x-3)$ has x-intercepts where $x + 1 = 0$ or $x - 3 = 0$, i.e. $(-1, 0)$ and $(3, 0)$, for every a. Through $(1, -8)$: $a(2)(-2) = -8 \Rightarrow -4a = -8 \Rightarrow a = 2$.



Q21. Set $mx + 1 = x^2 + 3 \Rightarrow x^2 - mx + 2 = 0$. Two distinct points $\Rightarrow \Delta > 0$: $m^2 - 8 > 0 \Rightarrow m^2 > 8 \Rightarrow m < -2\sqrt{2}$ or $m > 2\sqrt{2}$.

Topic 1 Test A — Technology-active

One bound reference and an approved CAS calculator are permitted. Reading time 5 minutes; writing time 50 minutes. Total 50 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

Let $f:[0,\infty)\to\mathbb{R}$, $f(x)=x^2-1$.

a. Find the rule for f^{-1} and state its domain. 3 marks

b. On the one set of axes, sketch the graphs of f and f^{-1}, showing the line $y=x$. 2 marks

c. Find the exact coordinates of the point where $f(x)=f^{-1}(x)$. 2 marks

Question 2 (7 marks)

Consider $p(x)=x^3-2x^2-5x+6$.

a. Show that $x=1$ is a solution of $p(x)=0$. 1 mark

b. Hence factorise $p(x)$ fully. 2 marks

c. Sketch the graph of $y=p(x)$, labelling all axis intercepts. 2 marks

d. Hence solve the inequality $p(x)\geq 0$. 2 marks

Question 3 (7 marks)

a. Solve $2e^{2x}-5e^{x}+2=0$ for x, giving exact values. 3 marks

b. For $g(x)=\log_e(x-2)+1$, find the rule for g^{-1} and state its range. 3 marks

c. Write down the equation of the horizontal asymptote of $y=g^{-1}(x)$. 1 mark

Question 4 (8 marks)

The function $f(x)=2\sin(2x)-1$ has domain $[0,2\pi]$.

a. State the amplitude, the period and the range of f. 3 marks

b. Solve $f(x)=0$ over the domain, giving exact values. 3 marks

c. Sketch the graph of $y=f(x)$. 2 marks

Question 5 (7 marks)

The graph of $y=\frac{1}{x^2}$ is transformed by a dilation of factor 2 from the x-axis, then a reflection in the x-axis, then a translation of 3 units to the right and 4 units up.

a. Write down the rule of the transformed graph. 2 marks

b. State the equations of the asymptotes and find the exact axis intercepts. 3 marks

c. State the domain and range. 2 marks

Question 6 (7 marks)

The line $y=mx+2$ meets the parabola $y=x^2+3x+4$.

a. Show that the x-coordinates of the intersection points satisfy $x^2+(3-m)x+2=0$. 1 mark

b. Find the exact values of m for which the line is a tangent to the parabola. 3 marks

c. For the larger value of m, find the exact coordinates of the point of tangency. 3 marks

Question 7 (7 marks)

Let $f(x)=\sqrt{x}$ and $g(x)=2x-1$.

