

Mathematical Methods Units 3 & 4

Chapter 6 — Circular functions

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Chapter 6 Circular functions — 6A Radian measure

Theory

Angles can be measured in **degrees** or in **radians**. Radian measure is the standard unit for Units 3 & 4 and is essential for calculus with circular functions.

The radian. One **radian** is the angle subtended at the centre of a circle by an arc equal in length to the radius. Since the full circumference is $2\pi r$, a complete revolution is 2π radians. Hence

$$2\pi \text{ radians} = 360^\circ, \therefore \pi \text{ radians} = 180^\circ.$$

Converting.

- **Degrees to radians:** multiply by $\frac{\pi}{180}$.
- **Radians to degrees:** multiply by $\frac{180}{\pi}$.

Common angles.

Degrees	30°	45°	60°	90°	120°	135°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

Arc length and sector area. For a sector of radius r subtending an angle θ in **radians** at the centre:

$$\text{arc length } s = r\theta, \text{ sector area } A = \frac{1}{2}r^2\theta.$$

Coterminal angles. Adding or subtracting 2π (or 360°) gives an angle with the same position. To express an angle in $[0, 2\pi)$, add or subtract multiples of 2π .

Worked examples

Example 1 — scaffolded

Convert 135° to radians, giving an exact answer.

Working

$$\text{Multiply by } \frac{\pi}{180}: 135^\circ = 135 \times \frac{\pi}{180}.$$

$$= \frac{135\pi}{180} = \frac{3\pi}{4}.$$

$$\therefore 135^\circ = \frac{3\pi}{4} \text{ radians.}$$

Comment

Degrees $\rightarrow$ radians uses the factor $\frac{\pi}{180}$.

Simplify the fraction (135 and 180 share a factor of 45).

Leave the answer exact, in terms of π .

Example 2 — scaffolded

Convert $\frac{5\pi}{6}$ radians to degrees.

Working

$$\text{Multiply by } \frac{180}{\pi}: \frac{5\pi}{6} \times \frac{180}{\pi}.$$

Comment

Radians $\rightarrow$ degrees uses the factor $\frac{180}{\pi}$.

$$= \frac{5 \times 180}{6} = \frac{900}{6} = 150.$$

The π cancels.

$$\therefore \frac{5\pi}{6} = 150^\circ.$$

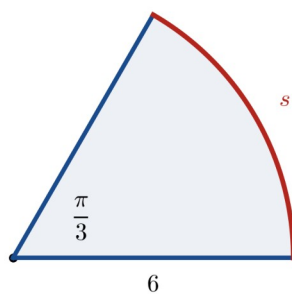
Example 3 — unscaffolded

A sector of a circle has radius 6 cm and central angle $\frac{\pi}{3}$. Find the exact arc length and the exact area of the sector.

$$\text{Arc length: } s = r\theta = 6 \times \frac{\pi}{3} = 2\pi \text{ cm.}$$

$$\text{Sector area: } A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 6^2 \times \frac{\pi}{3} = \frac{1}{2} \times 36 \times \frac{\pi}{3} = 6\pi \text{ cm}^2.$$

$\therefore$ the arc is 2π cm and the sector area is 6π cm².



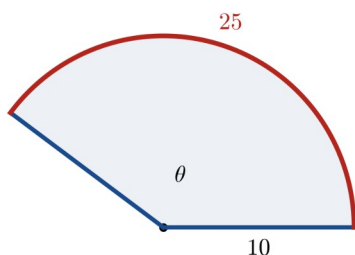
Example 4 — unscaffolded

A sector of radius 10 cm has an arc length of 25 cm. Find the central angle θ in radians and the area of the sector.

$$s = r\theta \Rightarrow 25 = 10\theta \Rightarrow \theta = \frac{25}{10} = \frac{5}{2} \text{ radians.}$$

$$A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 100 \times \frac{5}{2} = 125 \text{ cm}^2.$$

$\therefore \theta = \frac{5}{2}$ radians and the area is 125 cm².



Practice questions

Scaffolded

Q1. Convert each angle to radians, giving an exact answer: (a) 60° ; (b) 150° ; (c) 315° .

Q2. Convert each angle to degrees: (a) $\frac{\pi}{4}$; (b) $\frac{2\pi}{3}$; (c) $\frac{7\pi}{6}$.

Q3. A circle has radius 8 cm. A sector has central angle $\frac{\pi}{4}$. (a) Find the exact arc length. (b) Find the exact sector area.

Q4. A sector has radius 5 cm and central angle 72° . (a) Convert the angle to radians. (b) Find the exact arc length. (c) Find the exact area.

Q5. An arc of length 12 cm lies on a circle of radius 9 cm. (a) Find the central angle θ in radians. (b) Find θ in degrees (to one decimal place). (c) Find the area of the sector.

Q6. A sector of radius 12 cm has area 24π cm². (a) Find the central angle θ . (b) Find the exact arc length. (c) Find the exact perimeter of the sector.

Un scaffolded

Q7. Convert 40° to radians (exact).

Q8. Convert 225° to radians (exact).

Q9. Convert 330° to radians (exact).

Q10. Convert $\frac{5\pi}{12}$ radians to degrees.

Q11. Convert $\frac{3\pi}{2}$ radians to degrees.

Q12. Convert $\frac{11\pi}{9}$ radians to degrees.

Q13. A sector has radius 6 cm and central angle $\frac{2\pi}{3}$. Find the exact arc length and area.

Q14. An arc of length 15 cm lies on a circle of radius 10 cm. Find the central angle in radians and the sector area.

Q15. A sector has radius 4 cm and central angle 135° . Find the exact arc length and area.

Q16. A sector of central angle $\frac{\pi}{5}$ has area 10π cm². Find the radius and the exact arc length.

Q17. A sector has radius 7 cm and central angle $\frac{\pi}{2}$. Find the exact arc length, the exact perimeter and the exact area.

Q18. A sector has arc length 8π cm and central angle $\frac{2\pi}{3}$. Find the radius and the exact area.

Q19. Express each angle as an angle in $[0, 2\pi)$: (a) $\frac{7\pi}{3}$; (b) $-\frac{\pi}{6}$.

Q20. The minute hand of a clock is 14 cm long. Find the exact distance travelled by the tip of the hand in 40 minutes.

Q21. A sector of a circle has perimeter 20 cm and radius 6 cm. (a) Find the arc length. (b) Find the central angle θ in radians. (c) Find the area of the sector.

Answers

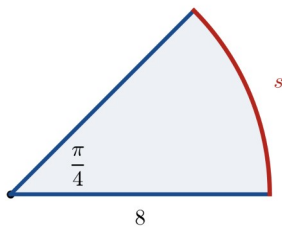
Q1. (a) $\frac{\pi}{3}$; (b) $\frac{5\pi}{6}$; (c) $\frac{7\pi}{4}$. **Q2.** (a) 45° ; (b) 120° ; (c) 210° . **Q3.** (a) 2π cm; (b) 8π cm². **Q4.** (a) $\frac{2\pi}{5}$; (b) 2π cm; (c) 5π cm². **Q5.** (a) $\frac{4}{3}$ rad; (b) $\approx 76.4^\circ$; (c) 54 cm². **Q6.** (a) $\frac{\pi}{3}$; (b) 4π cm; (c) $(24 + 4\pi)$ cm. **Q7.** $\frac{2\pi}{9}$. **Q8.** $\frac{5\pi}{4}$. **Q9.** $\frac{11\pi}{6}$. **Q10.** 75° . **Q11.** 270° . **Q12.** 220° . **Q13.** 4π cm; 12π cm². **Q14.** $\frac{3}{2}$ rad; 75 cm². **Q15.** 3π cm; 6π cm². **Q16.** $r = 10$ cm; arc 2π cm. **Q17.** arc $\frac{7\pi}{2}$ cm; perimeter $\left(14 + \frac{7\pi}{2}\right)$ cm; area $\frac{49\pi}{4}$ cm². **Q18.** $r = 12$ cm; area 48π cm². **Q19.** (a) $\frac{\pi}{3}$; (b) $\frac{11\pi}{6}$. **Q20.** $\frac{56\pi}{3}$ cm. **Q21.** (a) 8 cm; (b) $\frac{4}{3}$ rad; (c) 24 cm².

Fully-worked solutions

Q1. Multiply by $\frac{\pi}{180}$: (a) $60 \times \frac{\pi}{180} = \frac{\pi}{3}$; (b) $150 \times \frac{\pi}{180} = \frac{5\pi}{6}$; (c) $315 \times \frac{\pi}{180} = \frac{7\pi}{4}$.

Q2. Multiply by $\frac{180}{\pi}$: (a) $\frac{\pi}{4} \times \frac{180}{\pi} = 45^\circ$; (b) $\frac{2\pi}{3} \times \frac{180}{\pi} = 120^\circ$; (c) $\frac{7\pi}{6} \times \frac{180}{\pi} = 210^\circ$.

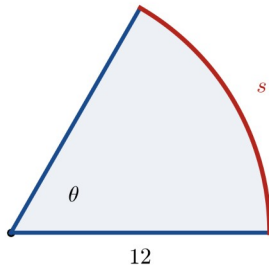
Q3. (a) $s = r\theta = 8 \times \frac{\pi}{4} = 2\pi$ cm. (b) $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 64 \times \frac{\pi}{4} = 8\pi$ cm².



Q4. (a) $72 \times \frac{\pi}{180} = \frac{2\pi}{5}$. (b) $s = 5 \times \frac{2\pi}{5} = 2\pi$ cm. (c) $A = \frac{1}{2} \times 25 \times \frac{2\pi}{5} = 5\pi$ cm².

Q5. (a) $s = r\theta \Rightarrow 12 = 9\theta \Rightarrow \theta = \frac{4}{3}$ rad. (b) $\frac{4}{3} \times \frac{180}{\pi} = \frac{240}{\pi} \approx 76.4^\circ$. (c) $A = \frac{1}{2} \times 81 \times \frac{4}{3} = 54$ cm².

Q6. (a) $A = \frac{1}{2}r^2\theta \Rightarrow 24\pi = \frac{1}{2} \times 144 \times \theta = 72\theta \Rightarrow \theta = \frac{24\pi}{72} = \frac{\pi}{3}$. (b) $s = 12 \times \frac{\pi}{3} = 4\pi$ cm. (c) perimeter $= 2r + s = 24 + 4\pi$ cm.



Q7. $40 \times \frac{\pi}{180} = \frac{2\pi}{9}$.

Q8. $225 \times \frac{\pi}{180} = \frac{5\pi}{4}$.

Q9. $330 \times \frac{\pi}{180} = \frac{11\pi}{6}$.

Q10. $\frac{5\pi}{12} \times \frac{180}{\pi} = \frac{5 \times 180}{12} = 75^\circ$.

Q11. $\frac{3\pi}{2} \times \frac{180}{\pi} = 270^\circ$.

Q12. $\frac{11\pi}{9} \times \frac{180}{\pi} = \frac{11 \times 180}{9} = 220^\circ$.

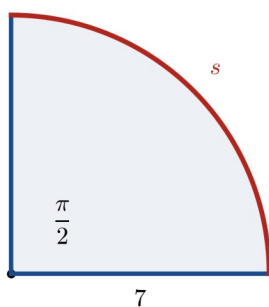
Q13. $s = 6 \times \frac{2\pi}{3} = 4\pi$ cm; $A = \frac{1}{2} \times 36 \times \frac{2\pi}{3} = 12\pi$ cm².

Q14. $15 = 10\theta \Rightarrow \theta = \frac{3}{2}$ rad; $A = \frac{1}{2} \times 100 \times \frac{3}{2} = 75$ cm².

Q15. $135^\circ = \frac{3\pi}{4}$; $s = 4 \times \frac{3\pi}{4} = 3\pi$ cm; $A = \frac{1}{2} \times 16 \times \frac{3\pi}{4} = 6\pi$ cm².

Q16. $A = \frac{1}{2} r^2 \theta \Rightarrow 10\pi = \frac{1}{2} r^2 \times \frac{\pi}{5} = \frac{\pi r^2}{10} \Rightarrow r^2 = 100 \Rightarrow r = 10$ cm; $s = 10 \times \frac{\pi}{5} = 2\pi$ cm.

Q17. $s = 7 \times \frac{\pi}{2} = \frac{7\pi}{2}$ cm; perimeter $= 2(7) + \frac{7\pi}{2} = 14 + \frac{7\pi}{2}$ cm; $A = \frac{1}{2} \times 49 \times \frac{\pi}{2} = \frac{49\pi}{4}$ cm².

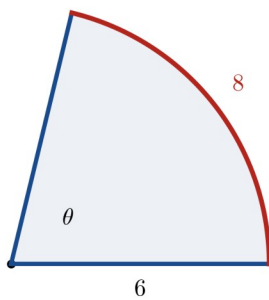


Q18. $s = r\theta \Rightarrow 8\pi = r \times \frac{2\pi}{3} \Rightarrow r = 8\pi \times \frac{3}{2\pi} = 12 \text{ cm}$; $A = 1/2 \times 144 \times \frac{2\pi}{3} = 48\pi \text{ cm}^2$.

Q19. (a) $\frac{7\pi}{3} - 2\pi = \frac{7\pi - 6\pi}{3} = \frac{\pi}{3}$. (b) $-\frac{\pi}{6} + 2\pi = \frac{-\pi + 12\pi}{6} = \frac{11\pi}{6}$.

Q20. In 40 minutes the minute hand turns through $\frac{40}{60}$ of a full revolution: $\theta = \frac{40}{60} \times 2\pi = \frac{4\pi}{3}$. Distance travelled by the tip $= r\theta = 14 \times \frac{4\pi}{3} = \frac{56\pi}{3} \text{ cm}$.

Q21. (a) Perimeter $= 2r + s \Rightarrow 20 = 12 + s \Rightarrow s = 8 \text{ cm}$. (b) $s = r\theta \Rightarrow 8 = 6\theta \Rightarrow \theta = \frac{4}{3} \text{ rad}$. (c) $A = 1/2 \times 36 \times \frac{4}{3} = 24 \text{ cm}^2$.



Chapter 6 Circular functions — 6B Sine, cosine and tangent on the unit circle

Theory

The **unit circle** is the circle of radius 1 centred at the origin, $x^2 + y^2 = 1$. Starting from the point $(1, 0)$ and rotating **anticlockwise** through an angle θ (measured in radians) gives a point $P(\theta)$ on the circle.

Definitions. The circular functions are defined by the coordinates of $P(\theta)$:

$$\cos \theta = x\text{-coordinate of } P(\theta), \sin \theta = y\text{-coordinate of } P(\theta),$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} (\cos \theta \neq 0).$$

So $P(\theta) = (\cos \theta, \sin \theta)$, and because the radius is 1 the identity $\cos^2 \theta + \sin^2 \theta = 1$ holds for every θ .

Exact values in the first quadrant.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Signs by quadrant (CAST). The sign of each function depends on the quadrant of $P(\theta)$:

- **Quadrant 1** ($0 < \theta < \pi/2$): all three positive — **All**.
- **Quadrant 2** ($\pi/2 < \theta < \pi$): only $\sin \theta > 0$ — **Sine**.
- **Quadrant 3** ($\pi < \theta < 3\pi/2$): only $\tan \theta > 0$ — **Tangent**.
- **Quadrant 4** ($3\pi/2 < \theta < 2\pi$): only $\cos \theta > 0$ — **Cosine**.

Angles beyond the first quadrant. Find the **reference angle** (the acute angle between the radius OP and the x -axis), read off the first-quadrant exact value, then attach the sign from CAST. For example $P(2\pi/3)$ lies in quadrant 2 with reference angle $\pi/3$, so $\sin 2\pi/3 = +\sin \pi/3 = \sqrt{3}/2$ and $\cos 2\pi/3 = -\cos \pi/3 = -1/2$.

Periodicity. Rotating by a full turn returns to the same point, so $\sin(\theta + 2\pi) = \sin \theta$, $\cos(\theta + 2\pi) = \cos \theta$ (period 2π), while $\tan(\theta + \pi) = \tan \theta$ (period π). Negative angles are measured **clockwise**.

Worked examples

Example 1 — scaffolded

Find the exact values of $\sin \frac{2\pi}{3}$, $\cos \frac{2\pi}{3}$ and $\tan \frac{2\pi}{3}$.

Working

$\frac{2\pi}{3}$ lies in **quadrant 2** (between $\pi/2$ and π).

Reference angle: $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

Comment

Locate $P(\theta)$ on the unit circle.

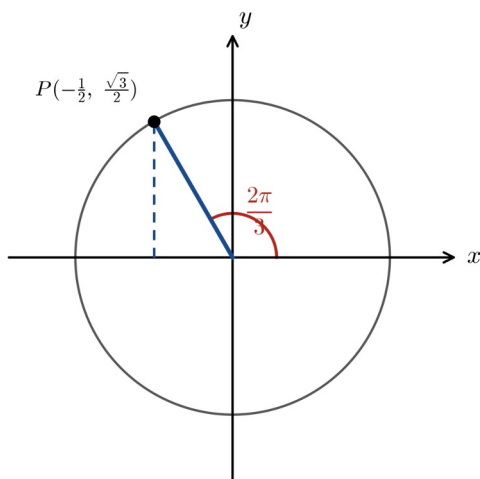
The acute angle to the x -axis.

Working

In quadrant 2, $\sin > 0$: $\sin \frac{2\pi}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$\cos < 0$: $\cos \frac{2\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$.

$\tan \frac{2\pi}{3} = \frac{\sin}{\cos} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$.



Comment

CAST: sine is positive.

CAST: cosine is negative.

$$\tan \theta = \frac{\sin \theta}{\cos \theta}.$$

Example 2 — scaffolded

Evaluate $\sin \frac{5\pi}{4}$, $\cos \frac{5\pi}{4}$ and $\tan \frac{5\pi}{4}$.

Working

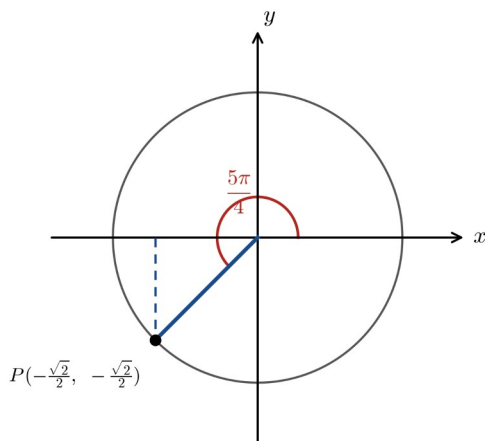
$\frac{5\pi}{4}$ lies in **quadrant 3** (between π and $3\pi/2$).

Reference angle: $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$.

In quadrant 3, $\sin < 0$ and $\cos < 0$: $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$,

$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.

$\tan \frac{5\pi}{4} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$.



Comment

Locate $P(\theta)$.

Angle past π .

CAST: only tangent positive.

Two negatives divide to a positive.

Example 3 — unscaffolded

Find the exact values of $\sin \frac{7\pi}{6}$, $\cos \frac{7\pi}{6}$ and $\tan \frac{7\pi}{6}$.

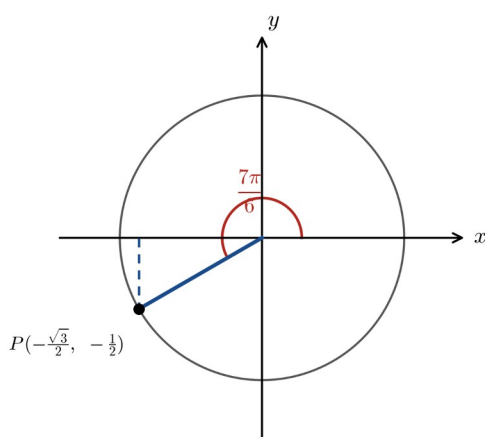
$\frac{7\pi}{6}$ is in quadrant 3, with reference angle $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$.

In quadrant 3, sine and cosine are negative:

$$\sin \frac{7\pi}{6} = -\frac{1}{2}, \cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}.$$

$$\tan \frac{7\pi}{6} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

$$\therefore \sin \frac{7\pi}{6} = -\frac{1}{2}, \cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}, \tan \frac{7\pi}{6} = \frac{\sqrt{3}}{3}.$$



Example 4 — unscaffolded

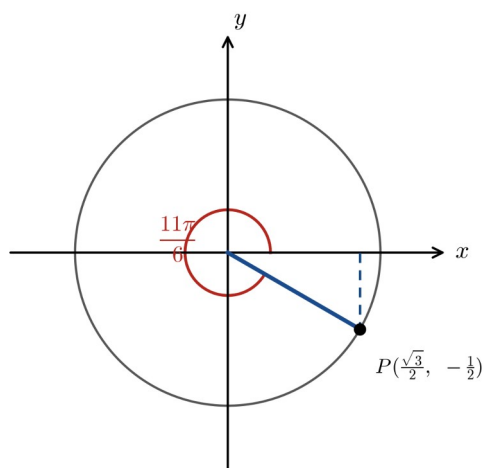
Find the exact values of the three circular functions of $\frac{11\pi}{6}$.

$\frac{11\pi}{6}$ is in quadrant 4, with reference angle $2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$.

In quadrant 4, only cosine is positive:

$$\sin \frac{11\pi}{6} = -\frac{1}{2}, \cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}, \tan \frac{11\pi}{6} = \frac{-1/2}{\sqrt{3}/2} = -\frac{\sqrt{3}}{3}.$$

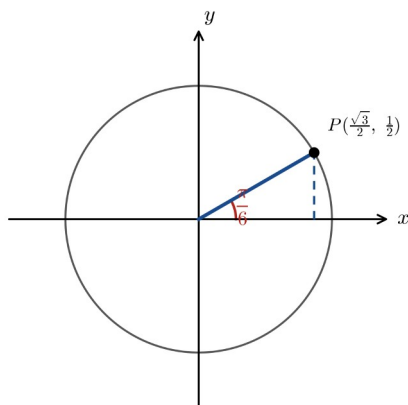
$$\therefore \text{the values are } -\frac{1}{2}, \frac{\sqrt{3}}{2} \text{ and } -\frac{\sqrt{3}}{3}.$$



Practice questions

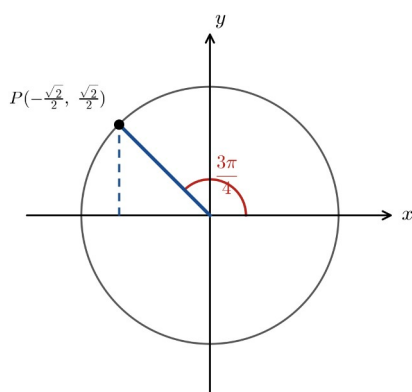
Scaffolded

Q1. For $\theta = \frac{\pi}{6}$: (a) state $\sin \theta$; (b) state $\cos \theta$; (c) state $\tan \theta$.



Q2. For $\theta = \pi$: (a) find $\sin \pi$; (b) find $\cos \pi$; (c) find $\tan \pi$.

Q3. For $\theta = \frac{3\pi}{4}$: (a) state the quadrant of $P(\theta)$ and the reference angle; (b) find $\sin \theta$ and $\cos \theta$; (c) hence find $\tan \theta$.



Q4. For $\theta = \frac{5\pi}{6}$: (a) write the reference angle; (b) find $\sin \theta$ and $\cos \theta$; (c) find $\tan \theta$.

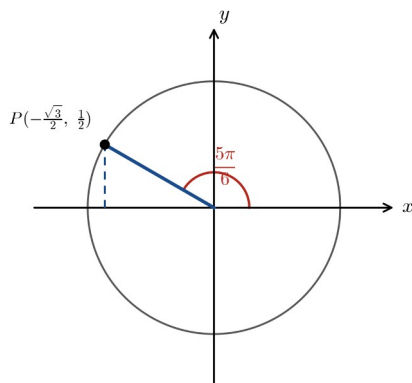
Q5. For $\theta = \frac{4\pi}{3}$: (a) state the quadrant; (b) find $\sin \theta$ and $\cos \theta$; (c) find $\tan \theta$.

Q6. An angle θ with $0 \leq \theta \leq 2\pi$ satisfies $\sin \theta > 0$ and $\cos \theta < 0$. (a) In which quadrant does θ lie? (b) Give a possible exact value of θ .

Un scaffolded

Find the exact value(s) requested. Give all answers in exact form.

Q7. $\sin \frac{5\pi}{6}$, $\cos \frac{5\pi}{6}$ and $\tan \frac{5\pi}{6}$.

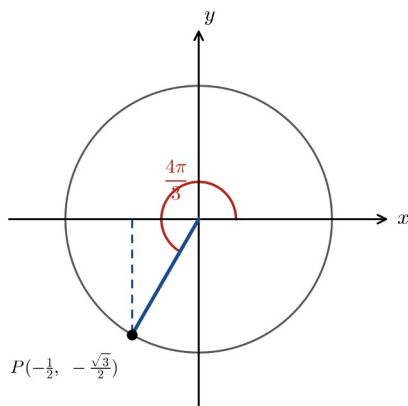


Q8. $\sin \frac{3\pi}{2}$ and $\cos \frac{3\pi}{2}$, and explain why $\tan \frac{3\pi}{2}$ is undefined.

Q9. The three circular functions of $\frac{7\pi}{4}$.

Q10. The three circular functions of $\frac{5\pi}{4}$.

Q11. The three circular functions of $\frac{4\pi}{3}$.

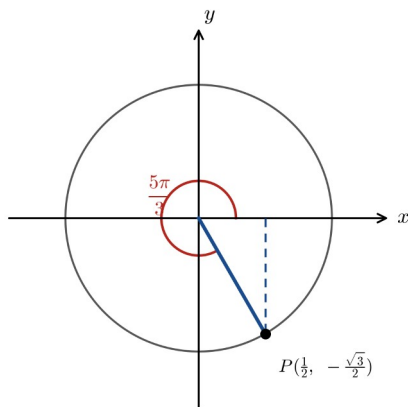


Q12. $\sin 2\pi$, $\cos 2\pi$ and $\tan 2\pi$.

Q13. $\sin\left(-\frac{\pi}{3}\right)$ and $\cos\left(-\frac{\pi}{3}\right)$.

Q14. $\tan \frac{11\pi}{6}$.

Q15. The three circular functions of $\frac{5\pi}{3}$.



Q16. Find $\tan \frac{2\pi}{3}$, and **verify** that it equals $\frac{\sin(2\pi/3)}{\cos(2\pi/3)}$.

Q17. The angle $\frac{7\pi}{3}$ is more than one full turn. Find the first-quadrant angle it corresponds to, and hence state $\sin \frac{7\pi}{3}$ and $\cos \frac{7\pi}{3}$.

Q18. Find $\cos \frac{9\pi}{4}$ and $\tan \frac{9\pi}{4}$.

Q19. An angle $\theta \in [0, 2\pi]$ has $\sin \theta = -\frac{1}{2}$ and $\cos \theta = \frac{\sqrt{3}}{2}$. Determine the quadrant of θ and find its exact value.

Q20. Show that $\sin \frac{5\pi}{6} = \sin \frac{\pi}{6}$, and explain the result using the unit circle.

Q21. State $\tan \frac{3\pi}{4}$, $\tan \frac{5\pi}{4}$ and $\tan \frac{7\pi}{4}$, and comment on the pattern of their signs.

Answers

Q1. (a) $\frac{1}{2}$; (b) $\frac{\sqrt{3}}{2}$; (c) $\frac{\sqrt{3}}{3}$. **Q2.** (a) 0; (b) -1 ; (c) 0. **Q3.** (a) quadrant 2, reference $\frac{\pi}{4}$; (b) $\sin = \frac{\sqrt{2}}{2}$, $\cos = -\frac{\sqrt{2}}{2}$; (c) -1 . **Q4.** (a) $\frac{\pi}{6}$; (b) $\sin = \frac{1}{2}$, $\cos = -\frac{\sqrt{3}}{2}$; (c) $-\frac{\sqrt{3}}{3}$. **Q5.** (a) quadrant 3; (b) $\sin = -\frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$; (c) $\sqrt{3}$. **Q6.** (a) quadrant 2; (b) e.g. $\theta = \frac{2\pi}{3}$ (any angle in $(\pi/2, \pi)$). **Q7.** $\sin = \frac{1}{2}$, $\cos = -\frac{\sqrt{3}}{2}$, $\tan = -\frac{\sqrt{3}}{3}$. **Q8.** $\sin = -1$, $\cos = 0$; $\tan$ is undefined because $\cos \frac{3\pi}{2} = 0$ (division by zero). **Q9.** $\sin = -\frac{\sqrt{2}}{2}$, $\cos = \frac{\sqrt{2}}{2}$, $\tan = -1$. **Q10.** $\sin = -\frac{\sqrt{2}}{2}$, $\cos = -\frac{\sqrt{2}}{2}$, $\tan = 1$. **Q11.** $\sin = -\frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$, $\tan = \sqrt{3}$. **Q12.** 0, 1, 0. **Q13.** $\sin = -\frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$. **Q14.** $-\frac{\sqrt{3}}{3}$. **Q15.** $\sin = -\frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = -\sqrt{3}$. **Q16.** $\tan \frac{2\pi}{3} = -\sqrt{3}$. **Q17.** $\frac{\pi}{3}$; $\sin = \frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$. **Q18.** $\cos \frac{9\pi}{4} = \frac{\sqrt{2}}{2}$, $\tan \frac{9\pi}{4} = 1$. **Q19.** Quadrant 4; $\theta = \frac{11\pi}{6}$. **Q20.** Both equal $\frac{1}{2}$. **Q21.** $-1, 1, -1$; the sign is $+$ in quadrant 3 and $-$ in quadrants 2 and 4.

Fully-worked solutions

Q1. $\frac{\pi}{6}$ is in quadrant 1, so all values are the standard positives: $\sin = \frac{1}{2}$, $\cos = \frac{\sqrt{3}}{2}$, $\tan = \frac{\sin}{\cos} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Q2. $P(\pi) = (-1, 0)$. So $\sin \pi = 0$, $\cos \pi = -1$, and $\tan \pi = \frac{0}{-1} = 0$.

Q3. (a) $\frac{3\pi}{4}$ is in quadrant 2; reference angle $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$. (b) $\sin \frac{3\pi}{4} = +\frac{\sqrt{2}}{2}$, $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$ (sine positive, cosine negative in Q2). (c) $\tan \frac{3\pi}{4} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$.

Q4. (a) Reference angle $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$. (b) Quadrant 2: $\sin \frac{5\pi}{6} = \frac{1}{2}$, $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$. (c) $\tan \frac{5\pi}{6} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

Q5. (a) $\frac{4\pi}{3}$ is in quadrant 3; reference angle $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$. (b) Both negative: $\sin = -\frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$. (c) $\tan = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$.

Q6. (a) $\sin > 0$, $\cos < 0$ is the CAST signature of **quadrant 2**. (b) Any θ with $\frac{\pi}{2} < \theta < \pi$, for example $\theta = \frac{2\pi}{3}$.

Q7. $\frac{5\pi}{6}$ is in quadrant 2, reference angle $\frac{\pi}{6}$. $\sin = \frac{1}{2}$, $\cos = -\frac{\sqrt{3}}{2}$, $\tan = \frac{1/2}{-\sqrt{3}/2} = -\frac{\sqrt{3}}{3}$.

Q8. $P\left(\frac{3\pi}{2}\right) = (0, -1)$, so $\sin \frac{3\pi}{2} = -1$ and $\cos \frac{3\pi}{2} = 0$. Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cos \frac{3\pi}{2} = 0$, the quotient is undefined (division by zero).

Q9. $\frac{7\pi}{4}$ is in quadrant 4, reference angle $2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$. $\sin = -\frac{\sqrt{2}}{2}$, $\cos = \frac{\sqrt{2}}{2}$, $\tan = -1$.

Q10. $\frac{5\pi}{4}$ is in quadrant 3, reference angle $\frac{\pi}{4}$. $\sin = -\frac{\sqrt{2}}{2}$, $\cos = -\frac{\sqrt{2}}{2}$, $\tan = 1$.

Q11. $\frac{4\pi}{3}$ is in quadrant 3, reference angle $\frac{\pi}{3}$. $\sin = -\frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$, $\tan = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$.

Q12. $P(2\pi) = (1, 0)$ (a full turn back to the start), so $\sin 2\pi = 0$, $\cos 2\pi = 1$, $\tan 2\pi = 0$.

Q13. $-\frac{\pi}{3}$ is measured clockwise into quadrant 4, reference angle $\frac{\pi}{3}$. $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$.

Q14. $\frac{11\pi}{6}$ is in quadrant 4, reference angle $\frac{\pi}{6}$. $\tan \frac{11\pi}{6} = \frac{-1/2}{\sqrt{3}/2} = -\frac{\sqrt{3}}{3}$.

Q15. $\frac{5\pi}{3}$ is in quadrant 4, reference angle $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$. $\sin = -\frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$.

Q16. $\frac{2\pi}{3}$ is in quadrant 2 with reference angle $\frac{\pi}{3}$, so $\tan \frac{2\pi}{3} = -\sqrt{3}$. Verification: $\frac{\sin(2\pi/3)}{\cos(2\pi/3)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$. The two agree. ✓

Q17. $\frac{7\pi}{3} = \frac{\pi}{3} + 2\pi$, so it is coterminal with $\frac{\pi}{3}$. Hence $\sin \frac{7\pi}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos \frac{7\pi}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$.

Q18. $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$, coterminal with $\frac{\pi}{4}$. $\cos \frac{9\pi}{4} = \frac{\sqrt{2}}{2}$ and $\tan \frac{9\pi}{4} = 1$.

Q19. $\sin \theta < 0$ and $\cos \theta > 0$ is the CAST signature of **quadrant 4**. The reference angle has $\sin = \frac{1}{2}$, $\cos = \frac{\sqrt{3}}{2}$, i.e. $\frac{\pi}{6}$.

In quadrant 4, $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Q20. $\frac{5\pi}{6}$ is in quadrant 2 with reference angle $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$, and sine is positive in quadrant 2, so $\sin \frac{5\pi}{6} = \sin \frac{\pi}{6} = \frac{1}{2}$.

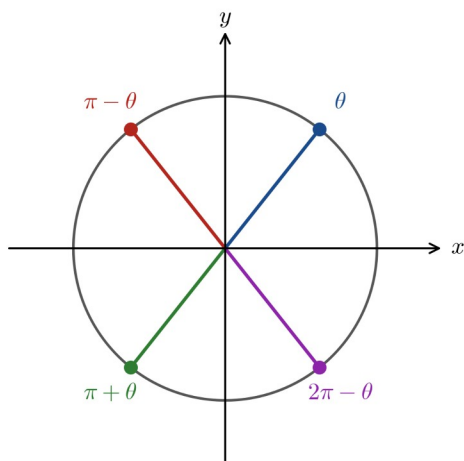
On the unit circle, $P\left(\frac{5\pi}{6}\right)$ and $P\left(\frac{\pi}{6}\right)$ are mirror images in the y -axis, so they have the **same** y -coordinate — and the y -coordinate is the sine.

Q21. $\tan \frac{3\pi}{4} = -1$ (quadrant 2), $\tan \frac{5\pi}{4} = 1$ (quadrant 3), $\tan \frac{7\pi}{4} = -1$ (quadrant 4). Tangent is **positive in quadrant**

3 and **negative in quadrants 2 and 4**, matching CAST; the magnitude is 1 each time because the reference angle is $\frac{\pi}{4}$ throughout.

Theory

The unit circle lets us relate the sine, cosine and tangent of an angle to those of the related acute angle θ in the first quadrant. If $P(\theta) = (\cos \theta, \sin \theta)$ is the point on the unit circle, the points for $\pi - \theta$, $\pi + \theta$ and $2\pi - \theta$ are reflections of $P(\theta)$ in the axes.

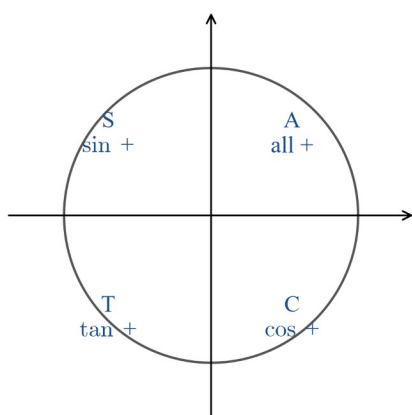


Related angles on the unit circle

Symmetry properties. For every angle θ :

Second quadrant	Third quadrant	Fourth quadrant	Negative angle
$\sin(\pi - \theta) = \sin \theta$	$\sin(\pi + \theta) = -\sin \theta$	$\sin(2\pi - \theta) = -\sin \theta$	$\sin(-\theta) = -\sin \theta$
$\cos(\pi - \theta) = -\cos \theta$	$\cos(\pi + \theta) = -\cos \theta$	$\cos(2\pi - \theta) = \cos \theta$	$\cos(-\theta) = \cos \theta$
$\tan(\pi - \theta) = -\tan \theta$	$\tan(\pi + \theta) = \tan \theta$	$\tan(2\pi - \theta) = -\tan \theta$	$\tan(-\theta) = -\tan \theta$

The signs follow the **ASTC** rule — which ratios are **positive** in each quadrant:



ASTC quadrant signs

Adding or subtracting a whole number of revolutions (2π) does not change a ratio, e.g.

$$\sin(3\pi - \theta) = \sin(2\pi + (\pi - \theta)) = \sin(\pi - \theta) = \sin \theta.$$

Complementary relationships (angles adding to $\frac{\pi}{2}$):

$$\sin(\pi/2 - \theta) = \cos \theta, \cos(\pi/2 - \theta) = \sin \theta, \sin(\pi/2 + \theta) = \cos \theta, \cos(\pi/2 + \theta) = -\sin \theta.$$

The Pythagorean identity. Because $P(\theta) = (\cos \theta, \sin \theta)$ lies on the unit circle $x^2 + y^2 = 1$,

$$\cos^2 \theta + \sin^2 \theta = 1,$$

which rearranges to $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos^2 \theta = 1 - \sin^2 \theta$. (Here $\sin^2 \theta$ means $(\sin \theta)^2$.) With $\tan \theta = \frac{\sin \theta}{\cos \theta}$, these let us find all three ratios from any one of them, together with the quadrant of θ .

To find the other ratios from one: use $\cos^2 \theta + \sin^2 \theta = 1$ to get the second ratio (as the $\pm$ square root), then choose the sign from the quadrant (ASTC), and finally use $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

Worked examples

Example 1 — scaffolded

Simplify $\frac{\sin(\pi - \theta)}{\cos(2\pi - \theta)} + \tan(\pi + \theta)$.

Working

$$\sin(\pi - \theta) = \sin \theta$$

$$\cos(2\pi - \theta) = \cos \theta$$

$$\tan(\pi + \theta) = \tan \theta$$

$$\frac{\sin \theta}{\cos \theta} + \tan \theta = \tan \theta + \tan \theta$$

$$= 2 \tan \theta$$

Comment

Second-quadrant symmetry.

Fourth-quadrant symmetry.

Third-quadrant symmetry.

$$\frac{\sin \theta}{\cos \theta} = \tan \theta.$$

Collect like terms.

Example 2 — scaffolded

Given $\cos \theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact values of $\sin \theta$ and $\tan \theta$.

Working

θ is in the **third quadrant**, so $\sin \theta < 0$ and $\tan \theta > 0$.

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{25}{169} = \frac{144}{169}$$

$$\sin \theta = -\frac{12}{13}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-12/13}{-5/13} = \frac{12}{5}$$

Comment

ASTC: only $\tan$ is positive there.

Pythagorean identity.

Take the negative root (third quadrant).

$$\tan \theta = \frac{\sin \theta}{\cos \theta}; \text{ positive, as expected.}$$

Example 3 — unscaffolded

Find the exact value of (a) $\sin \frac{2\pi}{3}$, (b) $\cos \frac{7\pi}{6}$, (c) $\tan \frac{5\pi}{4}$.

$$(a) \sin \frac{2\pi}{3} = \sin \left(\pi - \frac{\pi}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

$$(b) \cos \frac{7\pi}{6} = \cos \left(\pi + \frac{\pi}{6} \right) = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}.$$

$$(c) \tan \frac{5\pi}{4} = \tan \left(\pi + \frac{\pi}{4} \right) = \tan \frac{\pi}{4} = 1.$$

$\therefore$ the values are $\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$ and 1.

Example 4 — unscaffolded

Show that $(\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 = 2$.

$$(\sin \theta + \cos \theta)^2 = \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta$$

$$(\sin \theta - \cos \theta)^2 = \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta$$

Adding, the $2 \sin \theta \cos \theta$ terms cancel:

$$(\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 = 2 \sin^2 \theta + 2 \cos^2 \theta = 2(\sin^2 \theta + \cos^2 \theta) = 2(1) = 2.$$

$\therefore$ the identity holds.

Practice questions

Scaffolded

Q1. Write each in terms of $\sin \theta$, $\cos \theta$ or $\tan \theta$: (a) $\sin(\pi + \theta)$; (b) $\cos(\pi - \theta)$; (c) $\tan(2\pi - \theta)$; (d) $\cos(-\theta)$.

Q2. Find the exact value using symmetry: (a) $\sin \frac{5\pi}{6}$; (b) $\cos \frac{4\pi}{3}$; (c) $\tan \frac{2\pi}{3}$; (d) $\sin \frac{7\pi}{4}$.

Q3. Given $\sin \theta = \frac{3}{5}$ with $0 < \theta < \frac{\pi}{2}$: (a) find $\cos \theta$; (b) find $\tan \theta$; (c) find $\sin(\pi - \theta)$; (d) find $\cos(\pi + \theta)$.

Q4. Given $\tan \theta = -\frac{7}{24}$ with $\frac{\pi}{2} < \theta < \pi$: (a) state the signs of $\sin \theta$ and $\cos \theta$; (b) find $\sin \theta$; (c) find $\cos \theta$.

Q5. Simplify using complementary angles: (a) $\sin\left(\frac{\pi}{2} - \theta\right)$; (b) $\cos\left(\frac{\pi}{2} - \theta\right)$; (c) $\sin\left(\frac{\pi}{2} + \theta\right)$; (d) $\cos\left(\frac{\pi}{2} + \theta\right)$.

Q6. Use the Pythagorean identity to simplify: (a) $1 - \cos^2 \theta$; (b) $\frac{1 - \sin^2 \theta}{\cos \theta}$; (c) $\frac{\sin^2 \theta}{1 - \cos \theta}$.

Unscaffolded

Q7. Write $\sin(3\pi - \theta)$ in terms of $\sin \theta$.

Q8. Write $\cos(3\pi + \theta)$ in terms of $\cos \theta$.

Q9. Find the exact value of $\sin \frac{11\pi}{6}$.

Q10. Find the exact value of $\cos \frac{5\pi}{3}$.

Q11. Find the exact value of $\tan \frac{7\pi}{6}$.

Q12. Given $\cos \theta = \frac{2}{3}$ with $\frac{3\pi}{2} < \theta < 2\pi$, find the exact values of $\sin \theta$ and $\tan \theta$.

Q13. Given $\sin \theta = -\frac{1}{4}$ with $\pi < \theta < \frac{3\pi}{2}$, find the exact value of $\cos \theta$.

Q14. Simplify $\frac{\cos(\pi - \theta)}{\sin\left(\frac{\pi}{2} - \theta\right)}$.

Q15. Simplify $\sin(\pi + \theta)\cos(2\pi - \theta) + \cos(\pi + \theta)\sin(2\pi - \theta)$.

Q16. Show that $\frac{1 - \cos^2 \theta}{\sin \theta} = \sin \theta$, stating any restriction.

Q17. Show that $\cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta$.

Q18. Show that $(1 - \sin \theta)(1 + \sin \theta) = \cos^2 \theta$.

Q19. Given that $\sin \theta + \cos \theta = \frac{1}{2}$, find the exact value of $\sin \theta \cos \theta$.

Q20. Given $\tan \theta = \sqrt{3}$ with $\pi < \theta < \frac{3\pi}{2}$, find the exact values of $\sin \theta$ and $\cos \theta$.

Q21. Show that $\frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} = \frac{2}{\sin \theta}$.

Answers

Q1. (a) $-\sin \theta$ (b) $-\cos \theta$ (c) $-\tan \theta$ (d) $\cos \theta$. **Q2.** (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\sqrt{3}$ (d) $-\frac{\sqrt{2}}{2}$. **Q3.** (a) $\frac{4}{5}$ (b) $\frac{3}{4}$ (c) $\frac{3}{5}$ (d) $-\frac{4}{5}$.
Q4. (a) $\sin \theta > 0$, $\cos \theta < 0$ (b) $\frac{7}{25}$ (c) $-\frac{24}{25}$. **Q5.** (a) $\cos \theta$ (b) $\sin \theta$ (c) $\cos \theta$ (d) $-\sin \theta$. **Q6.** (a) $\sin^2 \theta$ (b) $\cos \theta$ (c) $1 + \cos \theta$. **Q7.** $\sin \theta$. **Q8.** $-\cos \theta$. **Q9.** $-\frac{1}{2}$. **Q10.** $\frac{1}{2}$. **Q11.** $\frac{\sqrt{3}}{3}$. **Q12.** $\sin \theta = -\frac{\sqrt{5}}{3}$, $\tan \theta = -\frac{\sqrt{5}}{2}$. **Q13.** $\cos \theta = -\frac{\sqrt{15}}{4}$.
Q14. -1 . **Q15.** 0 . **Q16.** proof ($\sin \theta \neq 0$). **Q17.** proof. **Q18.** proof. **Q19.** $-\frac{3}{8}$. **Q20.** $\sin \theta = -\frac{\sqrt{3}}{2}$, $\cos \theta = -\frac{1}{2}$. **Q21.** proof.

Fully-worked solutions

Q1. From the symmetry table: (a) $\sin(\pi + \theta) = -\sin \theta$; (b) $\cos(\pi - \theta) = -\cos \theta$; (c) $\tan(2\pi - \theta) = -\tan \theta$; (d) $\cos(-\theta) = \cos \theta$.

Q2. (a) $\sin \frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$. (b) $\cos \frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right) = -\cos \frac{\pi}{3} = -\frac{1}{2}$. (c) $\tan \frac{2\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\sqrt{3}$. (d) $\sin \frac{7\pi}{4} = \sin\left(2\pi - \frac{\pi}{4}\right) = -\sin \frac{\pi}{4} = -\frac{\sqrt{2}}{2}$.

Q3. θ is in the first quadrant (all ratios positive). (a) $\cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}$, so $\cos \theta = \frac{4}{5}$. (b) $\tan \theta = \frac{3/5}{4/5} = \frac{3}{4}$. (c) $\sin(\pi - \theta) = \sin \theta = \frac{3}{5}$. (d) $\cos(\pi + \theta) = -\cos \theta = -\frac{4}{5}$.

Q4. (a) In the second quadrant $\sin \theta > 0$ and $\cos \theta < 0$. (b),(c) With $\tan \theta = -\frac{7}{24}$, a right triangle with legs 7 and 24 has hypotenuse $\sqrt{7^2 + 24^2} = \sqrt{625} = 25$. Taking the second-quadrant signs, $\sin \theta = \frac{7}{25}$ and $\cos \theta = -\frac{24}{25}$ (check: $\frac{\sin \theta}{\cos \theta} = -\frac{7}{24}$ ✓).

