

Mathematical Methods Units 3 & 4

Chapter 4 — Polynomial functions

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Chapter 4 Polynomial functions — 4A Quadratic functions

Theory

A **quadratic function** has the form $y = ax^2 + bx + c$ with $a \neq 0$. Its graph is a **parabola**, which opens **upwards** when $a > 0$ (with a **minimum** turning point) and **downwards** when $a < 0$ (with a **maximum** turning point).

Three useful forms.

- **General (polynomial) form** $y = ax^2 + bx + c$: the y -intercept is $(0, c)$.
- **Turning-point form** $y = a(x - h)^2 + k$: the turning point is (h, k) and the axis of symmetry is $x = h$.
- **Factored form** $y = a(x - p)(x - q)$: the x -intercepts are $x = p$ and $x = q$.

Axis of symmetry and turning point (from general form). The axis of symmetry is

$$x = -\frac{b}{2a},$$

and substituting this value into the rule gives the y -coordinate of the turning point.

Completing the square converts general form to turning-point form. For a monic quadratic $x^2 + bx + c$, add and subtract $\left(\frac{b}{2}\right)^2$:

$$x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c.$$

The discriminant $\Delta = b^2 - 4ac$ determines the number of x -intercepts (real solutions of $ax^2 + bx + c = 0$):

Discriminant	Real solutions	x -intercepts
$\Delta > 0$	two	two
$\Delta = 0$	one (repeated)	one (the parabola touches the axis at its turning point)
$\Delta < 0$	none	none (the parabola lies entirely above or below the x -axis)

When $\Delta > 0$ but is not a perfect square, the roots are **surds** — give them exactly, e.g. $x = \frac{-b \pm \sqrt{\Delta}}{2a}$.

Maximum and minimum values. The turning point gives the greatest value of the function (when $a < 0$) or the least value (when $a > 0$).

To sketch a parabola: (1) decide the shape from the sign of a ; (2) find the turning point (from turning-point form, or via $x = -b/2a$); (3) find the y -intercept; (4) find the x -intercepts, if any; (5) mark and label these points and draw a smooth curve.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = x^2 - 6x + 5$, labelling the turning point and all axis intercepts.

Working

Comment

$a = 1 > 0$, so the parabola has a **minimum**.

The sign of a gives the shape.

Axis of symmetry: $x = -\frac{-6}{2 \times 1} = 3$.

Use $x = -\frac{b}{2a}$ with $b = -6$.

$y(3) = 3^2 - 6(3) + 5 = -4$, so the turning point is $(3, -4)$.

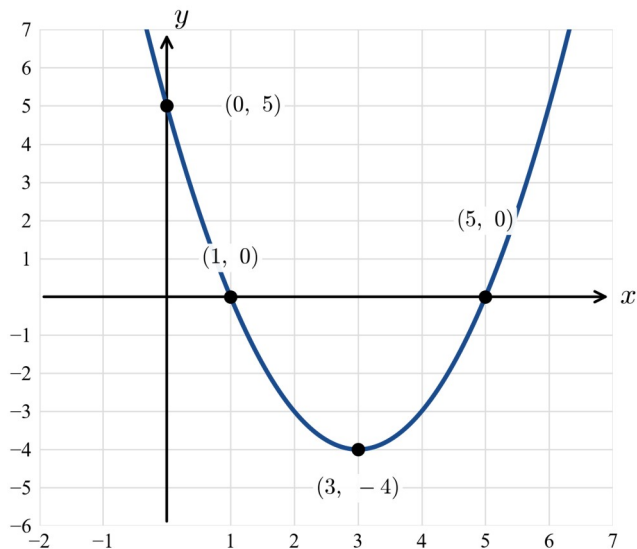
Substitute the axis value into the rule.

y -intercept: $x = 0 \Rightarrow y = 5$, giving $(0, 5)$.

The y -intercept is $(0, c)$.

x -intercepts: $x^2 - 6x + 5 = (x - 1)(x - 5) = 0$, so $x = 1$ or $x = 5$, giving $(1, 0)$ and $(5, 0)$.

Factorise, then apply the null factor law.



Example 2 — scaffolded

Express $y = x^2 + 4x + 1$ in turning-point form by completing the square, hence state the turning point and find the exact x -intercepts, and sketch the graph.

$$x^2 + 4x + 1 = (x^2 + 4x + 4) - 4 + 1$$

Add and subtract $(4/2)^2 = 4$.

$$= (x + 2)^2 - 3$$

Turning-point form; the turning point is $(-2, -3)$.

x -intercepts:

Keep the answer exact — the roots are surds.

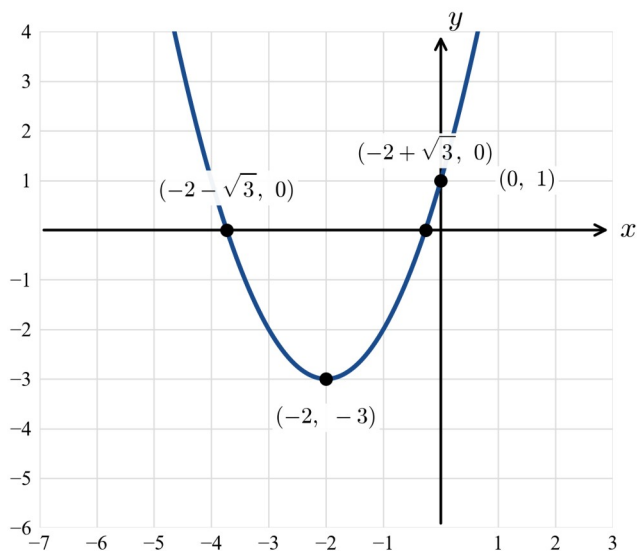
$$(x + 2)^2 - 3 = 0 \Rightarrow (x + 2)^2 = 3 \Rightarrow x + 2 = \pm\sqrt{3}, \text{ so}$$

$$x = -2 \pm \sqrt{3}.$$

y -intercept: $x = 0 \Rightarrow y = 1$, giving $(0, 1)$.

Substitute $x = 0$.

$\therefore$ turning point $(-2, -3)$; intercepts $(-2 - \sqrt{3}, 0)$, $(-2 + \sqrt{3}, 0)$, $(0, 1)$.



Example 3 — unscaffolded

For what values of m does the equation $x^2 + mx + 9 = 0$ have (i) two solutions, (ii) one solution, (iii) no real solutions?

The discriminant is $\Delta = m^2 - 4(1)(9) = m^2 - 36$.

(i) Two solutions when $\Delta > 0$: $m^2 - 36 > 0 \Rightarrow m^2 > 36 \Rightarrow m < -6$ or $m > 6$.

(ii) One solution when $\Delta = 0$: $m^2 = 36 \Rightarrow m = -6$ or $m = 6$.

(iii) No real solutions when $\Delta < 0$: $m^2 < 36 \Rightarrow -6 < m < 6$.

$\therefore$ two solutions for $m < -6$ or $m > 6$; one for $m = \pm 6$; none for $-6 < m < 6$.

Example 4 — unscaffolded

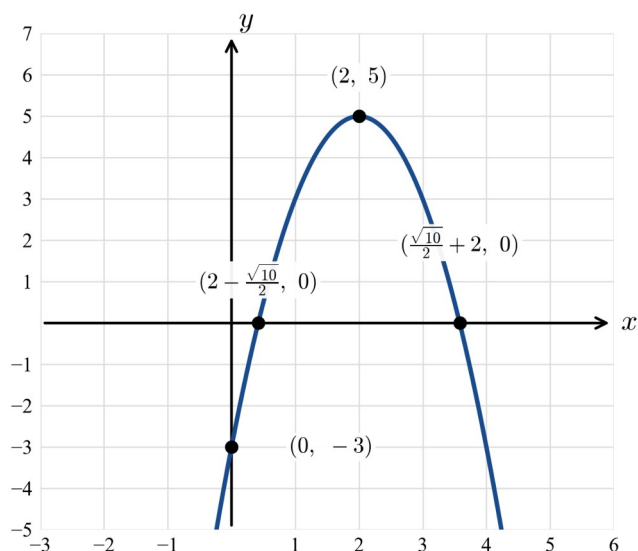
Find the maximum value of $y = -2x^2 + 8x - 3$ and the value of x at which it occurs, and sketch the graph.

$a = -2 < 0$, so the parabola has a **maximum**. Axis: $x = -\frac{8}{2 \times (-2)} = 2$.

$$y(2) = -2(2)^2 + 8(2) - 3 = -8 + 16 - 3 = 5.$$

x-intercepts: $\Delta = 8^2 - 4(-2)(-3) = 40$, so $x = \frac{-8 \pm \sqrt{40}}{2 \times (-2)} = \frac{-8 \pm 2\sqrt{10}}{-4} = 2 \pm \frac{\sqrt{10}}{2}$, giving $\left(2 - \frac{\sqrt{10}}{2}, 0\right)$ and $\left(2 + \frac{\sqrt{10}}{2}, 0\right)$.

$\therefore$ the maximum value is 5, occurring at $x = 2$ (turning point $(2, 5)$); intercepts $\left(2 - \frac{\sqrt{10}}{2}, 0\right)$, $\left(2 + \frac{\sqrt{10}}{2}, 0\right)$, $(0, -3)$.



Practice questions

Scaffolded

Q1. For $y = x^2 - 2x - 8$: (a) state the shape; (b) find the axis of symmetry and hence the turning point; (c) find the axis intercepts; (d) sketch the graph, labelling all key features.

Q2. For $y = (x + 3)^2 - 4$: (a) write the turning point; (b) find the y -intercept; (c) find the x -intercepts; (d) sketch the graph.

Q3. For $y = x^2 + 2x - 1$: (a) express the rule in turning-point form by completing the square; (b) state the turning point; (c) find the exact x -intercepts; (d) sketch the graph.

Q4. Consider $x^2 - 4x + k = 0$. (a) write the discriminant in terms of k ; (b) find the value of k for which the equation has exactly one solution; (c) state what this means for the graph of $y = x^2 - 4x + k$.

Q5. For $y = 2x^2 - 4x + 5$: (a) find the turning point; (b) evaluate the discriminant and explain why the graph has no x -intercepts; (c) state the minimum value; (d) sketch the graph.

Q6. For $y = -x^2 + 4x + 5$: (a) state the shape and the turning point; (b) find all axis intercepts; (c) state the maximum value and the range; (d) sketch the graph.

Un scaffolded

Sketch each of Q7–Q15 and Q21, labelling the turning point and all axis intercepts (exact values).

Q7. $y = x^2 - 4x + 3$

Q8. $y = x^2 + 6x + 8$

Q9. $y = -x^2 + 2x + 3$

Q10. $y = 2x^2 + 2x - 4$

Q11. $y = x^2 - 2x - 2$

Q12. $y = x^2 + 4x$

Q13. $y = x^2 + x - 6$

Q14. $y = x^2 - 6x + 7$

Q15. $y = 3x^2 - 6x$

Q16. Find the values of m for which $x^2 + mx + 4 = 0$ has exactly one solution.

Q17. Find the values of k for which $2x^2 + 3x + k = 0$ has no real solutions.

Q18. Show that $y = x^2 - 2x + 5$ is positive for all real x , and sketch the graph.

Q19. A ball is thrown so that its height h metres after t seconds is $h = -5t^2 + 20t$. Find the maximum height reached and the time at which it occurs.

Q20. Find the value(s) of p for which the graph of $y = x^2 + px + p$ has exactly one x -intercept.

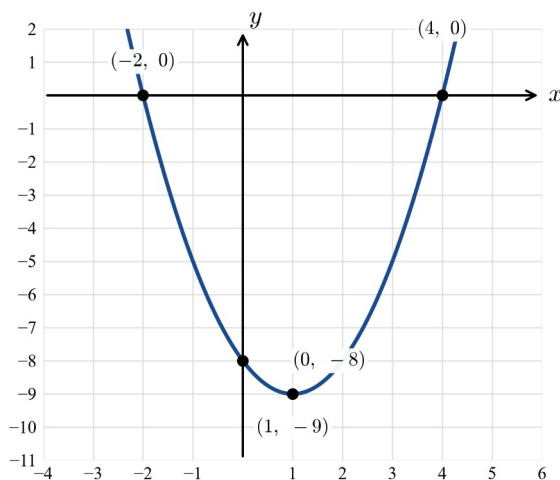
Q21. $y = -2x^2 - 4x + 6$

Answers

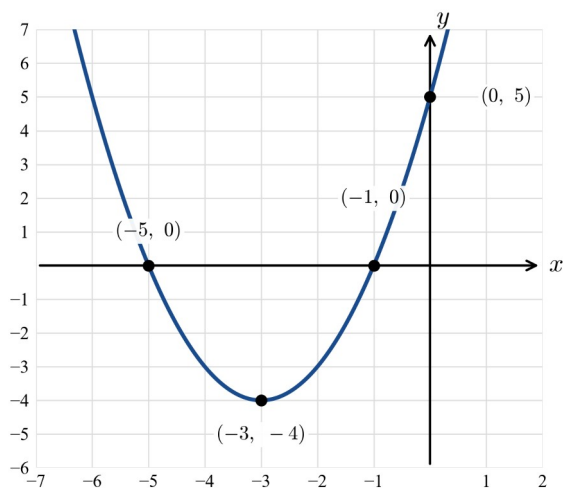
Q1. Minimum; axis $x=1$; TP $(1, -9)$; intercepts $(-2, 0)$, $(4, 0)$, $(0, -8)$. **Q2.** TP $(-3, -4)$; y -intercept $(0, 5)$; x -intercepts $(-5, 0)$, $(-1, 0)$. **Q3.** $y=(x+1)^2-2$; TP $(-1, -2)$; x -intercepts $(-1-\sqrt{2}, 0)$, $(-1+\sqrt{2}, 0)$. **Q4.** $\Delta=16-4k$; one solution when $k=4$; the graph then touches the x -axis at one point (its turning point). **Q5.** TP $(1, 3)$; $\Delta=-24<0$, so no x -intercepts; minimum value 3. **Q6.** Maximum; TP $(2, 9)$; intercepts $(-1, 0)$, $(5, 0)$, $(0, 5)$; maximum value 9; range $\{y: y \leq 9\}$. **Q7.** TP $(2, -1)$; intercepts $(1, 0)$, $(3, 0)$, $(0, 3)$. **Q8.** TP $(-3, -1)$; intercepts $(-4, 0)$, $(-2, 0)$, $(0, 8)$. **Q9.** TP $(1, 4)$; intercepts $(-1, 0)$, $(3, 0)$, $(0, 3)$. **Q10.** TP $\left(-\frac{1}{2}, -\frac{9}{2}\right)$; intercepts $(-2, 0)$, $(1, 0)$, $(0, -4)$. **Q11.** TP $(1, -3)$; intercepts $(1-\sqrt{3}, 0)$, $(1+\sqrt{3}, 0)$, $(0, -2)$. **Q12.** TP $(-2, -4)$; intercepts $(-4, 0)$, $(0, 0)$. **Q13.** TP $\left(-\frac{1}{2}, -\frac{25}{4}\right)$; intercepts $(-3, 0)$, $(2, 0)$, $(0, -6)$. **Q14.** TP $(3, -2)$; intercepts $(3-\sqrt{2}, 0)$, $(3+\sqrt{2}, 0)$, $(0, 7)$. **Q15.** TP $(1, -3)$; intercepts $(0, 0)$, $(2, 0)$. **Q16.** $m=-4$ or $m=4$. **Q17.** $k>\frac{9}{8}$. **Q18.** $\Delta=-16<0$ and $a>0$, so $y>0$ for all x (minimum value 4 at $(1, 4)$). **Q19.** Maximum height 20 m at $t=2$ s. **Q20.** $p=0$ or $p=4$. **Q21.** TP $(-1, 8)$; intercepts $(-3, 0)$, $(1, 0)$, $(0, 6)$.

Fully-worked solutions

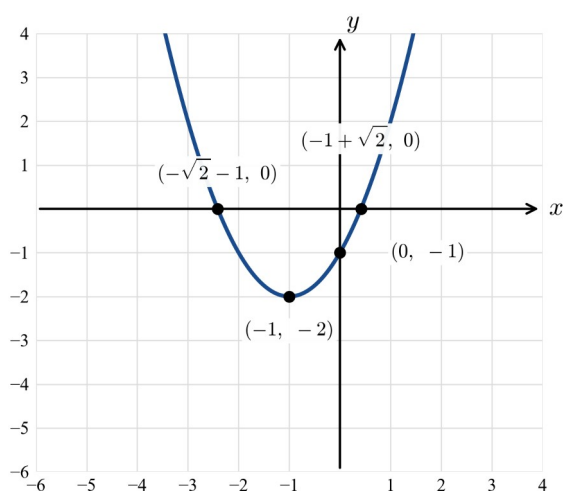
Q1. $y=x^2-2x-8$. $a=1>0$: minimum. Axis $x=-\frac{-2}{2}=1$; $y(1)=1-2-8=-9$, TP $(1, -9)$. y -intercept $(0, -8)$. x -intercepts: $x^2-2x-8=(x+2)(x-4)=0$, so $(-2, 0)$ and $(4, 0)$.



Q2. $y=(x+3)^2-4$. TP $(-3, -4)$. y -intercept: $x=0 \Rightarrow y=9-4=5$, giving $(0, 5)$. x -intercepts: $(x+3)^2=4 \Rightarrow x+3=\pm 2$, so $x=-5$ or $x=-1$, giving $(-5, 0)$ and $(-1, 0)$.



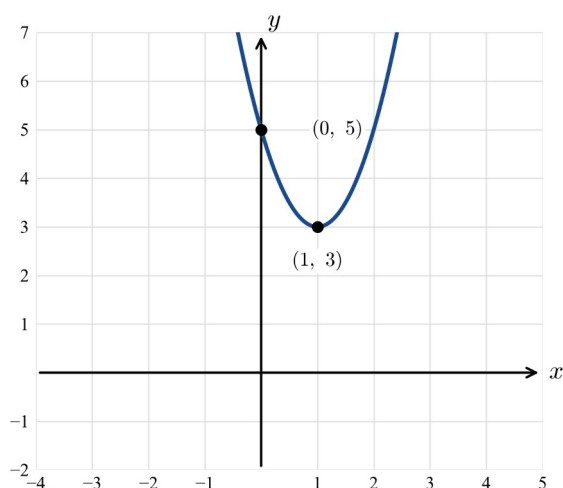
Q3. $y = x^2 + 2x - 1$. (a) $x^2 + 2x - 1 = (x+1)^2 - 1 - 1 = (x+1)^2 - 2$. (b) TP $(-1, -2)$. (c) $(x+1)^2 = 2 \Rightarrow x+1 = \pm\sqrt{2}$, so $x = -1 \pm \sqrt{2}$, giving $(-1 - \sqrt{2}, 0)$ and $(-1 + \sqrt{2}, 0)$; y-intercept $(0, -1)$.



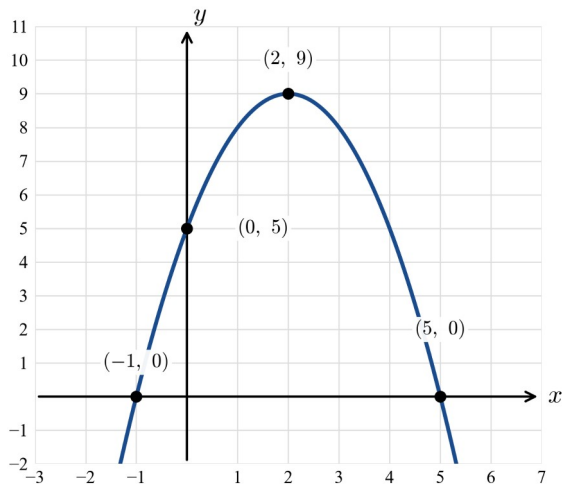
Q4. $x^2 - 4x + k = 0$. (a) $\Delta = (-4)^2 - 4(1)(k) = 16 - 4k$. (b) One solution when $\Delta = 0$: $16 - 4k = 0 \Rightarrow k = 4$. (c) The graph of $y = x^2 - 4x + 4$ touches the x-axis at exactly one point — its turning point $(2, 0)$.

Q5. $y = 2x^2 - 4x + 5$. Axis $x = -\frac{-4}{2 \times 2} = 1$; $y(1) = 2 - 4 + 5 = 3$, TP $(1, 3)$.

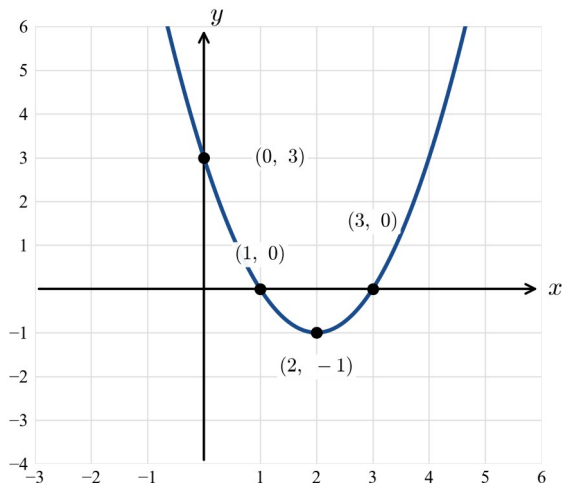
$\Delta = (-4)^2 - 4(2)(5) = 16 - 40 = -24 < 0$, so there are no x-intercepts. Since $a > 0$, the minimum value is 3. y-intercept $(0, 5)$.



