Mathematical Methods Units 3 & 4

Chapter 5 — Exponential and logarithmic functions

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Contents

Theory

An **exponential function** has the rule $y = a^x$, where the **base** a is a positive constant, $a \neq 1$, and the variable x is in the index (exponent).

Shape. The base decides whether the graph rises or falls:

- if $a > 1$ the function is **increasing** — *exponential growth* (e.g. $y = 2^x$);
- if $0 < a < 1$ the function is **decreasing** — *exponential decay* (e.g. $y = (1/2)^x = 2^{-x}$).

Key features of $y = a^x$.

- **Horizontal asymptote** $y = 0$ (the x-axis): as $x \to -\infty$ the graph approaches 0 for growth (and as $x \to +\infty$ for decay), but never reaches it.
- **y-intercept** $(0, 1)$, since $a^0 = 1$; the graph also passes through $(1, a)$ and $(-1, 1/a)$.
- **No x-intercept**: $a^x > 0$ for all x.
- **Domain** $\mathbb{R}$, **range** $\{y : y > 0\}$.

Transformations. The graph of

$$y = A\,a^{x-h} + k$$

is obtained from $y = a^x$ by a dilation of factor A from the x-axis (with a reflection in the x-axis when $A < 0$) and a translation of h units horizontally and k units vertically.

- The vertical translation moves the **horizontal asymptote to** $y = k$.
- The **range** becomes $\{y : y > k\}$ when $A > 0$, and $\{y : y < k\}$ when $A < 0$.
- A negative index, $y = a^{-x}$, is a **reflection in the y-axis** (and equals $(1/a)^x$).

Finding the features of a transformed exponential.

- **Asymptote:** $y = k$ (the constant added on the end).
- **y-intercept:** substitute $x = 0$.
- **x-intercept:** solve $y = 0$; this has a solution only when the graph crosses its asymptote — i.e. when A and k have opposite signs. Solve by writing both sides as powers of the same base.

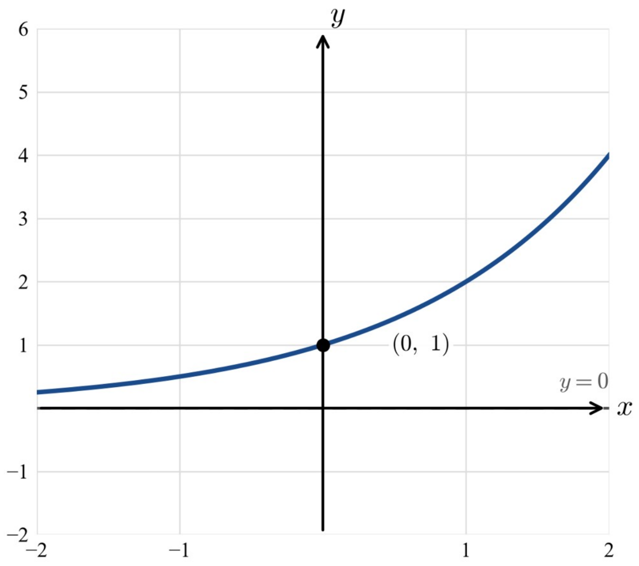
To sketch $y = A\,a^{x-h} + k$: (1) draw the asymptote $y = k$ (dashed); (2) decide growth/decay and whether it is reflected; (3) mark the y-intercept and the x-intercept (if any); (4) draw the curve approaching the asymptote.

Worked examples

Example 1 — scaffolded

Sketch $y = 2^x$, stating the equation of the asymptote, the y-intercept, the domain and the range.

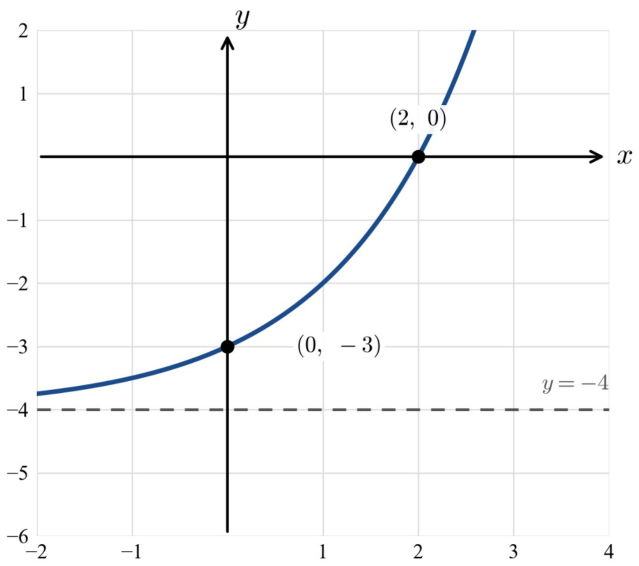
Working	Comment
Base $a = 2 > 1$, so the graph is **increasing** (growth).	The base decides the shape.
Asymptote $y = 0$.	For $y = a^x$ the asymptote is the x-axis.
y-intercept: $x = 0 \Rightarrow y = 2^0 = 1$, giving $(0, 1)$. It also passes through $(1, 2)$ and $(-1, 1/2)$.	Every $y = a^x$ passes through $(0, 1)$ and $(1, a)$.
Domain $\mathbb{R}$, **range** $\{y : y > 0\}$; no x-intercept.	$2^x > 0$ for all x.



Example 2 — scaffolded

Sketch $y=2^x-4$, labelling the asymptote and all axis intercepts, and state the range.

Working	Comment
This is $y=2^x$ translated 4 units **down**, so the **asymptote** is $y=-4$.	The constant -4 shifts the asymptote.
y**-intercept:** $x=0 \Rightarrow y=2^0-4=1-4=-3$, giving $(0,-3)$.	Substitute $x=0$.
x**-intercept:** $2^x-4=0 \Rightarrow 2^x=4=2^2$, so $x=2$, giving $(2,0)$.	Write both sides as powers of 2 and equate indices.
$A=1>0$, so the **range** is $\{y: y>-4\}$.	The graph sits above its asymptote.



Example 3 — unscaffolded

Sketch $y=3^{-x}+1$, stating the asymptote, the y-intercept, the domain and the range.

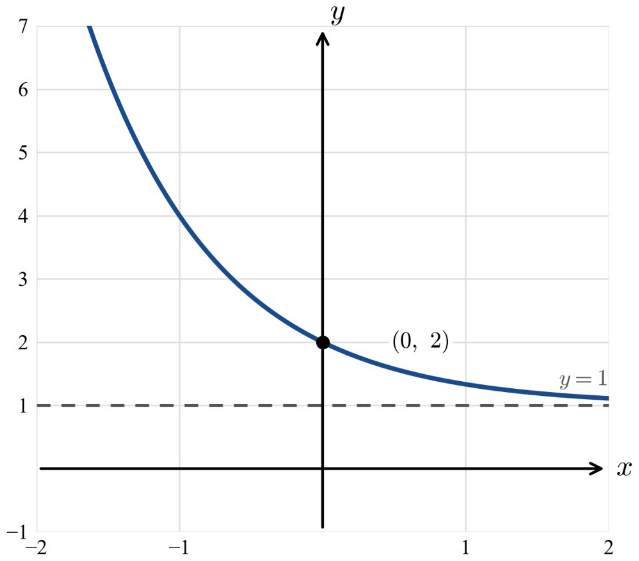
The index is $-x$, so this is $y=3^x$ reflected in the y-axis (a **decay** graph, equal to $(1/3)^x$), then translated 1 unit up.

Asymptote $y=1$; domain $\mathbb{R}$, range $\{y: y>1\}$.

y-intercept: $x=0 \Rightarrow y=3^0+1=2$, giving $(0,2)$.

Since $y > 1$ for all x, there is **no x-intercept**.

$\therefore$ the graph decreases from the upper left towards the asymptote $y = 1$.



Example 4 — unscaffolded

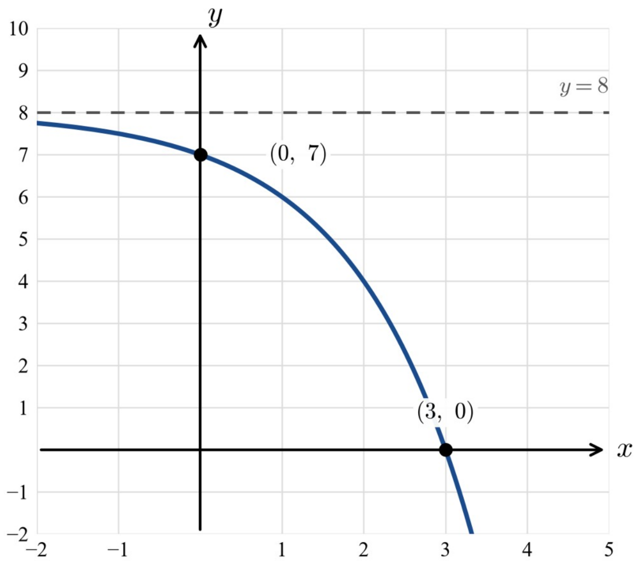
Sketch $y = -2^x + 8$, labelling the asymptote and all axis intercepts, and state the range.

The coefficient $A = -1$ reflects $y = 2^x$ in the x-axis; the $+8$ raises the asymptote to $y = 8$. So the graph is **decreasing** and lies **below** $y = 8$: range $\{y : y < 8\}$.

y-intercept: $x = 0 \Rightarrow y = -2^0 + 8 = -1 + 8 = 7$, giving $(0, 7)$.

x-intercept: $-2^x + 8 = 0 \Rightarrow 2^x = 8 = 2^3$, so $x = 3$, giving $(3, 0)$.

$\therefore$ the graph falls from just below $y = 8$ through $(0, 7)$ and $(3, 0)$.



Practice questions

Scaffolded

Q1. For $y = 3^x$: (a) state the equation of the asymptote; (b) find the y-intercept and the coordinates of the point where $x = 1$; (c) state the domain and range; (d) sketch the graph.

Q2. For $y=2^x+1$: (a) state the asymptote; (b) find the y-intercept; (c) explain why there is no x-intercept; (d) sketch the graph.

Q3. For $y=2^{x+1}-2$: (a) state the asymptote; (b) find the exact axis intercepts; (c) state the range; (d) sketch the graph.

Q4. For $y=2^{-x}$: (a) state whether the graph is increasing or decreasing, and why; (b) find the y-intercept; (c) state the domain and range; (d) sketch the graph.

Q5. For $y=-3^x+9$: (a) state the asymptote and the range; (b) find the exact axis intercepts; (c) sketch the graph.

Q6. For $y=2^{x-2}+1$: (a) describe the transformations that map $y=2^x$ onto this graph; (b) state the asymptote and find the y-intercept; (c) sketch the graph.

Unscaffolded

Sketch each of the following, labelling the asymptote and the exact coordinates of all axis intercepts, and state the range.

Q7. $y=4^x$

Q8. $y=2^x-8$

Q9. $y=3^{x-1}$

Q10. $y=-2^{x-1}+4$

Q11. $y=2^{-x}+3$

Q12. $y=5^x-1$

Q13. $y=2^{x+2}-8$

Q14. $y=-4^x+16$

Q15. $y=3^{-x}-1$

Q16. $y=2^{2-x}+1$

Q17. $y=-2^{-x}+4$

Q18. $y=10^x$

Q19. $y=2^{x-3}$

Q20. Describe the sequence of transformations that maps $y=3^x$ onto $y=2\cdot 3^x-6$, and hence sketch the graph, labelling the asymptote and axis intercepts.

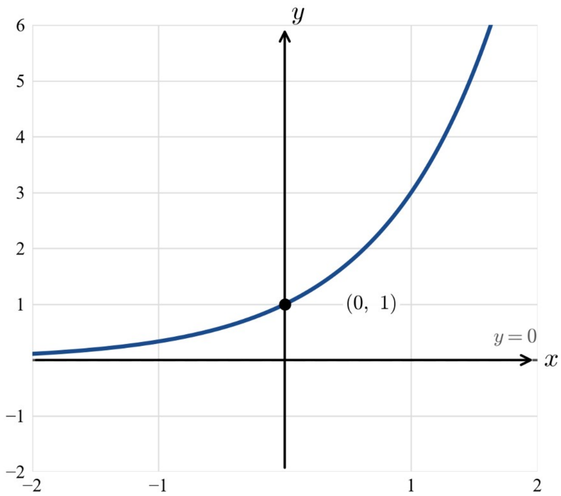
Q21. For $y=3^{x-2}-9$, state the equation of the asymptote, find the exact axis intercepts, and hence sketch the graph.

Answers

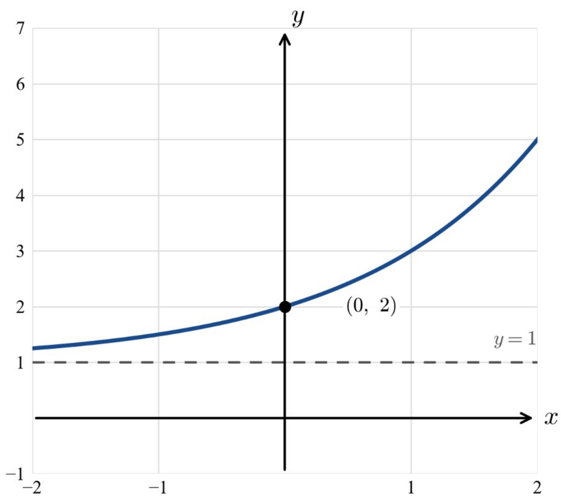
Q1. Asymptote $y=0$; y-intercept $(0,1)$, point $(1,3)$; domain $\mathbb{R}$, range $\{y : y>0\}$. **Q2.** Asymptote $y=1$; $(0,2)$; no x-intercept ($y>1$); range $\{y : y>1\}$. **Q3.** Asymptote $y=-2$; intercepts $(0,0)$; range $\{y : y>-2\}$. **Q4.** Decreasing (base $2^{-1}=1/2<1$); $(0,1)$; domain $\mathbb{R}$, range $\{y : y>0\}$. **Q5.** Asymptote $y=9$, range $\{y : y<9\}$; intercepts $(0,8)$, $(2,0)$. **Q6.** Translation 2 right and 1 up; asymptote $y=1$, y-intercept $(0,5/4)$. **Q7.** Asymptote $y=0$; $(0,1)$; range $\{y : y>0\}$. **Q8.** Asymptote $y=-8$; $(0,-7)$, $(3,0)$; range $\{y : y>-8\}$. **Q9.** Asymptote $y=0$; $(0,1/3)$; range $\{y : y>0\}$. **Q10.** Asymptote $y=4$; $(0,7/2)$, $(3,0)$; range $\{y : y<4\}$. **Q11.** Asymptote $y=3$; $(0,4)$; range $\{y : y>3\}$. **Q12.** Asymptote $y=-1$; $(0,0)$; range $\{y : y>-1\}$. **Q13.** Asymptote $y=-8$; $(0,-4)$, $(1,0)$; range $\{y : y>-8\}$. **Q14.** Asymptote $y=16$; $(0,15)$, $(2,0)$; range $\{y : y<16\}$. **Q15.** Asymptote $y=-1$; $(0,0)$; range $\{y : y>-1\}$. **Q16.** Asymptote $y=1$; $(0,5)$; range $\{y : y>1\}$. **Q17.** Asymptote $y=4$; $(0,3)$, $(-2,0)$; range $\{y : y<4\}$. **Q18.** Asymptote $y=0$; $(0,1)$; range $\{y : y>0\}$. **Q19.** Asymptote $y=0$; $(0,1/8)$; range $\{y : y>0\}$. **Q20.** Dilation of factor 2 from the x-axis then translation 6 down; asymptote $y=-6$; intercepts $(0,-4)$, $(1,0)$; range $\{y : y>-6\}$. **Q21.** Asymptote $y=-9$; intercepts $(0,-80/9)$, $(4,0)$; range $\{y : y>-9\}$.

Fully-worked solutions

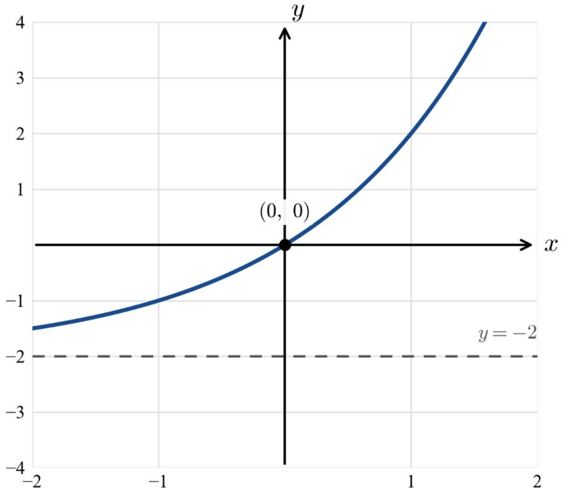
Q1. $y=3^x$. Asymptote $y=0$. y-intercept $x=0 \Rightarrow y=3^0=1$, giving $(0,1)$; and $x=1 \Rightarrow y=3$, giving $(1,3)$. Domain $\mathbb{R}$, range $\{y : y>0\}$; increasing.



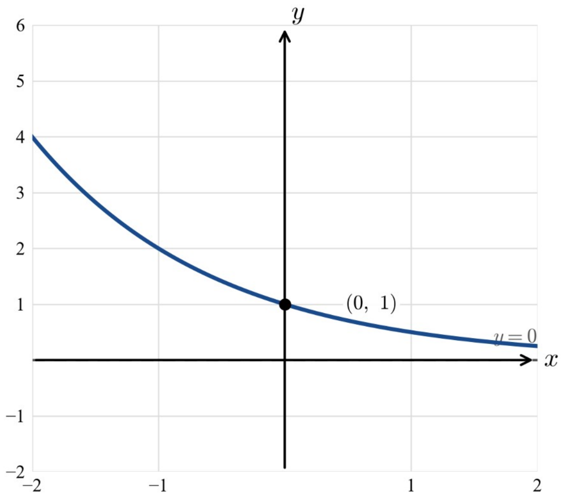
Q2. $y=2^x+1$. Asymptote $y=1$. y-intercept $x=0 \Rightarrow y=1+1=2$, giving $(0,2)$. Since $2^x>0$, $y=2^x+1>1$ for all x, so there is no x-intercept. Range $\{y : y>1\}$.



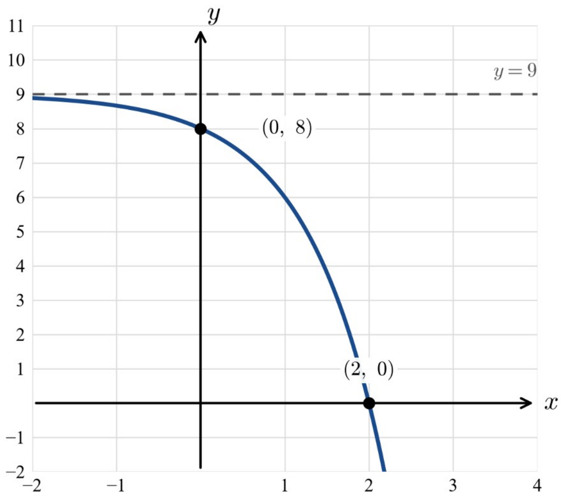
Q3. $y=2^{x+1}-2$. Asymptote $y=-2$. y-intercept $x=0 \Rightarrow y=2^1-2=0$, giving $(0,0)$. x-intercept $2^{x+1}-2=0 \Rightarrow 2^{x+1}=2^1 \Rightarrow x+1=1$, so $x=0$, giving $(0,0)$. Range $\{y : y>-2\}$.



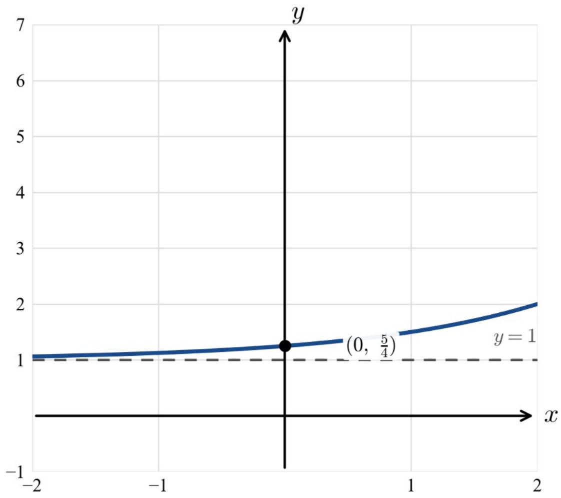
Q4. $y=2^{-x}=(1/2)^x$. The base $1/2<1$, so the graph is **decreasing** (equivalently, it is $y=2^x$ reflected in the y-axis). y-intercept $x=0 \Rightarrow y=2^0=1$, giving $(0,1)$. Domain $\mathbb{R}$, range $\{y : y>0\}$.



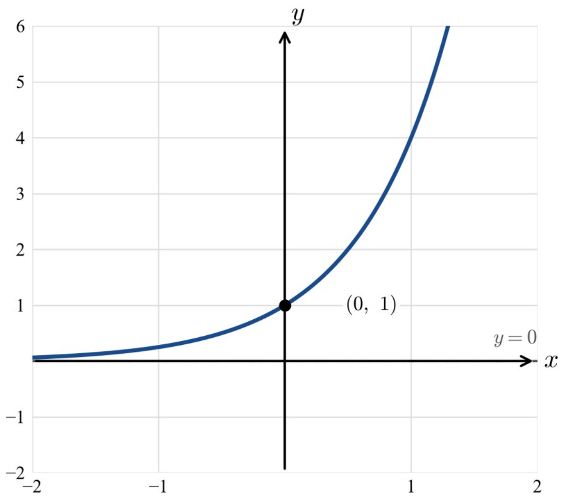
Q5. $y=-3^x+9$. The -3^x reflects $y=3^x$ in the x-axis; asymptote $y=9$, so range $\{y : y<9\}$. y-intercept $x=0 \Rightarrow y=-1+9=8$, giving $(0,8)$. x-intercept $3^x=9=3^2$, so $x=2$, giving $(2,0)$.



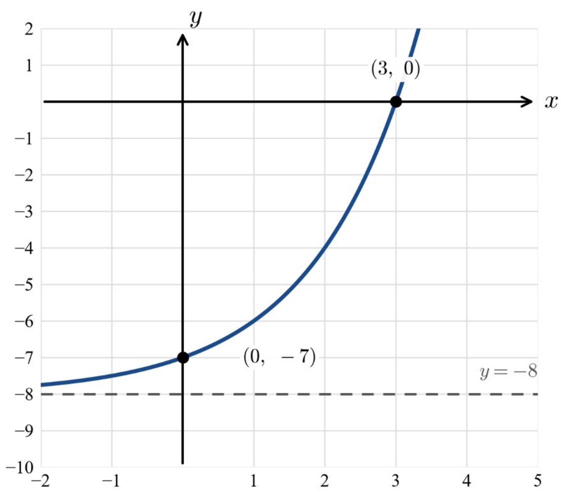
Q6. $y=2^{x-2}+1$. From $y=2^x$: a translation of 2 units right and 1 unit up. Asymptote $y=1$. y-intercept $x=0 \Rightarrow y=2^{-2}+1=1/4+1=5/4$, giving $(0,5/4)$. Range $\{y : y>1\}$.



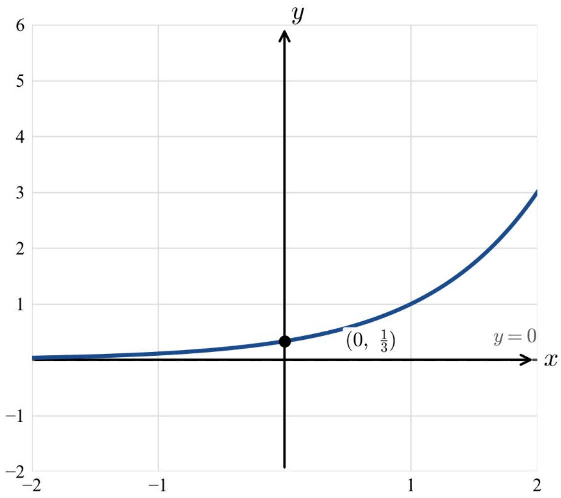
Q7. $y=4^x$. Asymptote $y=0$; y-intercept $(0,1)$; increasing; range $\{y : y>0\}$.



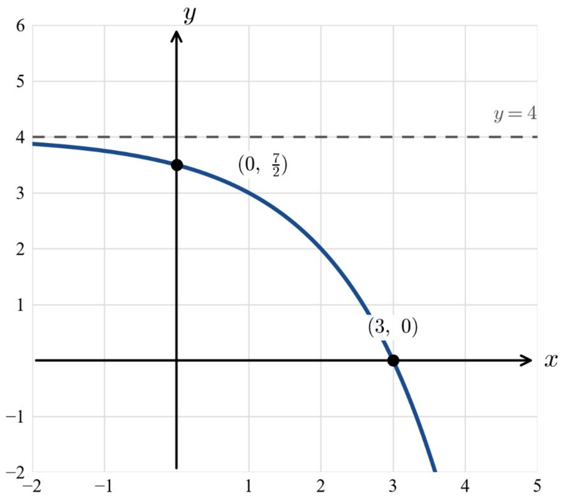
Q8. $y=2^x-8$. Asymptote $y=-8$. y-intercept $(0,1-8)=(0,-7)$. x-intercept $2^x=8=2^3$, so $x=3$, giving $(3,0)$. Range $\{y : y>-8\}$.



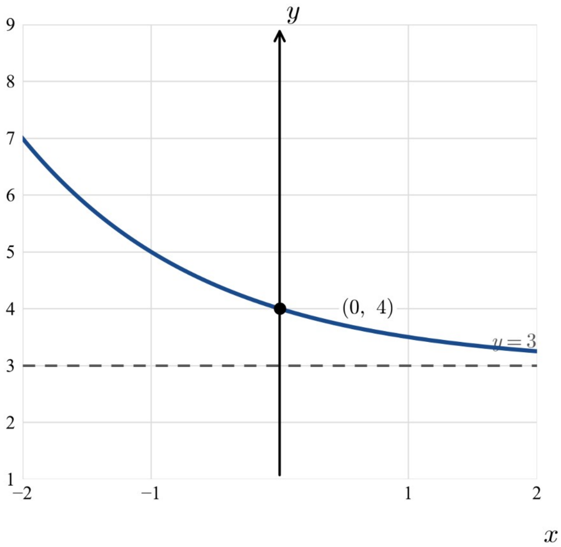
Q9. $y=3^{x-1}$. Asymptote $y=0$; y-intercept $x=0\Rightarrow y=3^{-1}=1/3$, giving $(0,1/3)$; passes through $(1,1)$. Range $\{y : y>0\}$.



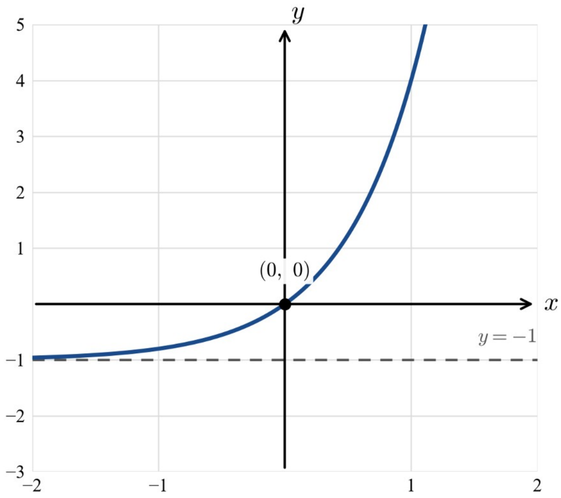
Q10. $y = -2^{x-1} + 4$. Reflected in the x-axis; asymptote $y = 4$, range $\{y : y < 4\}$. y-intercept $x = 0 \Rightarrow y = -2^{-1} + 4 = -1/2 + 4 = 7/2$, giving $(0, 7/2)$. x-intercept $2^{x-1} = 4 = 2^2 \Rightarrow x - 1 = 2$, so $x = 3$, giving $(3, 0)$.



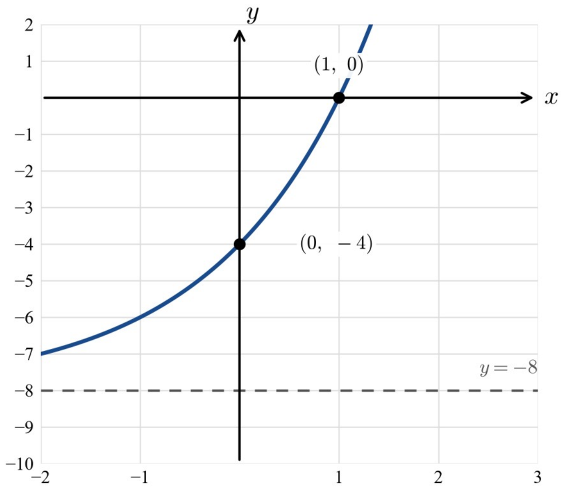
Q11. $y = 2^{-x} + 3$. Decreasing (reflected in y-axis), asymptote $y = 3$, range $\{y : y > 3\}$. y-intercept $x = 0 \Rightarrow y = 1 + 3 = 4$, giving $(0, 4)$. No x-intercept.



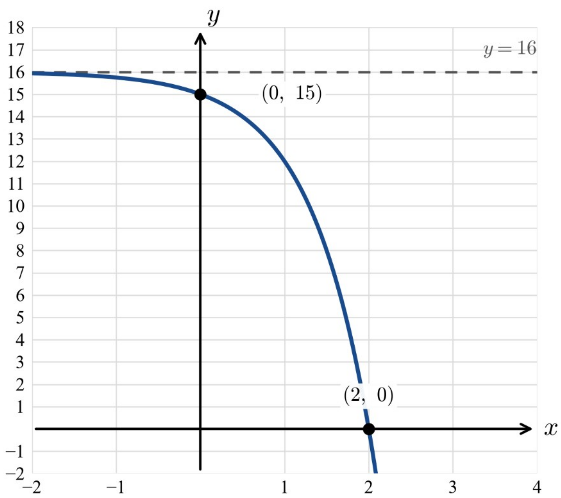
Q12. $y = 5^x - 1$. Asymptote $y = -1$. y-intercept $x = 0 \Rightarrow y = 1 - 1 = 0$, giving $(0, 0)$. x-intercept $5^x = 1 = 5^0$, so $x = 0$, giving $(0, 0)$. Range $\{y : y > -1\}$.



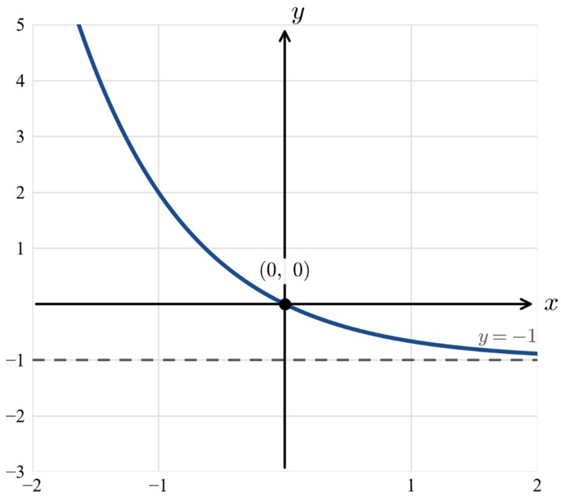
Q13. $y = 2^{x+2} - 8$. Asymptote $y = -8$. y-intercept $x = 0 \Rightarrow y = 2^2 - 8 = -4$, giving $(0, -4)$. x-intercept $2^{x+2} = 8 = 2^3 \Rightarrow x + 2 = 3$, so $x = 1$, giving $(1, 0)$. Range $\{y : y > -8\}$.



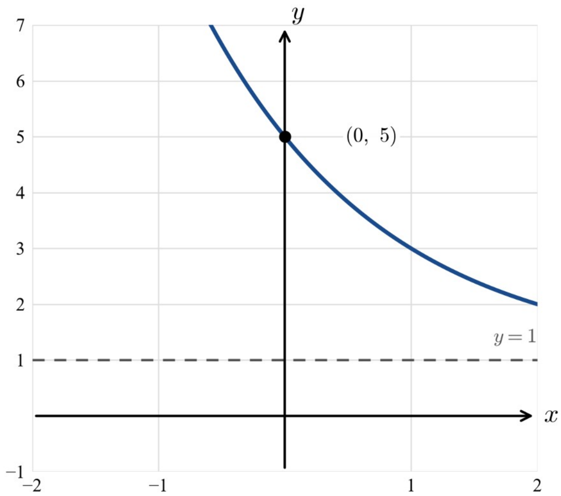
Q14. $y = -4^x + 16$. Reflected in the x-axis; asymptote $y = 16$, range $\{y : y < 16\}$. y-intercept $x = 0 \Rightarrow y = -1 + 16 = 15$, giving $(0, 15)$. x-intercept $4^x = 16 = 4^2$, so $x = 2$, giving $(2, 0)$.