Q5. Complementary relationships: (a) $\cos \theta$; (b) $\sin \theta$; (c) $\cos \theta$; (d) $-\sin \theta$.

Q6. (a) $1 - \cos^2 \theta = \sin^2 \theta$. (b) $\frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta}{\cos \theta} = \cos \theta$. (c) $\frac{\sin^2 \theta}{1 - \cos \theta} = \frac{1 - \cos^2 \theta}{1 - \cos \theta} = \frac{(1 - \cos \theta)(1 + \cos \theta)}{1 - \cos \theta} = 1 + \cos \theta$.

Q7. $\sin(3\pi - \theta) = \sin(2\pi + (\pi - \theta)) = \sin(\pi - \theta) = \sin \theta$.

Q8. $\cos(3\pi + \theta) = \cos(2\pi + (\pi + \theta)) = \cos(\pi + \theta) = -\cos \theta$.

Q9. $\sin \frac{11\pi}{6} = \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin \frac{\pi}{6} = -\frac{1}{2}$.

Q10. $\cos \frac{5\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$.

Q11. $\tan \frac{7\pi}{6} = \tan\left(\pi + \frac{\pi}{6}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Q12. Fourth quadrant: $\sin \theta < 0$, $\tan \theta < 0$. $\sin^2 \theta = 1 - \frac{4}{9} = \frac{5}{9}$, so $\sin \theta = -\frac{\sqrt{5}}{3}$; $\tan \theta = \frac{-\sqrt{5}/3}{2/3} = -\frac{\sqrt{5}}{2}$.

Q13. Third quadrant: $\cos \theta < 0$. $\cos^2 \theta = 1 - \frac{1}{16} = \frac{15}{16}$, so $\cos \theta = -\frac{\sqrt{15}}{4}$.

Q14. $\frac{\cos(\pi - \theta)}{\sin(\frac{\pi}{2} - \theta)} = \frac{-\cos \theta}{\cos \theta} = -1$ (for $\cos \theta \neq 0$).

Q15.
 $\sin(\pi + \theta) \cos(2\pi - \theta) + \cos(\pi + \theta) \sin(2\pi - \theta) = (-\sin \theta)(\cos \theta) + (-\cos \theta)(-\sin \theta) = -\sin \theta \cos \theta + \sin \theta \cos \theta = 0$.

Q16. $\frac{1 - \cos^2 \theta}{\sin \theta} = \frac{\sin^2 \theta}{\sin \theta} = \sin \theta$, provided $\sin \theta \neq 0$.

Q17. $\cos^2 \theta - \sin^2 \theta = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2\sin^2 \theta$.

Q18. $(1 - \sin \theta)(1 + \sin \theta) = 1 - \sin^2 \theta = \cos^2 \theta$.

Q19. Squaring: $(\sin \theta + \cos \theta)^2 = \frac{1}{4}$, so $\sin^2 \theta + 2\sin \theta \cos \theta + \cos^2 \theta = \frac{1}{4}$, i.e. $1 + 2\sin \theta \cos \theta = \frac{1}{4}$. Then

$2\sin \theta \cos \theta = -\frac{3}{4}$, so $\sin \theta \cos \theta = -\frac{3}{8}$.

Q20. Third quadrant: $\sin \theta < 0$, $\cos \theta < 0$. The reference angle has $\tan = \sqrt{3}$, i.e. $\frac{\pi}{3}$, so $\sin \theta = -\frac{\sqrt{3}}{2}$ and $\cos \theta = -\frac{1}{2}$

(check: $\frac{\sin \theta}{\cos \theta} = \sqrt{3}$ ✓).

Q21. Over the common denominator $\sin \theta(1 + \cos \theta)$:

$$\frac{\sin^2 \theta + (1 + \cos \theta)^2}{\sin \theta(1 + \cos \theta)} = \frac{\sin^2 \theta + 1 + 2\cos \theta + \cos^2 \theta}{\sin \theta(1 + \cos \theta)} = \frac{2 + 2\cos \theta}{\sin \theta(1 + \cos \theta)} = \frac{2(1 + \cos \theta)}{\sin \theta(1 + \cos \theta)} = \frac{2}{\sin \theta}.$$

Chapter 6 Circular functions — 6D Graphs of sine and cosine

Theory

The **sine** and **cosine** functions are defined using the unit circle: for an angle of x radians, the point on the unit circle is $(\cos x, \sin x)$, so $\cos x$ is its x -coordinate and $\sin x$ is its y -coordinate. As x increases, the point travels around the circle, and the coordinates trace out the two graphs below. All angles here are in **radians**.

The graph of $y = \sin x$.

- Domain $\mathbb{R}$, range $[-1, 1]$. The **amplitude** (half the distance between the maximum and minimum) is 1, and the **period** (the length of one complete cycle) is 2π .
- Over one cycle $[0, 2\pi]$ the key points are $(0, 0)$, $(\frac{\pi}{2}, 1)$, $(\pi, 0)$, $(\frac{3\pi}{2}, -1)$, $(2\pi, 0)$.
- $\sin x$ is an **odd** function: $\sin(-x) = -\sin x$, so the graph has point symmetry about the origin.

The graph of $y = \cos x$.

- Domain $\mathbb{R}$, range $[-1, 1]$, amplitude 1, period 2π .
- Over $[0, 2\pi]$ the key points are $(0, 1)$, $(\frac{\pi}{2}, 0)$, $(\pi, -1)$, $(\frac{3\pi}{2}, 0)$, $(2\pi, 1)$.
- $\cos x$ is an **even** function: $\cos(-x) = \cos x$, so the graph is symmetric about the y -axis.

Periodicity. Because both functions repeat every 2π , $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$. To evaluate $\sin$ or $\cos$ of an angle outside $[0, 2\pi]$, add or subtract multiples of 2π to bring it into a cycle you can read from the graph. The two graphs are related by $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ — the cosine graph is the sine graph translated $\frac{\pi}{2}$ to the left.

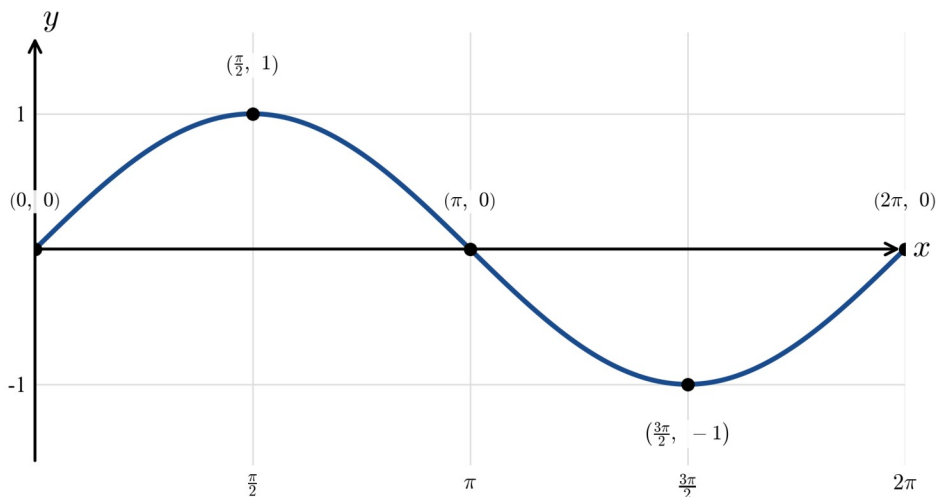
To sketch $y = \sin x$ or $y = \cos x$ over a given domain: mark the five key points of each cycle (a maximum, a minimum, and the x -intercepts), then draw a smooth periodic curve. A sketch shows the key features and is *not to scale*.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = \sin x$ for $x \in [0, 2\pi]$, labelling the axis intercepts and the coordinates of the maximum and minimum points.

Working	Comment
Amplitude 1, period 2π : one complete cycle fits exactly in $[0, 2\pi]$.	Read the amplitude and period from the standard graph.
x -intercepts: $\sin x = 0$ at $x = 0, \pi, 2\pi$.	$\sin x = 0$ at integer multiples of π .
Maximum: $(\frac{\pi}{2}, 1)$.	$\sin x$ reaches 1 a quarter of the way through the cycle.
Minimum: $(\frac{3\pi}{2}, -1)$.	$\sin x$ reaches -1 three-quarters of the way through.



Example 2 — scaffolded

Sketch the graph of $y = \cos x$ for $x \in [0, 2\pi]$, labelling the axis intercepts and the maximum and minimum points.

Working

Amplitude 1, period 2π .

y-intercept: $\cos 0 = 1$, giving $(0, 1)$.

x-intercepts: $\cos x = 0$ at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Maximum $(0, 1)$ and $(2\pi, 1)$; minimum $(\pi, -1)$.

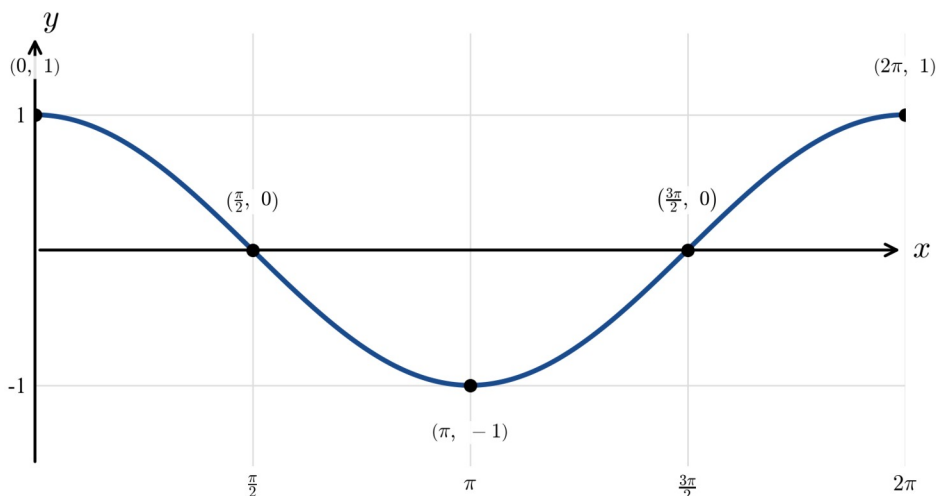
Comment

Same shape as sine, started from its maximum.

Cosine begins at its maximum.

$\cos x = 0$ at odd multiples of $\frac{\pi}{2}$.

Cosine is at -1 halfway through the cycle.



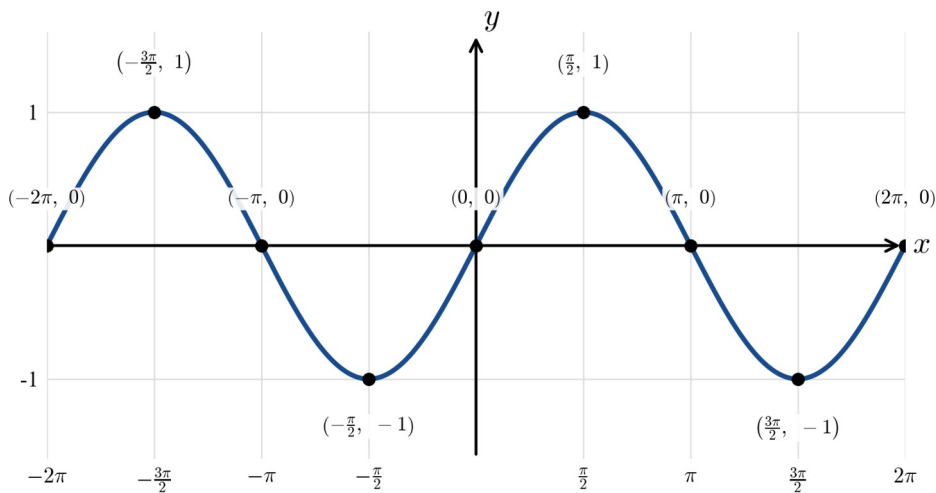
Example 3 — unscaffolded

For the graph of $y = \sin x$ on $x \in [-2\pi, 2\pi]$, state the coordinates of all maximum points and all x-intercepts.

The graph is two complete cycles. $\sin x = 1$ (maximum) at $x = -\frac{3\pi}{2}$ and $x = \frac{\pi}{2}$, giving $(-\frac{3\pi}{2}, 1)$ and $(\frac{\pi}{2}, 1)$.

x-intercepts: $\sin x = 0$ at $x = -2\pi, -\pi, 0, \pi, 2\pi$.

$\therefore$ the maxima are $(-\frac{3\pi}{2}, 1)$ and $(\frac{\pi}{2}, 1)$.



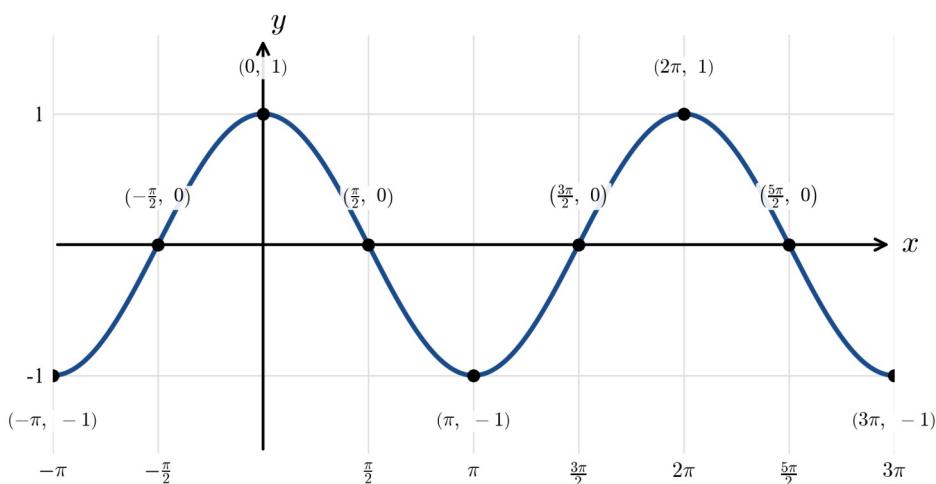
Example 4 — unscaffolded

Sketch $y = \cos x$ for $x \in [-\pi, 3\pi]$ and hence state the x -intercepts in this interval.

Cosine has period 2π , so this domain of length 4π shows two complete cycles.

x -intercepts: $\cos x = 0$ at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$.

$\therefore$ there are four x -intercepts in $[-\pi, 3\pi]$.



Practice questions

Scaffolded

Q1. For $y = \sin x$ on $x \in [0, 4\pi]$: (a) state the period; (b) find all x -intercepts; (c) give the coordinates of the maximum and minimum points; (d) sketch the graph.

Q2. For $y = \cos x$ on $x \in [-2\pi, 2\pi]$: (a) state whether the function is odd or even, and what symmetry the graph has; (b) find all x -intercepts; (c) sketch the graph.

Q3. For $y = \sin x$ on $x \in [-\pi, \pi]$: (a) state the symmetry of the graph; (b) state the range; (c) sketch the graph, labelling the maximum and minimum.

Q4. For $y = \cos x$ on $x \in [0, 4\pi]$: (a) how many complete cycles are shown? (b) give the coordinates of all maximum points; (c) sketch the graph.

Q5. Consider $y = \sin x$ on $x \in [0, 2\pi]$. (a) On the one set of axes, sketch $y = \sin x$ and the line $y = \frac{1}{2}$. (b) Hence state the values of x for which $\sin x = \frac{1}{2}$.

Q6. Consider $y = \cos x$ on $x \in [0, 2\pi]$. (a) On the one set of axes, sketch $y = \cos x$ and the line $y = -\frac{1}{2}$. (b) Hence state the values of x for which $\cos x = -\frac{1}{2}$.

Unscaffolded

Q7. Sketch $y = \sin x$ for $x \in [0, 3\pi]$ and state the coordinates of all maximum points.

Q8. Sketch $y = \cos x$ for $x \in [-\pi, 2\pi]$ and state all x -intercepts.

Q9. Sketch $y = \sin x$ for $x \in \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$ and state the range and the coordinates of all minimum points.

Q10. Sketch $y = \cos x$ for $x \in [0, 3\pi]$ and state the coordinates of all minimum points.

Q11. Sketch $y = \sin x$ for $x \in [-2\pi, 0]$ and state the x -intercepts and the coordinates of the maximum and minimum points.

Q12. For $y = \sin x$ on $x \in [0, 2\pi]$, find the x -intercepts and the coordinates of the maximum and minimum points.

Q13. For $y = \cos x$ on $x \in [0, 2\pi]$, find the x -intercepts and the coordinates of the maximum and minimum points.

Q14. Sketch $y = \sin x$ for $x \in [-\pi, 3\pi]$ and state the coordinates of all maximum points.

Q15. Using the graph of $y = \sin x$ and its periodicity, evaluate $\sin \frac{5\pi}{2}$ and $\sin 3\pi$.

Q16. Using the graph of $y = \cos x$ and its periodicity, evaluate $\cos \frac{9\pi}{2}$ and $\cos 4\pi$.

Q17. Using the graph of $y = \sin x$, evaluate $\sin\left(-\frac{3\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$.

Q18. Sketch $y = \cos x$ for $x \in [-2\pi, 2\pi]$ and use the graph to explain why $\cos$ is an even function.

Q19. (a) On the one set of axes sketch $y = \sin x$ and $y = \frac{1}{2}$ for $x \in [0, 2\pi]$. (b) Hence find the values of x for which $\sin x = \frac{1}{2}$.

Q20. (a) On the one set of axes sketch $y = \sin x$ and $y = \cos x$ for $x \in [0, 2\pi]$. (b) Hence find the values of x for which $\sin x = \cos x$.

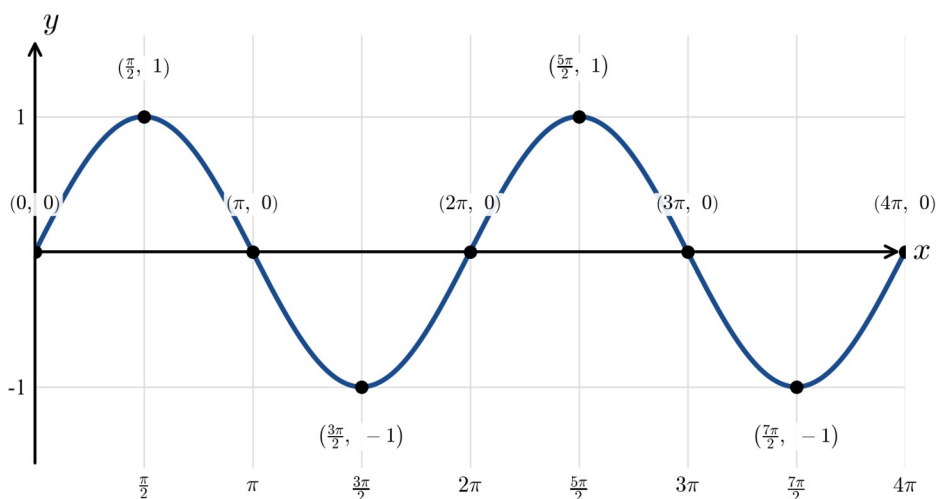
Q21. (a) On the one set of axes sketch $y = \cos x$ and $y = -\frac{1}{2}$ for $x \in [0, 2\pi]$. (b) Hence solve $\cos x = -\frac{1}{2}$ for $x \in [0, 2\pi]$.

Answers

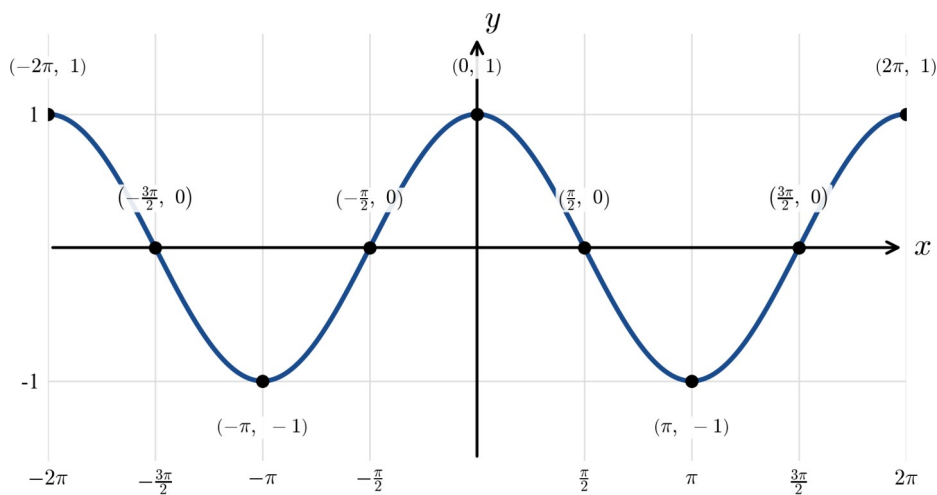
Q1. Period 2π ; intercepts $0, \pi, 2\pi, 3\pi, 4\pi$; maxima $\left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{2}, 1\right)$; minima $\left(\frac{3\pi}{2}, -1\right), \left(\frac{7\pi}{2}, -1\right)$. **Q2.** Even; symmetric about the y -axis; intercepts $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. **Q3.** Point symmetry about the origin (odd); range $[-1, 1]$; max $\left(\frac{\pi}{2}, 1\right)$, min $\left(-\frac{\pi}{2}, -1\right)$. **Q4.** Two complete cycles; maxima $(0, 1), (2\pi, 1), (4\pi, 1)$. **Q5.** $x = \frac{\pi}{6}, \frac{5\pi}{6}$. **Q6.** $x = \frac{2\pi}{3}, \frac{4\pi}{3}$. **Q7.** Maxima $\left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{2}, 1\right)$. **Q8.** $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. **Q9.** Range $[-1, 1]$; minima $\left(-\frac{\pi}{2}, -1\right), \left(\frac{3\pi}{2}, -1\right)$. **Q10.** Minima $(\pi, -1), (3\pi, -1)$. **Q11.** Intercepts $-2\pi, -\pi, 0$; max $\left(-\frac{3\pi}{2}, 1\right)$; min $\left(-\frac{\pi}{2}, -1\right)$. **Q12.** Intercepts $0, \pi, 2\pi$; max $\left(\frac{\pi}{2}, 1\right)$; min $\left(\frac{3\pi}{2}, -1\right)$. **Q13.** Intercepts $\frac{\pi}{2}, \frac{3\pi}{2}$; maxima $(0, 1), (2\pi, 1)$; min $(\pi, -1)$. **Q14.** Maxima $\left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{2}, 1\right)$. **Q15.** $\sin \frac{5\pi}{2} = 1$; $\sin 3\pi = 0$. **Q16.** $\cos \frac{9\pi}{2} = 0$; $\cos 4\pi = 1$. **Q17.** $\sin\left(-\frac{3\pi}{2}\right) = 1$; $\sin\left(-\frac{\pi}{2}\right) = -1$. **Q18.** The graph is symmetric about the y -axis, so $\cos(-x) = \cos x$ — $\cos$ is even. **Q19.** $x = \frac{\pi}{6}, \frac{5\pi}{6}$. **Q20.** $x = \frac{\pi}{4}, \frac{5\pi}{4}$. **Q21.** $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

Fully-worked solutions

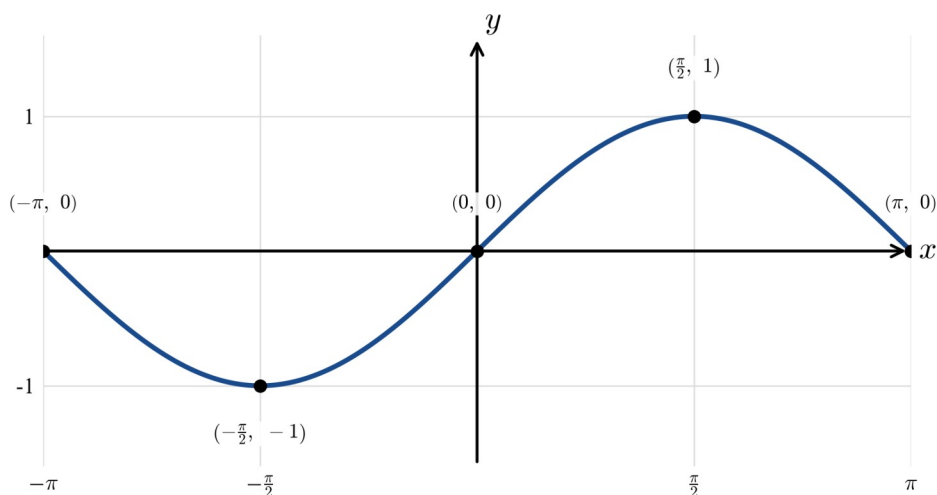
Q1. Period 2π , so $[0, 4\pi]$ shows two cycles. $\sin x = 0$ at $x = 0, \pi, 2\pi, 3\pi, 4\pi$; maxima where $\sin x = 1$ at $x = \frac{\pi}{2}, \frac{5\pi}{2}$; minima where $\sin x = -1$ at $x = \frac{3\pi}{2}, \frac{7\pi}{2}$.



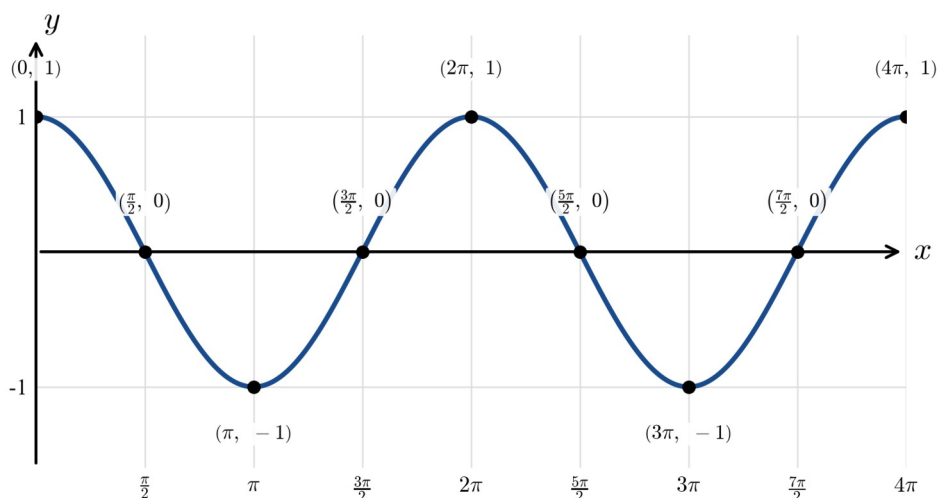
Q2. $\cos$ is even: $\cos(-x) = \cos x$, so the graph is symmetric about the y -axis. On $[-2\pi, 2\pi]$, $\cos x = 0$ at $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.



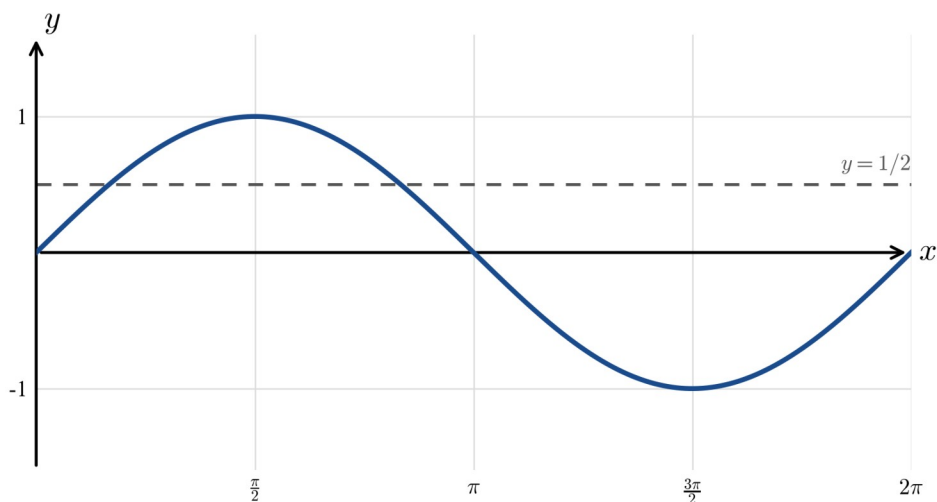
Q3. $\sin$ is **odd**, so the graph has point symmetry about the origin; range $[-1, 1]$. On $[-\pi, \pi]$ the maximum is $\left(\frac{\pi}{2}, 1\right)$ and the minimum is $\left(-\frac{\pi}{2}, -1\right)$.



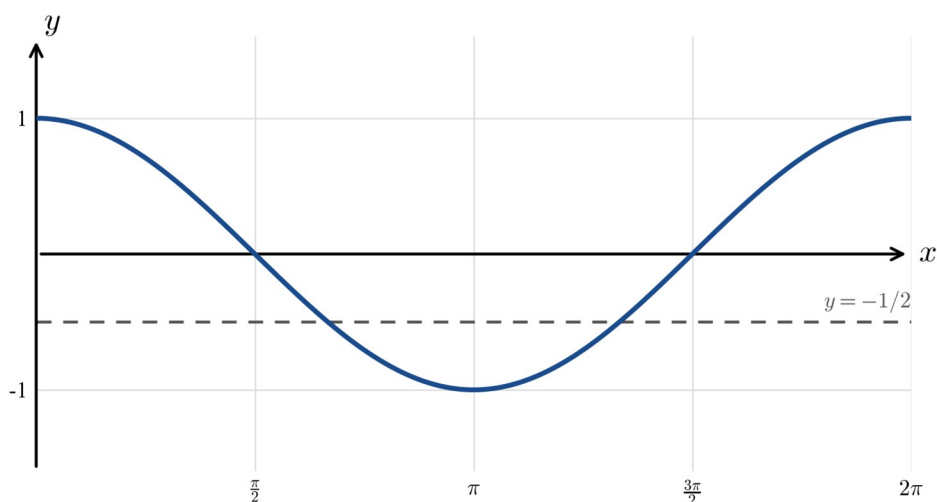
Q4. Period 2π , so $[0, 4\pi]$ shows **two** complete cycles. Maxima ($\cos x = 1$) at $x = 0, 2\pi, 4\pi$, giving $(0, 1), (2\pi, 1), (4\pi, 1)$.



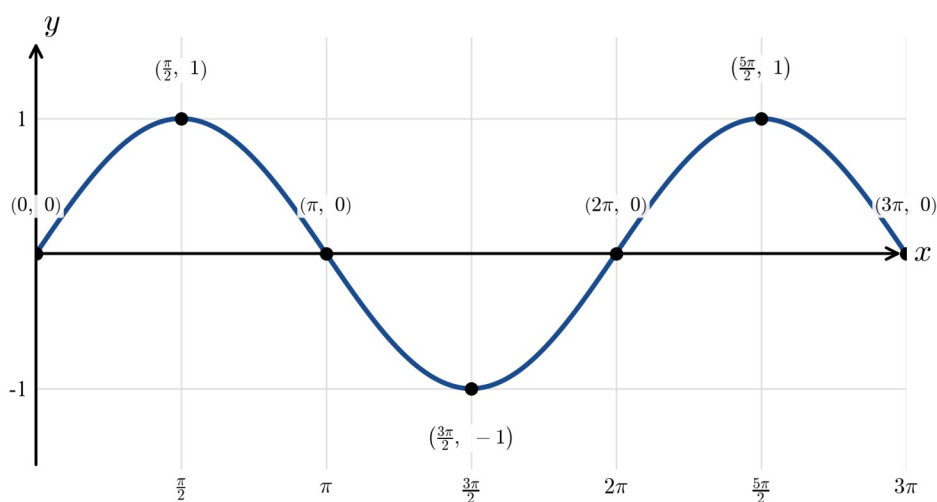
Q5. The line $y = \frac{1}{2}$ cuts the sine curve twice in $[0, 2\pi]$. Reading the intersections, $\sin x = \frac{1}{2}$ at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.



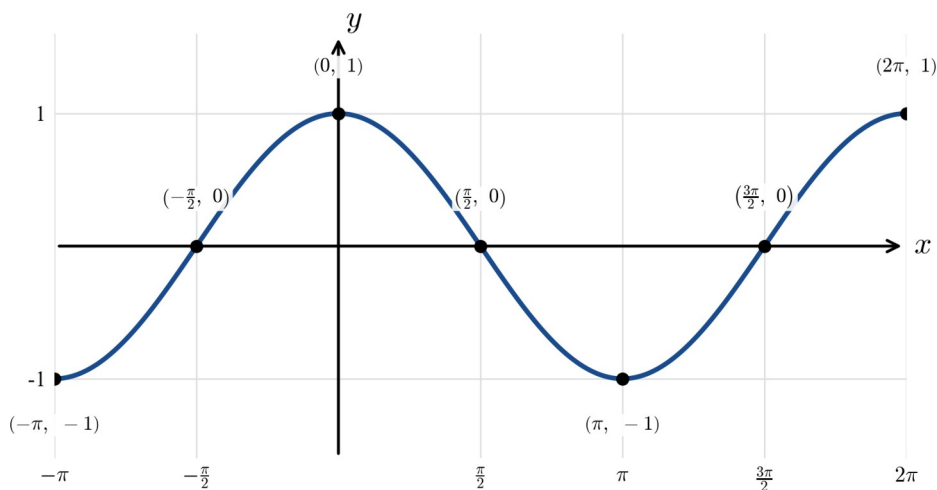
Q6. The line $y = -\frac{1}{2}$ cuts the cosine curve twice in $[0, 2\pi]$. Reading the intersections, $\cos x = -\frac{1}{2}$ at $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.



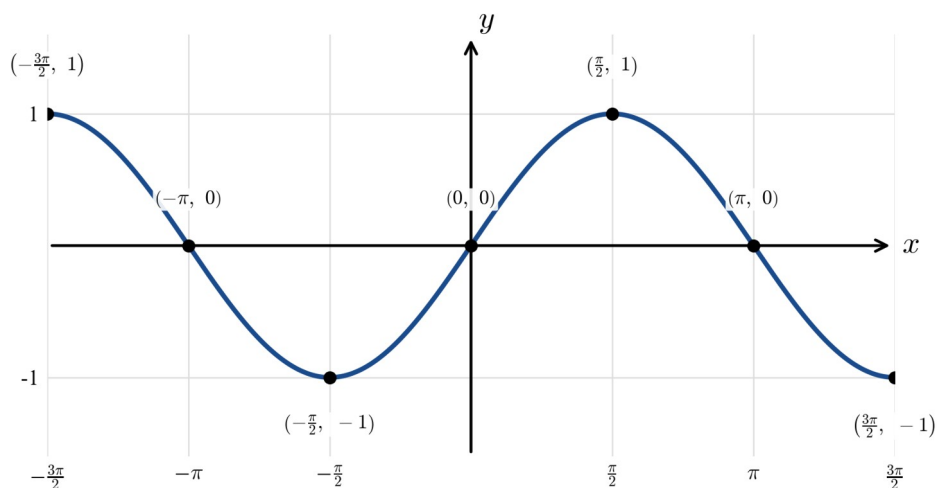
Q7. $[0, 3\pi]$ shows one and a half cycles. Maxima ($\sin x = 1$) at $x = \frac{\pi}{2}, \frac{5\pi}{2}$, giving $(\frac{\pi}{2}, 1)$ and $(\frac{5\pi}{2}, 1)$.



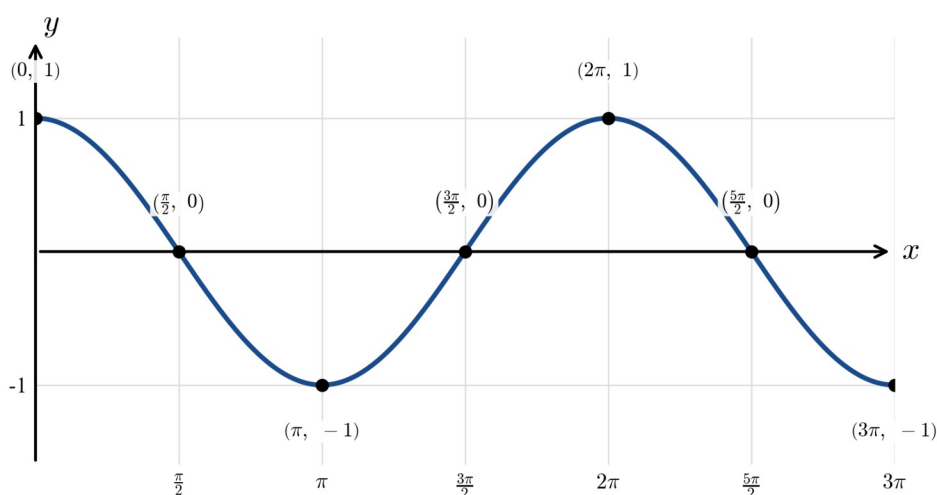
Q8. $\cos x = 0$ on $[-\pi, 2\pi]$ at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.



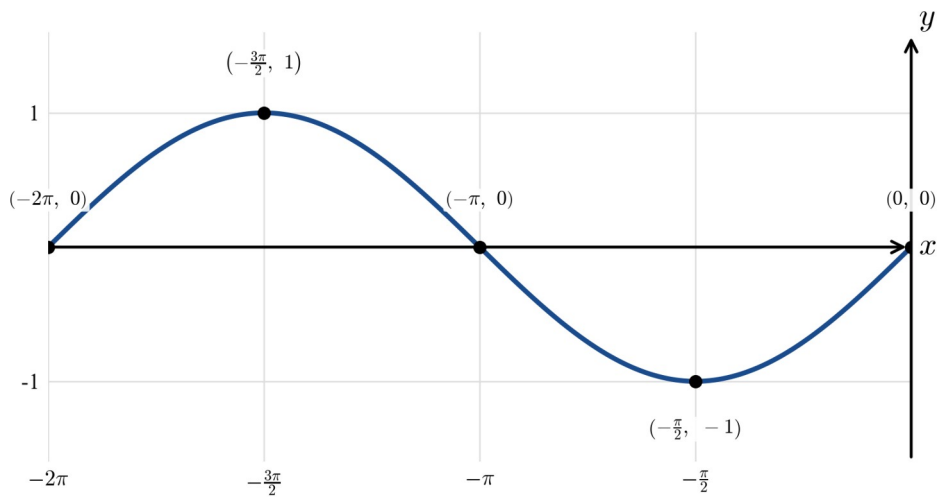
Q9. Range $[-1, 1]$. On $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$, $\sin x = -1$ at $x = -\frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, giving minima $\left(-\frac{\pi}{2}, -1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$.



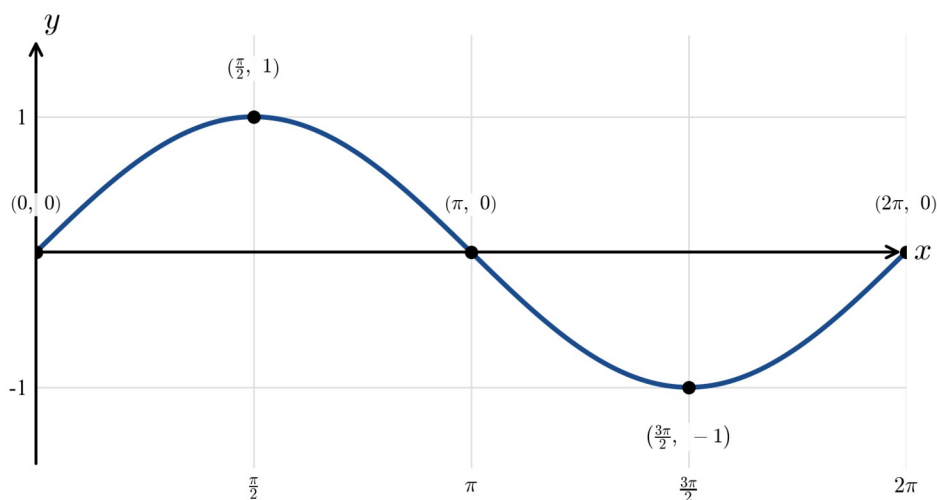
Q10. On $[0, 3\pi]$, $\cos x = -1$ at $x = \pi$ and $x = 3\pi$, giving minima $(\pi, -1)$ and $(3\pi, -1)$.



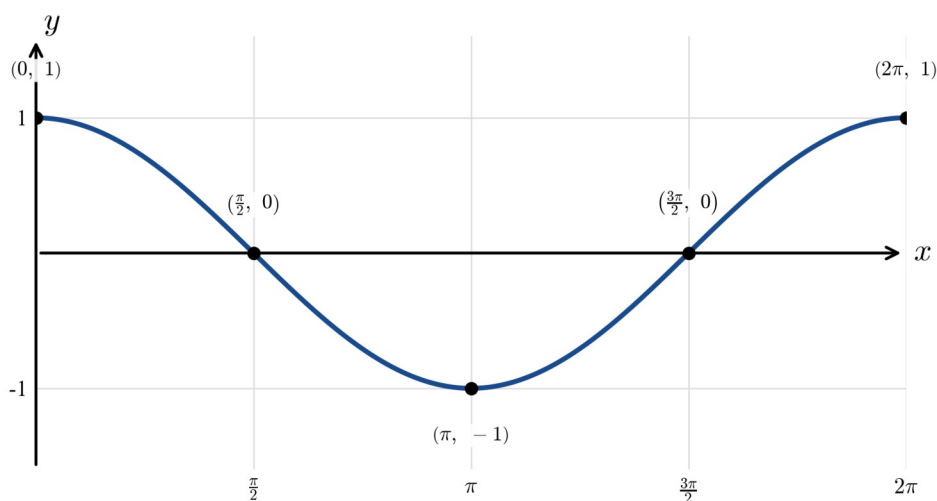
Q11. On $[-2\pi, 0]$: $\sin x = 0$ at $x = -2\pi, -\pi, 0$; maximum $\left(-\frac{3\pi}{2}, 1\right)$; minimum $\left(-\frac{\pi}{2}, -1\right)$.



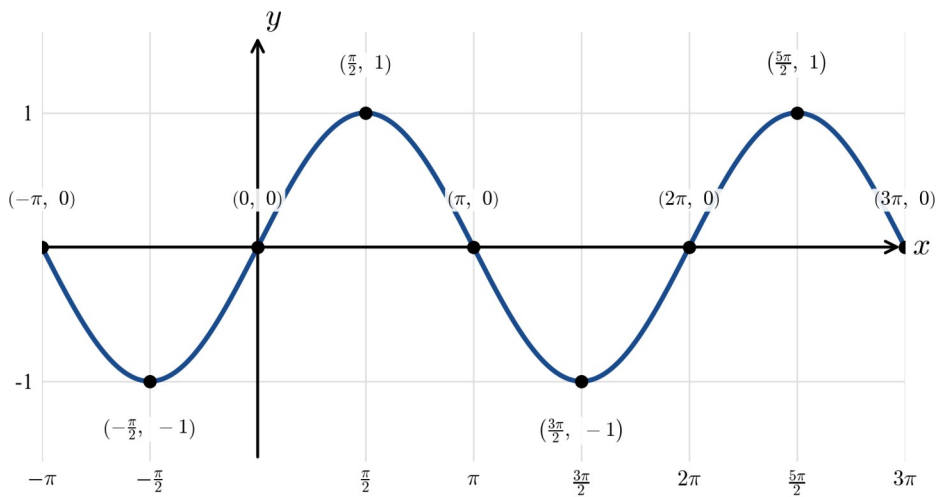
Q12. Intercepts $x = 0, \pi, 2\pi$; maximum $(\frac{\pi}{2}, 1)$; minimum $(\frac{3\pi}{2}, -1)$.



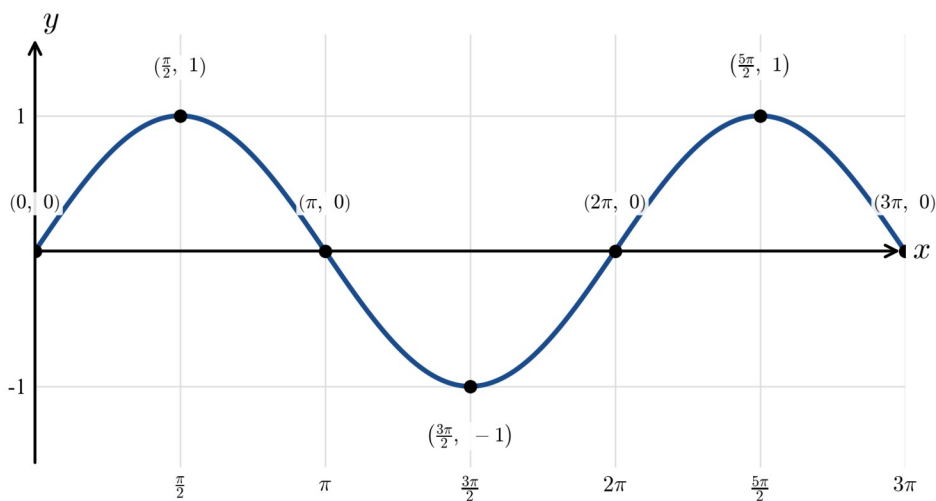
Q13. Intercepts $x = \frac{\pi}{2}, \frac{3\pi}{2}$; maxima $(0, 1)$ and $(2\pi, 1)$; minimum $(\pi, -1)$.



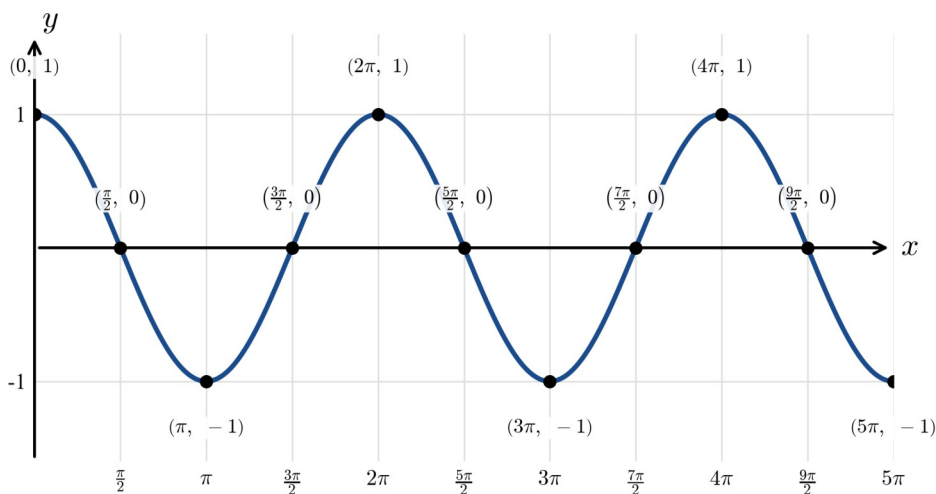
Q14. On $[-\pi, 3\pi]$, $\sin x = 1$ at $x = \frac{\pi}{2}, \frac{5\pi}{2}$, giving maxima $(\frac{\pi}{2}, 1)$ and $(\frac{5\pi}{2}, 1)$.