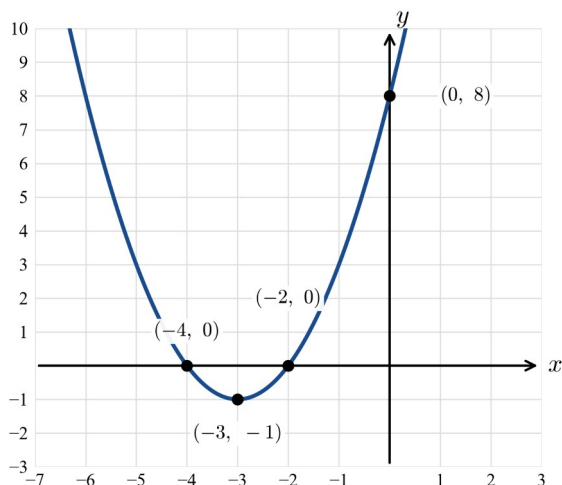
Q6. $y = -x^2 + 4x + 5$. $a = -1 < 0$: maximum. Axis $x = -\frac{4}{2 \times (-1)} = 2$; $y(2) = -4 + 8 + 5 = 9$, TP $(2, 9)$. x -intercepts: $-x^2 + 4x + 5 = 0 \Rightarrow x^2 - 4x - 5 = (x - 5)(x + 1) = 0$, so $(-1, 0)$ and $(5, 0)$; y -intercept $(0, 5)$. Maximum value 9; range $\{y : y \leq 9\}$.



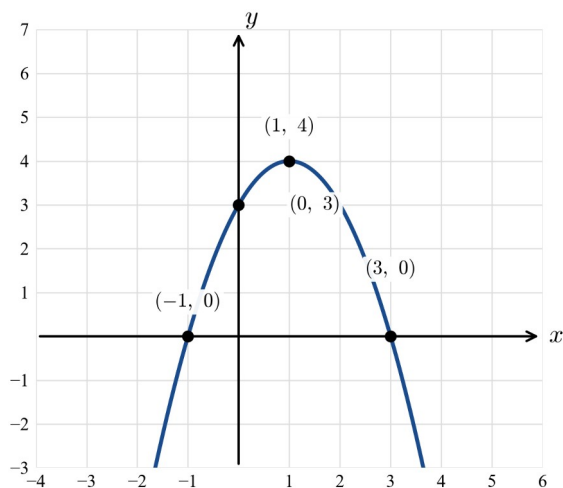
Q7. $y = x^2 - 4x + 3$. Axis $x = 2$; TP $(2, -1)$. $(x - 1)(x - 3) = 0$, intercepts $(1, 0)$, $(3, 0)$; y -intercept $(0, 3)$.



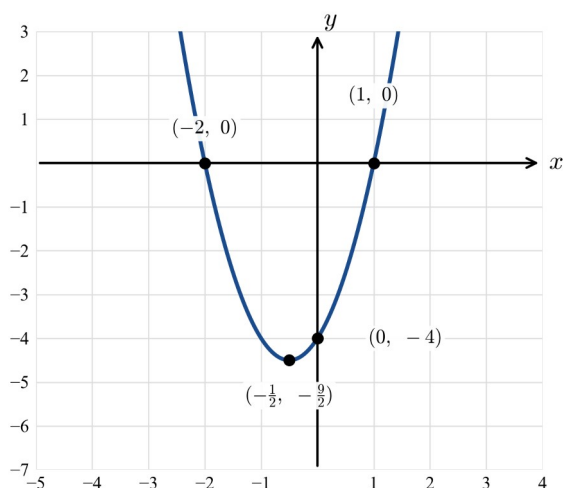
Q8. $y = x^2 + 6x + 8$. Axis $x = -3$; TP $(-3, -1)$. $(x + 4)(x + 2) = 0$, intercepts $(-4, 0)$, $(-2, 0)$; y -intercept $(0, 8)$.



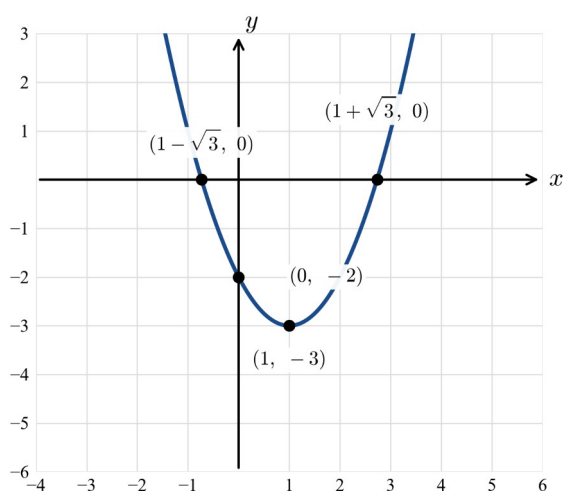
Q9. $y = -x^2 + 2x + 3$. $a < 0$: maximum. Axis $x = 1$; TP $(1, 4)$. $x^2 - 2x - 3 = (x - 3)(x + 1) = 0$, intercepts $(-1, 0)$, $(3, 0)$; y -intercept $(0, 3)$.



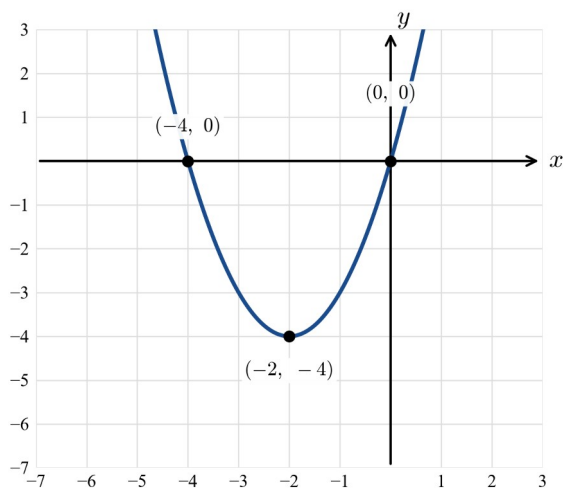
Q10. $y = 2x^2 + 2x - 4$. Axis $x = -\frac{2}{4} = -\frac{1}{2}$; $y(-1/2) = 2 \cdot 1/4 - 1 - 4 = -\frac{9}{2}$, TP $\left(-\frac{1}{2}, -\frac{9}{2}\right)$.
 $2(x^2 + x - 2) = 2(x + 2)(x - 1) = 0$, intercepts $(-2, 0)$, $(1, 0)$; y -intercept $(0, -4)$.



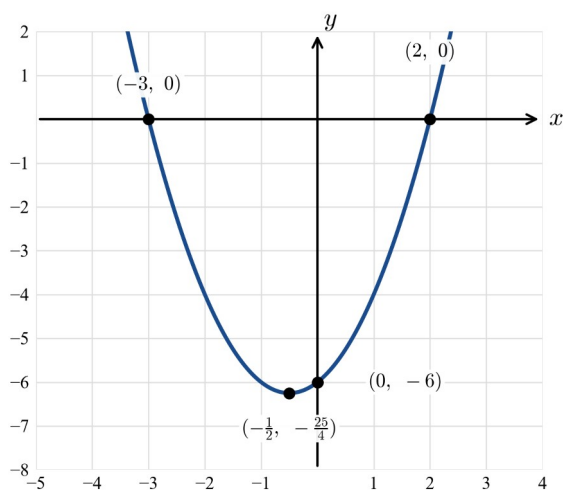
Q11. $y = x^2 - 2x - 2$. Axis $x = 1$; TP $(1, -3)$. $(x - 1)^2 = 3 \Rightarrow x = 1 \pm \sqrt{3}$, intercepts $(1 - \sqrt{3}, 0)$, $(1 + \sqrt{3}, 0)$; y -intercept $(0, -2)$.



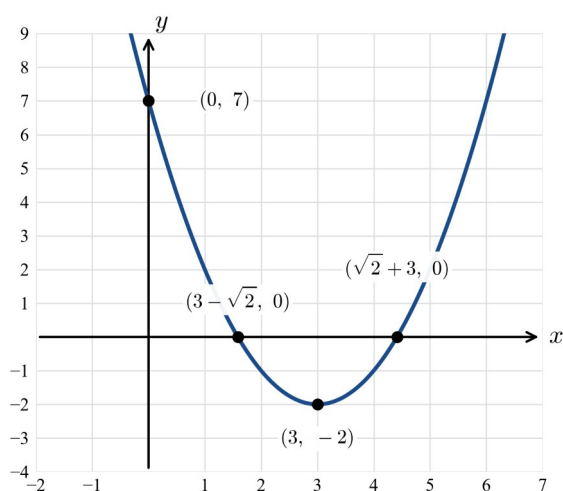
Q12. $y = x^2 + 4x = x(x + 4)$, intercepts $(0, 0)$, $(-4, 0)$; axis $x = -2$; TP $(-2, -4)$.



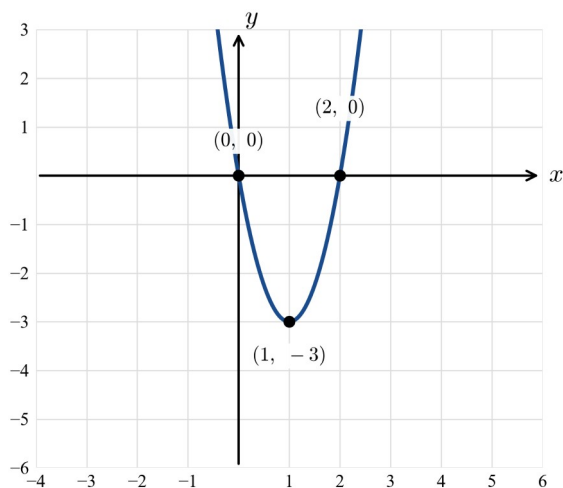
Q13. $y = x^2 + x - 6$. Axis $x = -\frac{1}{2}$; $y(-\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} - 6 = -\frac{25}{4}$, TP $(-\frac{1}{2}, -\frac{25}{4})$. $(x+3)(x-2)=0$, intercepts $(-3, 0), (2, 0)$; y-intercept $(0, -6)$.



Q14. $y = x^2 - 6x + 7$. Completing the square: $(x-3)^2 - 2$, TP $(3, -2)$. $(x-3)^2 = 2 \Rightarrow x = 3 \pm \sqrt{2}$, intercepts $(3 - \sqrt{2}, 0), (3 + \sqrt{2}, 0)$; y-intercept $(0, 7)$.



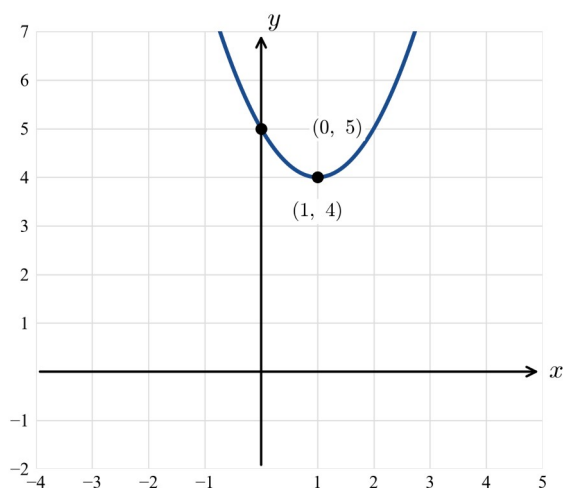
Q15. $y = 3x^2 - 6x = 3x(x-2)$, intercepts $(0, 0), (2, 0)$; axis $x = 1$; $y(1) = 3 - 6 = -3$, TP $(1, -3)$.



Q16. $x^2 + mx + 4 = 0$ has one solution when $\Delta = m^2 - 4(1)(4) = m^2 - 16 = 0$, so $m = -4$ or $m = 4$.

Q17. $2x^2 + 3x + k = 0$ has no real solutions when $\Delta = 3^2 - 4(2)(k) = 9 - 8k < 0$, i.e. $k > \frac{9}{8}$.

Q18. $y = x^2 - 2x + 5 = (x - 1)^2 + 4$. Since $(x - 1)^2 \geq 0$, $y \geq 4 > 0$ for all real x . Equivalently $\Delta = (-2)^2 - 4(1)(5) = -16 < 0$ with $a > 0$, so the parabola lies entirely above the x -axis (minimum value 4 at $(1, 4)$); y -intercept $(0, 5)$.

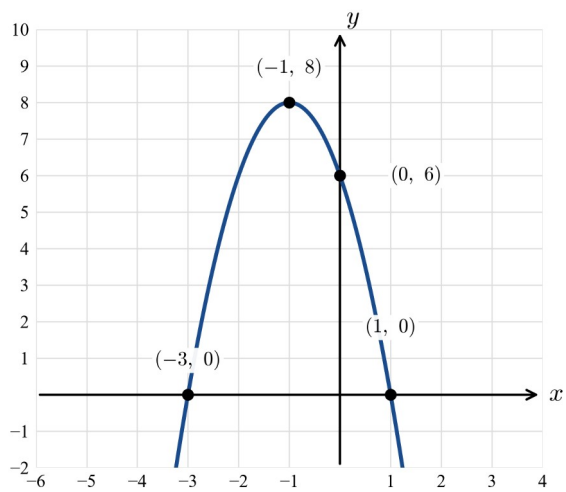


Q19. $h = -5t^2 + 20t$. Axis $t = -\frac{20}{2 \times (-5)} = 2$; $h(2) = -5(4) + 20(2) = -20 + 40 = 20$. $\therefore$ the maximum height is 20 m, reached at $t = 2$ s.

Q20. $y = x^2 + px + p$ has one x -intercept when $\Delta = p^2 - 4(1)(p) = p^2 - 4p = 0$, i.e. $p(p - 4) = 0$, so $p = 0$ or $p = 4$.

Q21. $y = -2x^2 - 4x + 6$. $a < 0$: maximum. Axis $x = -\frac{-4}{2 \times (-2)} = -1$; $y(-1) = -2 + 4 + 6 = 8$, TP $(-1, 8)$.

$-2(x^2 + 2x - 3) = -2(x + 3)(x - 1) = 0$, intercepts $(-3, 0)$, $(1, 0)$; y -intercept $(0, 6)$.



Chapter 4 Polynomial functions — 4B Finding the rule of a quadratic

Theory

Given enough information about a parabola — points it passes through, its turning point, or its x -intercepts — we can recover its rule. Choose the **form** that matches the information given.

The three forms of a quadratic.

- **General form** $y = ax^2 + bx + c$ — use when you are given **three points**. Substituting each point gives three linear equations in a, b, c ; solve simultaneously. (If one point is the y -intercept $(0, c)$, then c is read off immediately, leaving two equations.)
- **Turning-point form** $y = a(x - h)^2 + k$ — use when you are given the **turning point** (h, k) . Write h and k straight in, then substitute one more point to find a .
- **Factored form** $y = a(x - p)(x - q)$ — use when you are given the **x -intercepts** $x = p$ and $x = q$. Substitute one more point to find a .

Useful facts.

- The **axis of symmetry** is halfway between the x -intercepts, at $x = \frac{p+q}{2}$, and the turning point lies on it.
- A **maximum** or **minimum value** of the parabola is the y -coordinate k of the turning point (with $a < 0$ for a maximum, $a > 0$ for a minimum).
- Two points with the **same y -value** are symmetric about the axis, so the axis is halfway between their x -coordinates.

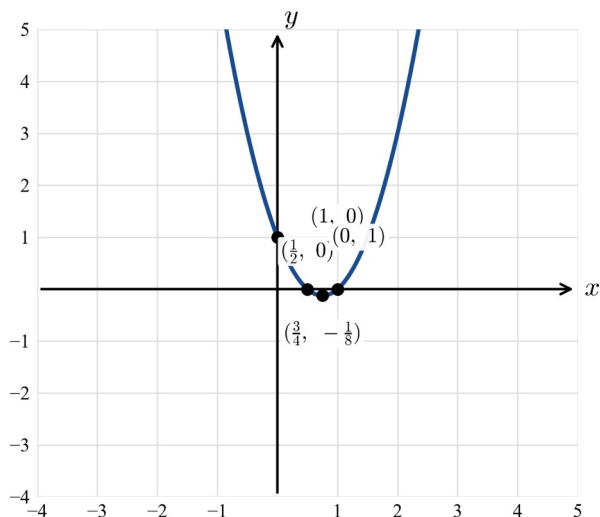
Always finish by **checking** the rule against a point you have not yet used.

Worked examples

Example 1 — scaffolded

A parabola passes through $(0, 1)$, $(1, 0)$ and $(2, 3)$. Find its rule.

Working	Comment
Use general form $y = ax^2 + bx + c$.	Three arbitrary points — general form is best.
$(0, 1)$: $c = 1$.	The point on the y -axis gives c directly.
$(1, 0)$: $a + b + c = 0 \Rightarrow a + b = -1$.	Substitute and use $c = 1$.
$(2, 3)$: $4a + 2b + c = 3 \Rightarrow 4a + 2b = 2 \Rightarrow 2a + b = 1$.	Second equation in a, b .
Subtract: $(2a + b) - (a + b) = 1 - (-1) \Rightarrow a = 2$, then $b = -3$.	Solve simultaneously.
$\therefore y = 2x^2 - 3x + 1$.	Check $(2, 3)$: $8 - 6 + 1 = 3$. ✓



Example 2 — scaffolded

A parabola has turning point $(2, -1)$ and passes through $(0, 3)$. Find its rule.

Working

Use turning-point form $y = a(x - h)^2 + k$ with $h = 2$,
 $k = -1$: $y = a(x - 2)^2 - 1$.

Substitute $(0, 3)$: $3 = a(0 - 2)^2 - 1 = 4a - 1$.

$4a = 4 \Rightarrow a = 1$.

$\therefore y = (x - 2)^2 - 1 = x^2 - 4x + 3$.

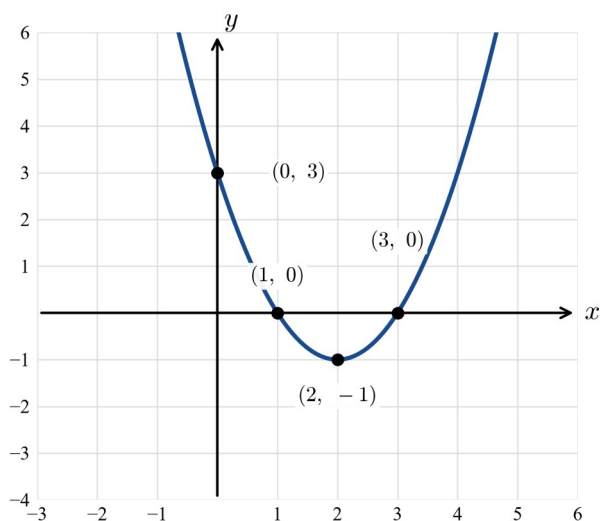
Comment

The turning point is given, so use turning-point form.

Use the extra point to find a .

Solve for a .

Expand if the general form is wanted.



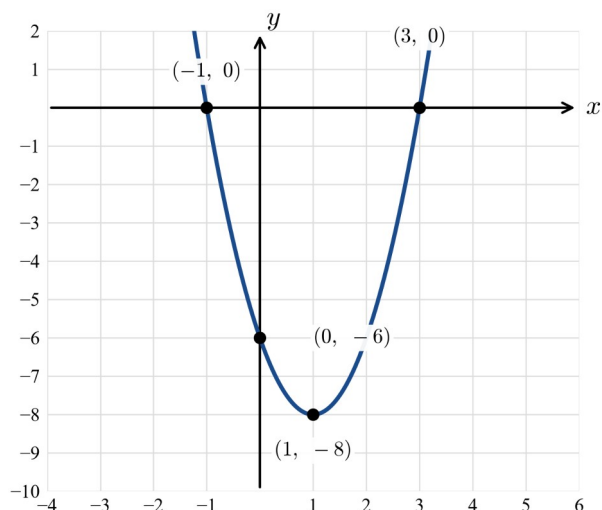
Example 3 — unscaffolded

A parabola has x -intercepts $x = -1$ and $x = 3$ and y -intercept -6 . Find its rule.

The x -intercepts give the factored form $y = a(x + 1)(x - 3)$.

Substitute the y -intercept $(0, -6)$: $-6 = a(0 + 1)(0 - 3) = -3a$, so $a = 2$.

$\therefore y = 2(x + 1)(x - 3) = 2x^2 - 4x - 6$.



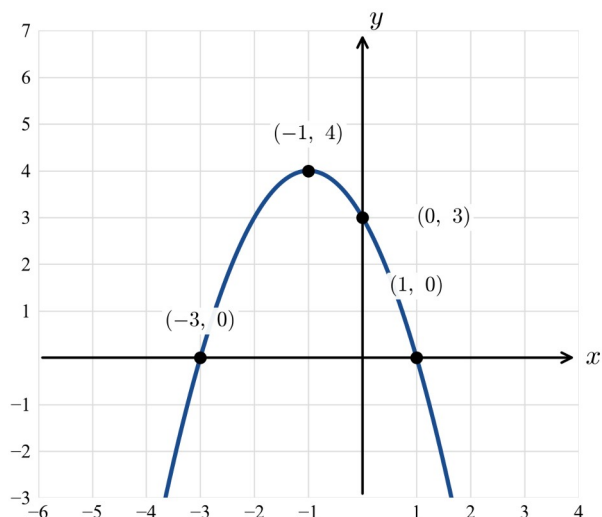
Example 4 — unscaffolded

The parabola shown has turning point $(-1, 4)$ and passes through $(0, 3)$. Find its rule.