a. Find the rule for $f(g(x))$ and state its maximal domain. 3 marks

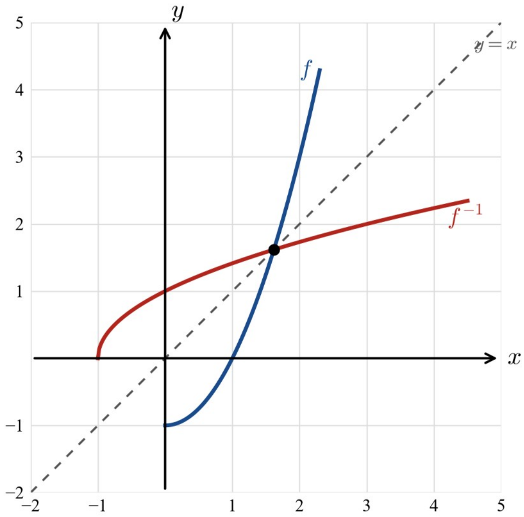
b. Find the rule for $g(f(x))$. 2 marks

c. Solve $f(g(x))=3$. 2 marks

End of Topic 1 Test A

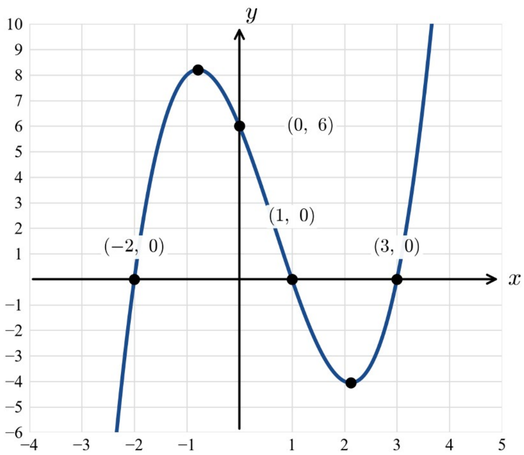
Test A — solutions

Q1. (a) Let $y=x^2-1$; swap: $x=y^2-1\Rightarrow y=\sqrt{x+1}$ (positive branch, since $ran\,f^{-1}=dom\,f=[0,\infty)$). So $f^{-1}(x)=\sqrt{x+1}$, domain $[-1,\infty)$ $(=ran\,f)$. (b) The graphs are reflections in $y=x$:



(c) f is increasing, so $f(x)=f^{-1}(x)$ on $y=x$: solve $x^2-1=x\Rightarrow x^2-x-1=0\Rightarrow x=\dfrac{1\pm\sqrt{5}}{2}$. Only $x=\dfrac{1+\sqrt{5}}{2}(\ge 0)$ is in the domain, giving $\left(\dfrac{1+\sqrt{5}}{2},\dfrac{1+\sqrt{5}}{2}\right)$.

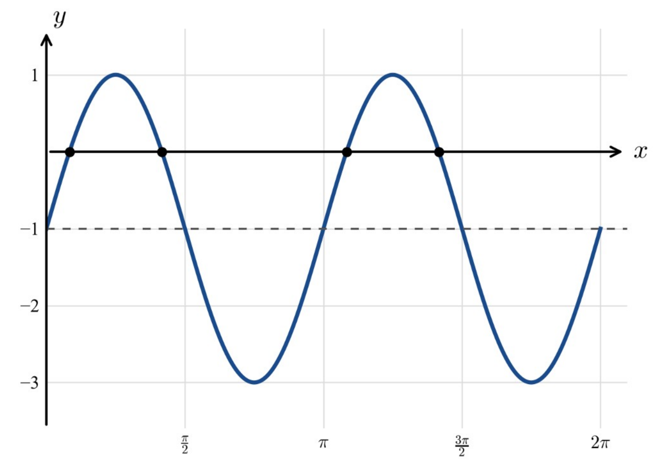
Q2. (a) $p(1)=1-2-5+6=0$, so $x=1$ is a solution. (b) $p(x)=(x-1)(x^2-x-6)=(x-1)(x-3)(x+2)$. (c) x-intercepts $(-2,0),(1,0),(3,0)$; y-intercept $(0,6)$.



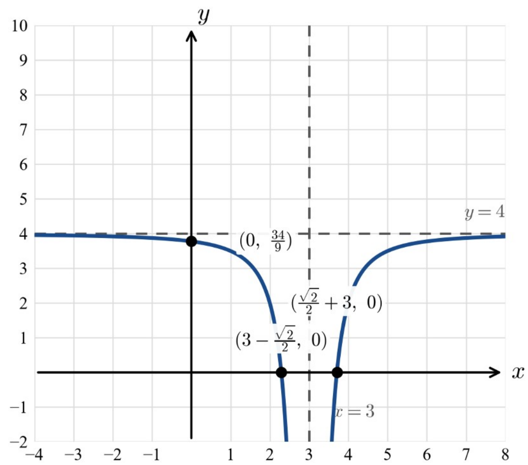
(d) From the sketch, $p(x)\ge 0$ for $x\in[-2,1]\cup[3,\infty)$.

Q3. (a) Let $u=e^x$: $2u^2-5u+2=0\Rightarrow(2u-1)(u-2)=0\Rightarrow u=1/2$ or $u=2$. So $e^x=1/2$ or $e^x=2$, giving $x=-\log_e 2$ or $x=\log_e 2$. (b) $y=\log_e(x-2)+1\Rightarrow x=\log_e(y-2)+1\Rightarrow y-2=e^{x-1}\Rightarrow g^{-1}(x)=e^{x-1}+2$. Range: $\{y:y>2\}$ $(=dom\,g)$. (c) Horizontal asymptote $y=2$.