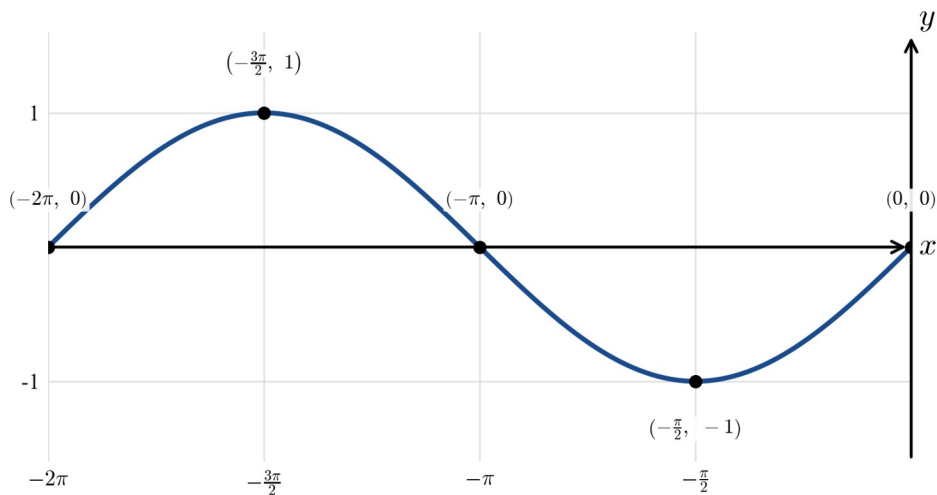
Q15. $\frac{5\pi}{2} = \frac{\pi}{2} + 2\pi$, so $\sin \frac{5\pi}{2} = \sin \frac{\pi}{2} = 1$. $3\pi = \pi + 2\pi$, so $\sin 3\pi = \sin \pi = 0$.



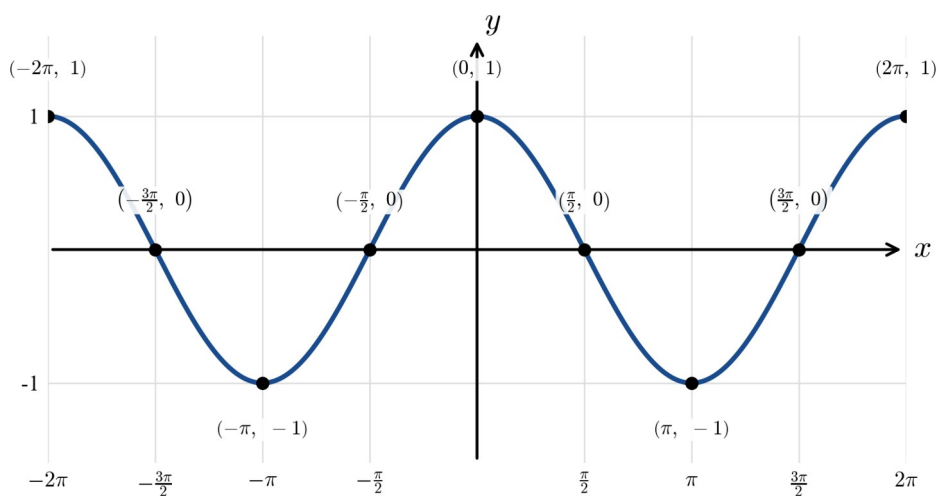
Q16. $\frac{9\pi}{2} = \frac{\pi}{2} + 4\pi$, so $\cos \frac{9\pi}{2} = \cos \frac{\pi}{2} = 0$. $4\pi = 0 + 4\pi$, so $\cos 4\pi = \cos 0 = 1$.



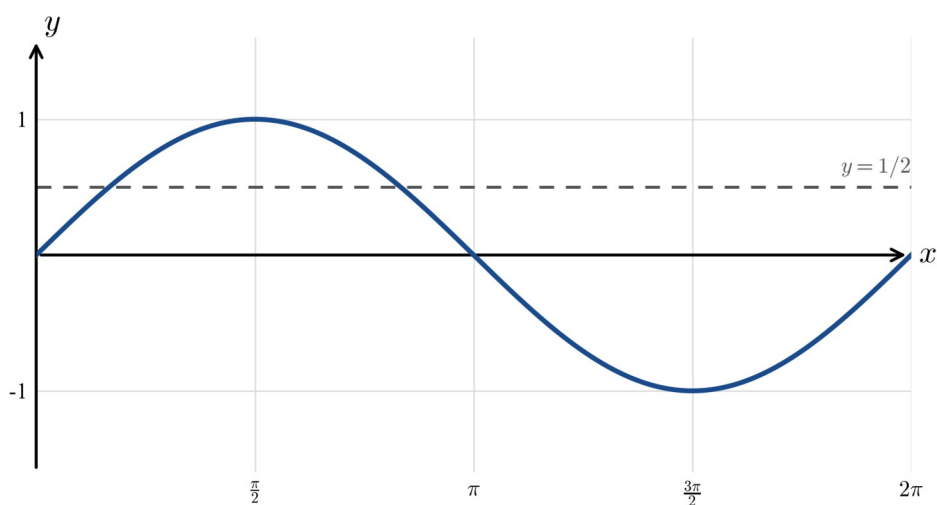
Q17. From the graph on $[-2\pi, 0]$, $\sin\left(-\frac{3\pi}{2}\right) = 1$ (a maximum) and $\sin\left(-\frac{\pi}{2}\right) = -1$ (a minimum). (Equivalently, $\sin$ is odd: $\sin\left(-\frac{\pi}{2}\right) = -\sin \frac{\pi}{2} = -1$.)



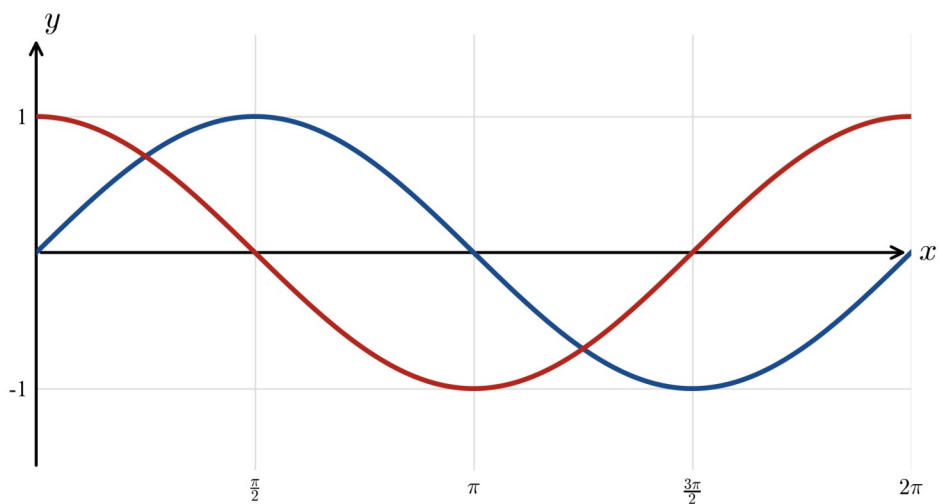
Q18. The graph is symmetric about the y -axis: reflecting any point $(x, \cos x)$ in the y -axis gives $(-x, \cos x)$, which is also on the graph. Hence $\cos(-x) = \cos x$ for all x , so $\cos$ is an **even** function.



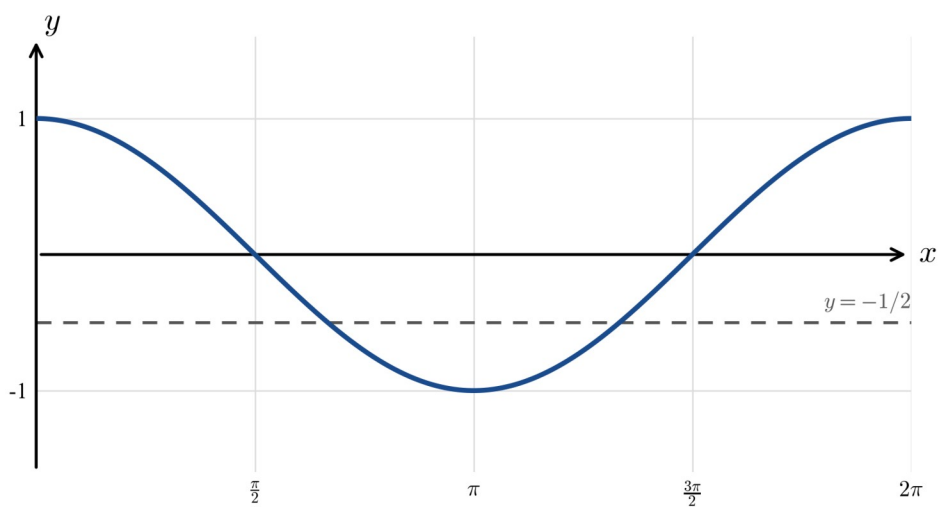
Q19. The line $y = \frac{1}{2}$ meets $y = \sin x$ where $\sin x = \frac{1}{2}$, i.e. at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ in $[0, 2\pi]$.



Q20. The curves $y = \sin x$ and $y = \cos x$ intersect where $\sin x = \cos x$, i.e. at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ in $[0, 2\pi]$.



Q21. The line $y = -\frac{1}{2}$ meets $y = \cos x$ where $\cos x = -\frac{1}{2}$, i.e. at $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$ in $[0, 2\pi]$.



Chapter 6 Circular functions — 6E Solving trigonometric equations

Theory

A trigonometric equation such as $\sin x = a$ usually has **many** solutions, because the circular functions are periodic. To solve one over a given domain we find the **base angle** and then use the **quadrant signs** to locate every solution in the domain.

The base (reference) angle. The base angle α is the *acute* angle (in $\left[0, \frac{\pi}{2}\right]$) whose sine, cosine or tangent equals the **magnitude** $|a|$. For the standard exact values:

$ a $	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
base angle for sin / cos	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	—	$\frac{\pi}{2}$	—
base angle for tan	—	—	—	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

Quadrant signs (CAST). In the four quadrants of $[0, 2\pi]$: All functions are positive in the 1st; Sine only in the 2nd; Tangent only in the 3rd; Cosine only in the 4th. So, using the base angle α :

- $\sin x = a$: solutions in the 1st & 2nd quadrants ($x = \alpha, \pi - \alpha$) when $a > 0$; in the 3rd & 4th ($x = \pi + \alpha, 2\pi - \alpha$) when $a < 0$.
- $\cos x = a$: 1st & 4th ($x = \alpha, 2\pi - \alpha$) when $a > 0$; 2nd & 3rd ($x = \pi - \alpha, \pi + \alpha$) when $a < 0$.
- $\tan x = a$: 1st & 3rd ($x = \alpha, \pi + \alpha$) when $a > 0$; 2nd & 4th ($x = \pi - \alpha, 2\pi - \alpha$) when $a < 0$.

Multiple-angle equations. For $\sin(nx) = a$ (or $\cos, \tan$), **substitute** $u = nx$. If $x \in [0, 2\pi]$ then $u \in [0, 2\pi n]$, so solve over the *larger* domain — there will be about n times as many solutions — and then divide each by n .

Method. (1) rearrange to $\sin x = a$, $\cos x = a$ or $\tan x = a$; (2) find the base angle α ; (3) decide the quadrants from the sign of a ; (4) write every solution in the domain, exactly.

Worked examples

Example 1 — scaffolded

Solve $2 \sin x = \sqrt{3}$ for $x \in [0, 2\pi]$.

Working	Comment
$\sin x = \frac{\sqrt{3}}{2}$	Divide both sides by 2.
Base angle: $\sin \alpha = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{3}$	The acute angle with this sine.
$\sin x > 0$, so x is in the 1st or 2nd quadrant.	Sine is positive in quadrants 1 and 2.
$x = \frac{\pi}{3}$ or $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$	1st-quadrant solution α ; 2nd-quadrant solution $\pi - \alpha$.
$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}$	Both lie in $[0, 2\pi]$.

Example 2 — scaffolded

Solve $\sqrt{2} \cos x + 1 = 0$ for $x \in [0, 2\pi]$.

Working	Comment
$\cos x = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$	Isolate $\cos x$ and rationalise.
Base angle: $\cos \alpha = \frac{\sqrt{2}}{2} \Rightarrow \alpha = \frac{\pi}{4}$	Use the magnitude for the base angle.
$\cos x < 0$, so x is in the 2nd or 3rd quadrant.	Cosine is negative in quadrants 2 and 3.
$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ or $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$	2nd quadrant $\pi - \alpha$; 3rd quadrant $\pi + \alpha$.
$\therefore x = \frac{3\pi}{4}, \frac{5\pi}{4}$	

Example 3 — unscaffolded

Solve $\tan x = \sqrt{3}$ for $x \in [0, 2\pi]$.

Base angle: $\tan \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$. Since $\tan x > 0$, x is in the 1st or 3rd quadrant, so $x = \frac{\pi}{3}$ or $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

$$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$$

Example 4 — unscaffolded

Solve $2 \sin(2x) = \sqrt{2}$ for $x \in [0, 2\pi]$.

$\sin(2x) = \frac{\sqrt{2}}{2}$. Let $u = 2x$; as $x \in [0, 2\pi]$, $u \in [0, 4\pi]$.

Base angle $\frac{\pi}{4}$, and $\sin u > 0$ (quadrants 1 and 2), so over $[0, 4\pi]$: $u = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
 $u = \frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$, $u = \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$.

Then $x = \frac{u}{2}$:

$$\therefore x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

Practice questions

Scaffolded

Q1. Solve $2 \cos x = 1$ for $x \in [0, 2\pi]$. (a) Write the equation in the form $\cos x = a$. (b) State the base angle and the quadrants in which the solutions lie. (c) Find all solutions in the domain.

Q2. Solve $2 \sin x + 1 = 0$ for $x \in [0, 2\pi]$. (a) Write in the form $\sin x = a$. (b) State the base angle and quadrants. (c) Find all solutions.

Q3. Solve $\sqrt{3} \tan x = 1$ for $x \in [0, 2\pi]$. (a) Write in the form $\tan x = a$. (b) State the base angle and quadrants. (c) Find all solutions.

Q4. Solve $\sqrt{2} \sin x = 1$ for $x \in [0, 2\pi]$. (a) Write in the form $\sin x = a$. (b) State the base angle and quadrants. (c) Find all solutions.

Q5. Solve $2 \cos x + \sqrt{3} = 0$ for $x \in [0, 2\pi]$. (a) Write in the form $\cos x = a$. (b) State the base angle and quadrants. (c) Find all solutions.

Q6. Solve $\tan x + 1 = 0$ for $x \in [0, 2\pi]$. (a) Write in the form $\tan x = a$. (b) State the base angle and quadrants. (c) Find all solutions.

Unscaffolded

Solve each equation for x over the given domain, giving exact values.

Q7. $\tan x = 1, x \in [0, 2\pi]$

Q8. $2 \cos x + 1 = 0, x \in [0, 2\pi]$

Q9. $\sqrt{3} \tan x + 1 = 0, x \in [0, 2\pi]$

Q10. $\sin x = 1, x \in [0, 2\pi]$

Q11. $\cos x = -1, x \in [0, 2\pi]$

Q12. $2 \sin(2x) = \sqrt{3}, x \in [0, 2\pi]$

Q13. $2 \cos(2x) = 1, x \in [0, 2\pi]$

Q14. $\tan(2x) = 1, x \in [0, 2\pi]$

Q15. $\sqrt{2} \sin x + 1 = 0, x \in [0, 2\pi]$

Q16. $2 \cos x = \sqrt{2}, x \in [0, 2\pi]$

Q17. $2 \sin(3x) = \sqrt{3}, x \in [0, 2\pi]$

Q18. $\tan x = 0, x \in [0, 2\pi]$

Q19. $2 \sin x + \sqrt{3} = 0, x \in [0, 2\pi]$

Q20. Solve $\cos x = \frac{\sqrt{3}}{2}$ for $x \in [-\pi, \pi]$, giving all solutions.

Q21. Find all solutions of $2 \sin x - 1 = 0$ for $x \in [0, 4\pi]$.

Answers

Q1. $x = \frac{\pi}{3}, \frac{5\pi}{3}$ **Q2.** $x = \frac{7\pi}{6}, \frac{11\pi}{6}$ **Q3.** $x = \frac{\pi}{6}, \frac{7\pi}{6}$ **Q4.** $x = \frac{\pi}{4}, \frac{3\pi}{4}$ **Q5.** $x = \frac{5\pi}{6}, \frac{7\pi}{6}$ **Q6.** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ **Q7.** $x = \frac{\pi}{4}, \frac{5\pi}{4}$
Q8. $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ **Q9.** $x = \frac{5\pi}{6}, \frac{11\pi}{6}$ **Q10.** $x = \frac{\pi}{2}$ **Q11.** $x = \pi$ **Q12.** $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ **Q13.** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
Q14. $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$ **Q15.** $x = \frac{5\pi}{4}, \frac{7\pi}{4}$ **Q16.** $x = \frac{\pi}{4}, \frac{7\pi}{4}$ **Q17.** $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$ **Q18.**
 $x = 0, \pi, 2\pi$ **Q19.** $x = \frac{4\pi}{3}, \frac{5\pi}{3}$ **Q20.** $x = -\frac{\pi}{6}, \frac{\pi}{6}$ **Q21.** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

Fully-worked solutions

Q1. $\cos x = \frac{1}{2}$; base angle $\frac{\pi}{3}$; $\cos x > 0 \rightarrow$ quadrants 1 & 4: $x = \frac{\pi}{3}$ or $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Q2. $\sin x = -\frac{1}{2}$; base angle $\frac{\pi}{6}$; $\sin x < 0 \rightarrow$ quadrants 3 & 4: $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ or $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Q3. $\tan x = \frac{1}{\sqrt{3}}$; base angle $\frac{\pi}{6}$; $\tan x > 0 \rightarrow$ quadrants 1 & 3: $x = \frac{\pi}{6}$ or $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

Q4. $\sin x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$; base angle $\frac{\pi}{4}$; $\sin x > 0 \rightarrow$ quadrants 1 & 2: $x = \frac{\pi}{4}$ or $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

Q5. $\cos x = -\frac{\sqrt{3}}{2}$; base angle $\frac{\pi}{6}$; $\cos x < 0 \rightarrow$ quadrants 2 & 3: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ or $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

Q6. $\tan x = -1$; base angle $\frac{\pi}{4}$; $\tan x < 0 \rightarrow$ quadrants 2 & 4: $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ or $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Q7. $\tan x = 1$; base $\frac{\pi}{4}$; $\tan$ positive in quadrants 1 & 3: $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

Q8. $\cos x = -\frac{1}{2}$; base $\frac{\pi}{3}$; quadrants 2 & 3: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$, $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Q9. $\tan x = -\frac{1}{\sqrt{3}}$; base $\frac{\pi}{6}$; $\tan x < 0 \rightarrow$ quadrants 2 & 4: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$, $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Q10. $\sin x = 1$ at the boundary between quadrants 1 and 2: $x = \frac{\pi}{2}$ only.

Q11. $\cos x = -1$ at the boundary between quadrants 2 and 3: $x = \pi$ only.

Q12. $\sin(2x) = \frac{\sqrt{3}}{2}$. Let $u = 2x \in [0, 4\pi]$, base $\frac{\pi}{3}$, quadrants 1 & 2: $u = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$. So
 $x = \frac{u}{2} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$.

Q13. $\cos(2x) = \frac{1}{2}$. $u = 2x \in [0, 4\pi]$, base $\frac{\pi}{3}$, quadrants 1 & 4: $u = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$. So $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Q14. $\tan(2x) = 1$. $u = 2x \in [0, 4\pi]$, base $\frac{\pi}{4}$, quadrants 1 & 3: $u = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$. So $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$.

Q15. $\sin x = -\frac{\sqrt{2}}{2}$; base $\frac{\pi}{4}$; quadrants 3 & 4: $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$, $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Q16. $\cos x = \frac{\sqrt{2}}{2}$; base $\frac{\pi}{4}$; quadrants 1 & 4: $x = \frac{\pi}{4}$, $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Q17. $\sin(3x) = \frac{\sqrt{3}}{2}$. Let $u = 3x \in [0, 6\pi]$, base $\frac{\pi}{3}$, quadrants 1 & 2 across three cycles:

$u = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}$. So $x = \frac{u}{3} = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$.

Q18. $\tan x = 0$ where $\sin x = 0$: $x = 0, \pi, 2\pi$ (the endpoints are included since the domain is closed).

Q19. $\sin x = -\frac{\sqrt{3}}{2}$; base $\frac{\pi}{3}$; quadrants 3 & 4: $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$, $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Q20. $\cos x = \frac{\sqrt{3}}{2}$; base $\frac{\pi}{6}$; $\cos x > 0$. Over $[-\pi, \pi]$ the solutions are $x = \frac{\pi}{6}$ and its reflection $x = -\frac{\pi}{6}$ (cosine is even).

$\therefore x = -\frac{\pi}{6}, \frac{\pi}{6}$.

Q21. $\sin x = \frac{1}{2}$; base $\frac{\pi}{6}$; quadrants 1 & 2. Over $[0, 4\pi]$ (two full cycles):

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}, \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$.

Chapter 6 Circular functions — 6F Sketching sine and cosine graphs

Theory

Every graph in this section is a **transformation of** $y = \sin t$ or $y = \cos t$. Writing the rule in the form $y = a \sin(n(t - \varepsilon))$ or $y = a \cos(n(t - \varepsilon))$ lets you read off each transformation directly.

Amplitude. The **amplitude** is $|a|$: the graph oscillates between $y = |a|$ and $y = -|a|$, so its range is $[-|a|, |a|]$. If $a < 0$ the basic graph is also **reflected in the t -axis**.

Period. The **period** is $T = \frac{2\pi}{n}$ — the horizontal length of one complete cycle. A larger n compresses the graph (more cycles); n is a dilation of factor $\frac{1}{n}$ from the y -axis.

Phase (horizontal translation). In the form $y = a \sin(n(t - \varepsilon))$ the graph of $y = a \sin(nt)$ is translated ε **units in the positive t -direction** (to the right if $\varepsilon > 0$, to the left if $\varepsilon < 0$). Always factor out n first so the translation can be read correctly — e.g. $\sin(2t - \pi/2) = \sin(2(t - \pi/4))$, a translation of $\frac{\pi}{4}$, not $\frac{\pi}{2}$.

Key points. One cycle of a sine or cosine graph has five evenly spaced key points — spaced $\frac{T}{4}$ apart — at the maximum, minimum and where the graph crosses its centre line ($y = 0$ here). Mark these across the required domain, together with the **exact coordinates of the endpoints**.

To sketch $y = a \sin(n(t - \varepsilon))$ (or cosine) over a stated domain:

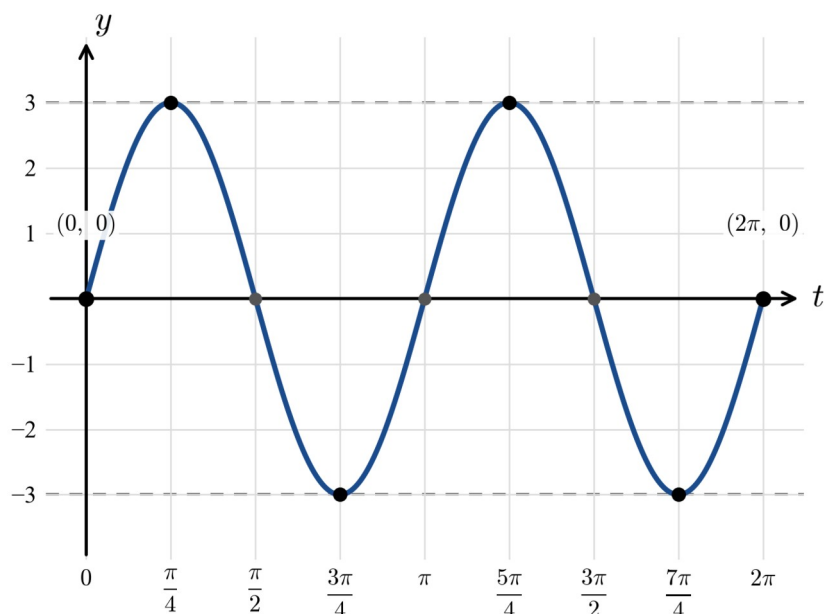
1. State the **amplitude** $|a|$ (note a reflection if $a < 0$) and the **period** $T = \frac{2\pi}{n}$.
2. Identify the **phase** ε (after factoring out n).
3. Locate the first cycle's key points from the starting point, then repeat every T .
4. Draw the curve through the key points and label the **endpoints** with exact coordinates; the sketch shows key features and is **not to scale**.

Worked examples

Example 1 — scaffolded

Sketch $y = 3 \sin(2t)$ for $t \in [0, 2\pi]$, stating the amplitude and period and labelling all key points.

Working	Comment
Compare with $y = a \sin(nt)$: $a = 3$, $n = 2$.	Read the parameters from the rule.
Amplitude $= a = 3$; period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$.	Range $[-3, 3]$; one cycle every π .
t -intercepts where $\sin(2t) = 0$: $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.	$2t = 0, \pi, 2\pi, \dots$
Maxima ($y = 3$) at $t = \frac{\pi}{4}, \frac{5\pi}{4}$; minima ($y = -3$) at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$.	Quarter-period spacing $\frac{T}{4} = \frac{\pi}{4}$.
Endpoints $(0, 0)$ and $(2\pi, 0)$.	Substitute the domain ends.



Example 2 — scaffolded

Sketch $y = 2 \cos(2(t - \pi/4))$ for $t \in [0, 2\pi]$, stating the amplitude, period and phase.

Working

Comment

$$a = 2, n = 2, \varepsilon = \frac{\pi}{4}.$$

Already in factored form $a \cos(n(t - \varepsilon))$.

Amplitude 2; period $\frac{2\pi}{2} = \pi$; phase $\frac{\pi}{4}$ to the right.

The basic $y = 2 \cos(2t)$ is shifted right by $\frac{\pi}{4}$.

At $t = 0$: $y = 2 \cos(-\pi/2) = 0$, so t -intercepts are $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

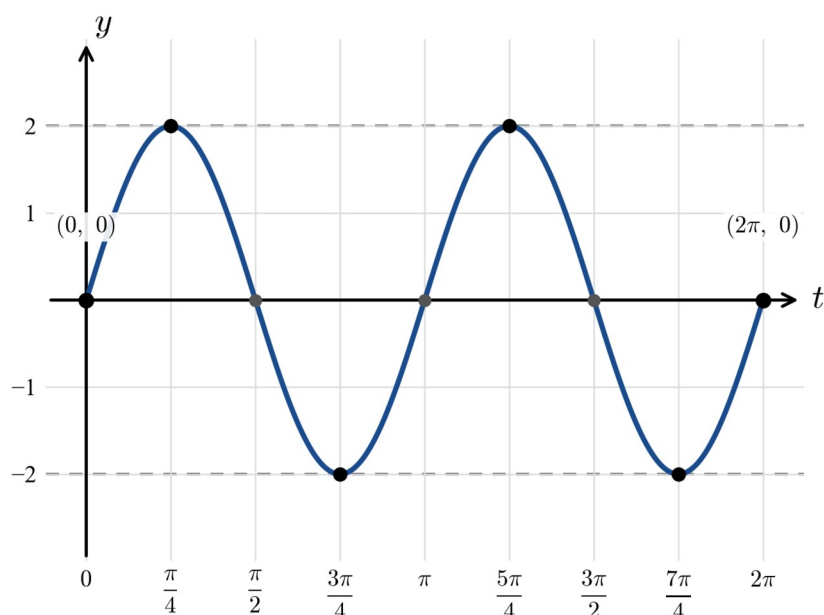
The shift moves the usual cosine maximum off $t = 0$.

Maxima at $t = \frac{\pi}{4}, \frac{5\pi}{4}$; minima at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$.

Maximum where $2(t - \pi/4) = 0, 2\pi, \dots$

Endpoints $(0, 0)$ and $(2\pi, 0)$.

Substitute the domain ends.



Example 3 — unscaffolded

Sketch $y = -2 \sin(t + \pi/3)$ for $t \in [0, 2\pi]$.

Writing the rule as $y = -2 \sin(t - (-\pi/3))$: $a = -2$, $n = 1$, $\varepsilon = -\frac{\pi}{3}$.

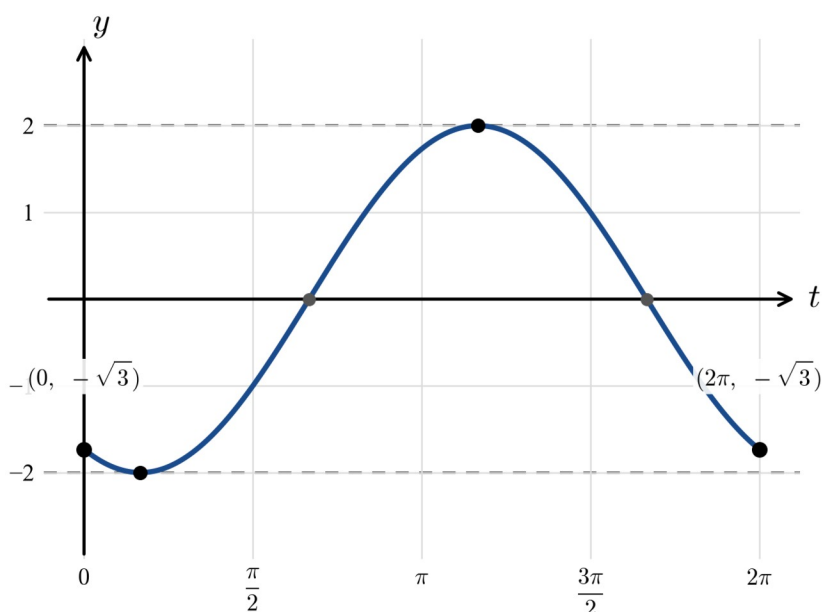
Amplitude 2 (graph **reflected** in the t -axis since $a < 0$); period 2π ; phase $\frac{\pi}{3}$ to the left.

t -intercepts: $\sin(t + \pi/3) = 0 \Rightarrow t = \frac{2\pi}{3}, \frac{5\pi}{3}$.

Because of the reflection, the **minimum** ($y = -2$) is at $t = \frac{\pi}{6}$ and the **maximum** ($y = 2$) at $t = \frac{7\pi}{6}$.

Endpoints: $t = 0 \Rightarrow y = -2 \sin \pi/3 = -\sqrt{3}$, and $t = 2\pi \Rightarrow y = -\sqrt{3}$, giving $(0, -\sqrt{3})$ and $(2\pi, -\sqrt{3})$.

$\therefore$ the graph runs from $(0, -\sqrt{3})$ down to the minimum, up through its intercepts to the maximum, and back to $(2\pi, -\sqrt{3})$.



Example 4 — unscaffolded

Sketch $y = 4 \cos(1/2 t)$ for $t \in [0, 4\pi]$.

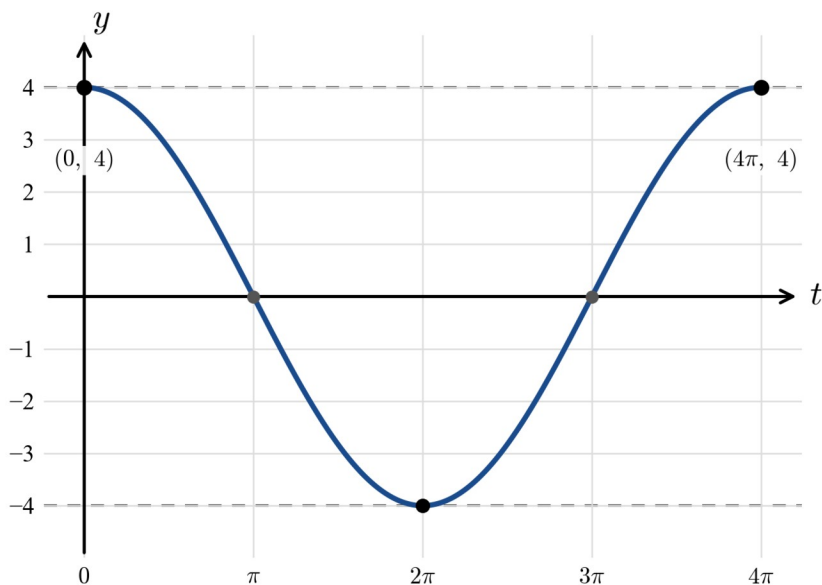
$a = 4$, $n = \frac{1}{2}$, $\varepsilon = 0$. Amplitude 4; period $\frac{2\pi}{1/2} = 4\pi$ — exactly one cycle over the domain.

t -intercepts: $\cos(1/2 t) = 0 \Rightarrow t = \pi, 3\pi$.

Maximum ($y = 4$) at $t = 0$ and $t = 4\pi$; minimum ($y = -4$) at $t = 2\pi$.

Endpoints $(0, 4)$ and $(4\pi, 4)$.

$\therefore$ the graph is a single cosine cycle stretched to width 4π and height 4.



Practice questions

Scaffolded

Q1. For $y = 2 \sin t$, $t \in [0, 2\pi]$: (a) state the amplitude and period; (b) find the t -intercepts and the coordinates of the maximum and minimum; (c) sketch the graph, labelling all key points.

Q2. For $y = 3 \cos(2t)$, $t \in [0, 2\pi]$: (a) state the amplitude and period; (b) find the t -intercepts and the maxima and minima; (c) sketch the graph.

Q3. For $y = \sin(t - \pi/2)$, $t \in [0, 2\pi]$: (a) state the amplitude, period and phase; (b) find the coordinates of the endpoints, maximum and minimum; (c) sketch the graph.

Q4. For $y = 2 \sin(3t)$, $t \in [0, 2\pi]$: (a) state the amplitude and period; (b) how many complete cycles occur in the domain? (c) sketch the graph, labelling the t -intercepts.

Q5. For $y = -3 \cos t$, $t \in [0, 2\pi]$: (a) state the amplitude, and describe the effect of the negative sign; (b) find the endpoints, maximum and minimum; (c) sketch the graph.

Q6. For $y = 2 \sin(2(t - \pi/6))$, $t \in [0, \pi]$: (a) state the amplitude, period and phase; (b) find the t -intercepts and the coordinates of the maximum and minimum; (c) sketch the graph, labelling the endpoints exactly.

Un scaffolded

Sketch each of the following over the stated domain, labelling the amplitude, period, key points and the exact coordinates of the endpoints.

Q7. $y = 4 \sin t$, $t \in [0, 2\pi]$

Q8. $y = 2 \cos(2t)$, $t \in [0, 2\pi]$

Q9. $y = 3 \cos(t - \pi/4)$, $t \in [0, 2\pi]$

Q10. $y = -2 \sin(2t)$, $t \in [0, 2\pi]$

Q11. $y = \sin(1/2t)$, $t \in [0, 4\pi]$

Q12. $y = 2 \cos(t + \pi/2)$, $t \in [0, 2\pi]$

Q13. $y = 3 \sin(3t)$, $t \in [0, \pi]$

Q14. $y = 2 \sin(2(t - \pi/4)), t \in [0, 2\pi]$

Q15. $y = -\sin t, t \in [0, 2\pi]$

Q16. $y = 2 \sin(t - \pi/3), t \in [0, 2\pi]$

Q17. $y = 3 \cos(2(t - \pi/2)), t \in [0, 2\pi]$

Q18. $y = 4 \cos(1/2(t - \pi)), t \in [0, 4\pi]$

Q19. State the amplitude and period of $y = 2 \cos(3(t - \pi/6))$, and hence sketch it for $t \in [0, \pi]$.

Q20. Describe the sequence of transformations that maps $y = \cos t$ onto $y = -2 \cos(2(t - \pi/4))$, and hence sketch it for $t \in [0, 2\pi]$.

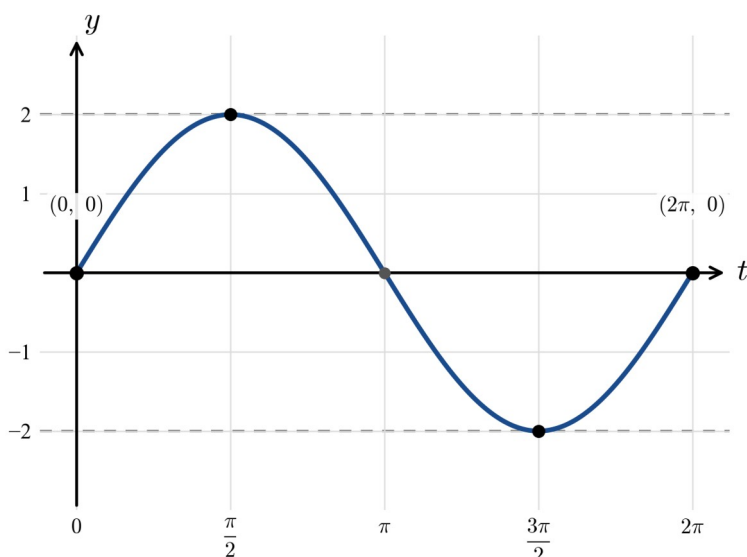
Q21. For $y = 3 \sin(t + \pi/6), t \in [0, 2\pi]$, state the amplitude, period and phase, find the exact endpoints, and hence sketch the graph.

Answers

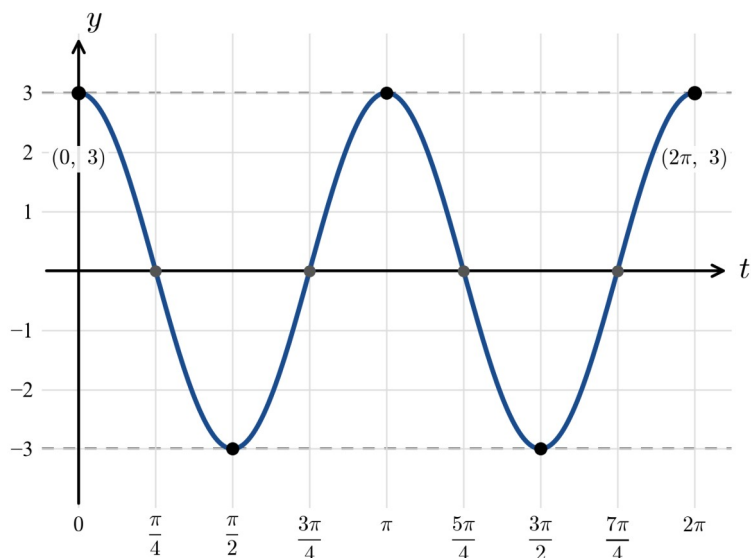
Q1. Amp 2, period 2π ; t -int $0, \pi, 2\pi$; max $(\pi/2, 2)$, min $(3\pi/2, -2)$. **Q2.** Amp 3, period π ; t -int $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$; max at $t=0, \pi, 2\pi$, min at $t=\pi/2, 3\pi/2$. **Q3.** Amp 1, period 2π , phase $\pi/2$; endpoints $(0, -1), (2\pi, -1)$; max $(\pi, 1)$, min at $t=0, 2\pi$. **Q4.** Amp 2, period $2\pi/3$; 3 complete cycles; t -int $0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3, 2\pi$. **Q5.** Amp 3, reflected in the t -axis; endpoints $(0, -3), (2\pi, -3)$; max $(\pi, 3)$, min at $t=0, 2\pi$. **Q6.** Amp 2, period π , phase $\pi/6$; t -int $\pi/6, 2\pi/3$; max $(5\pi/12, 2)$, min $(11\pi/12, -2)$; endpoints $(0, -\sqrt{3}), (\pi, -\sqrt{3})$. **Q7.** Amp 4, period 2π ; t -int $0, \pi, 2\pi$; max $(\pi/2, 4)$, min $(3\pi/2, -4)$. **Q8.** Amp 2, period π ; t -int $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$; max at $t=0, \pi, 2\pi$, min at $t=\pi/2, 3\pi/2$; endpoints $(0, 2), (2\pi, 2)$. **Q9.** Amp 3, period 2π , phase $\pi/4$; t -int $3\pi/4, 7\pi/4$; max $(\pi/4, 3)$, min $(5\pi/4, -3)$; endpoints $(0, 3\sqrt{2}/2), (2\pi, 3\sqrt{2}/2)$. **Q10.** Amp 2, period π , reflected; t -int $0, \pi/2, \pi, 3\pi/2, 2\pi$; max at $t=3\pi/4, 7\pi/4$, min at $t=\pi/4, 5\pi/4$. **Q11.** Amp 1, period 4π ; t -int $0, 2\pi, 4\pi$; max $(\pi, 1)$, min $(3\pi, -1)$. **Q12.** Amp 2, period 2π , phase $-\pi/2$; t -int $0, \pi, 2\pi$; max $(3\pi/2, 2)$, min $(\pi/2, -2)$; endpoints $(0, 0), (2\pi, 0)$. **Q13.** Amp 3, period $2\pi/3$; t -int $0, \pi/3, 2\pi/3, \pi$; max at $t=\pi/6, 5\pi/6$, min $(\pi/2, -3)$. **Q14.** Amp 2, period π , phase $\pi/4$; t -int $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$; max at $t=\pi/2, 3\pi/2$, min at $t=0, \pi, 2\pi$; endpoints $(0, -2), (2\pi, -2)$. **Q15.** Amp 1, reflected, period 2π ; t -int $0, \pi, 2\pi$; max $(3\pi/2, 1)$, min $(\pi/2, -1)$. **Q16.** Amp 2, period 2π , phase $\pi/3$; t -int $\pi/3, 4\pi/3$; max $(5\pi/6, 2)$, min $(11\pi/6, -2)$; endpoints $(0, -\sqrt{3}), (2\pi, -\sqrt{3})$. **Q17.** Amp 3, period π , phase $\pi/2$; t -int $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$; max at $t=\pi/2, 3\pi/2$, min at $t=0, \pi, 2\pi$; endpoints $(0, -3), (2\pi, -3)$. **Q18.** Amp 4, period 4π , phase π ; t -int $0, 2\pi, 4\pi$; max $(\pi, 4)$, min $(3\pi, -4)$; endpoints $(0, 0), (4\pi, 0)$. **Q19.** Amp 2, period $2\pi/3$; t -int $0, \pi/3, 2\pi/3, \pi$; max at $t=\pi/6, 5\pi/6$, min $(\pi/2, -2)$. **Q20.** Dilation factor 2 from the t -axis, reflection in the t -axis, dilation factor $1/2$ from the y -axis (period π), then translation $\pi/4$ right; t -int $0, \pi/2, \pi, 3\pi/2, 2\pi$; max at $t=3\pi/4, 7\pi/4$, min at $t=\pi/4, 5\pi/4$. **Q21.** Amp 3, period 2π , phase $-\pi/6$; t -int $5\pi/6, 11\pi/6$; max $(\pi/3, 3)$, min $(4\pi/3, -3)$; endpoints $(0, 3/2), (2\pi, 3/2)$.

Fully-worked solutions

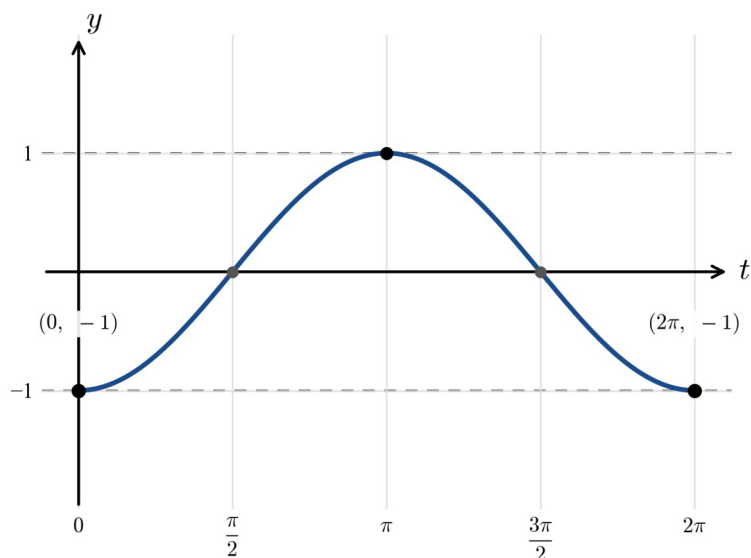
Q1. $y = 2 \sin t$. Amplitude 2, period $\frac{2\pi}{1} = 2\pi$. $\sin t = 0$ at $t = 0, \pi, 2\pi$; maximum $(\pi/2, 2)$, minimum $(3\pi/2, -2)$.



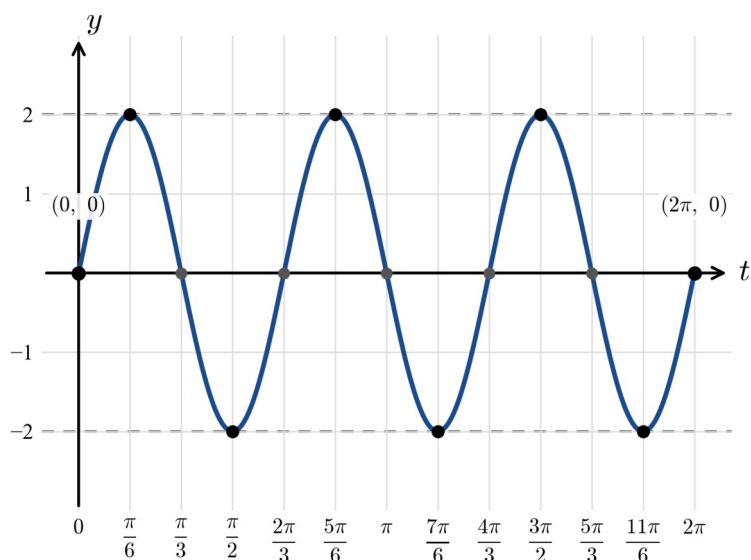
Q2. $y = 3 \cos(2t)$. Amplitude 3, period $\frac{2\pi}{2} = \pi$. $\cos(2t) = 0$ at $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; maxima at $t = 0, \pi, 2\pi$; minima at $t = \frac{\pi}{2}, \frac{3\pi}{2}$; endpoints $(0, 3), (2\pi, 3)$.



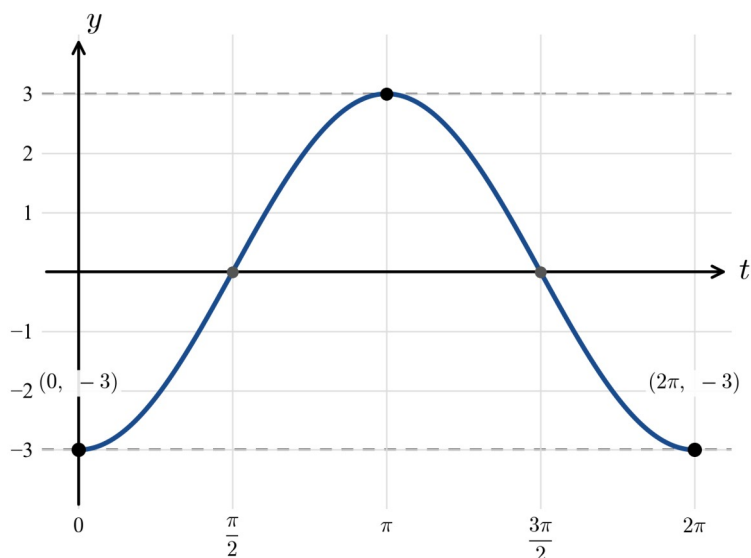
Q3. $y = \sin(t - \pi/2)$. Amplitude 1, period 2π , phase $\frac{\pi}{2}$ right. t -intercepts $t = \frac{\pi}{2}, \frac{3\pi}{2}$; maximum $(\pi, 1)$; minima at the ends $t = 0, 2\pi$; endpoints $(0, -1), (2\pi, -1)$.