Turning-point form: $y = a(x + 1)^2 + 4$.

Substitute $(0, 3)$: $3 = a(1) + 4 \Rightarrow a = -1$.

$\therefore y = -(x + 1)^2 + 4 = -x^2 - 2x + 3$. (The negative a agrees with the maximum turning point.)



Practice questions

Scaffolded

Q1. A parabola has turning point $(1, -4)$ and passes through $(3, 0)$. (a) Write the turning-point form with h and k filled in; (b) substitute $(3, 0)$ to find a ; (c) state the rule.

Q2. A parabola has x -intercepts -2 and 4 and y -intercept -8 . (a) Write the factored form $y = a(x - p)(x - q)$; (b) use the y -intercept to find a ; (c) state the rule.

Q3. A parabola passes through $(0, 6)$, $(1, 2)$ and $(3, 0)$. (a) Explain why $c = 6$; (b) form two equations in a and b ; (c) solve them and state the rule.

Q4. A parabola has turning point $(-2, 3)$ and passes through $(0, -5)$. (a) State h and k ; (b) find a ; (c) state the rule and its maximum value.

Q5. A parabola cuts the x -axis at -3 and 1 and passes through $(-1, -4)$. (a) Write the factored form; (b) find a ; (c) state the rule.

Q6. A parabola passes through $(-1, 6)$, $(0, 3)$ and $(1, 2)$. (a) Write down c ; (b) form and solve two equations in a and b ; (c) state the rule.

Unscaffolded

Determine the rule of each parabola.

Q7. Turning point $(3, 2)$, passing through $(1, 6)$.

Q8. Turning point $(-1, -2)$, passing through $(1, 6)$.

Q9. x -intercepts 2 and 5 , y -intercept 10 .

Q10. x -intercepts -4 and -1 , y -intercept -8 .

Q11. Passing through $(0, -1)$, $(1, 4)$ and $(-1, 0)$.

Q12. Passing through $(0, 2)$, $(2, 0)$ and $(3, 2)$.

Q13. Turning point $(2, 0)$, passing through $(0, 8)$.

Q14. x -intercepts -3 and 3 , passing through $(1, -8)$.

Q15. Turning point $(-2, -1)$, passing through $(-1, 2)$.

Q16. x -intercepts 0 and 2 , passing through $(1, -1)$.

Q17. x -intercepts 1 and 4 , passing through $(2, -3)$.

Q18. Turning point $(1, 2)$, passing through $(3, 0)$.

Q19. A parabola has axis of symmetry $x = 2$, a minimum value of 1 , and passes through $(0, 5)$. Determine its rule.

Q20. A parabola has x -intercepts -1 and 5 and a maximum value of 9 . Find its rule.

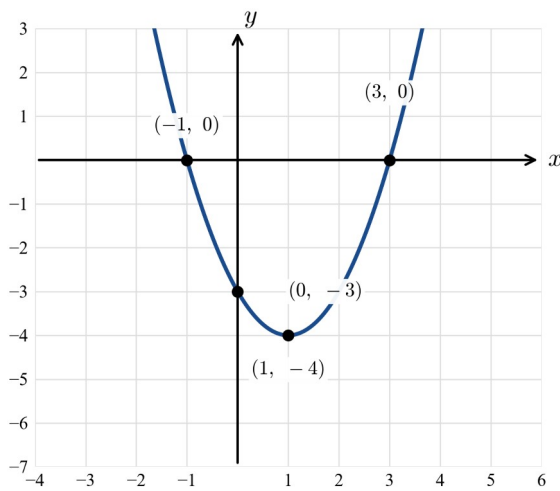
Q21. A parabola passes through $(0, 3)$, $(2, 3)$ and $(1, 1)$. Show that its axis of symmetry is $x = 1$ and its turning point is $(1, 1)$, and hence find its rule.

Answers

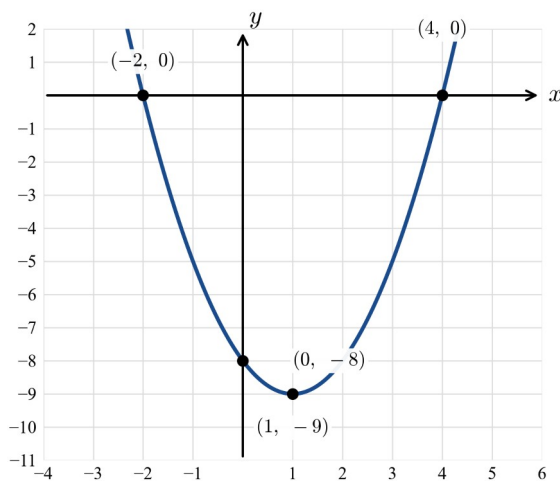
Q1. $y = (x-1)^2 - 4$ **Q2.** $y = (x+2)(x-4)$ **Q3.** $y = x^2 - 5x + 6$ **Q4.** $y = -2(x+2)^2 + 3$; maximum value 3 **Q5.**
 $y = (x+3)(x-1)$ **Q6.** $y = x^2 - 2x + 3$ **Q7.** $y = (x-3)^2 + 2$ **Q8.** $y = 2(x+1)^2 - 2$ **Q9.** $y = (x-2)(x-5)$ **Q10.**
 $y = -2(x+4)(x+1)$ **Q11.** $y = 3x^2 + 2x - 1$ **Q12.** $y = x^2 - 3x + 2$ **Q13.** $y = 2(x-2)^2$ **Q14.** $y = (x+3)(x-3)$ **Q15.**
 $y = 3(x+2)^2 - 1$ **Q16.** $y = x^2 - 2x$ **Q17.** $y = \frac{3}{2}(x-1)(x-4)$ **Q18.** $y = -\frac{1}{2}(x-1)^2 + 2$ **Q19.** $y = (x-2)^2 + 1$ **Q20.**
 $y = -(x+1)(x-5)$ **Q21.** $y = 2(x-1)^2 + 1$

Fully-worked solutions

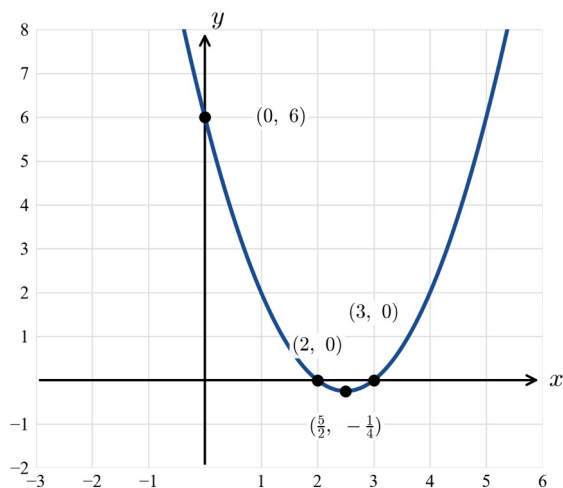
Q1. $y = a(x-1)^2 - 4$. Substitute $(3, 0)$: $0 = a(2)^2 - 4 = 4a - 4 \Rightarrow a = 1$. $\therefore y = (x-1)^2 - 4$.



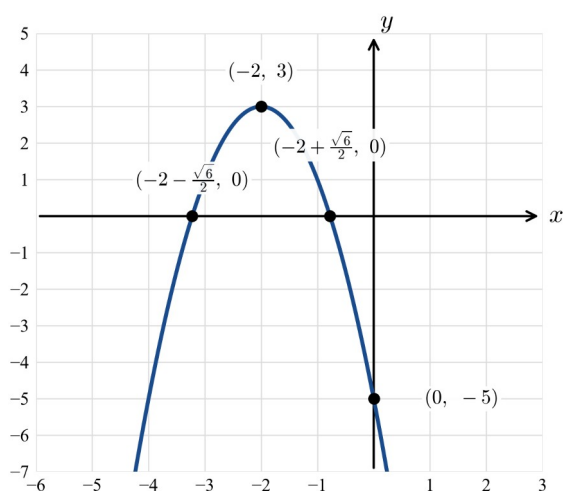
Q2. $y = a(x+2)(x-4)$. Substitute $(0, -8)$: $-8 = a(2)(-4) = -8a \Rightarrow a = 1$. $\therefore y = (x+2)(x-4)$.



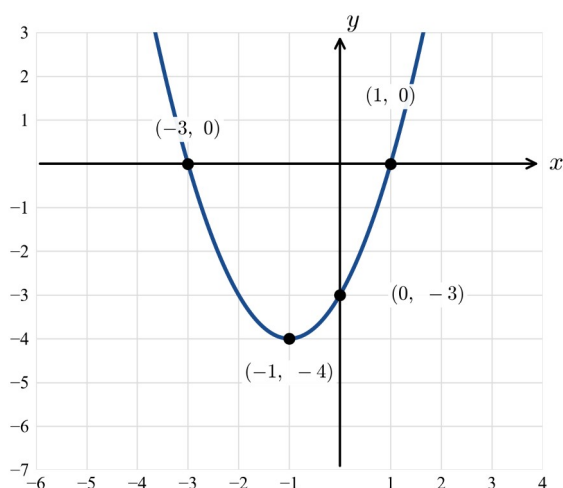
Q3. $c = 6$ from $(0, 6)$. $(1, 2)$: $a + b + 6 = 2 \Rightarrow a + b = -4$. $(3, 0)$: $9a + 3b + 6 = 0 \Rightarrow 3a + b = -2$. Subtract:
 $2a = 2 \Rightarrow a = 1, b = -5$. $\therefore y = x^2 - 5x + 6$.



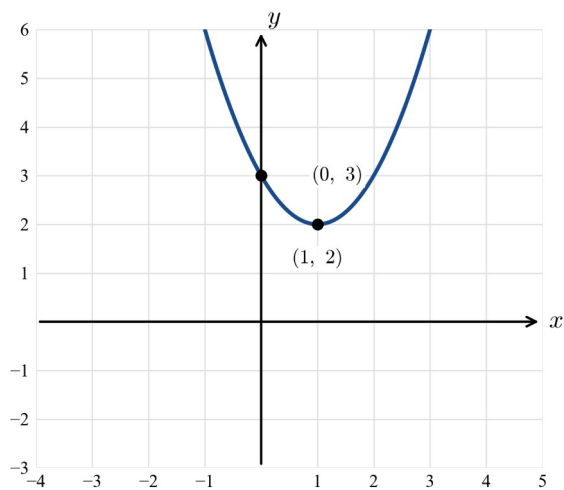
Q4. $y = a(x+2)^2 + 3$. Substitute $(0, -5)$: $-5 = 4a + 3 \Rightarrow a = -2$. $\therefore y = -2(x+2)^2 + 3$; since $a < 0$ the maximum value is 3.



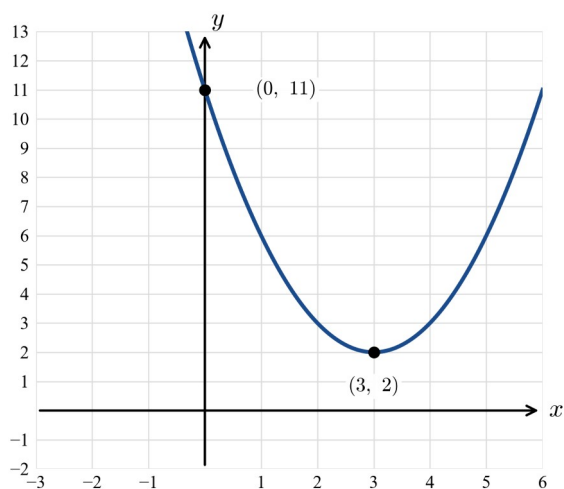
Q5. $y = a(x+3)(x-1)$. Substitute $(-1, -4)$: $-4 = a(2)(-2) = -4a \Rightarrow a = 1$. $\therefore y = (x+3)(x-1)$.



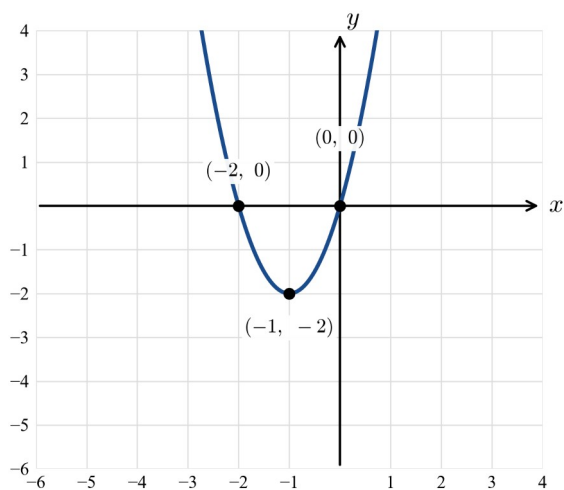
Q6. $c = 3$ from $(0, 3)$. $(-1, 6)$: $a - b + 3 = 6 \Rightarrow a - b = 3$. $(1, 2)$: $a + b + 3 = 2 \Rightarrow a + b = -1$. Add: $2a = 2 \Rightarrow a = 1$, $b = -2$. $\therefore y = x^2 - 2x + 3$.



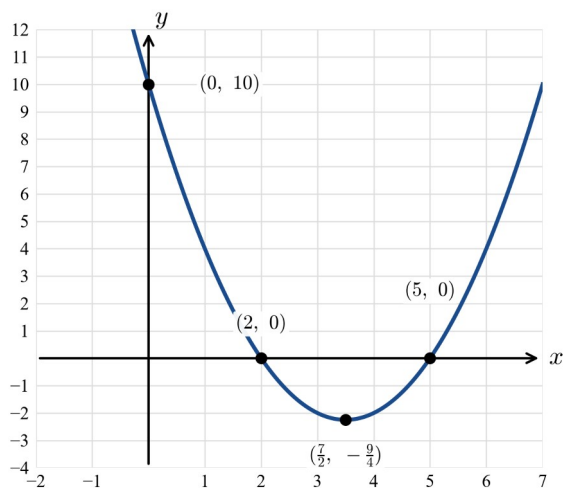
Q7. $y = a(x - 3)^2 + 2$. $(1, 6): 6 = 4a + 2 \Rightarrow a = 1. \therefore y = (x - 3)^2 + 2$.



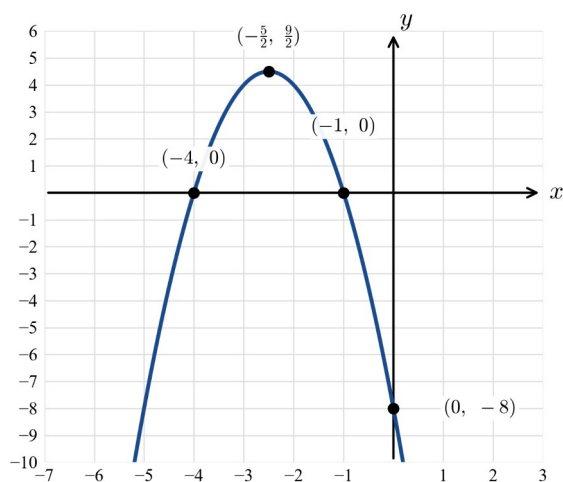
Q8. $y = a(x + 1)^2 - 2$. $(1, 6): 6 = 4a - 2 \Rightarrow a = 2. \therefore y = 2(x + 1)^2 - 2$.



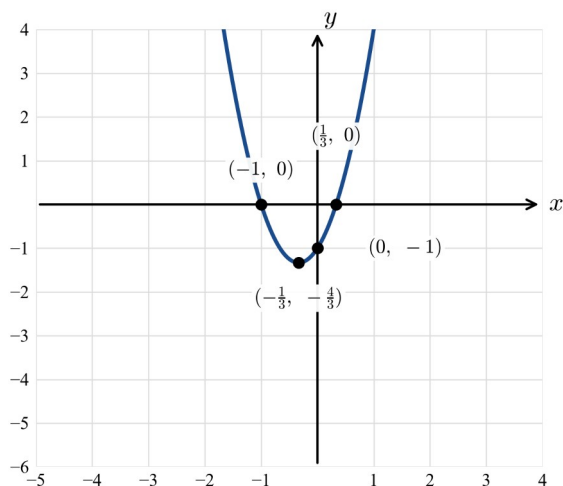
Q9. $y = a(x - 2)(x - 5)$. $(0, 10): 10 = a(-2)(-5) = 10a \Rightarrow a = 1. \therefore y = (x - 2)(x - 5)$.



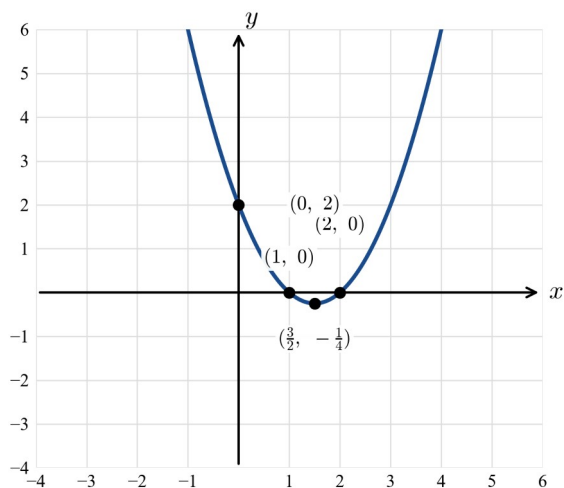
Q10. $y = a(x+4)(x+1)$. $(0, -8)$: $-8 = a(4)(1) = 4a \Rightarrow a = -2$. $\therefore y = -2(x+4)(x+1)$.



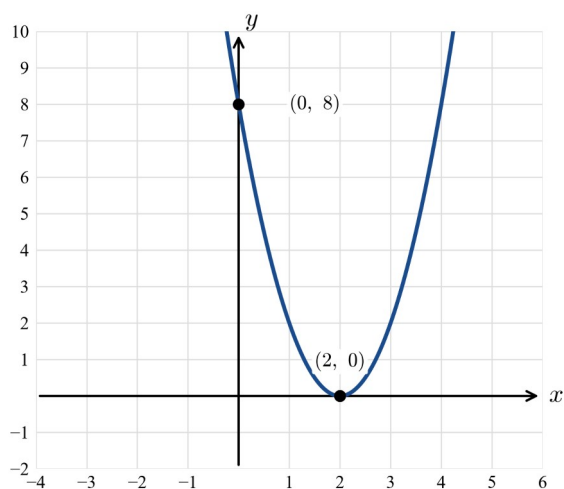
Q11. $c = -1$ from $(0, -1)$. $(1, 4)$: $a + b - 1 = 4 \Rightarrow a + b = 5$. $(-1, 0)$: $a - b - 1 = 0 \Rightarrow a - b = 1$. Add: $2a = 6 \Rightarrow a = 3$, $b = 2$. $\therefore y = 3x^2 + 2x - 1$.



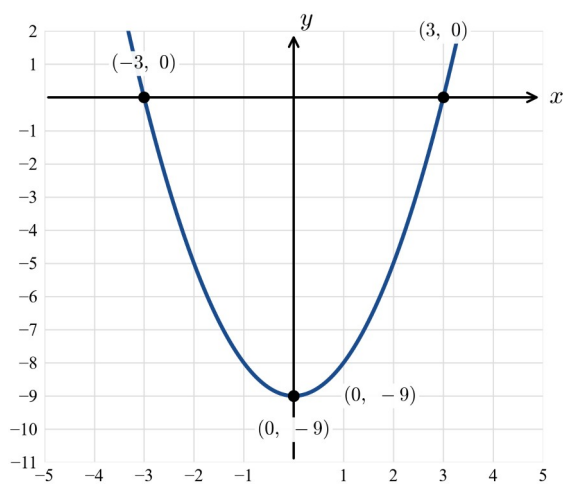
Q12. $c = 2$ from $(0, 2)$. $(2, 0)$: $4a + 2b + 2 = 0 \Rightarrow 2a + b = -1$. $(3, 2)$: $9a + 3b + 2 = 2 \Rightarrow 3a + b = 0$. Subtract: $a = 1$, $b = -3$. $\therefore y = x^2 - 3x + 2$.



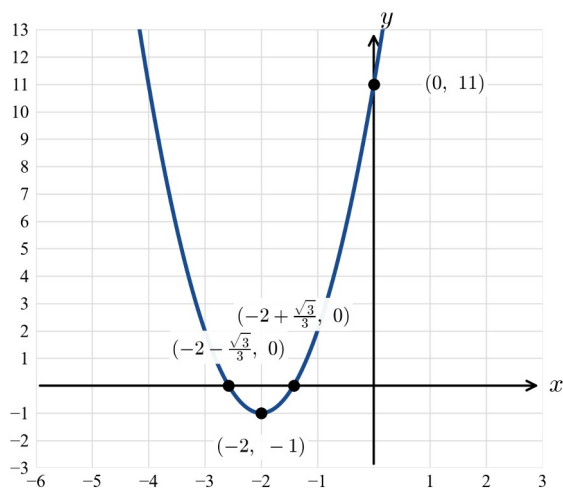
Q13. $y = a(x - 2)^2$ (repeated root at $x = 2$). $(0, 8)$: $8 = 4a \Rightarrow a = 2$. $\therefore y = 2(x - 2)^2$.