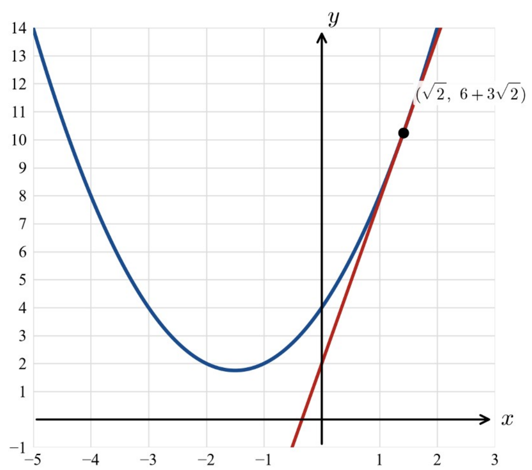
Q4. (a) Amplitude 2; period $\dfrac{2\pi}{2}=\pi$; range $[-3,1]$. (b) $2\sin(2x)-1=0\Rightarrow\sin(2x)=1/2$. With $2x\in[0,4\pi]$: $2x=\dfrac{\pi}{6},\dfrac{5\pi}{6},\dfrac{13\pi}{6},\dfrac{17\pi}{6}$, so $x=\dfrac{\pi}{12},\dfrac{5\pi}{12},\dfrac{13\pi}{12},\dfrac{17\pi}{12}$. (c)



Q5. (a) $y=-\dfrac{2}{(x-3)^2}+4$. (b) Asymptotes $x=3$ and $y=4$. y-intercept: $x=0\Rightarrow y=-\dfrac{2}{9}+4=\dfrac{34}{9}$, giving $\left(0,\dfrac{34}{9}\right)$. x-intercepts: $\dfrac{2}{(x-3)^2}=4\Rightarrow(x-3)^2=1/2\Rightarrow x=3\pm\dfrac{\sqrt{2}}{2}$. (c) Domain $\mathbb{R}\setminus\{3\}$, range $\{y:y<4\}$.



Q6. (a) At intersection $mx+2=x^2+3x+4\Rightarrow x^2+3x+4-mx-2=0\Rightarrow x^2+(3-m)x+2=0$. (b) Tangent $\Rightarrow$ discriminant $=0$: $(3-m)^2-8=0\Rightarrow 3-m=\pm2\sqrt{2}\Rightarrow m=3\mp2\sqrt{2}$, i.e. $m=3-2\sqrt{2}$ or $m=3+2\sqrt{2}$. (c) For $m=3+2\sqrt{2}$: the repeated root is $x=-\dfrac{3-m}{2}=\dfrac{2\sqrt{2}}{2}=\sqrt{2}$; then $y=(\sqrt{2})^2+3\sqrt{2}+4=6+3\sqrt{2}$. Point $(\sqrt{2},6+3\sqrt{2})$.



Q7. (a) $f(g(x))=\sqrt{2x-1}$; maximal domain $2x-1\ge0$, i.e. $\left\{x:x\ge\dfrac{1}{2}\right\}$. (b) $g(f(x))=2\sqrt{x}-1$. (c) $\sqrt{2x-1}=3\Rightarrow 2x-1=9\Rightarrow x=5$.

Topic 1 Test B — Technology-free

No calculator and no notes are permitted. Writing time 45 minutes. Total 40 marks. All numerical answers must be exact. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

a. Sketch the graph of $y=-\frac{1}{(x+2)^2}+1$, labelling the asymptotes and the exact axis intercepts. 4 marks

b. State the domain and range. 1 mark

c. Describe the sequence of transformations that maps $y=\frac{1}{x^2}$ onto this graph. 2 marks

Question 2 (6 marks)

a. Solve $e^{2x}-4e^{x}+3=0$ for x. 3 marks

b. Solve $\log_e(x)+\log_e(x-3)=\log_e 4$ for x. 3 marks

Question 3 (6 marks)

a. Solve $2\cos(x)+\sqrt{3}=0$ over $[0,2\pi]$. 3 marks

b. Find the exact value of $\sin\left(\frac{2\pi}{3}\right)+\cos\left(\frac{5\pi}{6}\right)$. 3 marks

Question 4 (7 marks)

Let $p(x)=2x^3+3x^2-8x+3$.

a. Show that $x=1$ is a solution of $p(x)=0$. 1 mark

b. Hence factorise $p(x)$ fully. 3 marks

c. State the x-intercepts of $y=p(x)$ and sketch the graph. 3 marks

Question 5 (7 marks)

Let $f(x)=\frac{x-1}{x+2}$.

a. Find the rule for $f^{-1}(x)$. 3 marks

b. State the domain of f^{-1}. 1 mark

c. Solve $f(x)=2$. 3 marks

Question 6 (7 marks)

Consider the simultaneous equations $x+ay=1$ and $ax+y=1$, where a is a real constant.