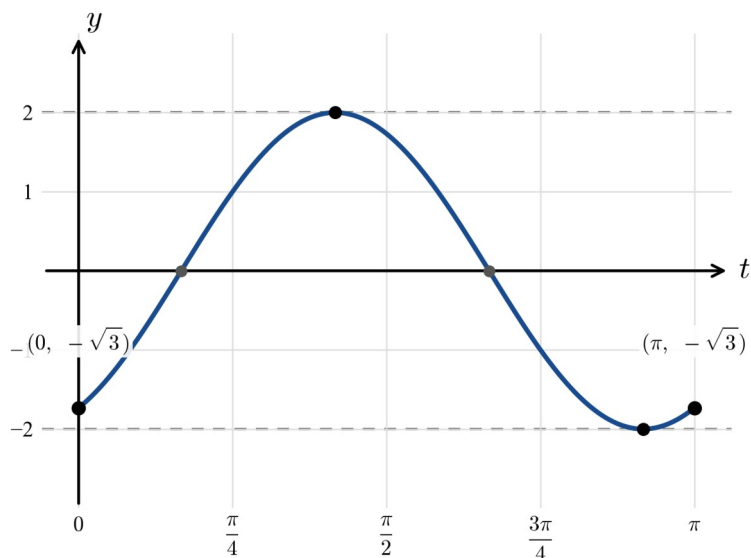
Q4. $y = 2 \sin(3t)$. Amplitude 2, period $\frac{2\pi}{3}$, so $\frac{2\pi}{2\pi/3} = 3$ complete cycles. $\sin(3t) = 0$ at $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$; maxima at $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$; minima at $t = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.



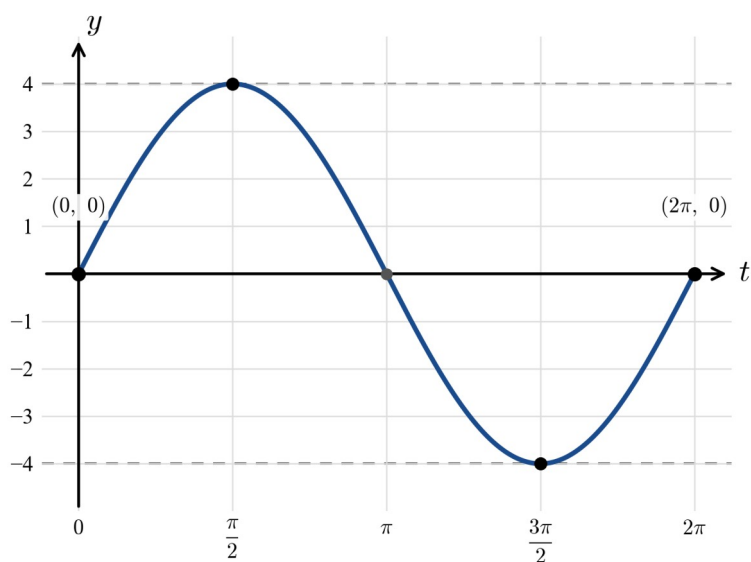
Q5. $y = -3 \cos t$. Amplitude 3; the negative sign reflects $y = 3 \cos t$ in the t -axis, so it starts at a **minimum**. $\cos t = 0$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$; maximum $(\pi, 3)$; minima at $t = 0, 2\pi$; endpoints $(0, -3), (2\pi, -3)$.



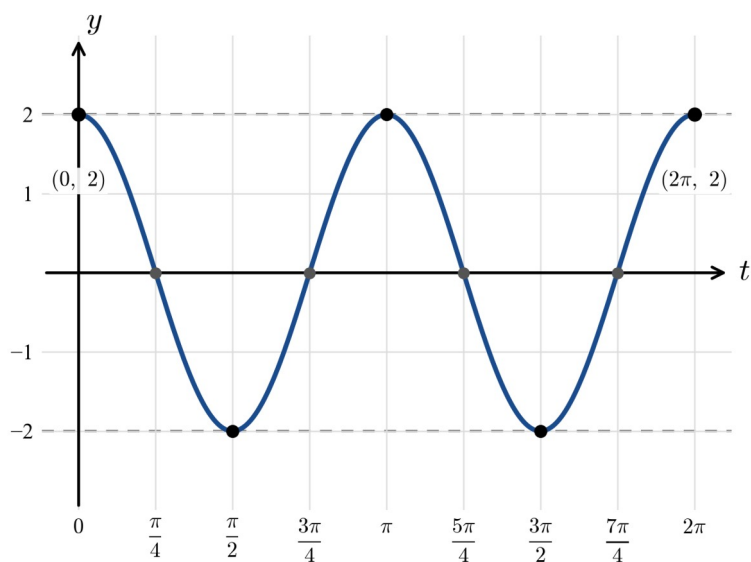
Q6. $y = 2 \sin(2(t - \pi/6))$. Amplitude 2, period π , phase $\frac{\pi}{6}$ right. $\sin(2(t - \pi/6)) = 0$ at $t = \frac{\pi}{6}, \frac{2\pi}{3}$; maximum $(5\pi/12, 2)$; minimum $(11\pi/12, -2)$; endpoints $t = 0 \Rightarrow y = 2 \sin(-\pi/3) = -\sqrt{3}$, and $t = \pi \Rightarrow y = -\sqrt{3}$, giving $(0, -\sqrt{3}), (\pi, -\sqrt{3})$.



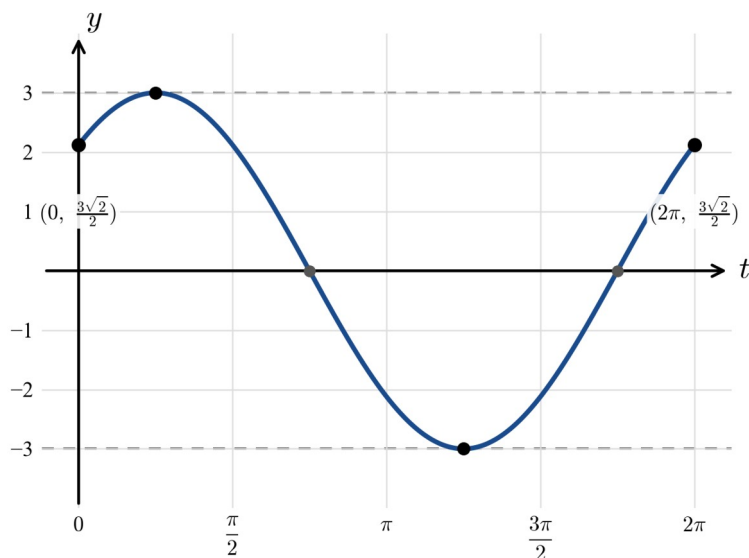
Q7. $y = 4 \sin t$. Amplitude 4, period 2π ; t -int $0, \pi, 2\pi$; maximum $(\pi/2, 4)$, minimum $(3\pi/2, -4)$.



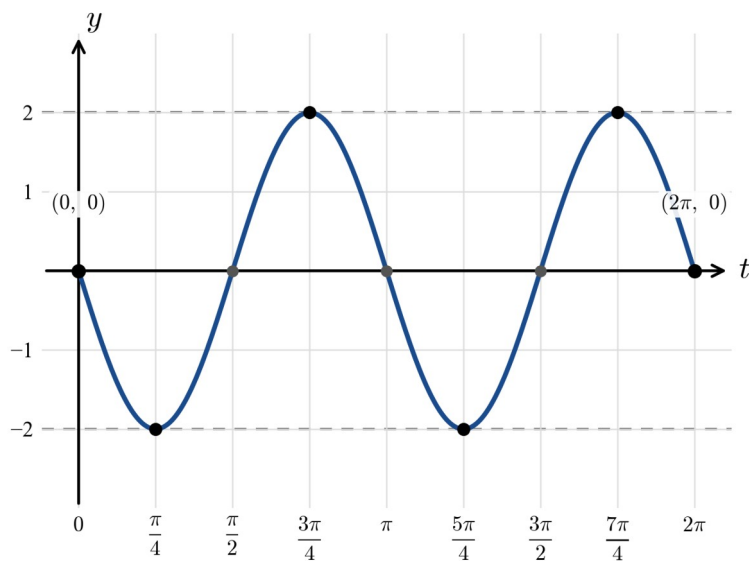
Q8. $y = 2 \cos(2t)$. Amplitude 2, period π ; t -int $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; maxima at $t = 0, \pi, 2\pi$; minima at $t = \frac{\pi}{2}, \frac{3\pi}{2}$; endpoints $(0, 2), (2\pi, 2)$.



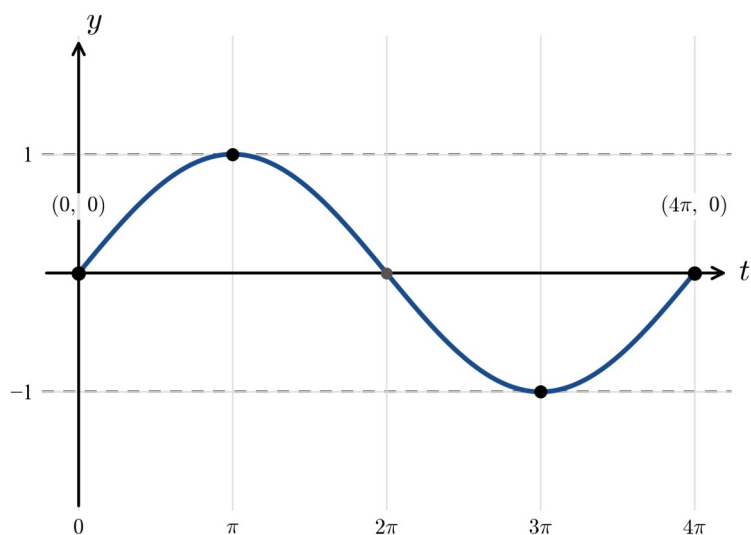
Q9. $y = 3 \cos(t - \pi/4)$. Amplitude 3, period 2π , phase $\frac{\pi}{4}$ right; t -int $\frac{3\pi}{4}, \frac{7\pi}{4}$; maximum $(\pi/4, 3)$, minimum $(5\pi/4, -3)$; endpoints $t = 0 \Rightarrow y = 3 \cos(-\pi/4) = \frac{3\sqrt{2}}{2}$, giving $(0, 3\sqrt{2}/2), (2\pi, 3\sqrt{2}/2)$.



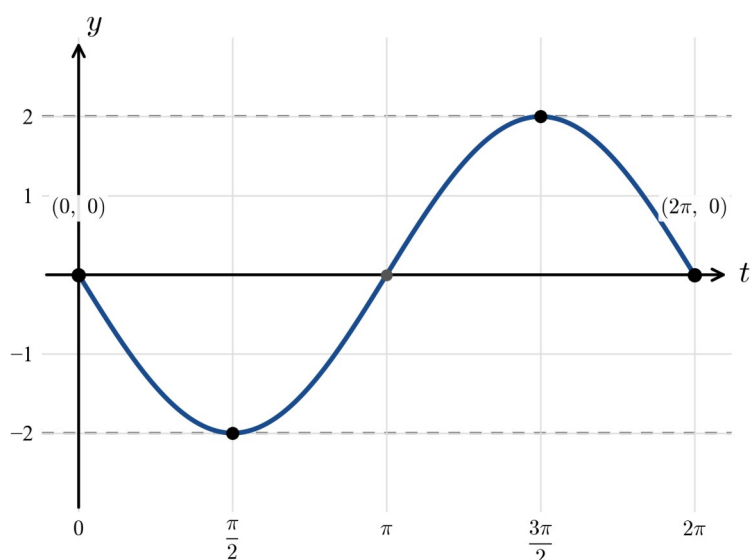
Q10. $y = -2 \sin(2t)$. Amplitude 2 (reflected), period π ; t -int $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; the reflection puts maxima at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$ and minima at $t = \frac{\pi}{4}, \frac{5\pi}{4}$.



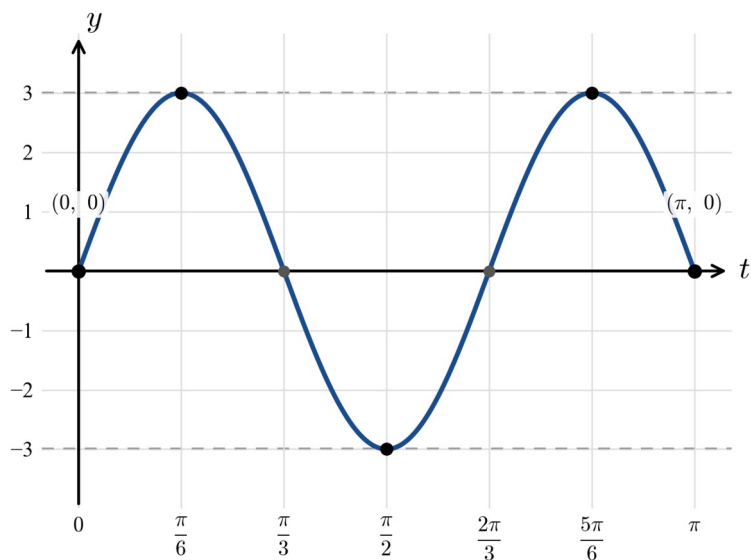
Q11. $y = \sin(1/2 t)$. Amplitude 1, period $\frac{2\pi}{1/2} = 4\pi$; t -int $0, 2\pi, 4\pi$; maximum $(\pi, 1)$, minimum $(3\pi, -1)$.



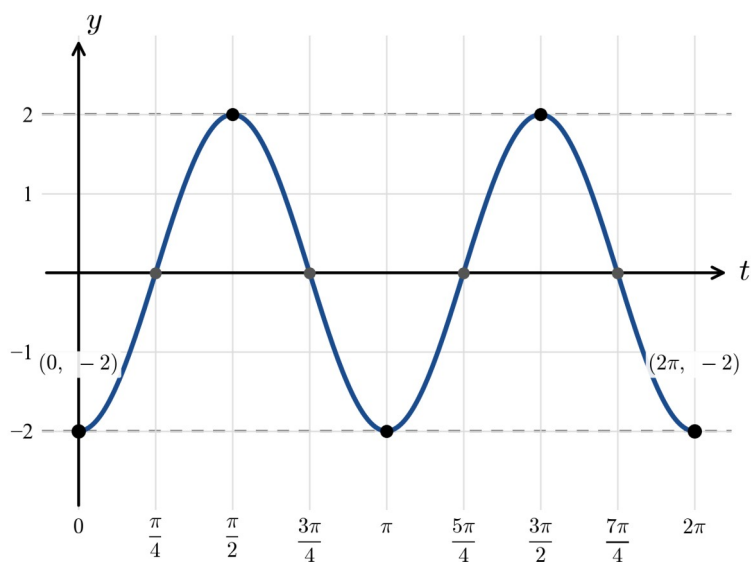
Q12. $y = 2 \cos(t + \pi/2)$. Amplitude 2, period 2π , phase $\frac{\pi}{2}$ left; t -int $0, \pi, 2\pi$; maximum $(3\pi/2, 2)$, minimum $(\pi/2, -2)$; endpoints $(0, 0), (2\pi, 0)$ (note $\cos(t + \pi/2) = -\sin t$).



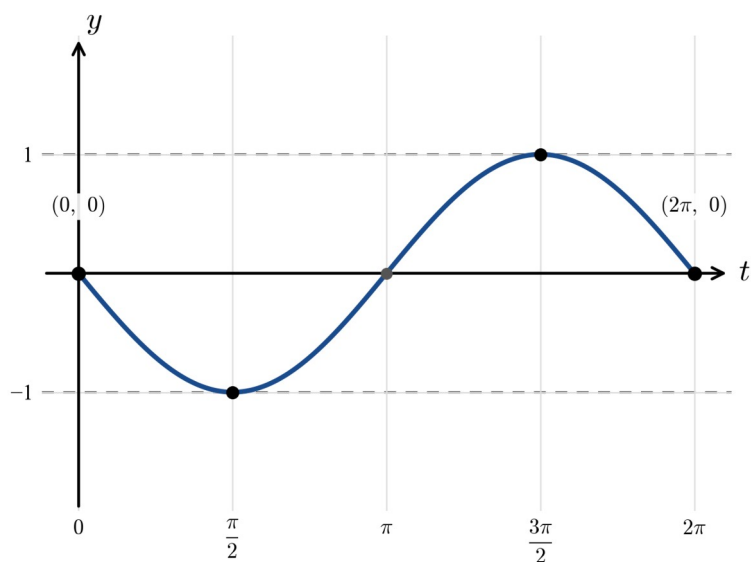
Q13. $y = 3 \sin(3t)$. Amplitude 3, period $\frac{2\pi}{3}$; over $[0, \pi]$ that is $1\frac{1}{2}$ cycles. t -int $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$; maxima at $t = \frac{\pi}{6}, \frac{5\pi}{6}$; minimum $(\pi/2, -3)$.



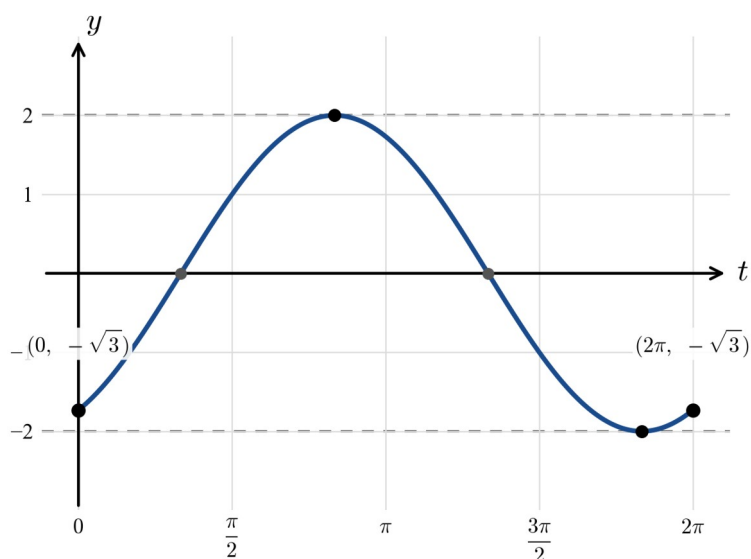
Q14. $y = 2 \sin(2(t - \pi/4))$. Amplitude 2, period π , phase $\frac{\pi}{4}$ right; t -int $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; maxima at $t = \frac{\pi}{2}, \frac{3\pi}{2}$; minima at $t = 0, \pi, 2\pi$; endpoints $(0, -2), (2\pi, -2)$.



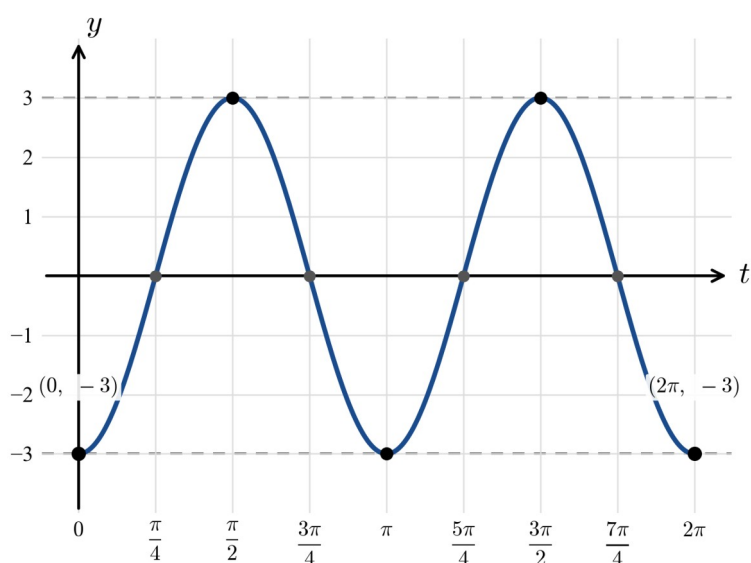
Q15. $y = -\sin t$. Amplitude 1 (reflected), period 2π ; t -int $0, \pi, 2\pi$; maximum $(3\pi/2, 1)$, minimum $(\pi/2, -1)$.



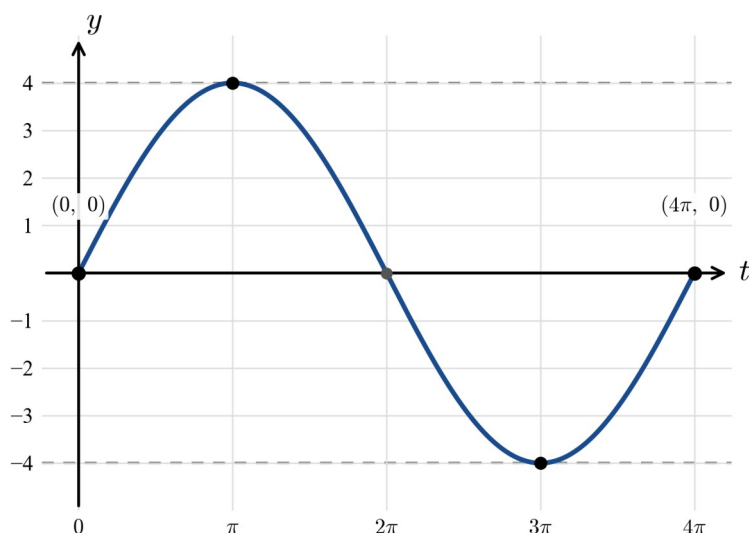
Q16. $y = 2 \sin(t - \pi/3)$. Amplitude 2, period 2π , phase $\frac{\pi}{3}$ right; t -int $\frac{\pi}{3}, \frac{4\pi}{3}$; maximum $(5\pi/6, 2)$, minimum $(11\pi/6, -2)$; endpoints $t = 0 \Rightarrow y = 2 \sin(-\pi/3) = -\sqrt{3}$, giving $(0, -\sqrt{3}), (2\pi, -\sqrt{3})$.



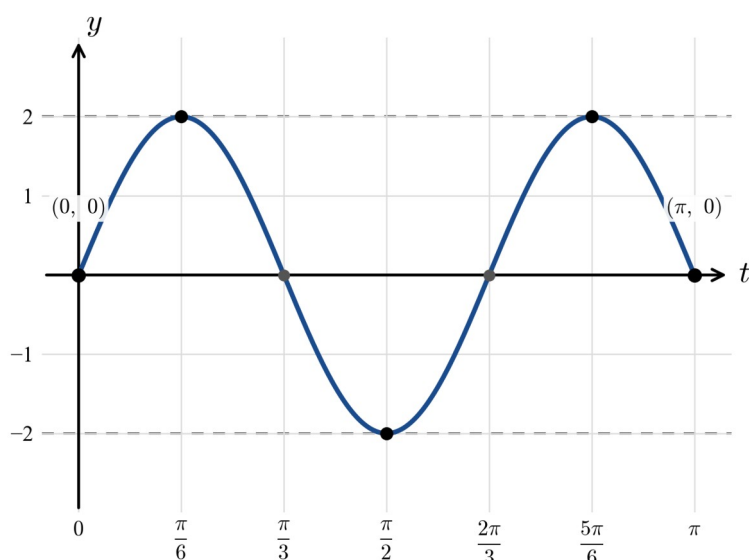
Q17. $y = 3 \cos(2(t - \pi/2))$. Amplitude 3, period π , phase $\frac{\pi}{2}$ right; t -int $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; maxima at $t = \frac{\pi}{2}, \frac{3\pi}{2}$; minima at $t = 0, \pi, 2\pi$; endpoints $(0, -3), (2\pi, -3)$.



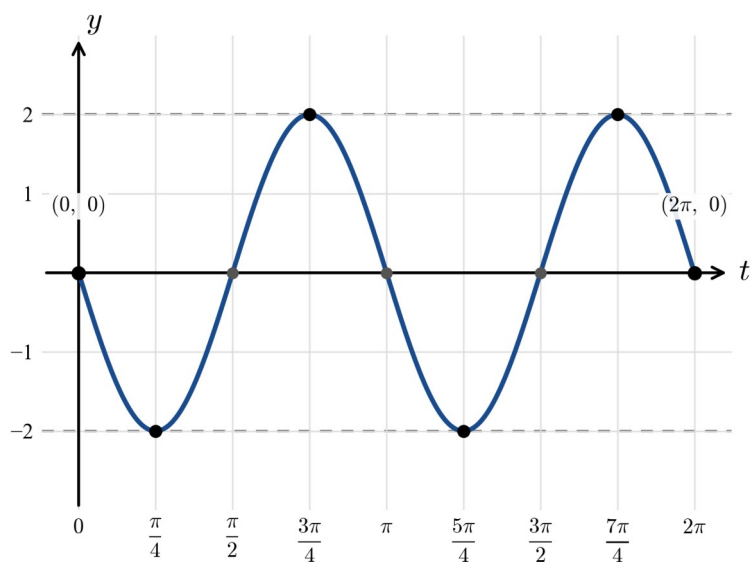
Q18. $y = 4 \cos(1/2(t - \pi))$. Amplitude 4, period 4π , phase π right; t -int $0, 2\pi, 4\pi$; maximum $(\pi, 4)$, minimum $(3\pi, -4)$; endpoints $(0, 0), (4\pi, 0)$.



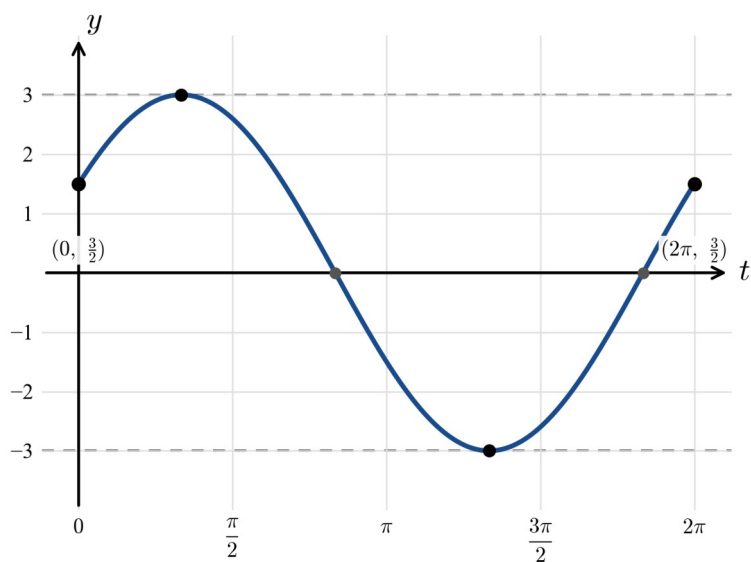
Q19. $y = 2 \cos(3(t - \pi/6))$. Amplitude 2, period $\frac{2\pi}{3}$; t -int $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$; maxima at $t = \frac{\pi}{6}, \frac{5\pi}{6}$; minimum $(\pi/2, -2)$; endpoints $(0, 0), (\pi, 0)$.



Q20. $y = -2 \cos(2(t - \pi/4))$. From $y = \cos t$: dilation of factor 2 from the t -axis, reflection in the t -axis, dilation of factor $\frac{1}{2}$ from the y -axis (period π), then translation $\frac{\pi}{4}$ right. t -int $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; maxima at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$; minima at $t = \frac{\pi}{4}, \frac{5\pi}{4}$; endpoints $(0, 0), (2\pi, 0)$.



Q21. $y = 3 \sin(t + \pi/6)$. Amplitude 3, period 2π , phase $\frac{\pi}{6}$ left; t -int $\frac{5\pi}{6}, \frac{11\pi}{6}$; maximum $(\pi/3, 3)$, minimum $(4\pi/3, -3)$; endpoints $t = 0 \Rightarrow y = 3 \sin \pi/6 = \frac{3}{2}$, giving $(0, 3/2), (2\pi, 3/2)$.



Chapter 6 Circular functions — 6G Sketching translated sine and cosine graphs

Theory

In Section 6F you sketched $y = a \sin(n(t - \varepsilon))$ and $y = a \cos(n(t - \varepsilon))$. Adding a constant b **translates the whole graph b units vertically**, moving the *mean position* (the line about which the curve oscillates) from $y=0$ to $y=b$.

For $y = a \sin(n(t - \varepsilon)) + b$ (and the cosine analogue), with $a \neq 0$ and $n > 0$:

- **Amplitude** $= |a|$ — the distance from the mean line to a maximum or minimum.
- **Period** $= \frac{2\pi}{n}$ — the horizontal length of one full cycle.
- **Phase (horizontal translation)** $= \varepsilon$.
- **Mean line** $y = b$; **maximum value** $= b + |a|$; **minimum value** $= b - |a|$.
- **Range** $= [b - |a|, b + |a|]$.

When $a < 0$ the graph is reflected in the mean line (a maximum of the basic graph becomes a minimum).

To sketch over a given domain: draw the dashed mean line $y = b$; mark the maximum $(b + |a|)$ and minimum $(b - |a|)$ levels; use the period to space the key points (mean–max–mean–min–mean each cycle); find the y -intercept by substituting $t = 0$; then draw a smooth curve. Give the maximum, minimum and endpoints; axis intercepts are labelled where they are exact.

Worked examples

Example 1 — scaffolded

Sketch $y = 2 \sin(x) + 1$ for $x \in [0, 2\pi]$, stating the amplitude, period, range and the coordinates of the maximum and minimum.

Working

Compare with $y = a \sin(nx) + b$: $a = 2$, $n = 1$, $b = 1$.

Amplitude $= |a| = 2$; period $= \frac{2\pi}{1} = 2\pi$.

Mean line $y = 1$; maximum $= 1 + 2 = 3$; minimum $= 1 - 2 = -1$; range $= [-1, 3]$.

y -intercept: $x = 0 \Rightarrow y = 2 \sin 0 + 1 = 1$, giving $(0, 1)$.

Maximum at $x = \frac{\pi}{2}$ (value 3); minimum at $x = \frac{3\pi}{2}$ (value -1).

Comment

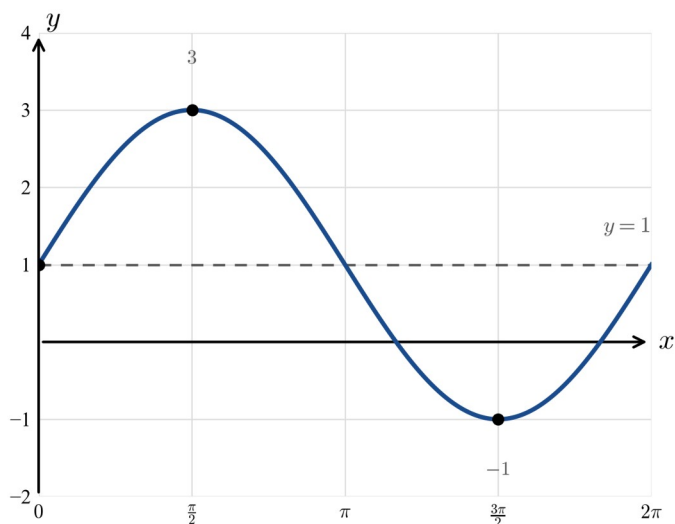
Read off the parameters.

Standard formulas.

The $+1$ lifts the mean to $y = 1$.

Substitute $x = 0$.

$\sin$ peaks a quarter-period after the mean.



Example 2 — scaffolded

Sketch $y = 3 \cos(2x) - 1$ for $x \in [0, 2\pi]$, stating the amplitude, period and range.

Working

Compare with $y = a \cos(nx) + b$: $a = 3$, $n = 2$, $b = -1$.

Amplitude = 3; period = $\frac{2\pi}{2} = \pi$ (two full cycles on $[0, 2\pi]$).

Mean line $y = -1$; maximum = $-1 + 3 = 2$; minimum = $-1 - 3 = -4$; range = $[-4, 2]$.

y-intercept: $x = 0 \Rightarrow y = 3 \cos 0 - 1 = 2$, giving $(0, 2)$ (a maximum).

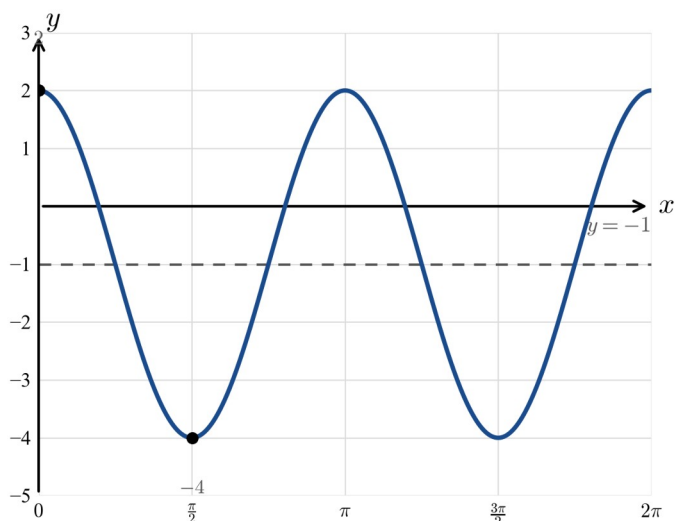
Comment

Read off the parameters.

$n = 2$ halves the period.

The -1 lowers the mean.

$\cos$ starts at a maximum.



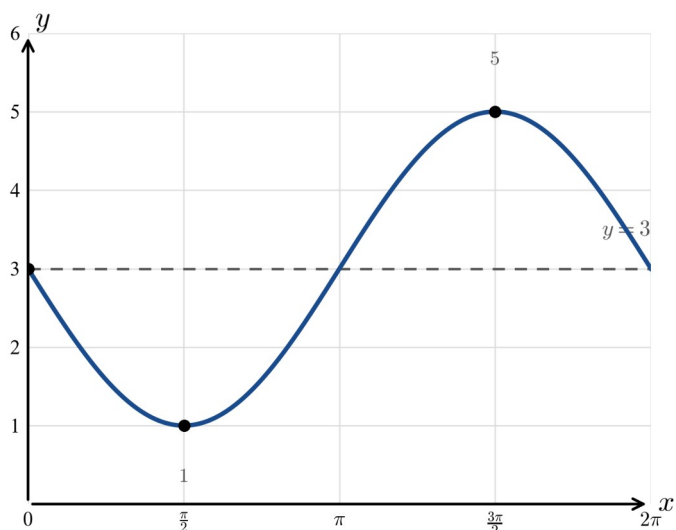
Example 3 — unscaffolded

Sketch $y = -2 \sin(x) + 3$ for $x \in [0, 2\pi]$, stating the range.

$a = -2$, $n = 1$, $b = 3$: amplitude 2, period 2π , mean line $y = 3$.

Because $a = -2 < 0$ the graph is reflected in the mean line, so it starts at the mean and goes **down** first. Maximum = $3 + 2 = 5$, minimum = $3 - 2 = 1$, range = $[1, 5]$. Since the minimum $1 > 0$, the graph has **no x-intercepts**. y-intercept $(0, 3)$.

$\therefore$ the curve runs from $(0, 3)$ down to a minimum 1, up to a maximum 5, and back to $(2\pi, 3)$.



Example 4 — unscaffolded

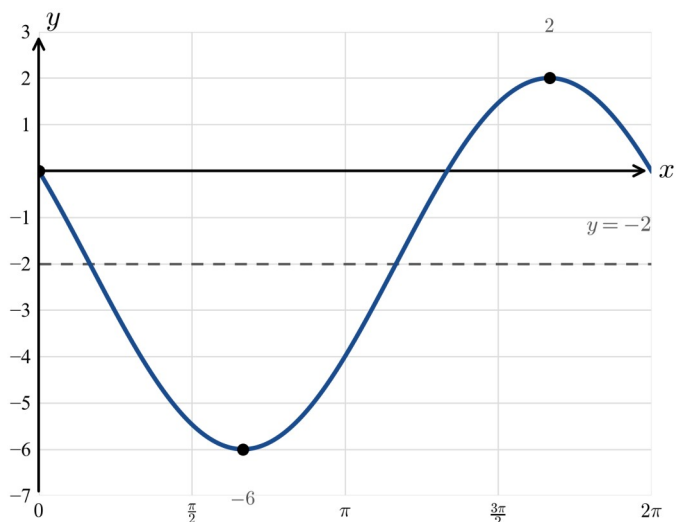
Sketch $y = 4 \cos\left(x + \frac{\pi}{3}\right) - 2$ for $x \in [0, 2\pi]$.

Writing $x + \frac{\pi}{3} = (x - (-\pi/3))$: $a = 4$, $n = 1$, $\varepsilon = -\frac{\pi}{3}$, $b = -2$. Amplitude 4, period 2π , phase $\frac{\pi}{3}$ to the **left**, mean line $y = -2$.

Maximum $= -2 + 4 = 2$, minimum $= -2 - 4 = -6$, range $[-6, 2]$. y-intercept:

$x = 0 \Rightarrow y = 4 \cos \frac{\pi}{3} - 2 = 4 \cdot \frac{1}{2} - 2 = 0$, giving $(0, 0)$. x-intercepts (where $\cos(x + \pi/3) = 1/2$): $x = 0, \frac{4\pi}{3}, 2\pi$.

$\therefore$ the basic cosine is stretched to amplitude 4, shifted $\frac{\pi}{3}$ left and 2 down.



Practice questions

Scaffolded

Q1. For $y = 3 \sin(x) - 1$, $x \in [0, 2\pi]$: (a) state the amplitude and period; (b) state the mean line, maximum and minimum; (c) find the y-intercept; (d) sketch the graph.

Q2. For $y = 2 \cos(x) + 2$, $x \in [0, 2\pi]$: (a) state the amplitude, period and range; (b) find the axis intercepts; (c) sketch the graph.

Q3. For $y = \sin(2x) + 1$, $x \in [0, 2\pi]$: (a) state the period; (b) state the range; (c) find the x -intercepts; (d) sketch the graph.

Q4. For $y = -3\cos(x) + 1$, $x \in [0, 2\pi]$: (a) state the amplitude and mean line; (b) explain the effect of the negative coefficient; (c) state the range; (d) sketch the graph.

Q5. For $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$, $x \in [0, 2\pi]$: (a) state the amplitude, period and phase; (b) state the range; (c) sketch the graph.

Q6. For $y = 4\sin(x) - 2$, $x \in [0, 2\pi]$: (a) state the range; (b) find the exact x -intercepts; (c) sketch the graph.

Unscaffolded

For each function, sketch the graph over $x \in [0, 2\pi]$, stating the amplitude, period, range and the coordinates of the maximum and minimum (and any exact axis intercepts).

Q7. $y = 2\sin(2x) + 1$ **Q8.** $y = 3\cos(2x) + 1$ **Q9.** $y = -2\sin(x) - 1$ **Q10.** $y = 2\cos(3x) - 1$

Q11. $y = 3\sin\left(x + \frac{\pi}{6}\right) + 2$ **Q12.** $y = 2\sin(3x) + 1$ **Q13.** $y = -\cos(x) + 3$

Q14. $y = 4\sin(x) - 1$ **Q15.** $y = 2\cos\left(x - \frac{\pi}{3}\right) + 2$ **Q16.** $y = 3\sin(2x) - 2$

Q17. $y = -2\cos(2x) + 1$ **Q18.** $y = \sin(3x) + 2$ **Q19.** $y = 2\cos\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$

Q20. For $y = 5\cos(x) - 3$, sketch the graph and **state the range**.

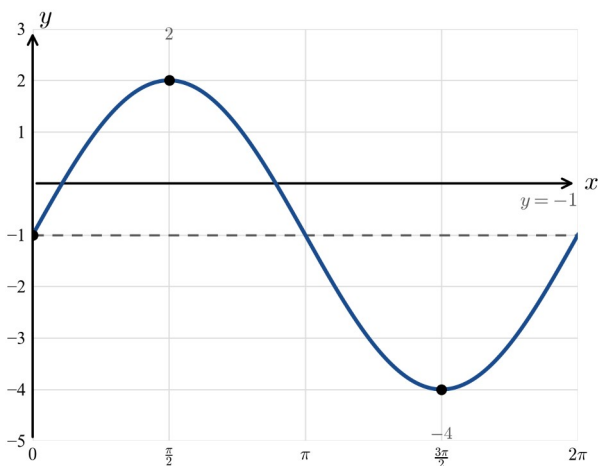
Q21. For $y = 3\cos\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$, **describe how it is obtained from $y = \cos x$** , then sketch it.

Answers

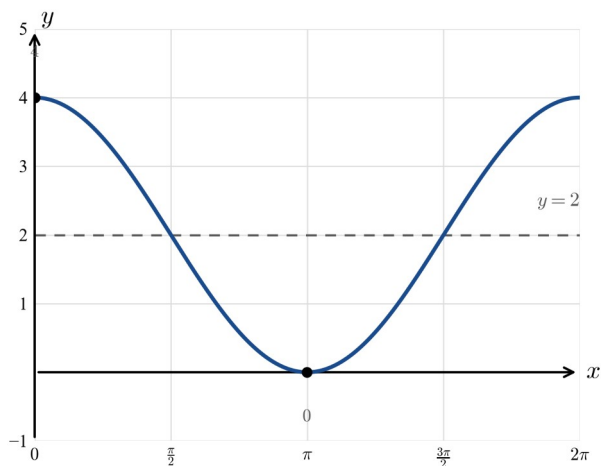
Q1. amp 3, period 2π ; mean $y = -1$, max 2, min -4 , range $[-4, 2]$; $(0, -1)$. **Q2.** amp 2, period 2π , range $[0, 4]$; intercepts $(0, 4)$ and $(\pi, 0)$. **Q3.** period π ; range $[0, 2]$; x -intercepts $\frac{3\pi}{4}, \frac{7\pi}{4}$. **Q4.** amp 3, mean $y = 1$; reflected (starts at a minimum); range $[-2, 4]$. **Q5.** amp 2, period 2π , phase $\frac{\pi}{2}$; range $[-1, 3]$. **Q6.** range $[-6, 2]$; x -intercepts $\frac{\pi}{6}, \frac{5\pi}{6}$. **Q7.** amp 2, period π , range $[-1, 3]$. **Q8.** amp 3, period π , range $[-2, 4]$; $(0, 4)$. **Q9.** amp 2, period 2π , range $[-3, 1]$. **Q10.** amp 2, period $\frac{2\pi}{3}$, range $[-3, 1]$. **Q11.** amp 3, period 2π , phase $\frac{\pi}{6}$ left, range $[-1, 5]$; $(0, 7/2)$. **Q12.** amp 2, period $\frac{2\pi}{3}$, range $[-1, 3]$. **Q13.** amp 1, period 2π , range $[2, 4]$; $(0, 2)$, no x -intercepts. **Q14.** amp 4, period 2π , range $[-5, 3]$. **Q15.** amp 2, period 2π , phase $\frac{\pi}{3}$, range $[0, 4]$; $(0, 3)$. **Q16.** amp 3, period π , range $[-5, 1]$. **Q17.** amp 2, period π , range $[-1, 3]$; x -intercepts $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. **Q18.** amp 1, period $\frac{2\pi}{3}$, range $[1, 3]$; $(0, 2)$, no x -intercepts. **Q19.** amp 2, period π , phase $\frac{\pi}{6}$, range $[-1, 3]$. **Q20.** amp 5, period 2π , range $[-8, 2]$. **Q21.** amp 3, period π , phase $\frac{\pi}{6}$, range $[-2, 4]$.

Fully-worked solutions

Q1. $y = 3 \sin(x) - 1$. Amplitude 3, period 2π , mean line $y = -1$. Maximum $-1 + 3 = 2$ (at $x = \pi/2$), minimum $-1 - 3 = -4$ (at $x = 3\pi/2$), range $[-4, 2]$. y -intercept $x = 0 \Rightarrow y = -1$, giving $(0, -1)$.

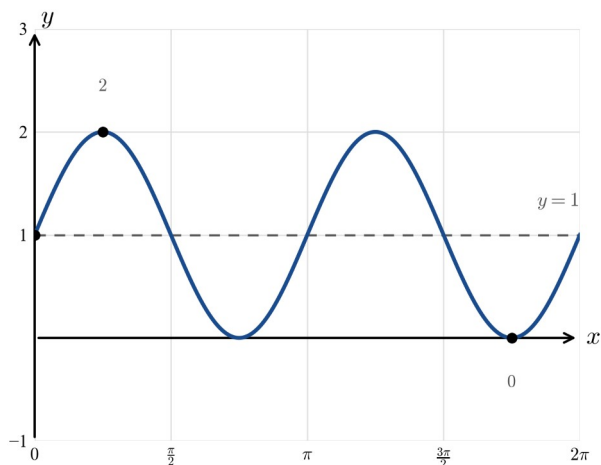


Q2. $y = 2 \cos(x) + 2$. Amplitude 2, period 2π , mean line $y = 2$, range $[0, 4]$. y -intercept $x = 0 \Rightarrow y = 4$, giving $(0, 4)$. Minimum 0 occurs at $x = \pi$, so $(\pi, 0)$ is the single x -intercept (the curve touches the axis).

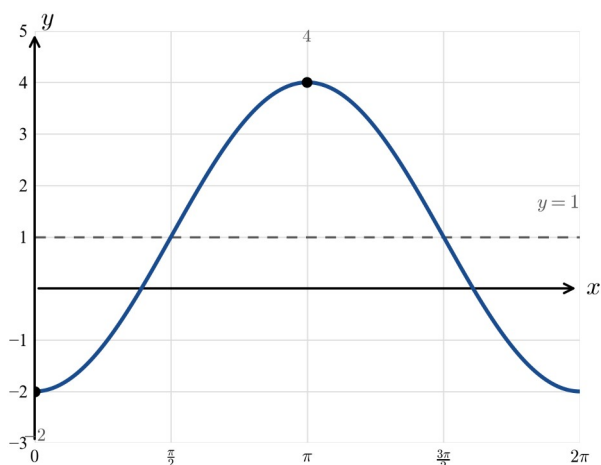


Q3. $y = \sin(2x) + 1$. Period $\frac{2\pi}{2} = \pi$, amplitude 1, mean $y = 1$, range $[0, 2]$. Setting $y = 0$:

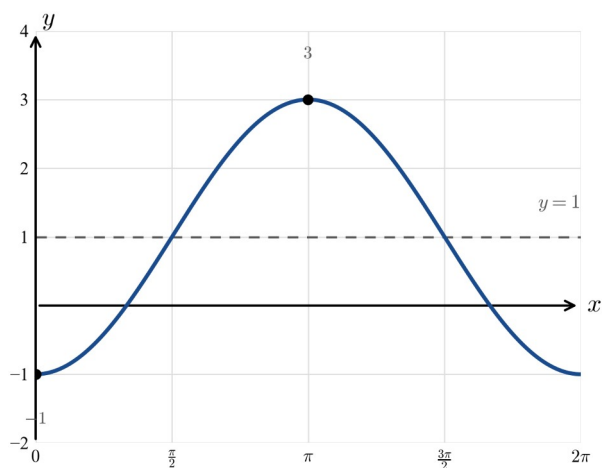
$\sin(2x) = -1 \Rightarrow 2x = \frac{3\pi}{2}, \frac{7\pi}{2} \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$ (the minima, where the curve touches the axis). $(0, 1)$.