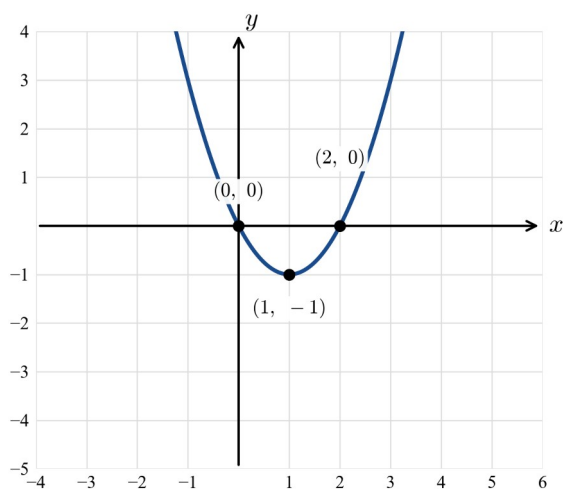
Q14. $y = a(x + 3)(x - 3)$. $(1, -8)$: $-8 = a(4)(-2) = -8a \Rightarrow a = 1$. $\therefore y = (x + 3)(x - 3)$.



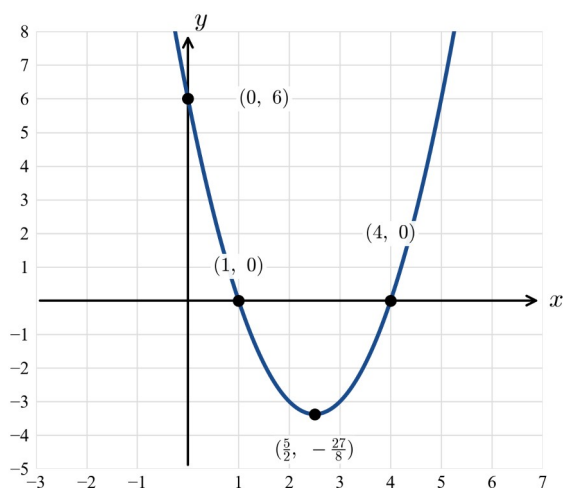
Q15. $y = a(x + 2)^2 - 1$. $(-1, 2)$: $2 = a(1) - 1 \Rightarrow a = 3$. $\therefore y = 3(x + 2)^2 - 1$.



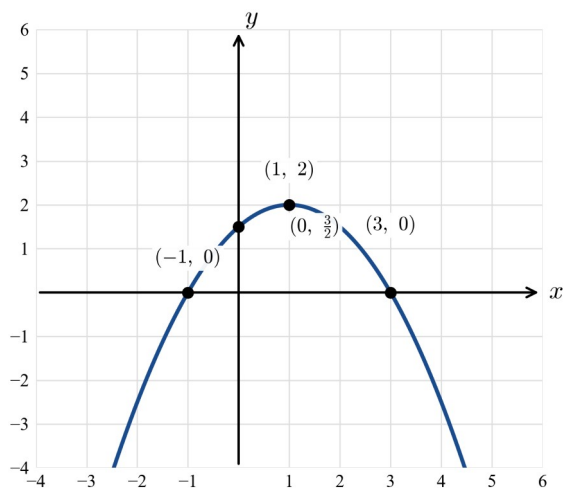
Q16. $y = a(x)(x - 2)$ (intercepts 0 and 2). $(1, -1)$: $-1 = a(1)(-1) = -a \Rightarrow a = 1$. $\therefore y = x(x - 2) = x^2 - 2x$.



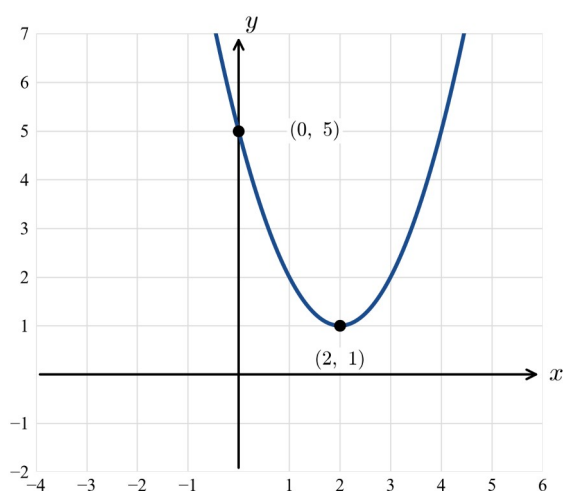
Q17. $y = a(x - 1)(x - 4)$. $(2, -3)$: $-3 = a(1)(-2) = -2a \Rightarrow a = \frac{3}{2}$. $\therefore y = \frac{3}{2}(x - 1)(x - 4)$.



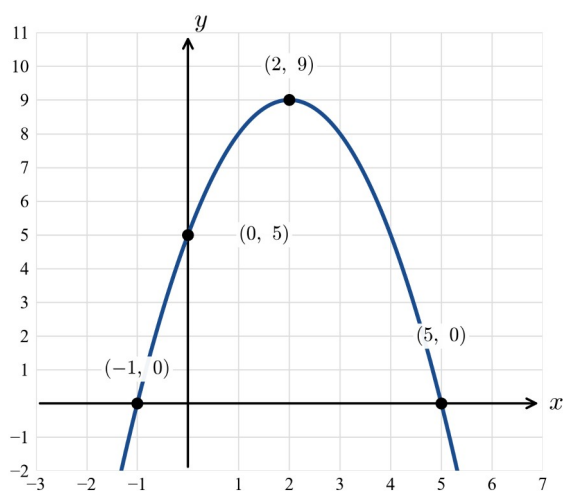
Q18. $y = a(x - 1)^2 + 2$. $(3, 0)$: $0 = 4a + 2 \Rightarrow a = -\frac{1}{2}$. $\therefore y = -\frac{1}{2}(x - 1)^2 + 2$.



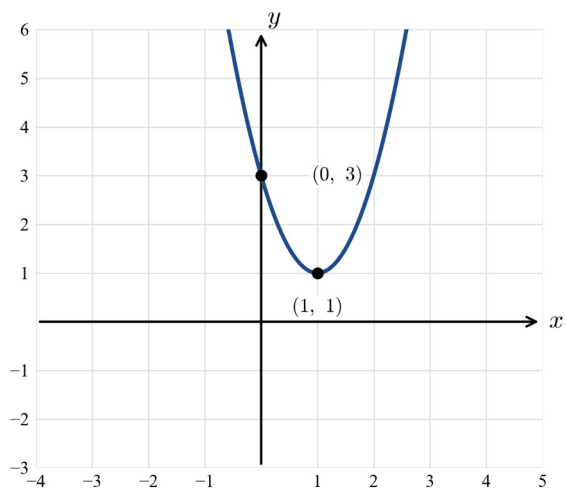
Q19. A minimum value of 1 on the axis $x=2$ means the turning point is $(2, 1)$, so $y = a(x - 2)^2 + 1$. Substitute $(0, 5)$: $5 = 4a + 1 \Rightarrow a = 1$. $\therefore y = (x - 2)^2 + 1$.



Q20. The x -intercepts give $y = a(x + 1)(x - 5)$, with axis of symmetry $x = \frac{-1+5}{2} = 2$. The maximum value 9 occurs there: $9 = a(2 + 1)(2 - 5) = -9a \Rightarrow a = -1$. $\therefore y = -(x + 1)(x - 5)$.



Q21. The points $(0, 3)$ and $(2, 3)$ have the same y -value, so the axis of symmetry is halfway between them, at $x = \frac{0+2}{2} = 1$. The third point $(1, 1)$ lies on this axis, so it is the turning point $(1, 1)$. Hence $y = a(x - 1)^2 + 1$; substituting $(0, 3)$: $3 = a + 1 \Rightarrow a = 2$. $\therefore y = 2(x - 1)^2 + 1$.



Chapter 4 Polynomial functions — 4C Polynomial terminology

Theory

A **polynomial** in x is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a non-negative integer and the **coefficients** $a_0, a_1, \dots, a_n$ are real numbers with $a_n \neq 0$.

- The **degree** of P is the highest power of x that appears, namely n .
- The **leading term** is $a_n x^n$ and the **leading coefficient** is a_n .
- The **constant term** is $a_0 = P(0)$.
- A polynomial is **monic** if its leading coefficient is 1.

Low-degree polynomials have special names: degree 0 is a (non-zero) **constant**, degree 1 is **linear**, degree 2 **quadratic**, degree 3 **cubic**, and degree 4 **quartic**.

Evaluating a polynomial. $P(a)$ is the number obtained by substituting $x = a$ into the rule. For example, if $P(x) = x^2 - 3x + 1$ then $P(2) = 4 - 6 + 1 = -1$.

Arithmetic of polynomials.

- To **add or subtract**, collect like terms. The degree of a sum is at most the larger of the two degrees.
- To **multiply**, expand and collect like terms. The degree of a product is the **sum** of the degrees.

Equality of polynomials. Two polynomials are **equal** (for all x) if and only if they have the same degree and **equal corresponding coefficients**. Comparing coefficients of like powers is called **equating coefficients**, and it is the key tool for finding unknown coefficients:

$$ax^2 + bx + c \equiv 2x^2 + 4x - 1 \Leftrightarrow a = 2, b = 4, c = -1.$$

The symbol $\equiv$ (“is identically equal to”) stresses that the two sides agree for **every** value of x , not just for particular solutions.

Worked examples

Example 1 — scaffolded

For $P(x) = 2x^3 - 5x^2 + 3x - 7$, state the degree, leading coefficient and constant term, and evaluate $P(2)$ and $P(-1)$.

Working

Comment

Highest power is x^3 , so the **degree is 3**.

The degree is the largest exponent with a non-zero coefficient.

Leading term $2x^3$, so the **leading coefficient is 2**; the **constant term is -7**.

Read the leading coefficient off the leading term; the constant term is $P(0)$.

$$P(2) = 2(2)^3 - 5(2)^2 + 3(2) - 7 = 16 - 20 + 6 - 7 = -5$$

Substitute $x = 2$ and evaluate carefully.

$$P(-1) = 2(-1)^3 - 5(-1)^2 + 3(-1) - 7 = -2 - 5 - 3 - 7 = -17$$

Watch the signs of the odd and even powers.

Example 2 — scaffolded

Find the values of a , b and c such that $x^2 - 4x + 7 \equiv a(x - 1)^2 + b(x - 1) + c$.

Working	Comment
Expand the right-hand side: $a(x^2 - 2x + 1) + b(x - 1) + c = ax^2 + (-2a + b)x + (a - b + c)$	Expand and group by powers of x .
Equate coefficients of x^2 : $a = 1$.	The two sides are identically equal, so like coefficients match.
Equate coefficients of x : $-2a + b = -4 \Rightarrow -2 + b = -4 \Rightarrow b = -2$.	Substitute the known a .
Equate constants: $a - b + c = 7 \Rightarrow 1 + 2 + c = 7 \Rightarrow c = 4$.	Solve for the last unknown.
$\therefore a = 1, b = -2, c = 4$.	State the result.

Example 3 — unscaffolded

Expand $(2x - 3)(x^2 + x - 4)$ and state the degree, leading coefficient and constant term of the result.

$$(2x - 3)(x^2 + x - 4) = 2x^3 + 2x^2 - 8x - 3x^2 - 3x + 12 = 2x^3 - x^2 - 11x + 12.$$

The product of a degree-1 and a degree-2 polynomial has degree $1 + 2 = 3$.

$\therefore$ degree 3, leading coefficient 2, constant term 12.

Example 4 — unscaffolded

The polynomial $P(x) = x^3 + ax^2 + bx - 6$ satisfies $P(1) = 0$ and $P(2) = 0$. Find a and b .

$$P(1) = 1 + a + b - 6 = 0 \Rightarrow a + b = 5.$$

$$P(2) = 8 + 4a + 2b - 6 = 0 \Rightarrow 4a + 2b = -2 \Rightarrow 2a + b = -1.$$

Subtracting the first equation from the second: $a = -6$, and then $b = 5 - (-6) = 11$.

$$\therefore a = -6, b = 11, \text{ and } P(x) = x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3).$$

Practice questions

Scaffolded

Q1. For $P(x) = 3x^4 - 2x^2 + x - 5$: (a) state the degree; (b) state the leading coefficient; (c) state the constant term; (d) evaluate $P(1)$.

Q2. For $P(x) = x^3 - 4x + 1$, evaluate: (a) $P(0)$; (b) $P(2)$; (c) $P(-1)$; (d) $P\left(\frac{1}{2}\right)$.

Q3. Let $P(x) = 2x^2 - x + 3$ and $Q(x) = x^2 + 4x - 1$. Find: (a) $P(x) + Q(x)$; (b) $P(x) - Q(x)$; (c) the degree of $P(x) + Q(x)$; (d) $2P(x)$.

Q4. Expand $(x - 2)(x^2 + 3x - 1)$ and hence state: (a) the expanded polynomial; (b) its degree; (c) its leading coefficient; (d) its constant term.

Q5. Find a , b and c such that $ax^2 + bx + c \equiv 2(x + 1)^2 - 3$.

Q6. The polynomial $P(x) = x^3 + kx^2 - 2x + 1$ satisfies $P(2) = 13$. (a) Write an equation in k ; (b) hence find the value of k .

Unscaffolded

Q7. State the degree, leading coefficient and constant term of $P(x) = 5 - 3x + 2x^2 - x^5$.

Q8. Evaluate $P(2)$ for $P(x) = x^4 - 3x^3 + 2$.

Q9. For $P(x) = 2x^3 + x^2 - 5$ and $Q(x) = x^3 - 3x^2 + x + 4$, find $P(x) - Q(x)$.

Q10. Expand and simplify $(2x + 1)(x - 3)(x + 2)$, and state the degree and constant term.

Q11. Find a , b and c such that $x^2 - 6x + 11 \equiv a(x - 3)^2 + b(x - 3) + c$.

Q12. Find a , b , c and d such that $ax^3 + bx^2 + cx + d \equiv 2(x - 1)^3$.

Q13. The polynomial $P(x) = x^3 + ax + b$ satisfies $P(1) = 0$ and $P(-2) = 0$. Find a and b , and hence factorise $P(x)$.

Q14. Given that $(x^2 + ax + b)(x - 1) \equiv x^3 - 2x^2 - 5x + 6$, find a and b .

Q15. Express $P(x) = 3x^2 - 12x + 7$ in the form $a(x - h)^2 + k$, stating a , h and k .

Q16. For $P(x) = x^4 + x^3 - 7x^2 - x + 6$, evaluate $P(1)$ and $P(-1)$, and state what your answers tell you about the factors of $P(x)$.

Q17. A quadratic polynomial $P(x)$ satisfies $P(0) = 5$, $P(1) = 4$ and $P(2) = 1$. **Find the rule of $P(x)$.**

Q18. **Find the values of a , b and c** such that $ax^2 + bx + c \equiv (x - 2)(x + 3) + (2x - 1)$.

Q19. The polynomial $P(x) = 2x^3 - 3x^2 + ax + b$ satisfies $P(1) = 3$ and $P(-1) = -5$. Find a and b .

Q20. The polynomials $x^3 + ax^2 + bx + c$ and $(x + 1)(x - 2)(x - 3)$ are equal for all x . **Find a , b and c .**

Q21. The polynomial $P(x) = x^2 + px + q$ satisfies $P(2) = 0$ and $P(-1) = 0$. Find p and q , and hence write $P(x)$ in factorised form.

Answers

Q1. (a) 4; (b) 3; (c) -5; (d) -3. **Q2.** (a) 1; (b) 1; (c) 4; (d) $-\frac{7}{8}$. **Q3.** (a) $3x^2 + 3x + 2$; (b) $x^2 - 5x + 4$; (c) 2; (d) $4x^2 - 2x + 6$. **Q4.** (a) $x^3 + x^2 - 7x + 2$; (b) 3; (c) 1; (d) 2. **Q5.** $a=2, b=4, c=-1$. **Q6.** (a) $5 + 4k = 13$; (b) $k=2$. **Q7.** Degree 5, leading coefficient -1, constant term 5. **Q8.** -6. **Q9.** $x^3 + 4x^2 - x - 9$. **Q10.** $2x^3 - x^2 - 13x - 6$; degree 3, constant term -6. **Q11.** $a=1, b=0, c=2$. **Q12.** $a=2, b=-6, c=6, d=-2$. **Q13.** $a=-3, b=2$; $P(x) = (x-1)^2(x+2)$. **Q14.** $a=-1, b=-6$. **Q15.** $a=3, h=2, k=-5$. **Q16.** $P(1)=0$ and $P(-1)=0$, so $(x-1)$ and $(x+1)$ are factors of $P(x)$. **Q17.** $P(x) = -x^2 + 5$. **Q18.** $a=1, b=3, c=-7$. **Q19.** $a=2, b=2$. **Q20.** $a=-4, b=1, c=6$. **Q21.** $p=-1, q=-2$; $P(x) = (x-2)(x+1)$.

Fully-worked solutions

Q1. $P(x) = 3x^4 - 2x^2 + x - 5$. (a) Highest power $x^4 \Rightarrow$ degree 4. (b) Leading term $3x^4 \Rightarrow$ leading coefficient 3. (c) Constant term -5. (d) $P(1) = 3 - 2 + 1 - 5 = -3$.

Q2. $P(x) = x^3 - 4x + 1$. (a) $P(0) = 1$. (b) $P(2) = 8 - 8 + 1 = 1$. (c) $P(-1) = -1 + 4 + 1 = 4$. (d) $P\left(\frac{1}{2}\right) = \frac{1}{8} - 2 + 1 = -\frac{7}{8}$.

Q3. $P+Q = (2x^2 - x + 3) + (x^2 + 4x - 1) = 3x^2 + 3x + 2$. $P-Q = (2x^2 - x + 3) - (x^2 + 4x - 1) = x^2 - 5x + 4$. (c) Degree 2. (d) $2P = 4x^2 - 2x + 6$.

Q4. $(x-2)(x^2 + 3x - 1) = x^3 + 3x^2 - x - 2x^2 - 6x + 2 = x^3 + x^2 - 7x + 2$. Degree 3, leading coefficient 1, constant term 2.

Q5. $2(x+1)^2 - 3 = 2(x^2 + 2x + 1) - 3 = 2x^2 + 4x - 1$. Equating with $ax^2 + bx + c$: $a=2, b=4, c=-1$.

Q6. $P(2) = 8 + 4k - 4 + 1 = 5 + 4k$. (a) $5 + 4k = 13$. (b) $4k = 8 \Rightarrow k = 2$.

Q7. Writing $P(x) = -x^5 + 2x^2 - 3x + 5$: degree 5, leading coefficient -1, constant term 5.

Q8. $P(2) = 2^4 - 3(2)^3 + 2 = 16 - 24 + 2 = -6$.

Q9. $P-Q = (2x^3 + x^2 - 5) - (x^3 - 3x^2 + x + 4) = x^3 + 4x^2 - x - 9$.

Q10. $(2x+1)(x-3) = 2x^2 - 5x - 3$; then $(2x^2 - 5x - 3)(x+2) = 2x^3 + 4x^2 - 5x^2 - 10x - 3x - 6 = 2x^3 - x^2 - 13x - 6$. Degree 3, constant term -6.

Q11. $a(x-3)^2 + b(x-3) + c = ax^2 + (-6a+b)x + (9a-3b+c)$. Equate with $x^2 - 6x + 11$: $a=1$; $-6a+b = -6 \Rightarrow b=0$; $9-0+c = 11 \Rightarrow c=2$. (Indeed $x^2 - 6x + 11 = (x-3)^2 + 2$.)

Q12. $2(x-1)^3 = 2(x^3 - 3x^2 + 3x - 1) = 2x^3 - 6x^2 + 6x - 2$. So $a=2, b=-6, c=6, d=-2$.

Q13. $P(1) = 1 + a + b = 0 \Rightarrow a + b = -1$. $P(-2) = -8 - 2a + b = 0 \Rightarrow -2a + b = 8$. Subtracting: $3a = -9 \Rightarrow a = -3$, then $b = 2$. So $P(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$.

Q14. $(x^2 + ax + b)(x-1) = x^3 + (a-1)x^2 + (b-a)x - b$. Equate with $x^3 - 2x^2 - 5x + 6$: $-b = 6 \Rightarrow b = -6$; $a-1 = -2 \Rightarrow a = -1$ (and $b-a = -6+1 = -5$ ✓).

Q15. $3x^2 - 12x + 7 = 3(x^2 - 4x) + 7 = 3((x-2)^2 - 4) + 7 = 3(x-2)^2 - 12 + 7 = 3(x-2)^2 - 5$. So $a=3, h=2, k=-5$.

Q16. $P(1)=1+1-7-1+6=0$ and $P(-1)=1-1-7+1+6=0$. By the factor theorem, $P(1)=0$ and $P(-1)=0$ mean $(x-1)$ and $(x+1)$ are both factors of $P(x)$.

Q17. Let $P(x)=ax^2+bx+c$. $P(0)=c=5$. $P(1)=a+b+5=4 \Rightarrow a+b=-1$.
 $P(2)=4a+2b+5=1 \Rightarrow 2a+b=-2$. Subtracting: $a=-1$, then $b=0$. $\therefore P(x)=-x^2+5$.

Q18. $(x-2)(x+3)+(2x-1)=(x^2+x-6)+(2x-1)=x^2+3x-7$. So $a=1$, $b=3$, $c=-7$.