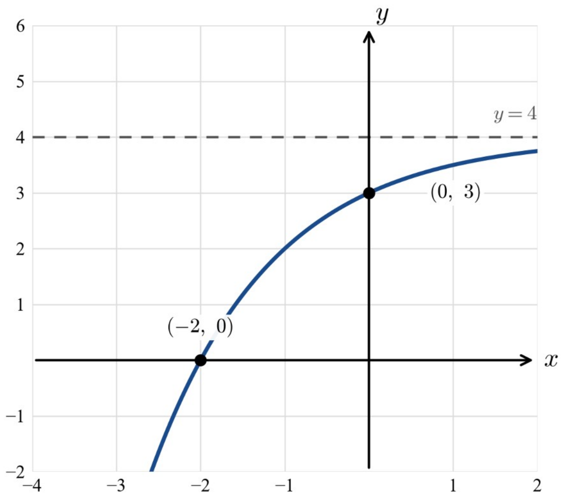
Q15. $y = 3^{-x} - 1$. Decreasing, asymptote $y = -1$, range $\{y : y > -1\}$. y-intercept $x = 0 \Rightarrow y = 1 - 1 = 0$, giving $(0, 0)$. x-intercept $3^{-x} = 1 = 3^0$, so $x = 0$, giving $(0, 0)$.



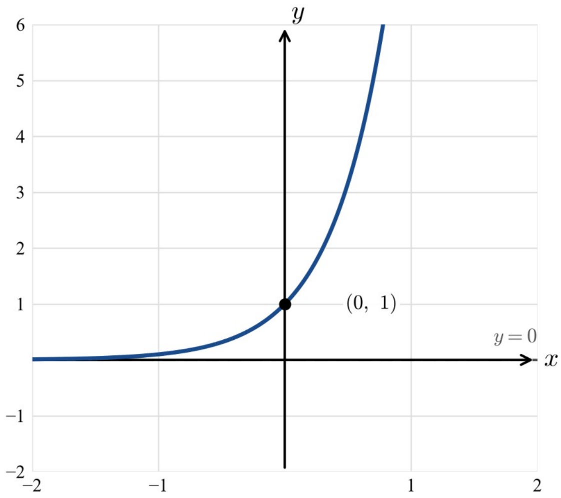
Q16. $y=2^{2-x}+1$. The index $2-x=-(x-2)$, so the graph is **decreasing**; asymptote $y=1$, range $\{y:y>1\}$. y-intercept $x=0 \Rightarrow y=2^2+1=5$, giving $(0,5)$. No x-intercept.



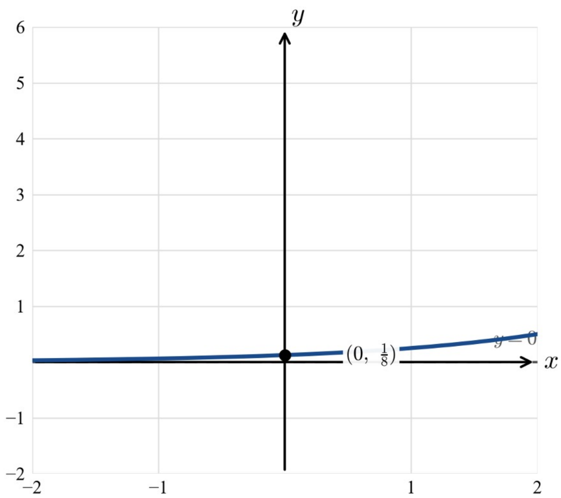
Q17. $y=-2^{-x}+4$. Reflected in both axes; asymptote $y=4$, range $\{y:y<4\}$; it is **increasing**. y-intercept $x=0 \Rightarrow y=-1+4=3$, giving $(0,3)$. x-intercept $2^{-x}=4=2^2 \Rightarrow -x=2$, so $x=-2$, giving $(-2,0)$.



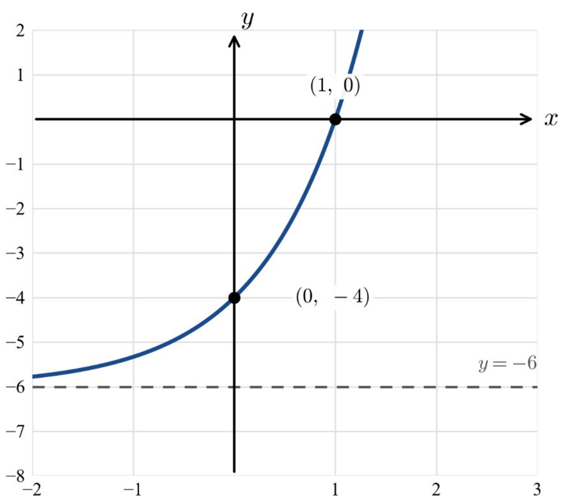
Q18. $y=10^x$. Asymptote $y=0$; y-intercept $(0,1)$; increasing (steeply); range $\{y:y>0\}$.



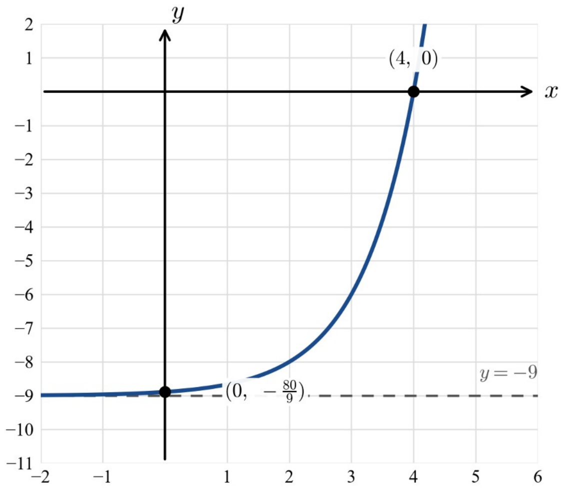
Q19. $y=2^{x-3}$. Asymptote $y=0$; y-intercept $x=0 \Rightarrow y=2^{-3}=1/8$, giving $(0,1/8)$; passes through $(3,1)$. Range $\{y : y>0\}$.



Q20. $y=2\cdot 3^{x}-6$. From $y=3^{x}$: a dilation of factor 2 from the x-axis, then a translation of 6 units down. Asymptote $y=-6$, range $\{y : y>-6\}$. y-intercept $x=0 \Rightarrow y=2(1)-6=-4$, giving $(0,-4)$. x-intercept $2\cdot 3^{x}=6 \Rightarrow 3^{x}=3=3^{1}$, so $x=1$, giving $(1,0)$.



Q21. $y=3^{x-2}-9$. Asymptote $y=-9$, range $\{y : y>-9\}$. y-intercept $x=0 \Rightarrow y=3^{-2}-9=1/9-9=-80/9$, giving $(0,-80/9)$. x-intercept $3^{x-2}=9=3^{2} \Rightarrow x-2=2$, so $x=4$, giving $(4,0)$.


y
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(4, 0)
(0, −80/9)
y = −9
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Theory

Among all exponential functions $y=a^x$, one base is special. **Euler's number** $e\approx 2.71828$ is the base for which the graph of $y=e^x$ has a gradient of exactly 1 at the point $(0,1)$. The function $f(x)=e^x$ is called **the exponential function** (or the *natural* exponential function) and is central to all of the calculus that follows.

The graph of $y=e^x$.

- Domain $\mathbb{R}$, range $\{y:y>0\}$.
- **Horizontal asymptote** $y=0$ (the curve approaches the x-axis as $x\to-\infty$ but never reaches it).
- y-intercept $(0,1)$; there is **no x-intercept**.
- The function is **strictly increasing** for all x, and $e^x>0$ always.

Transformations. Applying a dilation of factor a from the x-axis (and a reflection in the x-axis when $a<0$), a dilation of factor $1/n$ from the y-axis (a reflection in the y-axis when $n<0$), and a translation of h horizontally and k vertically gives the general form

$$y=a\,e^{n(x-h)}+k.$$

- The **horizontal asymptote** moves to $y=k$.
- The **range** is $\{y:y>k\}$ if $a>0$, and $\{y:y<k\}$ if $a<0$.
- If $a>0$ and $n>0$ the graph is increasing; a negative a (reflection in the x-axis) or a negative n (reflection in the y-axis) makes it decreasing.

Finding the intercepts (give exact values).

- **y-intercept:** substitute $x=0$ — this always exists (the domain is $\mathbb{R}$).
- **x-intercept:** solve $y=0$. This requires $a\,e^{n(x-h)}=-k$, i.e. $e^{n(x-h)}=-\frac{k}{a}$, which has a solution only when $-\frac{k}{a}>0$ (that is, when the asymptote and the curve are on opposite sides of the x-axis). When it exists, take $\log_e$ of both sides: $x=h+\frac{1}{n}\log_e\left(-\frac{k}{a}\right)$.

To sketch: (1) identify the asymptote $y=k$ and draw it dashed; (2) state the domain and range; (3) find the exact y-intercept and the x-intercept (if any); (4) draw the curve approaching $y=k$ on one side and growing away from it on the other.

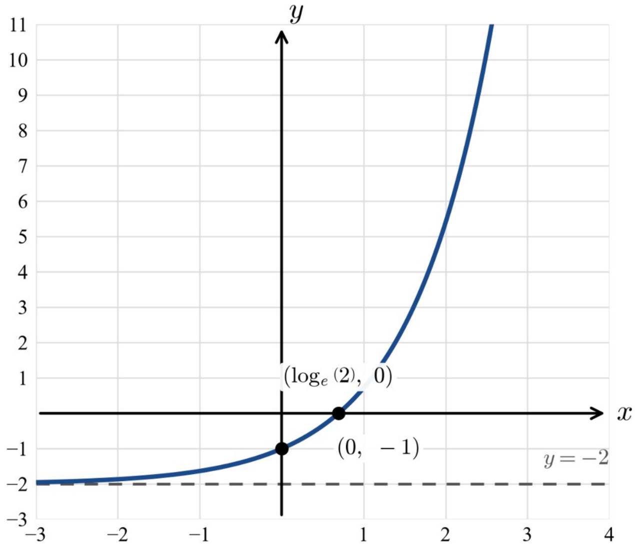
Worked examples

Example 1 — scaffolded

Sketch the graph of $y=e^x-2$, stating the equation of the asymptote and the exact coordinates of the axis intercepts, and give the range.

Working	Comment
Base graph $y=e^x$ translated **down** 2 units, so $k=-2$.	Compare with $y=a\,e^{n(x-h)}+k$; here $a=1$, $n=1$, $h=0$, $k=-2$.
Asymptote $y=-2$; range $\{y:y>-2\}$.	The horizontal asymptote is $y=k$; $a>0$ so $y>k$.

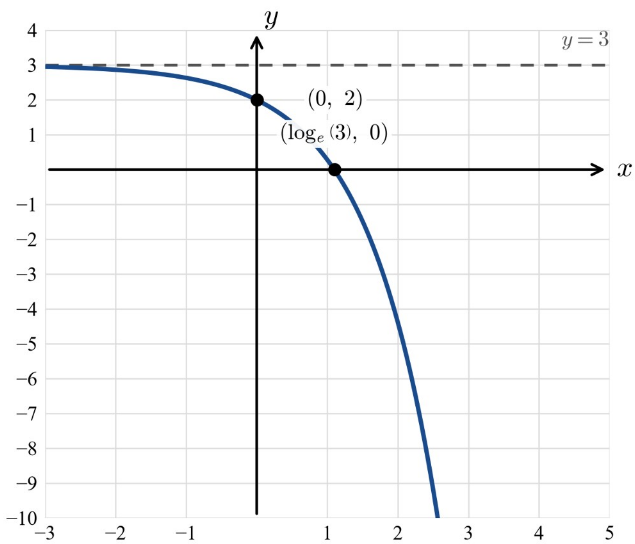
Working	Comment
y-intercept: $x=0 \Rightarrow y=e^0-2=1-2=-1$, giving $(0,-1)$.	Substitute $x=0$.
x-intercept: $e^x-2=0 \Rightarrow e^x=2 \Rightarrow x=\log_e 2$, giving $(\log_e 2,0)$.	Take $\log_e$ of both sides; keep the answer exact.



Example 2 — scaffolded

Sketch the graph of $y=3-e^x$, labelling the asymptote and the axis intercepts, and state the range.

Working	Comment
Write as $y=-e^x+3$: a reflection in the x-axis ($a=-1$) then a translation **up** 3, so $k=3$.	Compare with $y=a\,e^{n(x-h)}+k$.
Asymptote $y=3$; because $a=-1<0$, range $\{y: y<3\}$ and the graph is decreasing.	Reflected in the x-axis, the curve lies **below** $y=3$.
y-intercept: $x=0 \Rightarrow y=3-e^0=3-1=2$, giving $(0,2)$.	Substitute $x=0$.
x-intercept: $3-e^x=0 \Rightarrow e^x=3 \Rightarrow x=\log_e 3$, giving $(\log_e 3,0)$.	Isolate e^x, then take $\log_e$.



Example 3 — unscaffolded

Sketch the graph of $y = e^{2x} - 1$, stating the asymptote, intercepts and range.

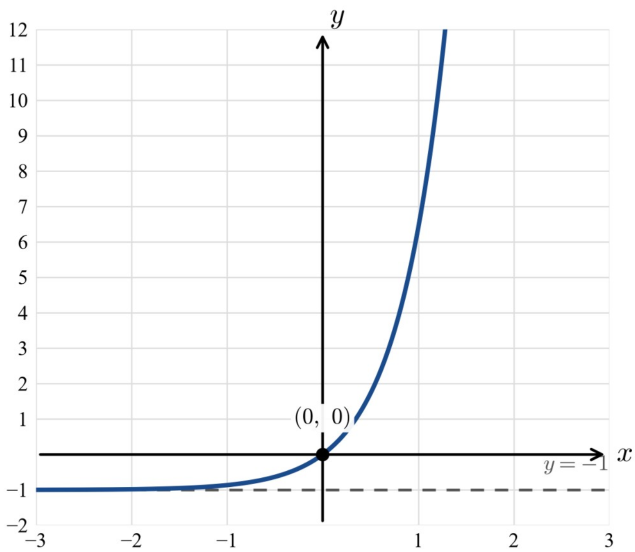
Here $n = 2$ (a dilation of factor $1/2$ from the y-axis) and $k = -1$.

Asymptote $y = -1$; since $a = 1 > 0$, range $\{y : y > -1\}$, increasing.

y-intercept: $x = 0 \Rightarrow y = e^0 - 1 = 0$, giving $(0, 0)$.

x-intercept: $e^{2x} - 1 = 0 \Rightarrow e^{2x} = 1 \Rightarrow 2x = 0 \Rightarrow x = 0$, giving $(0, 0)$.

$\therefore$ the graph passes through the origin and rises steeply, approaching $y = -1$ as $x \to -\infty$.



Example 4 — unscaffolded

Sketch the graph of $y = 2e^{-x} + 1$, stating the asymptote, intercepts and range.

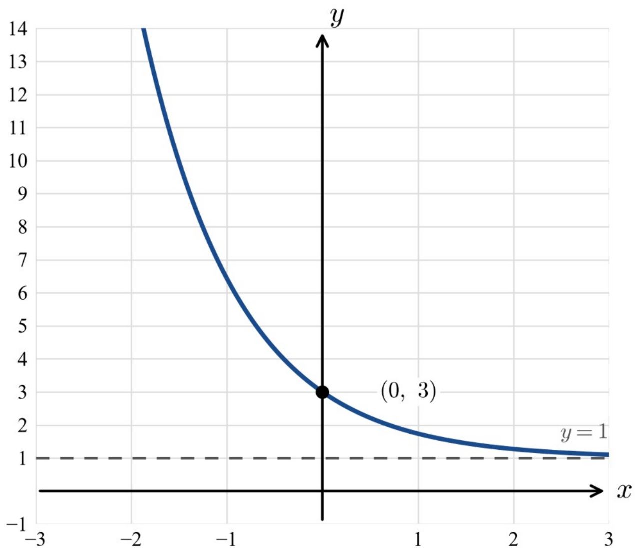
Here $n = -1$ (a reflection in the y-axis), $a = 2$ (dilation of factor 2 from the x-axis) and $k = 1$.

Asymptote $y = 1$; since $a = 2 > 0$, range $\{y : y > 1\}$. As $n < 0$ the graph is **decreasing** (it approaches $y = 1$ as $x \to +\infty$).

y-intercept: $x = 0 \Rightarrow y = 2e^0 + 1 = 2 + 1 = 3$, giving $(0, 3)$.

x-intercept: $2e^{-x} + 1 = 0 \Rightarrow e^{-x} = -1/2$, which has **no solution**, so there is no x-intercept (the whole curve lies above $y = 1$).

$\therefore$ the graph decreases from the upper left, through $(0, 3)$, approaching $y = 1$.



Practice questions

Scaffolded

Q1. For $y=e^x+1$: (a) state the equation of the asymptote; (b) find the exact axis intercepts; (c) state the domain and range; (d) sketch the graph.

Q2. For $y=e^x-3$: (a) state the asymptote and range; (b) find the exact axis intercepts; (c) sketch the graph.

Q3. For $y=2-e^x$: (a) describe the two transformations of $y=e^x$ that produce this graph; (b) state the asymptote and range; (c) find the exact axis intercepts; (d) sketch the graph.

Q4. For $y=e^{x+1}$: (a) state the asymptote; (b) find the exact y-intercept and explain why there is no x-intercept; (c) sketch the graph.

Q5. For $y=e^{-x}-1$: (a) state the asymptote and range; (b) find the exact axis intercepts; (c) state whether the graph is increasing or decreasing, and sketch it.

Q6. For $y=2e^x-4$: (a) state the asymptote and range; (b) find the exact axis intercepts; (c) sketch the graph.

Unscaffolded

Sketch each of the following, labelling the asymptote and the exact coordinates of all axis intercepts, and state the range.

Q7. $y=e^x-4$

Q8. $y=4-e^x$

Q9. $y=e^{x-2}$

Q10. $y=e^x+2$

Q11. $y=2e^x-6$

Q12. $y=e^{-x}+1$

Q13. $y=2-e^{-x}$

Q14. $y=e^{2x}-4$

Q15. $y=3e^x-3$

Q16. $y = e^{x+1} - 1$

Q17. $y = 2 - 2e^{x}$

Q18. $y = e^{-2x}$

Q19. $y = e^{x-1} + 1$

Q20. Describe the sequence of transformations that maps $y = e^{x}$ onto $y = 3e^{x} + 1$, and hence sketch the graph, labelling the asymptote and y-intercept.

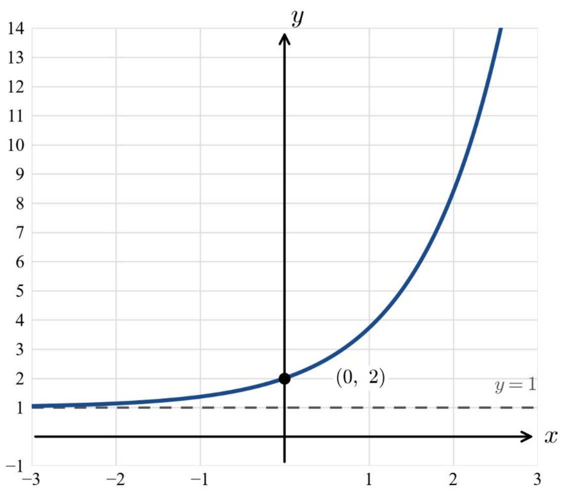
Q21. For $y = e^{2x} - 9$, show that the graph crosses the x-axis at $x = \log_e 3$, and hence sketch the graph, labelling the asymptote and both axis intercepts.

Answers

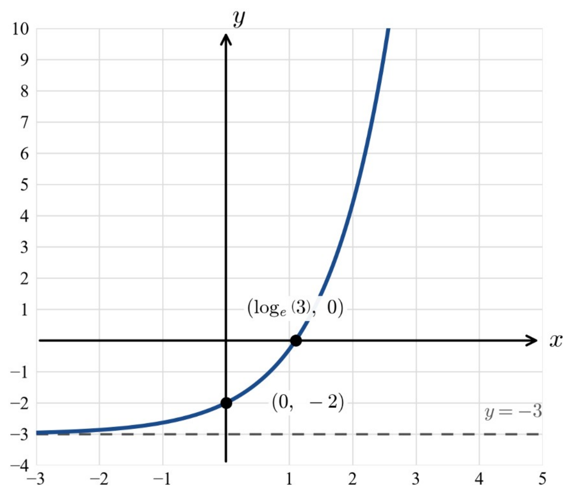
Q1. Asymptote $y=1$; intercept $(0,2)$ (no x-intercept); domain $\mathbb{R}$, range $\{y: y>1\}$. **Q2.** Asymptote $y=-3$, range $\{y: y>-3\}$; intercepts $(0,-2)$, $(\log_e 3,0)$. **Q3.** Reflection in the x-axis then translation up 2; asymptote $y=2$, range $\{y: y<2\}$; intercepts $(0,1)$, $(\log_e 2,0)$. **Q4.** Asymptote $y=0$; y-intercept $(0,e)$; no x-intercept since $e^{x+1}>0$. **Q5.** Asymptote $y=-1$, range $\{y: y>-1\}$; intercept $(0,0)$; decreasing. **Q6.** Asymptote $y=-4$, range $\{y: y>-4\}$; intercepts $(0,-2)$, $(\log_e 2,0)$. **Q7.** Asymptote $y=-4$, range $\{y: y>-4\}$; intercepts $(0,-3)$, $(\log_e 4,0)$. **Q8.** Asymptote $y=4$, range $\{y: y<4\}$; intercepts $(0,3)$, $(\log_e 4,0)$. **Q9.** Asymptote $y=0$, range $\{y: y>0\}$; y-intercept $(0,e^{-2})$; no x-intercept. **Q10.** Asymptote $y=2$, range $\{y: y>2\}$; y-intercept $(0,3)$; no x-intercept. **Q11.** Asymptote $y=-6$, range $\{y: y>-6\}$; intercepts $(0,-4)$, $(\log_e 3,0)$. **Q12.** Asymptote $y=1$, range $\{y: y>1\}$; y-intercept $(0,2)$; no x-intercept; decreasing. **Q13.** Asymptote $y=2$, range $\{y: y<2\}$; intercepts $(0,1)$, $(-\log_e 2,0)$; increasing. **Q14.** Asymptote $y=-4$, range $\{y: y>-4\}$; intercepts $(0,-3)$, $(\log_e 2,0)$. **Q15.** Asymptote $y=-3$, range $\{y: y>-3\}$; intercept $(0,0)$. **Q16.** Asymptote $y=-1$, range $\{y: y>-1\}$; intercepts $(0,e-1)$, $(-1,0)$. **Q17.** Asymptote $y=2$, range $\{y: y<2\}$; intercept $(0,0)$; decreasing. **Q18.** Asymptote $y=0$, range $\{y: y>0\}$; y-intercept $(0,1)$; no x-intercept; decreasing. **Q19.** Asymptote $y=1$, range $\{y: y>1\}$; y-intercept $(0,e^{-1}+1)$; no x-intercept. **Q20.** Dilation of factor 3 from the x-axis, then translation up 1; asymptote $y=1$, y-intercept $(0,4)$, no x-intercept. **Q21.** Asymptote $y=-9$, range $\{y: y>-9\}$; intercepts $(0,-8)$, $(\log_e 3,0)$.

Fully-worked solutions

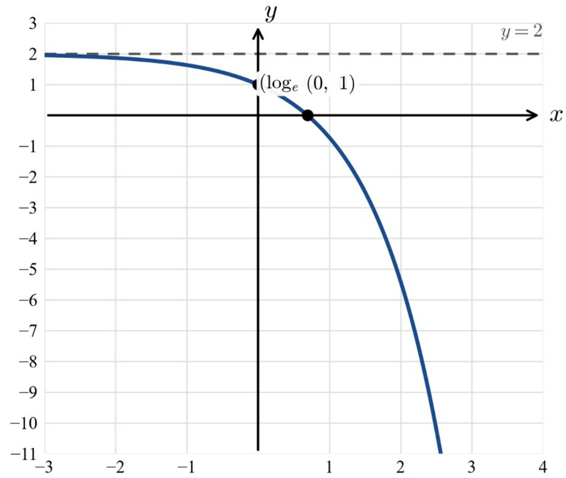
Q1. $y=e^x+1$. Asymptote $y=1$. y-intercept $x=0 \Rightarrow y=1+1=2$, giving $(0,2)$. Since $e^x>0$, $y>1$ always, so there is no x-intercept. Range $\{y: y>1\}$.



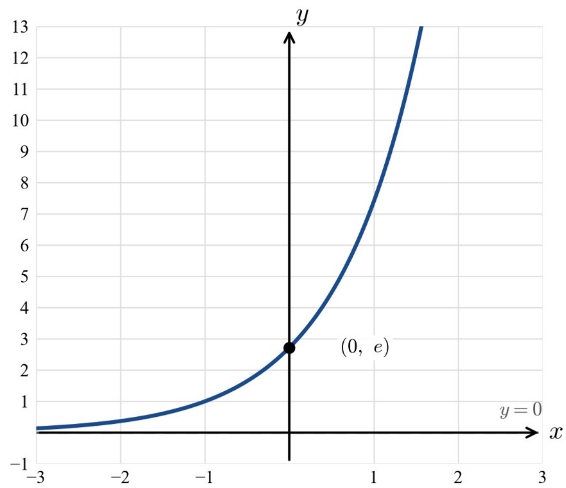
Q2. $y=e^x-3$. Asymptote $y=-3$, range $\{y: y>-3\}$. y-intercept $(0,-2)$. x-intercept: $e^x=3 \Rightarrow x=\log_e 3$, giving $(\log_e 3,0)$.



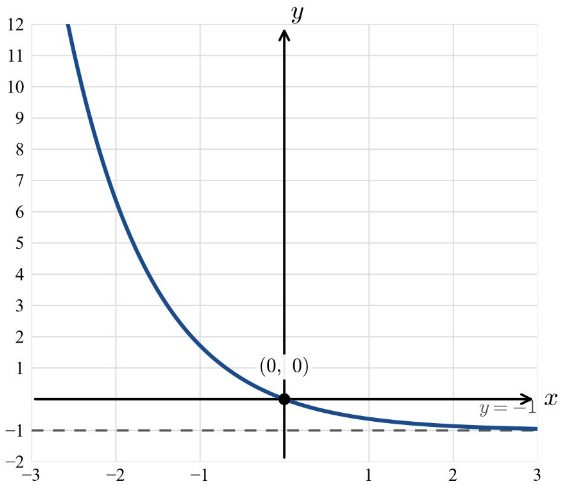
Q3. $y=2-e^{x}=-e^{x}+2$. (a) Reflection in the x-axis, then translation up 2. (b) Asymptote $y=2$, range $\{y : y<2\}$. (c) y-intercept $(0,1)$; x-intercept $e^{x}=2 \Rightarrow x=\log_e 2$, giving $(\log_e 2, 0)$.



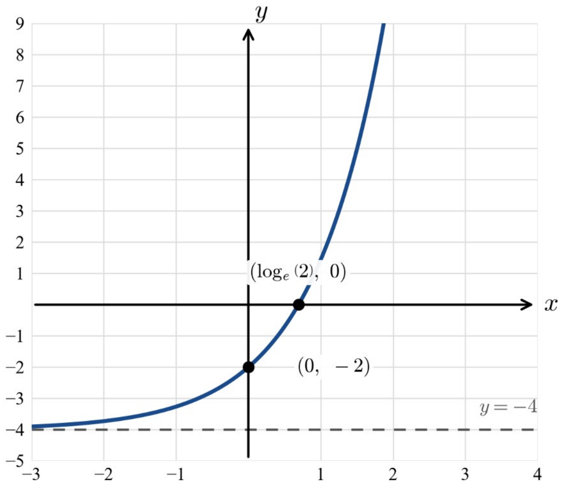
Q4. $y=e^{x+1}$. Asymptote $y=0$. y-intercept $x=0 \Rightarrow y=e^{1}=e$, giving $(0,e)$. There is no x-intercept because $e^{x+1}>0$ for all x.



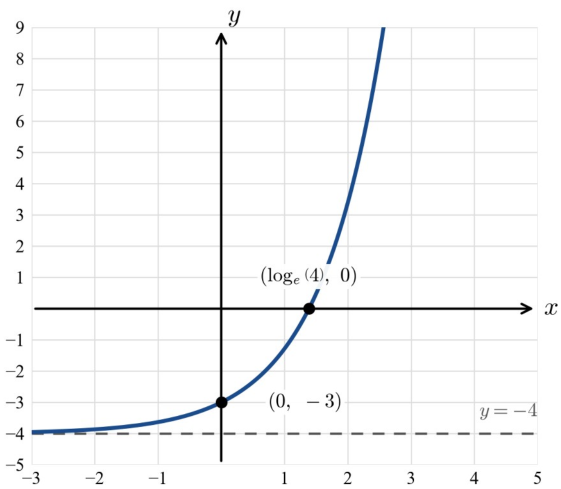
Q5. $y=e^{-x}-1$. Asymptote $y=-1$, range $\{y : y>-1\}$. y-intercept $(0,0)$; x-intercept $e^{-x}=1 \Rightarrow -x=0 \Rightarrow x=0$, giving $(0,0)$. As $n=-1<0$ the graph is **decreasing**.



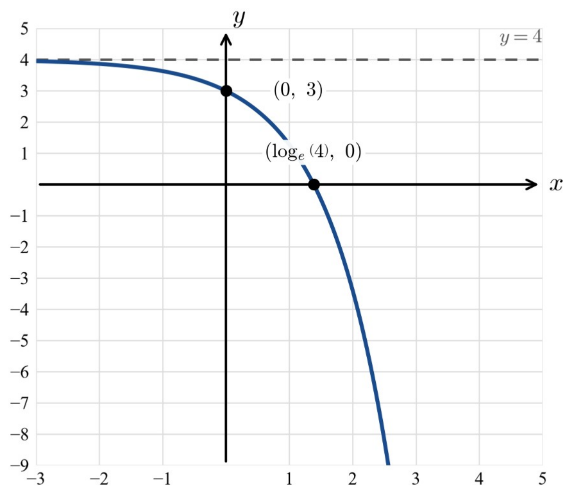
Q6. $y = 2e^x - 4$. Asymptote $y = -4$, range $\{y : y > -4\}$. y-intercept $x = 0 \Rightarrow y = 2 - 4 = -2$, giving $(0, -2)$. x-intercept $2e^x = 4 \Rightarrow e^x = 2 \Rightarrow x = \log_e 2$, giving $(\log_e 2, 0)$.



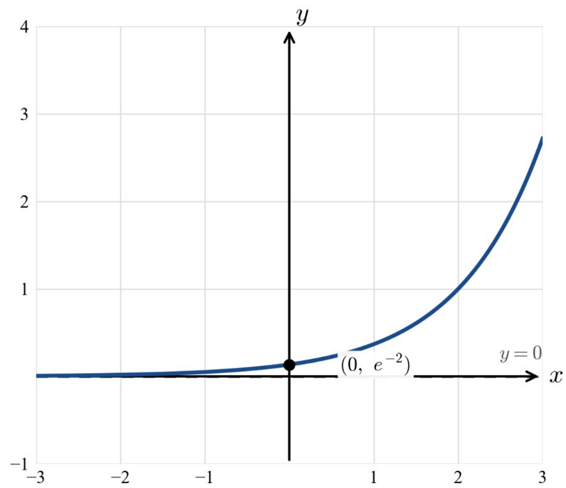
Q7. $y = e^x - 4$. Asymptote $y = -4$, range $\{y : y > -4\}$. y-intercept $(0, -3)$; x-intercept $e^x = 4 \Rightarrow x = \log_e 4$, giving $(\log_e 4, 0)$.



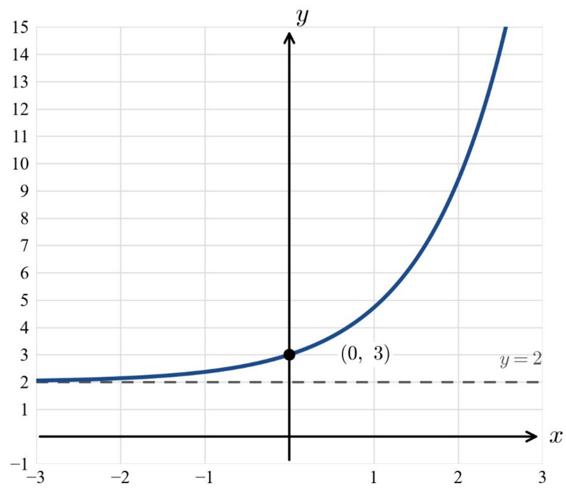
Q8. $y = 4 - e^x$. Asymptote $y = 4$, range $\{y : y < 4\}$; decreasing. y-intercept $(0, 3)$; x-intercept $e^x = 4 \Rightarrow x = \log_e 4$, giving $(\log_e 4, 0)$.