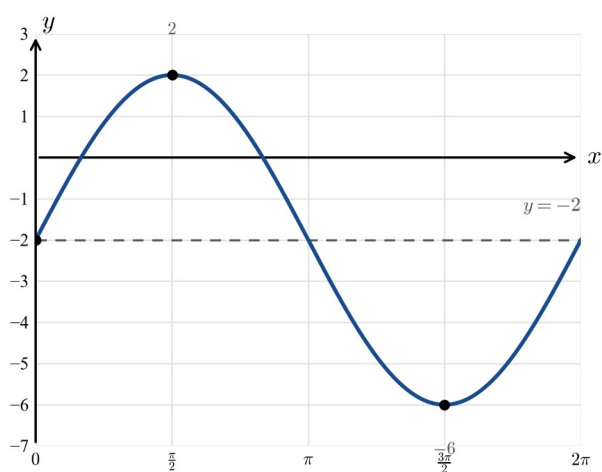
Q4. $y = -3 \cos(x) + 1$. Amplitude 3, mean line $y = 1$. The coefficient $-3 < 0$ reflects the cosine in the mean line, so the graph starts at a **minimum**: $x = 0 \Rightarrow y = -3 + 1 = -2$, giving $(0, -2)$. Maximum 4, minimum -2 , range $[-2, 4]$.



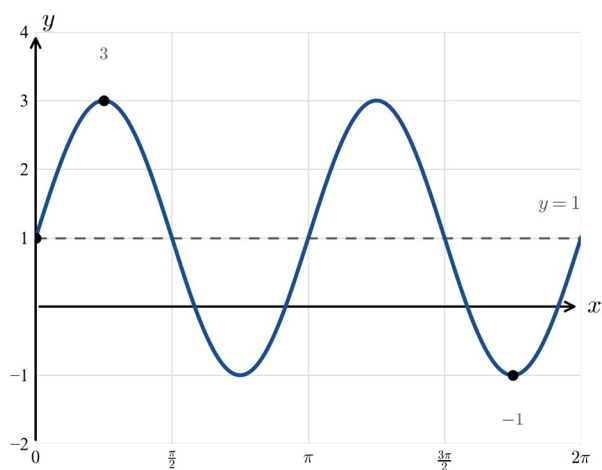
Q5. $y = 2 \sin(x - \pi/2) + 1$. Amplitude 2, period 2π , phase $\frac{\pi}{2}$ to the right, mean line $y = 1$, range $[-1, 3]$. y -intercept $x = 0 \Rightarrow y = 2 \sin(-\pi/2) + 1 = -2 + 1 = -1$, giving $(0, -1)$.



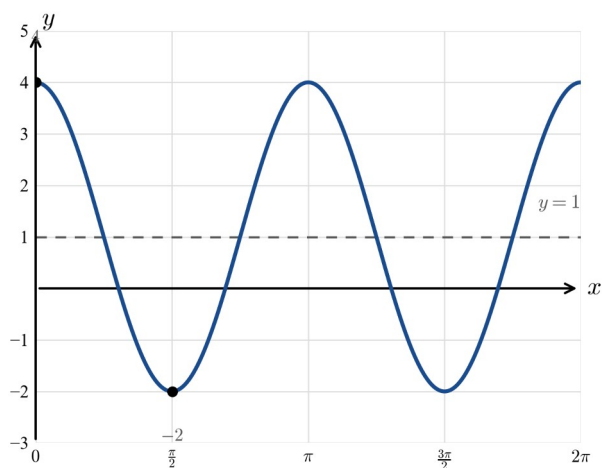
Q6. $y = 4 \sin(x) - 2$. Amplitude 4, mean $y = -2$, range $[-6, 2]$. Setting $y = 0$: $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$. y -intercept $(0, -2)$.



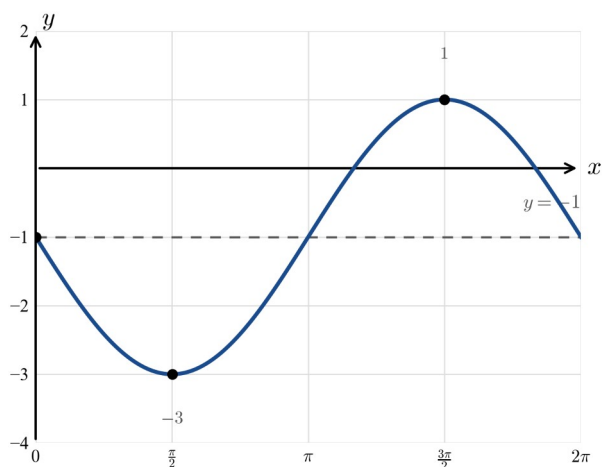
Q7. $y = 2 \sin(2x) + 1$. Amplitude 2, period π , mean $y = 1$, max 3, min -1 , range $[-1, 3]$; $(0, 1)$.



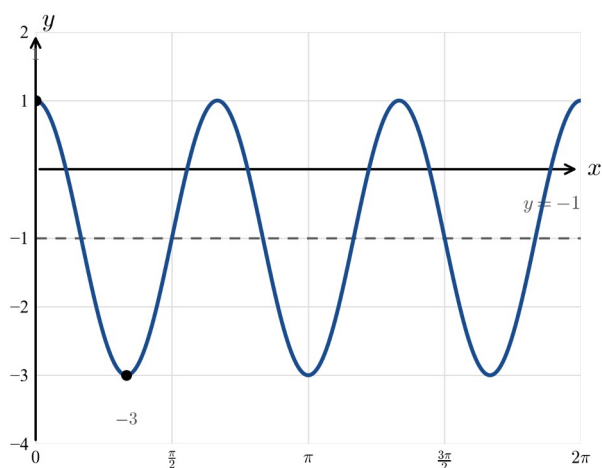
Q8. $y = 3 \cos(2x) + 1$. Amplitude 3, period π , mean $y = 1$, max 4, min -2 , range $[-2, 4]$; $(0, 4)$.



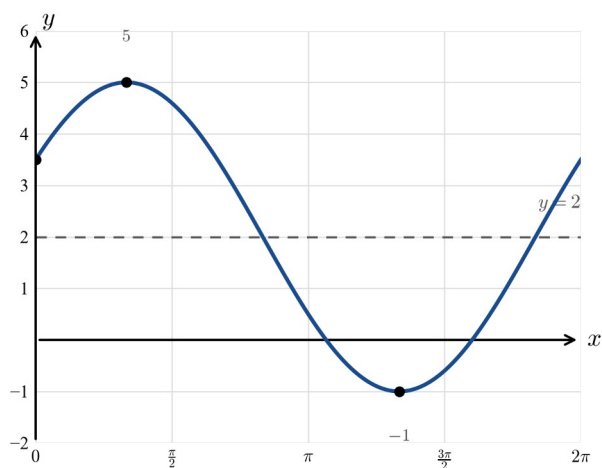
Q9. $y = -2 \sin(x) - 1$. Amplitude 2, period 2π , mean $y = -1$ (reflected), max 1, min -3 , range $[-3, 1]$; $(0, -1)$.



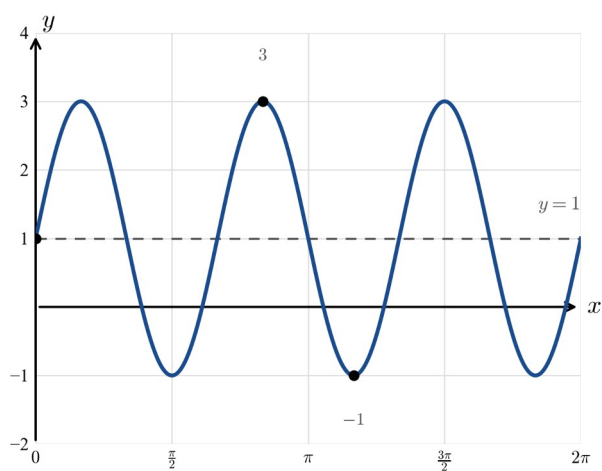
Q10. $y = 2 \cos(3x) - 1$. Amplitude 2, period $\frac{2\pi}{3}$ (three cycles on $[0, 2\pi]$), mean $y = -1$, max 1, min -3 , range $[-3, 1]$; $(0, 1)$.



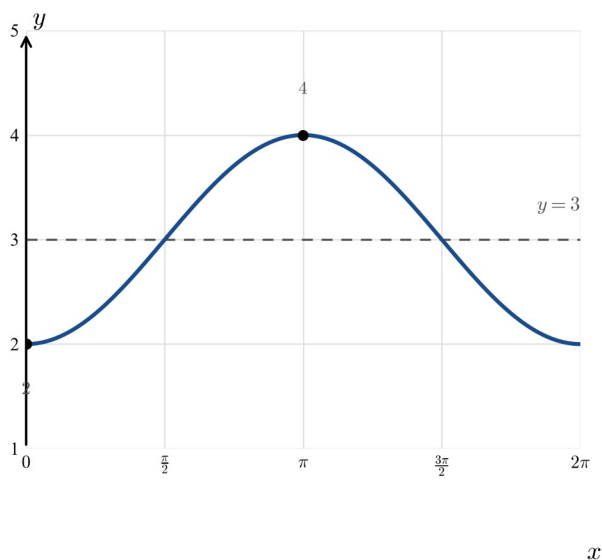
Q11. $y = 3 \sin(x + \pi/6) + 2$. Amplitude 3, period 2π , phase $\frac{\pi}{6}$ left, mean $y = 2$, max 5, min -1 , range $[-1, 5]$. y -intercept $x = 0 \Rightarrow y = 3 \sin \pi/6 + 2 = 3/2 + 2 = 7/2$, giving $(0, 7/2)$.



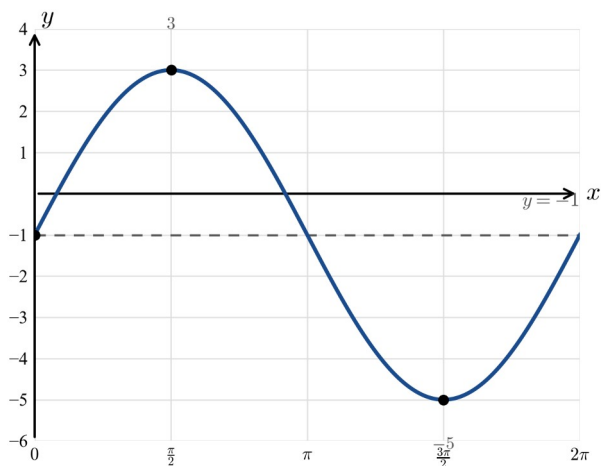
Q12. $y = 2 \sin(3x) + 1$. Amplitude 2, period $\frac{2\pi}{3}$, mean $y = 1$, max 3, min -1 , range $[-1, 3]$; $(0, 1)$.



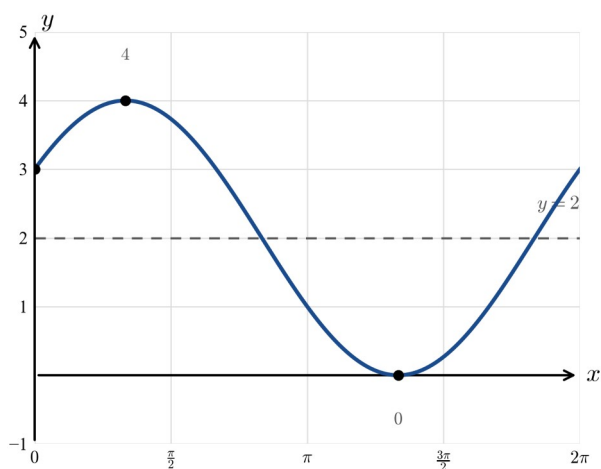
Q13. $y = -\cos(x) + 3$. Amplitude 1, period 2π , mean $y = 3$ (reflected), max 4, min 2, range $[2, 4]$. Since the minimum $2 > 0$ there are no x -intercepts. y -intercept $x = 0 \Rightarrow y = -1 + 3 = 2$, giving $(0, 2)$.



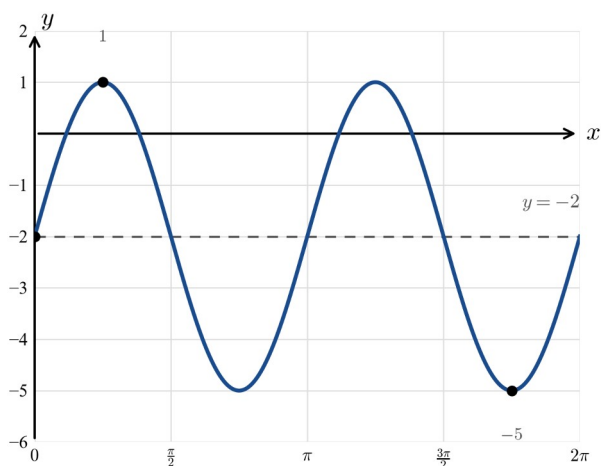
Q14. $y = 4 \sin(x) - 1$. Amplitude 4, period 2π , mean $y = -1$, max 3, min -5 , range $[-5, 3]$; $(0, -1)$.



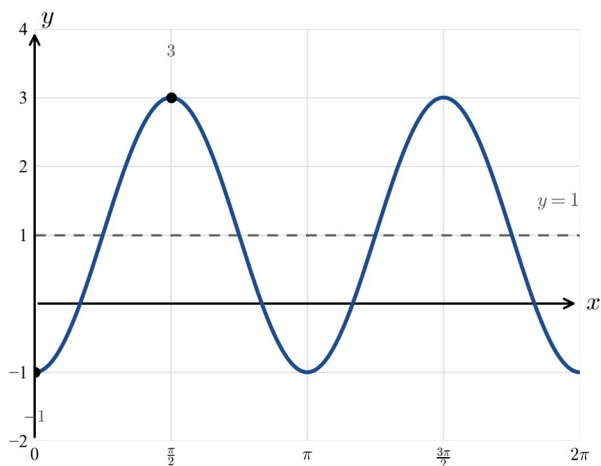
Q15. $y = 2 \cos(x - \pi/3) + 2$. Amplitude 2, period 2π , phase $\frac{\pi}{3}$ right, mean $y = 2$, max 4, min 0, range $[0, 4]$. y -intercept $x = 0 \Rightarrow y = 2 \cos(-\pi/3) + 2 = 1 + 2 = 3$, giving $(0, 3)$.



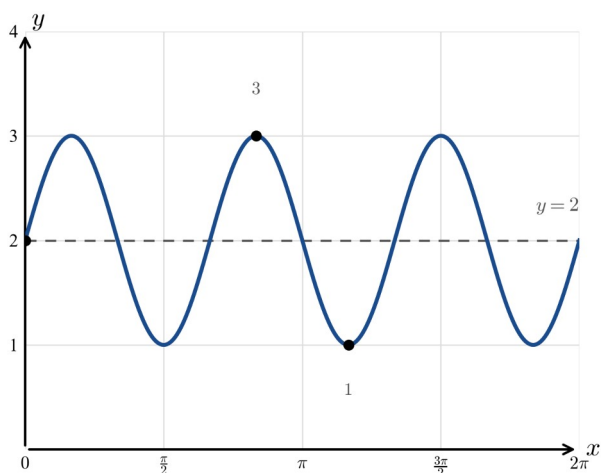
Q16. $y = 3 \sin(2x) - 2$. Amplitude 3, period π , mean $y = -2$, max 1, min -5 , range $[-5, 1]$; $(0, -2)$.



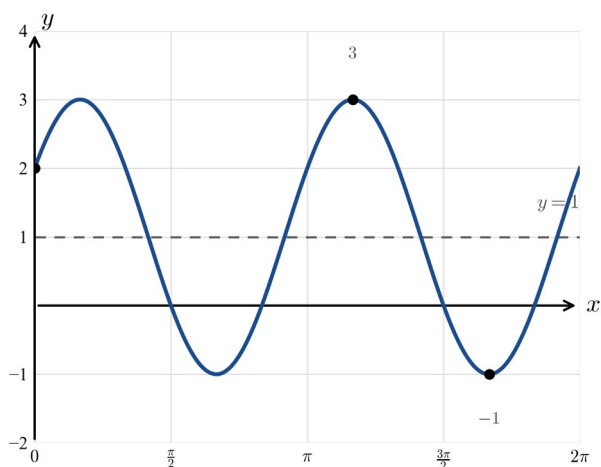
Q17. $y = -2 \cos(2x) + 1$. Amplitude 2, period π , mean $y = 1$ (reflected, starts at a minimum), max 3, min -1 , range $[-1, 3]$. Setting $y = 0$: $\cos(2x) = 1/2 \Rightarrow 2x = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$, so $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. y -intercept $(0, -1)$.



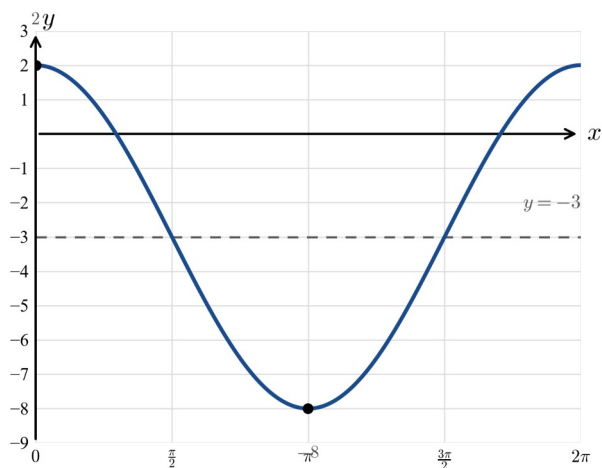
Q18. $y = \sin(3x) + 2$. Amplitude 1, period $\frac{2\pi}{3}$, mean $y = 2$, max 3, min 1, range $[1, 3]$; no x -intercepts ($\min = 1 > 0$); $(0, 2)$.



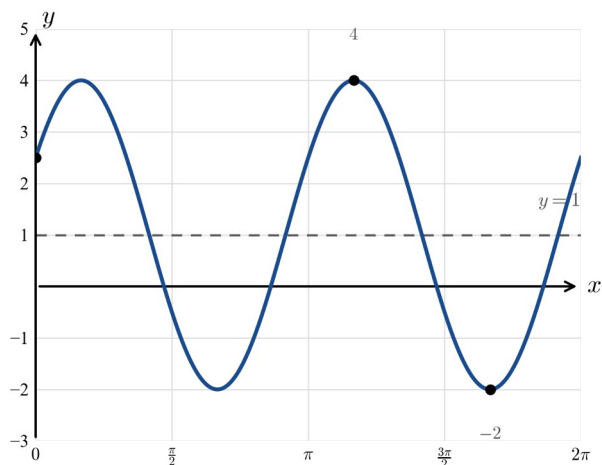
Q19. $y = 2 \cos(2(x - \pi/6)) + 1$. Amplitude 2, period $\frac{2\pi}{2} = \pi$, phase $\frac{\pi}{6}$ right, mean $y = 1$, max 3, min -1 , range $[-1, 3]$. y -intercept $x = 0 \Rightarrow y = 2 \cos(-\pi/3) + 1 = 1 + 1 = 2$, giving $(0, 2)$.



Q20. $y = 5 \cos(x) - 3$. Amplitude 5, period 2π , mean line $y = -3$, maximum $-3 + 5 = 2$, minimum $-3 - 5 = -8$, so the range is $[-8, 2]$. y -intercept $x = 0 \Rightarrow y = 2$, giving $(0, 2)$.



Q21. $y = 3 \cos(2(x - \pi/6)) + 1$. From $y = \cos x$: a **dilation of factor 3 from the x -axis** (amplitude 3), a **dilation of factor $1/2$ from the y -axis** (period π), a **translation of $\frac{\pi}{6}$ to the right** and **1 unit up**. Mean line $y = 1$, max 4, min -2 , range $[-2, 4]$. y -intercept $x = 0 \Rightarrow y = 3 \cos(-\pi/3) + 1 = 3/2 + 1 = 5/2$, giving $(0, 5/2)$.



Chapter 6 Circular functions — 6H Adding circular functions

Extension (in part). Writing $a \sin x + b \cos x$ as a single sinusoid $R \sin(x + \varphi)$ goes beyond the Mathematical Methods study design. It is included because it explains the graphs produced when sine and cosine curves are added, and it is not assessed in the graded topic tests or practice examinations.

Theory

Addition of ordinates is a graphical method for sketching the sum of two functions $y = f(x) + g(x)$. At each value of x the **ordinate** (the y -value) of the sum is found by **adding the ordinates** of the two component graphs:

$$(f + g)(x) = f(x) + g(x).$$

To sketch $y = f(x) + g(x)$ by addition of ordinates:

1. Lightly sketch the two components $y = f(x)$ and $y = g(x)$ on the same axes.
2. At convenient values of x — usually $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ and any points where a component is 0 or where the two components are equal — **add the two ordinates**.
3. Where one component is **zero**, the sum equals the other component (the sum meets that graph there).
4. Where the two components are **equal and opposite**, the sum is **zero** (an x -intercept of the sum).
5. Plot the summed points and join them with a smooth curve.

Two special cases occur often.

- **Sum of two circular functions of the same period**, $y = a \sin x + b \cos x$: the result is a single sinusoid $y = R \sin(x + \varphi)$ with **amplitude** $R = \sqrt{a^2 + b^2}$ and **period** 2π .
- **A circular function plus a straight line**, e.g. $y = x + \sin x$: the sine graph oscillates about the line $y = x$; the sum crosses the line wherever the circular part is 0, and its gradient is that of the line plus that of the wave.

Give exact values where possible (using $\sqrt{2}, \sqrt{3}, \pi$); a sketch shows **key features, not to scale**.

Worked examples

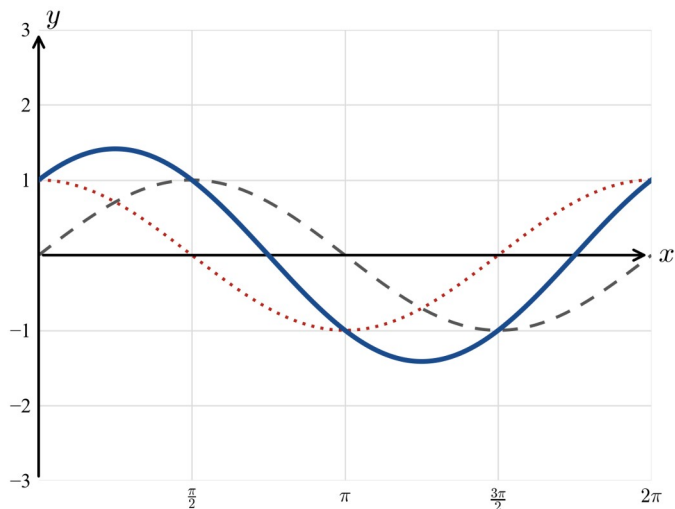
Example 1 — scaffolded

By addition of ordinates, sketch $y = \sin x + \cos x$ for $x \in [0, 2\pi]$, and state its amplitude and the exact coordinates of its x -intercepts.

Working	Comment
Components: $y = \sin x$ and $y = \cos x$, both with period 2π .	Sketch each lightly first.
Ordinates: $x = 0 \rightarrow 0 + 1 = 1$; $\frac{\pi}{2} \rightarrow 1 + 0 = 1$; $\pi \rightarrow 0 - 1 = -1$; $\frac{3\pi}{2} \rightarrow -1 + 0 = -1$; $2\pi \rightarrow 1$.	Add the two y -values at each key angle.
At $x = \frac{\pi}{4}$: $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$ (maximum); at $x = \frac{5\pi}{4}$: $-\sqrt{2}$ (minimum).	The greatest sum occurs where both components are positive and equal.
x -intercepts: $\sin x + \cos x = 0 \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$.	The sum is 0 where the ordinates are equal and opposite.

$\therefore$ amplitude $\sqrt{2}$, period 2π (indeed $y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$).

A sum of sin and cos of equal period is a single sinusoid, $R = \sqrt{1^2 + 1^2} = \sqrt{2}$.



Example 2 — scaffolded

By addition of ordinates, sketch $y = x + \sin x$ for $x \in [0, 2\pi]$.

Components: the line $y = x$ and the wave $y = \sin x$.

The wave oscillates about the line.

Ordinates: $x = 0 \rightarrow 0$; $\frac{\pi}{2} \rightarrow \frac{\pi}{2} + 1$; $\pi \rightarrow \pi + 0 = \pi$;

At $x = 0, \pi, 2\pi$ the sine is 0, so the sum meets the line $y = x$.

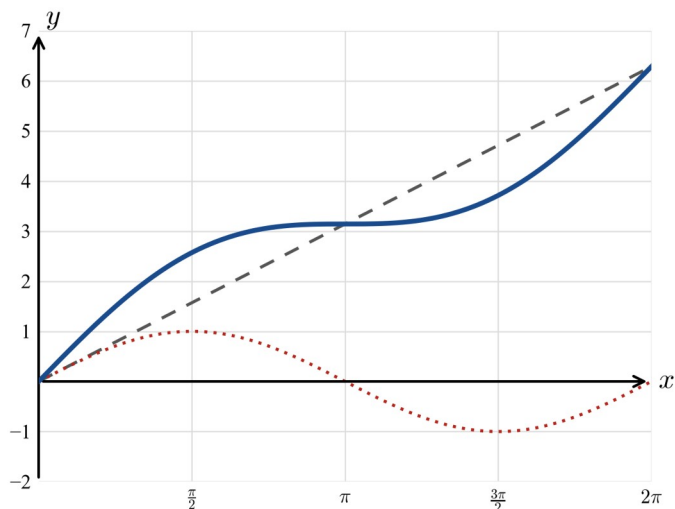
$\frac{3\pi}{2} \rightarrow \frac{3\pi}{2} - 1$; $2\pi \rightarrow 2\pi$.

Gradient $\frac{dy}{dx} = 1 + \cos x \geq 0$, and $= 0$ only at $x = \pi$.

So the graph is increasing, with a horizontal **stationary point of inflection** at (π, π) .

$\therefore$ the curve rises from $(0, 0)$, has a flat spot at (π, π) , and reaches $(2\pi, 2\pi)$.

It weaves above the line then below it, meeting it at $0, \pi, 2\pi$.



Example 3 — unscaffolded

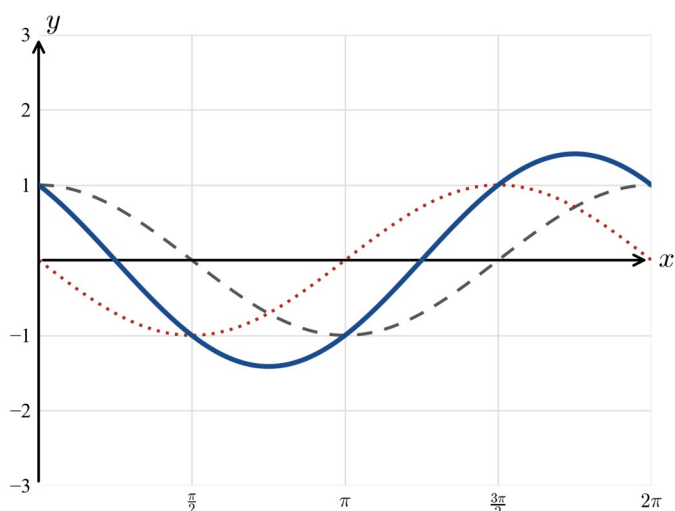
By addition of ordinates, sketch $y = \cos x - \sin x$ for $x \in [0, 2\pi]$, giving the amplitude and the exact x -intercepts.

Components $y = \cos x$ and $y = -\sin x$. Ordinates: $x = 0 \rightarrow 1$; $\frac{\pi}{2} \rightarrow -1$; $\pi \rightarrow -1$; $\frac{3\pi}{2} \rightarrow 1$; $2\pi \rightarrow 1$.

x -intercepts: $\cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$.

Maximum $\sqrt{2}$ at $x = \frac{7\pi}{4}$; minimum $-\sqrt{2}$ at $x = \frac{3\pi}{4}$.

$\therefore y = \cos x - \sin x = \sqrt{2} \cos\left(x + \frac{\pi}{4}\right)$: amplitude $\sqrt{2}$, period 2π .



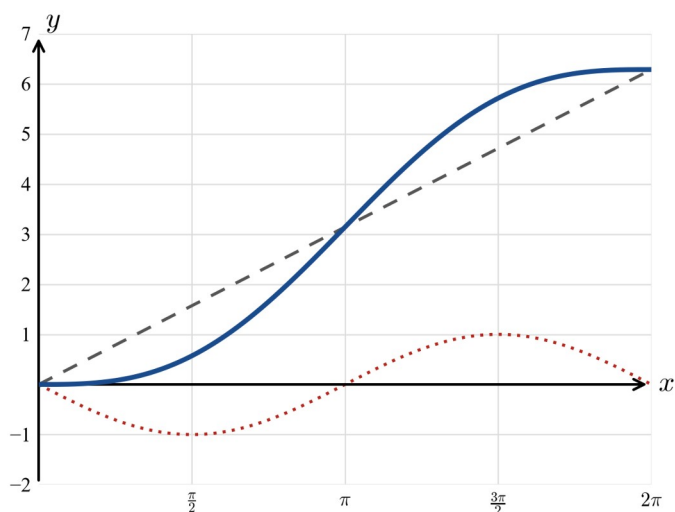
Example 4 — unscaffolded

By addition of ordinates, sketch $y = x - \sin x$ for $x \in [0, 2\pi]$.

Components: the line $y = x$ and $y = -\sin x$. Ordinates: $x = 0 \rightarrow 0$; $\frac{\pi}{2} \rightarrow \frac{\pi}{2} - 1$; $\pi \rightarrow \pi$; $\frac{3\pi}{2} \rightarrow \frac{3\pi}{2} + 1$; $2\pi \rightarrow 2\pi$.

Gradient $\frac{dy}{dx} = 1 - \cos x \geq 0$, equal to 0 at $x = 0$ and $x = 2\pi$ (stationary points of inflection there). The graph meets the line $y = x$ at $x = 0, \pi, 2\pi$.

$\therefore$ the curve is increasing throughout, flat at the ends, weaving about $y = x$.



Practice questions

Scaffolded

Q1. For $y = \sqrt{3} \sin x + \cos x$, $x \in [0, 2\pi]$: (a) state the amplitude $R = \sqrt{a^2 + b^2}$; (b) find the exact x -intercepts; (c) state the coordinates of the maximum and minimum; (d) sketch the graph by addition of ordinates.

Q2. For $y = \sin x - \sqrt{3} \cos x$, $x \in [0, 2\pi]$: (a) state the amplitude; (b) find the exact x -intercepts; (c) sketch the graph by addition of ordinates.

Q3. For $y = x + \cos x$, $x \in [0, 2\pi]$: (a) evaluate the ordinate at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; (b) explain why the graph is not always increasing; (c) sketch the graph by addition of ordinates.

Q4. For $y = \frac{x}{2} + \sin x$, $x \in [0, 2\pi]$: (a) find the ordinates at the quarter-period points; (b) state where the graph meets the line $y = \frac{x}{2}$; (c) sketch the graph by addition of ordinates.

Q5. For $y = \sin x + \sin 2x$, $x \in [0, 2\pi]$: (a) state the period of each component; (b) find the exact x -intercepts; (c) sketch the graph by addition of ordinates.

Q6. For $y = 3 \sin x + 4 \cos x$, $x \in [0, 2\pi]$: (a) state the amplitude; (b) find the maximum and minimum values; (c) sketch the graph by addition of ordinates.

Unscaffolded

Sketch each of the following for $x \in [0, 2\pi]$ by addition of ordinates, labelling the key features (and, where the sum is a single sinusoid, stating its amplitude and exact x -intercepts).

Q7. $y = \sin x + \sqrt{3} \cos x$

Q8. $y = \sqrt{3} \sin x - \cos x$

Q9. $y = x - \cos x$

Q10. $y = 2x + \sin x$

Q11. $y = \sin x + \cos 2x$

Q12. $y = \cos x + \sin 2x$

Q13. $y = 2 \sin x - 2 \cos x$

Q14. $y = x + 2 \sin x$

Q15. $y = \sin x + \frac{1}{2} \sin 2x$

Q16. $y = \cos x - \cos 2x$

Q17. $y = 4 \sin x - 3 \cos x$

Q18. $y = \frac{x}{2} + \cos x$

Q19. $y = \sin x - \sin 2x$

Q20. $y = x + 2 \cos x$

Q21. $y = \sin x + 2 \cos x$

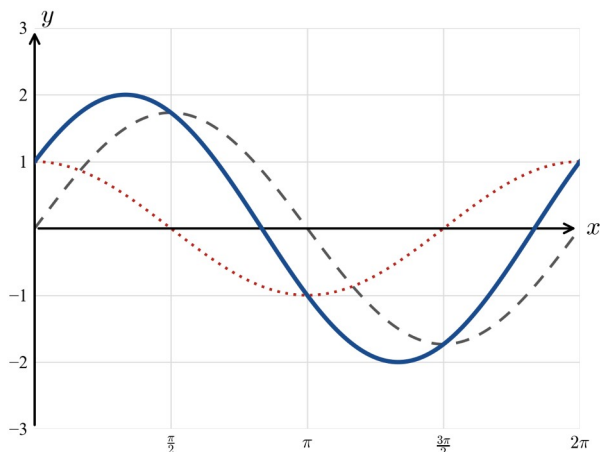
Answers

Q1. Amplitude 2; intercepts $\left(\frac{5\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$; max $\left(\frac{\pi}{3}, 2\right)$, min $\left(\frac{4\pi}{3}, -2\right)$. **Q2.** Amplitude 2; intercepts $\left(\frac{\pi}{3}, 0\right), \left(\frac{4\pi}{3}, 0\right)$. **Q3.** Ordinates $1, \frac{\pi}{2}, \pi - 1, \frac{3\pi}{2}, 2\pi + 1$; not always increasing since $\frac{dy}{dx} = 1 - \sin x$ is 0 at $x = \frac{\pi}{2}$ (a flat spot). **Q4.** Ordinates $0, \frac{\pi}{4} + 1, \frac{\pi}{2}, \frac{3\pi}{4} - 1, \pi$; meets $y = \frac{x}{2}$ at $x = 0, \pi, 2\pi$. **Q5.** Periods 2π and π ; intercepts $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$. **Q6.** Amplitude 5; maximum 5, minimum -5 . **Q7.** Amplitude 2; intercepts $\left(\frac{2\pi}{3}, 0\right), \left(\frac{5\pi}{3}, 0\right)$. **Q8.** Amplitude 2; intercepts $\left(\frac{\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right)$. **Q9.** Increasing; meets $y = x$ at $x = \frac{\pi}{2}, \frac{3\pi}{2}$; y-intercept $(0, -1)$. **Q10.** Increasing about the line $y = 2x$; meets it at $x = 0, \pi, 2\pi$. **Q11.** Intercepts $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$; minimum -2 at $x = \frac{3\pi}{2}$. **Q12.** Intercepts $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$. **Q13.** Amplitude $2\sqrt{2}$; intercepts $\left(\frac{\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right)$; max $2\sqrt{2}$ at $x = \frac{3\pi}{4}$. **Q14.** Increasing about $y = x$; meets it at $x = 0, \pi, 2\pi$; a flat spot does not occur ($1 + 2\cos x = 0$ gives local turning points). **Q15.** Intercepts $x = 0, \pi, 2\pi$; oscillates about the x-axis with the half-amplitude second harmonic. **Q16.** Intercepts $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$; minimum -2 at $x = \pi$. **Q17.** Amplitude 5; intercepts $\left(\arctan \frac{3}{4}, 0\right), \left(\pi + \arctan \frac{3}{4}, 0\right)$ (i.e. $x \approx 0.64, 3.79$). **Q18.** Increasing about $y = \frac{x}{2}$; y-intercept $(0, 1)$; meets the line at $x = \frac{\pi}{2}, \frac{3\pi}{2}$. **Q19.** Intercepts $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$. **Q20.** Increasing overall about $y = x$; y-intercept $(0, 2)$; local turning points where $1 - 2\sin x = 0$. **Q21.** Amplitude $\sqrt{5}$; intercepts $(\pi - \arctan 2, 0), (2\pi - \arctan 2, 0)$ (i.e. $x \approx 2.03, 5.18$).

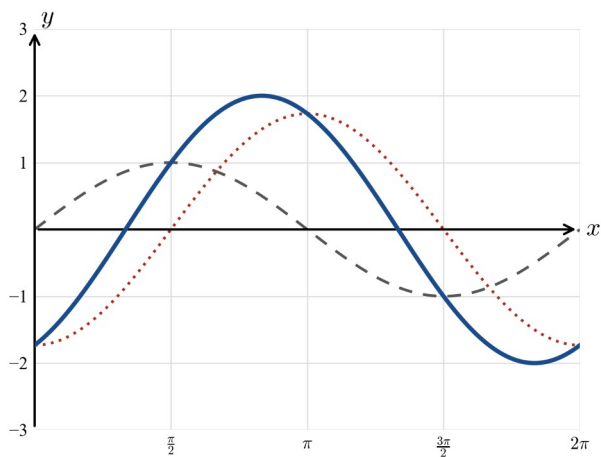
Fully-worked solutions

Q1. $y = \sqrt{3}\sin x + \cos x$. Amplitude $R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$, so $y = 2\sin\left(x + \frac{\pi}{6}\right)$. Intercepts:

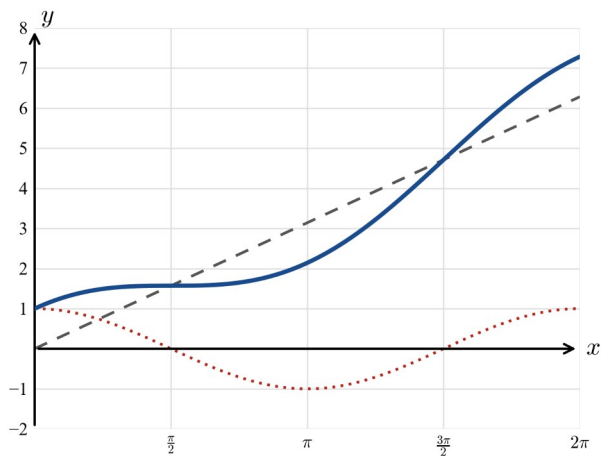
$\tan x = -\frac{1}{\sqrt{3}} \Rightarrow x = \frac{5\pi}{6}, \frac{11\pi}{6}$. Maximum 2 at $x = \frac{\pi}{3}$; minimum -2 at $x = \frac{4\pi}{3}$.



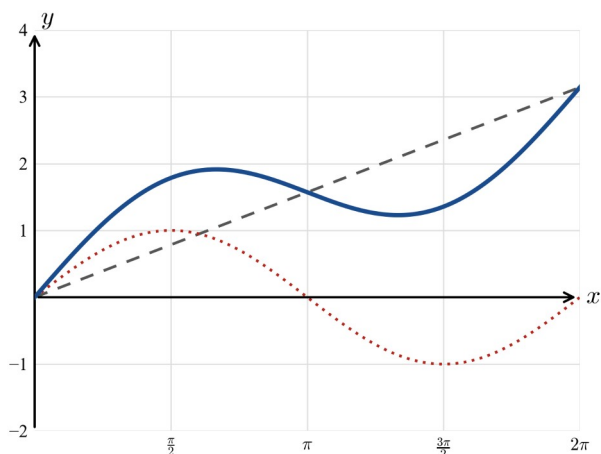
Q2. $y = \sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$, amplitude 2. Intercepts: $\tan x = \sqrt{3} \Rightarrow x = \frac{\pi}{3}, \frac{4\pi}{3}$.



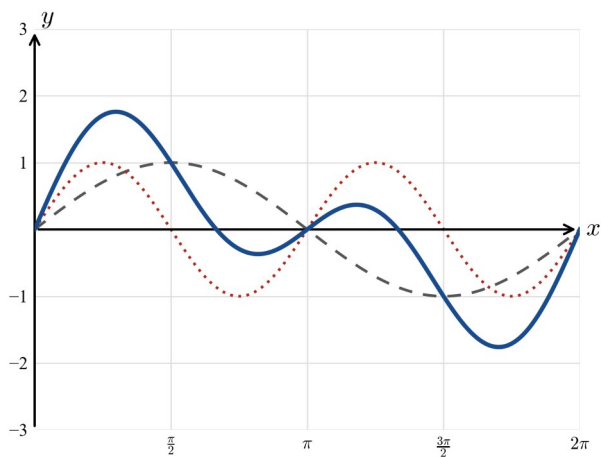
Q3. $y = x + \cos x$. Ordinates $1, \frac{\pi}{2}, \pi - 1, \frac{3\pi}{2}, 2\pi + 1$. Since $\frac{dy}{dx} = 1 - \sin x \geq 0$ with equality at $x = \frac{\pi}{2}$, the graph is increasing but has a horizontal flat spot at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$; it weaves about the line $y = x$, meeting it where $\cos x = 0$.



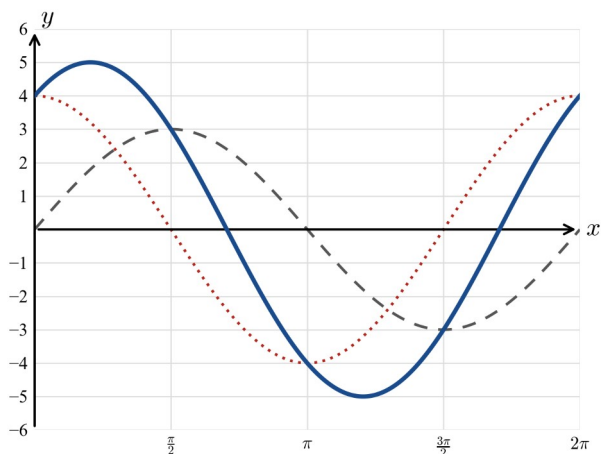
Q4. $y = \frac{x}{2} + \sin x$. Ordinates at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$: $0, \frac{\pi}{4} + 1, \frac{\pi}{2}, \frac{3\pi}{4} - 1, \pi$. It meets $y = \frac{x}{2}$ where $\sin x = 0$, i.e. $x = 0, \pi, 2\pi$, oscillating about that line.



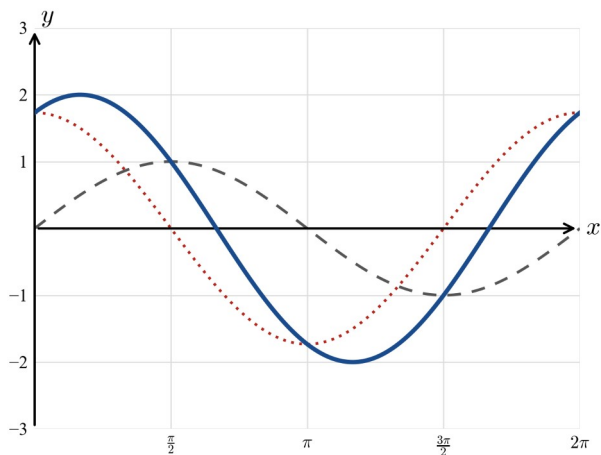
Q5. $y = \sin x + \sin 2x$. Periods 2π and π . x -intercepts: $\sin x + 2 \sin x \cos x = \sin x (1 + 2 \cos x) = 0 \Rightarrow \sin x = 0$ or $\cos x = -\frac{1}{2}$, giving $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$.



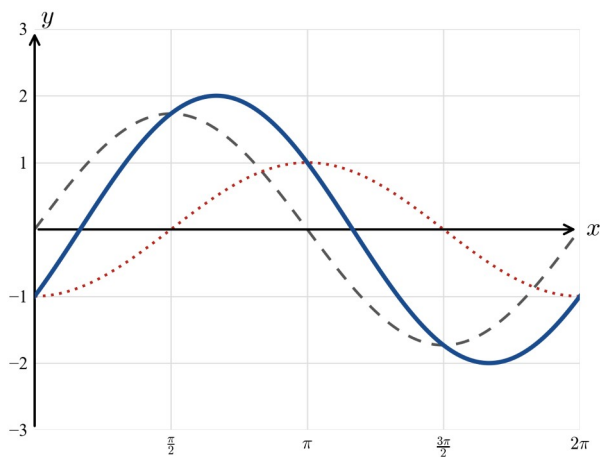
Q6. $y = 3 \sin x + 4 \cos x$. Amplitude $R = \sqrt{3^2 + 4^2} = 5$, so the maximum value is 5 and the minimum is -5 ; period 2π .



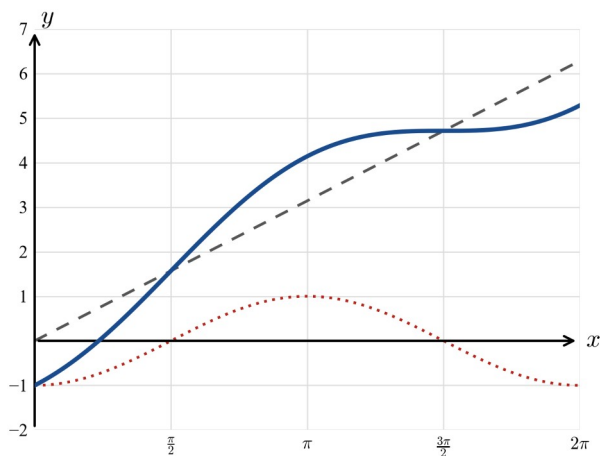
Q7. $y = \sin x + \sqrt{3} \cos x = 2 \sin\left(x + \frac{\pi}{3}\right)$, amplitude 2. Intercepts: $\tan x = -\sqrt{3} \Rightarrow x = \frac{2\pi}{3}, \frac{5\pi}{3}$.



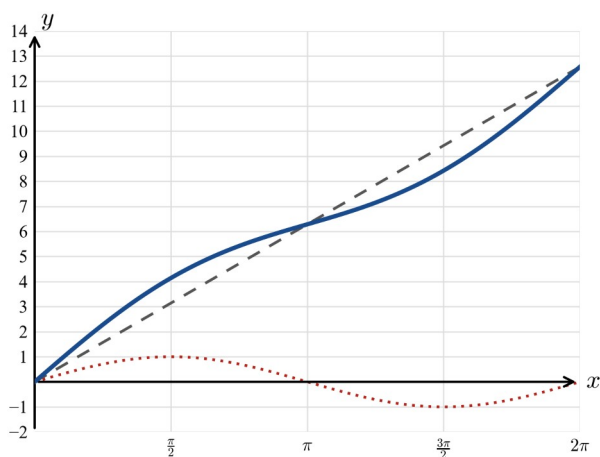
Q8. $y = \sqrt{3} \sin x - \cos x = 2 \sin\left(x - \frac{\pi}{6}\right)$, amplitude 2. Intercepts: $\tan x = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6}, \frac{7\pi}{6}$.