Q19. $P(1)=2-3+a+b=a+b-1=3 \Rightarrow a+b=4$. $P(-1)=-2-3-a+b=b-a-5=-5 \Rightarrow b-a=0$, so $b=a$.
 Then $2a=4 \Rightarrow a=2$, $b=2$.

Q20. $(x+1)(x-2)(x-3)=(x+1)(x^2-5x+6)=x^3-5x^2+6x+x^2-5x+6=x^3-4x^2+x+6$. Equating: $a=-4$, $b=1$, $c=6$.

Q21. $P(2)=4+2p+q=0$ and $P(-1)=1-p+q=0$. Subtracting the second from the first: $3+3p=0 \Rightarrow p=-1$;
 then $q=p-1=-2$. $\therefore P(x)=x^2-x-2=(x-2)(x+1)$.

Chapter 4 Polynomial functions — 4D Dividing and factorising polynomials

Theory

A **polynomial** of degree n has the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ with $a_n \neq 0$. This section develops the tools for **factorising** polynomials of degree 3 and 4, so their graphs can be sketched and their equations solved.

Polynomial long division. Dividing a polynomial $P(x)$ by a linear divisor $x - a$ gives a **quotient** $Q(x)$ and a constant **remainder** r :

$$P(x) = (x - a)Q(x) + r.$$

The process mirrors long division of numbers: divide the leading terms, multiply back, subtract, and bring down the next term.

The remainder theorem. Substituting $x = a$ into $P(x) = (x - a)Q(x) + r$ gives $P(a) = r$. So:

the remainder when $P(x)$ is divided by $(x - a)$ is $P(a)$.

For a divisor $bx - c$, the remainder is $P\left(\frac{c}{b}\right)$.

The factor theorem. If the remainder is 0 then $(x - a)$ divides $P(x)$ exactly:

$$(x - a) \text{ is a factor of } P(x) \Leftrightarrow P(a) = 0.$$

This is the key tool for finding a first factor: test small values $x = \pm 1, \pm 2, \pm 3, \dots$ (for a **monic** polynomial, the integer factors of the constant term). For a non-monic polynomial $a_n x^n + \dots + a_0$, also test rational values $\frac{\text{factor of } a_0}{\text{factor of } a_n}$.

Factorising fully. Once one factor $(x - a)$ is found by the factor theorem, divide to obtain a quotient of degree one lower, then factorise the quotient (a quadratic can be factorised by inspection, the null factor law, or the quadratic formula). Repeat for a quartic.

Solving polynomial equations. Factorise fully, then apply the **null factor law**: a product is zero exactly when one of its factors is zero.

Grouping is sometimes quicker: e.g. $x^3 - 3x^2 - 4x + 12 = x^2(x - 3) - 4(x - 3) = (x - 3)(x^2 - 4)$.

Worked examples

Example 1 — scaffolded

Divide $2x^3 - 3x^2 + 4x - 5$ by $x - 2$, stating the quotient and remainder.

Working

Comment

$2x^3 \div x = 2x^2$. Multiply: $2x^2(x - 2) = 2x^3 - 4x^2$;
subtract: $(-3x^2) - (-4x^2) = x^2$.

Divide leading terms, multiply back, subtract.

Bring down: $x^2 + 4x$. $x^2 \div x = x$; $x(x - 2) = x^2 - 2x$;
subtract: $4x - (-2x) = 6x$.

Repeat with the new leading term.

Bring down: $6x - 5$. $6x \div x = 6$; $6(x - 2) = 6x - 12$;
subtract: $-5 - (-12) = 7$.

The degree of 7 is less than the divisor, so stop.

$$\therefore 2x^3 - 3x^2 + 4x - 5 = (x - 2)(2x^2 + x + 6) + 7.$$

Quotient $2x^2 + x + 6$, remainder 7.

Check by the remainder theorem: $P(2) = 2(8) - 3(4) + 8 - 5 = 16 - 12 + 8 - 5 = 7$. ✓

Example 2 — scaffolded

Factorise $P(x) = x^3 - 4x^2 + x + 6$ completely.

Working	Comment
$P(-1) = -1 - 4 - 1 + 6 = 0$, so $(x + 1)$ is a factor.	Test factors of 6; the factor theorem gives a first factor.
Divide: $x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$.	Long division (or inspection) drops the degree by one.
$x^2 - 5x + 6 = (x - 2)(x - 3)$.	Factorise the quadratic quotient.
$\therefore P(x) = (x + 1)(x - 2)(x - 3)$.	Fully factorised.

Example 3 — unscaffolded

Factorise $2x^3 + 3x^2 - 11x - 6$ completely.

Testing values: $P(2) = 16 + 12 - 22 - 6 = 0$, so $(x - 2)$ is a factor.

Dividing gives $2x^3 + 3x^2 - 11x - 6 = (x - 2)(2x^2 + 7x + 3)$.

The quadratic factorises as $2x^2 + 7x + 3 = (2x + 1)(x + 3)$.

$\therefore 2x^3 + 3x^2 - 11x - 6 = (x - 2)(2x + 1)(x + 3)$.

Example 4 — unscaffolded

Solve $x^3 - 3x^2 - 4x + 12 = 0$.

Group in pairs: $x^2(x - 3) - 4(x - 3) = (x - 3)(x^2 - 4) = (x - 3)(x - 2)(x + 2)$.

By the null factor law, $x - 3 = 0$ or $x - 2 = 0$ or $x + 2 = 0$.

$\therefore x = -2, 2$ or 3 .

Practice questions

Scaffolded

Q1. For $P(x) = x^3 - x^2 - 10x - 8$: (a) show that $(x + 2)$ is a factor; (b) divide to find the quadratic quotient; (c) factorise $P(x)$ completely.

Q2. Use the remainder theorem to find the remainder when $x^3 + 3x^2 - 2x + 4$ is divided by $(x - 1)$.

Q3. For $P(x) = x^3 - 2x^2 - 5x + 6$: (a) show that $(x - 3)$ is a factor; (b) hence factorise $P(x)$ completely.

Q4. The polynomial $x^3 - kx^2 + 2x - 8$ has $(x - 2)$ as a factor. Find the value of k .

Q5. Factorise $2x^3 - 5x^2 - x + 6$ completely.

Q6. Solve $x^3 + 2x^2 - 9x - 18 = 0$ by first factorising by grouping.

Unscaffolded

Q7. Divide $3x^3 + 2x^2 - x + 4$ by $(x + 1)$, stating the quotient and remainder.

Q8. Find the remainder when $2x^3 - x^2 + 3x - 5$ is divided by $(x - 2)$.

Q9. Find the remainder when $x^3 + 2x^2 + 3$ is divided by $(x - 2)$.

- Q10.** Find the value of a for which $(x+2)$ is a factor of $x^3 + ax^2 - x - 2$.
- Q11.** The polynomial $x^3 + ax^2 + bx - 2$ has both $(x-1)$ and $(x+2)$ as factors. Find a and b .
- Q12.** Factorise $x^3 - 7x - 6$ completely.
- Q13.** Factorise $3x^3 - x^2 - 12x + 4$ completely.
- Q14.** Factorise $x^4 - 5x^2 + 4$ completely.
- Q15.** Factorise $x^4 - 2x^3 - 7x^2 + 8x + 12$ completely.
- Q16.** Solve $x^3 + 2x^2 - 5x - 6 = 0$.
- Q17.** Solve $2x^3 - 3x^2 - 11x + 6 = 0$.
- Q18.** Solve $x^4 - 13x^2 + 36 = 0$.
- Q19.** Show that $(x-1)$ is a factor of $x^3 - 6x^2 + 11x - 6$, and hence solve $x^3 - 6x^2 + 11x - 6 = 0$.
- Q20.** The polynomial $x^3 - 2x^2 + ax + b$ is divisible by $(x-1)$ and leaves a remainder of -6 when divided by $(x+1)$. Find a and b .
- Q21.** For $P(x) = x^3 + x^2 - 10x + 8$: (a) show that $(x-1)$ is a factor; (b) hence factorise $P(x)$ completely; (c) hence solve $P(x) = 0$.
-

Answers

Q1. (c) $(x+2)(x+1)(x-4)$. **Q2.** 6. **Q3.** $(x-3)(x-1)(x+2)$. **Q4.** $k=1$. **Q5.** $(x-2)(2x-3)(x+1)$. **Q6.** $x=-3, -2, 3$. **Q7.** quotient $3x^2 - x$, remainder 4. **Q8.** 13. **Q9.** 19. **Q10.** $a=2$. **Q11.** $a=2, b=-1$. **Q12.** $(x+1)(x+2)(x-3)$. **Q13.** $(x-2)(x+2)(3x-1)$. **Q14.** $(x-1)(x+1)(x-2)(x+2)$. **Q15.** $(x+1)(x+2)(x-2)(x-3)$. **Q16.** $x=-3, -1, 2$. **Q17.** $x=-2, 1/2, 3$. **Q18.** $x=\pm 2, \pm 3$. **Q19.** $x=1, 2, 3$. **Q20.** $a=2, b=-1$. **Q21.** (c) $x=-4, 1, 2$.

Fully-worked solutions

Q1. (a) $P(-2) = -8 - 4 + 20 - 8 = 0$, so $(x+2)$ is a factor. (b) Dividing, $x^3 - x^2 - 10x - 8 = (x+2)(x^2 - 3x - 4)$. (c) $x^2 - 3x - 4 = (x+1)(x-4)$, so $P(x) = (x+2)(x+1)(x-4)$.

Q2. By the remainder theorem the remainder is $P(1) = 1 + 3 - 2 + 4 = 6$.

Q3. (a) $P(3) = 27 - 18 - 15 + 6 = 0$, so $(x-3)$ is a factor. (b) $x^3 - 2x^2 - 5x + 6 = (x-3)(x^2 + x - 2) = (x-3)(x-1)(x+2)$.

Q4. $(x-2)$ is a factor $\Rightarrow P(2) = 0$: $8 - 4k + 4 - 8 = 0 \Rightarrow 4 - 4k = 0 \Rightarrow k = 1$.

Q5. $P(2) = 16 - 20 - 2 + 6 = 0$, so $(x-2)$ is a factor. $2x^3 - 5x^2 - x + 6 = (x-2)(2x^2 - x - 3) = (x-2)(2x-3)(x+1)$.

Q6. $x^2(x+2) - 9(x+2) = (x+2)(x^2 - 9) = (x+2)(x-3)(x+3)$. Null factor law: $x = -2, 3, -3$.

Q7. Dividing $3x^3 + 2x^2 - x + 4$ by $(x+1)$: quotient $3x^2 - x$, remainder 4; that is, $3x^3 + 2x^2 - x + 4 = (x+1)(3x^2 - x) + 4$. (Check: $P(-1) = -3 + 2 + 1 + 4 = 4$. ✓)

Q8. $P(2) = 16 - 4 + 6 - 5 = 13$.

Q9. $P(2) = 8 + 8 + 3 = 19$.

Q10. $(x+2)$ a factor $\Rightarrow P(-2) = 0$: $-8 + 4a + 2 - 2 = 0 \Rightarrow 4a - 8 = 0 \Rightarrow a = 2$.

Q11. $P(1) = 1 + a + b - 2 = 0 \Rightarrow a + b = 1$; $P(-2) = -8 + 4a - 2b - 2 = 0 \Rightarrow 4a - 2b = 10 \Rightarrow 2a - b = 5$. Adding, $3a = 6 \Rightarrow a = 2$, so $b = -1$.

Q12. $P(-1) = -1 + 7 - 6 = 0$, so $(x+1)$ is a factor. $x^3 - 7x - 6 = (x+1)(x^2 - x - 6) = (x+1)(x+2)(x-3)$.

Q13. $P(2) = 24 - 4 - 24 + 4 = 0$, so $(x-2)$ is a factor. $3x^3 - x^2 - 12x + 4 = (x-2)(3x^2 + 5x - 2) = (x-2)(3x-1)(x+2)$.

Q14. Treat as a quadratic in x^2 : $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x-1)(x+1)(x-2)(x+2)$.

Q15. Testing: $P(-1) = 1 + 2 - 7 - 8 + 12 = 0$ and $P(2) = 16 - 16 - 28 + 16 + 12 = 0$. So $(x+1)$ and $(x-2)$ are factors; dividing by $(x+1)(x-2) = x^2 - x - 2$ gives $x^2 - x - 6 = (x+2)(x-3)$. Hence $x^4 - 2x^3 - 7x^2 + 8x + 12 = (x+1)(x-2)(x+2)(x-3)$.

Q16. $P(2) = 8 + 8 - 10 - 6 = 0$, so $(x-2)$ is a factor. $x^3 + 2x^2 - 5x - 6 = (x-2)(x^2 + 4x + 3) = (x-2)(x+1)(x+3)$. $\therefore x = -3, -1, 2$.

Q17. $P(3) = 54 - 27 - 33 + 6 = 0$, so $(x-3)$ is a factor. $2x^3 - 3x^2 - 11x + 6 = (x-3)(2x^2 + 3x - 2) = (x-3)(2x-1)(x+2)$. $\therefore x = -2, 1/2, 3$.

Q18. Quadratic in x^2 : $x^4 - 13x^2 + 36 = (x^2 - 4)(x^2 - 9) = 0 \Rightarrow x^2 = 4$ or $x^2 = 9$. $\therefore x = \pm 2, \pm 3$.

Q19. $P(1) = 1 - 6 + 11 - 6 = 0$, so $(x - 1)$ is a factor.

$$x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6) = (x - 1)(x - 2)(x - 3). \therefore x = 1, 2, 3.$$

Q20. Divisible by $(x - 1)$: $P(1) = 1 - 2 + a + b = 0 \Rightarrow a + b = 1$. Remainder -6 at $x = -1$:

$$P(-1) = -1 - 2 - a + b = -6 \Rightarrow -a + b = -3 \Rightarrow a - b = 3. \text{ Adding, } 2a = 4 \Rightarrow a = 2, \text{ so } b = -1.$$

Q21. (a) $P(1) = 1 + 1 - 10 + 8 = 0$, so $(x - 1)$ is a factor. (b)

$$x^3 + x^2 - 10x + 8 = (x - 1)(x^2 + 2x - 8) = (x - 1)(x + 4)(x - 2). \text{ (c) } \therefore x = -4, 1, 2.$$

Chapter 4 Polynomial functions — 4E Cubic functions

Theory

A **cubic function** has degree 3; its general form is $y = ax^3 + bx^2 + cx + d$ with $a \neq 0$. This section sketches cubics from their **factored form**, using the x -intercepts, the behaviour at any repeated factor, the y -intercept and the end behaviour.

Factored form. If $y = a(x - p)(x - q)(x - r)$ then the x -intercepts are $x = p$, $x = q$ and $x = r$, and the y -intercept is $y = a(-p)(-q)(-r)$ (put $x = 0$).

Repeated factors — the multiplicity of a factor controls the shape at that intercept:

- a **single** factor $(x - p)$: the graph **crosses** the x -axis at p ;
- a **double** factor $(x - p)^2$: the graph **touches** the x -axis at p — there is a turning point on the axis;
- a **triple** factor $(x - p)^3$: the graph has a **stationary point of inflection** at p .

End behaviour. For large $|x|$ the leading term ax^3 dominates. If $a > 0$ the curve falls from the lower left and rises to the upper right; if $a < 0$ it is the other way. A cubic always has range $\mathbb{R}$.

The form $y = a(x - h)^3 + k$. This is the basic cubic $y = x^3$ dilated by factor a (and reflected in the x -axis if $a < 0$) then translated h right and k up. It has a **stationary point of inflection** at (h, k) and exactly **one** x -intercept.

To sketch a cubic:

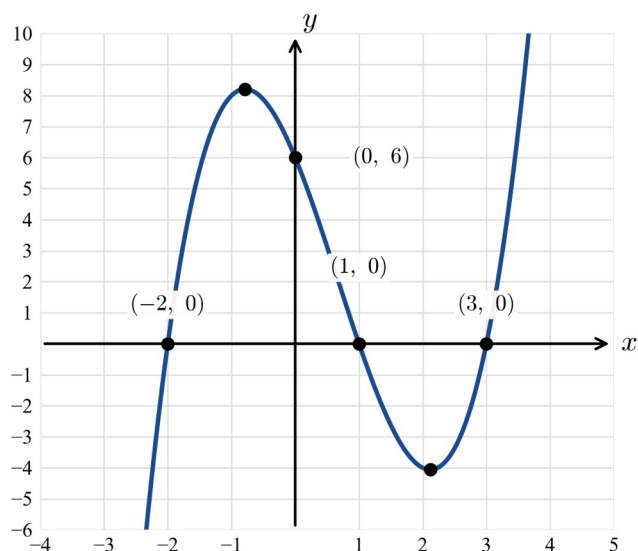
1. Find the **x -intercepts** from the factors and note the **multiplicity** (cross / touch / inflection).
2. Find the **y -intercept** ($x = 0$).
3. Decide the **end behaviour** from the sign of a .
4. Draw a smooth curve with the correct behaviour at each intercept. (Exact coordinates of the turning points of a cubic with three distinct roots need calculus — Chapter 10 — and are not required here.)

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = (x - 1)(x + 2)(x - 3)$, labelling the axis intercepts.

Working	Comment
Leading coefficient $a = 1 > 0$: the curve falls from the lower left and rises to the upper right.	The sign of a fixes the end behaviour.
x-intercepts: $x = 1$, $x = -2$, $x = 3$ — all single factors, so the graph crosses at each.	Set each factor to 0; single factors give crossings.
y-intercept: $x = 0 \Rightarrow y = (-1)(2)(-3) = 6$, giving $(0, 6)$.	Substitute $x = 0$.
Mark $(-2, 0)$, $(1, 0)$, $(3, 0)$ and $(0, 6)$ and draw a smooth cubic crossing at each intercept.	Three distinct roots $\Rightarrow$ two turning points (found exactly with calculus later).



Example 2 — scaffolded

Sketch the graph of $y = (x - 2)^2(x + 1)$, labelling the axis intercepts and the turning point on the axis.

Working

$a = 1 > 0$: falls from lower left, rises to upper right.

x-intercepts: $x = 2$ is a **double** factor $\Rightarrow$ the graph **touches** at $(2, 0)$; $x = -1$ is single $\Rightarrow$ it **crosses** at $(-1, 0)$.

y-intercept: $x = 0 \Rightarrow y = (-2)^2(1) = 4$, giving $(0, 4)$.

The touch at $(2, 0)$ is a **local minimum on the axis**; there is a local maximum at $(0, 4)$.

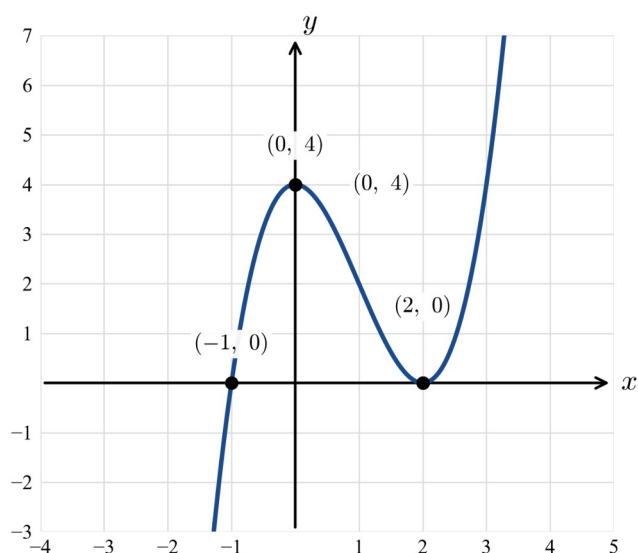
Comment

End behaviour from the sign of a .

Multiplicity: $(x - 2)^2$ touches, $(x + 1)$ crosses.

Substitute $x = 0$.

A double root sits at a turning point.



Example 3 — unscaffolded

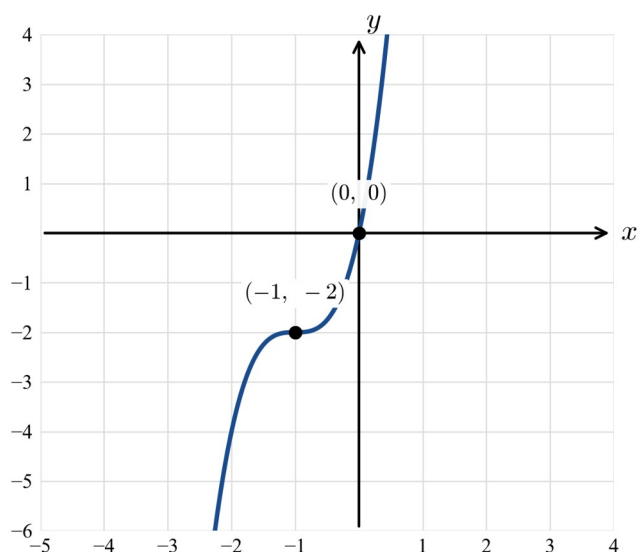
Sketch the graph of $y = 2(x + 1)^3 - 2$, labelling the stationary point of inflection and the axis intercepts.

This is $y = x^3$ dilated by factor 2 from the x -axis, then translated 1 left and 2 down, so it has a **stationary point of inflection** at $(-1, -2)$ and rises from lower left to upper right.

x-intercept: $2(x + 1)^3 - 2 = 0 \Rightarrow (x + 1)^3 = 1 \Rightarrow x + 1 = 1$, so $x = 0$, giving $(0, 0)$.

y-intercept: $x=0 \Rightarrow y=2(1)^3-2=0$, giving $(0,0)$ (the same point).