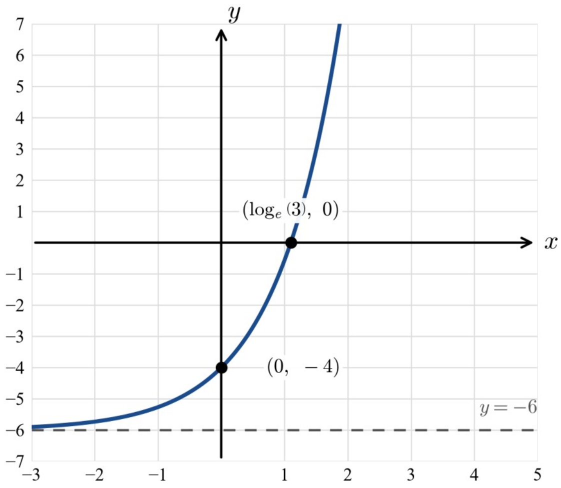
Q9. $y=e^{x-2}$. A translation of $y=e^x$ right 2. Asymptote $y=0$, range $\{y:y>0\}$. y-intercept $x=0 \Rightarrow y=e^{-2}$, giving $(0,e^{-2})$. No x-intercept.



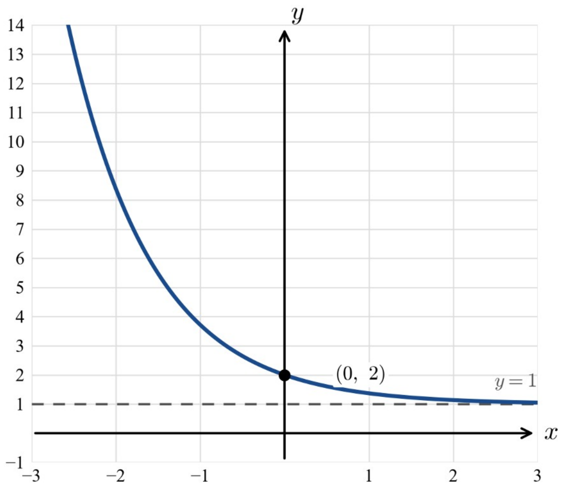
Q10. $y=e^x+2$. Asymptote $y=2$, range $\{y:y>2\}$. y-intercept $(0,3)$. No x-intercept ($y>2$).



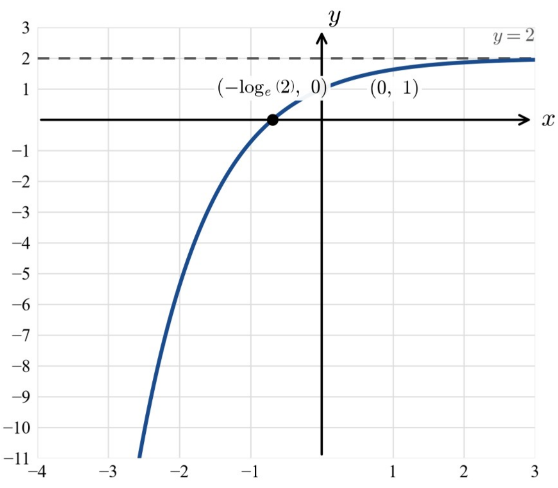
Q11. $y=2e^x-6$. Asymptote $y=-6$, range $\{y:y>-6\}$. y-intercept $(0,-4)$; x-intercept $2e^x=6 \Rightarrow e^x=3 \Rightarrow x=\log_e 3$, giving $(\log_e 3,0)$.



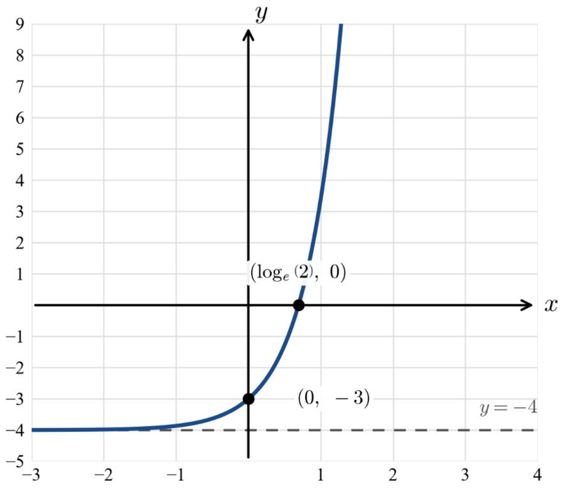
Q12. $y=e^{-x}+1$. Asymptote $y=1$, range $\{y: y>1\}$; decreasing. y-intercept $(0,2)$. No x-intercept.



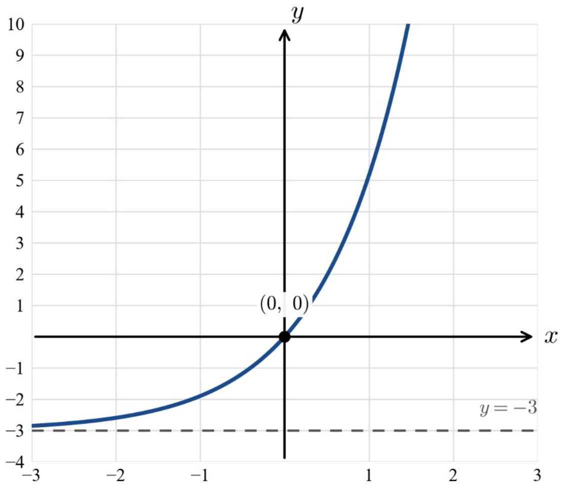
Q13. $y=2-e^{-x}$. Asymptote $y=2$, range $\{y: y<2\}$. y-intercept $(0,1)$; x-intercept $e^{-x}=2 \Rightarrow -x=\log_e 2 \Rightarrow x=-\log_e 2$, giving $(-\log_e 2, 0)$. Increasing.



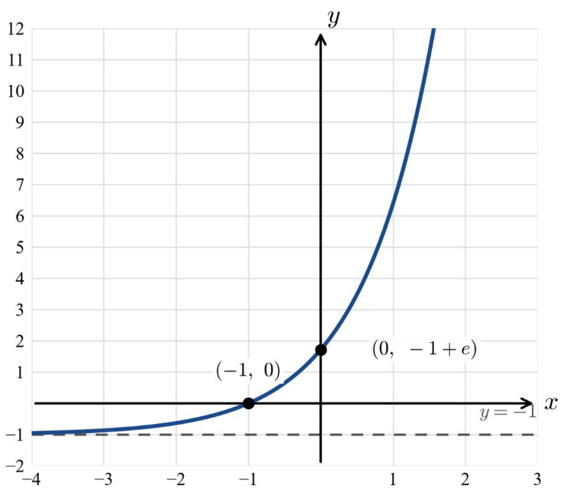
Q14. $y=e^{2x}-4$. Asymptote $y=-4$, range $\{y: y>-4\}$. y-intercept $(0,-3)$; x-intercept $e^{2x}=4 \Rightarrow 2x=\log_e 4 \Rightarrow x=1/2\log_e 4=\log_e 2$, giving $(\log_e 2, 0)$.



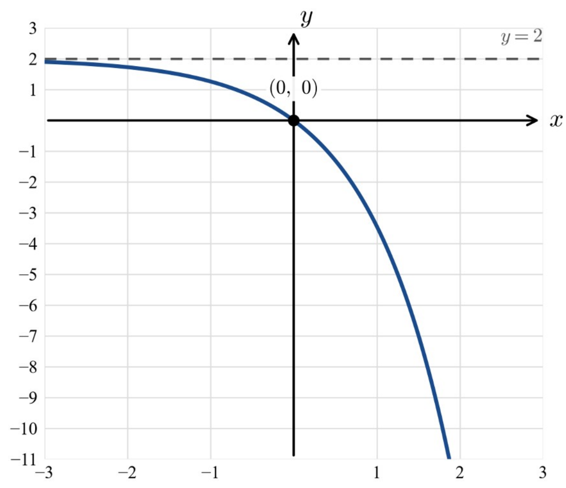
Q15. $y = 3e^x - 3$. Asymptote $y = -3$, range $\{y : y > -3\}$. y-intercept $x = 0 \Rightarrow y = 3 - 3 = 0$, giving $(0, 0)$; x-intercept $3e^x = 3 \Rightarrow e^x = 1 \Rightarrow x = 0$, also $(0, 0)$.



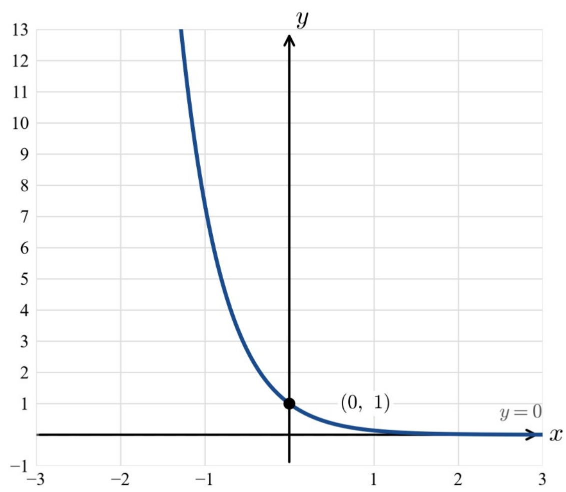
Q16. $y = e^{x+1} - 1$. Asymptote $y = -1$, range $\{y : y > -1\}$. y-intercept $x = 0 \Rightarrow y = e - 1$, giving $(0, e - 1)$; x-intercept $e^{x+1} = 1 \Rightarrow x + 1 = 0 \Rightarrow x = -1$, giving $(-1, 0)$.



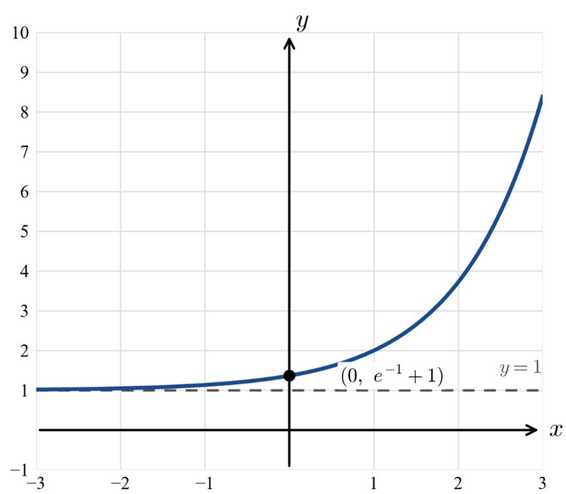
Q17. $y = 2 - 2e^x$. Asymptote $y = 2$, range $\{y : y < 2\}$; decreasing. y-intercept $(0, 0)$; x-intercept $2e^x = 2 \Rightarrow e^x = 1 \Rightarrow x = 0$, also $(0, 0)$.



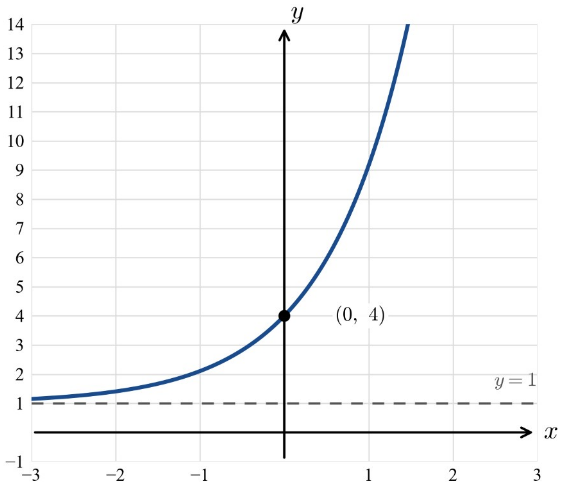
Q18. $y=e^{-2x}$. A reflection in the y-axis with a dilation of factor $1/2$ from the y-axis. Asymptote $y=0$, range $\{y: y>0\}$; decreasing. y-intercept $(0,1)$; no x-intercept.



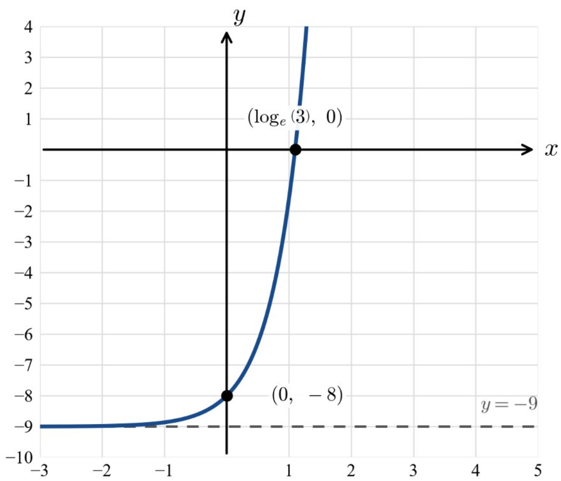
Q19. $y=e^{x-1}+1$. Asymptote $y=1$, range $\{y: y>1\}$. y-intercept $x=0 \Rightarrow y=e^{-1}+1$, giving $(0, e^{-1}+1)$. No x-intercept ($y>1$).



Q20. $y=3e^{x}+1$. From $y=e^{x}$: a **dilation of factor** 3 **from the** x**-axis**, then a **translation of** 1 **unit up**. Asymptote $y=1$, range $\{y: y>1\}$. y-intercept $x=0 \Rightarrow y=3+1=4$, giving $(0,4)$. No x-intercept, since $3e^{x}>0 \Rightarrow y>1$.



Q21. $y = e^{2x} - 9$. Setting $y = 0$: $e^{2x} - 9 = 0 \Rightarrow e^{2x} = 9 \Rightarrow 2x = \log_e 9 \Rightarrow x = 1/2 \log_e 9 = 1/2 \log_e 3^2 = \log_e 3$, as required, giving $(\log_e 3, 0)$. Asymptote $y = -9$, range $\{y : y > -9\}$; y-intercept $x = 0 \Rightarrow y = 1 - 9 = -8$, giving $(0, -8)$.



Theory

An **exponential equation** has the unknown in an index (exponent), for example $2^{x+1}=8^{x-1}$ or $e^{2x}-8e^{x}+7=0$. This section develops the by-hand techniques; §5G adds the use of logarithms.

The one-to-one property. For a fixed base $a>0$, $a\neq 1$, the function a^x is one-to-one, so

$$a^m=a^n \Leftrightarrow m=n.$$

To use this, **write both sides as powers of the same base**, then equate the indices.

Index laws (used to rewrite each side to a common base):

$$a^m a^n=a^{m+n},\frac{a^m}{a^n}=a^{m-n},\left(a^m\right)^n=a^{mn},a^{-n}=\frac{1}{a^n},a^0=1.$$

Equations reducible to a quadratic. An equation such as $e^{2x}-8e^{x}+7=0$ becomes a quadratic under the substitution $u=e^x$: since $e^{2x}=\left(e^x\right)^2=u^2$, it reads $u^2-8u+7=0$. Solve for u, then return to x with $e^x=u$.

A crucial check. For any base $a>0$, the value a^x is **always positive**. So after substituting $u=a^x$, any solution $u\leq 0$ **must be rejected** — only positive values of u give a value of x.

Method.

1. Use index laws to write each side (or each term) with a common base.
2. If the equation is linear in the index, equate indices and solve.
3. If it is quadratic in a^x, substitute $u=a^x$, solve the quadratic, **reject** $u\leq 0$, then solve $a^x=u$ for each remaining u.

Worked examples

Example 1 — scaffolded

Solve $2^{x+1}=8^{x-1}$ for x.

Working	Comment
$8=2^3$, so $8^{x-1}=\left(2^3\right)^{x-1}=2^{3(x-1)}=2^{3x-3}$.	Write both sides to the common base 2 using $\left(a^m\right)^n=a^{mn}$.
$2^{x+1}=2^{3x-3}$	Same base on both sides.
$x+1=3x-3$	Equate the indices (the one-to-one property).
$4=2x\Rightarrow x=2$	Solve the linear equation.
$\therefore x=2$	Check: $2^3=8$ and $8^1=8$. ✓

Example 2 — scaffolded

Solve $e^{2x}-8e^{x}+7=0$.

Working	Comment
Let $u=e^x$. Then $e^{2x}=u^2$, and note $u>0$.	Substitute to make the equation quadratic.
$u^2-8u+7=0$	The equation in u.
$(u-1)(u-7)=0\Rightarrow u=1$ or $u=7$	Factorise; both values are positive, so both are kept.

Working	Comment
$e^x = 1 \Rightarrow x = 0$; $e^x = 7 \Rightarrow x = \log_e 7$	Return to x. Recall $\log_e = \log_e$.
$\therefore x = 0$ or $x = \log_e 7$	Two exact solutions.

Example 3 — unscaffolded

Solve $9^x = 3^{x+1}$.

Write $9 = 3^2$, so $9^x = \left(3^2\right)^x = 3^{2x}$. The equation becomes $3^{2x} = 3^{x+1}$.

Equating indices: $2x = x + 1$, so $x = 1$.

$\therefore x = 1$.

Example 4 — unscaffolded

Solve $e^{2x} - 2e^x - 3 = 0$.

Let $u = e^x$ (so $u > 0$): $u^2 - 2u - 3 = 0 \Rightarrow (u-3)(u+1) = 0$, giving $u = 3$ or $u = -1$.

Since $u = e^x > 0$, **reject** $u = -1$. From $u = 3$: $e^x = 3 \Rightarrow x = \log_e 3$.

$\therefore x = \log_e 3$ (the only solution).

Practice questions

Scaffolded

Q1. Solve $3^x = 81$: (a) write 81 as a power of 3; (b) equate the indices; (c) state x.

Q2. Solve $2^{3x} = 16^{x-1}$: (a) write 16 as a power of 2 and simplify the right-hand side; (b) equate the indices; (c) solve for x.

Q3. Solve $e^{2x} - 4e^x + 3 = 0$: (a) with $u = e^x$, write the quadratic in u; (b) factorise and solve for u; (c) hence find all values of x.

Q4. Solve $5^{x^2} = 5^{3x-2}$: (a) equate the indices to form a quadratic in x; (b) solve the quadratic; (c) state the solutions.

Q5. Solve $e^{2x} - e^x - 6 = 0$: (a) substitute $u = e^x$ and solve for u; (b) explain which value of u must be rejected; (c) find x.

Q6. Solve $4^x - 2^{x+1} - 8 = 0$: (a) write 4^x and 2^{x+1} in terms of $u = 2^x$; (b) solve the quadratic in u; (c) hence find x.

Unscaffolded

Solve each equation for x, giving exact values.

Q7. $2^x = 32$

Q8. $3^{x-1} = 27$

Q9. $2^{2x+1} = 8^x$

Q10. $5^{2x} = 125^{x-1}$

Q11. $e^{2x} - 7e^x + 12 = 0$

Q12. $e^{2x} - 5e^x + 4 = 0$

Q13. $4^x - 9 \cdot 2^x + 8 = 0$

Q14. $9^x - 4 \cdot 3^x + 3 = 0$

Q15. $e^{2x} - e^x - 2 = 0$

Q16. $2^{x^2 - 1} = 8$

Q17. $3^{x^2} = 9^x$

Q18. $2^{2x} - 2^x - 12 = 0$

Q19. $25^x - 6 \cdot 5^x + 5 = 0$

Q20. Show that $e^{2x} - 4e^x + 4 = 0$ has exactly one solution, and find it.

Q21. Solve $2^{x+2} + 2^x = 40$.

Answers

Q1. $x=4$. **Q2.** $x=4$. **Q3.** $x=0$ or $x=\log_e 3$. **Q4.** $x=1$ or $x=2$. **Q5.** $x=\log_e 3$. **Q6.** $x=2$. **Q7.** $x=5$. **Q8.** $x=4$. **Q9.** $x=1$. **Q10.** $x=3$. **Q11.** $x=\log_e 3$ or $x=\log_e 4$. **Q12.** $x=0$ or $x=\log_e 4$. **Q13.** $x=0$ or $x=3$. **Q14.** $x=0$ or $x=1$. **Q15.** $x=\log_e 2$. **Q16.** $x=-2$ or $x=2$. **Q17.** $x=0$ or $x=2$. **Q18.** $x=2$. **Q19.** $x=0$ or $x=1$. **Q20.** $x=\log_e 2$ (a repeated solution). **Q21.** $x=3$.

Fully-worked solutions

Q7. $2^x=32=2^5$, so $x=5$.

Q8. $3^{x-1}=27=3^3$, so $x-1=3$, giving $x=4$.

Q9. $8^x=\left(2^3\right)^x=2^{3x}$, so $2^{2x+1}=2^{3x}\Rightarrow 2x+1=3x\Rightarrow x=1$.

Q10. $125^{x-1}=\left(5^3\right)^{x-1}=5^{3x-3}$, so $5^{2x}=5^{3x-3}\Rightarrow 2x=3x-3\Rightarrow x=3$.

Q11. Let $u=e^x>0$: $u^2-7u+12=(u-3)(u-4)=0$, so $u=3$ or $u=4$ (both positive). $e^x=3\Rightarrow x=\log_e 3$; $e^x=4\Rightarrow x=\log_e 4$. $\therefore x=\log_e 3$ or $x=\log_e 4$.

Q12. Let $u=e^x$: $u^2-5u+4=(u-1)(u-4)=0$, so $u=1$ or $u=4$. $e^x=1\Rightarrow x=0$; $e^x=4\Rightarrow x=\log_e 4$. $\therefore x=0$ or $x=\log_e 4$.

Q13. $4^x=\left(2^2\right)^x=\left(2^x\right)^2$. Let $u=2^x>0$: $u^2-9u+8=(u-1)(u-8)=0$, so $u=1$ or $u=8$. $2^x=1\Rightarrow x=0$; $2^x=8=2^3\Rightarrow x=3$. $\therefore x=0$ or $x=3$.

Q14. $9^x=\left(3^x\right)^2$. Let $u=3^x>0$: $u^2-4u+3=(u-1)(u-3)=0$, so $u=1$ or $u=3$. $3^x=1\Rightarrow x=0$; $3^x=3\Rightarrow x=1$. $\therefore x=0$ or $x=1$.

Q15. Let $u=e^x>0$: $u^2-u-2=(u-2)(u+1)=0$, so $u=2$ or $u=-1$. Reject $u=-1$ (since $e^x>0$). $e^x=2\Rightarrow x=\log_e 2$. $\therefore x=\log_e 2$.

Q16. $2^{x^2-1}=8=2^3$, so $x^2-1=3\Rightarrow x^2=4\Rightarrow x=\pm 2$. $\therefore x=-2$ or $x=2$.

Q17. $9^x=\left(3^2\right)^x=3^{2x}$, so $3^{x^2}=3^{2x}\Rightarrow x^2=2x\Rightarrow x^2-2x=0\Rightarrow x(x-2)=0$. $\therefore x=0$ or $x=2$.

Q18. $2^{2x}=\left(2^x\right)^2$. Let $u=2^x>0$: $u^2-u-12=(u-4)(u+3)=0$, so $u=4$ or $u=-3$. Reject $u=-3$. $2^x=4=2^2\Rightarrow x=2$. $\therefore x=2$.

Q19. $25^x=\left(5^x\right)^2$. Let $u=5^x>0$: $u^2-6u+5=(u-1)(u-5)=0$, so $u=1$ or $u=5$. $5^x=1\Rightarrow x=0$; $5^x=5\Rightarrow x=1$. $\therefore x=0$ or $x=1$.

Q20. Let $u=e^x>0$: $u^2-4u+4=(u-2)^2=0$, which has the single (repeated) root $u=2$. Since $u=2>0$ it is accepted, and $e^x=2\Rightarrow x=\log_e 2$. As there is only one value of u, and $e^x=u$ has exactly one solution for each $u>0$, the equation has **exactly one solution**, $x=\log_e 2$.

Q21. $2^{x+2}=2^2\cdot 2^x=4\cdot 2^x$, so $2^{x+2}+2^x=4\cdot 2^x+2^x=5\cdot 2^x$. Then $5\cdot 2^x=40\Rightarrow 2^x=8=2^3\Rightarrow x=3$. $\therefore x=3$.

Theory

The **logarithm** answers the question *"to what power must the base be raised to give this number?"*

Definition. For a base $a>0$, $a\neq 1$, and $x>0$,

$$\log_a(x)=y \Leftrightarrow a^y=x.$$

So $\log_a(x)$ is the index to which a must be raised to produce x. Because $a^y>0$ for every y, only **positive** numbers have logarithms.

Immediate consequences (from the definition):

- $\log_a(1)=0$ (since $a^0=1$) and $\log_a(a)=1$ (since $a^1=a$).
- $\log_a(a^n)=n$ for every real n, and $a^{\log_a(x)}=x$ for $x>0$. (Exponentials and logarithms of the same base are inverse operations.)

The natural logarithm is the logarithm to base e: $\log_e x=\log_e(x)$. Base 10 logarithms are often written $\log x$.

Logarithm laws. For $a>0$, $a\neq 1$, and $m,n>0$:

Law	Statement
Product	$\log_a(mn)=\log_a(m)+\log_a(n)$
Quotient	$\log_a\left(\frac{m}{n}\right)=\log_a(m)-\log_a(n)$
Power	$\log_a(m^p)=p\log_a(m)$

Each law is the logarithmic form of the corresponding index law; for example, $a^p a^q=a^{p+q}$ gives the product law.

Change of base. To evaluate a logarithm on a calculator, or to compare logarithms of different bases,

$$\log_a(x)=\frac{\log_b(x)}{\log_b(a)}=\frac{\log_e x}{\log_e a}.$$

In particular $\log_a(x)=\dfrac{1}{\log_x(a)}$.

Solving logarithmic equations.

1. Use the laws to combine every logarithm into a **single** logarithm on each side.
2. Convert to exponential form (or, if $\log_a(P)=\log_a(Q)$, equate $P=Q$).
3. Solve, then **check each solution in the original equation** — a value that makes the argument of any logarithm zero or negative must be rejected.

Worked examples

Example 1 — scaffolded

Evaluate (a) $\log_2(32)$, (b) $\log_3\left(\frac{1}{9}\right)$, (c) $\log_4(8)$.

Working	Comment
(a) $\log_2(32)$: since $32=2^5$, $\log_2(2^5)=5$.	Write the number as a power of the base, then read off the index.
(b) $\frac{1}{9}=3^{-2}$, so $\log_3\left(\frac{1}{9}\right)=-2$.	A reciprocal gives a negative index.
(c) $8=2^3$ and $4=2^2$, so $\log_4(8)=y$ means $4^y=8$, i.e. $2^{2y}=2^3$, giving $2y=3$, $y=\frac{3}{2}$.	When base and number are powers of a common number, equate indices.

Example 2 — scaffolded

Simplify $\log_2(12)+\log_2(8)-\log_2(3)$ to a single number.

Working	Comment
$=\log_2\left(\frac{12\times 8}{3}\right)$	Product law on the sum, quotient law on the difference.
$=\log_2(32)$	$\frac{12\times 8}{3}=32$.
$=5$	$32=2^5$.

Example 3 — unscaffolded

Express $\log_a\left(\frac{x^3\sqrt{y}}{z^2}\right)$ in terms of $\log_a(x)$, $\log_a(y)$ and $\log_a(z)$.

$$\log_a\left(\frac{x^3\sqrt{y}}{z^2}\right)=\log_a(x^3)+\log_a(y^{1/2})-\log_a(z^2)$$

Applying the power law to each term:

$$\therefore \log_a\left(\frac{x^3\sqrt{y}}{z^2}\right)=3\log_a(x)+1/2\log_a(y)-2\log_a(z).$$

Example 4 — unscaffolded

Solve $\log_2(x)+\log_2(x-2)=3$ for x.

Combining the left side with the product law: $\log_2(x(x-2))=3$.

Converting to exponential form: $x(x-2)=2^3=8$, so $x^2-2x-8=0$, giving $(x-4)(x+2)=0$.

Thus $x=4$ or $x=-2$. The original equation requires $x>0$ **and** $x-2>0$, i.e. $x>2$, so $x=-2$ is rejected.

$$\therefore x=4.$$

Practice questions

Scaffolded

Q1. Evaluate each logarithm. (a) $\log_5(\sqrt{5})$ (b) $\log_2\left(\frac{1}{16}\right)$ (c) $\log_8(2)$ (d) $\log_{25}(5)$

Q2. Simplify each expression to a single number. (a) $\log_5(50)-\log_5(2)$ (b) $2\log_3(6)-\log_3(4)$ (c) $1/2\log_2(36)-\log_2(3)$

Q3. Express in terms of $\log_a(x)$ and $\log_a(y)$. (a) $\log_a(x^2 y^3)$ (b) $\log_a\left(\frac{\sqrt{x}}{y}\right)$

Q4. Write each as a single logarithm. (a) $2\log_a(x)+3\log_a(y)$ (b) $1/2\log_a(x)-\log_a(y)$

Q5. Solve for x. (a) $\log_3(x+2)=2$ (b) $\log_e(x)=2$

Q6. (a) Using the change of base rule, express $\log_8(x)$ in terms of $\log_2$. (b) Hence evaluate $\log_8(2)$.

Unscaffolded

Q7. Evaluate $\log_{10}\left(\frac{1}{1000}\right)$.

Q8. Evaluate $\log_9(27)$.

Q9. Evaluate $\log_{1/2}(8)$.

Q10. Simplify $3\log_2(2)+\log_2(4)$ to a single number.

Q11. Express $\log_a\left(\frac{x^3\sqrt{y}}{z^2}\right)$ in terms of $\log_a(x)$, $\log_a(y)$ and $\log_a(z)$.

Q12. Write $1/2\log_a(x)-3\log_a(y)$ as a single logarithm.

Q13. Solve $\log_2(x+3)-\log_2(x-1)=2$.

Q14. Solve $\log_{10}(x)+\log_{10}(x-3)=1$.

Q15. Solve $2\log_2(x)-\log_2(x+3)=1$, giving the exact value.

Q16. Solve $\log_x(16)=2$.

Q17. Show that $\log_a(a^2 b)-\log_a(b)=2$ for all $a>0$, $a\neq 1$, $b>0$.

Q18. Given that $\log_{10}(2)=0.3010$ and $\log_{10}(3)=0.4771$ (to 4 decimal places), find, without a calculator, (a) $\log_{10}(6)$, (b) $\log_{10}(12)$, (c) $\log_{10}(1.5)$.

Q19. Solve $\log_3(x)+\log_3(x+6)=3$.

Q20. Solve $\log_2(x)=3+\log_2(5)$.

Q21. Solve $\log_5(x)+\log_5(x-4)=1$.

Answers

Q1. (a) $1/2$ (b) -4 (c) $1/3$ (d) $1/2$. **Q2.** (a) 2 (b) 2 (c) 1. **Q3.** (a) $2\log_a(x)+3\log_a(y)$ (b) $1/2\log_a(x)-\log_a(y)$. **Q4.** (a) $\log_a(x^2y^3)$ (b) $\log_a\left(\frac{\sqrt{x}}{y}\right)$. **Q5.** (a) $x=7$ (b) $x=e^2$. **Q6.** (a) $\log_8(x)=\frac{\log_2(x)}{3}$ (b) $1/3$. **Q7.** -3. **Q8.** $3/2$. **Q9.** -3. **Q10.** 5. **Q11.** $3\log_a(x)+1/2\log_a(y)-2\log_a(z)$. **Q12.** $\log_a\left(\frac{\sqrt{x}}{y^3}\right)$. **Q13.** $x=7/3$. **Q14.** $x=5$. **Q15.** $x=1+\sqrt{7}$. **Q16.** $x=4$. **Q17.** (proof — see solutions). **Q18.** (a) 0.7781 (b) 1.0791 (c) 0.1761. **Q19.** $x=3$. **Q20.** $x=40$. **Q21.** $x=5$.

Fully-worked solutions

Q1. (a) $\sqrt{5}=5^{1/2}$, so $\log_5(\sqrt{5})=1/2$. (b) $1/16=2^{-4}$, so $\log_2(1/16)=-4$. (c) $\log_8(2)=y\Rightarrow 8^y=2\Rightarrow 2^{3y}=2^1\Rightarrow y=1/3$. (d) $\log_{25}(5)=y\Rightarrow 25^y=5\Rightarrow 5^{2y}=5^1\Rightarrow y=1/2$.

Q2. (a) $\log_5(50)-\log_5(2)=\log_5(50/2)=\log_5(25)=2$. (b) $2\log_3(6)-\log_3(4)=\log_3(36)-\log_3(4)=\log_3(9)=2$. (c) $1/2\log_2(36)-\log_2(3)=\log_2(6)-\log_2(3)=\log_2(2)=1$.

Q3. (a) $\log_a(x^2y^3)=\log_a(x^2)+\log_a(y^3)=2\log_a(x)+3\log_a(y)$. (b) $\log_a(\sqrt{x}/y)=\log_a(x^{1/2})-\log_a(y)=1/2\log_a(x)-\log_a(y)$.

Q4. (a) $2\log_a(x)+3\log_a(y)=\log_a(x^2)+\log_a(y^3)=\log_a(x^2y^3)$. (b) $1/2\log_a(x)-\log_a(y)=\log_a(x^{1/2})-\log_a(y)=\log_a(\sqrt{x}/y)$.