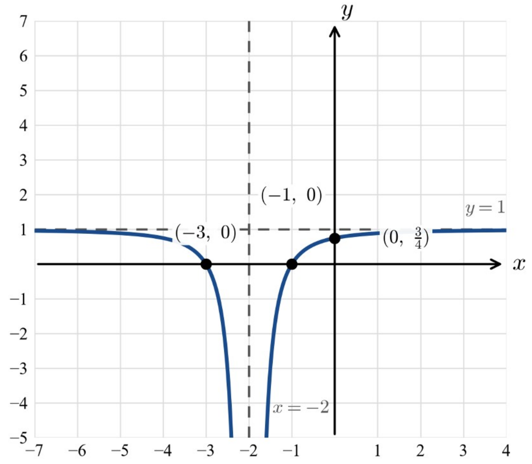
a. Solve for x and y in terms of a, stating the values of a for which your solution is valid. 4 marks

b. Describe the solution set when $a=1$ and when $a=-1$, interpreting each geometrically. 3 marks

End of Topic 1 Test B

Test B — solutions

Q1. (a) Base graph $y=\frac{1}{x^2}$, with $a=-1, h=-2, k=1$. Asymptotes $x=-2$, $y=1$. y-intercept: $x=0 \Rightarrow y=-1/4+1=3/4$, giving $(0, 3/4)$. x-intercepts: $(x+2)^2=1 \Rightarrow x=-1$ or $x=-3$.



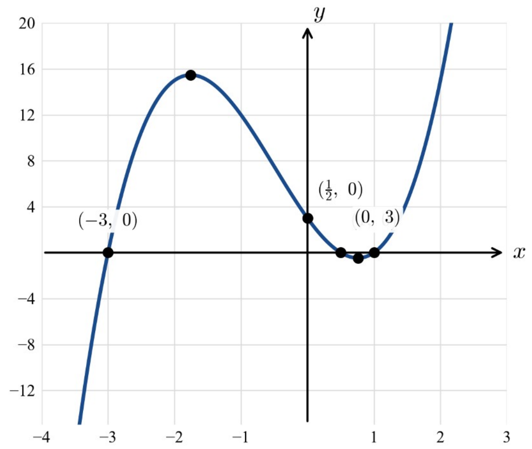
(b) Domain $\mathbb{R} \setminus \{-2\}$, range $\{y : y<1\}$.

(c) A reflection in the x-axis, then a translation of 2 units to the left and 1 unit up.

Q2. (a) Let $u=e^x$: $u^2-4u+3=0 \Rightarrow (u-1)(u-3)=0 \Rightarrow u=1$ or $u=3$, so $x=0$ or $x=\log_e 3$. (b) $\log_e(x(x-3))=\log_e 4 \Rightarrow x^2-3x=4 \Rightarrow x^2-3x-4=0 \Rightarrow (x-4)(x+1)=0$. Need $x>3$ (domain of the logs), so $x=4$ ($x=-1$ rejected).

Q3. (a) $\cos x=-\frac{\sqrt{3}}{2}$; over $[0, 2\pi]$ (2nd and 3rd quadrants) $x=\frac{5\pi}{6}, \frac{7\pi}{6}$. (b) $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$, so the sum is 0.

Q4. (a) $p(1)=2+3-8+3=0$. (b) $p(x)=(x-1)(2x^2+5x-3)=(x-1)(2x-1)(x+3)$. (c) x-intercepts $x=-3, 1/2, 1$; y-intercept $(0, 3)$.



Q5. (a) $y=\frac{x-1}{x+2} \Rightarrow y(x+2)=x-1 \Rightarrow x(y-1)=-1-2y \Rightarrow x=\frac{2y+1}{1-y}$. So $f^{-1}(x)=\frac{2x+1}{1-x}$. (b) $dom\, f^{-1}=ran\, f=\mathbb{R} \setminus \{1\}$ (the horizontal asymptote of f is $y=1$). (c) $\frac{x-1}{x+2}=2 \Rightarrow x-1=2(x+2) \Rightarrow x-1=2x+4 \Rightarrow x=-5$.

Q6. (a) Subtracting: $(x+a y)-(a x+y)=0 \Rightarrow (1-a)x-(1-a)y=0 \Rightarrow (1-a)(x-y)=0$. For $a \neq 1$, $x=y$; substituting into $x+a y=1$ gives $x(1+a)=1$, so $x=y=\frac{1}{a+1}$ (valid for $a \neq -1$ and $a \neq 1$). (b) When $a=1$ both equations are $x+y=1$: infinitely many solutions — the two lines are coincident. When $a=-1$ the equations are $x-y=1$ and $-x+y=1$, i.e. $x-y=1$ and $x-y=-1$: no solution — the lines are parallel and distinct.