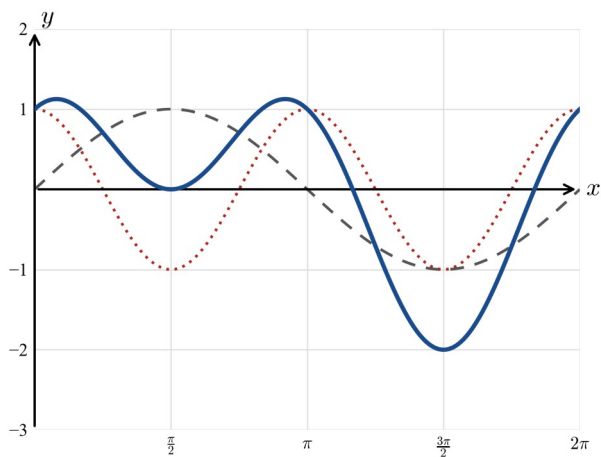
Q9. $y = x - \cos x$. y-intercept $(0, -1)$; ordinates increase overall ($\frac{dy}{dx} = 1 + \sin x \geq 0$). It meets the line $y = x$ where $\cos x = 0$, i.e. $x = \frac{\pi}{2}, \frac{3\pi}{2}$.



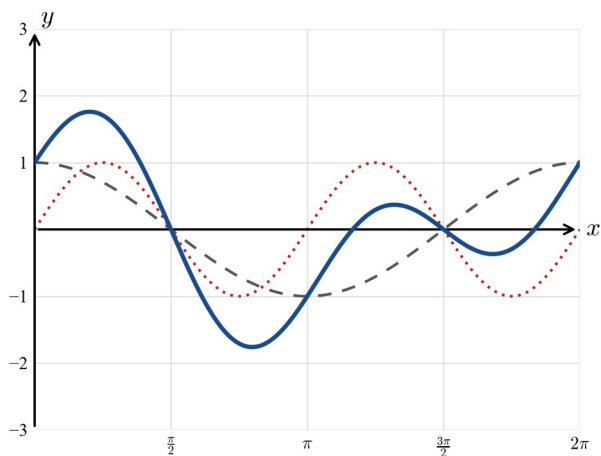
Q10. $y = 2x + \sin x$. The wave oscillates about the line $y = 2x$; since $\frac{dy}{dx} = 2 + \cos x > 0$ the graph is strictly increasing, meeting the line where $\sin x = 0$, i.e. $x = 0, \pi, 2\pi$.



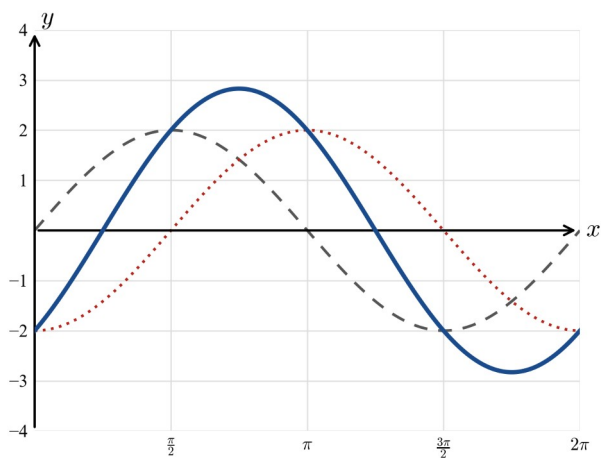
Q11. $y = \sin x + \cos 2x$. Ordinates $1, 0, 1, -2, 1$ at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; minimum -2 at $x = \frac{3\pi}{2}$. Intercepts (from $\sin x + 1 - 2\sin^2 x = 0$): $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.



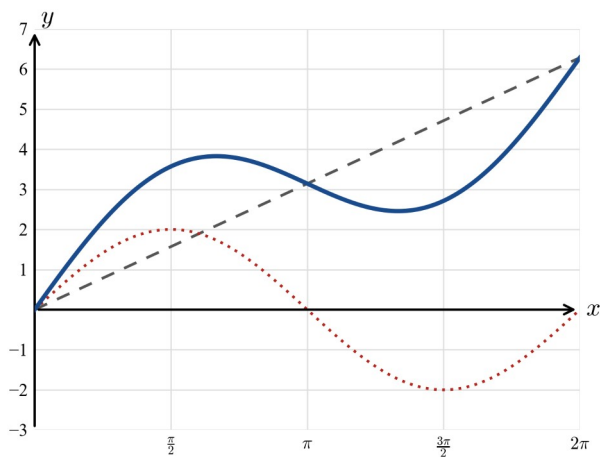
Q12. $y = \cos x + \sin 2x$. Intercepts: $\cos x + 2 \sin x \cos x = \cos x(1 + 2 \sin x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = -\frac{1}{2}$,
i.e. $x = \frac{7\pi}{6}, \frac{11\pi}{6}$.



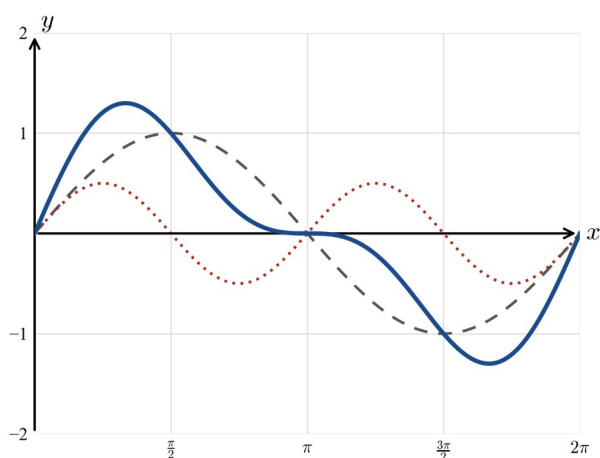
Q13. $y = 2 \sin x - 2 \cos x = 2\sqrt{2} \sin\left(x - \frac{\pi}{4}\right)$, amplitude $2\sqrt{2}$. Intercepts: $\tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$; maximum $2\sqrt{2}$ at $x = \frac{3\pi}{4}$.



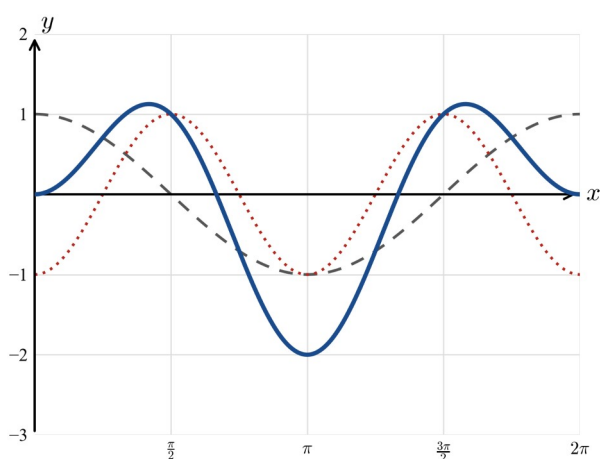
Q14. $y = x + 2 \sin x$. The wave (amplitude 2) oscillates about $y = x$, meeting it at $x = 0, \pi, 2\pi$. Since $\frac{dy}{dx} = 1 + 2 \cos x = 0$ at $x = \frac{2\pi}{3}, \frac{4\pi}{3}$, there are a local maximum and local minimum there.



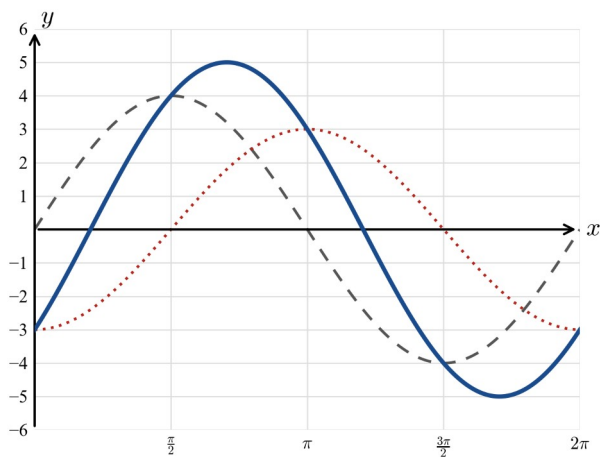
Q15. $y = \sin x + \frac{1}{2} \sin 2x = \sin x (1 + \cos x)$, so x -intercepts $x = 0, \pi, 2\pi$. The small second harmonic makes the positive hump taller than the negative one.



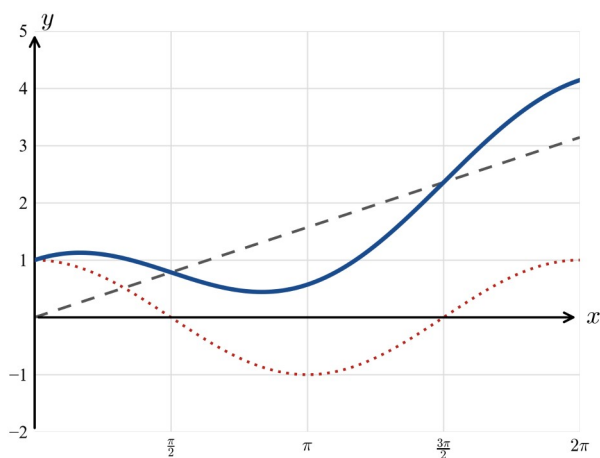
Q16. $y = \cos x - \cos 2x$. Ordinates $0, 1, -2, 1, 0$; minimum -2 at $x = \pi$. Intercepts (from $\cos x - (2\cos^2 x - 1) = 0$): $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$.



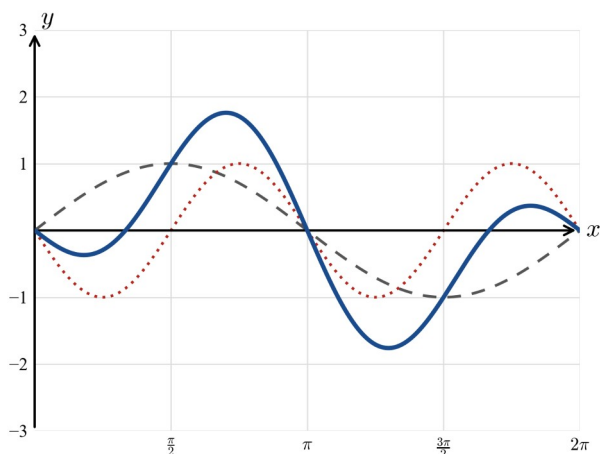
Q17. $y = 4 \sin x - 3 \cos x = 5 \sin(x - \varphi)$ with $\varphi = \arctan \frac{3}{4}$, amplitude 5. Intercepts: $\tan x = \frac{3}{4} \Rightarrow x = \arctan \frac{3}{4} \approx 0.64$ and $x = \pi + \arctan \frac{3}{4} \approx 3.79$.



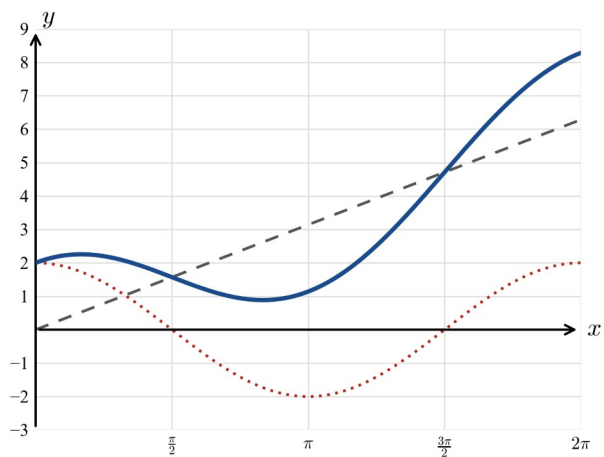
Q18. $y = \frac{x}{2} + \cos x$. y -intercept $(0, 1)$; the wave oscillates about $y = \frac{x}{2}$, meeting it where $\cos x = 0$, i.e. $x = \frac{\pi}{2}, \frac{3\pi}{2}$.



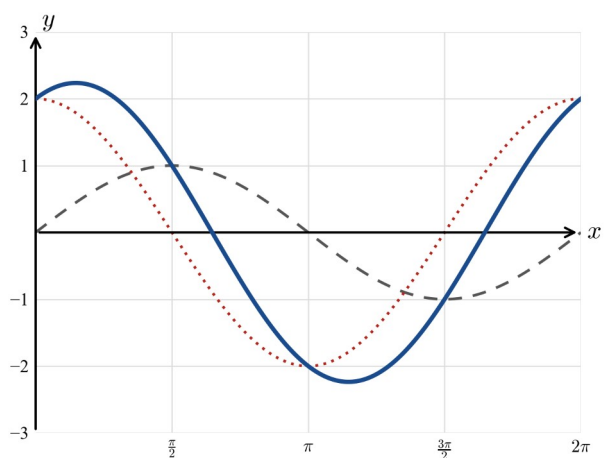
Q19. $y = \sin x - \sin 2x = \sin x(1 - 2\cos x)$, so x -intercepts $\sin x = 0$ or $\cos x = \frac{1}{2}$: $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.



Q20. $y = x + 2 \cos x$. y -intercept $(0, 2)$; the amplitude-2 wave oscillates about $y = x$. Since $\frac{dy}{dx} = 1 - 2 \sin x = 0$ at $x = \frac{\pi}{6}, \frac{5\pi}{6}$, there are a local maximum and minimum there before the line dominates.



Q21. $y = \sin x + 2 \cos x = \sqrt{5} \sin(x + \varphi)$ with $\varphi = \arctan 2$, amplitude $\sqrt{5}$. Intercepts:
 $\tan x = -2 \Rightarrow x = \pi - \arctan 2 \approx 2.03$ and $x = 2\pi - \arctan 2 \approx 5.18$.



Chapter 6 Circular functions — 6I Finding rules for circular-function graphs

Theory

Given the graph of a sinusoidal function, we can find a rule for it in the form

$$y = a \cos(n(t - \varepsilon)) + b \text{ or } y = a \sin(n(t - \varepsilon)) + b.$$

Four features of the graph give the four constants.

Amplitude a . The distance from the centre line to a maximum (or minimum):

$$a = \frac{\text{maximum} - \text{minimum}}{2}.$$

Central value b (the mean position, the dashed line about which the graph oscillates):

$$b = \frac{\text{maximum} + \text{minimum}}{2}.$$

Value of n . Read the **period T** from the graph (the horizontal length of one complete cycle — for example the distance between successive maxima). Then

$$n = \frac{2\pi}{T}.$$

Phase ε . This depends on which of $\sin$ or $\cos$ you use.

- Using **cosine**, $y = a \cos(n(t - \varepsilon)) + b$ has a **maximum at $t = \varepsilon$** . So read the t -coordinate of any maximum and take that as ε .
- Using **sine**, $y = a \sin(n(t - \varepsilon)) + b$ crosses its centre line **going upward at $t = \varepsilon$** .

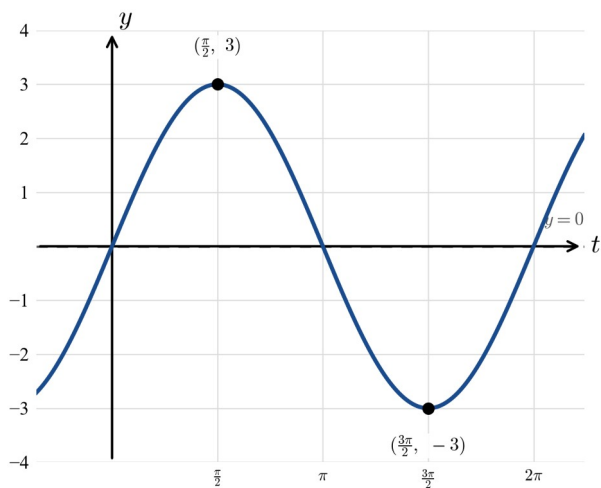
Because a maximum is usually the easiest point to read, the **cosine form keyed to a maximum** is often quickest. Any equivalent rule is acceptable — a sine form, or a cosine form using a different maximum (they differ by a whole number of periods).

Method. (1) $a = 1/2(\max - \min)$; (2) $b = 1/2(\max + \min)$; (3) $n = \frac{2\pi}{T}$ from the period; (4) ε = the t -coordinate of a maximum (cosine form).

Worked examples

Example 1 — scaffolded

Determine a rule for the graph shown.



Working

Maximum 3, minimum -3 . $a = \frac{3 - (-3)}{2} = 3$.

$$b = \frac{3 + (-3)}{2} = 0.$$

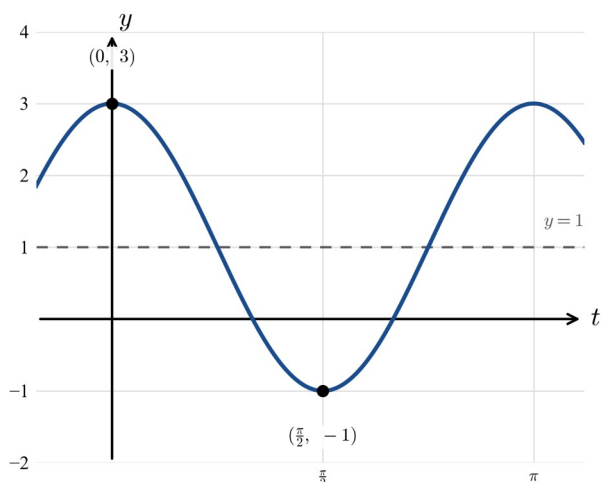
One cycle has length 2π , so $T = 2\pi$ and $n = \frac{2\pi}{2\pi} = 1$.

There is a maximum at $t = \frac{\pi}{2}$, so take $\varepsilon = \frac{\pi}{2}$.

$$\therefore y = 3 \cos\left(t - \frac{\pi}{2}\right)$$

Example 2 — scaffolded

Determine a rule for the graph shown.



Working

Maximum 3, minimum -1 : $a = \frac{3 - (-1)}{2} = 2$,

$$b = \frac{3 + (-1)}{2} = 1.$$

One cycle has length π , so $T = \pi$ and $n = \frac{2\pi}{\pi} = 2$.

A maximum occurs at $t = 0$, so $\varepsilon = 0$.

Comment

Amplitude = half the peak-to-trough distance.

The graph oscillates about $y = 0$.

Read the period, then $n = 2\pi / T$.

For the cosine form, a maximum occurs at $t = \varepsilon$.

Substitute a, n, ε, b . (Equivalently $y = 3 \sin t$.)

Comment

Amplitude and centre line.

Period is π here (two cycles in 2π).

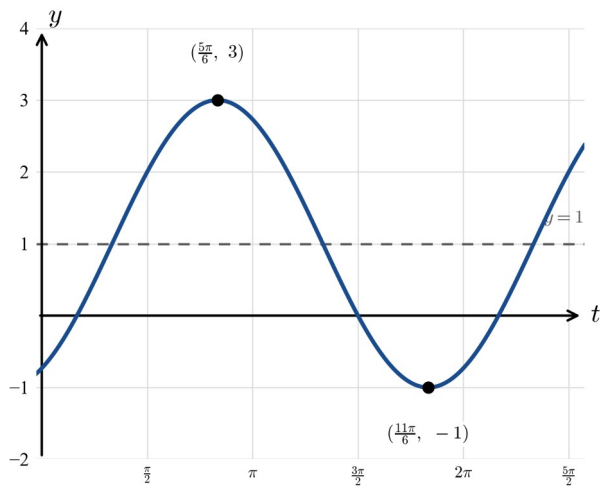
Cosine has its maximum at $t = \varepsilon$.

$$\therefore y = 2 \cos(2t) + 1$$

With $\varepsilon = 0$ the phase term vanishes.

Example 3 — unscaffolded

Determine a rule for the graph shown.



Maximum 3, minimum -1 : $a = \frac{3 - (-1)}{2} = 2$ and $b = \frac{3 + (-1)}{2} = 1$.

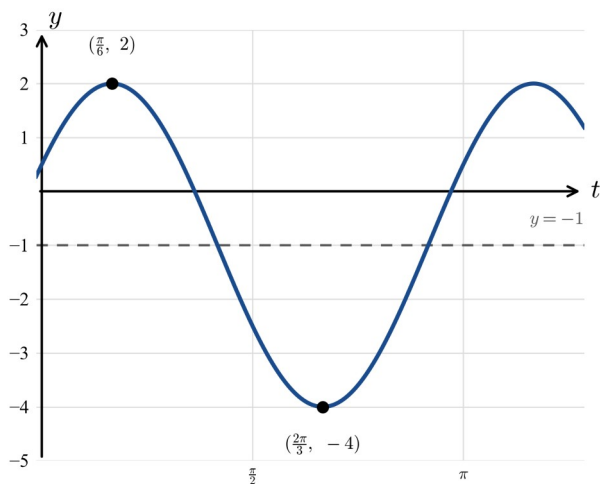
One cycle has length 2π , so $T = 2\pi$ and $n = \frac{2\pi}{2\pi} = 1$.

There is a maximum at $t = \frac{5\pi}{6}$, so $\varepsilon = \frac{5\pi}{6}$.

$$\therefore y = 2 \cos\left(t - \frac{5\pi}{6}\right) + 1$$

Example 4 — unscaffolded

Determine a rule for the graph shown.



Maximum 2, minimum -4 : $a = \frac{2 - (-4)}{2} = 3$ and $b = \frac{2 + (-4)}{2} = -1$.

One cycle has length π , so $T = \pi$ and $n = \frac{2\pi}{\pi} = 2$.

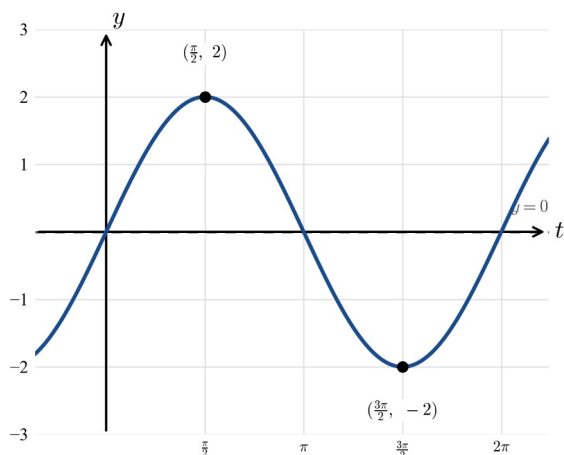
There is a maximum at $t = \frac{\pi}{6}$, so $\varepsilon = \frac{\pi}{6}$.

$$\therefore y = 3 \cos \left(2 \left(t - \frac{\pi}{6} \right) \right) - 1$$

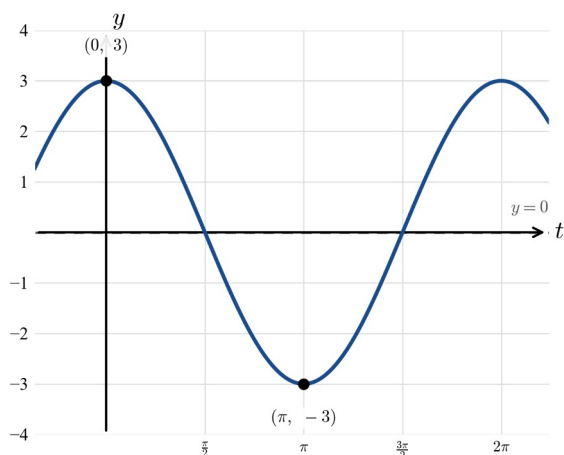
Practice questions

Scaffolded

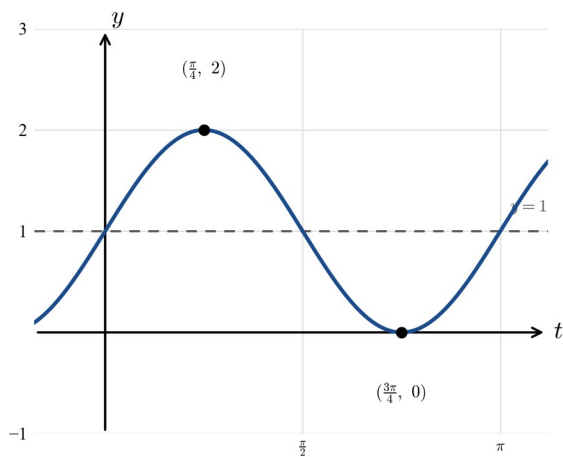
Q1. For the graph shown: (a) state the amplitude a ; (b) state the central value b ; (c) find the period and hence n ; (d) write a rule for the graph.



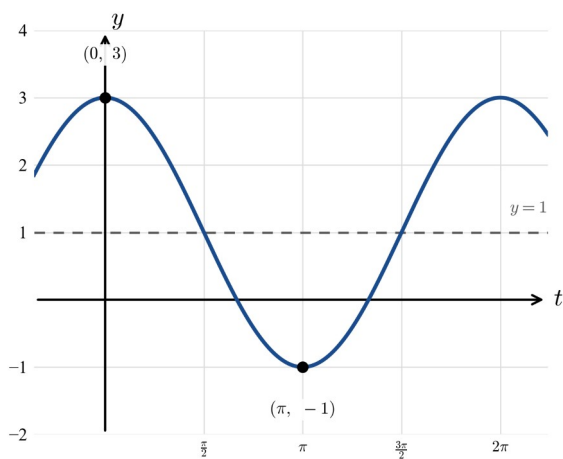
Q2. For the graph shown: (a) state a and b ; (b) find n ; (c) write a rule.



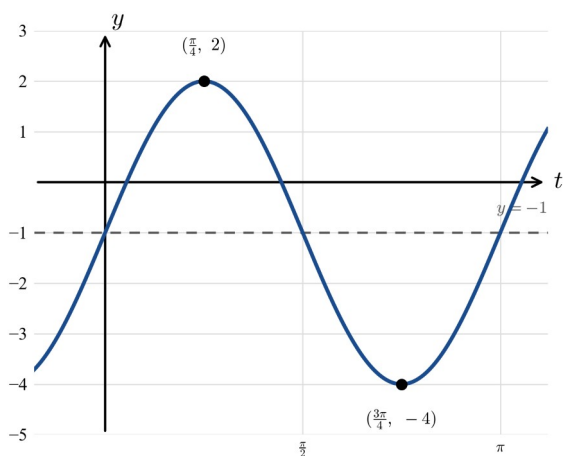
Q3. For the graph shown: (a) find a and b ; (b) find the period and n ; (c) using a maximum, state ε ; (d) write a rule.



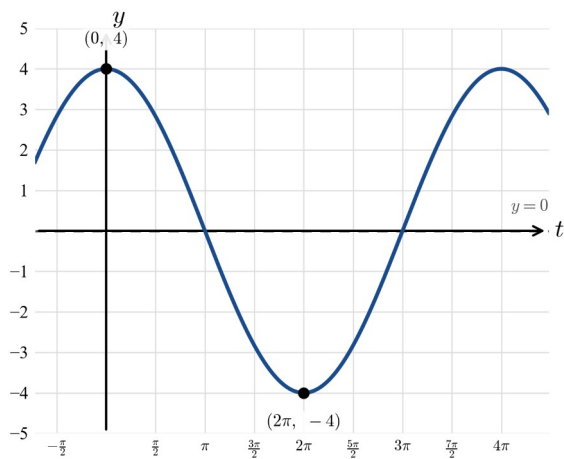
Q4. For the graph shown: (a) find a and b ; (b) find n ; (c) write a rule.



Q5. For the graph shown: (a) find a and b ; (b) find n ; (c) using a maximum, write a rule.

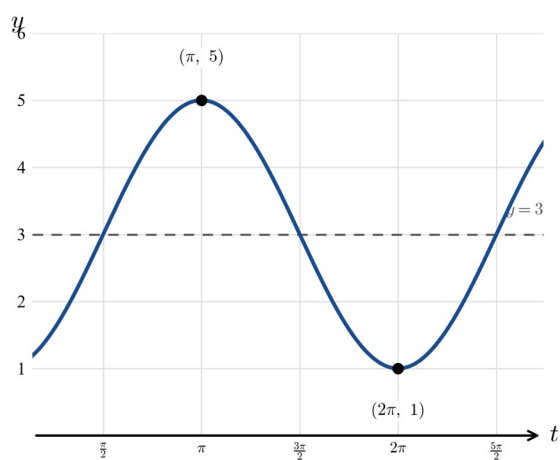


Q6. For the graph shown: (a) find a and b ; (b) find the period and n ; (c) write a rule.

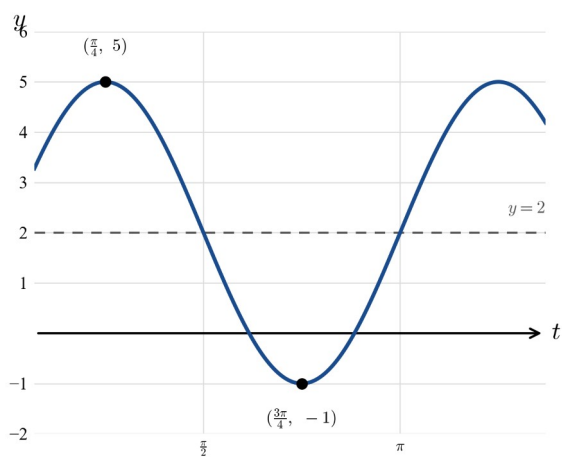


Unscaffolded

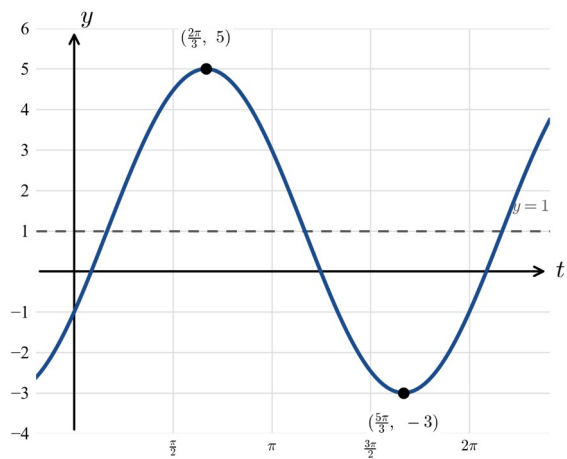
For each graph, **determine a possible rule** of the form $y = a \cos(n(t - \varepsilon)) + b$ (or an equivalent sine form).



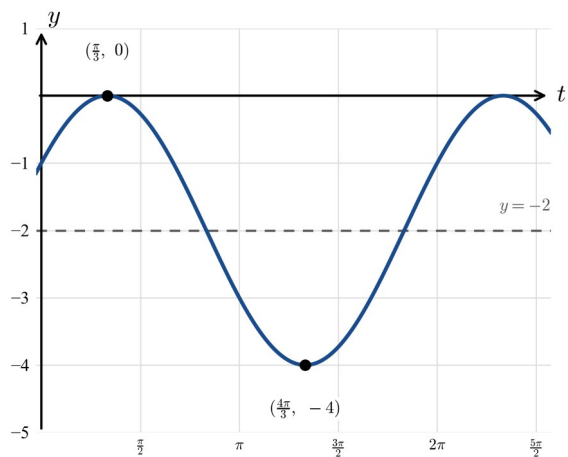
Q7.



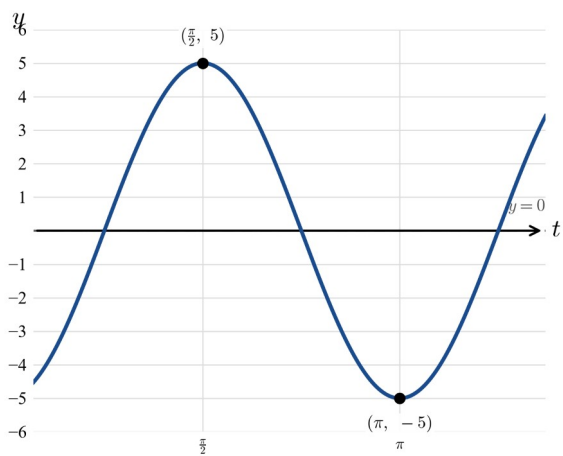
Q8.



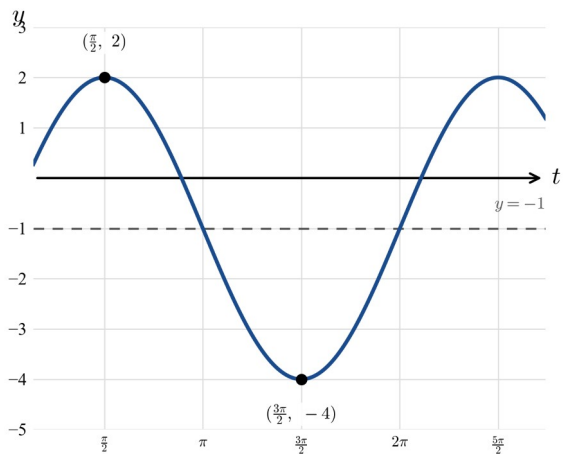
Q9.



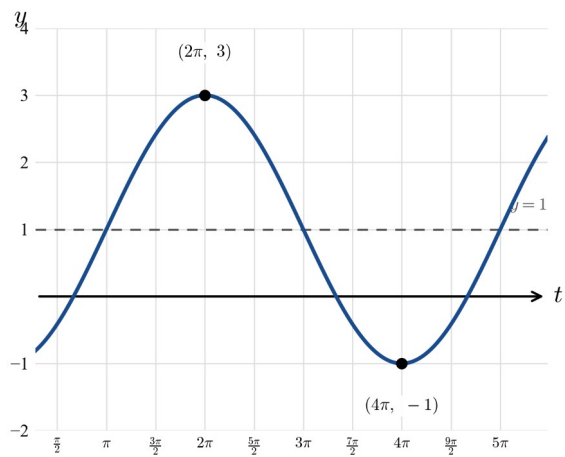
Q10.



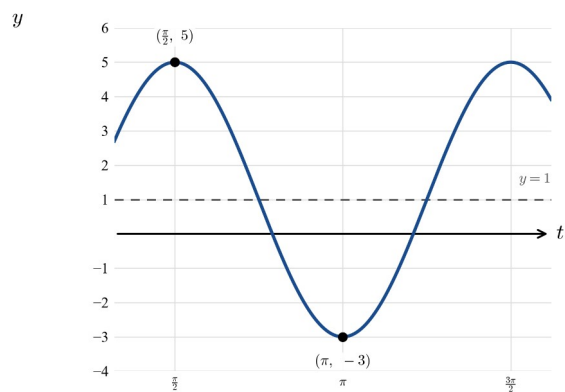
Q11.



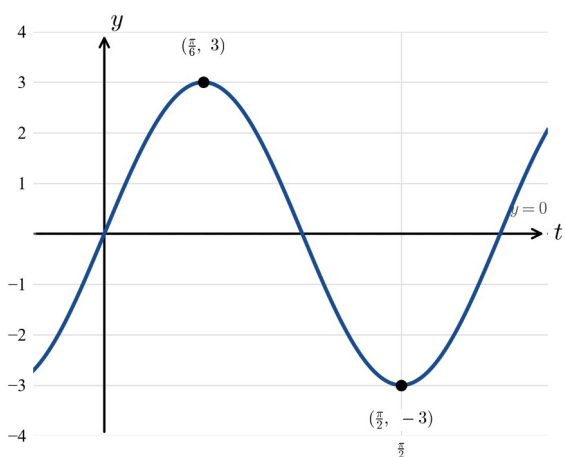
Q12.



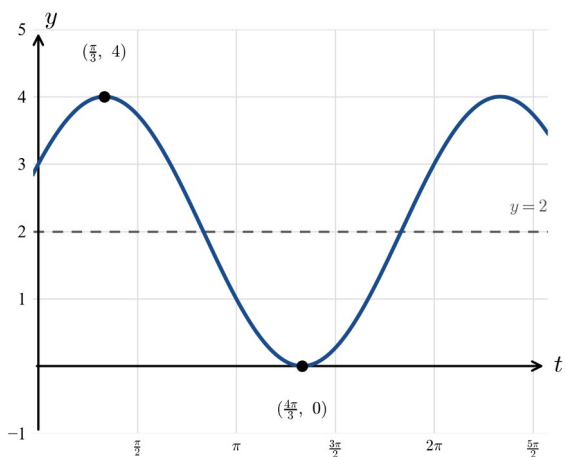
Q13.



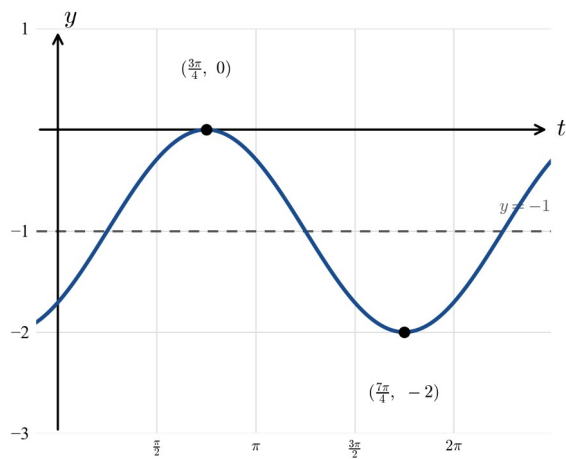
Q14.



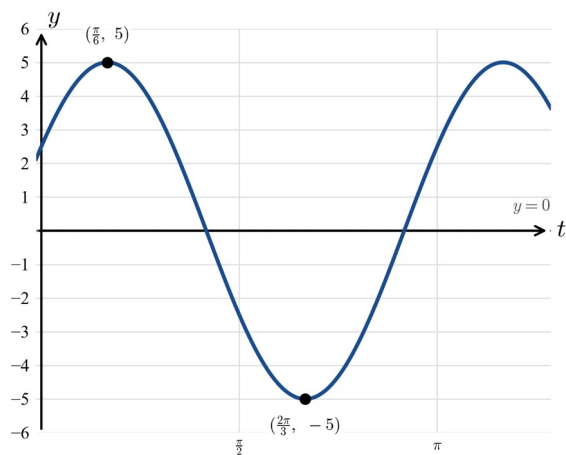
Q15.



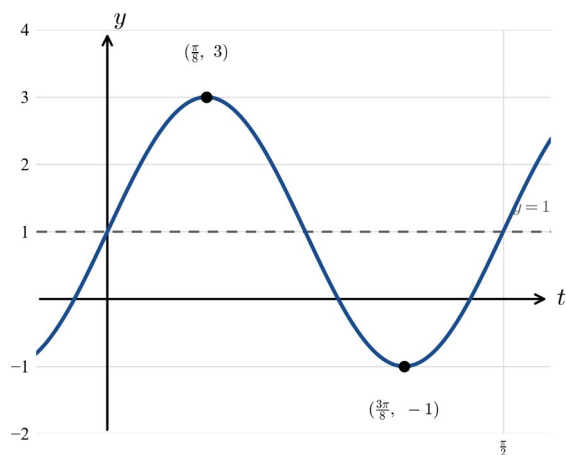
Q16.



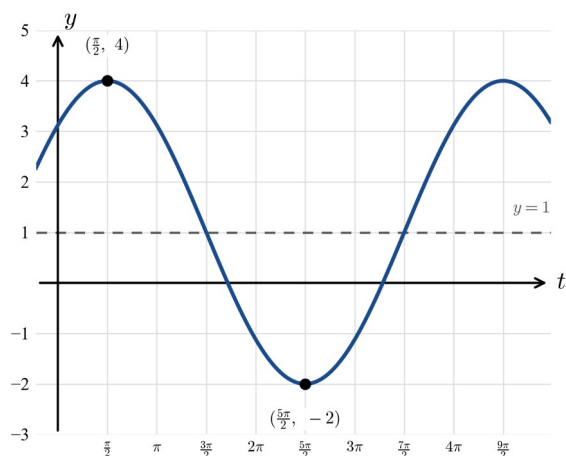
Q17.



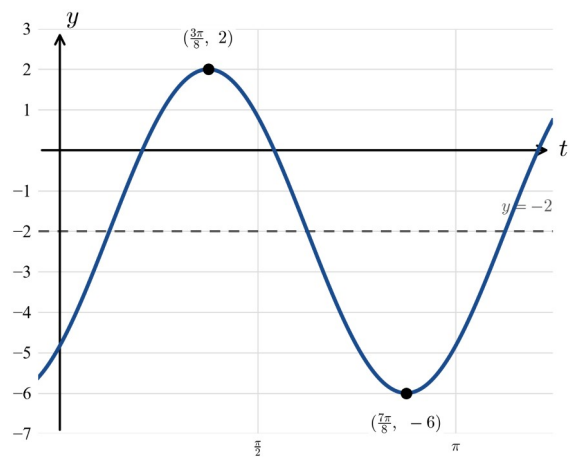
Q18.



Q19.



Q20.



Q21.

Answers

Q1. $a=2, b=0, T=2\pi$ so $n=1$; $y=2\cos\left(t-\frac{\pi}{2}\right)$. **Q2.** $a=3, b=0, n=1$; $y=3\cos t$. **Q3.** $a=1, b=1, T=\pi$ so $n=2$, $\varepsilon=\frac{\pi}{4}$; $y=\cos\left(2\left(t-\frac{\pi}{4}\right)\right)+1$. **Q4.** $a=2, b=1, n=1$; $y=2\cos t+1$. **Q5.** $a=3, b=-1, n=2$; $y=3\cos\left(2\left(t-\frac{\pi}{4}\right)\right)-1$. **Q6.** $a=4, b=0, T=4\pi$ so $n=\frac{1}{2}$; $y=4\cos\left(\frac{t}{2}\right)$. **Q7.** $y=2\cos(t-\pi)+3$. **Q8.** $y=3\cos\left(2\left(t-\frac{\pi}{4}\right)\right)+2$. **Q9.** $y=4\cos\left(t-\frac{2\pi}{3}\right)+1$. **Q10.** $y=2\cos\left(t-\frac{\pi}{3}\right)-2$. **Q11.** $y=5\cos\left(2\left(t-\frac{\pi}{2}\right)\right)$. **Q12.** $y=3\cos\left(t-\frac{\pi}{2}\right)-1$. **Q13.** $y=2\cos\left(\frac{1}{2}(t-2\pi)\right)+1$. **Q14.** $y=4\cos\left(2\left(t-\frac{\pi}{2}\right)\right)+1$. **Q15.** $y=3\cos\left(3\left(t-\frac{\pi}{6}\right)\right)$. **Q16.** $y=2\cos\left(t-\frac{\pi}{3}\right)+2$. **Q17.** $y=\cos\left(t-\frac{3\pi}{4}\right)-1$. **Q18.** $y=5\cos\left(2\left(t-\frac{\pi}{6}\right)\right)$. **Q19.** $y=2\cos\left(4\left(t-\frac{\pi}{8}\right)\right)+1$. **Q20.** $y=3\cos\left(\frac{1}{2}\left(t-\frac{\pi}{2}\right)\right)+1$. **Q21.** $y=4\cos\left(2\left(t-\frac{3\pi}{8}\right)\right)-2$.

Fully-worked solutions

Q1. Max 2, min -2: $a=2, b=0$. One cycle is 2π , so $n=1$. Maximum at $t=\frac{\pi}{2}$, so $\varepsilon=\frac{\pi}{2}$. $\therefore y=2\cos\left(t-\frac{\pi}{2}\right)$.

Q2. Max 3, min -3: $a=3, b=0$. Period 2π , so $n=1$. Maximum at $t=0$, so $\varepsilon=0$. $\therefore y=3\cos t$.

Q3. Max 2, min 0: $a=1, b=1$. Period π , so $n=2$. Maximum at $t=\frac{\pi}{4}$, so $\varepsilon=\frac{\pi}{4}$. $\therefore y=\cos\left(2\left(t-\frac{\pi}{4}\right)\right)+1$.

Q4. Max 3, min -1: $a=2, b=1$. Period 2π , so $n=1$. Maximum at $t=0$, so $\varepsilon=0$. $\therefore y=2\cos t+1$.

Q5. Max 2, min -4: $a=3, b=-1$. Period π , so $n=2$. Maximum at $t=\frac{\pi}{4}$. $\therefore y=3\cos\left(2\left(t-\frac{\pi}{4}\right)\right)-1$.

Q6. Max 4, min -4: $a=4, b=0$. One cycle is 4π , so $n=\frac{2\pi}{4\pi}=\frac{1}{2}$. Maximum at $t=0$. $\therefore y=4\cos\left(\frac{t}{2}\right)$.

Q7. Max 5, min 1: $a=2, b=3$. Period 2π , so $n=1$. Maximum at $t=\pi$. $\therefore y=2\cos(t-\pi)+3$.

Q8. Max 5, min -1: $a=3, b=2$. Period π , so $n=2$. Maximum at $t=\frac{\pi}{4}$. $\therefore y=3\cos\left(2\left(t-\frac{\pi}{4}\right)\right)+2$.

Q9. Max 5, min -3: $a=4, b=1$. Period 2π , so $n=1$. Maximum at $t=\frac{2\pi}{3}$. $\therefore y=4\cos\left(t-\frac{2\pi}{3}\right)+1$.

Q10. Max 0, min -4: $a=2, b=-2$. Period 2π , so $n=1$. Maximum at $t=\frac{\pi}{3}$. $\therefore y=2\cos\left(t-\frac{\pi}{3}\right)-2$.

Q11. Max 5, min -5: $a=5, b=0$. Period π , so $n=2$. Maximum at $t=\frac{\pi}{2}$. $\therefore y=5\cos\left(2\left(t-\frac{\pi}{2}\right)\right)$.

Q12. Max 2, min -4 : $a=3$, $b=-1$. Period 2π , so $n=1$. Maximum at $t=\frac{\pi}{2}$. $\therefore y=3\cos\left(t-\frac{\pi}{2}\right)-1$.

Q13. Max 3, min -1 : $a=2$, $b=1$. One cycle is 4π , so $n=\frac{1}{2}$. Maximum at $t=2\pi$. $\therefore y=2\cos\left(\frac{1}{2}(t-2\pi)\right)+1$.

Q14. Max 5, min -3 : $a=4$, $b=1$. Period π , so $n=2$. Maximum at $t=\frac{\pi}{2}$. $\therefore y=4\cos\left(2\left(t-\frac{\pi}{2}\right)\right)+1$.

Q15. Max 3, min -3 : $a=3$, $b=0$. One cycle is $\frac{2\pi}{3}$, so $n=3$. Maximum at $t=\frac{\pi}{6}$. $\therefore y=3\cos\left(3\left(t-\frac{\pi}{6}\right)\right)$.

Q16. Max 4, min 0 : $a=2$, $b=2$. Period 2π , so $n=1$. Maximum at $t=\frac{\pi}{3}$. $\therefore y=2\cos\left(t-\frac{\pi}{3}\right)+2$.

Q17. Max 0, min -2 : $a=1$, $b=-1$. Period 2π , so $n=1$. Maximum at $t=\frac{3\pi}{4}$. $\therefore y=\cos\left(t-\frac{3\pi}{4}\right)-1$.