Q5. (a) $\log_3(x+2)=2\Rightarrow x+2=3^2=9\Rightarrow x=7$ (and $x+2=9>0$ ✓). (b) $\log_e(x)=2\Rightarrow x=e^2$.

Q6. (a) By the change of base rule, $\log_8(x)=\frac{\log_2(x)}{\log_2(8)}=\frac{\log_2(x)}{3}$. (b) $\log_8(2)=\frac{\log_2(2)}{3}=\frac{1}{3}$.

Q7. $1/1000=10^{-3}$, so $\log_{10}(1/1000)=-3$.

Q8. $\log_9(27)=y\Rightarrow 9^y=27\Rightarrow 3^{2y}=3^3\Rightarrow 2y=3\Rightarrow y=3/2$.

Q9. $\log_{1/2}(8)=y\Rightarrow(1/2)^y=8\Rightarrow 2^{-y}=2^3\Rightarrow -y=3\Rightarrow y=-3$.

Q10. $3\log_2(2)+\log_2(4)=3(1)+2=5$.

Q11. $\log_a\left(\frac{x^3\sqrt{y}}{z^2}\right)=\log_a(x^3)+\log_a(y^{1/2})-\log_a(z^2)=3\log_a(x)+1/2\log_a(y)-2\log_a(z)$.

Q12. $1/2\log_a(x)-3\log_a(y)=\log_a(x^{1/2})-\log_a(y^3)=\log_a\left(\frac{\sqrt{x}}{y^3}\right)$.

Q13. $\log_2(x+3)-\log_2(x-1)=2\Rightarrow\log_2\left(\frac{x+3}{x-1}\right)=2\Rightarrow\frac{x+3}{x-1}=4$. So $x+3=4(x-1)=4x-4\Rightarrow 3x=7\Rightarrow x=7/3$ (and $x>1$ ✓).

Q14. $\log_{10}(x)+\log_{10}(x-3)=1\Rightarrow\log_{10}(x(x-3))=1\Rightarrow x(x-3)=10$. So $x^2-3x-10=0\Rightarrow(x-5)(x+2)=0$, giving $x=5$ or $x=-2$. Require $x>3$, so $x=5$.

Q15. $2\log_2(x)-\log_2(x+3)=1 \Rightarrow \log_2\left(\frac{x^2}{x+3}\right)=1 \Rightarrow \frac{x^2}{x+3}=2$. So $x^2=2(x+3) \Rightarrow x^2-2x-6=0 \Rightarrow x=\frac{2\pm\sqrt{4+24}}{2}=1\pm\sqrt{7}$. Require $x>0$, so $x=1+\sqrt{7}$.

Q16. $\log_x(16)=2 \Rightarrow x^2=16 \Rightarrow x=\pm 4$. A logarithm base must satisfy $x>0$, $x\neq 1$, so $x=4$.

Q17. $\log_a(a^2 b)-\log_a(b)=\log_a\left(\frac{a^2 b}{b}\right)=\log_a(a^2)=2\log_a(a)=2$, as required.

Q18. (a) $\log_{10}(6)=\log_{10}(2\times 3)=\log_{10}(2)+\log_{10}(3)=0.3010+0.4771=0.7781$. (b) $\log_{10}(12)=\log_{10}(2^2\times 3)=2\log_{10}(2)+\log_{10}(3)=0.6020+0.4771=1.0791$. (c) $\log_{10}(1.5)=\log_{10}(3/2)=\log_{10}(3)-\log_{10}(2)=0.4771-0.3010=0.1761$.

Q19. $\log_3(x)+\log_3(x+6)=3 \Rightarrow \log_3(x(x+6))=3 \Rightarrow x(x+6)=27$. So $x^2+6x-27=0 \Rightarrow (x+9)(x-3)=0$, giving $x=-9$ or $x=3$. Require $x>0$, so $x=3$.

Q20. $\log_2(x)=3+\log_2(5) \Rightarrow \log_2(x)-\log_2(5)=3 \Rightarrow \log_2\left(\frac{x}{5}\right)=3 \Rightarrow \frac{x}{5}=2^3=8 \Rightarrow x=40$.

Q21. $\log_5(x)+\log_5(x-4)=1 \Rightarrow \log_5(x(x-4))=1 \Rightarrow x(x-4)=5$. So $x^2-4x-5=0 \Rightarrow (x-5)(x+1)=0$, giving $x=5$ or $x=-1$. Require $x>4$, so $x=5$.

Chapter 5 Exponential and logarithmic functions — 5E Logarithmic functions and their graphs

Theory

The logarithmic function $y=\log_a(x)$ (with base $a>1$) is the **inverse** of the exponential function $y=a^x$. Its graph has these key features:

- a **vertical asymptote** at $x=0$ (the y-axis): as $x \to 0^+$, $y \to -\infty$;
- an x**-intercept** at $(1,0)$, and it passes through $(a,1)$;
- **domain** $\{x : x>0\}$ and **range** $\mathbb{R}$;
- it is **strictly increasing** (slowly — $y \to \infty$ as $x \to \infty$).

The **natural logarithm** $y=\log_e(x)$ (base $e \approx 2.718$) has the same shape and passes through $(e,1)$.

The transformed graph. Applying dilations, reflections and translations to $y=\log_a(x)$ gives graphs of the form $y=A\log_a(b(x-h))+k$. Reading the features:

Feature	How to find it
Vertical asymptote	set the argument of the log to 0: $b(x-h)=0$, i.e. $x=h$
Domain	the argument must be **positive**: $x>h$ (or $x<h$ if the graph is reflected in the y-axis)
x**-intercept**	put $y=0$ and solve $\log_a(\text{arg})=-\dfrac{k}{A}$, so $\text{arg}=a^{-k/A}$
y**-intercept**	substitute $x=0$ — **only if** $x=0$ lies in the domain
Range	always $\mathbb{R}$

- The **horizontal translation** h moves the asymptote to $x=h$.
- A negative coefficient **outside**, $y=-A\log_a(\dots)$, is a **reflection in the** x**-axis** (the graph decreases). A negative coefficient **inside**, e.g. $y=\log_a(h-x)$, is a **reflection in the** y**-axis**: the asymptote is still $x=h$ but the domain becomes $x<h$.

To sketch: (1) find and draw the asymptote (dashed); (2) state the domain; (3) find the exact x-intercept; (4) find the y-intercept if $x=0$ is in the domain; (5) note whether the graph increases or decreases; (6) draw the curve rising/falling steeply near the asymptote.

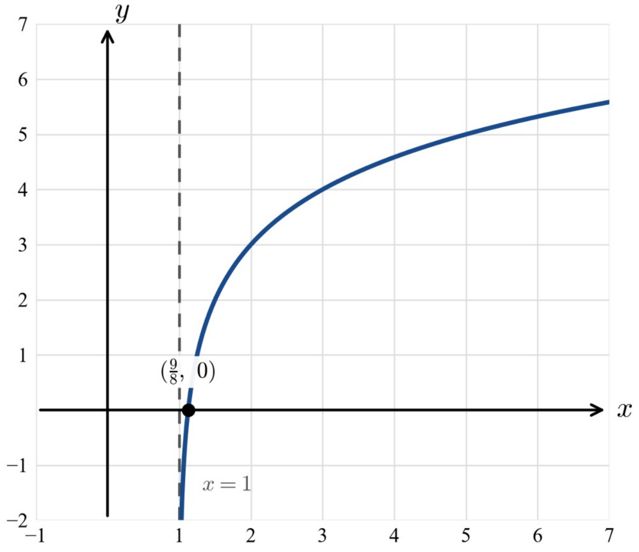
Worked examples

Example 1 — scaffolded

Sketch the graph of $y=\log_2(x-1)+3$, stating the asymptote, domain and range and the exact axis intercepts.

Working	Comment
Base graph $y=\log_2 x$, translated. Argument $x-1=0 \Rightarrow x=1$.	The asymptote is where the argument is zero.
Asymptote $x=1$; **domain** $\{x : x>1\}$; **range** $\mathbb{R}$.	The argument $x-1$ must be positive.
x**-intercept:**	Isolate the log, then rewrite in index form. Keep it exact.
$\log_2(x-1)+3=0 \Rightarrow \log_2(x-1)=-3 \Rightarrow x-1=2^{-3}=\dfrac{1}{8}$	

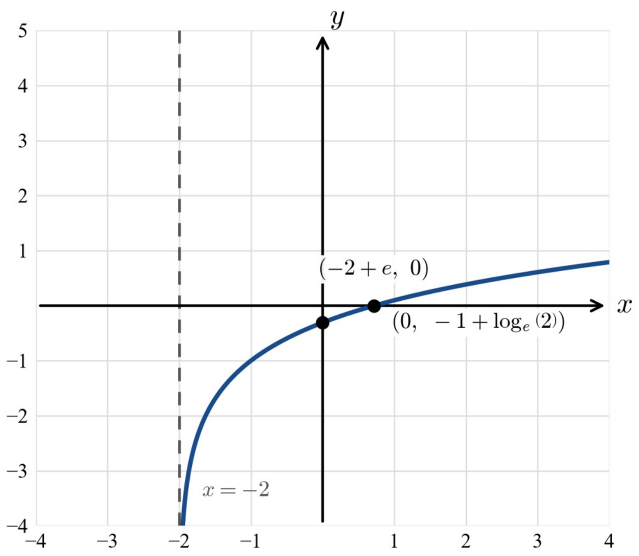
Working	Comment
, so $x=\frac{9}{8}$, giving $\left(\frac{9}{8},0\right)$.	
No y-intercept, since $x=0$ is not in the domain.	$x=0<1$.
Check: when $x-1=1$, i.e. $x=2$, $y=\log_2 1+3=3$, so the graph passes through $(2,3)$.	A convenient point for the sketch.



Example 2 — scaffolded

Sketch the graph of $y=\log_e(x+2)-1$, labelling the asymptote and the exact axis intercepts.

Working	Comment
Argument $x+2=0\Rightarrow x=-2$. **Asymptote** $x=-2$; **domain** $\{x:x>-2\}$.	Natural-log graph translated 2 left and 1 down.
x-intercept: $\log_e(x+2)-1=0\Rightarrow\log_e(x+2)=1\Rightarrow x+2=e$, so $x=e-2$, giving $(e-2,0)$.	$\log_e(x+2)=1$ means $x+2=e^1$.
y-intercept: $x=0$ is in the domain, so $y=\log_e 2-1$, giving $(0,\log_e 2-1)$.	Substitute $x=0$; leave the answer exact.



Example 3 — unscaffolded

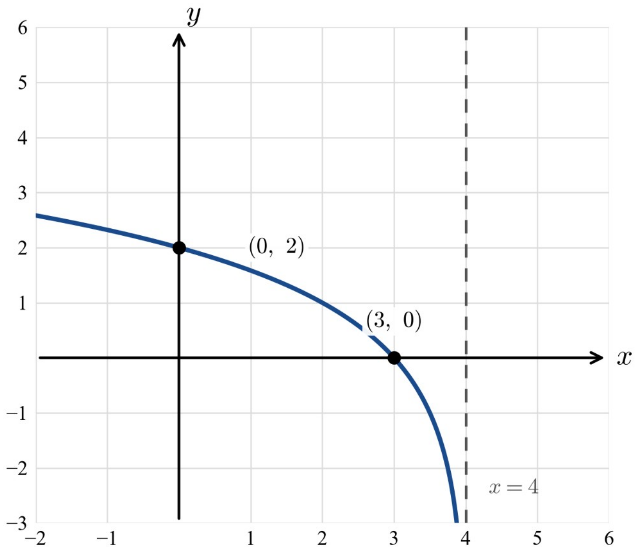
Sketch the graph of $y = \log_2(4 - x)$, stating the domain and the exact axis intercepts.

The argument is $4 - x$, so this is $y = \log_2 x$ **reflected in the y-axis** and translated. Argument $4 - x = 0 \Rightarrow x = 4$: asymptote $x = 4$, and since $4 - x > 0$ requires $x < 4$, the domain is $\{x : x < 4\}$ (the graph lies to the **left** of the asymptote and decreases).

x-intercept: $\log_2(4 - x) = 0 \Rightarrow 4 - x = 1 \Rightarrow x = 3$, giving $(3, 0)$.

y-intercept: $x = 0$ is in the domain, so $y = \log_2 4 = 2$, giving $(0, 2)$.

∴ the curve falls from top-left, through $(0, 2)$ and $(3, 0)$, down the asymptote $x = 4$.



Example 4 — unscaffolded

Sketch the graph of $y = 3 - 2\log_2(x)$, stating the asymptote, domain and the exact x-intercept.

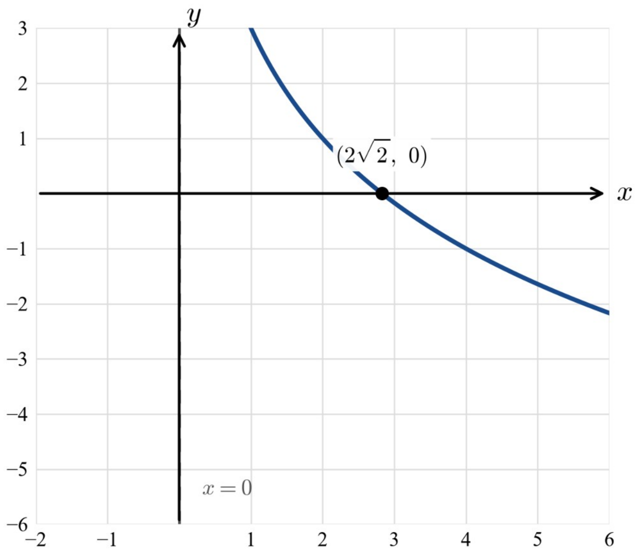
Rewrite as $y = -2\log_2(x) + 3$: the base graph $y = \log_2 x$ is dilated by factor 2 from the x-axis, **reflected in the x-axis** ($-2 < 0$, so the graph decreases) and translated 3 up.

Asymptote $x = 0$; domain $\{x : x > 0\}$; range $\mathbb{R}$.

x-intercept: $3 - 2\log_2 x = 0 \Rightarrow \log_2 x = \dfrac{3}{2} \Rightarrow x = 2^{3/2} = 2\sqrt{2}$, giving $(2\sqrt{2}, 0)$.

No y-intercept ($x = 0$ is not in the domain). The graph passes through $(1, 3)$.

∴ the curve falls from top-left near $x = 0$, through $(1, 3)$ and $(2\sqrt{2}, 0)$.



Practice questions

Scaffolded

Q1. For $y = \log_2(x) + 2$: (a) state the equation of the asymptote and the domain; (b) find the exact x-intercept; (c) state the range; (d) sketch the graph.

Q2. For $y = \log_e(x - 3)$: (a) state the asymptote and domain; (b) find the exact axis intercept(s); (c) sketch the graph.

Q3. For $y = \log_2(x + 1) - 3$: (a) state the asymptote and domain; (b) find the exact axis intercepts; (c) sketch the graph.

Q4. For $y = -\log_2(x)$: (a) state the asymptote and domain; (b) explain why the graph is **decreasing**; (c) find the x-intercept; (d) sketch the graph.

Q5. For $y = \log_e(2x)$: (a) state the asymptote and domain; (b) find the exact x-intercept; (c) sketch the graph.

Q6. For $y = \log_2(x - 2) + 1$: (a) state the asymptote and domain; (b) find the exact x-intercept; (c) sketch the graph.

Unscaffolded

Sketch each of the following, labelling the asymptote and the exact coordinates of all axis intercepts, and state the domain and range.

Q7. $y = \log_2(x) - 3$

Q8. $y = \log_e(x + 1)$

Q9. $y = \log_2(x - 4) + 2$

Q10. $y = 1 - \log_e(x)$

Q11. $y = \log_2(-x)$

Q12. $y = \log_e(x) + 2$

Q13. $y = 2\log_2(x)$

Q14. $y = \log_2(x + 8) - 3$

Q15. $y = \log_e(3 - x)$

Q16. $y = 1 - \log_2(x)$

Q17. $y = \log_2(2x - 4)$

Q18. $y = 3\log_e(x) - 3$

Q19. $y = \log_2(4x)$

Q20. Describe the sequence of transformations that maps $y = \log_2(x)$ onto $y = -\log_2(x-1) + 2$, and hence sketch the graph.

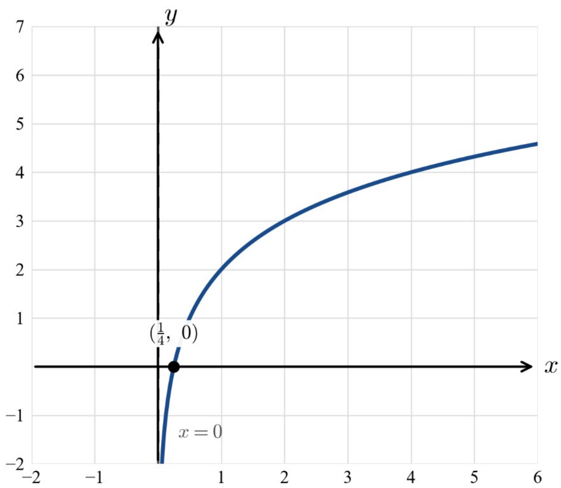
Q21. For $y = \log_e(2x+6) - 2$, state the maximal domain and the equation of the asymptote, find the exact coordinates of the axis intercepts, and hence sketch the graph.

Answers

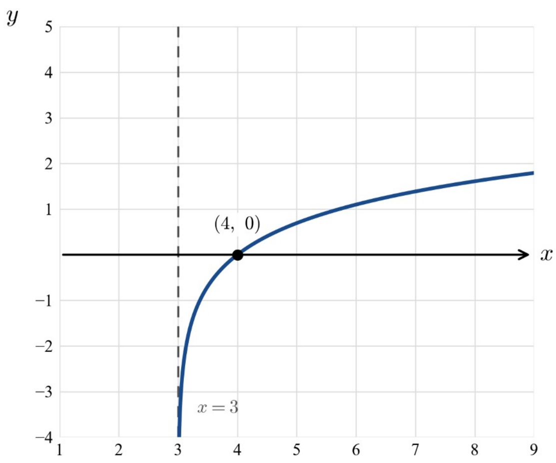
Q1. Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $\left(\frac{1}{4},0\right)$; range $\mathbb{R}$. **Q2.** Asymptote $x=3$, domain $\{x:x>3\}$; x-intercept $(4,0)$ (no y-intercept). **Q3.** Asymptote $x=-1$, domain $\{x:x>-1\}$; intercepts $(7,0)$ and $(0,-3)$. **Q4.** Asymptote $x=0$, domain $\{x:x>0\}$; decreasing; x-intercept $(1,0)$. **Q5.** Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $\left(\frac{1}{2},0\right)$. **Q6.** Asymptote $x=2$, domain $\{x:x>2\}$; x-intercept $\left(\frac{5}{2},0\right)$. **Q7.** Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $(8,0)$. **Q8.** Asymptote $x=-1$, domain $\{x:x>-1\}$; intercept $(0,0)$. **Q9.** Asymptote $x=4$, domain $\{x:x>4\}$; x-intercept $\left(\frac{17}{4},0\right)$. **Q10.** Asymptote $x=0$, domain $\{x:x>0\}$; decreasing; x-intercept $(e,0)$. **Q11.** Asymptote $x=0$, domain $\{x:x<0\}$; x-intercept $(-1,0)$. **Q12.** Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $(e^{-2},0)$. **Q13.** Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $(1,0)$. **Q14.** Asymptote $x=-8$, domain $\{x:x>-8\}$; intercept $(0,0)$. **Q15.** Asymptote $x=3$, domain $\{x:x<3\}$; intercepts $(2,0)$ and $(0,\log_e 3)$. **Q16.** Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $(2,0)$. **Q17.** Asymptote $x=2$, domain $\{x:x>2\}$; x-intercept $\left(\frac{5}{2},0\right)$. **Q18.** Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $(e,0)$. **Q19.** Asymptote $x=0$, domain $\{x:x>0\}$; x-intercept $\left(\frac{1}{4},0\right)$. **Q20.** Reflection in the x-axis, then translation 1 right and 2 up; asymptote $x=1$, domain $\{x:x>1\}$; x-intercept $(5,0)$. **Q21.** Domain $\{x:x>-3\}$, asymptote $x=-3$; intercepts $\left(\frac{e^2-6}{2},0\right)$ and $(0,\log_e 6-2)$.

Fully-worked solutions

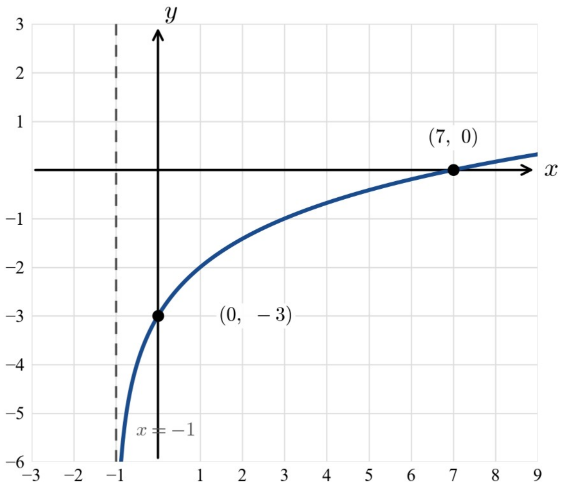
Q1. $y=\log_2 x+2$. Asymptote $x=0$, domain $\{x:x>0\}$, range $\mathbb{R}$. x-intercept: $\log_2 x=-2 \Rightarrow x=2^{-2}=\frac{1}{4}$, giving $\left(\frac{1}{4},0\right)$.



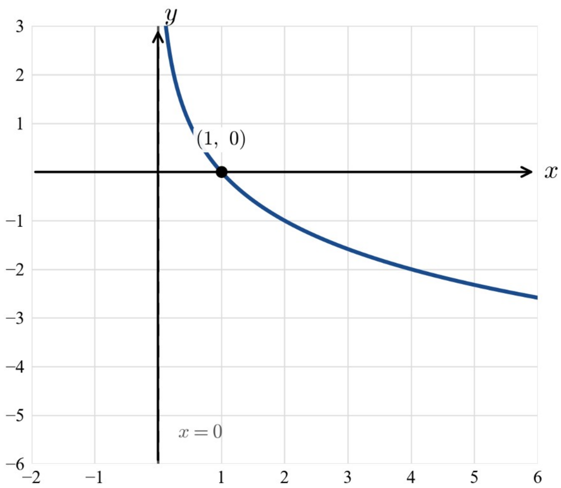
Q2. $y=\log_e(x-3)$. Asymptote $x=3$, domain $\{x:x>3\}$. x-intercept: $\log_e(x-3)=0 \Rightarrow x-3=1 \Rightarrow x=4$, giving $(4,0)$. No y-intercept ($x=0<3$).



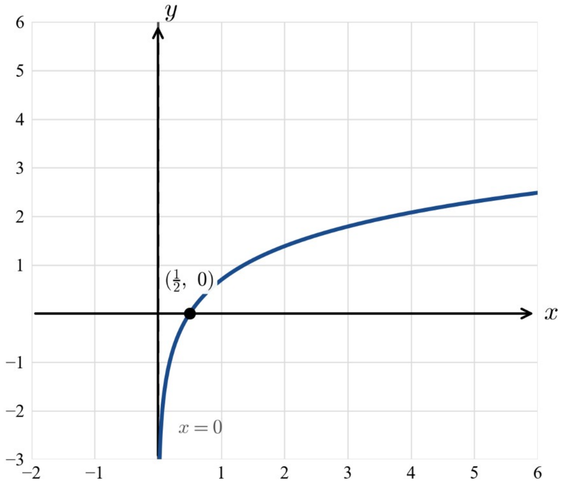
Q3. $y=\log_2(x+1)-3$. Asymptote $x=-1$, domain $\{x : x>-1\}$. x-intercept: $\log_2(x+1)=3 \Rightarrow x+1=8 \Rightarrow x=7$, giving $(7,0)$. y-intercept: $x=0 \Rightarrow y=\log_2 1-3=-3$, giving $(0,-3)$.



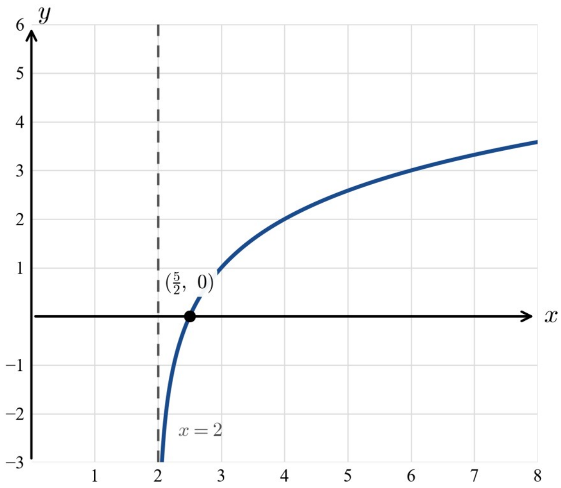
Q4. $y=-\log_2 x$. Asymptote $x=0$, domain $\{x : x>0\}$. The factor -1 reflects $y=\log_2 x$ in the x-axis, so the graph is **decreasing**. x-intercept: $-\log_2 x=0 \Rightarrow \log_2 x=0 \Rightarrow x=1$, giving $(1,0)$.



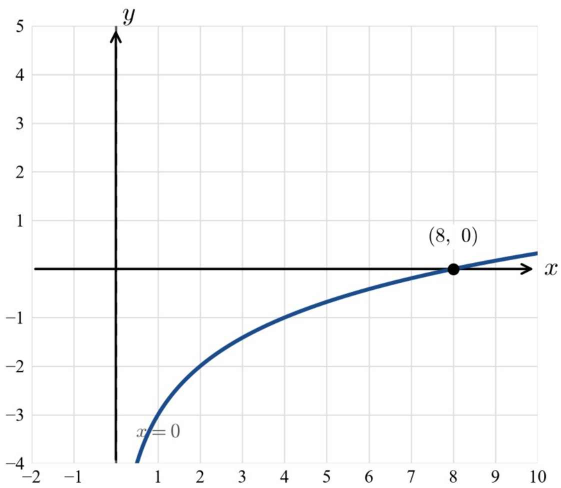
Q5. $y=\log_e(2x)$. Asymptote $2x=0 \Rightarrow x=0$, domain $\{x : x>0\}$. x-intercept: $\log_e(2x)=0 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$, giving $\left(\frac{1}{2},0\right)$.



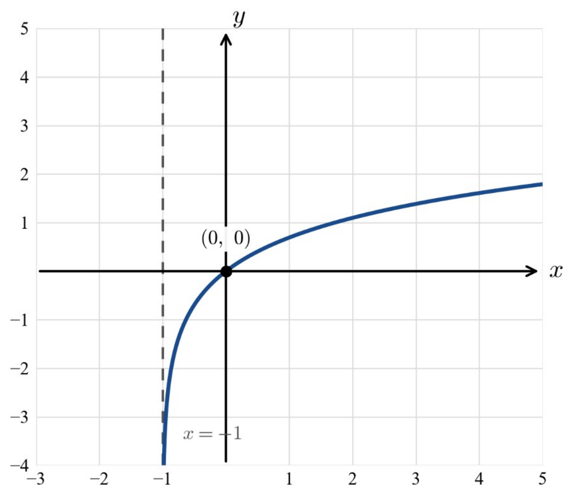
Q6. $y=\log_2(x-2)+1$. Asymptote $x=2$, domain $\{x : x>2\}$. x-intercept: $\log_2(x-2)=-1 \Rightarrow x-2=2^{-1}=\dfrac{1}{2} \Rightarrow x=\dfrac{5}{2}$, giving $\left(\dfrac{5}{2},0\right)$.



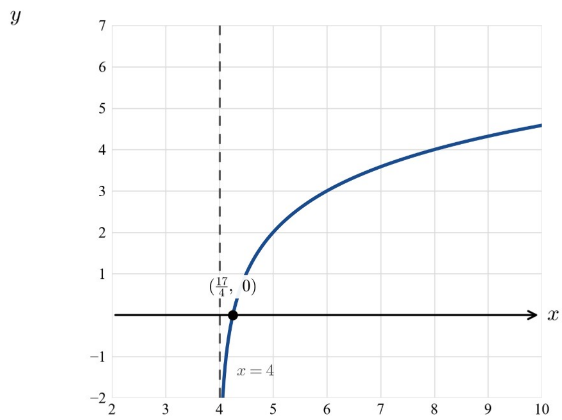
Q7. $y=\log_2 x-3$. Asymptote $x=0$, domain $\{x : x>0\}$. x-intercept: $\log_2 x=3 \Rightarrow x=8$, giving $(8,0)$.



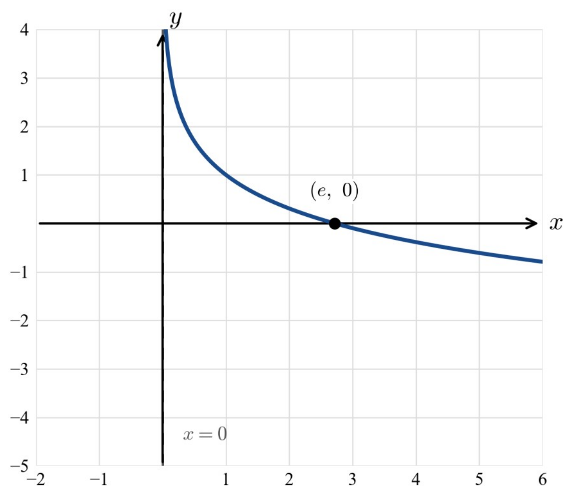
Q8. $y=\log_e(x+1)$. Asymptote $x=-1$, domain $\{x : x>-1\}$. x-intercept: $x+1=1 \Rightarrow x=0$, giving $(0,0)$, which is also the y-intercept.



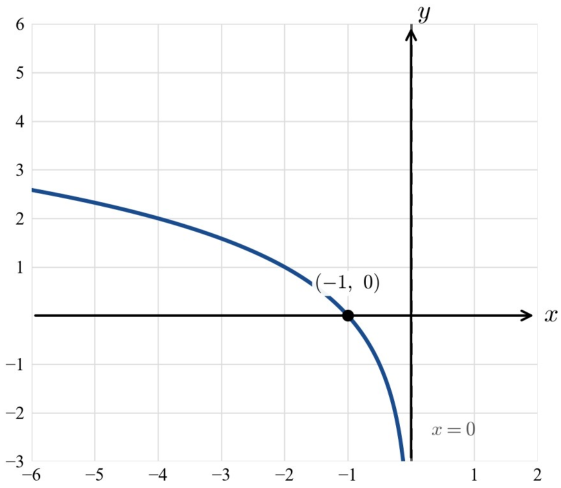
Q9. $y=\log_2(x-4)+2$. Asymptote $x=4$, domain $\{x : x>4\}$. x-intercept: $\log_2(x-4)=-2 \Rightarrow x-4=\frac{1}{4} \Rightarrow x=\frac{17}{4}$, giving $\left(\frac{17}{4},0\right)$.



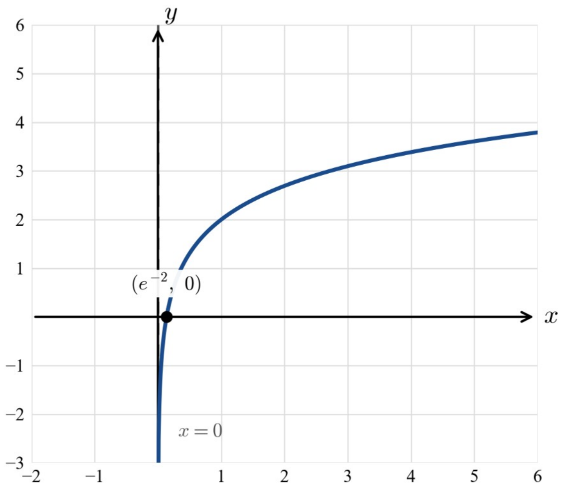
Q10. $y=1-\log_e x$. Asymptote $x=0$, domain $\{x : x>0\}$; the $-\log_e x$ term makes it **decreasing**. x-intercept: $\log_e x=1 \Rightarrow x=e$, giving $(e,0)$.



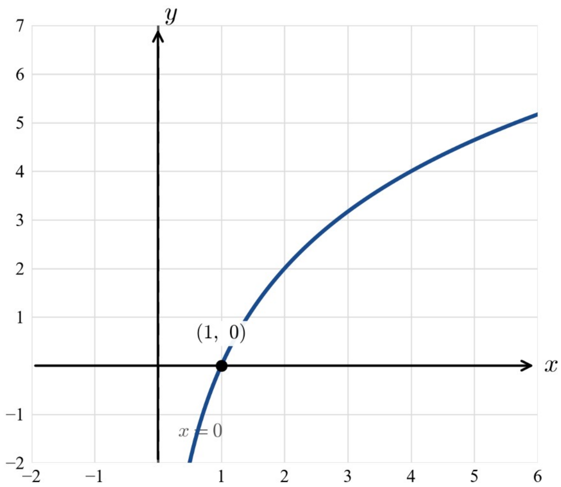
Q11. $y=\log_2(-x)$. Argument $-x>0 \Rightarrow x<0$: asymptote $x=0$, domain $\{x : x<0\}$ (reflection in the y-axis). x-intercept: $-x=1 \Rightarrow x=-1$, giving $(-1,0)$.