$\therefore$ the curve passes through the origin with a stationary point of inflection at $(-1, -2)$.



Example 4 — unscaffolded

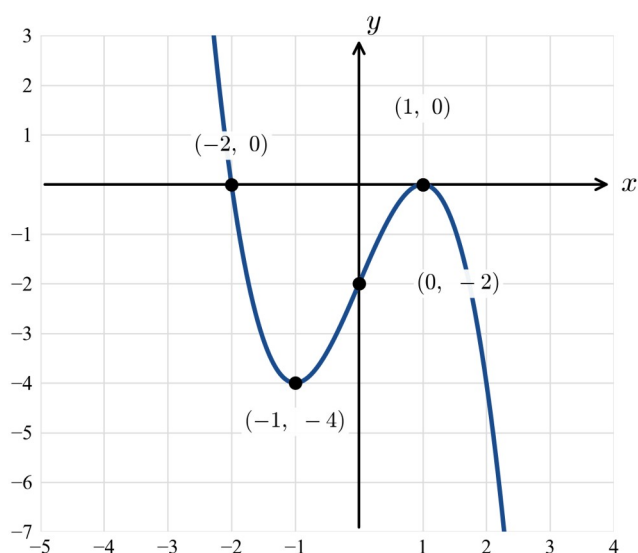
Sketch the graph of $y = -(x+2)(x-1)^2$, labelling the axis intercepts.

$a = -1 < 0$: the curve rises from the upper left and falls to the lower right.

x-intercepts: $x = -2$ is single (the graph **crosses**); $x = 1$ is a **double** factor (the graph **touches** at $(1, 0)$).

y-intercept: $x=0 \Rightarrow y = -(2)(-1)^2 = -2$, giving $(0, -2)$.

$\therefore$ the touch at $(1, 0)$ is a **local maximum on the axis** (because $a < 0$).



Practice questions

Scaffolded

Q1. For $y = (x+3)(x-1)(x-2)$: (a) state the x-intercepts and whether the graph crosses or touches at each; (b) find the y-intercept; (c) state the end behaviour; (d) sketch the graph.

Q2. For $y = (x - 1)^2(x - 4)$: (a) state the x -intercepts and their multiplicity; (b) find the y -intercept; (c) which intercept is a turning point on the axis? (d) sketch the graph.

Q3. For $y = (x + 2)^3$: (a) name the base graph and the transformation; (b) state the coordinates of the stationary point of inflection; (c) find the y -intercept; (d) sketch the graph.

Q4. For $y = -(x - 2)(x + 1)(x + 3)$: (a) state the x -intercepts; (b) state the end behaviour (note $a < 0$); (c) find the y -intercept; (d) sketch the graph.

Q5. For $y = (x - 1)^3 + 2$: (a) state the coordinates of the stationary point of inflection; (b) find the exact x -intercept; (c) find the y -intercept; (d) sketch the graph.

Q6. For $y = 2(x + 2)(x - 1)(x - 3)$: (a) state the x -intercepts; (b) find the y -intercept; (c) how does the factor of 2 affect the graph? (d) sketch the graph.

Unscaffolded

Sketch each of the following, labelling all axis intercepts (with the correct cross/touch/inflection behaviour), the y -intercept, and any stationary point of inflection.

Q7. $y = (x + 2)(x - 1)(x - 4)$

Q8. $y = (x + 1)(x - 2)(x - 4)$

Q9. $y = -(x + 2)(x - 1)(x - 3)$

Q10. $y = (x - 2)^2(x + 2)$

Q11. $y = (x + 1)^2(x - 3)$

Q12. $y = -(x - 1)^2(x + 2)$

Q13. $y = (x - 1)^3$

Q14. $y = (x + 2)^3 - 1$

Q15. $y = 2(x - 1)^3$

Q16. $y = x^3 - 4x$

Q17. $y = x^3 - 3x^2 - x + 3$ (*factorise first*)

Q18. $y = -x(x - 3)(x + 3)$

Q19. $y = (x - 2)^2(x - 5)$

Q20. Show that $x = 2$ is a solution of $2x^3 - 4x^2 - 2x + 4 = 0$, hence sketch $y = 2(x + 1)(x - 1)(x - 2)$.

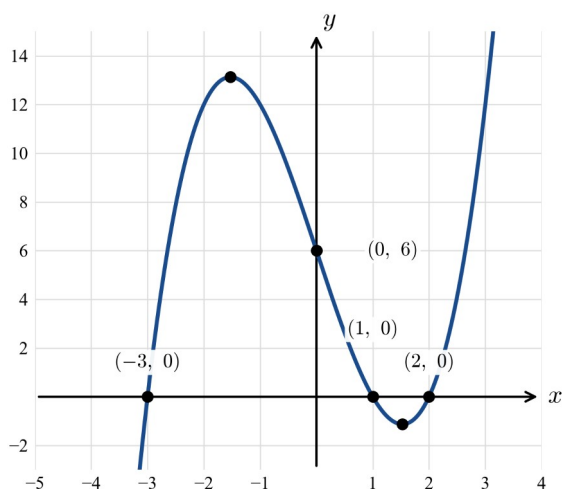
Q21. Describe the sequence of transformations that maps $y = x^3$ onto $y = (x + 1)^3 + 3$, and hence sketch it.

Answers

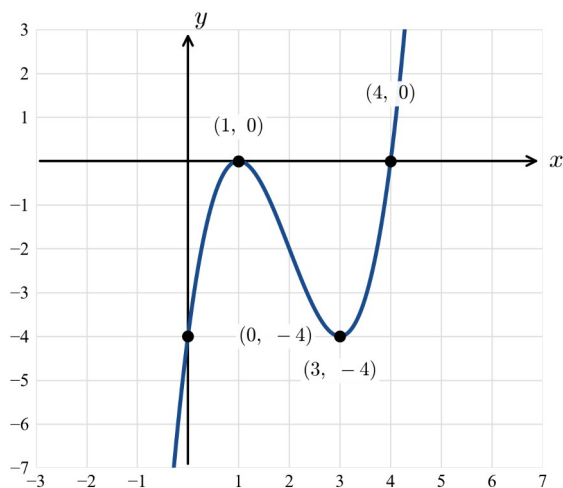
Q1. Crosses at $(-3, 0)$, $(1, 0)$, $(2, 0)$; y -intercept $(0, 6)$; $a > 0$ (falls left, rises right). **Q2.** Touches at $(1, 0)$ (double), crosses at $(4, 0)$; y -intercept $(0, -4)$. **Q3.** $y = x^3$ translated 2 left; stationary point of inflection $(-2, 0)$; y -intercept $(0, 8)$. **Q4.** Crosses at $(2, 0)$, $(-1, 0)$, $(-3, 0)$; rises left, falls right; y -intercept $(0, 6)$. **Q5.** Stationary point of inflection $(1, 2)$; x -intercept $(1 - \sqrt[3]{2}, 0)$; y -intercept $(0, 1)$. **Q6.** Crosses at $(-2, 0)$, $(1, 0)$, $(3, 0)$; y -intercept $(0, 12)$; the factor 2 dilates by factor 2 from the x -axis (steeper). **Q7.** Crosses at $(-2, 0)$, $(1, 0)$, $(4, 0)$; y -intercept $(0, 8)$. **Q8.** Crosses at $(-1, 0)$, $(2, 0)$, $(4, 0)$; y -intercept $(0, 8)$. **Q9.** Crosses at $(-2, 0)$, $(1, 0)$, $(3, 0)$; rises left, falls right; y -intercept $(0, -6)$. **Q10.** Crosses at $(-2, 0)$, touches at $(2, 0)$; y -intercept $(0, 8)$. **Q11.** Touches at $(-1, 0)$, crosses at $(3, 0)$; y -intercept $(0, -3)$. **Q12.** Touches at $(1, 0)$, crosses at $(-2, 0)$; rises left, falls right; y -intercept $(0, -2)$. **Q13.** Stationary point of inflection $(1, 0)$; y -intercept $(0, -1)$. **Q14.** Stationary point of inflection $(-2, -1)$; x -intercept $(-1, 0)$; y -intercept $(0, 7)$. **Q15.** Stationary point of inflection $(1, 0)$; y -intercept $(0, -2)$. **Q16.** $x(x-2)(x+2)$: crosses at $(-2, 0)$, $(0, 0)$, $(2, 0)$. **Q17.** $(x-1)(x+1)(x-3)$: crosses at $(-1, 0)$, $(1, 0)$, $(3, 0)$; y -intercept $(0, 3)$. **Q18.** $-x(x-3)(x+3)$: crosses at $(-3, 0)$, $(0, 0)$, $(3, 0)$; rises left, falls right. **Q19.** Touches at $(2, 0)$, crosses at $(5, 0)$; y -intercept $(0, -20)$; local minimum $(4, -4)$. **Q20.** $x = 2$: $2(8) - 4(4) - 2(2) + 4 = 16 - 16 - 4 + 4 = 0$. Crosses at $(-1, 0)$, $(1, 0)$, $(2, 0)$; y -intercept $(0, 4)$. **Q21.** Translation 1 left and 3 up; stationary point of inflection $(-1, 3)$; y -intercept $(0, 4)$.

Fully-worked solutions

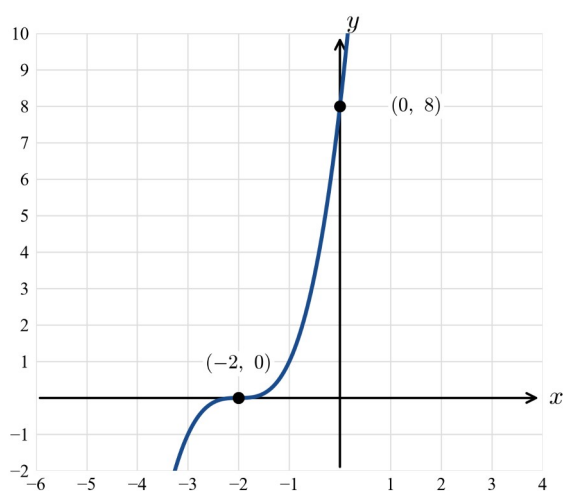
Q1. $y = (x+3)(x-1)(x-2)$. All single factors, so the graph crosses at $(-3, 0)$, $(1, 0)$ and $(2, 0)$. y -intercept: $x = 0 \Rightarrow y = (3)(-1)(-2) = 6$, giving $(0, 6)$. Since $a = 1 > 0$, it falls from the lower left and rises to the upper right.



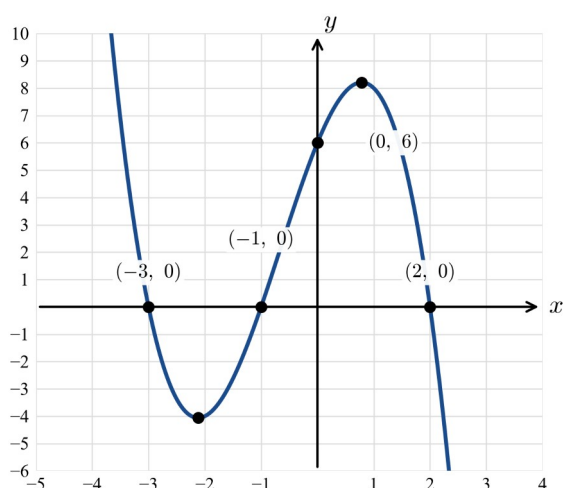
Q2. $y = (x-1)^2(x-4)$. $(x-1)^2$ is a double factor, so the graph **touches** at $(1, 0)$; $(x-4)$ is single, so it **crosses** at $(4, 0)$. y -intercept: $x = 0 \Rightarrow y = (-1)^2(-4) = -4$, giving $(0, -4)$. The touch at $(1, 0)$ is a local maximum on the axis.



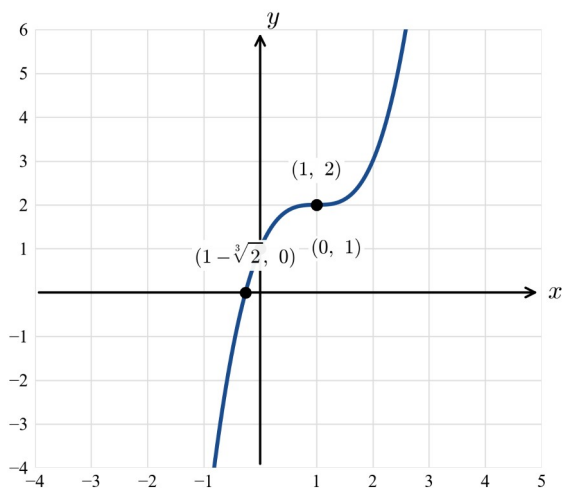
Q3. $y = (x+2)^3$ is $y = x^3$ translated 2 units left. Stationary point of inflection at $(-2, 0)$; y-intercept $x=0 \Rightarrow y=2^3=8$, giving $(0, 8)$.



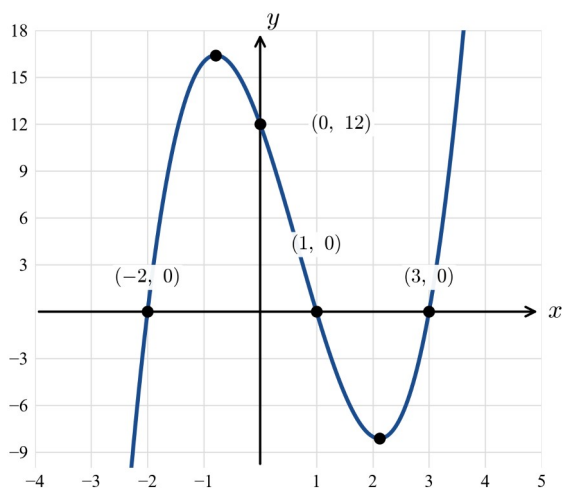
Q4. $y = -(x-2)(x+1)(x+3)$. Crosses at $(2, 0)$, $(-1, 0)$, $(-3, 0)$. Since $a = -1 < 0$ it rises from the upper left and falls to the lower right. y-intercept: $x=0 \Rightarrow y = -(-2)(1)(3) = 6$, giving $(0, 6)$.



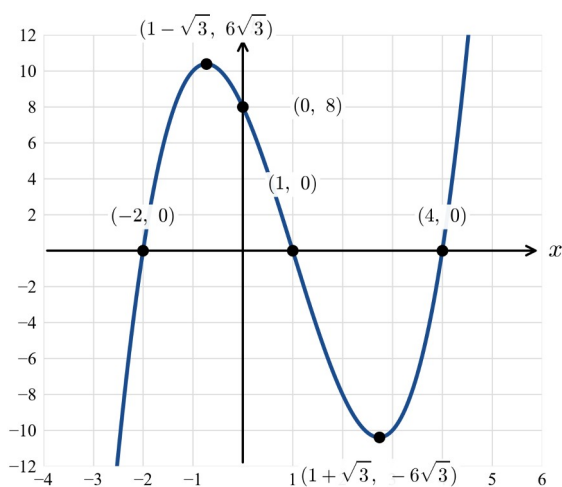
Q5. $y = (x-1)^3 + 2$. Stationary point of inflection at $(1, 2)$. x-intercept: $(x-1)^3 = -2 \Rightarrow x-1 = -\sqrt[3]{2}$, so $x = 1 - \sqrt[3]{2}$, giving $(1 - \sqrt[3]{2}, 0)$. y-intercept: $x=0 \Rightarrow y = (-1)^3 + 2 = 1$, giving $(0, 1)$.



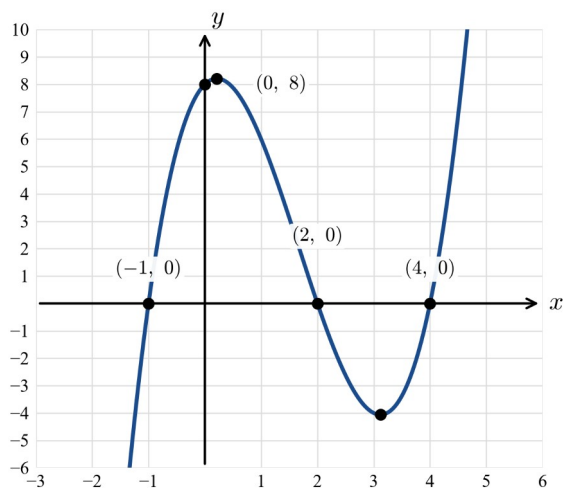
Q6. $y = 2(x + 2)(x - 1)(x - 3)$. Crosses at $(-2, 0)$, $(1, 0)$, $(3, 0)$. y -intercept: $x = 0 \Rightarrow y = 2(2)(-1)(-3) = 12$, giving $(0, 12)$. The factor 2 is a dilation of factor 2 from the x -axis, making the curve steeper than the monic cubic with the same roots.



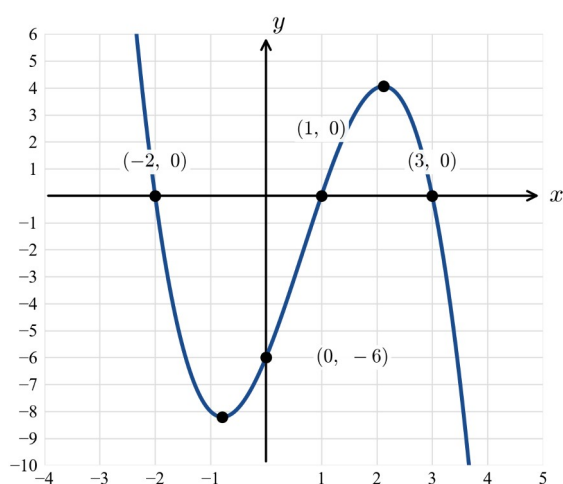
Q7. $y = (x + 2)(x - 1)(x - 4)$. Crosses at $(-2, 0)$, $(1, 0)$, $(4, 0)$; y -intercept $x = 0 \Rightarrow (2)(-1)(-4) = 8$, giving $(0, 8)$; $a > 0$.



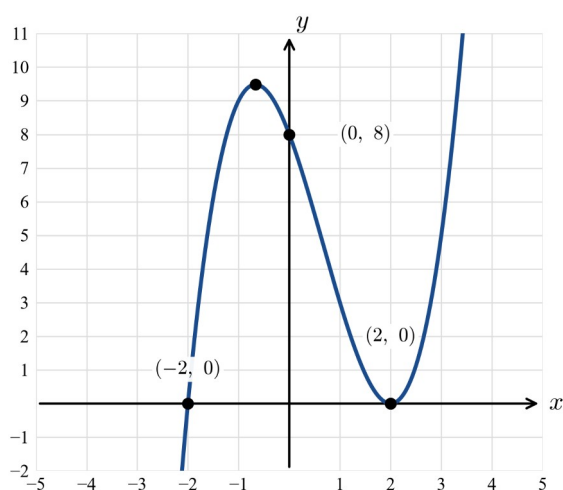
Q8. $y = (x + 1)(x - 2)(x - 4)$. Crosses at $(-1, 0)$, $(2, 0)$, $(4, 0)$; y -intercept $x = 0 \Rightarrow (1)(-2)(-4) = 8$, giving $(0, 8)$.



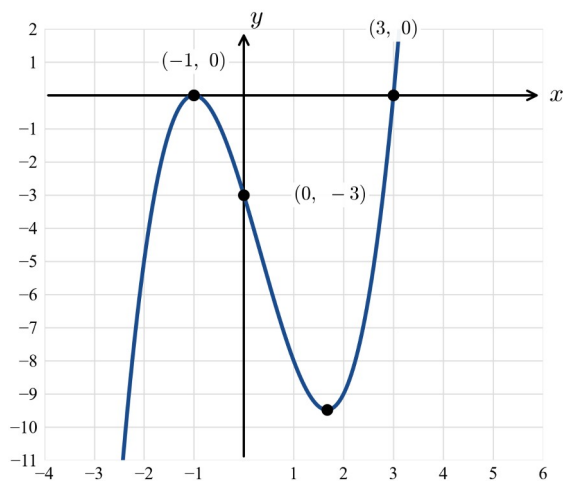
Q9. $y = -(x+2)(x-1)(x-3)$. Crosses at $(-2, 0)$, $(1, 0)$, $(3, 0)$; $a < 0$ so it rises left and falls right; y -intercept $x=0 \Rightarrow -(2)(-1)(-3) = -6$, giving $(0, -6)$.



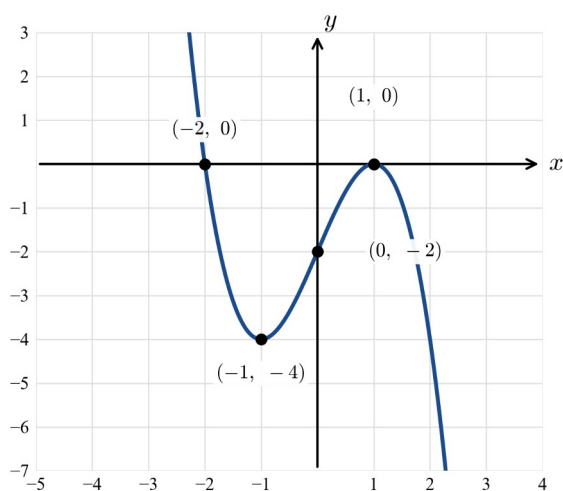
Q10. $y = (x-2)^2(x+2)$. Touches at $(2, 0)$ (double), crosses at $(-2, 0)$; y -intercept $x=0 \Rightarrow (-2)^2(2) = 8$, giving $(0, 8)$.