Q18. Max 5, min -5 : $a=5$, $b=0$. Period π , so $n=2$. Maximum at $t=\frac{\pi}{6}$. $\therefore y=5\cos\left(2\left(t-\frac{\pi}{6}\right)\right)$.

Q19. Max 3, min -1 : $a=2$, $b=1$. One cycle is $\frac{\pi}{2}$, so $n=4$. Maximum at $t=\frac{\pi}{8}$. $\therefore y=2\cos\left(4\left(t-\frac{\pi}{8}\right)\right)+1$.

Q20. Max 4, min -2 : $a=3$, $b=1$. One cycle is 4π , so $n=\frac{1}{2}$. Maximum at $t=\frac{\pi}{2}$. $\therefore y=3\cos\left(\frac{1}{2}\left(t-\frac{\pi}{2}\right)\right)+1$.

Q21. Max 2, min -6 : $a=4$, $b=-2$. Period π , so $n=2$. Maximum at $t=\frac{3\pi}{8}$. $\therefore y=4\cos\left(2\left(t-\frac{3\pi}{8}\right)\right)-2$.

Chapter 6 Circular functions — 6J The tangent function

Theory

The **tangent function** $y = \tan x$ is defined by $\tan x = \frac{\sin x}{\cos x}$, so it is undefined wherever $\cos x = 0$.

Features of $y = \tan x$:

- **Period π** — the graph repeats every π units.
- **Vertical asymptotes** where $\cos x = 0$, i.e. $x = \frac{\pi}{2} + k\pi$ for integer k .
- **x -intercepts** where $\sin x = 0$, i.e. $x = k\pi$.
- **Range $\mathbb{R}$** ; the graph is increasing on each interval between consecutive asymptotes.
- $\tan$ is an **odd** function: $\tan(-x) = -\tan x$.

Transformations. For $y = a \tan(n(x - \varepsilon))$ (with $n > 0$):

- a is a dilation of factor $|a|$ from the x -axis (a reflection in the x -axis as well when $a < 0$); it changes the steepness but **not** the asymptotes or intercepts.
- n changes the **period to $\frac{\pi}{n}$** ; the asymptotes and intercepts are packed n times as closely.
- ε is a horizontal translation of ε units.

Finding the key features of $y = a \tan(n(x - \varepsilon))$:

- **Asymptotes:** solve $n(x - \varepsilon) = \frac{\pi}{2} + k\pi$, giving $x = \varepsilon + \frac{\pi/2 + k\pi}{n}$.
- **x -intercepts:** solve $n(x - \varepsilon) = k\pi$, giving $x = \varepsilon + \frac{k\pi}{n}$.

Solving tan equations. Because $\tan$ has period π , once one solution x_0 of $\tan x = c$ is found, **all** solutions are $x = x_0 + k\pi$. Find every solution that lies in the required domain. The base angles come from the exact values, e.g. $\tan \frac{\pi}{4} = 1$, $\tan \frac{\pi}{3} = \sqrt{3}$, $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

To sketch: mark the asymptotes (dashed), plot the x -intercepts, and draw the increasing (or, if $a < 0$, decreasing) branch in each strip between asymptotes.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = \tan x$ for $x \in [0, 2\pi]$, showing the asymptotes and x -intercepts.

Working

Comment

Period = π .

For $y = \tan x$, $n = 1$, so the period is π .

Asymptotes: $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ in $[0, 2\pi]$.

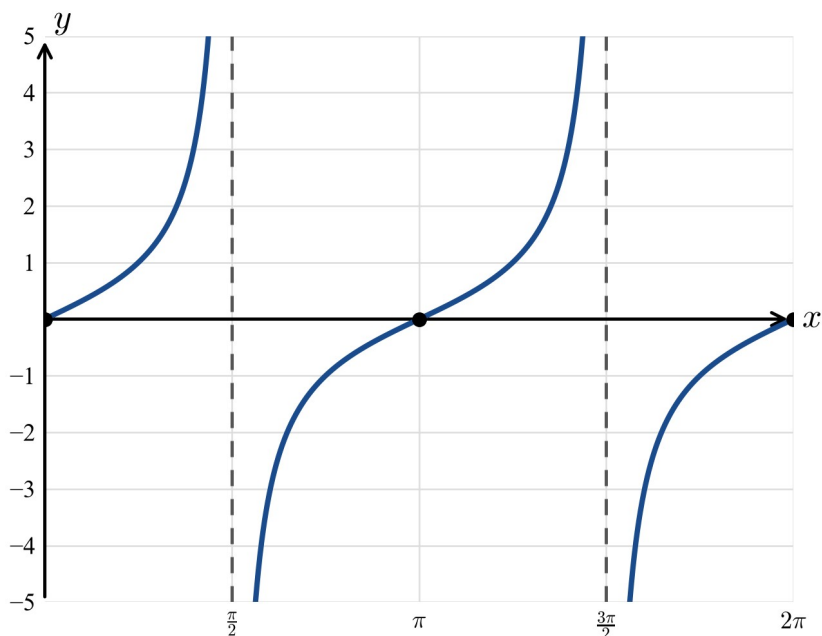
$x = \frac{\pi}{2} + k\pi$; take $k = 0, 1$.

x -intercepts: $\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$.

$x = k\pi$ within the domain.

Draw an increasing branch in each strip between asymptotes.

$\tan$ increases from $-\infty$ to $+\infty$ across each period.



Example 2 — scaffolded

Sketch the graph of $y = \tan 2x$ for $x \in [0, \pi]$.

Working

Comment

Here $n = 2$, so the period is $\frac{\pi}{2}$.

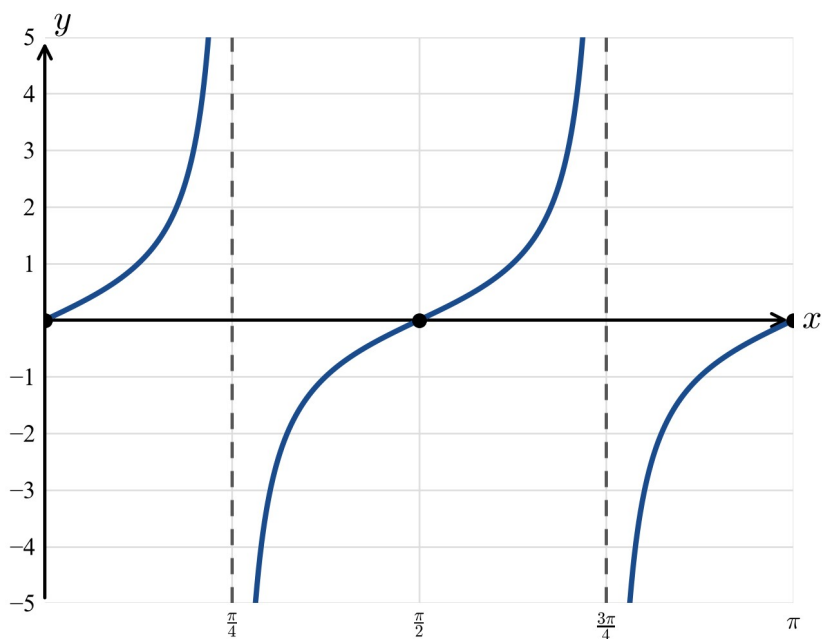
Period = $\frac{\pi}{n} = \frac{\pi}{2}$.

Asymptotes: $2x = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$.

Solve for x in $[0, \pi]$.

x -intercepts: $2x = k\pi \Rightarrow x = 0, \frac{\pi}{2}, \pi$.

Within the domain.



Example 3 — unscaffolded

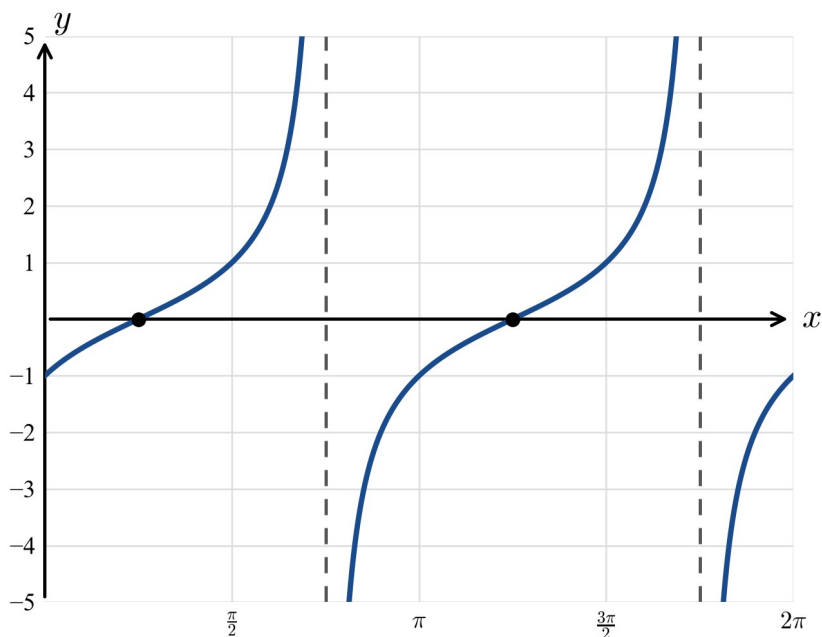
Sketch the graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ for $x \in [0, 2\pi]$.

This is $y = \tan x$ translated $\frac{\pi}{4}$ to the right; the period is still π .

Asymptotes: $x - \frac{\pi}{4} = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

x -intercepts: $x - \frac{\pi}{4} = k\pi \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$\therefore$ the whole graph of $y = \tan x$ shifts right by $\frac{\pi}{4}$.



Example 4 — unscaffolded

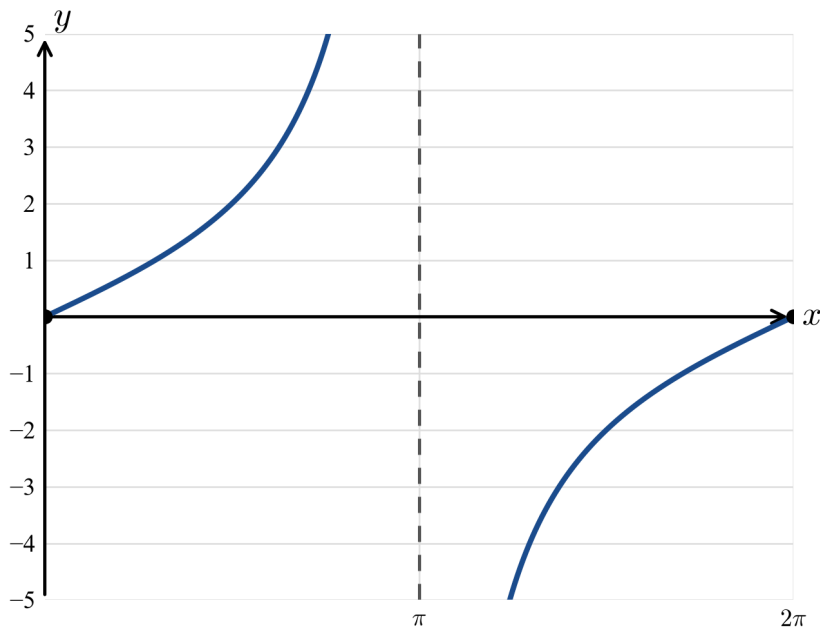
Sketch the graph of $y = 2 \tan \frac{x}{2}$ for $x \in [0, 2\pi]$.

Here $n = \frac{1}{2}$, so the period is $\frac{\pi}{1/2} = 2\pi$; the factor 2 makes the graph steeper but does not move the asymptotes or intercepts.

Asymptotes: $\frac{x}{2} = \frac{\pi}{2} + k\pi \Rightarrow x = \pi$ (the only one in $[0, 2\pi]$).

x -intercepts: $\frac{x}{2} = k\pi \Rightarrow x = 0, 2\pi$.

$\therefore$ over $[0, 2\pi]$ the graph shows exactly **one** complete branch.



Practice questions

Scaffolded

Q1. For $y = \tan x$, $x \in [-\pi, \pi]$: (a) state the period; (b) write the equations of the asymptotes; (c) find the x -intercepts; (d) sketch the graph.

Q2. For $y = \tan 3x$, $x \in [0, \pi]$: (a) state the period; (b) find the asymptotes; (c) find the x -intercepts; (d) sketch the graph.

Q3. For $y = \tan\left(x + \frac{\pi}{3}\right)$, $x \in [0, 2\pi]$: (a) describe the transformation from $y = \tan x$; (b) find the asymptotes; (c) find the x -intercepts; (d) sketch the graph.

Q4. For $y = -\tan x$, $x \in [0, 2\pi]$: (a) state how the graph differs from $y = \tan x$; (b) write the asymptotes and x -intercepts; (c) sketch the graph.

Q5. For $y = \tan \frac{x}{2}$, $x \in [0, 2\pi]$: (a) state the period; (b) find the asymptote and x -intercepts; (c) sketch the graph.

Q6. For $y = 3 \tan 2x$, $x \in [0, \pi]$: (a) state the period; (b) find the asymptotes and x -intercepts; (c) sketch the graph.

Un scaffolded

For **Q7–Q9, Q13, Q14, Q17**, sketch the graph over the stated domain, showing the asymptotes and x -intercepts. For **Q10–Q12, Q15, Q16, Q18–Q21**, solve the equation over the stated domain, giving exact values.

Q7. $y = \tan\left(x - \frac{\pi}{6}\right)$, $x \in [0, 2\pi]$

Q8. $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$, $x \in [0, \pi]$

Q9. $y = 2 \tan x$, $x \in [0, 2\pi]$

Q10. $\tan x = 1$, $x \in [0, 2\pi]$

Q11. $\tan x = \sqrt{3}$, $x \in [0, 2\pi]$

Q12. $\tan x = -1, x \in [0, 2\pi]$

Q13. $y = \tan\left(x + \frac{\pi}{6}\right), x \in [0, 2\pi]$

Q14. $y = -2 \tan\left(x - \frac{\pi}{3}\right), x \in [0, 2\pi]$

Q15. $\tan x = -\sqrt{3}, x \in [0, 2\pi]$

Q16. $\tan x = \frac{1}{\sqrt{3}}, x \in [0, 2\pi]$

Q17. $y = \tan\left(3\left(x - \frac{\pi}{3}\right)\right), x \in [0, \pi]$

Q18. $\tan 2x = 1, x \in [0, \pi]$

Q19. $\tan\left(x + \frac{\pi}{4}\right) = 1, x \in [0, 2\pi]$

Q20. $\tan x = 0, x \in [0, 2\pi]$

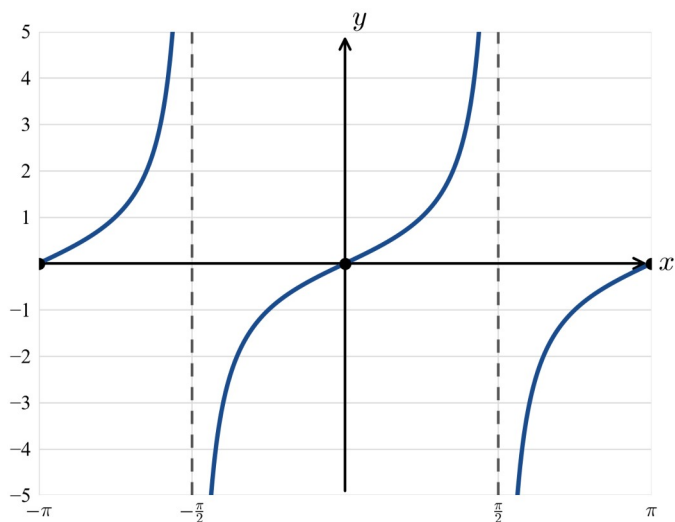
Q21. $\tan x = -\frac{1}{\sqrt{3}}, x \in [0, 2\pi]$

Answers

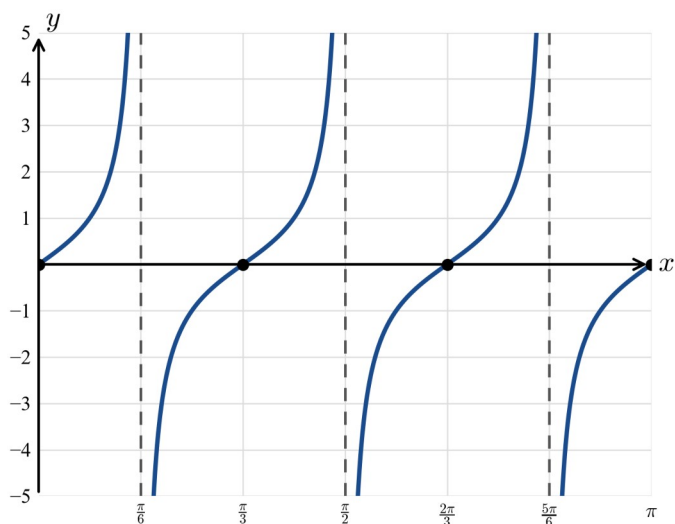
Q1. Period π ; asymptotes $x = -\frac{\pi}{2}, \frac{\pi}{2}$; x -intercepts $-\pi, 0, \pi$. **Q2.** Period $\frac{\pi}{3}$; asymptotes $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$; x -intercepts $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$. **Q3.** Translation $\frac{\pi}{3}$ left; asymptotes $x = \frac{\pi}{6}, \frac{7\pi}{6}$; x -intercepts $\frac{2\pi}{3}, \frac{5\pi}{3}$. **Q4.** Reflection in the x -axis; asymptotes $x = \frac{\pi}{2}, \frac{3\pi}{2}$; x -intercepts $0, \pi, 2\pi$. **Q5.** Period 2π ; asymptote $x = \pi$; x -intercepts $0, 2\pi$. **Q6.** Period $\frac{\pi}{2}$; asymptotes $x = \frac{\pi}{4}, \frac{3\pi}{4}$; x -intercepts $0, \frac{\pi}{2}, \pi$. **Q7.** Asymptotes $x = \frac{2\pi}{3}, \frac{5\pi}{3}$; x -intercepts $\frac{\pi}{6}, \frac{7\pi}{6}$. **Q8.** Period $\frac{\pi}{2}$; asymptotes $x = 0, \frac{\pi}{2}, \pi$; x -intercepts $\frac{\pi}{4}, \frac{3\pi}{4}$. **Q9.** Asymptotes $x = \frac{\pi}{2}, \frac{3\pi}{2}$; x -intercepts $0, \pi, 2\pi$ (steeper than $y = \tan x$). **Q10.** $x = \frac{\pi}{4}, \frac{5\pi}{4}$. **Q11.** $x = \frac{\pi}{3}, \frac{4\pi}{3}$. **Q12.** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. **Q13.** Asymptotes $x = \frac{\pi}{3}, \frac{4\pi}{3}$; x -intercepts $\frac{5\pi}{6}, \frac{11\pi}{6}$. **Q14.** Reflected & steeper; asymptotes $x = \frac{5\pi}{6}, \frac{11\pi}{6}$; x -intercepts $\frac{\pi}{3}, \frac{4\pi}{3}$. **Q15.** $x = \frac{2\pi}{3}, \frac{5\pi}{3}$. **Q16.** $x = \frac{\pi}{6}, \frac{7\pi}{6}$. **Q17.** Period $\frac{\pi}{3}$; asymptotes $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$; x -intercepts $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$. **Q18.** $x = \frac{\pi}{8}, \frac{5\pi}{8}$. **Q19.** $x = 0, \pi, 2\pi$. **Q20.** $x = 0, \pi, 2\pi$. **Q21.** $x = \frac{5\pi}{6}, \frac{11\pi}{6}$.

Fully-worked solutions

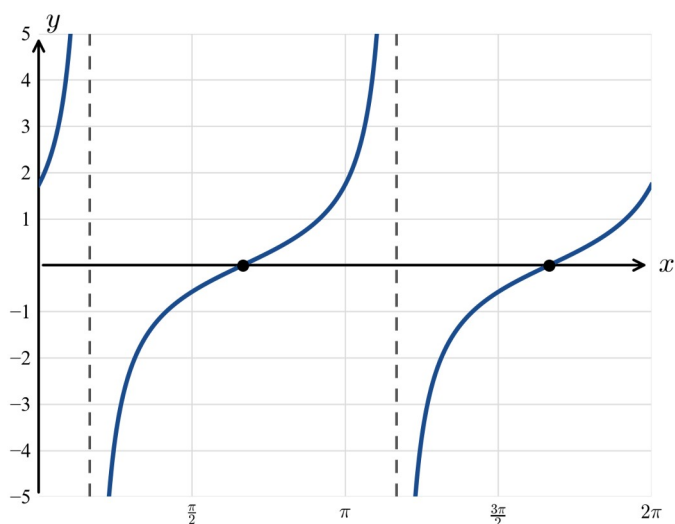
Q1. $y = \tan x$, period π . Asymptotes $x = \frac{\pi}{2} + k\pi \Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}$ in $[-\pi, \pi]$. x -intercepts $x = k\pi \Rightarrow -\pi, 0, \pi$.



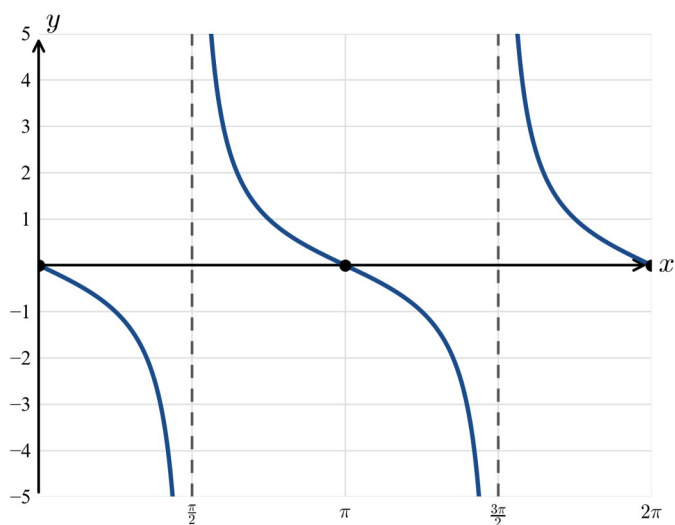
Q2. $n = 3$, period $\frac{\pi}{3}$. Asymptotes $3x = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$. x -intercepts $3x = k\pi \Rightarrow x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$.



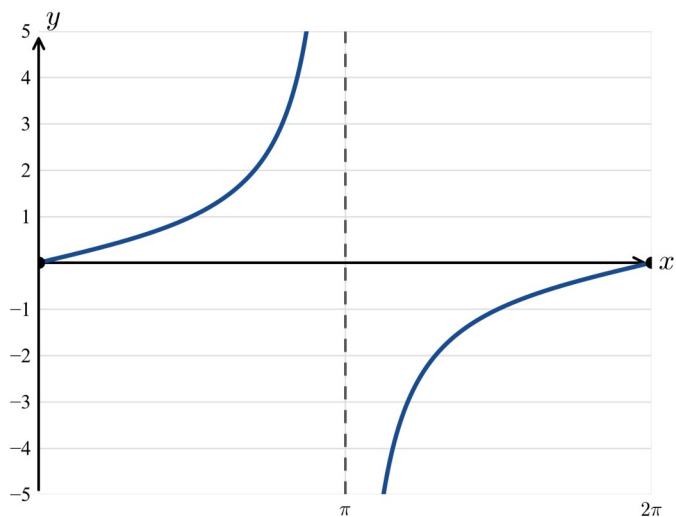
Q3. $y = \tan x$ translated $\frac{\pi}{3}$ left. Asymptotes $x + \frac{\pi}{3} = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{\pi}{6}, \frac{7\pi}{6}$. x -intercepts $x + \frac{\pi}{3} = k\pi \Rightarrow x = \frac{2\pi}{3}, \frac{5\pi}{3}$.



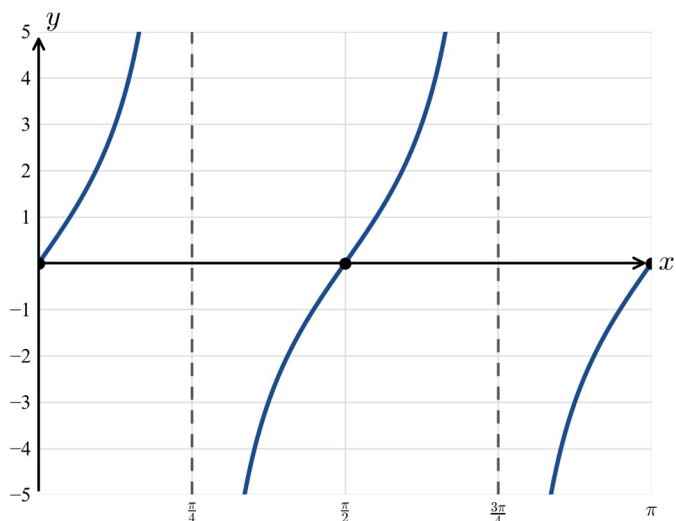
Q4. $y = -\tan x$ is $y = \tan x$ **reflected in the x -axis**, so each branch is decreasing. Asymptotes $x = \frac{\pi}{2}, \frac{3\pi}{2}$; x -intercepts $0, \pi, 2\pi$.



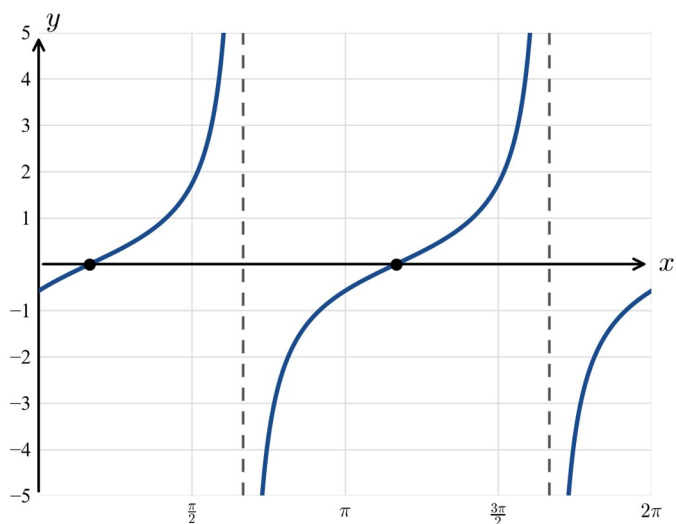
Q5. $n = \frac{1}{2}$, period 2π . Asymptote $\frac{x}{2} = \frac{\pi}{2} + k\pi \Rightarrow x = \pi$. x -intercepts $\frac{x}{2} = k\pi \Rightarrow x = 0, 2\pi$.



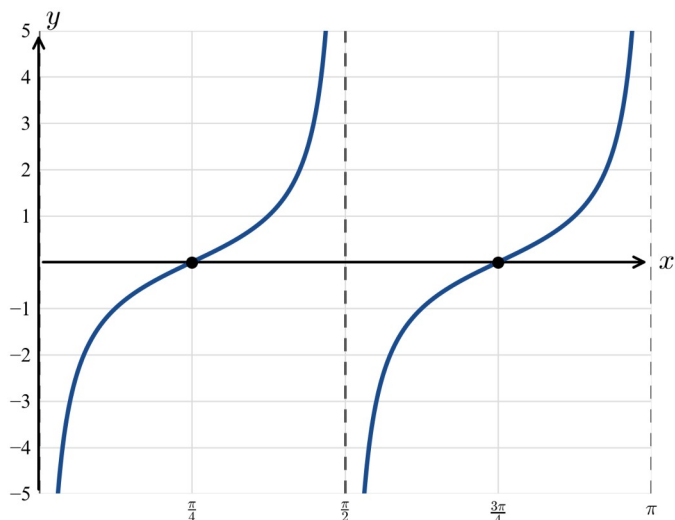
Q6. $n=2$, period $\frac{\pi}{2}$; the factor 3 makes the graph steeper. Asymptotes $2x = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$. x -intercepts $x = 0, \frac{\pi}{2}, \pi$.



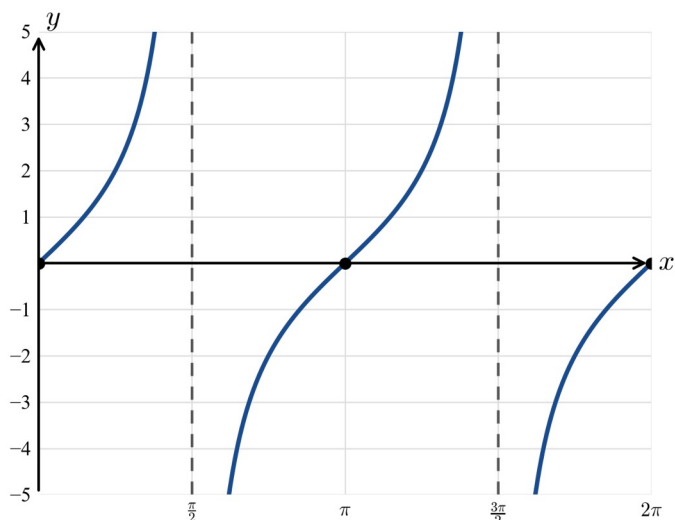
Q7. Translation $\frac{\pi}{6}$ right, period π . Asymptotes $x - \frac{\pi}{6} = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{2\pi}{3}, \frac{5\pi}{3}$. x -intercepts $x = \frac{\pi}{6}, \frac{7\pi}{6}$.



Q8. $y = \tan\left(2\left(x - \pi/4\right)\right)$: period $\frac{\pi}{2}$, shift $\frac{\pi}{4}$ right. Asymptotes $2\left(x - \pi/4\right) = \frac{\pi}{2} + k\pi \Rightarrow x = 0, \frac{\pi}{2}, \pi$. x -intercepts $2\left(x - \pi/4\right) = k\pi \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$.



Q9. $y = 2 \tan x$: same asymptotes ($x = \frac{\pi}{2}, \frac{3\pi}{2}$) and x -intercepts ($0, \pi, 2\pi$) as $y = \tan x$, but twice as steep.

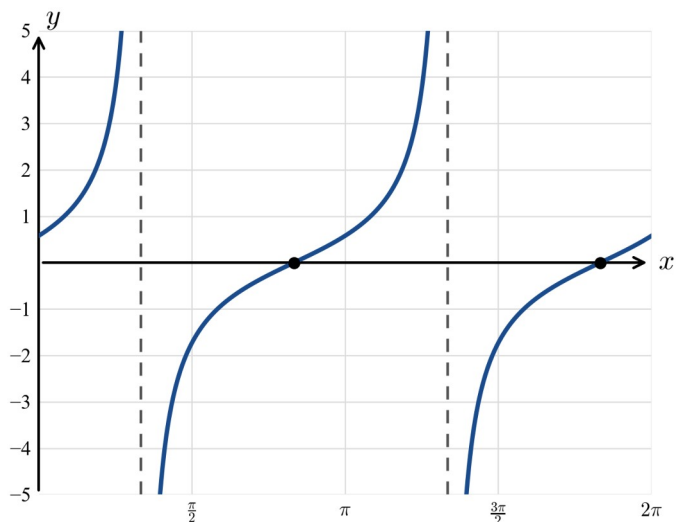


Q10. $\tan x = 1$. Base angle $\frac{\pi}{4}$; solutions $x = \frac{\pi}{4} + k\pi$, so in $[0, 2\pi]$, $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

Q11. $\tan x = \sqrt{3}$. Base angle $\frac{\pi}{3}$; $x = \frac{\pi}{3} + k\pi$, so $x = \frac{\pi}{3}, \frac{4\pi}{3}$.

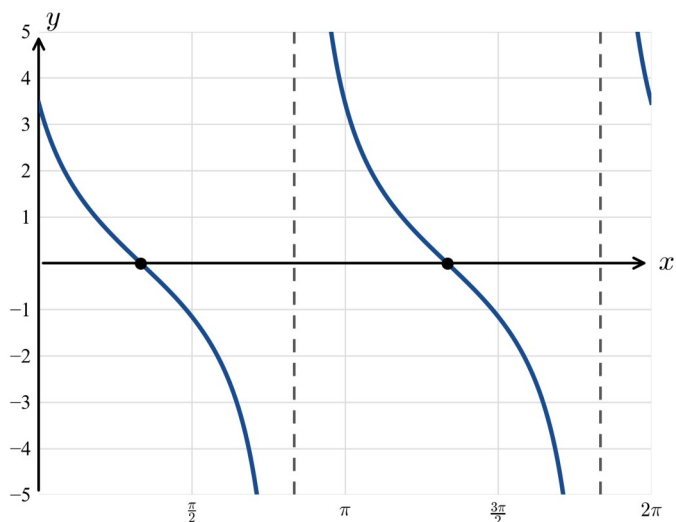
Q12. $\tan x = -1$. $\tan$ is negative in the 2nd and 4th quadrants; base angle $\frac{\pi}{4}$, so $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ and $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Q13. Translation $\frac{\pi}{6}$ left, period π . Asymptotes $x + \frac{\pi}{6} = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{\pi}{3}, \frac{4\pi}{3}$. x -intercepts $x = \frac{5\pi}{6}, \frac{11\pi}{6}$.



Q14. $y = -2 \tan(x - \pi/3)$: reflected in the x -axis (decreasing branches) and steeper; shift $\frac{\pi}{3}$ right. Asymptotes

$$x - \frac{\pi}{3} = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{5\pi}{6}, \frac{11\pi}{6}. \text{ } x\text{-intercepts } x = \frac{\pi}{3}, \frac{4\pi}{3}.$$

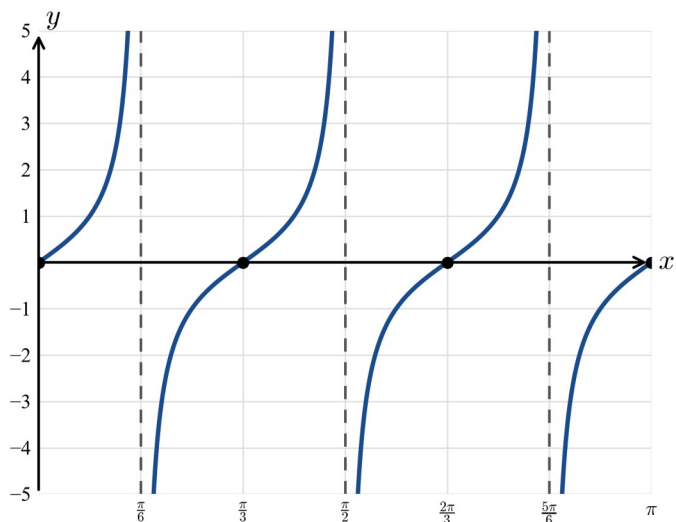


Q15. $\tan x = -\sqrt{3}$. Base angle $\frac{\pi}{3}$; negative in the 2nd and 4th quadrants: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Q16. $\tan x = \frac{1}{\sqrt{3}}$. Base angle $\frac{\pi}{6}$; $x = \frac{\pi}{6} + k\pi$, so $x = \frac{\pi}{6}, \frac{7\pi}{6}$.

Q17. $y = \tan(3(x - \pi/3))$: period $\frac{\pi}{3}$, shift $\frac{\pi}{3}$ right. Asymptotes $3(x - \pi/3) = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$. x -intercepts

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi.$$



Q18. $\tan 2x = 1$. Let $u = 2x$, $u \in [0, 2\pi]$: $u = \frac{\pi}{4} + k\pi \Rightarrow u = \frac{\pi}{4}, \frac{5\pi}{4}$. So $x = \frac{u}{2} = \frac{\pi}{8}, \frac{5\pi}{8}$.

Q19. $\tan(x + \pi/4) = 1$. Then $x + \frac{\pi}{4} = \frac{\pi}{4} + k\pi \Rightarrow x = k\pi$, so in $[0, 2\pi]$, $x = 0, \pi, 2\pi$.

Q20. $\tan x = 0 \Rightarrow x = k\pi$, so $x = 0, \pi, 2\pi$.

Q21. $\tan x = -\frac{1}{\sqrt{3}}$. Base angle $\frac{\pi}{6}$; negative in the 2nd and 4th quadrants: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ and $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Chapter 6 Circular functions — 6K General solution of trigonometric equations

Theory

In Section 6E we solved trigonometric equations over a **restricted domain**. Because the circular functions are **periodic**, an equation such as $\cos x = 1/2$ has **infinitely many** solutions over $\mathbb{R}$. The **general solution** is a formula, in terms of an integer $k \in \mathbb{Z}$, that lists every one of them.

Let α be a known (usually exact) solution — the **reference solution**. Then:

$$\cos x = a \ (|a| \leq 1): x = 2k\pi \pm \alpha, k \in \mathbb{Z},$$

where α is the solution with $0 \leq \alpha \leq \pi$ (i.e. $\alpha = \cos^{-1} a$).

$$\sin x = a \ (|a| \leq 1): x = 2k\pi + \alpha \text{ or } x = 2k\pi + (\pi - \alpha), k \in \mathbb{Z},$$

where $\alpha = \sin^{-1} a$ (with $-\pi/2 \leq \alpha \leq \pi/2$).

$$\tan x = a: x = k\pi + \alpha, k \in \mathbb{Z},$$

where $\alpha = \tan^{-1} a$. (Tangent has period π , so a single family covers all solutions.)

Method.

1. Find the exact **reference solution** α from the exact-value table.
2. Write the **general solution** using the correct form above (mind the sign of a and the quadrants).
3. If particular solutions in a domain are required, substitute successive integers k and keep only those values that lie in the domain.

Functions of nx . For $\cos(nx) = a$, $\sin(nx) = a$ or $\tan(nx) = a$, first write the general solution for the argument nx , **then divide by n** . Dividing the period by n produces more solutions in a given domain.

Worked examples

Example 1 — scaffolded

Find the general solution of $\cos x = \frac{1}{2}$.

Working

Comment

$\cos x = \frac{1}{2}$ is a cosine equation, so use $x = 2k\pi \pm \alpha$.

Identify the form.

Reference solution: $\cos \alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$.

The first-quadrant exact value.

$\therefore x = 2k\pi \pm \frac{\pi}{3}, k \in \mathbb{Z}$.

Both + and - are needed for cosine.

Example 2 — scaffolded

Find the general solution of $\sin x = \frac{\sqrt{3}}{2}$, and hence find all solutions in $[0, 2\pi]$.

Working

Comment

$\sin x = \frac{\sqrt{3}}{2}$ is a sine equation: $x = 2k\pi + \alpha$ or

Sine needs two families (2nd-quadrant partner).

$$x = 2k\pi + (\pi - \alpha).$$

$$\sin \alpha = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{3}; \text{ the partner is } \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Reference angle and its supplement.

$$\text{General solution: } x = 2k\pi + \frac{\pi}{3} \text{ or}$$

$$x = 2k\pi + \frac{2\pi}{3}, k \in \mathbb{Z}.$$

$$k=0: x = \frac{\pi}{3}, \frac{2\pi}{3}; \text{ other } k \text{ leave } [0, 2\pi].$$

Substitute k and keep those in the domain.

$$\therefore \text{ solutions in } [0, 2\pi] \text{ are } x = \frac{\pi}{3}, \frac{2\pi}{3}.$$

Example 3 — unscaffolded

Find the general solution of $\tan x = -1$.

$$\text{Tangent: } x = k\pi + \alpha \text{ with } \alpha = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

$$\therefore x = k\pi - \frac{\pi}{4}, k \in \mathbb{Z}.$$

(Equivalently $x = k\pi + \frac{3\pi}{4}$, since the two differ by π .)

Example 4 — unscaffolded

Find the general solution of $\cos 2x = -\frac{1}{2}$, and hence all solutions in $[0, 2\pi]$.

Let the argument be $2x$. Since $\cos 2x = -\frac{1}{2}$, the reference solution is $\alpha = \frac{2\pi}{3}$ (as $\cos 2\pi/3 = -1/2$, $0 \leq \alpha \leq \pi$), so

$$2x = 2k\pi \pm \frac{2\pi}{3} \Rightarrow x = k\pi \pm \frac{\pi}{3}, k \in \mathbb{Z}.$$

Substituting $k=0, 1$ and keeping values in $[0, 2\pi]$:

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

$\therefore$ dividing the argument by 2 has doubled the number of solutions in $[0, 2\pi]$.

Practice questions

Scaffolded

For each, (a) find the general solution, and (b) hence find all solutions in $[0, 2\pi]$.

Q1. $\sin x = \frac{1}{2}$

Q2. $\cos x = \frac{\sqrt{3}}{2}$

Q3. $\tan x = \sqrt{3}$

Q4. $\cos x = -\frac{1}{2}$

Q5. $\sin x = -\frac{1}{2}$

Q6. $\tan x = -\sqrt{3}$

Unscaffolded

Find the general solution, and hence all solutions in $[0, 2\pi]$ (unless another domain is stated).

Q7. $\cos x = \frac{\sqrt{2}}{2}$

Q8. $\sin x = 1$

Q9. $\tan x = \frac{1}{\sqrt{3}}$

Q10. $\cos x = -1$

Q11. $\sin x = -\frac{\sqrt{3}}{2}$

Q12. $\tan x = 0$

Q13. $\cos 2x = \frac{1}{2}$

Q14. $\sin 2x = \frac{\sqrt{2}}{2}$

Q15. $\tan 2x = 1$

Q16. $\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$

Q17. $2 \sin x = \sqrt{2}$

Q18. $\cos x = -\frac{\sqrt{3}}{2}$

Q19. $\tan x = -\frac{1}{\sqrt{3}}$

Q20. Find the general solution of $\sqrt{2} \cos x + 1 = 0$, and hence find all solutions in $[-2\pi, 2\pi]$.

Q21. Find the general solution of $\sin 2x = -\frac{1}{2}$, and hence find all solutions in $[0, 2\pi]$.

Answers

Q1. $x = 2k\pi + \frac{\pi}{6}$ or $2k\pi + \frac{5\pi}{6}$; in $[0, 2\pi]$: $\frac{\pi}{6}, \frac{5\pi}{6}$. **Q2.** $x = 2k\pi \pm \frac{\pi}{6}$; in $[0, 2\pi]$: $\frac{\pi}{6}, \frac{11\pi}{6}$. **Q3.** $x = k\pi + \frac{\pi}{3}$; in $[0, 2\pi]$: $\frac{\pi}{3}, \frac{4\pi}{3}$. **Q4.** $x = 2k\pi \pm \frac{2\pi}{3}$; in $[0, 2\pi]$: $\frac{2\pi}{3}, \frac{4\pi}{3}$. **Q5.** $x = 2k\pi + \frac{7\pi}{6}$ or $2k\pi + \frac{11\pi}{6}$; in $[0, 2\pi]$: $\frac{7\pi}{6}, \frac{11\pi}{6}$. **Q6.** $x = k\pi - \frac{\pi}{3}$; in $[0, 2\pi]$: $\frac{2\pi}{3}, \frac{5\pi}{3}$. **Q7.** $x = 2k\pi \pm \frac{\pi}{4}$; in $[0, 2\pi]$: $\frac{\pi}{4}, \frac{7\pi}{4}$. **Q8.** $x = 2k\pi + \frac{\pi}{2}$; in $[0, 2\pi]$: $\frac{\pi}{2}$. **Q9.** $x = k\pi + \frac{\pi}{6}$; in $[0, 2\pi]$: $\frac{\pi}{6}, \frac{7\pi}{6}$. **Q10.** $x = (2k+1)\pi$; in $[0, 2\pi]$: π . **Q11.** $x = 2k\pi + \frac{4\pi}{3}$ or $2k\pi + \frac{5\pi}{3}$; in $[0, 2\pi]$: $\frac{4\pi}{3}, \frac{5\pi}{3}$. **Q12.** $x = k\pi$; in $[0, 2\pi]$: $0, \pi, 2\pi$. **Q13.** $x = k\pi \pm \frac{\pi}{6}$; in $[0, 2\pi]$: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. **Q14.** $x = k\pi + \frac{\pi}{8}$ or $k\pi + \frac{3\pi}{8}$; in $[0, 2\pi]$: $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$. **Q15.** $x = \frac{k\pi}{2} + \frac{\pi}{8}$; in $[0, 2\pi]$: $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$. **Q16.** $x = 4k\pi \pm \frac{\pi}{3}$; in $[0, 2\pi]$: $\frac{\pi}{3}$. **Q17.** $x = 2k\pi + \frac{\pi}{4}$ or $2k\pi + \frac{3\pi}{4}$; in $[0, 2\pi]$: $\frac{\pi}{4}, \frac{3\pi}{4}$. **Q18.** $x = 2k\pi \pm \frac{5\pi}{6}$; in $[0, 2\pi]$: $\frac{5\pi}{6}, \frac{7\pi}{6}$. **Q19.** $x = k\pi - \frac{\pi}{6}$; in $[0, 2\pi]$: $\frac{5\pi}{6}, \frac{11\pi}{6}$. **Q20.** $x = 2k\pi \pm \frac{3\pi}{4}$; in $[-2\pi, 2\pi]$: $-\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$. **Q21.** $x = k\pi + \frac{7\pi}{12}$ or $k\pi + \frac{11\pi}{12}$; in $[0, 2\pi]$: $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

Fully-worked solutions

Q1. $\sin x = \frac{1}{2}$: $\alpha = \frac{\pi}{6}$, partner $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. General: $x = 2k\pi + \frac{\pi}{6}$ or $x = 2k\pi + \frac{5\pi}{6}$. $k=0$ gives $\frac{\pi}{6}, \frac{5\pi}{6}$ in $[0, 2\pi]$.

Q2. $\cos x = \frac{\sqrt{3}}{2}$: $\alpha = \frac{\pi}{6}$. General: $x = 2k\pi \pm \frac{\pi}{6}$. $k=0$: $\frac{\pi}{6}$; $k=1$ with the $-$: $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$. So $\frac{\pi}{6}, \frac{11\pi}{6}$.

Q3. $\tan x = \sqrt{3}$: $\alpha = \frac{\pi}{3}$. General: $x = k\pi + \frac{\pi}{3}$. $k=0, 1$: $\frac{\pi}{3}, \frac{4\pi}{3}$.

Q4. $\cos x = -\frac{1}{2}$: $\alpha = \frac{2\pi}{3}$. General: $x = 2k\pi \pm \frac{2\pi}{3}$. $k=0$: $\frac{2\pi}{3}$; $k=1$ with $-$: $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$. So $\frac{2\pi}{3}, \frac{4\pi}{3}$.

Q5. $\sin x = -\frac{1}{2}$: solutions are in the 3rd and 4th quadrants, $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. General: $x = 2k\pi + \frac{7\pi}{6}$ or $x = 2k\pi + \frac{11\pi}{6}$. $k=0$: $\frac{7\pi}{6}, \frac{11\pi}{6}$.

Q6. $\tan x = -\sqrt{3}$: $\alpha = -\frac{\pi}{3}$. General: $x = k\pi - \frac{\pi}{3}$. $k=1$: $\frac{2\pi}{3}$; $k=2$: $\frac{5\pi}{3}$. So $\frac{2\pi}{3}, \frac{5\pi}{3}$.

Q7. $\cos x = \frac{\sqrt{2}}{2}$: $\alpha = \frac{\pi}{4}$. General: $x = 2k\pi \pm \frac{\pi}{4}$. In $[0, 2\pi]$: $\frac{\pi}{4}$ and $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Q8. $\sin x = 1$: the maximum, reached once per period at $\frac{\pi}{2}$. General: $x = 2k\pi + \frac{\pi}{2}$. In $[0, 2\pi]$: $\frac{\pi}{2}$.