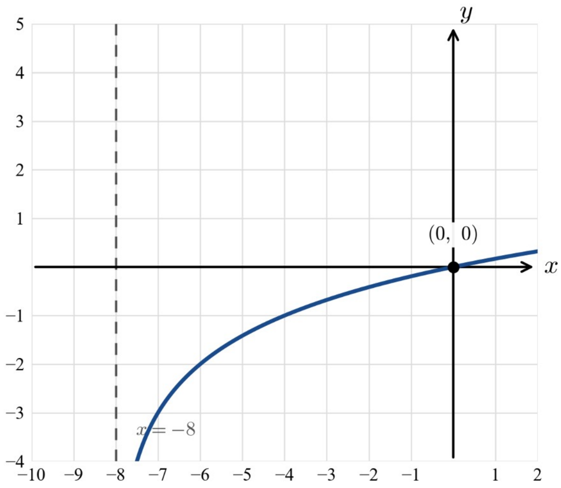
Q12. $y=\log_e x+2$. Asymptote $x=0$, domain $\{x:x>0\}$. x-intercept: $\log_e x=-2\Rightarrow x=e^{-2}$, giving $\left(e^{-2},0\right)$.



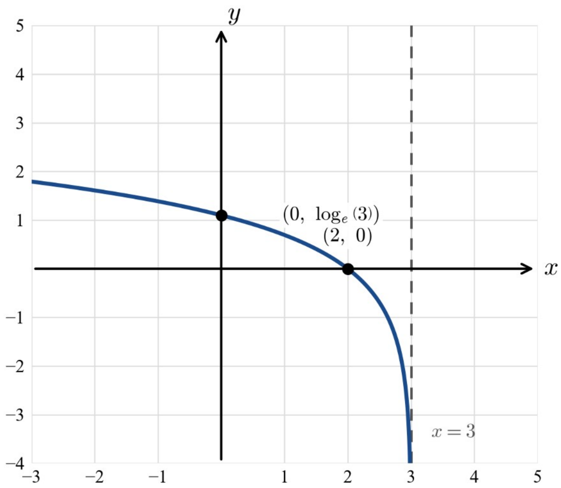
Q13. $y=2\log_2 x$. Asymptote $x=0$, domain $\{x:x>0\}$ (a dilation of factor 2 from the x-axis). x-intercept: $\log_2 x=0\Rightarrow x=1$, giving $(1,0)$.



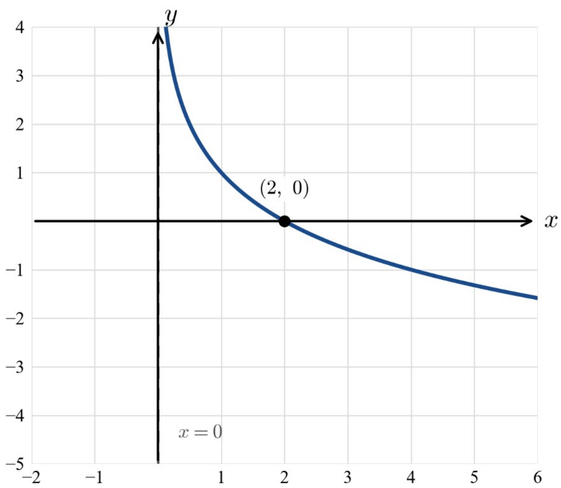
Q14. $y=\log_2(x+8)-3$. Asymptote $x=-8$, domain $\{x:x>-8\}$. x-intercept: $\log_2(x+8)=3\Rightarrow x+8=8\Rightarrow x=0$, giving $(0,0)$ (also the y-intercept).



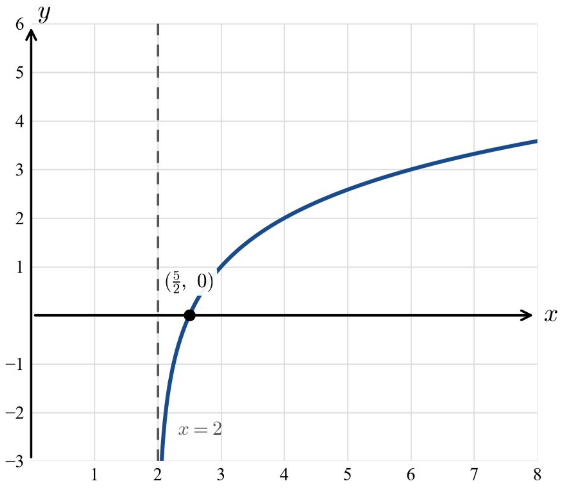
Q15. $y=\log_e(3-x)$. Argument $3-x>0 \Rightarrow x<3$: asymptote $x=3$, domain $\{x: x<3\}$ (reflection in the y-axis). x-intercept: $3-x=1 \Rightarrow x=2$, giving $(2,0)$. y-intercept: $x=0 \Rightarrow y=\log_e 3$, giving $(0, \log_e 3)$.



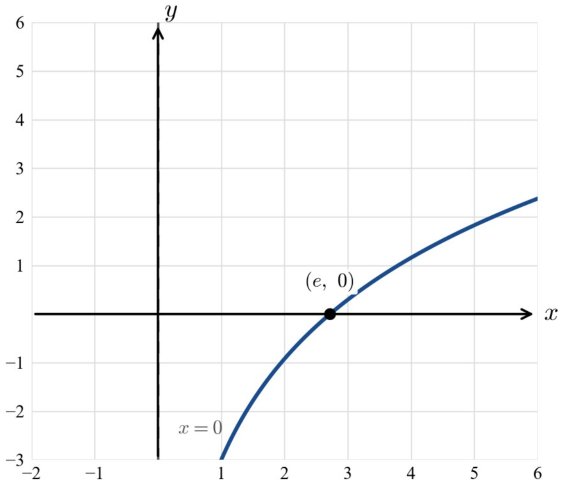
Q16. $y=1-\log_2 x$. Asymptote $x=0$, domain $\{x: x>0\}$; decreasing. x-intercept: $\log_2 x=1 \Rightarrow x=2$, giving $(2,0)$.



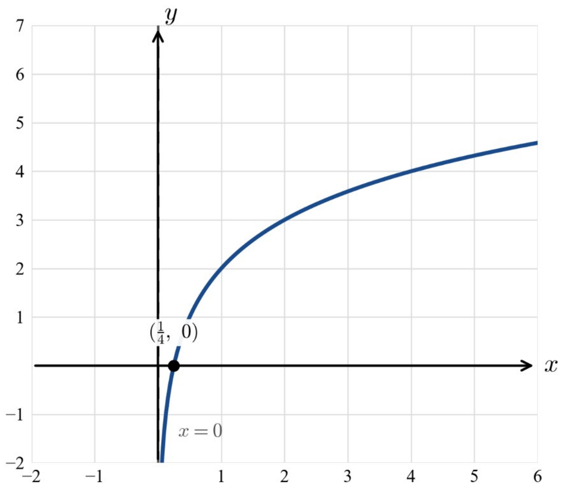
Q17. $y=\log_2(2x-4)$. Argument $2x-4=0 \Rightarrow x=2$: asymptote $x=2$, domain $\{x: x>2\}$. x-intercept: $2x-4=1 \Rightarrow x=\frac{5}{2}$, giving $\left(\frac{5}{2}, 0\right)$.



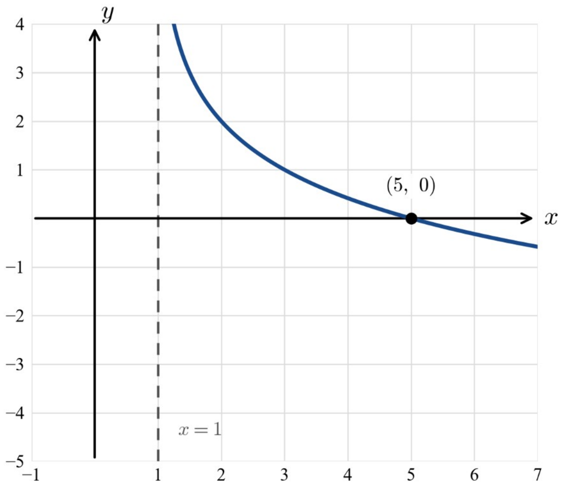
Q18. $y = 3\log_e x - 3$. Asymptote $x = 0$, domain $\{x : x > 0\}$. x-intercept: $3\log_e x = 3 \Rightarrow \log_e x = 1 \Rightarrow x = e$, giving $(e, 0)$.



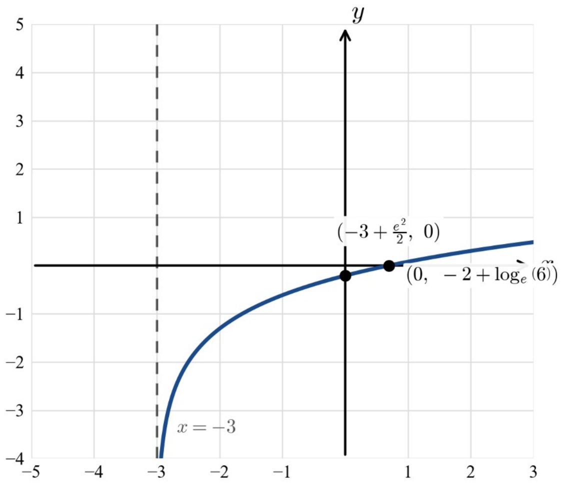
Q19. $y = \log_2(4x)$. Asymptote $x = 0$, domain $\{x : x > 0\}$. x-intercept: $4x = 1 \Rightarrow x = \frac{1}{4}$, giving $\left(\frac{1}{4}, 0\right)$. (Note $\log_2(4x) = 2 + \log_2 x$, a translation 2 up.)



Q20. $y = -\log_2(x-1) + 2$. From $y = \log_2 x$: a **reflection in the x-axis** (the $-$ sign), then a **translation of 1 unit right and 2 units up**. Asymptote $x = 1$, domain $\{x : x > 1\}$; the graph decreases. x-intercept: $-\log_2(x-1) + 2 = 0 \Rightarrow \log_2(x-1) = 2 \Rightarrow x - 1 = 4 \Rightarrow x = 5$, giving $(5, 0)$.



Q21. $y = \log_e(2x+6) - 2$. Argument $2x+6>0 \Rightarrow x>-3$: **domain** $\{x : x>-3\}$, **asymptote** $x=-3$. x-intercept: $\log_e(2x+6)=2 \Rightarrow 2x+6=e^2 \Rightarrow x=\dfrac{e^2-6}{2}$, giving $\left(\dfrac{e^2-6}{2}, 0\right)$. y-intercept: $x=0 \Rightarrow y=\log_e 6-2$, giving $(0, \log_e 6-2)$.



Chapter 5 Exponential and logarithmic functions — 5F Finding exponential and logarithmic rules

Theory

When a graph is known to be exponential or logarithmic, its rule can be recovered from a few key features — an **asymptote** and one or two **points**.

Exponential graphs $y = a\,b^x + c$ (with $b > 0$, $b \neq 1$).

- The horizontal **asymptote** is $y = c$, so read c straight from the asymptote.
- Substitute a point to find the rest. The y**-intercept** is convenient: at $x = 0$, $b^0 = 1$, so $y = a + c$, giving a immediately.
- Use a second point to find the base b: substitute and solve $b^x = \dfrac{y - c}{a}$.

A base $0 < b < 1$ describes **decay**; $b > 1$ describes **growth**; $a < 0$ reflects the graph in its asymptote.

Logarithmic graphs $y = a\log_b(x - h) + k$ (base b given).

- The vertical **asymptote** is $x = h$, so read h from the asymptote.
- At the point where $x - h = 1$ we have $\log_b(1) = 0$, so $y = k$ there — this gives k.
- Substitute a second point and use $\log_b(b^n) = n$ to find a.

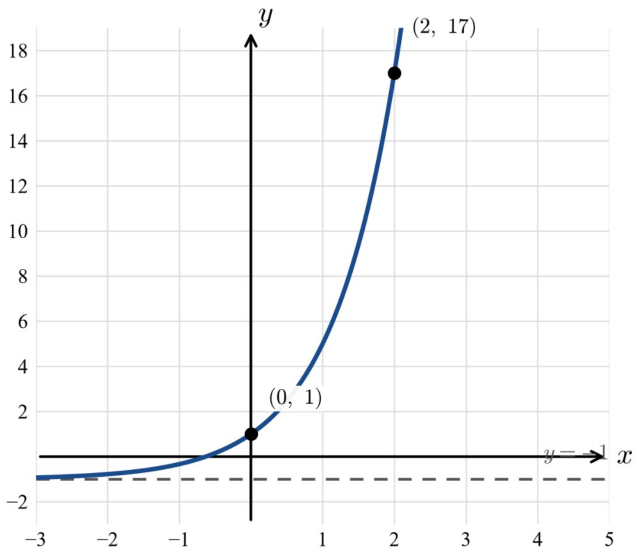
Method (both cases): (1) read the constant from the asymptote; (2) use the simplest point to find the leading coefficient; (3) use the remaining point to find the base (exponential) or the dilation a (logarithm); (4) state the rule and check it against every given point.

Worked examples

Example 1 — scaffolded

The graph of $y = a\,b^x + c$ has horizontal asymptote $y = -1$ and passes through $(0, 1)$ and $(2, 17)$. Determine the values of a, b and c.

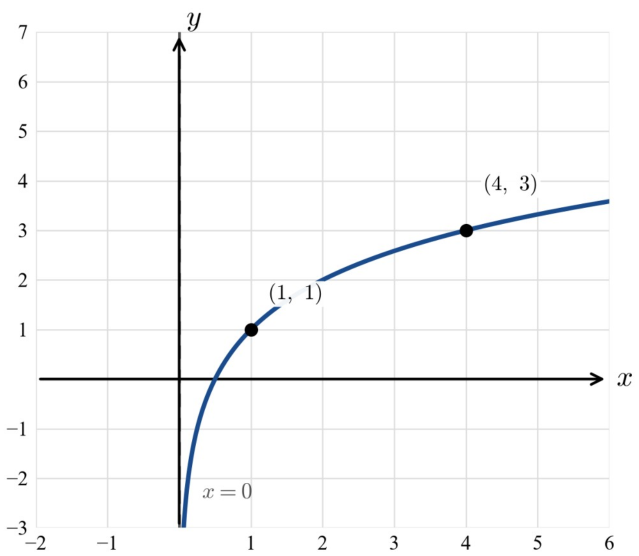
Working	Comment
Asymptote $y = -1$, so $c = -1$.	The horizontal asymptote of $y = a\,b^x + c$ is $y = c$.
At $(0, 1)$: $a\,b^0 + c = 1 \Rightarrow a - 1 = 1$, so $a = 2$.	$b^0 = 1$, so the y-intercept gives $a + c$.
At $(2, 17)$: $2b^2 - 1 = 17 \Rightarrow b^2 = 9$, so $b = 3$ (as $b > 0$).	Substitute the second point and solve for b.
$\therefore y = 2 \cdot 3^x - 1$.	State the rule; it checks against both points.



Example 2 — scaffolded

The graph of $y = a\log_2(x-h)+k$ has vertical asymptote $x=0$ and passes through $(1,1)$ and $(4,3)$. Determine the values of a, h and k.

Working	Comment
Asymptote $x=0$, so $h=0$.	The vertical asymptote of $y=a\log_2(x-h)+k$ is $x=h$.
At $(1,1)$: $a\log_2(1)+k=1 \Rightarrow k=1$.	$\log_2(1)=0$, so this point gives k.
At $(4,3)$: $a\log_2(4)+1=3 \Rightarrow 2a+1=3$, so $a=1$.	$\log_2(4)=2$; solve for a.
$\therefore y=\log_2(x)+1$.	State the rule.



Example 3 — unscaffolded

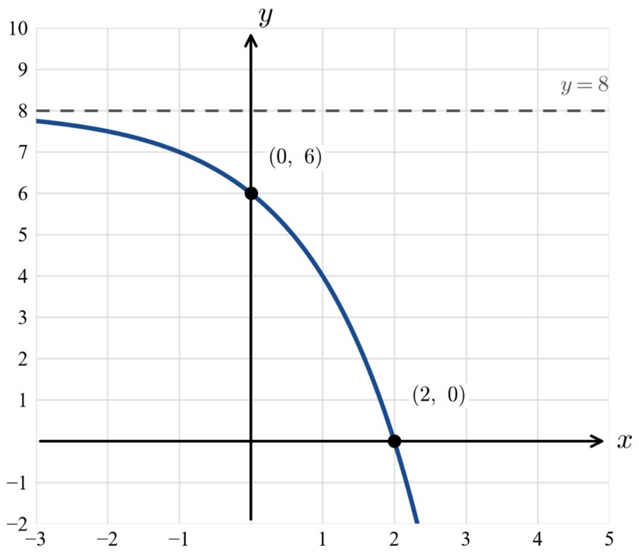
The graph of $y=ab^x+c$ has horizontal asymptote $y=8$ and passes through $(0,6)$ and $(2,0)$. Determine the rule.

Asymptote $y=8$, so $c=8$.

At $(0,6)$: $a+8=6$, so $a=-2$.

At $(2,0)$: $-2b^2+8=0 \Rightarrow b^2=4$, so $b=2$.

$\therefore y=-2\cdot 2^x+8$ (a decay-shaped reflection, since $a<0$).



Example 4 — unscaffolded

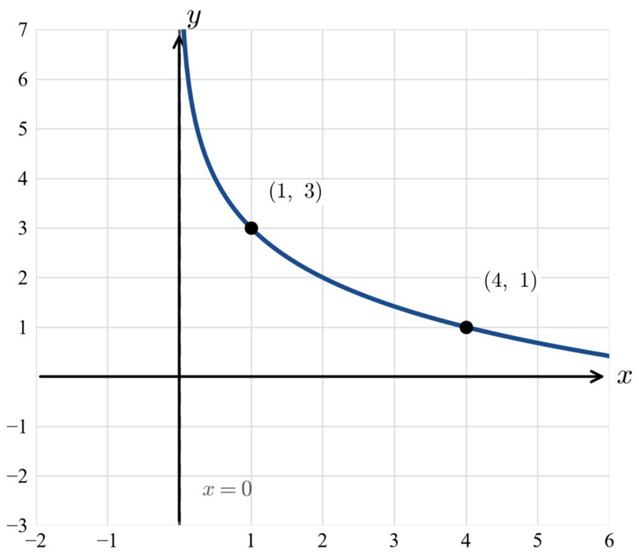
The graph of $y = a\log_2(x-h)+k$ has vertical asymptote $x=0$ and passes through $(1,3)$ and $(4,1)$. Determine the rule.

Asymptote $x=0$, so $h=0$.

At $(1,3)$: $\log_2(1)=0$, so $k=3$.

At $(4,1)$: $a\log_2(4)+3=1 \Rightarrow 2a=-2$, so $a=-1$.

$\therefore\ y=-\log_2(x)+3$.



Practice questions

Scaffolded

Q1. The graph of $y=ab^x+c$ has asymptote $y=1$ and passes through $(0,4)$ and $(2,13)$. (a) State c; (b) use $(0,4)$ to find a; (c) use $(2,13)$ to find b; (d) write the rule.

Q2. The graph of $y=a\log_3(x-h)+k$ has asymptote $x=0$ and passes through $(1,0)$ and $(9,4)$. (a) State h; (b) use $(1,0)$ to find k; (c) use $(9,4)$ to find a; (d) write the rule.

Q3. The graph of $y=ab^x+c$ has asymptote $y=1$ and passes through $(0,5)$ and $(2,2)$. (a) State c and find a; (b) find b; (c) write the rule and state whether it is growth or decay.

Q4. The graph of $y=a\log_2(x-h)+k$ has asymptote $x=1$ and passes through $(2,0)$ and $(5,2)$. (a) State h; (b) find k; (c) find a; (d) write the rule.

Q5. The graph of $y=a\log_2(x-h)+k$ has asymptote $x=-2$ and passes through $(-1,-1)$ and $(2,1)$. (a) State h; (b) find k; (c) find a; (d) write the rule.

Q6. The graph of $y=a\log_2(x-h)+k$ has asymptote $x=0$ and passes through $(1,-2)$ and $(4,2)$. (a) State h and k ; (b) find a; (c) write the rule.

Unscaffolded

Determine the rule of each graph. Exponentials have the form $y=ab^x+c$; logarithms have the stated base.

Q7. Exponential, asymptote $y=-3$, through $(0,-2)$ and $(2,1)$.

Q8. Exponential, asymptote $y=5$, through $(0,4)$ and $(2,-4)$.

Q9. Exponential, asymptote $y=-3$, through $(0,2)$ and $(2,17)$.

Q10. Exponential, asymptote $y=5$, through $(0,2)$ and $(2,-7)$.

Q11. Exponential, asymptote $y=0$, through $(0,8)$ and $(2,2)$.

Q12. Exponential, asymptote $y=3$, through $(0,7)$ and $(2,4)$.

Q13. $y=a\log_3(x-h)+k$, asymptote $x=0$, through $(1,0)$ and $(9,2)$.

Q14. $y=a\log_2(x-h)+k$, asymptote $x=0$, through $(1,1)$ and $(4,-3)$.

Q15. $y=a\log_2(x-h)+k$, asymptote $x=3$, through $(4,0)$ and $(7,2)$.

Q16. $y=a\log_2(x-h)+k$, asymptote $x=0$, through $(1,-3)$ and $(8,0)$.

Q17. Exponential, asymptote $y=0$, through $(0,2)$ and $(2,18)$.

Q18. $y=a\log_2(x-h)+k$, asymptote $x=0$, through $(1,0)$ and $(4,6)$.

Q19. $y=a\log_2(x-h)+k$, asymptote $x=-1$, through $(0,2)$ and $(3,4)$.

Q20. $y=a\log_3(x-h)+k$, asymptote $x=0$, through $(1,2)$ and $(9,0)$.

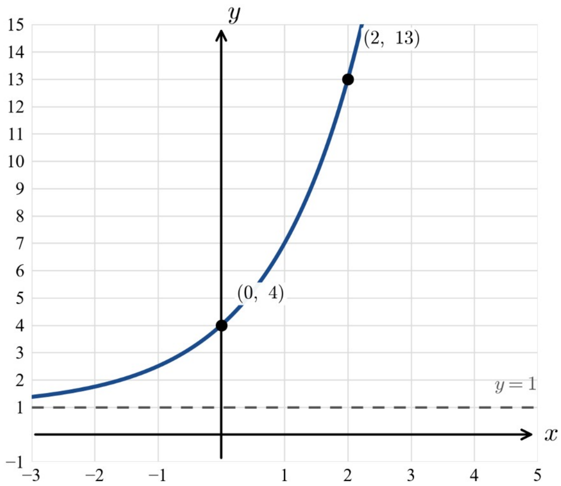
Q21. $y=a\log_2(x-h)+k$, asymptote $x=2$, through $(3,1)$ and $(6,3)$.

Answers

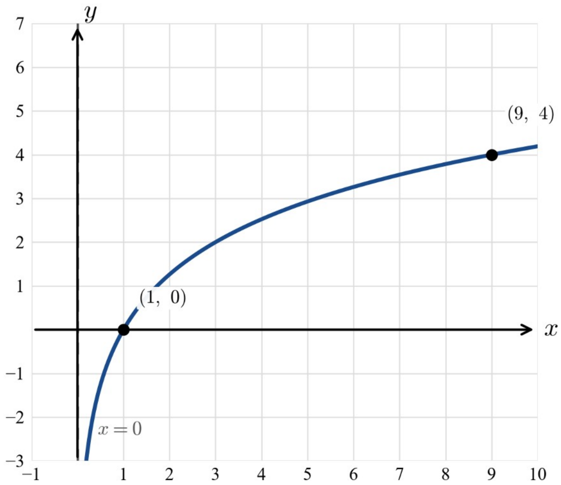
Q1. $y = 3 \cdot 2^x + 1$. **Q2.** $y = 2\log_3(x)$. **Q3.** $y = 4(1/2)^x + 1$ (decay). **Q4.** $y = \log_2(x-1)$. **Q5.** $y = \log_2(x+2) - 1$. **Q6.** $y = 2\log_2(x) - 2$. **Q7.** $y = 2^x - 3$. **Q8.** $y = -3^x + 5$. **Q9.** $y = 5 \cdot 2^x - 3$. **Q10.** $y = -3 \cdot 2^x + 5$. **Q11.** $y = 8(1/2)^x$. **Q12.** $y = 4(1/2)^x + 3$. **Q13.** $y = \log_3(x)$. **Q14.** $y = -2\log_2(x) + 1$. **Q15.** $y = \log_2(x-3)$. **Q16.** $y = \log_2(x) - 3$. **Q17.** $y = 2 \cdot 3^x$. **Q18.** $y = 3\log_2(x)$. **Q19.** $y = \log_2(x+1) + 2$. **Q20.** $y = -\log_3(x) + 2$. **Q21.** $y = \log_2(x-2) + 1$.

Fully-worked solutions

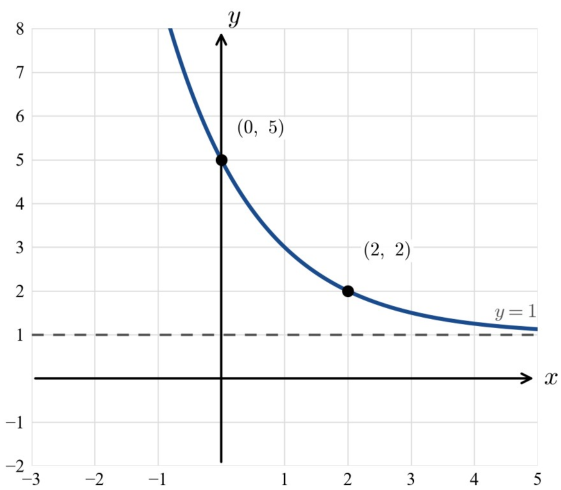
Q1. $c = 1$. At $(0,4)$: $a + 1 = 4 \Rightarrow a = 3$. At $(2,13)$: $3b^2 + 1 = 13 \Rightarrow b^2 = 4 \Rightarrow b = 2$. $\therefore y = 3 \cdot 2^x + 1$.



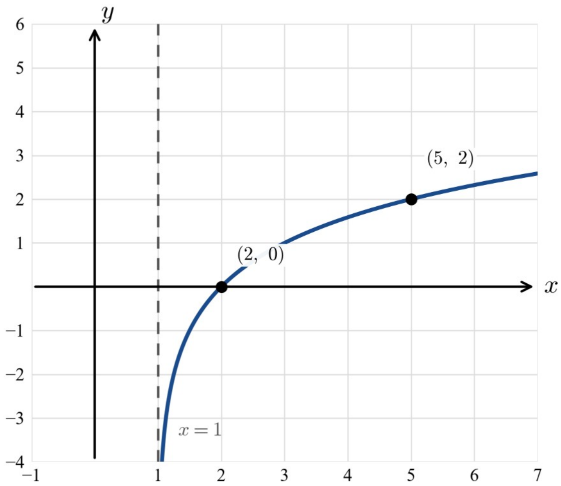
Q2. $h = 0$. At $(1,0)$: $k = 0$. At $(9,4)$: $a\log_3(9) = 4 \Rightarrow 2a = 4 \Rightarrow a = 2$. $\therefore y = 2\log_3(x)$.



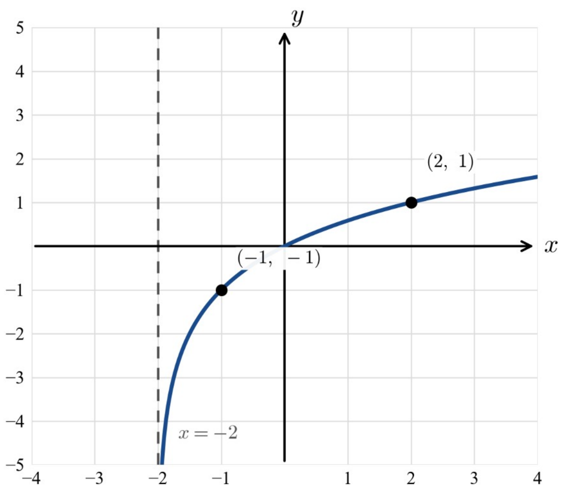
Q3. $c = 1$; at $(0,5)$: $a + 1 = 5 \Rightarrow a = 4$. At $(2,2)$: $4b^2 + 1 = 2 \Rightarrow b^2 = 1/4 \Rightarrow b = 1/2$. $\therefore y = 4(1/2)^x + 1$; since $0 < b < 1$ this is **decay**.



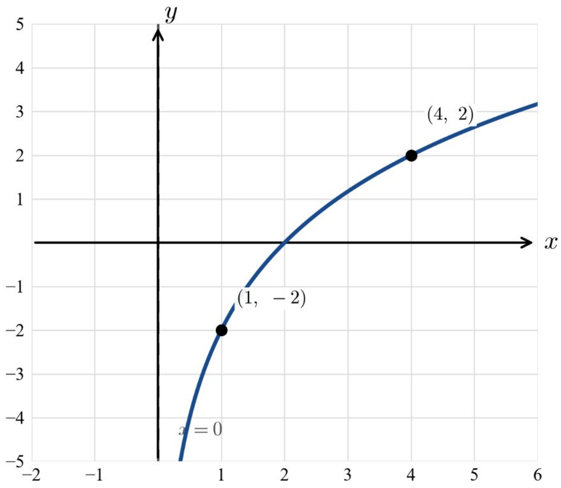
Q4. $h = 1$. At $(2, 0)$: $\log_2(1) = 0$, so $k = 0$. At $(5, 2)$: $a\log_2(4) = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$. $\therefore y = \log_2(x - 1)$.



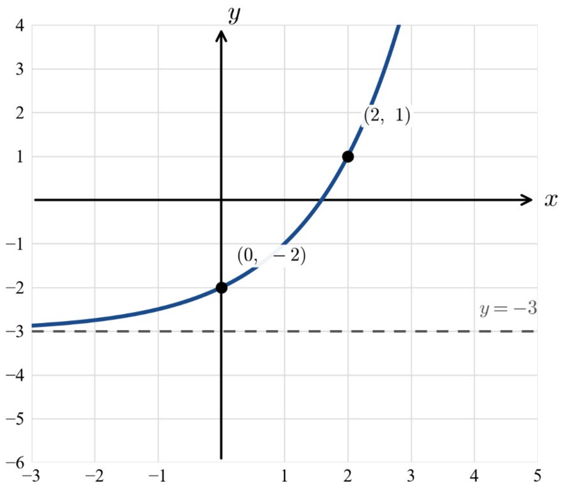
Q5. $h = -2$. At $(-1, -1)$: $\log_2(1) = 0$, so $k = -1$. At $(2, 1)$: $a\log_2(4) - 1 = 1 \Rightarrow 2a = 2 \Rightarrow a = 1$. $\therefore y = \log_2(x + 2) - 1$.



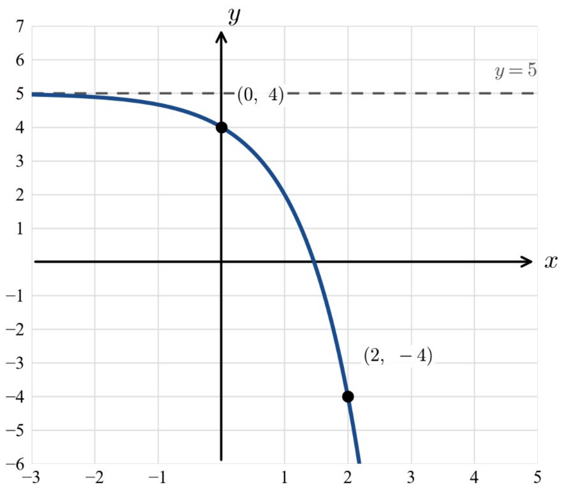
Q6. $h = 0$; at $(1, -2)$: $k = -2$. At $(4, 2)$: $a\log_2(4) - 2 = 2 \Rightarrow 2a = 4 \Rightarrow a = 2$. $\therefore y = 2\log_2(x) - 2$.



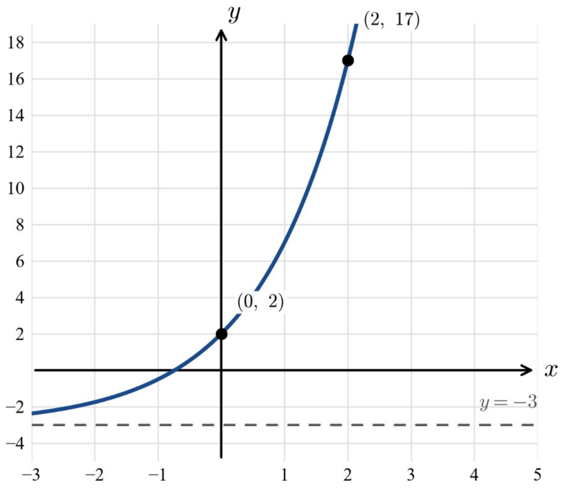
Q7. $c = -3$; at $(0, -2)$: $a - 3 = -2 \Rightarrow a = 1$. At $(2, 1)$: $b^2 - 3 = 1 \Rightarrow b^2 = 4 \Rightarrow b = 2$. $\therefore\ y = 2^x - 3$.