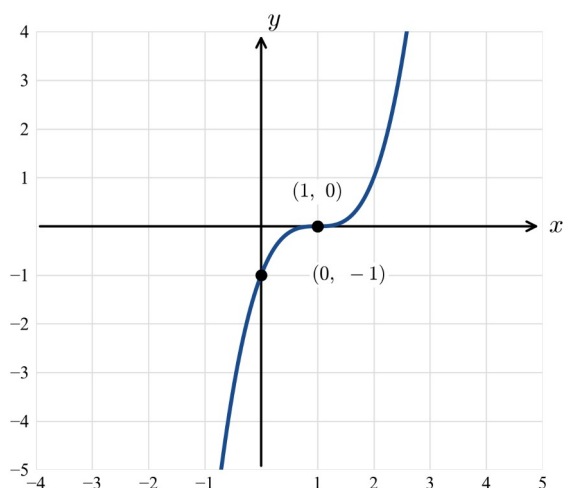
Q11. $y = (x+1)^2(x-3)$. Touches at $(-1, 0)$ (double), crosses at $(3, 0)$; y -intercept $x=0 \Rightarrow (1)^2(-3) = -3$, giving $(0, -3)$.



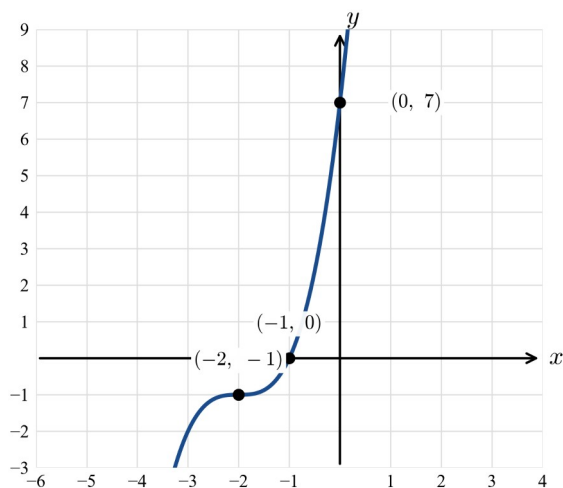
Q12. $y = -(x-1)^2(x+2)$. Touches at $(1, 0)$ (double), crosses at $(-2, 0)$; $a < 0$ so it rises left and falls right; y -intercept $x=0 \Rightarrow -(-1)^2(2) = -2$, giving $(0, -2)$.



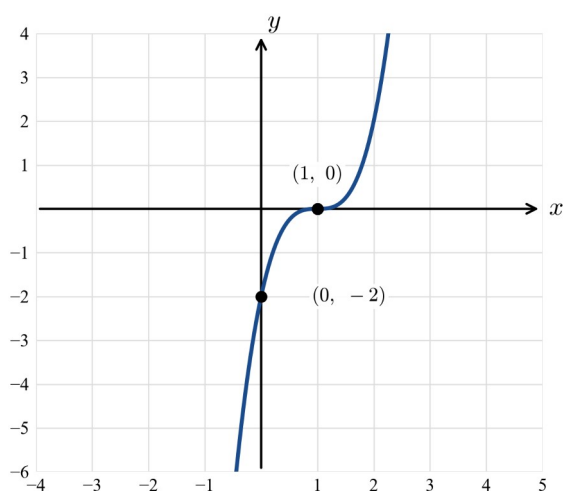
Q13. $y = (x-1)^3$ is $y = x^3$ translated 1 right; stationary point of inflection at $(1, 0)$; y -intercept $x=0 \Rightarrow (-1)^3 = -1$, giving $(0, -1)$.



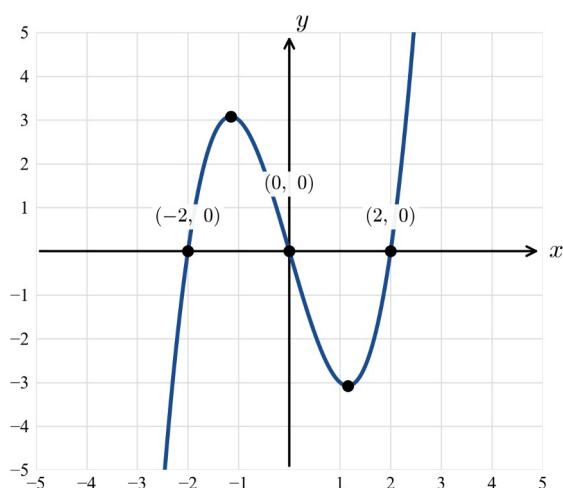
Q14. $y = (x+2)^3 - 1$. Stationary point of inflection at $(-2, -1)$; x -intercept $(x+2)^3 = 1 \Rightarrow x+2 = 1 \Rightarrow x = -1$, giving $(-1, 0)$; y -intercept $x=0 \Rightarrow 2^3 - 1 = 7$, giving $(0, 7)$.



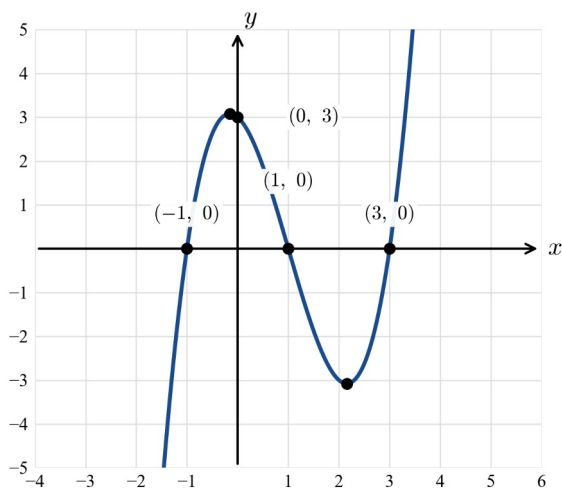
Q15. $y = 2(x - 1)^3$. Stationary point of inflection at $(1, 0)$; the factor 2 dilates by factor 2 from the x -axis; y -intercept $x = 0 \Rightarrow 2(-1)^3 = -2$, giving $(0, -2)$.



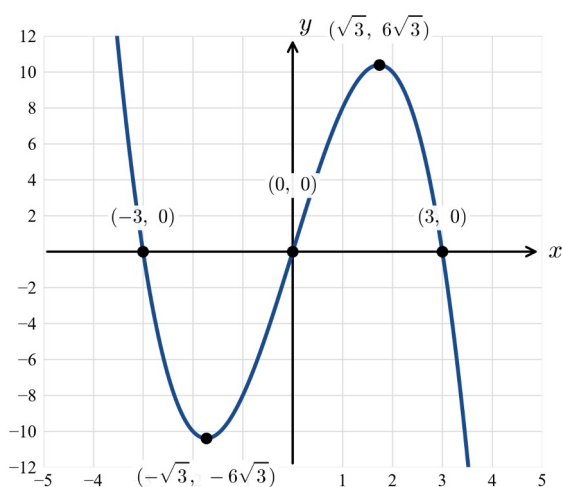
Q16. $y = x^3 - 4x = x(x - 2)(x + 2)$. Crosses at $(-2, 0)$, $(0, 0)$, $(2, 0)$; the y -intercept is the origin.



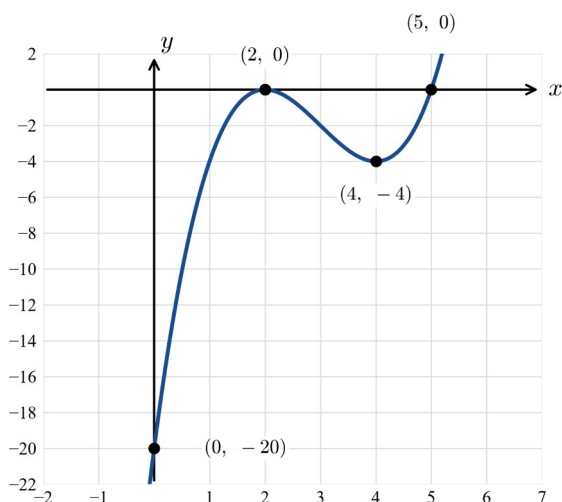
Q17. $y = x^3 - 3x^2 - x + 3$. Group: $x^2(x - 3) - (x - 3) = (x - 3)(x^2 - 1) = (x - 3)(x - 1)(x + 1)$. Crosses at $(-1, 0)$, $(1, 0)$, $(3, 0)$; y -intercept $(0, 3)$.



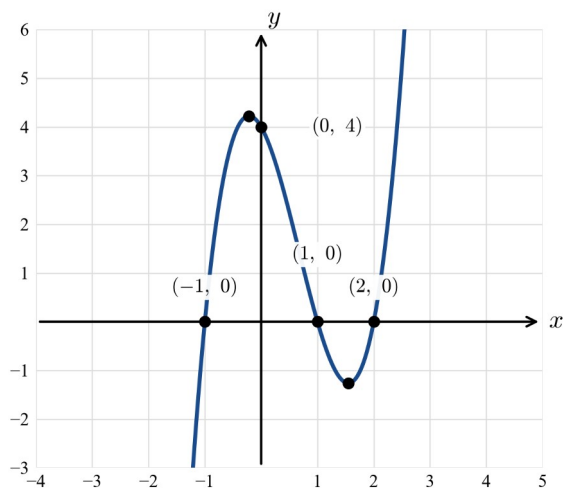
Q18. $y = -x(x-3)(x+3)$. Crosses at $(-3, 0)$, $(0, 0)$, $(3, 0)$; $a < 0$ so it rises left and falls right; the y -intercept is the origin.



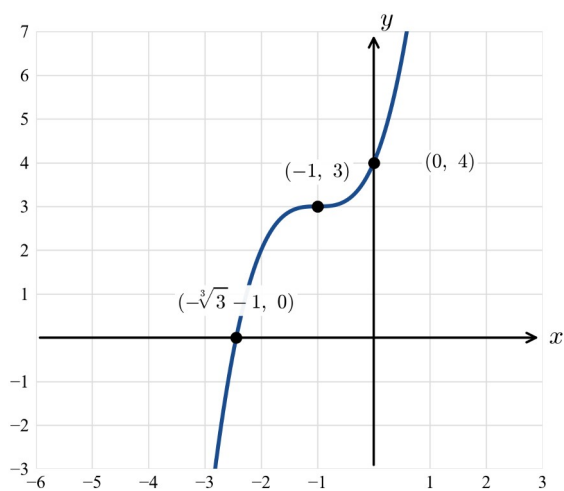
Q19. $y = (x-2)^2(x-5)$. Touches at $(2, 0)$ (double), crosses at $(5, 0)$; y -intercept $x=0 \Rightarrow (-2)^2(-5) = -20$, giving $(0, -20)$. The touch at $(2, 0)$ is a local maximum on the axis, with a local minimum at $(4, -4)$.



Q20. At $x=2$: $2(2)^3 - 4(2)^2 - 2(2) + 4 = 16 - 16 - 4 + 4 = 0$, so $x=2$ is a solution. Factorising, $2x^3 - 4x^2 - 2x + 4 = 2(x+1)(x-1)(x-2)$, so $y = 2(x+1)(x-1)(x-2)$ crosses at $(-1, 0)$, $(1, 0)$, $(2, 0)$; y -intercept $x=0 \Rightarrow 2(1)(-1)(-2) = 4$, giving $(0, 4)$.



Q21. $y = (x+1)^3 + 3$ is $y = x^3$ translated 1 unit left and 3 units up. Stationary point of inflection at $(-1, 3)$; y-intercept $x=0 \Rightarrow 1^3 + 3 = 4$, giving $(0, 4)$.



Chapter 4 Polynomial functions — 4F Quartics and higher-degree polynomials

Theory

A **polynomial of degree n** has the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ with $a_n \neq 0$. This section sketches polynomials of degree 3 and above from their **factored form**, where the key features can be read off directly.

x -intercepts and the behaviour at each factor. If $P(x) = a(x-r)^m \times (\text{other factors})$, then $x=r$ is an x -intercept, and the graph's behaviour there depends on the **multiplicity m** :

- $m=1$ (**simple factor**): the graph **crosses** the x -axis straight through.
- m **even** (**double, quadruple, ...**): the graph **touches** the axis and turns back (like a parabola vertex).
- m **odd and $m \geq 3$** (**triple, ...**): the graph **flattens** as it crosses (a stationary point of inflection on the axis).

End behaviour is fixed by the **leading term $a_n x^n$** :

Degree n	$a_n > 0$	$a_n < 0$
even	both arms up	both arms down
odd	down (left) to up (right)	up (left) to down (right)

y -intercept: substitute $x=0$ (this is a_0 , the constant term).

To sketch: (1) read the degree and leading coefficient for the end behaviour; (2) mark each x -intercept and decide cross/touch/flatten from its multiplicity; (3) mark the y -intercept; (4) join with a smooth curve, consistent with the arms.

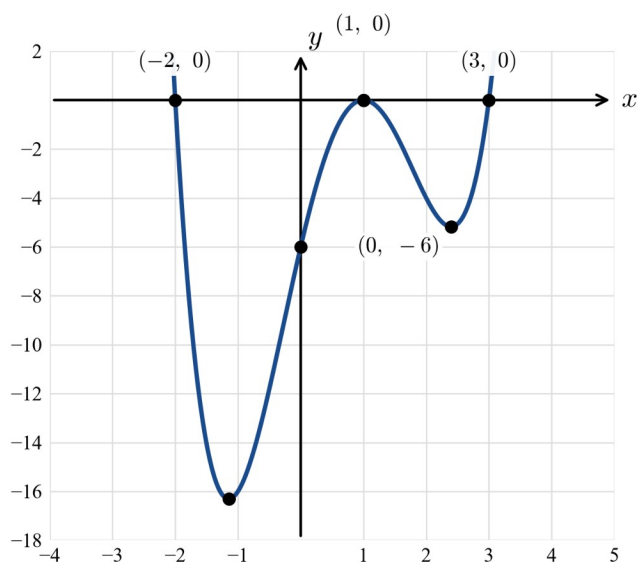
Sign diagrams and inequalities. Because a polynomial only changes sign at a **simple or odd-multiplicity** root (never at an even one), a sketch or a sign diagram lets you solve $P(x) \geq 0$ or $P(x) \leq 0$ by reading off the intervals where the curve is above or below the axis. Remember to **include** the roots themselves for $\geq / \leq$ (and any touch point where $P=0$).

Worked examples

Example 1 — scaffolded

Sketch $y = (x+2)(x-1)^2(x-3)$, labelling the intercepts.

Working	Comment
Degree 4 (add the factor powers: $1+2+1$), leading coefficient $+1$.	The expanded leading term is x^4 , so both arms point up .
x -intercepts: $x = -2$ (simple, cross), $x = 1$ (double, touch), $x = 3$ (simple, cross).	The power of each factor gives the multiplicity.
y -intercept: $x=0 \Rightarrow y = (2)(-1)^2(-3) = -6$, giving $(0, -6)$.	Substitute $x=0$.
Curve rises on both sides; crosses at -2 , comes up to touch at $(1, 0)$, turns down, then crosses at 3 .	Consistent with up/up arms and the touch at $x=1$.



Example 2 — scaffolded

Sketch $y = -(x+1)(x-2)^3$, labelling the intercepts.

Working

Degree 4 ($1+3$), leading coefficient -1 .

x -intercepts: $x = -1$ (simple, cross), $x = 2$ (triple, **flattens** as it crosses).

y -intercept: $x = 0 \Rightarrow y = -(1)(-2)^3 = -(-8) = 8$, giving $(0, 8)$.

From the lower left, up across $(-1, 0)$, over a maximum, then flattening down through $(2, 0)$ to the lower right.

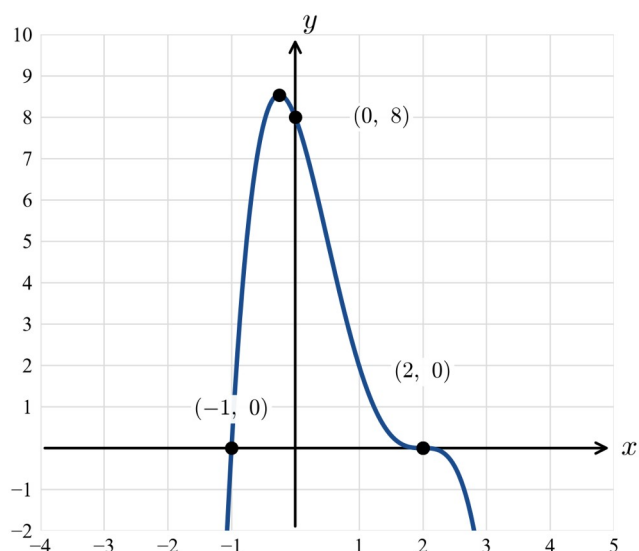
Comment

Expanded leading term $-x^4$, so **both arms point down**.

Odd multiplicity 3 $\Rightarrow$ stationary point of inflection on the axis.

Substitute $x = 0$.

Consistent with down/down arms.



Example 3 — unscaffolded

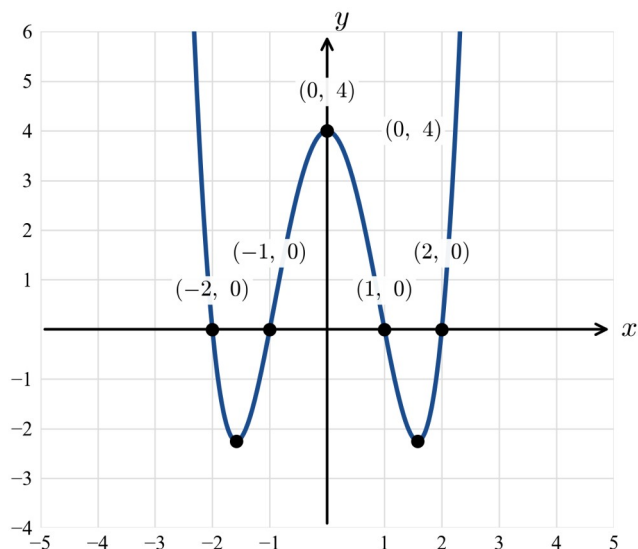
Sketch $y = (x-1)(x+1)(x-2)(x+2)$.

Degree 4, leading coefficient $+1$: both arms up. Expanding gives $y = x^4 - 5x^2 + 4$, an **even function** (symmetric about the y -axis).

x -intercepts (all simple, all cross): $x = -2, -1, 1, 2$.

y-intercept: $x=0 \Rightarrow y=4$, giving $(0, 4)$.

$\therefore$ a “W”-shaped quartic crossing the axis four times, symmetric about the y-axis.

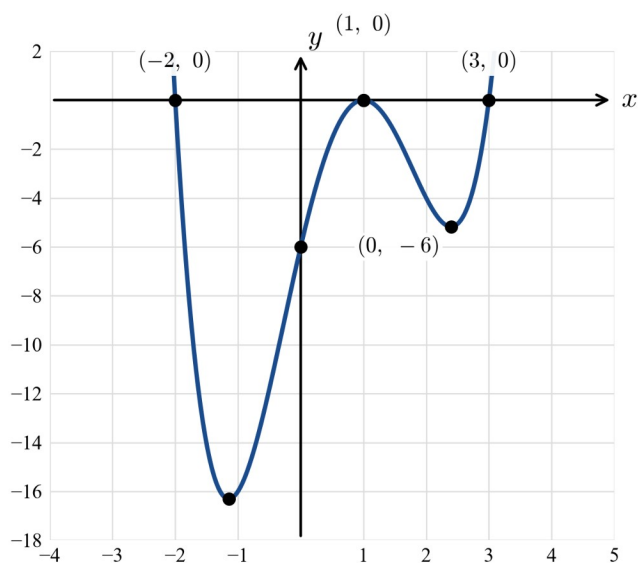


Example 4 — unscaffolded

Using the graph of $y = (x+2)(x-1)^2(x-3)$ from Example 1, solve $(x+2)(x-1)^2(x-3) \geq 0$.

From the sketch the curve is **on or above** the x-axis for $x \leq -2$, only at the single touch point $x = 1$, and for $x \geq 3$; it is below the axis on the rest of $(-2, 3)$.

$\therefore$ the solution is $(-\infty, -2] \cup \{1\} \cup [3, \infty)$.



Practice questions

Scaffolded

Q1. For $y = (x+3)(x+1)(x-1)(x-3)$: (a) state the degree and the end behaviour; (b) write the x- and y-intercepts; (c) sketch the graph.

Q2. For $y = (x-2)^2(x+1)^2$: (a) state the multiplicity of each factor and whether the graph crosses or touches; (b) find the y-intercept; (c) sketch the graph.

Q3. For $y = x(x - 3)^3$: (a) state the end behaviour; (b) describe the behaviour at each x -intercept; (c) sketch the graph.

Q4. For $y = -(x + 2)(x - 1)(x - 2)(x - 4)$: (a) state the degree and end behaviour; (b) find all intercepts; (c) sketch the graph.

Q5. For $y = (x + 1)(x - 2)^2(x - 4)$: (a) sketch the graph; (b) hence solve $(x + 1)(x - 2)^2(x - 4) \leq 0$.