Q9. $\tan x = \frac{1}{\sqrt{3}}$: $\alpha = \frac{\pi}{6}$. General: $x = k\pi + \frac{\pi}{6}$. $k=0, 1$: $\frac{\pi}{6}, \frac{7\pi}{6}$.

Q10. $\cos x = -1$: the minimum, reached at π . General: $x = 2k\pi + \pi = (2k+1)\pi$. In $[0, 2\pi]$: π .

Q11. $\sin x = -\frac{\sqrt{3}}{2}$: 3rd/4th-quadrant solutions $\frac{4\pi}{3}, \frac{5\pi}{3}$. General: $x = 2k\pi + \frac{4\pi}{3}$ or $x = 2k\pi + \frac{5\pi}{3}$. $k=0$: $\frac{4\pi}{3}, \frac{5\pi}{3}$.

Q12. $\tan x = 0$: solutions where $\sin x = 0$, i.e. $x = k\pi$. In $[0, 2\pi]$: $0, \pi, 2\pi$.

Q13. $\cos 2x = \frac{1}{2}$: $2x = 2k\pi \pm \frac{\pi}{3} \Rightarrow x = k\pi \pm \frac{\pi}{6}$. $k=0$: $\pm \frac{\pi}{6} \rightarrow$ keep $\frac{\pi}{6}$ and $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$; $k=1$: $\pi \pm \frac{\pi}{6} = \frac{5\pi}{6}, \frac{7\pi}{6}$.
So $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Q14. $\sin 2x = \frac{\sqrt{2}}{2}$: $2x = 2k\pi + \frac{\pi}{4}$ or $2k\pi + \frac{3\pi}{4} \Rightarrow x = k\pi + \frac{\pi}{8}$ or $k\pi + \frac{3\pi}{8}$. $k=0, 1$: $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$.

Q15. $\tan 2x = 1$: $2x = k\pi + \frac{\pi}{4} \Rightarrow x = \frac{k\pi}{2} + \frac{\pi}{8}$. $k=0, 1, 2, 3$: $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$.

Q16. $\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$: $\frac{x}{2} = 2k\pi \pm \frac{\pi}{6} \Rightarrow x = 4k\pi \pm \frac{\pi}{3}$. In $[0, 2\pi]$ only $k=0$ with $+$ works: $\frac{\pi}{3}$. (Dividing by $1/2$ stretches the period to 4π , so there is just one solution.)

Q17. $2\sin x = \sqrt{2} \Rightarrow \sin x = \frac{\sqrt{2}}{2}$: $\alpha = \frac{\pi}{4}$, partner $\frac{3\pi}{4}$. General: $x = 2k\pi + \frac{\pi}{4}$ or $2k\pi + \frac{3\pi}{4}$. $k=0$: $\frac{\pi}{4}, \frac{3\pi}{4}$.

Q18. $\cos x = -\frac{\sqrt{3}}{2}$: $\alpha = \frac{5\pi}{6}$. General: $x = 2k\pi \pm \frac{5\pi}{6}$. $k=0$: $\frac{5\pi}{6}$; $k=1$ with $-$: $2\pi - \frac{5\pi}{6} = \frac{7\pi}{6}$. So $\frac{5\pi}{6}, \frac{7\pi}{6}$.

Q19. $\tan x = -\frac{1}{\sqrt{3}}$: $\alpha = -\frac{\pi}{6}$. General: $x = k\pi - \frac{\pi}{6}$. $k=1$: $\frac{5\pi}{6}$; $k=2$: $\frac{11\pi}{6}$. So $\frac{5\pi}{6}, \frac{11\pi}{6}$.

Q20. $\sqrt{2}\cos x + 1 = 0 \Rightarrow \cos x = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$: $\alpha = \frac{3\pi}{4}$. General: $x = 2k\pi \pm \frac{3\pi}{4}$. Taking $k = -1, 0, 1$ and keeping values in $[-2\pi, 2\pi]$: $k=0$ gives $\pm \frac{3\pi}{4}$; $k=-1$ with $+$ gives $-2\pi + \frac{3\pi}{4} = -\frac{5\pi}{4}$; $k=1$ with $-$ gives $2\pi - \frac{3\pi}{4} = \frac{5\pi}{4}$.
 $\therefore x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$.

Q21. $\sin 2x = -\frac{1}{2}$: the 3rd/4th-quadrant partners of $\frac{1}{2}$ give $2x = 2k\pi + \frac{7\pi}{6}$ or $2k\pi + \frac{11\pi}{6}$, so $x = k\pi + \frac{7\pi}{12}$ or $k\pi + \frac{11\pi}{12}$. $k=0, 1$: $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

Chapter 6 Circular functions — 6L Modelling with circular functions

Theory

Many quantities in nature and technology **repeat** at regular intervals — the depth of the tide, the temperature over a day, the height of a seat on a Ferris wheel, an alternating voltage. These are modelled by a **sinusoidal function**

$$y = a \sin(n(t - \varepsilon)) + b \text{ or } y = a \cos(n(t - \varepsilon)) + b,$$

where t is usually **time**. Each parameter has a physical meaning:

- $|a|$ is the **amplitude** — half the distance between the maximum and minimum;
- b is the **mean** (equilibrium) value — the centre line $y = b$ about which the quantity oscillates;
- n fixes the **period** $T = \frac{2\pi}{n}$ — the time for one complete cycle;
- ε is the **phase** (horizontal translation), which sets where the cycle starts.

From these, the **maximum** value is $b + |a|$ and the **minimum** value is $b - |a|$, so the **range** is $[b - |a|, b + |a|]$.

To answer applied questions:

1. Read off a , b , n (and ε) to state the amplitude, mean, period and range.
2. The maximum $b + |a|$ occurs when the $\sin/\cos$ part equals $+1$ (for $+a$); the minimum when it equals -1 . Solve for the corresponding time.
3. To find **when** the quantity takes a particular value, substitute and **solve the resulting trigonometric equation over the given time interval** (find all solutions in the domain, §6E).
4. Sketch by marking the mean line $y = b$, the maximum and minimum, and one period.

Worked examples

Example 1 — scaffolded

The depth of water at a jetty, t hours after midnight, is $d(t) = 10 + 4 \sin\left(\frac{\pi t}{6}\right)$ metres, for $0 \leq t \leq 12$.

Working

Comment

$a = 4$, $b = 10$, $n = \frac{\pi}{6}$. Amplitude 4 m, mean 10 m, period

Read the parameters from the rule; $T = 2\pi / n$.

$$T = \frac{2\pi}{\pi/6} = 12 \text{ h.}$$

Maximum $= 10 + 4 = 14$ m; minimum $= 10 - 4 = 6$ m.

Max $= b + |a|$, min $= b - |a|$.

Max when $\sin\left(\frac{\pi t}{6}\right) = 1$: $\frac{\pi t}{6} = \frac{\pi}{2} \Rightarrow t = 3$. So high tide

The sine is $+1$ at $\pi/2$.

14 m at $t = 3$ (3:00 am).

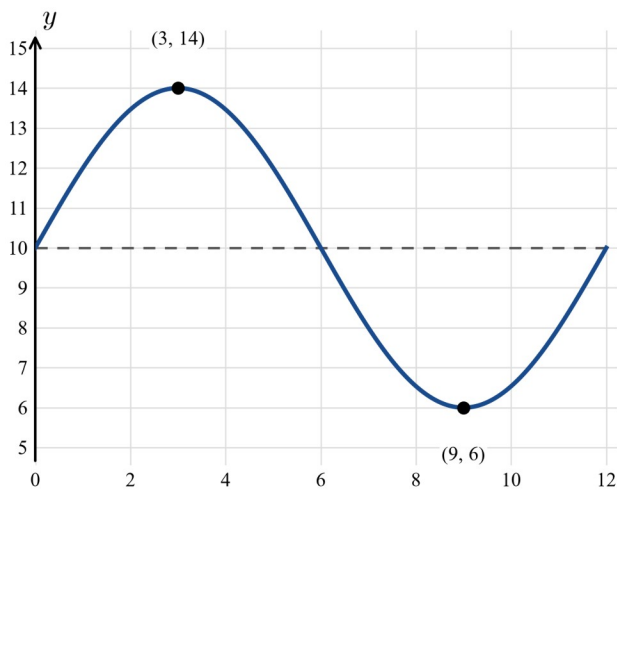
Min when $\sin = -1$: $\frac{\pi t}{6} = \frac{3\pi}{2} \Rightarrow t = 9$. Low tide 6 m at $t = 9$.

The sine is -1 at $3\pi/2$.

Depth $= 12$: $4 \sin\left(\frac{\pi t}{6}\right) = 2 \Rightarrow \sin\left(\frac{\pi t}{6}\right) = \frac{1}{2}$, so $\frac{\pi t}{6} = \frac{\pi}{6}$

Solve over $[0, 12]$ using the two first-cycle solutions.

or $\frac{5\pi}{6}$, giving $t = 1$ and $t = 5$.



Example 2 — scaffolded

The temperature in a greenhouse, t hours after midnight, is $T(t) = 18 - 6 \cos\left(\frac{\pi t}{12}\right)$ °C, for $0 \leq t \leq 24$.

Working

Comment

$a = -6$, $b = 18$, $n = \frac{\pi}{12}$. Amplitude 6°C, mean 18°C,

Amplitude is $|a| = 6$.

period $\frac{2\pi}{\pi/12} = 24$ h.

Maximum $= 18 + 6 = 24$ °C; minimum $= 18 - 6 = 12$ °C.

Max $= b + |a|$, min $= b - |a|$.

Range $[12, 24]$.

At $t = 0$: $T = 18 - 6 \cos 0 = 18 - 6 = 12$ °C (the minimum, at midnight).

$\cos 0 = 1$.

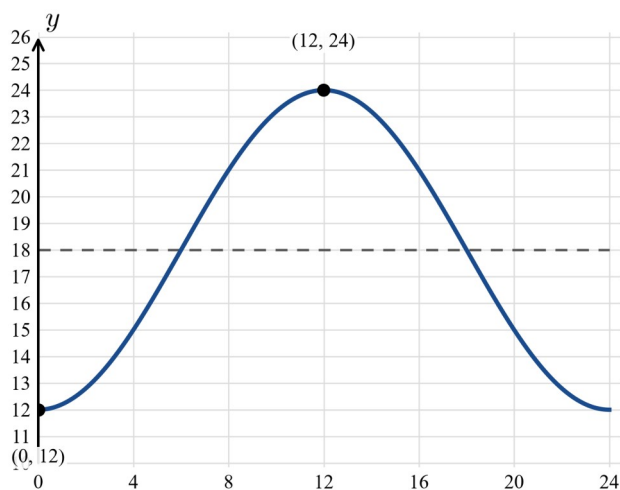
Max when $\cos\left(\frac{\pi t}{12}\right) = -1$: $\frac{\pi t}{12} = \pi \Rightarrow t = 12$, i.e. 24°C at noon.

Because of the -6 , the max is where cosine $= -1$.

$T = 21$: $-6 \cos\left(\frac{\pi t}{12}\right) = 3 \Rightarrow \cos\left(\frac{\pi t}{12}\right) = -\frac{1}{2}$, so

Solve over $[0, 24]$.

$\frac{\pi t}{12} = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$, giving $t = 8$ and $t = 16$.



t

Example 3 — unscaffolded

A seat on a Ferris wheel is at height $h(t) = 65 - 60 \cos\left(\frac{\pi t}{5}\right)$ metres, t minutes after boarding, $0 \leq t \leq 10$.

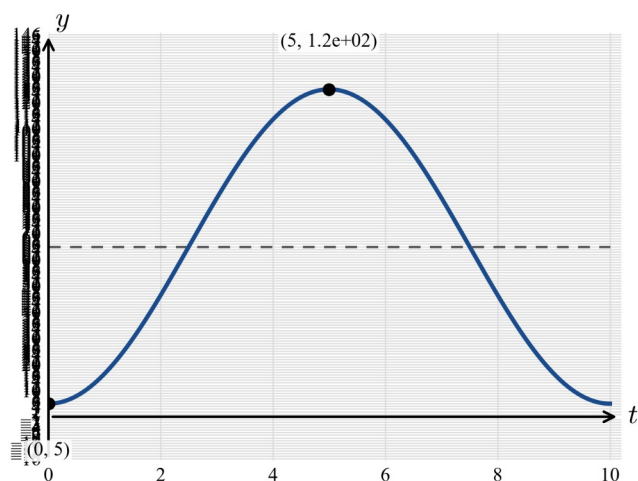
Amplitude 60 m, mean (axle height) 65 m, period $T = \frac{2\pi}{\pi/5} = 10$ min.

At $t=0$: $h = 65 - 60 = 5$ m — the boarding height. Maximum $65 + 60 = 125$ m when $\cos\left(\frac{\pi t}{5}\right) = -1$,

i.e. $\frac{\pi t}{5} = \pi \Rightarrow t = 5$ min (top of the wheel).

Height = 95: $-60 \cos\left(\frac{\pi t}{5}\right) = 30 \Rightarrow \cos\left(\frac{\pi t}{5}\right) = -\frac{1}{2}$, so $\frac{\pi t}{5} = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$, giving $t = \frac{10}{3}$ and $t = \frac{20}{3}$ minutes.

$\therefore$ the seat is 95 m high at $t = \frac{10}{3}$ min (rising) and $t = \frac{20}{3}$ min (falling).



Example 4 — unscaffolded

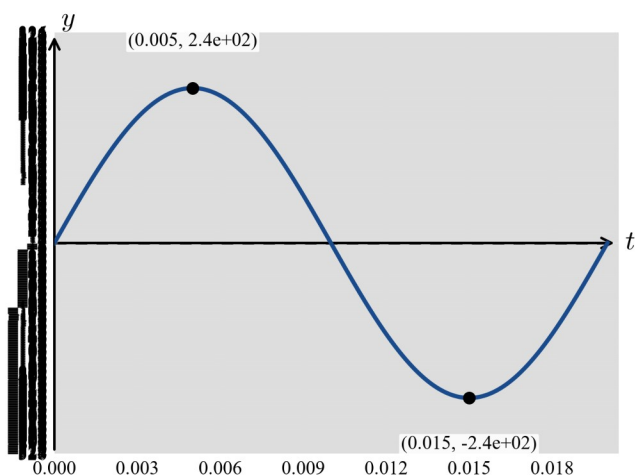
An alternating voltage is $V(t) = 240 \sin(100\pi t)$ volts, t seconds.

Amplitude (peak voltage) 240 V, mean 0 V, period $T = \frac{2\pi}{100\pi} = \frac{1}{50}$ s (frequency 50 Hz).

Maximum 240 V when $100\pi t = \frac{\pi}{2} \Rightarrow t = \frac{1}{200}$ s.

$V = 120$: $\sin(100\pi t) = \frac{1}{2}$, so $100\pi t = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, giving $t = \frac{1}{600}$ and $t = \frac{1}{120}$ s in the first period $\left[0, \frac{1}{100}\right]$.

$\therefore$ within one period the voltage first reaches 120 V at $t = \frac{1}{600}$ s.



Practice questions

Scaffolded

Q1. The height of water is $h(t) = 5 + 3 \sin\left(\frac{\pi t}{6}\right)$ m, t hours after 6 am. (a) State the amplitude, mean and period. (b)

Find the maximum height and when it first occurs. (c) Find the times in $[0, 12]$ when $h = \frac{13}{2}$ m. (d) Sketch the graph.

Q2. Temperature $T(t) = 20 - 8 \cos\left(\frac{\pi t}{12}\right)$ °C, t hours after midnight. (a) State the period and range. (b) Find the minimum temperature and when it occurs. (c) Find when $T = 24$ °C in $[0, 24]$. (d) Sketch the graph.

Q3. A Ferris-wheel seat is at $h(t) = 35 - 30 \cos\left(\frac{\pi t}{4}\right)$ m, t minutes after boarding. (a) State the period and the maximum height. (b) Find the boarding height. (c) Find when $h = 50$ m in $[0, 8]$. (d) Sketch the graph.

Q4. A mass on a spring has displacement $x(t) = 6 \cos(2\pi t)$ cm, t seconds. (a) State the amplitude and period. (b) Find $x(0)$ and $x(1/4)$. (c) Find when $x = 3$ cm in $[0, 1]$. (d) Sketch the graph.

Q5. Tide depth $d(t) = 8 + 2 \sin\left(\frac{\pi t}{6}\right)$ m, t hours after midnight. (a) State the amplitude and mean. (b) Find $d(0)$. (c) Find when $d = 9$ m in $[0, 12]$. (d) Sketch.

Q6. Daylight hours over a period are modelled by $H(t) = 12 + 2 \sin\left(\frac{\pi t}{6}\right)$, t months. (a) State the maximum daylight and when it occurs. (b) Find when $H = 13$ in $[0, 12]$. (c) Sketch.

Un scaffolded

For each model, state the amplitude, mean, period and range; find the maximum and minimum and the required time(s); and sketch one period.

Q7. $T(t) = 15 + 5 \sin\left(\frac{\pi t}{12}\right)$ °C, t hours; find when $T = \frac{35}{2}$ °C in $[0, 24]$.

Q8. $h(t) = 40 - 35 \cos\left(\frac{\pi t}{3}\right)$ m, t min; find when $h = \frac{115}{2}$ m in $[0, 6]$.

Q9. $d(t) = 12 + 5 \cos\left(\frac{\pi t}{6}\right)$ m, t h; find when $d = \frac{19}{2}$ m in $[0, 12]$.

Q10. $V(t) = 100 \sin(50\pi t)$ V, t s; find when $V = 50$ V in $\left[0, \frac{1}{25}\right]$.

Q11. $T(t) = 22 - 7 \cos\left(\frac{\pi(t-2)}{12}\right)$ °C, t h; find when $T = 22$ °C in $[0, 24]$.

Q12. $h(t) = 6 + 4 \sin\left(\frac{\pi t}{6}\right)$ m, t h; find when $h = 8$ m in $[0, 12]$.

Q13. $h(t) = 50 + 45 \sin\left(\frac{\pi(t-5/2)}{5}\right)$ m, t min; find when $h = 50$ m in $[0, 10]$.

Q14. $P(t) = 500 + 200 \sin\left(\frac{\pi t}{6}\right)$ animals, t months; find when $P = 600$ in $[0, 12]$.

Q15. $T(t) = 25 - 10 \cos\left(\frac{\pi t}{6}\right)$ °C, t h; find when $T = 30$ °C in $[0, 12]$.

Q16. $d(t) = 9 + 3 \sin\left(\frac{\pi t}{6}\right)$ m, t h; find when $d = \frac{21}{2}$ m in $[0, 12]$.

Q17. $y(t) = 4 \sin(\pi t)$, t s; find when $y = 2$ in $[0, 2]$.

Q18. $h(t) = 32 - 28 \cos\left(\frac{\pi t}{6}\right)$ m, t min; find when $h = 46$ m in $[0, 12]$.

Q19. $T(t) = 16 + 6 \sin\left(\frac{\pi(t-3)}{12}\right)$ °C, t h; find when $T = 19$ °C in $[0, 24]$.

Q20. $h(t) = 7 + 5 \cos\left(\frac{\pi t}{6}\right)$ m, t h; find when $h = \frac{9}{2}$ m in $[0, 12]$.

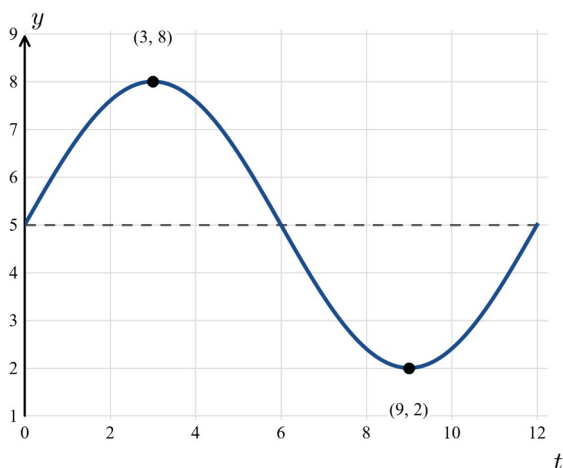
Q21. Blood pressure $p(t) = 100 + 20 \sin(2\pi t)$ mmHg, t s; find when $p = 110$ mmHg in $[0, 1]$.

Answers

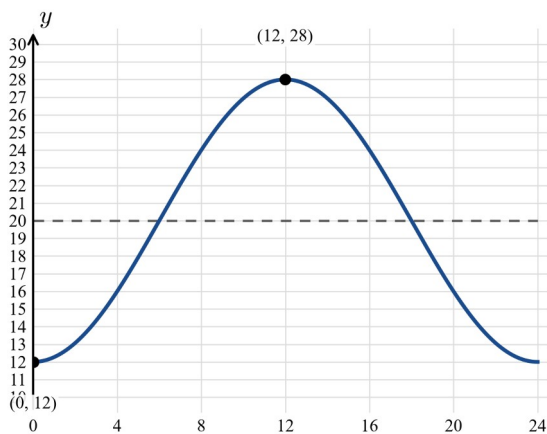
Q1. Amp 3, mean 5, period 12; max 8 m at $t=3$; $t=1, 5$. **Q2.** Period 24, range $[12, 28]$; min 12°C at $t=0$; $t=12$; ($T=24$ at $t=12$). **Q3.** Period 8, max 65 m; boarding height 5 m; $t=\frac{8}{3}, \frac{16}{3}$. **Q4.** Amp 6, period 1; $x(0)=6$, $x(1/4)=0$; $t=\frac{1}{6}, \frac{5}{6}$. **Q5.** Amp 2, mean 8; $d(0)=8$; $t=1, 5$. **Q6.** Max 14 at $t=3$; $t=1, 5$. **Q7.** Amp 5, mean 15, period 24, range $[10, 20]$; $t=2, 10$. **Q8.** Amp 35, mean 40, period 6, range $[5, 75]$; $t=2, 4$. **Q9.** Amp 5, mean 12, period 12, range $[7, 17]$; $t=4, 8$. **Q10.** Amp 100, mean 0, period $\frac{1}{25}$; $t=\frac{1}{300}, \frac{1}{60}$. **Q11.** Amp 7, mean 22, period 24, range $[15, 29]$; $t=8, 20$. **Q12.** Amp 4, mean 6, period 12, range $[2, 10]$; $t=1, 5$. **Q13.** Amp 45, mean 50, period 10, range $[5, 95]$; $t=\frac{5}{2}, \frac{15}{2}$. **Q14.** Amp 200, mean 500, period 12, range $[300, 700]$; $t=1, 5$. **Q15.** Amp 10, mean 25, period 12, range $[15, 35]$; $t=4, 8$. **Q16.** Amp 3, mean 9, period 12, range $[6, 12]$; $t=1, 5$. **Q17.** Amp 4, mean 0, period 2, range $[-4, 4]$; $t=\frac{1}{6}, \frac{5}{6}$. **Q18.** Amp 28, mean 32, period 12, range $[4, 60]$; $t=4, 8$. **Q19.** Amp 6, mean 16, period 24, range $[10, 22]$; $t=5, 13$. **Q20.** Amp 5, mean 7, period 12, range $[2, 12]$; $t=4, 8$. **Q21.** Amp 20, mean 100, period 1, range $[80, 120]$; $t=\frac{1}{12}, \frac{5}{12}$.

Fully-worked solutions

Q1. $h=5+3\sin\left(\frac{\pi t}{6}\right)$: amp 3, mean 5, period $\frac{2\pi}{\pi/6}=12$. Max $5+3=8$ m when $\sin=1$, i.e. $\frac{\pi t}{6}=\frac{\pi}{2} \Rightarrow t=3$. $h=\frac{13}{2}$: $3\sin\left(\frac{\pi t}{6}\right)=\frac{3}{2} \Rightarrow \sin\left(\frac{\pi t}{6}\right)=\frac{1}{2}$, so $\frac{\pi t}{6}=\frac{\pi}{6}, \frac{5\pi}{6}$, giving $t=1, 5$.

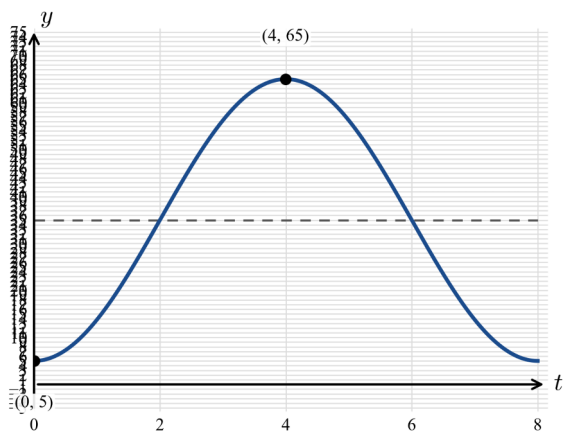


Q2. $T=20-8\cos\left(\frac{\pi t}{12}\right)$: period 24, range $[12, 28]$. Min 12°C when $\cos=1$, i.e. $t=0$. $T=24$: $-8\cos\left(\frac{\pi t}{12}\right)=4 \Rightarrow \cos\left(\frac{\pi t}{12}\right)=-\frac{1}{2}$, so $\frac{\pi t}{12}=\frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t=8, 16$; also the maximum 28°C is at $t=12$ (where $\cos=-1$), so $T=24$ again — over $[0, 24]$ the solutions of $T=24$ are $t=8, 16$.

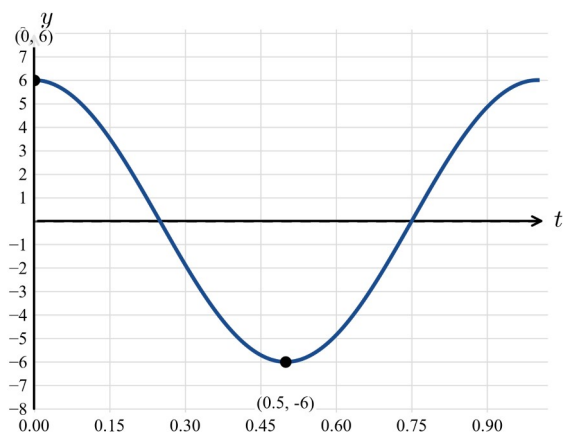


t

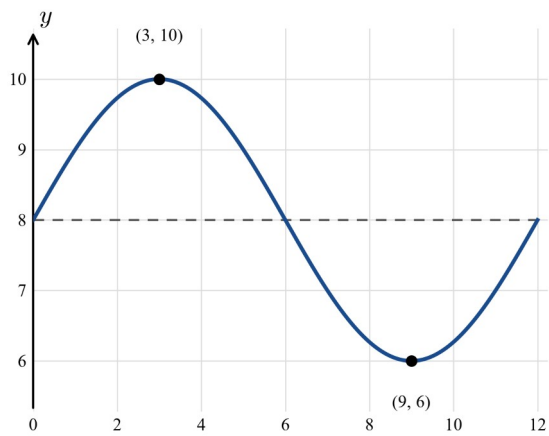
Q3. $h = 35 - 30 \cos\left(\frac{\pi t}{4}\right)$: period 8, max 65 m (when $\cos = -1$, $t = 4$). Boarding height $h(0) = 35 - 30 = 5$ m. $h = 50$
 $\therefore \cos\left(\frac{\pi t}{4}\right) = -\frac{1}{2}$, so $\frac{\pi t}{4} = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = \frac{8}{3}, \frac{16}{3}$.



Q4. $x = 6 \cos(2\pi t)$: amp 6, period $\frac{2\pi}{2\pi} = 1$. $x(0) = 6$, $x(1/4) = 6 \cos\left(\frac{\pi}{2}\right) = 0$. $x = 3$: $\cos(2\pi t) = \frac{1}{2}$, so
 $2\pi t = \frac{\pi}{3}, \frac{5\pi}{3} \Rightarrow t = \frac{1}{6}, \frac{5}{6}$.

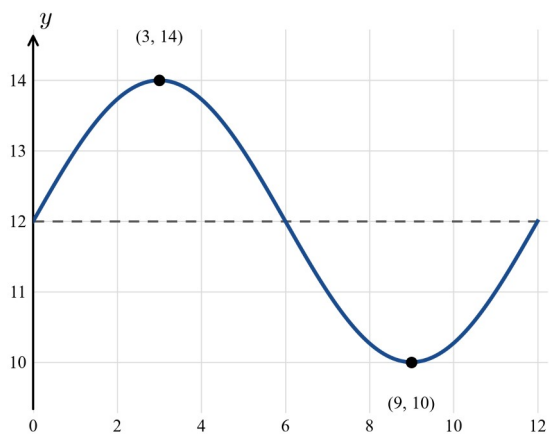


Q5. $d = 8 + 2 \sin\left(\frac{\pi t}{6}\right)$: amp 2, mean 8. $d(0) = 8$. $d = 9$: $\sin\left(\frac{\pi t}{6}\right) = \frac{1}{2} \Rightarrow t = 1, 5$.



t

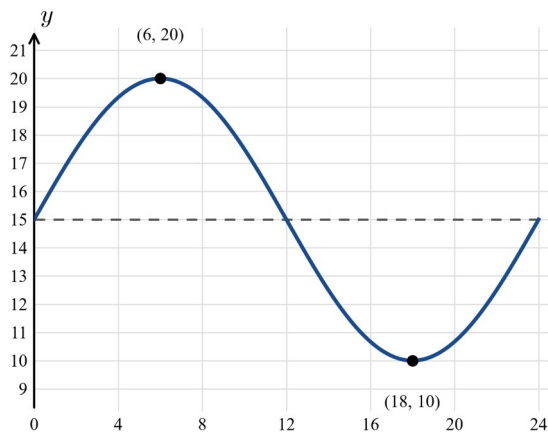
Q6. $H = 12 + 2 \sin\left(\frac{\pi t}{6}\right)$: max 14 when $\sin = 1$, $t = 3$. $H = 10$: $\sin\left(\frac{\pi t}{6}\right) = -1 \Rightarrow t = 9$.



t

Q7. $T = 15 + 5 \sin\left(\frac{\pi t}{12}\right)$: amp 5, mean 15, period 24, range $[10, 20]$. $T = \frac{35}{2}$: $\sin\left(\frac{\pi t}{12}\right) = \frac{1}{2}$, so

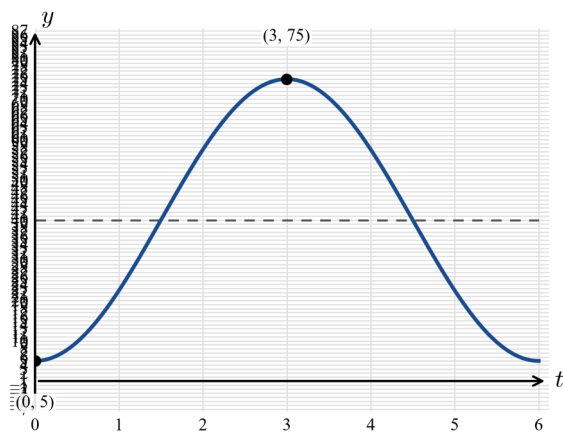
$$\frac{\pi t}{12} = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow t = 2, 10.$$



t

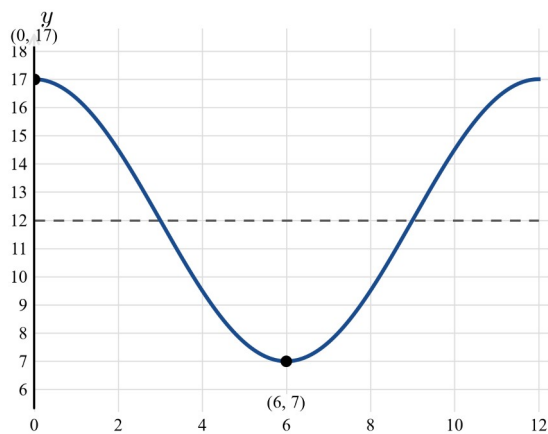
Q8. $h = 40 - 35 \cos\left(\frac{\pi t}{3}\right)$: amp 35, mean 40, period 6, range $[5, 75]$. $h = \frac{115}{2}$:

$$-35 \cos\left(\frac{\pi t}{3}\right) = \frac{35}{2} \Rightarrow \cos\left(\frac{\pi t}{3}\right) = -\frac{1}{2}, \text{ so } \frac{\pi t}{3} = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 2, 4.$$



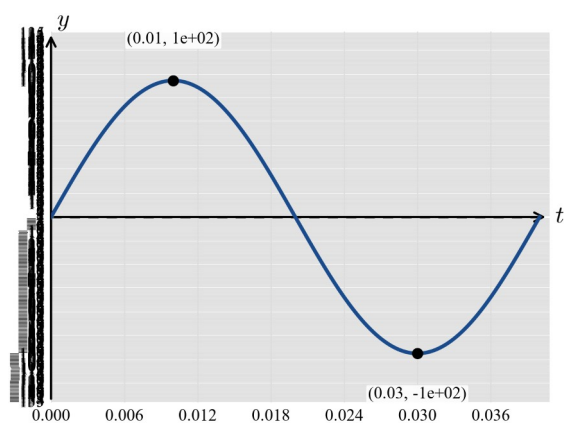
Q9. $d = 12 + 5 \cos\left(\frac{\pi t}{6}\right)$: amp 5, mean 12, period 12, range $[7, 17]$. $d = \frac{19}{2}$: $\cos\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$, so

$$\frac{\pi t}{6} = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 4, 8.$$

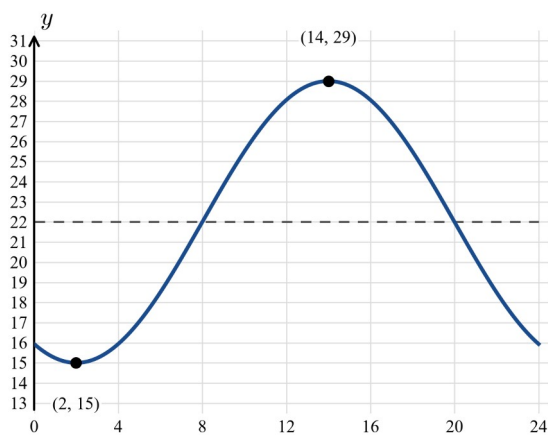


t

Q10. $V = 100 \sin(50\pi t)$: amp 100, period $\frac{2\pi}{50\pi} = \frac{1}{25}$. $V = 50$: $\sin(50\pi t) = \frac{1}{2}$, so $50\pi t = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow t = \frac{1}{300}, \frac{1}{60}$.

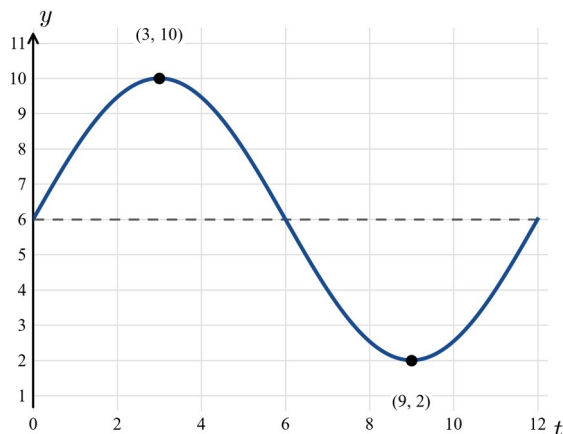


Q11. $T = 22 - 7 \cos\left(\frac{\pi(t-2)}{12}\right)$: amp 7, mean 22, period 24, range $[15, 29]$. $T = 22$: $\cos\left(\frac{\pi(t-2)}{12}\right) = 0$, so $\frac{\pi(t-2)}{12} = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow t - 2 = 6, 18 \Rightarrow t = 8, 20$.

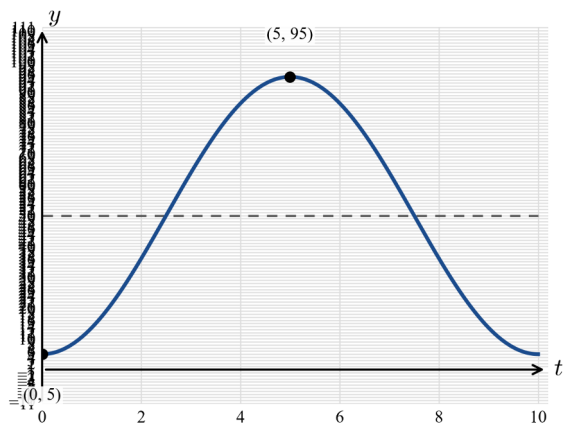


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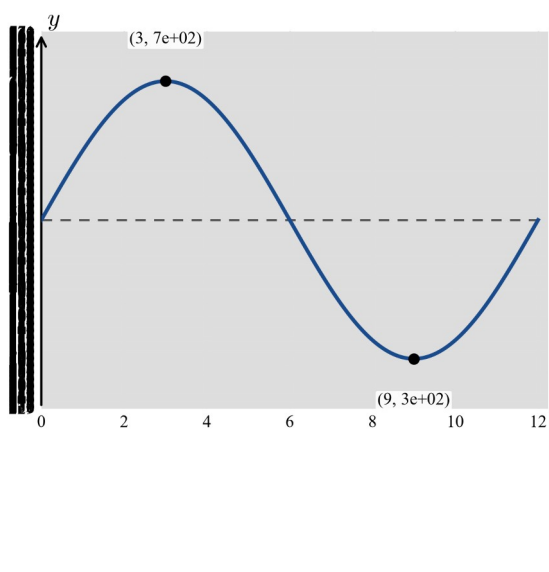
Q12. $h = 6 + 4 \sin\left(\frac{\pi t}{6}\right)$: amp 4, mean 6, period 12, range $[2, 10]$. $h = 8$: $\sin\left(\frac{\pi t}{6}\right) = \frac{1}{2} \Rightarrow t = 1, 5$.



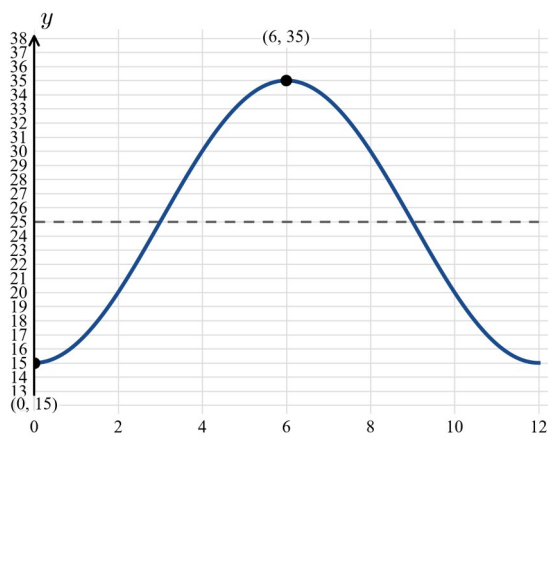
Q13. $h = 50 + 45 \sin\left(\frac{\pi(t-5/2)}{5}\right)$: amp 45, mean 50, period 10, range $[5, 95]$. $h = 50$: $\sin\left(\frac{\pi(t-5/2)}{5}\right) = 0$, so $\frac{\pi(t-5/2)}{5} = 0, \pi \Rightarrow t - \frac{5}{2} = 0, 5 \Rightarrow t = \frac{5}{2}, \frac{15}{2}$.



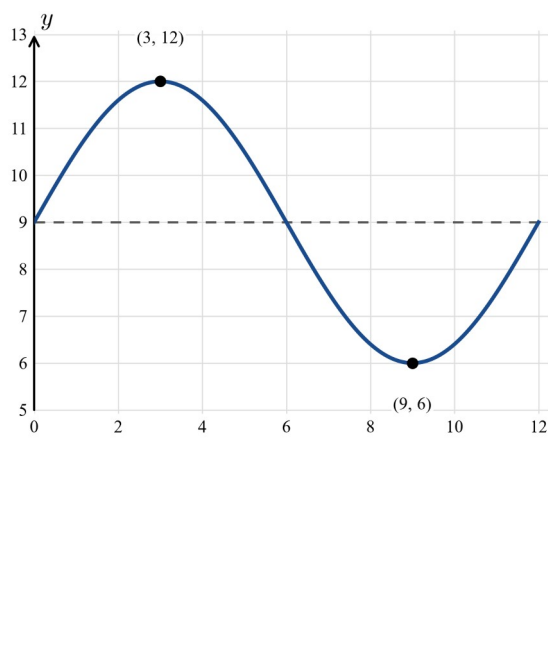
Q14. $P = 500 + 200 \sin\left(\frac{\pi t}{6}\right)$: amp 200, mean 500, period 12, range $[300, 700]$. $P = 600$: $\sin\left(\frac{\pi t}{6}\right) = \frac{1}{2} \Rightarrow t = 1, 5$.



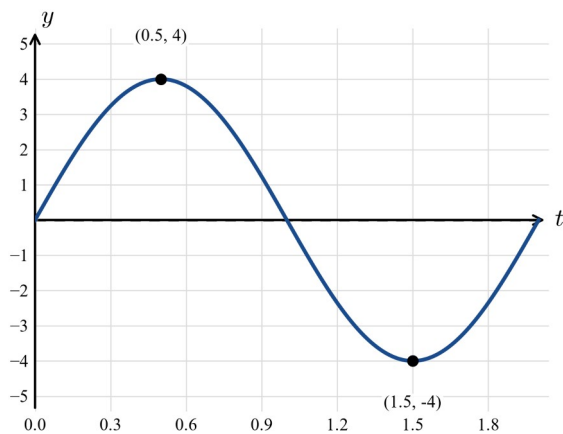
Q15. $T = 25 - 10 \cos\left(\frac{\pi t}{6}\right)$: amp 10, mean 25, period 12, range $[15, 35]$. $T = 30$: $\cos\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$, so $\frac{\pi t}{6} = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 4, 8$.



Q16. $d = 9 + 3 \sin\left(\frac{\pi t}{6}\right)$: amp 3, mean 9, period 12, range $[6, 12]$. $d = \frac{21}{2}$: $\sin\left(\frac{\pi t}{6}\right) = \frac{1}{2} \Rightarrow t = 1, 5$.

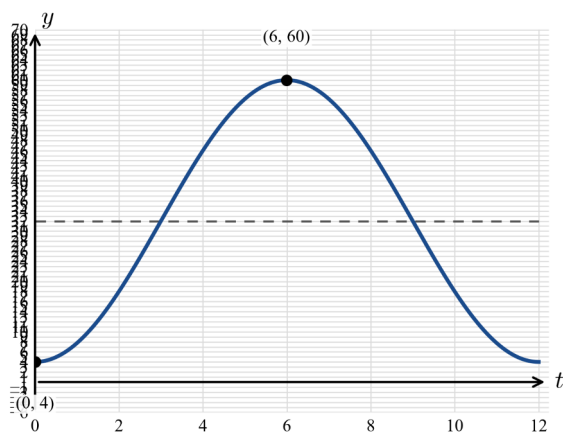


Q17. $y = 4 \sin(\pi t)$: amp 4, period 2, range $[-4, 4]$. $y = 2$: $\sin(\pi t) = \frac{1}{2}$, so $\pi t = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow t = \frac{1}{6}, \frac{5}{6}$.



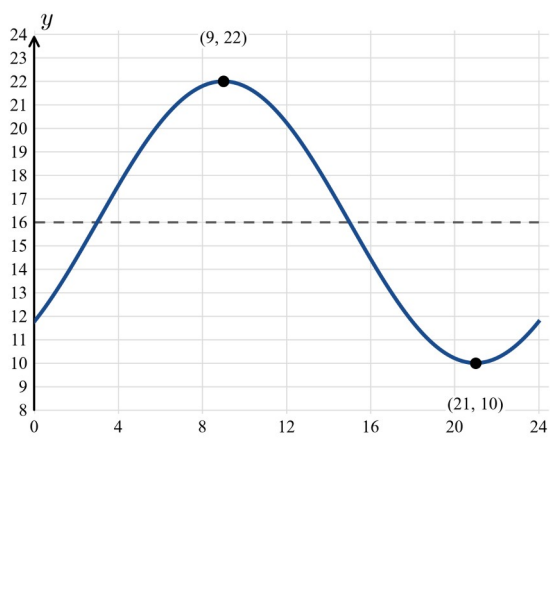
Q18. $h = 32 - 28 \cos\left(\frac{\pi t}{6}\right)$: amp 28, mean 32, period 12, range $[4, 60]$. $h = 46$: $\cos\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$, so

$$\frac{\pi t}{6} = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 4, 8.$$



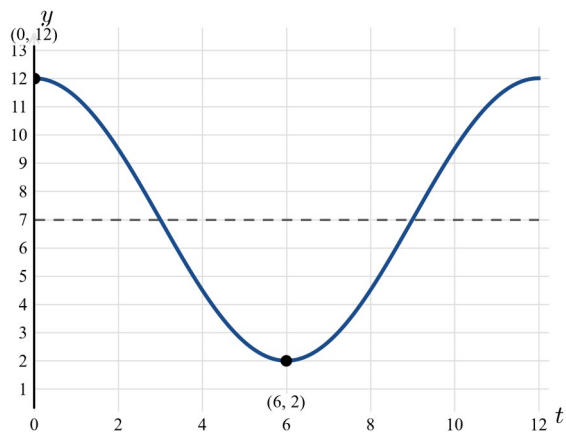
Q19. $T = 16 + 6 \sin\left(\frac{\pi(t-3)}{12}\right)$: amp 6, mean 16, period 24, range $[10, 22]$. $T = 19$: $\sin\left(\frac{\pi(t-3)}{12}\right) = \frac{1}{2}$, so

$$\frac{\pi(t-3)}{12} = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow t-3 = 2, 10 \Rightarrow t = 5, 13.$$



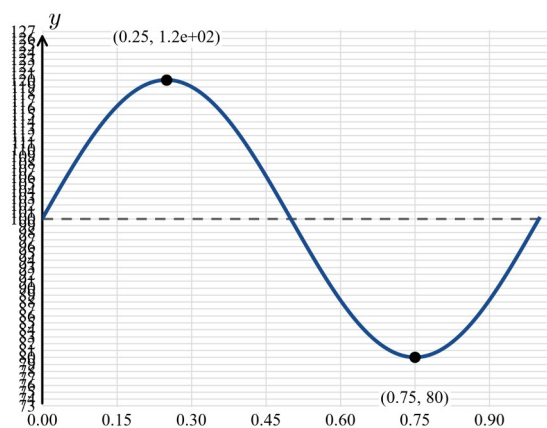
Q20. $h = 7 + 5 \cos\left(\frac{\pi t}{6}\right)$: amp 5, mean 7, period 12, range $[2, 12]$. $h = \frac{9}{2}$: $\cos\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$, so

$$\frac{\pi t}{6} = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 4, 8.$$



Q21. $p = 100 + 20 \sin(2\pi t)$: amp 20, mean 100, period 1, range $[80, 120]$. $p = 110$: $\sin(2\pi t) = \frac{1}{2}$, so

$$2\pi t = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow t = \frac{1}{12}, \frac{5}{12}.$$



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