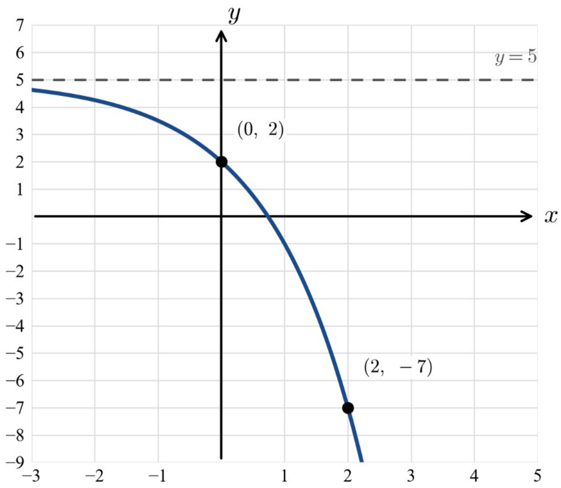
Q8. $c = 5$; at $(0, 4)$: $a + 5 = 4 \Rightarrow a = -1$. At $(2, -4)$: $-b^2 + 5 = -4 \Rightarrow b^2 = 9 \Rightarrow b = 3$. $\therefore\ y = -3^x + 5$.



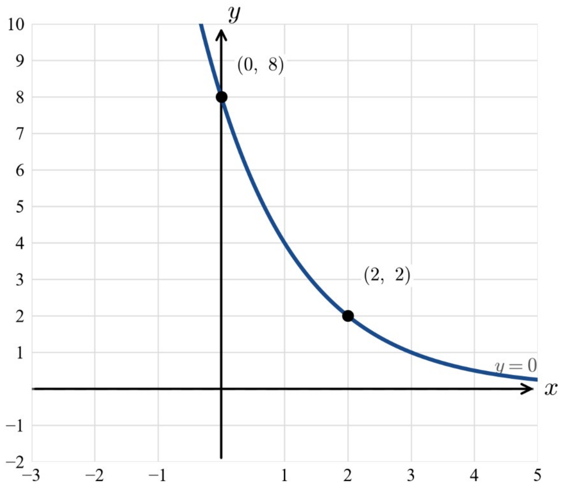
Q9. $c = -3$; at $(0, 2)$: $a - 3 = 2 \Rightarrow a = 5$. At $(2, 17)$: $5b^2 - 3 = 17 \Rightarrow b^2 = 4 \Rightarrow b = 2$. $\therefore\ y = 5 \cdot 2^x - 3$.



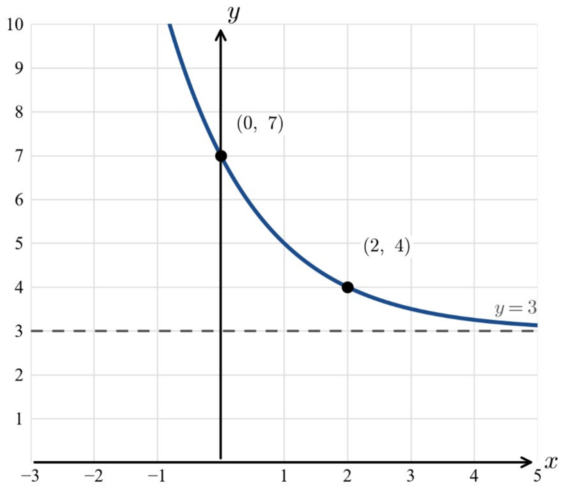
Q10. $c=5$; at $(0,2)$: $a+5=2 \Rightarrow a=-3$. At $(2,-7)$: $-3b^2+5=-7 \Rightarrow b^2=4 \Rightarrow b=2$. $\therefore y=-3\cdot 2^x+5$.



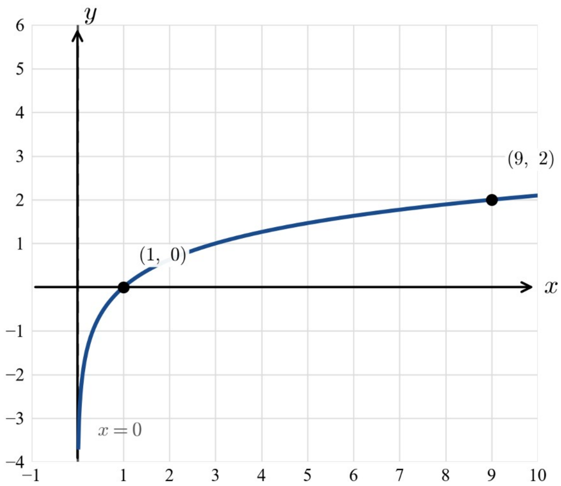
Q11. $c=0$; at $(0,8)$: $a=8$. At $(2,2)$: $8b^2=2 \Rightarrow b^2=1/4 \Rightarrow b=1/2$. $\therefore y=8(1/2)^x$.



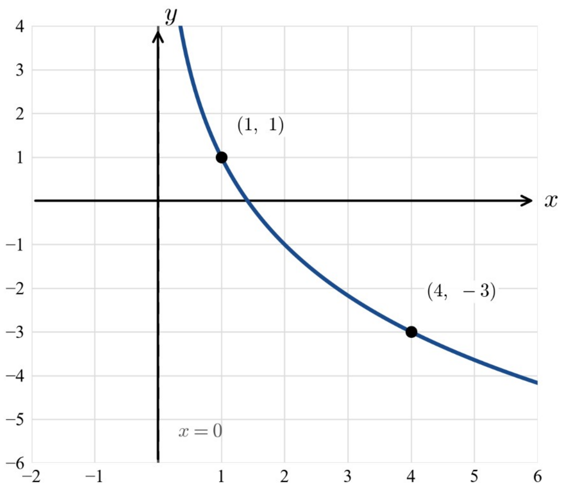
Q12. $c=3$; at $(0,7)$: $a+3=7 \Rightarrow a=4$. At $(2,4)$: $4b^2+3=4 \Rightarrow b^2=1/4 \Rightarrow b=1/2$. $\therefore y=4(1/2)^x+3$.



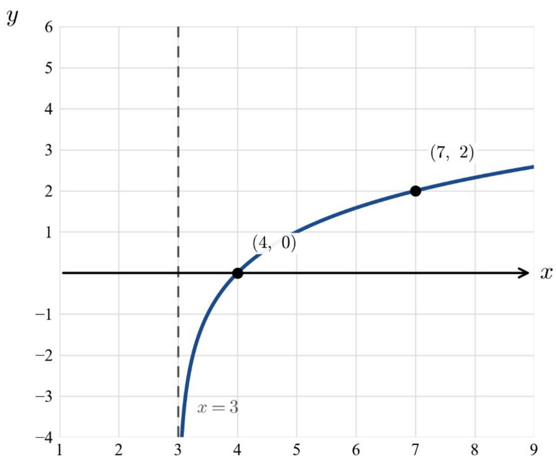
Q13. $h=0$; at $(1,0)$: $k=0$. At $(9,2)$: $a\log_3(9)=2 \Rightarrow 2a=2 \Rightarrow a=1$. $\therefore y=\log_3(x)$.



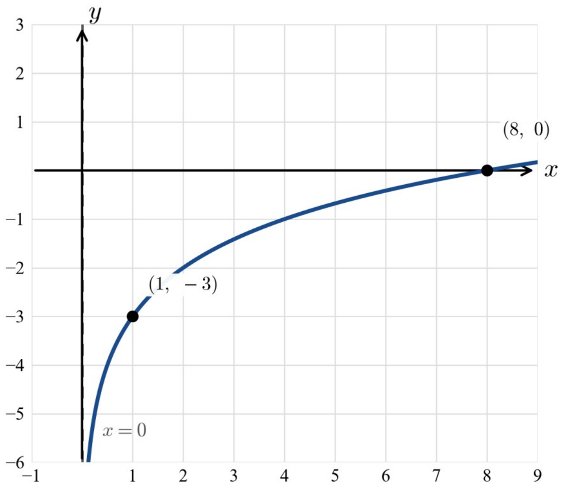
Q14. $h=0$; at $(1,1)$: $k=1$. At $(4,-3)$: $a\log_2(4)+1=-3 \Rightarrow 2a=-4 \Rightarrow a=-2$. $\therefore y=-2\log_2(x)+1$.



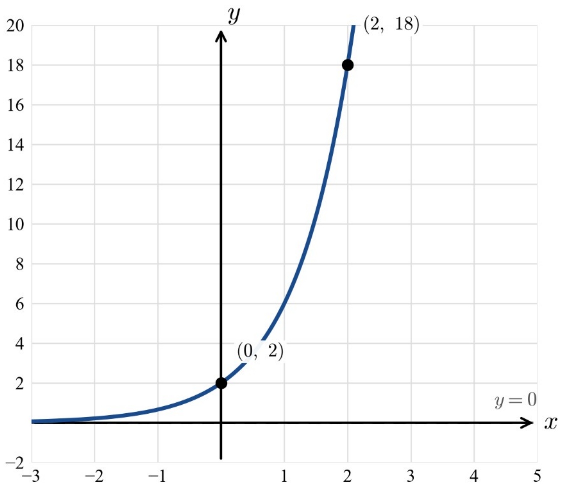
Q15. $h=3$; at $(4,0)$: $\log_2(1)=0$, so $k=0$. At $(7,2)$: $a\log_2(4)=2 \Rightarrow 2a=2 \Rightarrow a=1$. $\therefore y=\log_2(x-3)$.



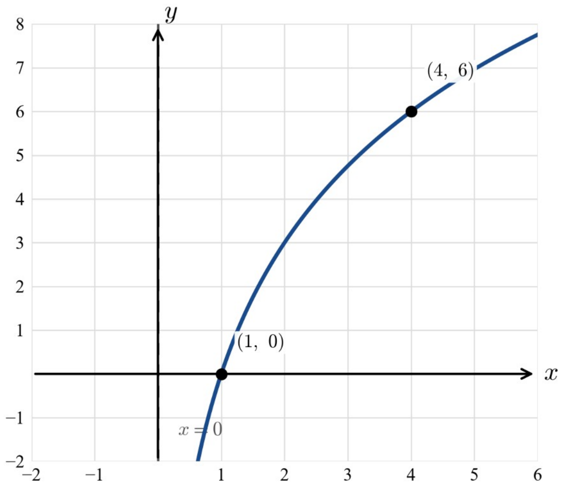
Q16. $h=0$; at $(1,-3)$: $k=-3$. At $(8,0)$: $a\log_2(8)-3=0 \Rightarrow 3a=3 \Rightarrow a=1$. $\therefore y=\log_2(x)-3$.



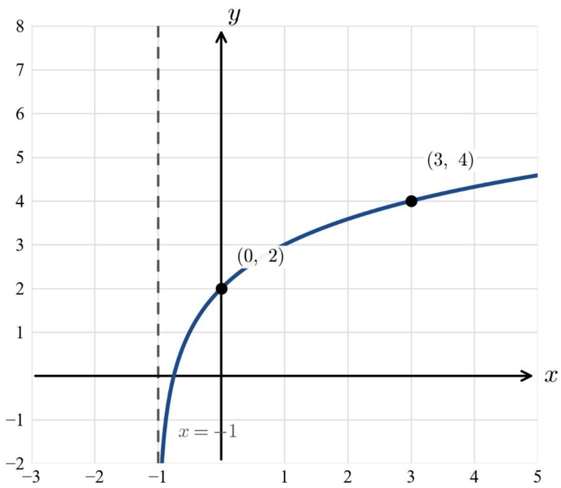
Q17. $c=0$; at $(0,2)$: $a=2$. At $(2,18)$: $2b^2=18 \Rightarrow b^2=9 \Rightarrow b=3$. $\therefore y=2\cdot 3^x$.



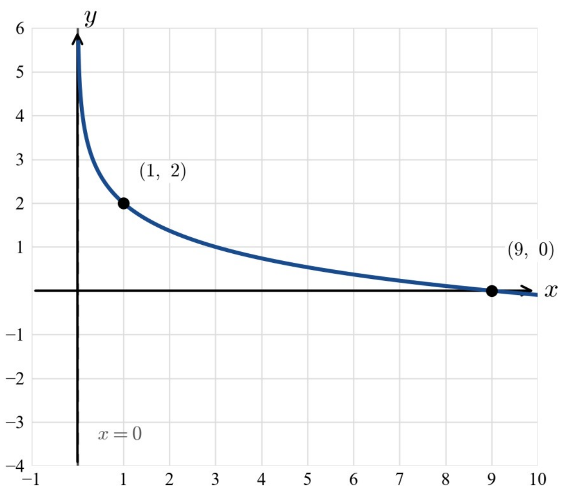
Q18. $h=0$; at $(1,0)$: $k=0$. At $(4,6)$: $a\log_2(4)=6 \Rightarrow 2a=6 \Rightarrow a=3$. $\therefore y=3\log_2(x)$.



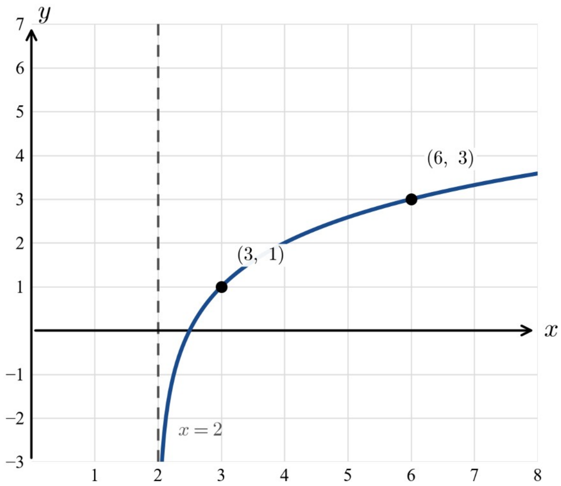
Q19. $h=-1$; at $(0,2)$: $\log_2(1)=0$, so $k=2$. At $(3,4)$: $a\log_2(4)+2=4 \Rightarrow 2a=2 \Rightarrow a=1$. $\therefore y=\log_2(x+1)+2$.



Q20. $h=0$; at $(1,2)$: $k=2$. At $(9,0)$: $a\log_3(9)+2=0 \Rightarrow 2a=-2 \Rightarrow a=-1$. $\therefore y=-\log_3(x)+2$.



Q21. $h=2$; at $(3,1)$: $\log_2(1)=0$, so $k=1$. At $(6,3)$: $a\log_2(4)+1=3 \Rightarrow 2a=2 \Rightarrow a=1$. $\therefore y=\log_2(x-2)+1$.


y
x
7
6
5
4
3
2
1
−1
−2
−3
1
2
3
4
5
6
7
8
(6, 3)
(3, 1)
x = 2

Chapter 5 Exponential and logarithmic functions — 5G Solving exponential equations with logarithms

Theory

When the unknown is in the **index** and the two sides cannot be written with the **same base**, we solve by **taking logarithms of both sides**. The key tool is the logarithm power law $\log\left(a^x\right)=x\log a$, which brings the index down as a factor.

The basic method. To solve $a^x=b$ (with $a>0$, $a\neq 1$, $b>0$):

$$a^x=b\Rightarrow\log\left(a^x\right)=\log b\Rightarrow x\log a=\log b\Rightarrow x=\frac{\log b}{\log a}.$$

This is the change-of-base result $x=\log_a b$. Any base of logarithm may be used; $\log_e$ (base e) and $\log_{10}$ are the usual choices because they are on the calculator.

- **Exact vs approximate.** $x=\dfrac{\log_e b}{\log_e a}$ (equivalently $\log_a b$) is the **exact** answer. A decimal such as $x\approx 2.81$ is an **approximation** — only give one when the question asks for it (e.g. “correct to two decimal places”).
- **First isolate the power.** If the equation is $c\cdot a^x=d$, divide by c to get $a^x=\dfrac{d}{c}$ **before** taking logarithms.
- **An index that is a linear expression.** For $a^{px+q}=b$, take logs to get $px+q=\log_a b$, then solve the linear equation for x.
- **The unknown in two different bases.** For $a^x=b^{x+c}$, take logs of both sides and collect the x-terms: $x\log_e a=\left(x+c\right)\log_e b\Rightarrow x\left(\log_e a-\log_e b\right)=c\log_e b\Rightarrow x=\dfrac{c\log_e b}{\log_e a-\log_e b}$. (Such a solution can be negative.)

To solve an exponential equation with logarithms: (1) isolate the power $a^{(\cdots)}$; (2) take $\log_e$ (or $\log$) of both sides; (3) use $\log\left(a^x\right)=x\log a$ to free the index; (4) solve the resulting linear equation, leaving the answer in exact log form unless a decimal is requested.

Worked examples

Example 1 — scaffolded

Solve $2^x=7$ for x, giving your answer in exact form and correct to two decimal places.

Working	Comment
$2^x=7$	The bases cannot be equated (7 is not a power of 2).
$\log_e\left(2^x\right)=\log_e 7$	Take the natural logarithm of both sides.
$x\log_e 2=\log_e 7$	Power law: $\log_e\left(2^x\right)=x\log_e 2$.
$x=\dfrac{\log_e 7}{\log_e 2}$	Divide by $\log_e 2$. This is the exact value ($=\log_2 7$).
$x\approx 2.81$	Evaluate on the calculator for the approximation.

Example 2 — scaffolded

Solve $5\times 3^x=20$.

Working	Comment
$3^x = \dfrac{20}{5} = 4$	Isolate the power first by dividing by 5.
$\log_e(3^x) = \log_e 4$	Take logarithms of both sides.
$x\log_e 3 = \log_e 4 = 2\log_e 2$	Power law; $\log_e 4 = \log_e 2^2 = 2\log_e 2$.
$x = \dfrac{2\log_e 2}{\log_e 3}$	Exact value ($= \log_3 4$), ≈ 1.26.

Example 3 — unscaffolded

Solve $2^{x+1} = 5$, giving the exact value of x.

$\log_e(2^{x+1}) = \log_e 5$, so $(x+1)\log_e 2 = \log_e 5$.

Then $x + 1 = \dfrac{\log_e 5}{\log_e 2}$, giving $x = \dfrac{\log_e 5}{\log_e 2} - 1$.

$\therefore x = \dfrac{\log_e 5}{\log_e 2} - 1 \approx 1.32$.

Example 4 — unscaffolded

Solve $3 \times 2^{2x} = 15$.

Divide by 3: $2^{2x} = 5$. Taking logarithms, $2x\log_e 2 = \log_e 5$, so $2x = \dfrac{\log_e 5}{\log_e 2}$.

$\therefore x = \dfrac{\log_e 5}{2\log_e 2} \approx 1.16$.

Practice questions

Scaffolded

Q1. Solve $2^x = 10$: (a) take logarithms of both sides; (b) use the power law to free the index; (c) write x in exact form; (d) give x correct to two decimal places.

Q2. Solve $3^x = 8$: (a) write x as a single logarithm; (b) show that $x = \dfrac{3\log_e 2}{\log_e 3}$; (c) evaluate to two decimal places.

Q3. Solve $5^x = 2$, giving the exact value and a decimal to two places.

Q4. Solve $4 \times 2^x = 30$: (a) isolate the power; (b) hence find x in exact form.

Q5. Solve $2^{x-1} = 6$: (a) take logarithms; (b) solve the resulting linear equation for x (exact form).

Q6. Solve $10^x = 45$, giving $x = \log_{10} 45$ and its value to two decimal places.

Unscaffolded

Solve each equation for x, giving the exact value. Give a decimal (to two decimal places) where indicated.

Q7. $2^x = 20$

Q8. $3^x = 100$

Q9. $7^x = 3$

Q10. $2^{3x} = 9$

Q11. $5^{x+1} = 30$

Q12. $2^x = 3^{x-1}$

Q13. $4^x = 5^{x+1}$

Q14. $3 \times 2^{2x} = 21$

Q15. $2^{x+2} = 7$

Q16. $100 \times (1/2)^x = 15$

Q17. $e^x = 10$

Q18. $e^{2x} = 5$

Q19. A sum of $500 grows as 500×1.05^x dollars after x years. Find, correct to two decimal places, the value of x for which the sum reaches $800.

Q20. Solve $2^x = 5$, giving the answer in exact form and correct to two decimal places.

Q21. A population is modelled by $P = 2000 \times 2^{t/5}$. Find the exact value of t (years) for which $P = 10\,000$, and give it correct to two decimal places.

Answers

Q1. $x=\dfrac{\log_e 10}{\log_e 2}=\log_2 10\approx 3.32$. **Q2.** $x=\dfrac{3\log_e 2}{\log_e 3}=\log_3 8\approx 1.89$. **Q3.** $x=\dfrac{\log_e 2}{\log_e 5}=\log_5 2\approx 0.43$. **Q4.** $x=\dfrac{\log_e 15}{\log_e 2}-1=\log_2 15/2\approx 2.91$. **Q5.** $x=\dfrac{\log_e 12}{\log_e 2}=\log_2 12\approx 3.58$. **Q6.** $x=\log_{10} 45\approx 1.65$. **Q7.** $x=\dfrac{\log_e 20}{\log_e 2}=\log_2 20\approx 4.32$. **Q8.** $x=\dfrac{2\log_e 10}{\log_e 3}=\log_3 100\approx 4.19$. **Q9.** $x=\dfrac{\log_e 3}{\log_e 7}=\log_7 3\approx 0.56$. **Q10.** $x=\dfrac{2\log_e 3}{3\log_e 2}=\dfrac{1}{3}\log_2 9\approx 1.06$. **Q11.** $x=\dfrac{\log_e 6}{\log_e 5}=\log_5 6\approx 1.11$. **Q12.** $x=\dfrac{\log_e 3}{\log_e 3-\log_e 2}\approx 2.71$. **Q13.** $x=\dfrac{\log_e 5}{\log_e 4-\log_e 5}\approx -7.21$. **Q14.** $x=\dfrac{\log_e 7}{2\log_e 2}\approx 1.40$. **Q15.** $x=\dfrac{\log_e 7}{\log_e 2}-2\approx 0.81$. **Q16.** $x=\dfrac{\log_e 20-\log_e 3}{\log_e 2}=\log_2 20/3\approx 2.74$. **Q17.** $x=\log_e 10\approx 2.30$. **Q18.** $x=\dfrac{\log_e 5}{2}\approx 0.80$. **Q19.** $x=\dfrac{\log_e 1.6}{\log_e 1.05}\approx 9.63$ years. **Q20.** $x=\dfrac{\log_e 5}{\log_e 2}=\log_2 5\approx 2.32$. **Q21.** $t=5\cdot\dfrac{\log_e 5}{\log_e 2}=5\log_2 5\approx 11.61$ years.

Fully-worked solutions

Q1. $2^x=10\Rightarrow x\log_e 2=\log_e 10\Rightarrow x=\dfrac{\log_e 10}{\log_e 2}=\log_2 10\approx 3.32$.

Q2. $3^x=8\Rightarrow x\log_e 3=\log_e 8=3\log_e 2\Rightarrow x=\dfrac{3\log_e 2}{\log_e 3}=\log_3 8\approx 1.89$.

Q3. $5^x=2\Rightarrow x\log_e 5=\log_e 2\Rightarrow x=\dfrac{\log_e 2}{\log_e 5}=\log_5 2\approx 0.43$.

Q4. $4\times 2^x=30\Rightarrow 2^x=\dfrac{15}{2}\Rightarrow x\log_e 2=\log_e 15/2\Rightarrow x=\dfrac{\log_e 15}{\log_e 2}-1\approx 2.91$.

Q5. $2^{x-1}=6\Rightarrow(x-1)\log_e 2=\log_e 6\Rightarrow x-1=\log_2 6\Rightarrow x=\dfrac{\log_e 12}{\log_e 2}=\log_2 12\approx 3.58$ (since $1+\log_2 6=\log_2 2+\log_2 6=\log_2 12$).

Q6. $10^x=45\Rightarrow x=\log_{10} 45=\dfrac{\log_e 45}{\log_e 10}\approx 1.65$.

Q7. $2^x=20\Rightarrow x=\dfrac{\log_e 20}{\log_e 2}=\log_2 20\approx 4.32$.

Q8. $3^x=100\Rightarrow x\log_e 3=\log_e 100=2\log_e 10\Rightarrow x=\dfrac{2\log_e 10}{\log_e 3}=\log_3 100\approx 4.19$.

Q9. $7^x=3\Rightarrow x=\dfrac{\log_e 3}{\log_e 7}=\log_7 3\approx 0.56$.

Q10. $2^{3x}=9\Rightarrow 3x\log_e 2=\log_e 9=2\log_e 3\Rightarrow x=\dfrac{2\log_e 3}{3\log_e 2}=1/3\log_2 9\approx 1.06$.

Q11. $5^{x+1}=30 \Rightarrow (x+1)\log_e 5=\log_e 30 \Rightarrow x+1=\log_5 30 \Rightarrow x=\log_5 30-1=\dfrac{\log_e 6}{\log_e 5}=\log_5 6\approx 1.11$ (as $\log_5 30-1=\log_5 30-\log_5 5=\log_5 6$).

Q12. $2^x=3^{x-1} \Rightarrow x\log_e 2=(x-1)\log_e 3 \Rightarrow x\log_e 2-x\log_e 3=-\log_e 3 \Rightarrow x(\log_e 2-\log_e 3)=-\log_e 3$.
$\therefore x=\dfrac{-\log_e 3}{\log_e 2-\log_e 3}=\dfrac{\log_e 3}{\log_e 3-\log_e 2}\approx 2.71$.

Q13. $4^x=5^{x+1} \Rightarrow x\log_e 4=(x+1)\log_e 5 \Rightarrow x(\log_e 4-\log_e 5)=\log_e 5$. $\therefore x=\dfrac{\log_e 5}{\log_e 4-\log_e 5}\approx -7.21$ (negative — the graphs meet to the left of the y-axis).

Q14. $3\times 2^{2x}=21 \Rightarrow 2^{2x}=7 \Rightarrow 2x\log_e 2=\log_e 7 \Rightarrow x=\dfrac{\log_e 7}{2\log_e 2}\approx 1.40$.

Q15. $2^{x+2}=7 \Rightarrow (x+2)\log_e 2=\log_e 7 \Rightarrow x+2=\log_2 7 \Rightarrow x=\dfrac{\log_e 7}{\log_e 2}-2\approx 0.81$.

Q16.
$100\times(1/2)^x=15 \Rightarrow (1/2)^x=\dfrac{3}{20} \Rightarrow x\log_e 1/2=\log_e 3/20 \Rightarrow x=\dfrac{\log_e(3/20)}{\log_e(1/2)}=\dfrac{\log_e 20-\log_e 3}{\log_e 2}=\log_2 20/3\approx 2.74$.

Q17. $e^x=10 \Rightarrow x=\log_e 10\approx 2.30$.

Q18. $e^{2x}=5 \Rightarrow 2x=\log_e 5 \Rightarrow x=\dfrac{\log_e 5}{2}\approx 0.80$.

Q19. $500\times 1.05^x=800 \Rightarrow 1.05^x=1.6 \Rightarrow x\log_e 1.05=\log_e 1.6 \Rightarrow x=\dfrac{\log_e 1.6}{\log_e 1.05}\approx 9.63$ years.

Q20. $2^x=5 \Rightarrow x=\dfrac{\log_e 5}{\log_e 2}=\log_2 5\approx 2.32$.

Q21. $2000\times 2^{t/5}=10\,000 \Rightarrow 2^{t/5}=5 \Rightarrow \dfrac{t}{5}\log_e 2=\log_e 5 \Rightarrow \dfrac{t}{5}=\log_2 5$. $\therefore t=5\log_2 5=\dfrac{5\log_e 5}{\log_e 2}\approx 11.61$ years.

Chapter 5 Exponential and logarithmic functions — 5H Exponentials and logarithms as inverses

Theory

The exponential and logarithmic functions are **inverses of each other**. For a base $a>0, a\neq 1$:

$$f(x)=a^x \Leftrightarrow f^{-1}(x)=\log_a(x), f(x)=\log_a(x) \Leftrightarrow f^{-1}(x)=a^x.$$

In particular e^x and $\log_e(x)$ are inverses.

Existence. A function has an inverse function iff it is **one-to-one**. Every exponential and every logarithmic function is one-to-one, so an inverse function always exists (after any domain restriction implied by a transformation).

Finding an inverse. Write $y=f(x)$, **interchange** x **and** y, then solve for y; this rule is $f^{-1}(x)$. To undo an exponential, take a logarithm; to undo a logarithm, exponentiate.

Domain and range swap.

$$\operatorname{dom} f^{-1}=\operatorname{ran} f, \operatorname{ran} f^{-1}=\operatorname{dom} f.$$

Graphs. The graph of f^{-1} is the reflection of the graph of f in the line $y=x$. A reflection swaps the asymptotes: an exponential's **horizontal** asymptote $y=k$ becomes the logarithm's **vertical** asymptote $x=k$, and the y-intercept of f becomes the x-intercept of f^{-1}.

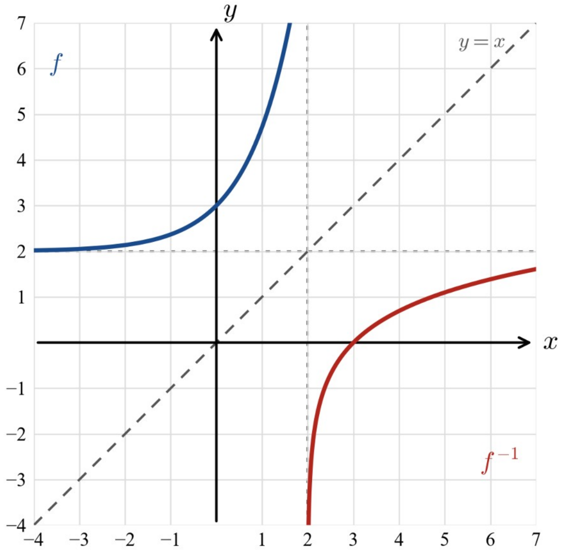
Check. For all valid x, $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$.

Worked examples

Example 1 — scaffolded

For $f(x)=e^x+2$, find f^{-1}, state the domain and range of f^{-1}, and sketch f and f^{-1}.

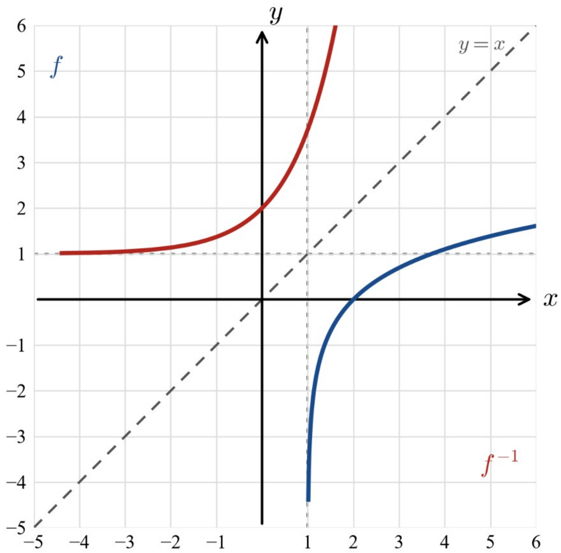
Working	Comment
$y=e^x+2$	Write $y=f(x)$.
Swap: $x=e^y+2$	Interchange x and y for the inverse.
$e^y=x-2$, so $y=\log_e(x-2)$	Isolate e^y, then take $\log_e$ of both sides.
$\therefore f^{-1}(x)=\log_e(x-2)$	The inverse rule.
f: dom $\mathbb{R}$, ran $(2,\infty)$ (since $e^x>0$, $f>2$; asymptote $y=2$).	Range of f from the asymptote.
f^{-1}: dom $(2,\infty)$, ran $\mathbb{R}$; asymptote $x=2$.	Domain/range are the swap of f's.



Example 2 — scaffolded

For $f(x)=\log_e(x-1)$, find f^{-1} and state the domain and range of both functions.

Working	Comment
$y=\log_e(x-1)$	Write $y=f(x)$.
Swap: $x=\log_e(y-1)$	Interchange x and y.
$y-1=e^x$, so $y=e^x+1$	Exponentiate both sides (base e).
$\therefore f^{-1}(x)=e^x+1$	The inverse rule.
f: dom $(1,\infty)$, ran $\mathbb{R}$; asymptote $x=1$.	$x-1>0$.
f^{-1}: dom $\mathbb{R}$, ran $(1,\infty)$; asymptote $y=1$.	Swap; asymptote reflects.

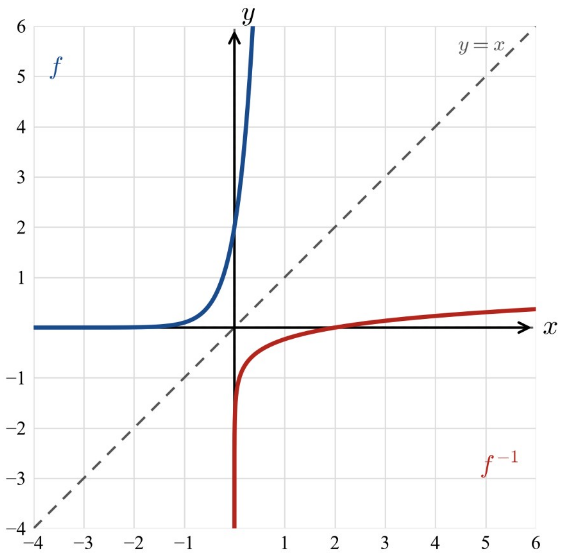


Example 3 — unscaffolded

Find the inverse of $f(x)=2e^{3x}$ and state the domain of f^{-1}.

$y=2e^{3x}$. Swap: $x=2e^{3y}\Rightarrow e^{3y}=\dfrac{x}{2}\Rightarrow 3y=\log_e\left(\dfrac{x}{2}\right)$, so $y=\dfrac{1}{3}\log_e\left(\dfrac{x}{2}\right)$.

$\therefore f^{-1}(x)=\dfrac{1}{3}\log_e\left(\dfrac{x}{2}\right)$, with dom $f^{-1}=(0,\infty)$ (the range of f), and ran $f^{-1}=\mathbb{R}$.

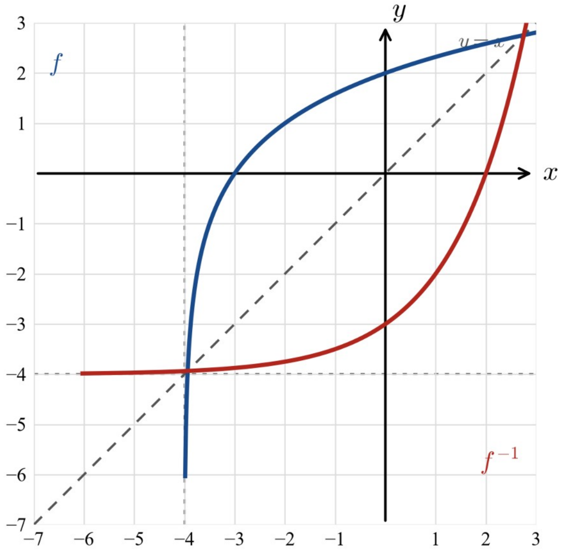


Example 4 — unscaffolded

Find the inverse of $f(x) = \log_2(x+4)$.

$y = \log_2(x+4)$. Swap: $x = \log_2(y+4) \Rightarrow y+4 = 2^x \Rightarrow y = 2^x - 4$.

$\therefore f^{-1}(x) = 2^x - 4$, with dom $f^{-1} = \mathbb{R}$ and ran $f^{-1} = (-4, \infty)$ (the domain of f was $(-4, \infty)$).



Practice questions

Scaffolded

For each function: (a) find $f^{-1}(x)$; (b) state the domain and range of f^{-1}; (c) sketch f, f^{-1} and the line $y = x$ on one set of axes.

Q1. $f(x) = e^x - 4$

Q2. $f(x) = \log_e(x+3)$

Q3. $f(x) = e^{2x}$

Q4. $f(x) = 3e^x$

Q5. $f(x)=2\log_e(x)$

Q6. $f(x)=10^x$

Unscaffolded

For **Q7–Q18** and **Q20**: find $f^{-1}(x)$, state its domain and range, and sketch f and f^{-1} (with $y=x$).

Q7. $f(x)=e^x+5$

Q8. $f(x)=e^{x-2}$

Q9. $f(x)=2e^x-6$

Q10. $f(x)=e^{-x}$

Q11. $f(x)=\log_e(x)+4$

Q12. $f(x)=\log_e(2x)$

Q13. $f(x)=\log_e(x-5)$

Q14. $f(x)=3\log_e(x-1)$

Q15. $f(x)=\log_2(x)$

Q16. $f(x)=\log_{10}(x+1)$

Q17. $f(x)=1-e^x$

Q18. $f(x)=e^{2x-1}$

Q19. Show that $f(x)=e^x-1$ and $g(x)=\log_e(x+1)$ are inverse functions.

Q20. For $f(x)=2-e^x$, find $f^{-1}(x)$ and state its domain.

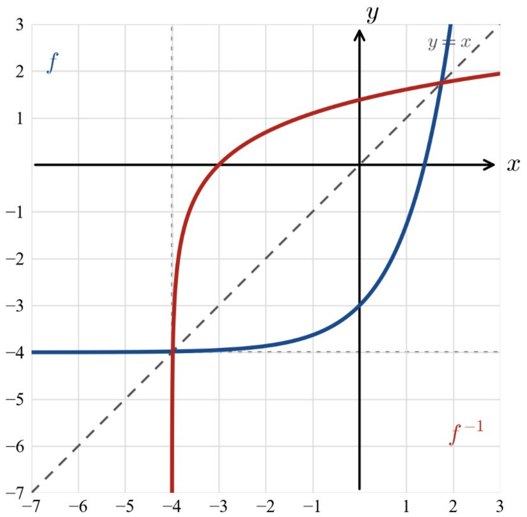
Q21. Using the inverse relationship, solve $e^{2x}-1=7$, giving the exact value of x.

Answers

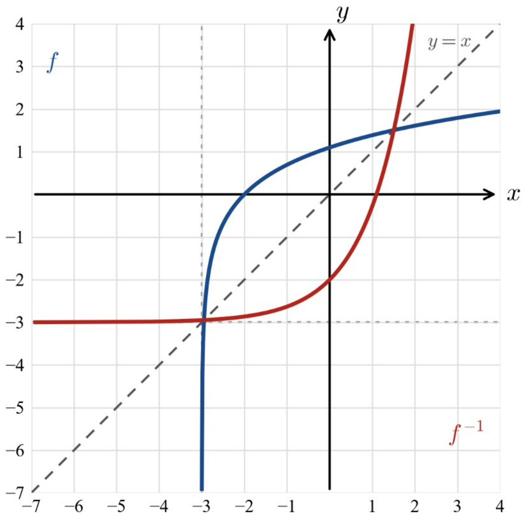
Q1. $f^{-1}(x)=\log_e(x+4)$; dom $(-4,\infty)$, ran $\mathbb{R}$. **Q2.** $f^{-1}(x)=e^x-3$; dom $\mathbb{R}$, ran $(-3,\infty)$. **Q3.** $f^{-1}(x)=\frac{1}{2}\log_e(x)$; dom $(0,\infty)$, ran $\mathbb{R}$. **Q4.** $f^{-1}(x)=\log_e\left(\frac{x}{3}\right)$; dom $(0,\infty)$, ran $\mathbb{R}$. **Q5.** $f^{-1}(x)=e^{x/2}$; dom $\mathbb{R}$, ran $(0,\infty)$. **Q6.** $f^{-1}(x)=\log_{10}(x)$; dom $(0,\infty)$, ran $\mathbb{R}$. **Q7.** $f^{-1}(x)=\log_e(x-5)$; dom $(5,\infty)$, ran $\mathbb{R}$. **Q8.** $f^{-1}(x)=\log_e(x)+2$; dom $(0,\infty)$, ran $\mathbb{R}$. **Q9.** $f^{-1}(x)=\log_e\left(\frac{x+6}{2}\right)$; dom $(-6,\infty)$, ran $\mathbb{R}$. **Q10.** $f^{-1}(x)=-\log_e(x)$; dom $(0,\infty)$, ran $\mathbb{R}$. **Q11.** $f^{-1}(x)=e^{x-4}$; dom $\mathbb{R}$, ran $(0,\infty)$. **Q12.** $f^{-1}(x)=\frac{1}{2}e^x$; dom $\mathbb{R}$, ran $(0,\infty)$. **Q13.** $f^{-1}(x)=e^x+5$; dom $\mathbb{R}$, ran $(5,\infty)$. **Q14.** $f^{-1}(x)=e^{x/3}+1$; dom $\mathbb{R}$, ran $(1,\infty)$. **Q15.** $f^{-1}(x)=2^x$; dom $\mathbb{R}$, ran $(0,\infty)$. **Q16.** $f^{-1}(x)=10^x-1$; dom $\mathbb{R}$, ran $(-1,\infty)$. **Q17.** $f^{-1}(x)=\log_e(1-x)$; dom $(-\infty,1)$, ran $\mathbb{R}$. **Q18.** $f^{-1}(x)=\frac{1}{2}(\log_e(x)+1)$; dom $(0,\infty)$, ran $\mathbb{R}$. **Q19.** $f(g(x))=x$ and $g(f(x))=x$ (shown below), so they are inverses. **Q20.** $f^{-1}(x)=\log_e(2-x)$; dom $(-\infty,2)$. **Q21.** $x=\frac{3\log_e 2}{2}$.

Fully-worked solutions

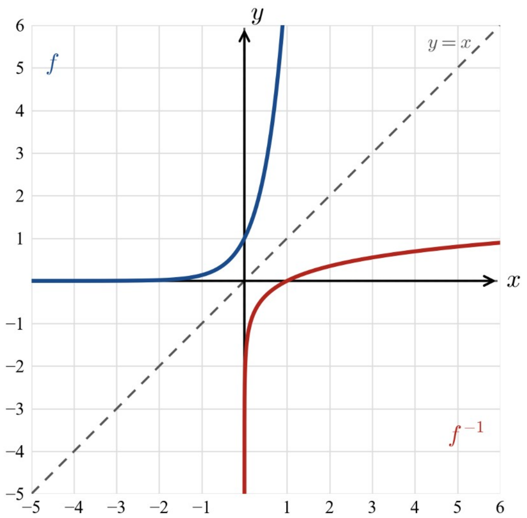
Q1. $y=e^x-4$; swap $x=e^y-4 \Rightarrow e^y=x+4 \Rightarrow y=\log_e(x+4)$. Since ran $f=(-4,\infty)$, dom $f^{-1}=(-4,\infty)$, ran $f^{-1}=\mathbb{R}$; asymptote $x=-4$.



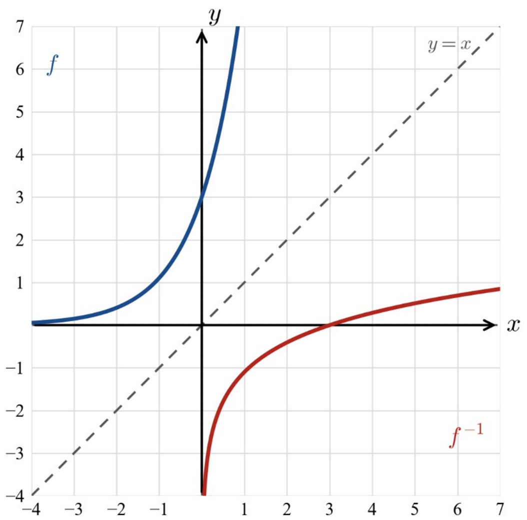
Q2. $y=\log_e(x+3)$; swap $x=\log_e(y+3) \Rightarrow y+3=e^x \Rightarrow y=e^x-3$. dom $f^{-1}=\mathbb{R}$, ran $(-3,\infty)$; asymptote $y=-3$.



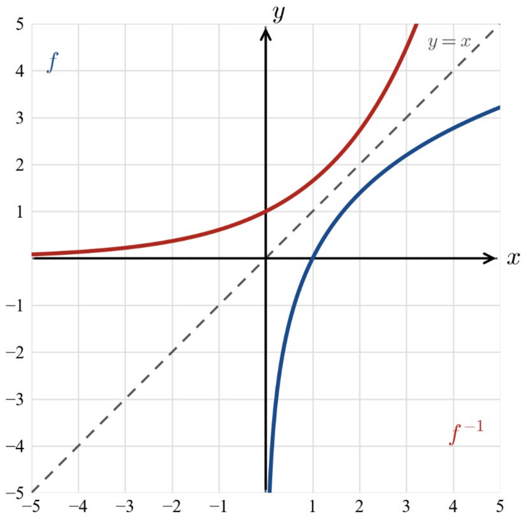
Q3. $y = e^{2x}$; swap $x = e^{2y} \Rightarrow 2y = \log_e(x) \Rightarrow y = \frac{1}{2}\log_e(x)$. dom $f^{-1} = (0, \infty)$, ran $\mathbb{R}$.



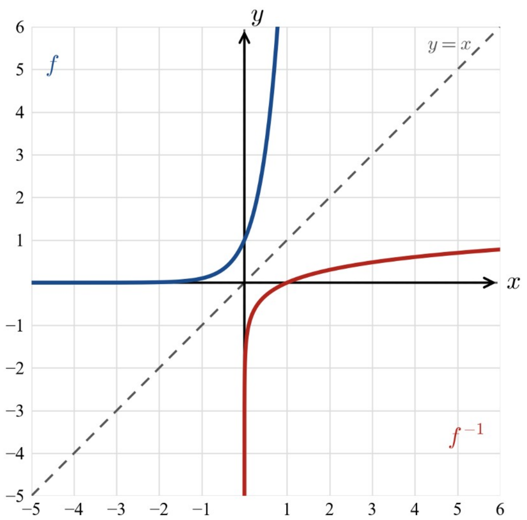
Q4. $y = 3e^x$; swap $x = 3e^y \Rightarrow e^y = \frac{x}{3} \Rightarrow y = \log_e\left(\frac{x}{3}\right)$. dom $f^{-1} = (0, \infty)$, ran $\mathbb{R}$.



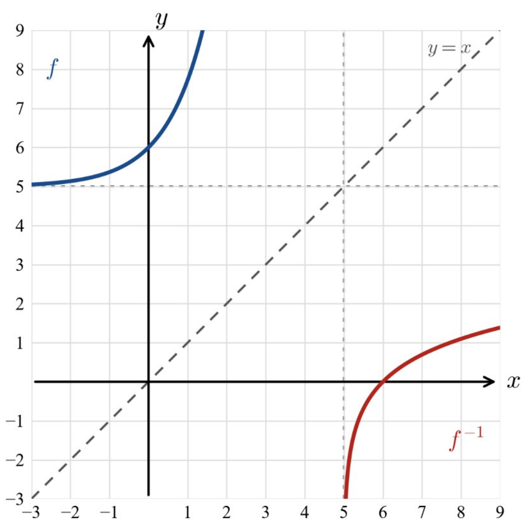
Q5. $y = 2\log_e(x)$; swap $x = 2\log_e(y) \Rightarrow \log_e(y) = \frac{x}{2} \Rightarrow y = e^{x/2}$. dom $f^{-1} = \mathbb{R}$, ran $(0, \infty)$.



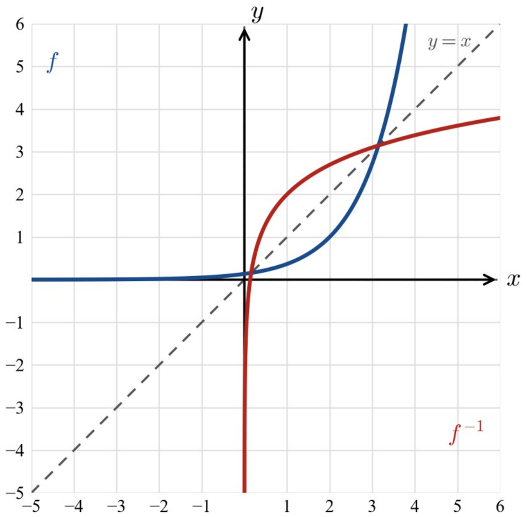
Q6. $y=10^x$; swap $x=10^y \Rightarrow y=\log_{10}(x)$. dom $f^{-1}=(0,\infty)$, ran $\mathbb{R}$.



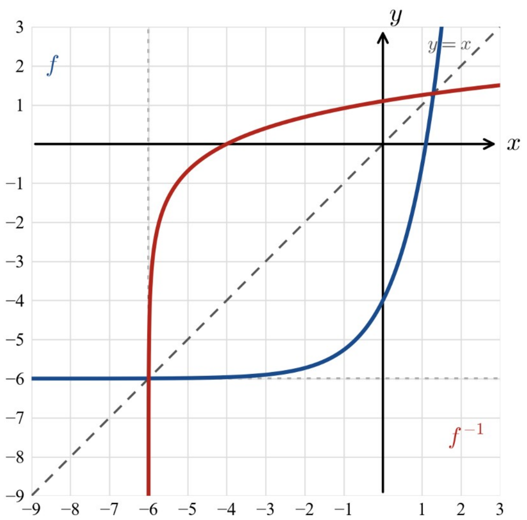
Q7. $x=e^y+5 \Rightarrow y=\log_e(x-5)$. dom $(5,\infty)$, ran $\mathbb{R}$; asymptote $x=5$.



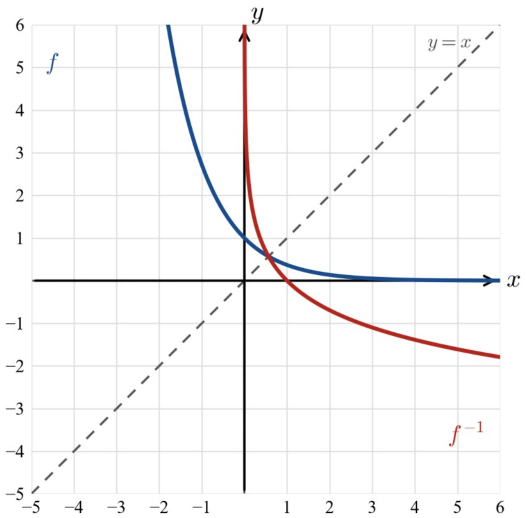
Q8. $x=e^{y-2} \Rightarrow y-2=\log_e(x) \Rightarrow y=\log_e(x)+2$. dom $(0,\infty)$, ran $\mathbb{R}$.



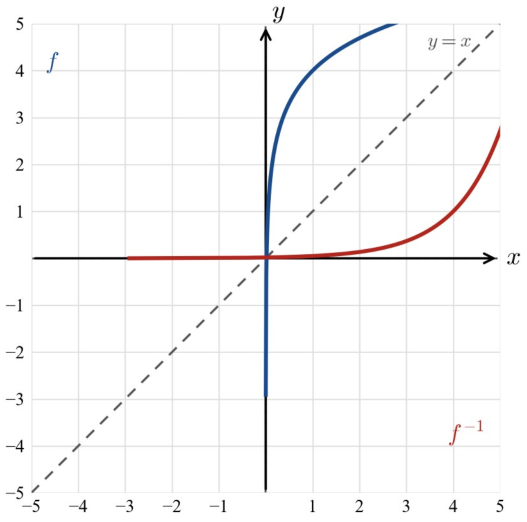
Q9. $x = 2e^{y} - 6 \Rightarrow e^{y} = \dfrac{x+6}{2} \Rightarrow y = \log_e\left(\dfrac{x+6}{2}\right)$. dom $(-6, \infty)$, ran $\mathbb{R}$.



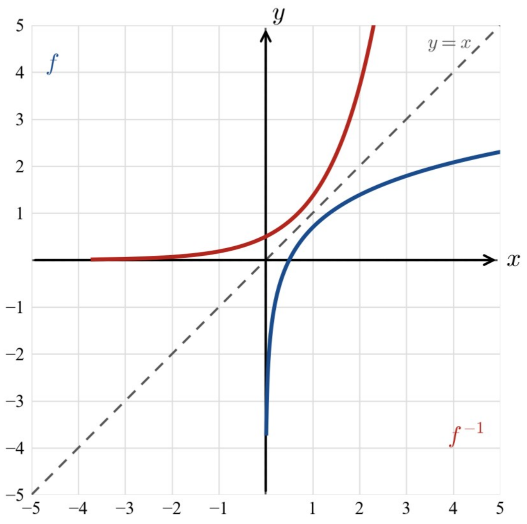
Q10. $x = e^{-y} \Rightarrow -y = \log_e(x) \Rightarrow y = -\log_e(x)$. dom $(0, \infty)$, ran $\mathbb{R}$. (f is decreasing, so is f^{-1}.)



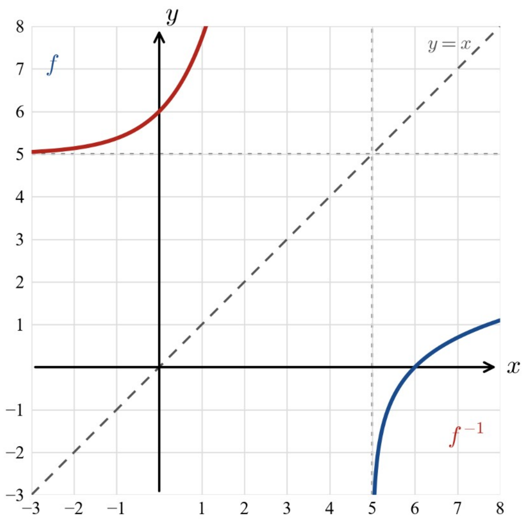
Q11. $x = \log_e(y) + 4 \Rightarrow \log_e(y) = x - 4 \Rightarrow y = e^{x-4}$. dom $\mathbb{R}$, ran $(0, \infty)$.



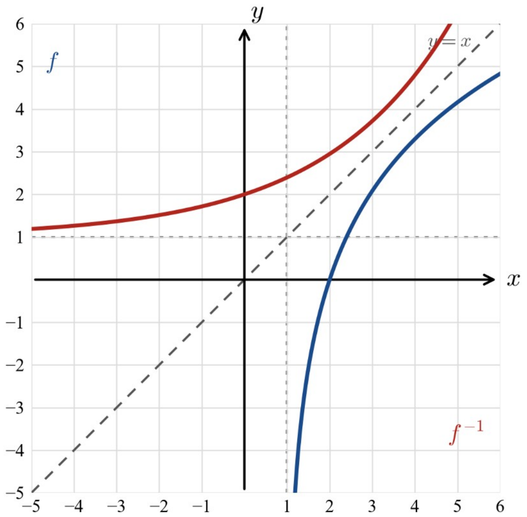
Q12. $x=\log_e(2y)\Rightarrow 2y=e^x\Rightarrow y=\frac{1}{2}e^x$. dom $\mathbb{R}$, ran $(0,\infty)$.



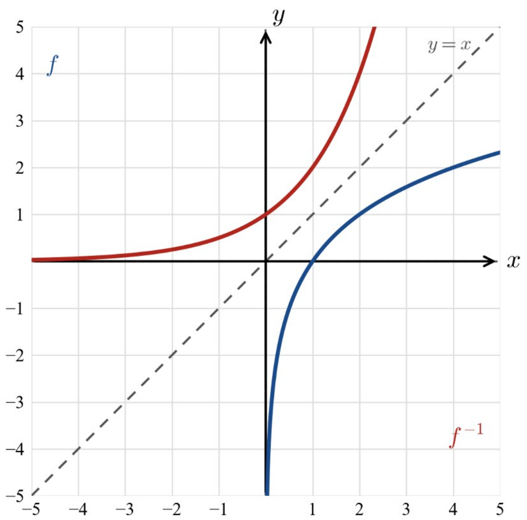
Q13. $x=\log_e(y-5)\Rightarrow y-5=e^x\Rightarrow y=e^x+5$. dom $\mathbb{R}$, ran $(5,\infty)$; asymptote $y=5$.



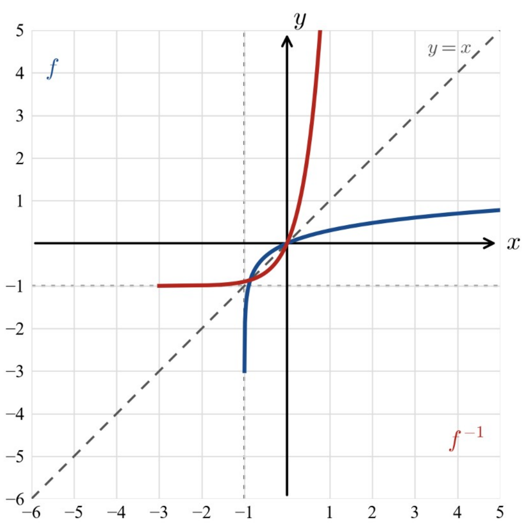
Q14. $x=3\log_e(y-1)\Rightarrow\log_e(y-1)=\frac{x}{3}\Rightarrow y=e^{x/3}+1$. dom $\mathbb{R}$, ran $(1,\infty)$.



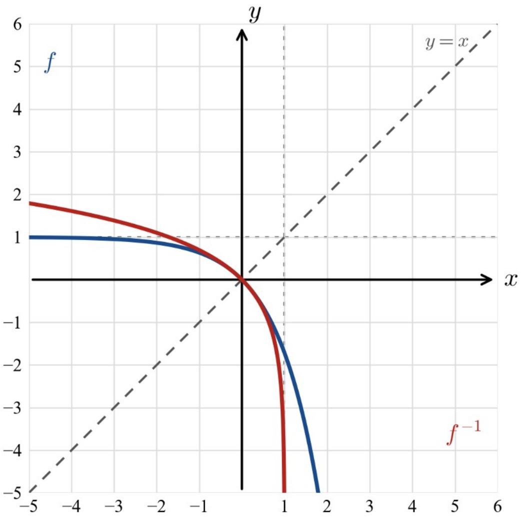
Q15. $x = \log_2(y) \Rightarrow y = 2^x$. dom $\mathbb{R}$, ran $(0, \infty)$.



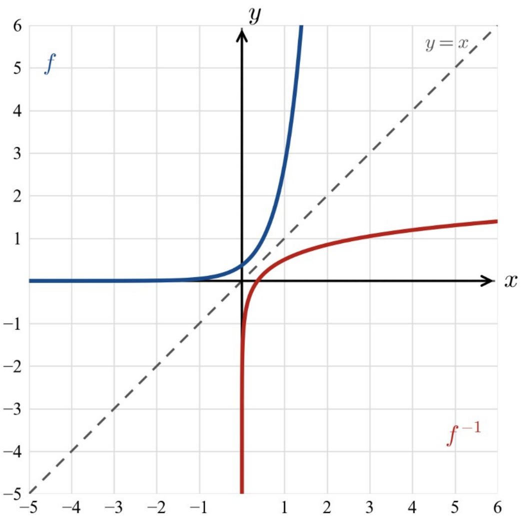
Q16. $x = \log_{10}(y+1) \Rightarrow y+1 = 10^x \Rightarrow y = 10^x - 1$. dom $\mathbb{R}$, ran $(-1, \infty)$.



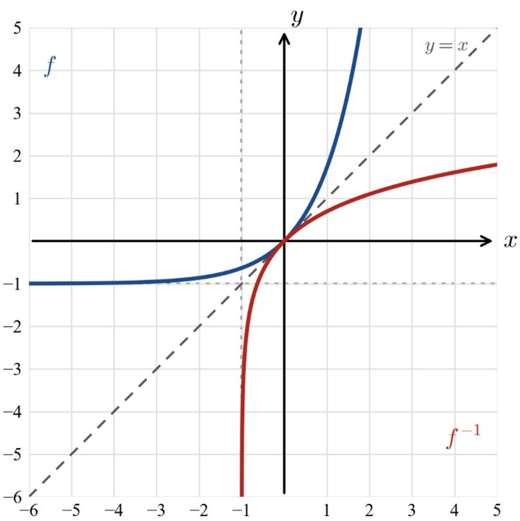
Q17. $x = 1 - e^y \Rightarrow e^y = 1 - x \Rightarrow y = \log_e(1-x)$. dom $(-\infty, 1)$, ran $\mathbb{R}$ (need $1 - x > 0$).



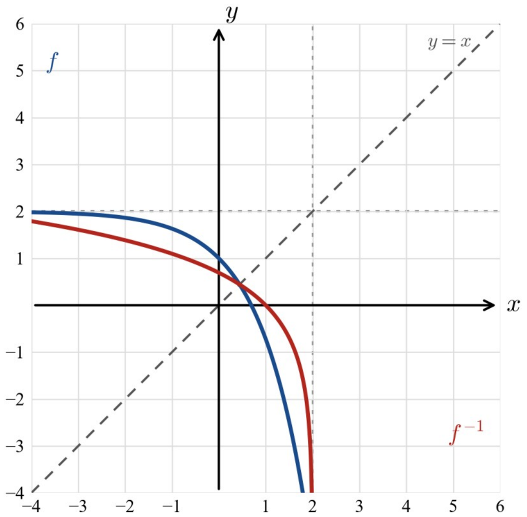
Q18. $x=e^{2y-1}\Rightarrow 2y-1=\log_e(x)\Rightarrow y=\frac{1}{2}(\log_e(x)+1)$. dom $(0,\infty)$, ran $\mathbb{R}$.



Q19. $f(g(x))=e^{\log_e(x+1)}-1=(x+1)-1=x$, and $g(f(x))=\log_e((e^x-1)+1)=\log_e(e^x)=x$. Since both compositions give x, f and g are inverse functions.



Q20. $y=2-e^x$; swap $x=2-e^y\Rightarrow e^y=2-x\Rightarrow y=\log_e(2-x)$. $\therefore f^{-1}(x)=\log_e(2-x)$, with dom $f^{-1}=(-\infty,2)$ (need $2-x>0$; this is the range of f).



Q21. Add 1: $e^{2x} = 8$. Apply the inverse of the exponential (take $\log_e$): $2x = \log_e 8 = \log_e 2^3 = 3\log_e 2$. $\therefore x = \dfrac{3\log_e 2}{2}$.

Theory

Many quantities change at a rate proportional to their current size. Such a quantity grows or decays **exponentially** and can be modelled by

$$A = A_0 e^{kt} \text{ or equivalently } A = A_0 b^t,$$

where A_0 is the **initial amount** (the value when $t=0$), t is time, and k (or the base b) fixes the rate.

- **Growth:** $k>0$ (equivalently $b>1$). The quantity increases without bound.
- **Decay:** $k<0$ (equivalently $0<b<1$). The quantity decreases towards the asymptote $A=0$.

Reading a model. Put $t=0$ to get A_0. The continuous rate k is the proportional rate of change: $\frac{dA}{dt}=kA$. A model given in the form $A_0 b^t$ has continuous rate $k=\log_e b$.

Doubling time and half-life. The time for the quantity to double (growth) or halve (decay) is *constant*:

$$t_{\text{double}}=\frac{\log_e 2}{k}, t_{\text{half}}=\frac{\log_e 2}{|k|}.$$

If a half-life h (or doubling time) is given, the model is $A=A_0(1/2)^{t/h}=A_0 2^{-t/h}$ (decay) or $A=A_0 2^{t/d}$ (growth).

Finding a model from data. Given A_0 and one further reading $A(t_1)=A_1$, solve $A_1=A_0 e^{kt_1}$ for k (take logs), or write $A=A_0 b^t$ with $b=(A_1/A_0)^{1/t_1}$.

Solving for time. To find when A reaches a target value, substitute and take logarithms: $A_0 e^{kt}=A \Rightarrow t=\frac{1}{k}\log_e \frac{A}{A_0}$. Give an exact value (in terms of $\log_e$) unless a decimal is asked for.

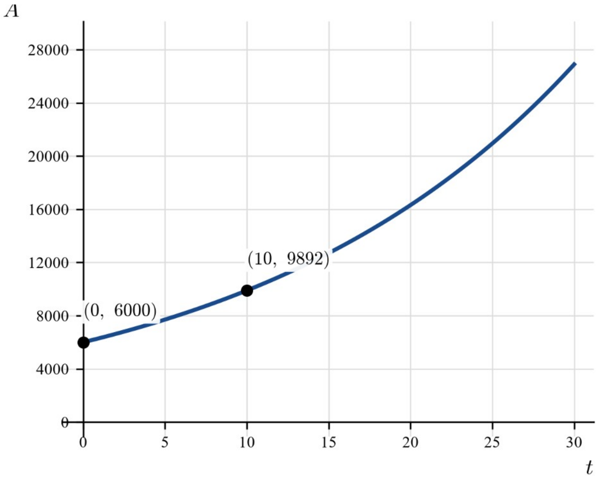
Bounded models (Newton's law of cooling). When a quantity approaches a limiting value A_∞ rather than 0, use $A=A_\infty+(A_0-A_\infty)e^{-kt}$; here it is the **difference** $A-A_\infty$ that decays exponentially.

Worked examples

Example 1 — scaffolded

A colony of bacteria is modelled by $A=6000e^{t/20}$, where A is the number present after t hours.

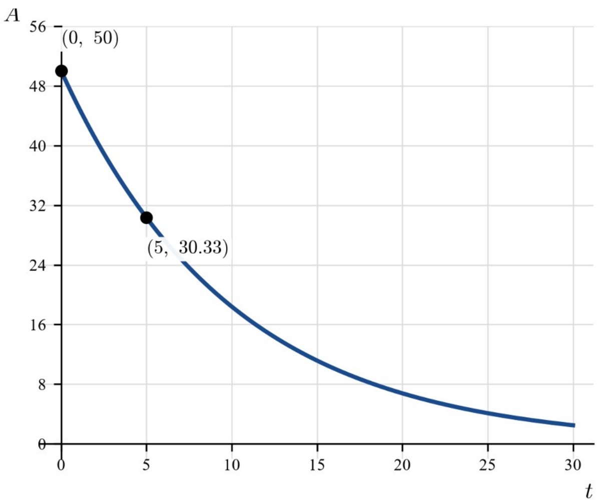
Working	Comment
$A(0)=6000e^0=6000$.	The initial count is A_0 (put $t=0$).
Continuous rate $k=\frac{1}{20}=0.05$, so this is **growth**.	$k>0$; the exponent is $kt=t/20$.
$A(10)=6000e^{10/20}=6000e^{1/2}\approx 9892$.	Substitute $t=10$; $e^{0.5}\approx 1.6487$.
Doubling time $=\frac{\log_e 2}{0.05}=20\log_e 2\approx 13.86$ hours.	Constant for a growth model.



Example 2 — scaffolded

A radioactive sample has mass $A = 50e^{-t/10}$ mg after t years.

Working	Comment
$A(0) = 50$ mg.	Initial mass.
$k = -\dfrac{1}{10} = -0.1 < 0$, so this is **decay**.	Negative rate.
$A(5) = 50e^{-1/2} \approx 30.33$ mg.	Substitute $t = 5$.
Half-life $= \dfrac{\log_e 2}{0.1} = 10\log_e 2 \approx 6.93$ years.	Time to halve.



Example 3 — unscaffolded

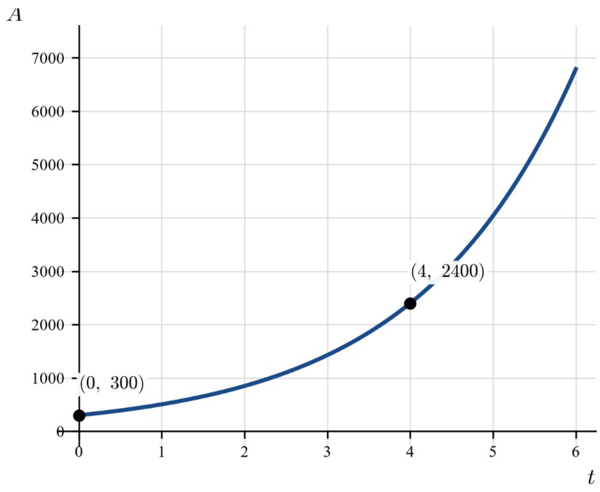
A population is $N_0 = 300$ at $t = 0$ and $N = 2400$ after 4 years, growing exponentially. Find a model and the time for the population to reach 4800.

Since the population multiplies by $2400/300 = 8$ over 4 years, $N = 300 \cdot 8^{t/4}$.

(In base e: $N = 300e^{kt}$ with $e^{4k} = 8$, so $k = \dfrac{\log_e 8}{4} = \dfrac{3\log_e 2}{4} \approx 0.520$.)

$N = 4800$: $8^{t/4} = \dfrac{4800}{300} = 16 \Rightarrow 2^{3t/4} = 2^4 \Rightarrow \dfrac{3t}{4} = 4$.

$\therefore t = \dfrac{16}{3} \approx 5.33$ years.



Example 4 — unscaffolded

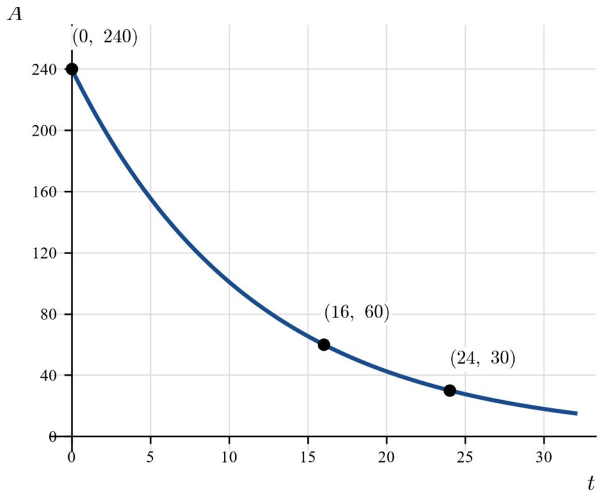
A substance with half-life 8 days has initial mass 240 g. Write a model and find when 30 g remains.

Half-life 8 days $\Rightarrow A = 240 \cdot 2^{-t/8}$ (equivalently $A = 240\, e^{-(\log_e 2/8)t}$).

Check: $A(16) = 240 \cdot 2^{-2} = 60$ g after two half-lives. ✓

$A = 30$: $2^{-t/8} = \dfrac{30}{240} = \dfrac{1}{8} = 2^{-3} \Rightarrow \dfrac{t}{8} = 3$.

$\therefore t = 24$ days (three half-lives).



Practice questions

Scaffolded

Q1. A population is $P = 2000\, e^{t/25}$ after t years. (a) State the initial population; (b) find P after 50 years (to the nearest whole number); (c) find the doubling time (to 2 dp).

Q2. A sample decays as $A = 80\, e^{-t/5}$ mg after t hours. (a) State the initial mass; (b) find the mass after 10 hours (to 2 dp); (c) find the half-life (to 2 dp).

Q3. A culture grows exponentially from 500 to 4000 in 3 hours. (a) By what factor does it multiply each 3 hours? (b) write a model $P = 500\, b^t$; (c) find the number present after 6 hours; (d) find the doubling time.

Q4. \$8000 is invested at 5% per annum compounded annually, so $A = 8000(1.05)^t$ dollars after t years. (a) State the initial investment; (b) find the value after 10 years (to the nearest cent); (c) explain why the growth is exponential.

Q5. A cup of coffee cools according to $T=20+70e^{-t/8}$ (°C) after t minutes. (a) State the initial temperature; (b) find the temperature after 8 minutes (to 2 dp); (c) find the limiting (room) temperature; (d) find, to 2 dp, when the coffee reaches 40°C.

Q6. A quantity decays as $A=100\cdot 2^{-t/12}$. (a) State the half-life; (b) find when $A=12.5$.

Unscaffolded

Q7. For $P=1500e^{t/30}$: state A_0, find $P(60)$ (nearest whole number), and the doubling time (2 dp).

Q8. A substance of half-life 6 hours starts at 160 g. Write the model and find when 20 g remains.

Q9. A colony grows from 250 to 2000 in 5 hours. Write a model $N=250b^t$ and find when $N=16000$.

Q10. Caffeine (200 mg) has half-life 6 hours: $A=200\cdot 2^{-t/6}$. Find $A(12)$ and the time until 25 mg remains.

Q11. Bacteria double every 3 hours from 400: $A=400\cdot 2^{t/3}$. Find $A(9)$ and the time to reach $12\,800$.

Q12. Soup cools as $T=18+62e^{-t/10}$ (°C). Find the initial temperature and the temperature after 10 minutes (2 dp).

Q13. $5000 grows as $A=5000e^{0.04t}$. Find the value after 15 years (nearest cent) and the exact time for it to double.

Q14. For $A=90e^{-t/15}$: find $A(30)$ (2 dp) and the half-life (2 dp).

Q15. For $A=1200e^{t/40}$, find the exact time for the amount to reach 3600.

Q16. A radioisotope of half-life 4 hours has initial mass 320 mg. Find when 5 mg remains.

Q17. A town of 8000 grows at 3% per year: $P=8000(1.03)^t$. Find the population after 20 years (nearest whole number).

Q18. A radioactive mass is $A=500e^{-t/20}$ g. Find the exact time until 125 g remains.

Q19. A quantity is 64 at $t=0$ and 16 at $t=2$, decaying exponentially. Show that $A=64\cdot 4^{-t/2}$ (equivalently $64\cdot 2^{-t}$) and find $A(1)$.

Q20. A drug (150 mg, half-life 8 hours) satisfies $A=150\cdot 2^{-t/8}$. Find $A(24)$ and hence the time until 18.75 mg remains.

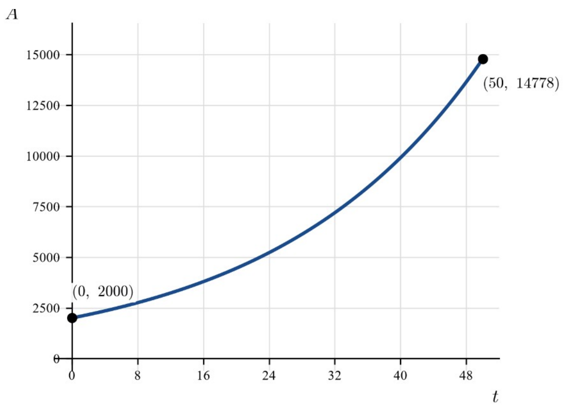
Q21. For $A=3000e^{t/50}$: find $A(100)$ (nearest whole number) and the exact time for the amount to reach 6000.

Answers

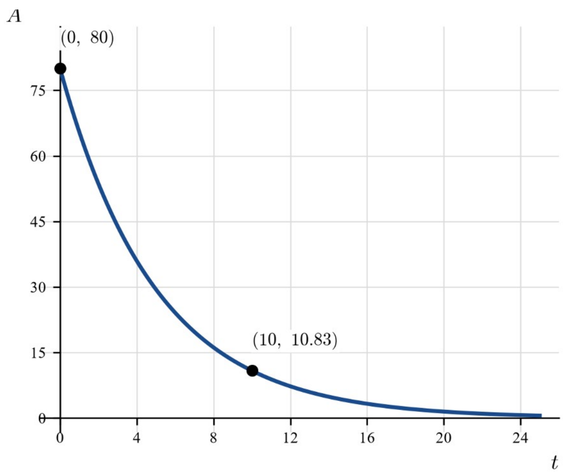
Q1. (a) 2000; (b) 14 778; (c) $25\log_e 2 \approx 17.33$ yr. **Q2.** (a) 80 mg; (b) $80e^{-2} \approx 10.83$ mg; (c) $5\log_e 2 \approx 3.47$ h. **Q3.** (a) 8; (b) $P = 500 \cdot 2^t$ (since $b = 8^{1/3} = 2$); (c) 32 000; (d) 1 hour. **Q4.** (a) \$8000; (b) \$13,031.16; (c) each year the value is multiplied by the constant factor 1.05. **Q5.** (a) 90°C; (b) $20 + 70e^{-1} \approx 45.75$°C; (c) 20°C; (d) $8\log_e(7/2) \approx 10.02$ min. **Q6.** (a) 12; (b) $t = 36$. **Q7.** $A_0 = 1500$; $P(60) = 1500e^2 \approx 11\,084$; doubling $30\log_e 2 \approx 20.79$. **Q8.** $A = 160 \cdot 2^{-t/6}$; 20 g at $t = 18$ h. **Q9.** $N = 250 \cdot 2^{3t/5}$ (i.e. $b = 8^{1/5}$); $N = 16000$ at $t = 10$ h. **Q10.** $A(12) = 50$ mg; 25 mg at $t = 18$ h. **Q11.** $A(9) = 3200$; 12 800 at $t = 15$ h. **Q12.** 80°C; $18 + 62e^{-1} \approx 40.81$°C. **Q13.** $A(15) \approx \$9110.59$; doubles at $t = 25\log_e 2 \approx 17.33$ yr. **Q14.** $A(30) = 90e^{-2} \approx 12.18$; half-life $15\log_e 2 \approx 10.40$. **Q15.** $t = 40\log_e 3 \approx 43.94$. **Q16.** $t = 24$ h. **Q17.** $\approx 14\,449$. **Q18.** $t = 20\log_e 4 = 40\log_e 2 \approx 27.73$. **Q19.** $A = 64 \cdot 4^{-t/2} = 64 \cdot 2^{-t}$; $A(1) = 32$. **Q20.** $A(24) = 18.75$ mg; $t = 24$ h. **Q21.** $A(100) = 3000e^2 \approx 22\,167$; 6000 at $t = 50\log_e 2 \approx 34.66$.

Fully-worked solutions

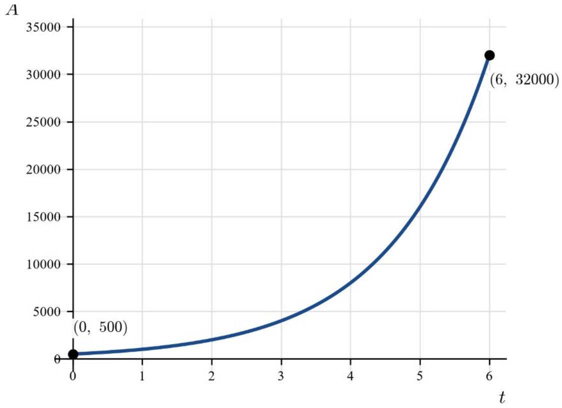
Q1. $P = 2000e^{t/25}$. (a) $P(0) = 2000$. (b) $P(50) = 2000e^{50/25} = 2000e^2 \approx 14\,778$. (c) doubling $= \dfrac{\log_e 2}{1/25} = 25\log_e 2 \approx 17.33$ yr.



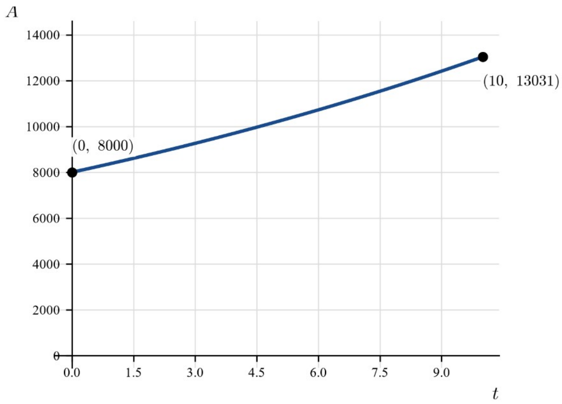
Q2. $A = 80e^{-t/5}$. (a) 80 mg. (b) $A(10) = 80e^{-2} \approx 10.83$ mg. (c) half-life $= \dfrac{\log_e 2}{1/5} = 5\log_e 2 \approx 3.47$ h.



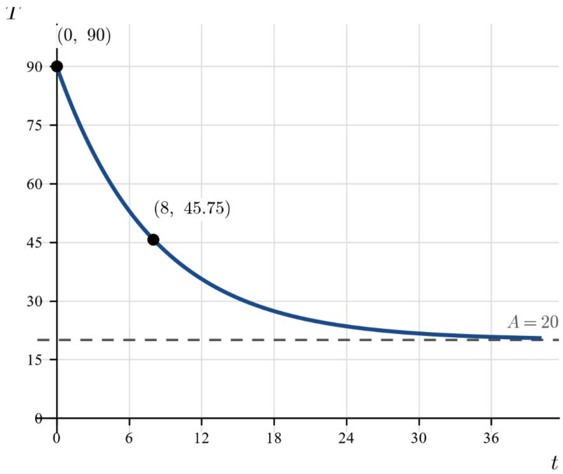
Q3. (a) $4000/500 = 8$. (b) $b^3 = 8 \Rightarrow b = 2$, so $P = 500 \cdot 2^t$. (c) $P(6) = 500 \cdot 2^6 = 32\,000$. (d) $2^t = 2$ at $t = 1$ hour.



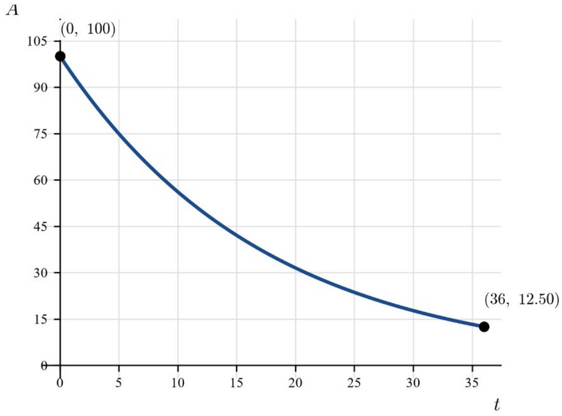
Q4. $A = 8000(1.05)^t$. (a) \$8000. (b) $A(10) = 8000(1.05)^{10} \approx \$13\,031.16$. (c) each year multiplies by the constant 1.05, i.e. $\frac{dA}{dt} \propto A$ — exponential growth.



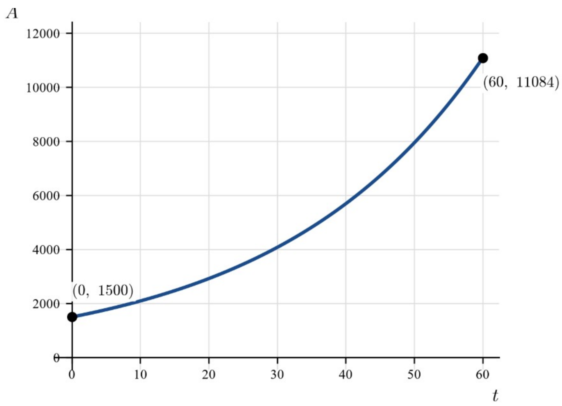
Q5. $T = 20 + 70e^{-t/8}$. (a) $T(0) = 20 + 70 = 90$°C. (b) $T(8) = 20 + 70e^{-1} \approx 45.75$°C. (c) as $t \to \infty$, $e^{-t/8} \to 0$, so $T \to 20$°C. (d) $20 + 70e^{-t/8} = 40 \Rightarrow e^{-t/8} = \frac{2}{7} \Rightarrow t = 8\log_e\left(\frac{7}{2}\right) \approx 10.02$ min.



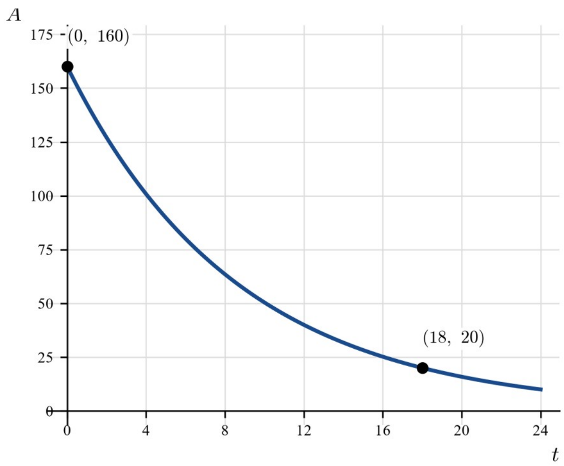
Q6. $A = 100 \cdot 2^{-t/12}$. (a) half-life 12. (b) $2^{-t/12} = \frac{12.5}{100} = \frac{1}{8} = 2^{-3} \Rightarrow \frac{t}{12} = 3 \Rightarrow t = 36$.



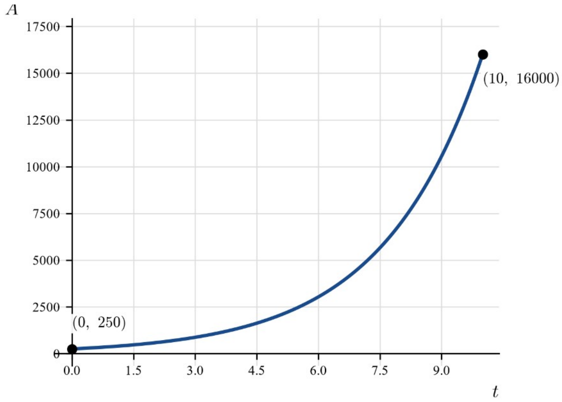
Q7. $A_0 = 1500$; $P(60) = 1500\, e^{60/30} = 1500\, e^2 \approx 11\,084$; doubling $= 30 \log_e 2 \approx 20.79$.



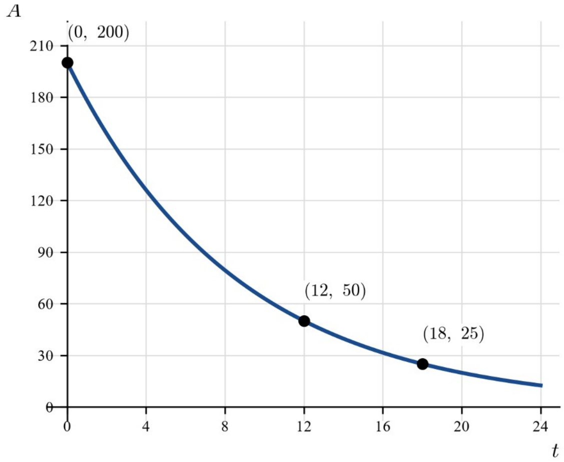
Q8. Half-life 6 h $\Rightarrow A = 160 \cdot 2^{-t/6}$. $20 = 160 \cdot 2^{-t/6} \Rightarrow 2^{-t/6} = \dfrac{1}{8} = 2^{-3} \Rightarrow t = 18$ h.



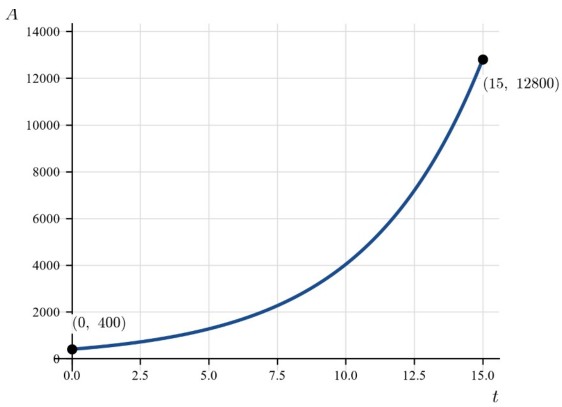
Q9. $b^5 = 2000/250 = 8 \Rightarrow b = 8^{1/5} = 2^{3/5}$, so $N = 250 \cdot 2^{3t/5}$. $16000 = 250 \cdot 2^{3t/5} \Rightarrow 2^{3t/5} = 64 = 2^6 \Rightarrow \dfrac{3t}{5} = 6 \Rightarrow t = 10$ h.



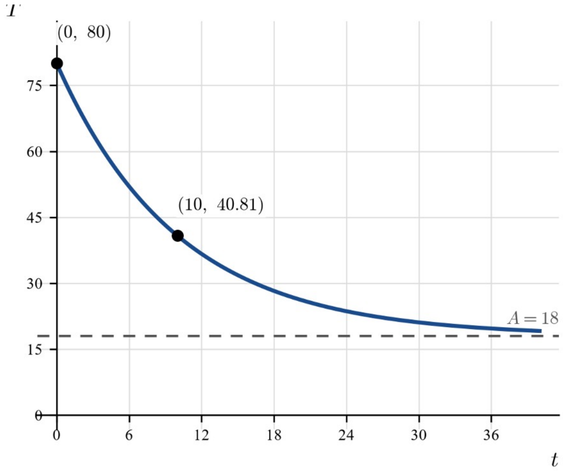
Q10. $A = 200 \cdot 2^{-t/6}$. $A(12) = 200 \cdot 2^{-2} = 50$ mg. $25 = 200 \cdot 2^{-t/6} \Rightarrow 2^{-t/6} = \frac{1}{8} = 2^{-3} \Rightarrow t = 18$ h.



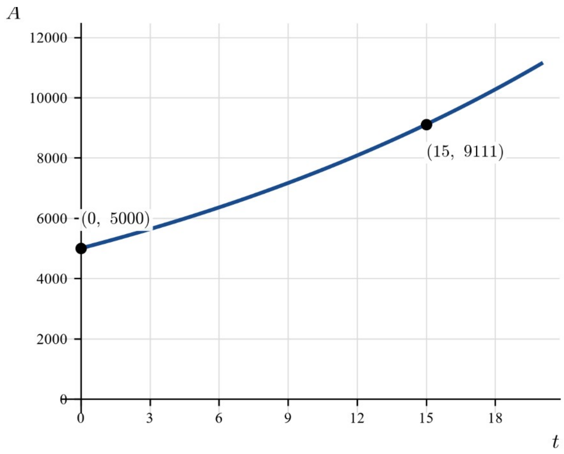
Q11. $A = 400 \cdot 2^{t/3}$. $A(9) = 400 \cdot 2^3 = 3200$. $12800 = 400 \cdot 2^{t/3} \Rightarrow 2^{t/3} = 32 = 2^5 \Rightarrow t = 15$ h.



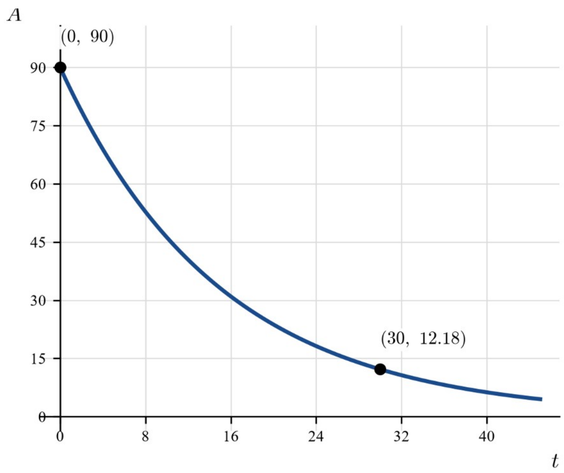
Q12. $T = 18 + 62e^{-t/10}$. $T(0) = 18 + 62 = 80$°C; $T(10) = 18 + 62e^{-1} \approx 40.81$°C.



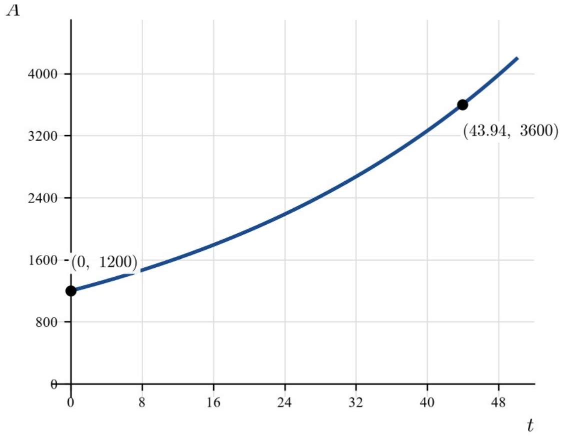
Q13. $A=5000e^{0.04t}$. $A(15)=5000e^{0.6}\approx\$9110.59$. Doubles when $e^{0.04t}=2\Rightarrow t=\dfrac{\log_e 2}{0.04}=25\log_e 2\approx17.33$ yr.



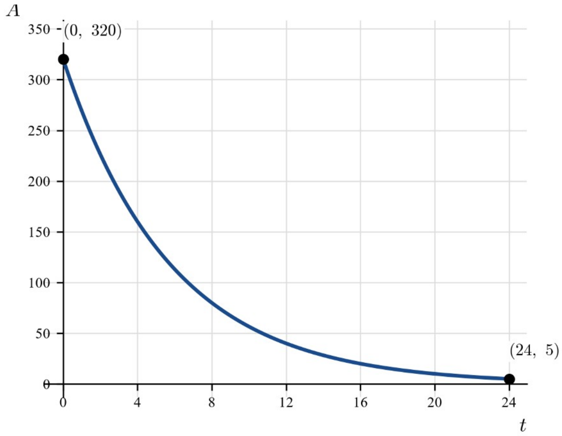
Q14. $A=90e^{-t/15}$. $A(30)=90e^{-2}\approx12.18$. Half-life $=15\log_e 2\approx10.40$.



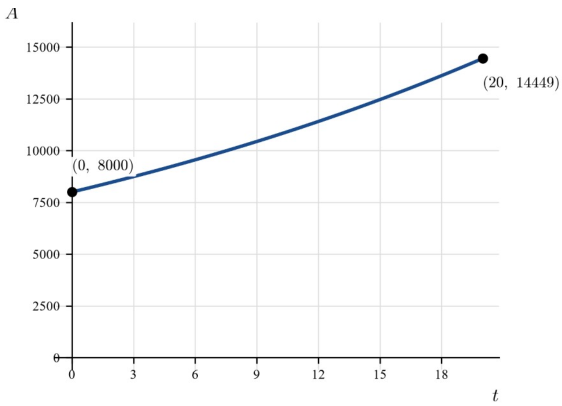
Q15. $1200e^{t/40}=3600\Rightarrow e^{t/40}=3\Rightarrow\dfrac{t}{40}=\log_e 3\Rightarrow t=40\log_e 3\approx43.94$.



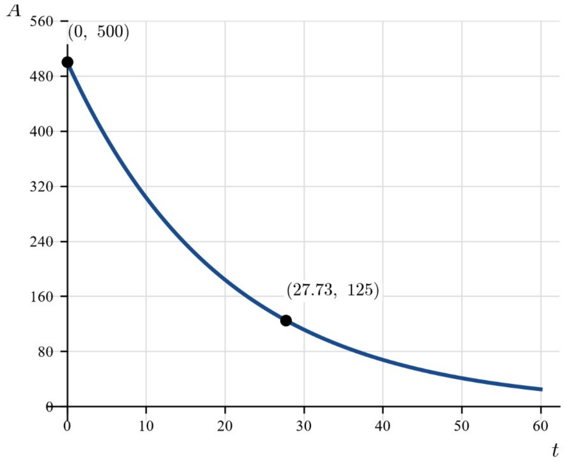
Q16. $A = 320 \cdot 2^{-t/4}$. $5 = 320 \cdot 2^{-t/4} \Rightarrow 2^{-t/4} = \dfrac{1}{64} = 2^{-6} \Rightarrow \dfrac{t}{4} = 6 \Rightarrow t = 24$ h.



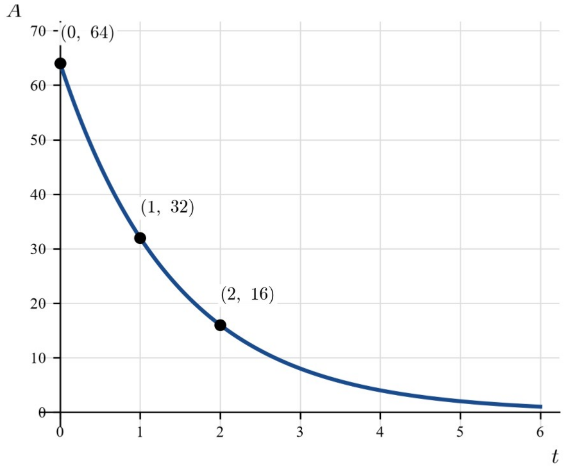
Q17. $P = 8000(1.03)^{20} \approx 14\,449$.



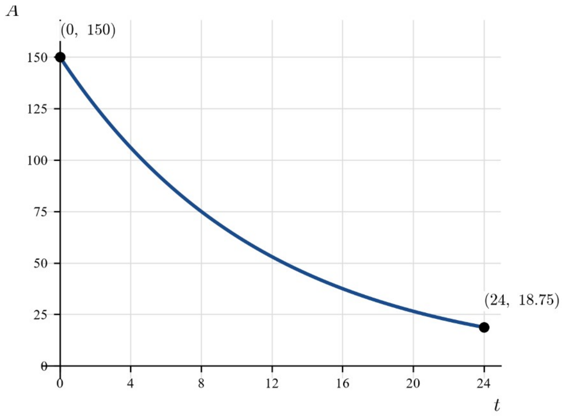
Q18. $500e^{-t/20} = 125 \Rightarrow e^{-t/20} = \dfrac{1}{4} \Rightarrow \dfrac{t}{20} = \log_e 4 \Rightarrow t = 20\log_e 4 = 40\log_e 2 \approx 27.73$.



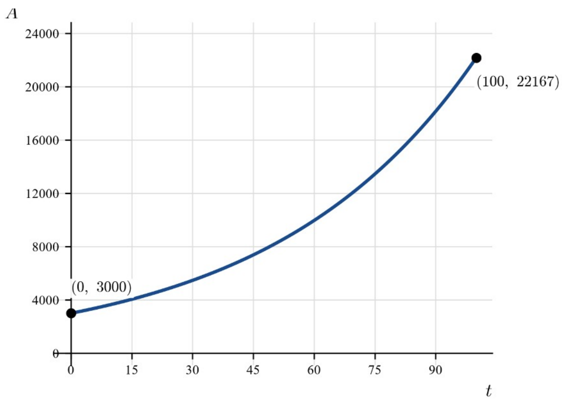
Q19. $A = A_0 b^t$ with $A_0 = 64$. From $A(2) = 16$: $64b^2 = 16 \Rightarrow b^2 = \frac{1}{4} \Rightarrow b = \frac{1}{2}$. So $A = 64\left(\frac{1}{2}\right)^t = 64 \cdot 2^{-t} = 64 \cdot 4^{-t/2}$ (as $4^{-t/2} = 2^{-t}$). Then $A(1) = 64 \cdot 2^{-1} = 32$.



Q20. $A = 150 \cdot 2^{-t/8}$. $A(24) = 150 \cdot 2^{-3} = 18.75$ mg. So 18.75 mg remains at $t = 24$ h (three half-lives).



Q21. $A = 3000e^{t/50}$. $A(100) = 3000e^2 \approx 22\,167$. $6000 = 3000e^{t/50} \Rightarrow e^{t/50} = 2 \Rightarrow t = 50\log_e 2 \approx 34.66$.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