Q6. For $y = x(x + 2)(x - 2)(x - 3)$: (a) find all intercepts; (b) state the end behaviour; (c) sketch the graph.

Un scaffolded

Sketch each of the following (Q7–Q15, Q18, Q19, Q21), labelling all intercepts and showing the correct behaviour at each; answer the inequality questions (Q16, Q17) from a sketch; and answer Q20 directly.

Q7. $y = (x + 1)(x - 2)(x - 3)(x + 3)$

Q8. $y = -(x - 1)^2(x + 2)(x - 4)$

Q9. $y = (x + 1)^2(x - 3)^2$

Q10. $y = x^2(x - 2)(x + 3)$

Q11. $y = (x - 1)(x + 2)^3$

Q12. $y = (x^2 - 4)(x^2 - 9)$

Q13. $y = -(x + 1)(x - 1)(x - 2)(x - 3)$

Q14. $y = x(x - 2)^2(x + 2)$

Q15. $y = (x + 2)(x + 1)(x - 1)(x - 2)(x - 3)$

Q16. Solve $x(x - 1)(x - 3)(x + 2) > 0$.

Q17. Solve $(x + 1)(x - 2)^2(x - 3) \geq 0$.

Q18. $y = -(x^2 - 1)(x^2 - 4)$

Q19. $y = (x - 2)^4$

Q20. State the degree, the leading coefficient, and the end behaviour of $y = -2x^5 + 3x^2 - 1$.

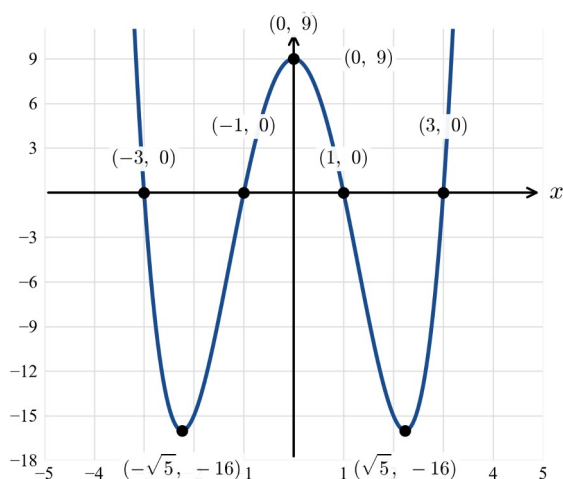
Q21. $y = x(x - 2)(x + 2)(x - 1)(x + 1)$

Answers

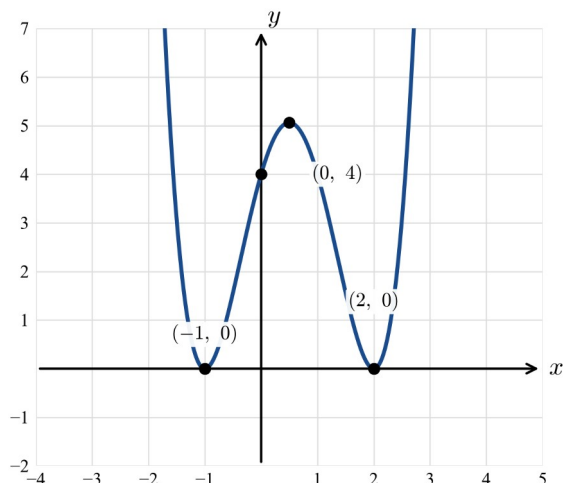
Q1. Degree 4, both arms up; intercepts $(\pm 1, 0), (\pm 3, 0), (0, 9)$. **Q2.** $x = -1$ and $x = 2$ both double (touch); $(0, 4)$; both arms up. **Q3.** Both arms up; crosses at $(0, 0)$, flattens through $(3, 0)$ (triple). **Q4.** Degree 4, both arms down; intercepts $(-2, 0), (1, 0), (2, 0), (4, 0), (0, 16)$. **Q5.** Intercepts $(-1, 0), (2, 0)$ touch, $(4, 0)$; solution of the inequality: $[-1, 4]$. **Q6.** Intercepts $(-2, 0), (0, 0), (2, 0), (3, 0)$; both arms up. **Q7.** Intercepts $(-3, 0), (-1, 0), (2, 0), (3, 0), (0, 18)$; both arms up. **Q8.** Intercepts $(-2, 0), (1, 0)$ touch, $(4, 0), (0, 8)$; both arms down. **Q9.** $x = -1, 3$ both touch; $(0, 9)$; both arms up. **Q10.** Intercepts $(-3, 0), (0, 0)$ touch, $(2, 0)$; both arms up. **Q11.** $x = -2$ triple (flattens), $x = 1$ simple; $(0, -8)$; both arms up. **Q12.** Intercepts $(\pm 2, 0), (\pm 3, 0), (0, 36)$; both arms up. **Q13.** Intercepts $(-1, 0), (1, 0), (2, 0), (3, 0), (0, 6)$; both arms down. **Q14.** Intercepts $(-2, 0), (0, 0), (2, 0)$ touch; both arms up. **Q15.** Degree 5; intercepts $(-2, 0), (-1, 0), (1, 0), (2, 0), (3, 0), (0, -12)$; left arm down, right arm up. **Q16.** $(-\infty, -2) \cup (0, 1) \cup (3, \infty)$. **Q17.** $(-\infty, -1] \cup [2, \infty)$. **Q18.** Degree 4, both arms down; intercepts $(\pm 1, 0), (\pm 2, 0), (0, -4)$. **Q19.** $x = 2$ quadruple (flat touch); $(0, 16)$; both arms up. **Q20.** Degree 5, leading coefficient -2 ; left arm up, right arm down. **Q21.** Degree 5; intercepts $(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)$; left arm down, right arm up.

Fully-worked solutions

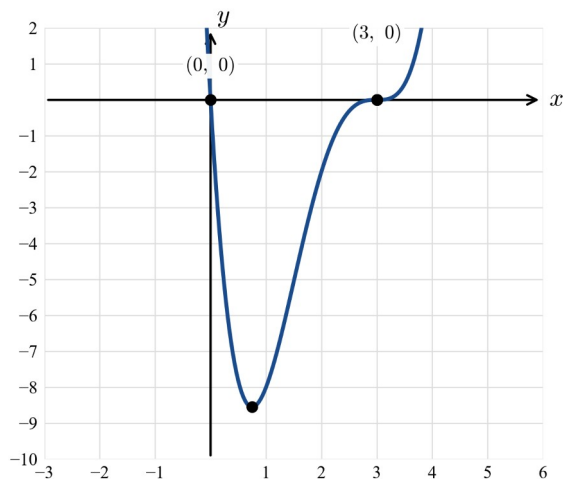
Q1. $y = (x+3)(x+1)(x-1)(x-3)$. Degree 4, leading coefficient $+1$: both arms up. All four factors are simple, so the graph crosses at $x = -3, -1, 1, 3$. y -intercept $x = 0 \Rightarrow (3)(1)(-1)(-3) = 9$, giving $(0, 9)$.



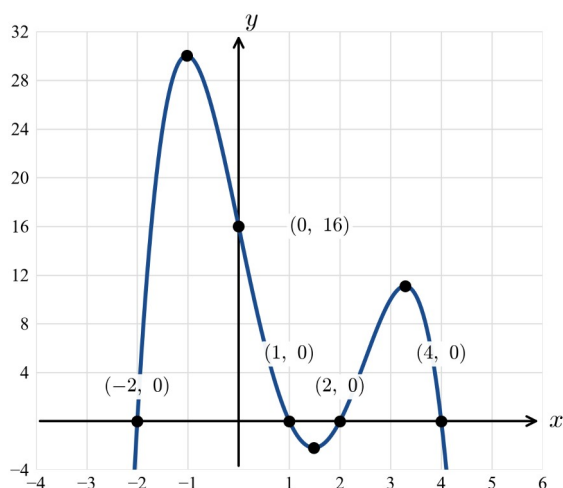
Q2. $y = (x-2)^2(x+1)^2$. Both factors are squared, so the graph **touches** at $x = 2$ and $x = -1$. Degree 4, leading coefficient $+1$: both arms up, and since $y = [(x-2)(x+1)]^2 \geq 0$ the curve never goes below the axis. y -intercept $(0, 4)$.



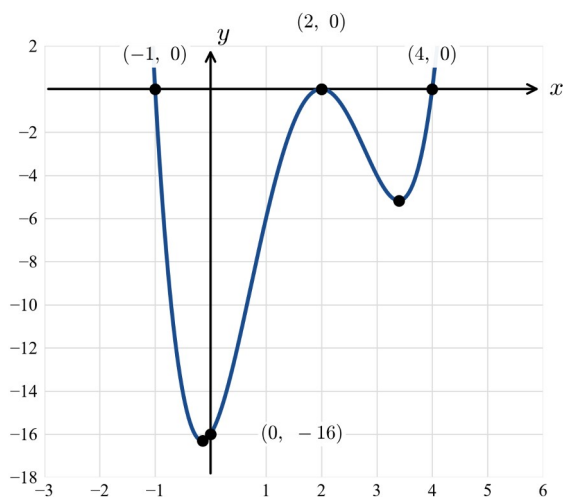
Q3. $y = x(x - 3)^3$. Degree 4, leading coefficient +1: both arms up. At $x = 0$ (simple) it crosses; at $x = 3$ (triple) it flattens as it crosses. y -intercept $(0, 0)$.



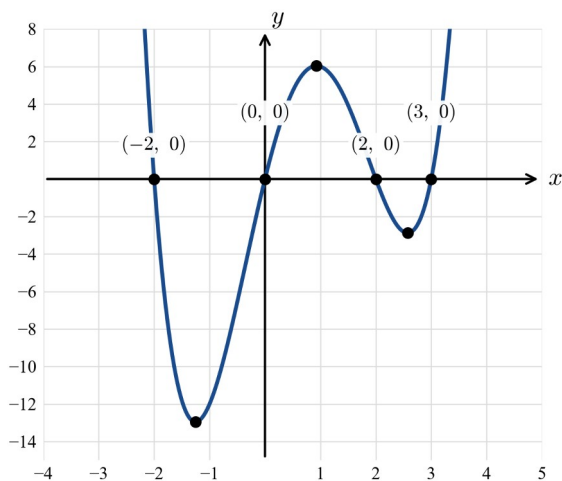
Q4. $y = -(x + 2)(x - 1)(x - 2)(x - 4)$. Degree 4, leading coefficient -1: both arms down. Four simple roots: crosses at $x = -2, 1, 2, 4$. y -intercept $x = 0 \Rightarrow -(2)(-1)(-2)(-4) = -(-16) = 16$, giving $(0, 16)$.



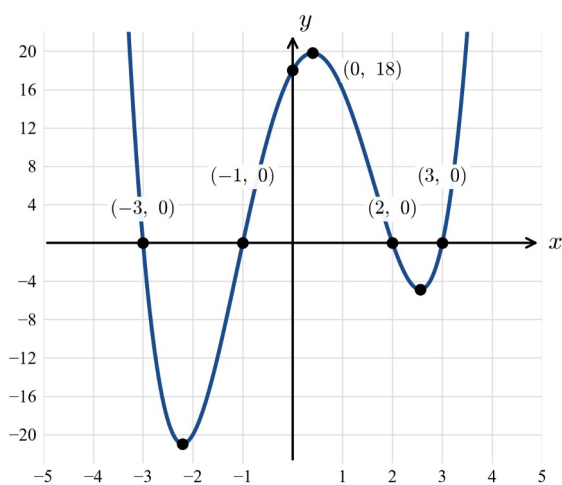
Q5. $y = (x + 1)(x - 2)^2(x - 4)$. Crosses at $x = -1$ and $x = 4$; touches at $x = 2$. Degree 4, leading coefficient +1: both arms up. (a) sketch below. (b) The curve is **on or below** the axis between the outer crossings, and only touches (does not go above) at $x = 2$, so $(x + 1)(x - 2)^2(x - 4) \leq 0$ for $-1 \leq x \leq 4$, i.e. $[-1, 4]$.



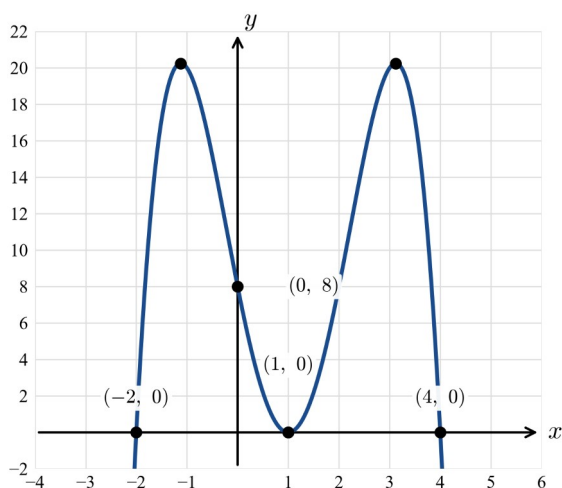
Q6. $y = x(x + 2)(x - 2)(x - 3)$. Four simple roots: $x = -2, 0, 2, 3$. Degree 4, leading coefficient +1: both arms up. y -intercept $(0, 0)$.



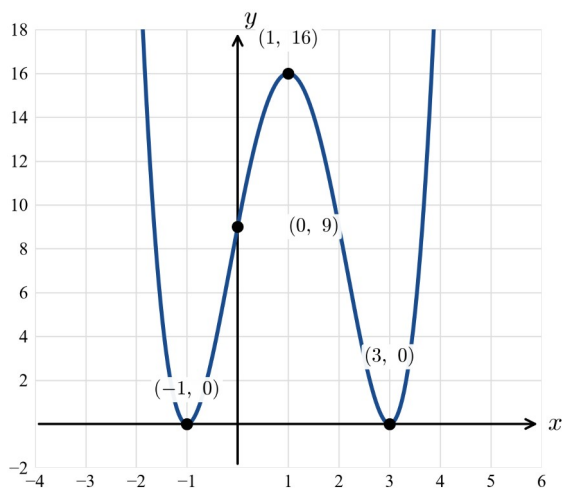
Q7. $y = (x+1)(x-2)(x-3)(x+3)$. Simple roots $-3, -1, 2, 3$; both arms up. y -intercept $x=0 \Rightarrow (1)(-2)(-3)(3)=18$, giving $(0, 18)$.



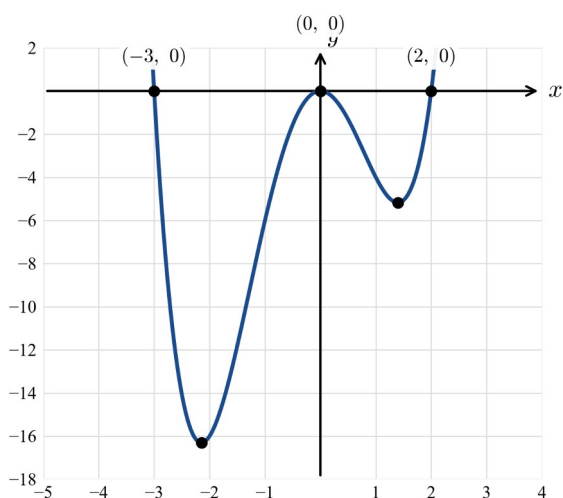
Q8. $y = -(x-1)^2(x+2)(x-4)$. Touches at $x=1$ (double); crosses at $x=-2$ and $x=4$. Degree 4, leading coefficient -1 : both arms down. y -intercept $x=0 \Rightarrow -(-1)^2(2)(-4) = -(1)(2)(-4)=8$, giving $(0, 8)$.



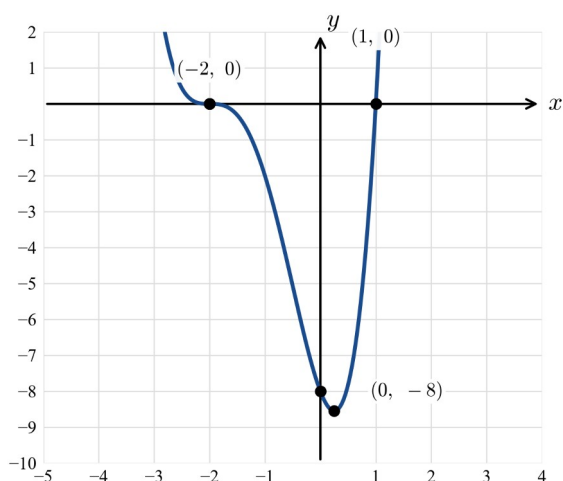
Q9. $y = (x+1)^2(x-3)^2$. Touches at $x=-1$ and $x=3$ (both double); both arms up; $y = [(x+1)(x-3)]^2 \geq 0$. y -intercept $(0, 9)$.



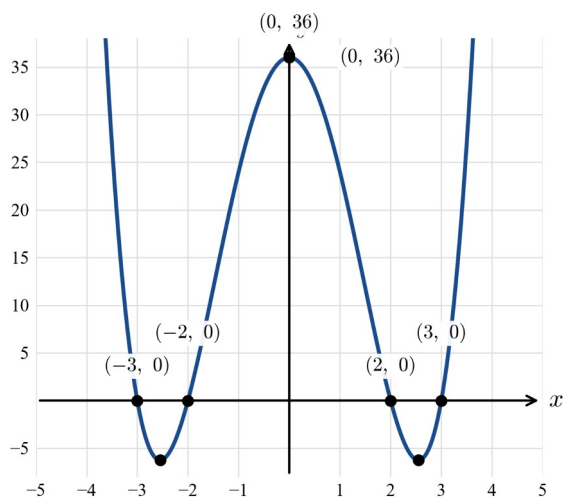
Q10. $y = x^2(x - 2)(x + 3)$. Touches at $x = 0$ (double); crosses at $x = 2$ and $x = -3$. Degree 4, leading coefficient $+1$: both arms up. y -intercept $(0, 0)$.



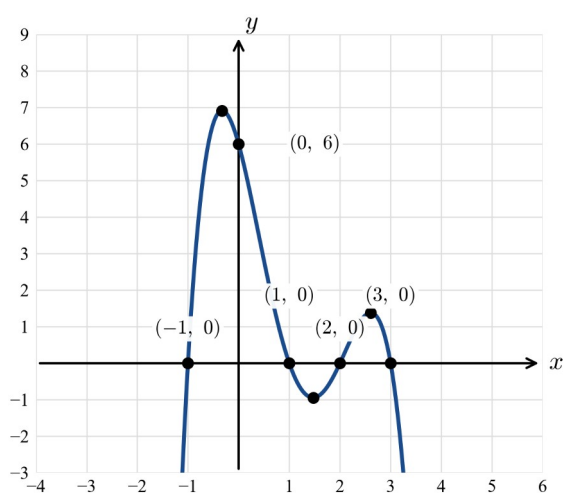
Q11. $y = (x - 1)(x + 2)^3$. Crosses at $x = 1$ (simple); flattens through $x = -2$ (triple). Both arms up. y -intercept $x = 0 \Rightarrow (-1)(2)^3 = -8$, giving $(0, -8)$.



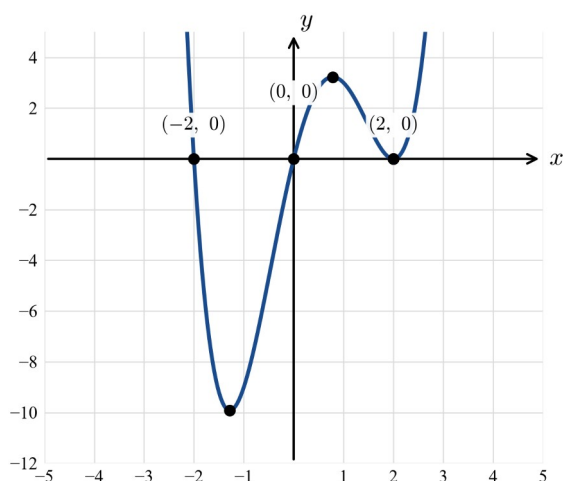
Q12. $y = (x^2 - 4)(x^2 - 9) = (x - 2)(x + 2)(x - 3)(x + 3)$. Four simple roots $\pm 2, \pm 3$; both arms up; even function. y -intercept $(0, 36)$.



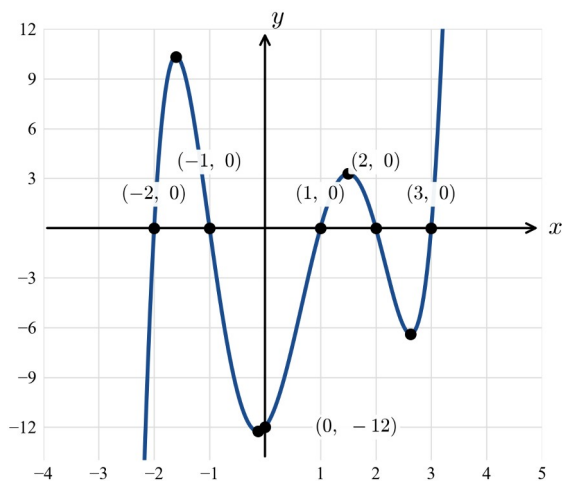
Q13. $y = -(x+1)(x-1)(x-2)(x-3)$. Four simple roots $-1, 1, 2, 3$; degree 4, leading coefficient -1 : both arms down. y -intercept $x=0 \Rightarrow -(1)(-1)(-2)(-3) = -(-6) = 6$, giving $(0, 6)$.



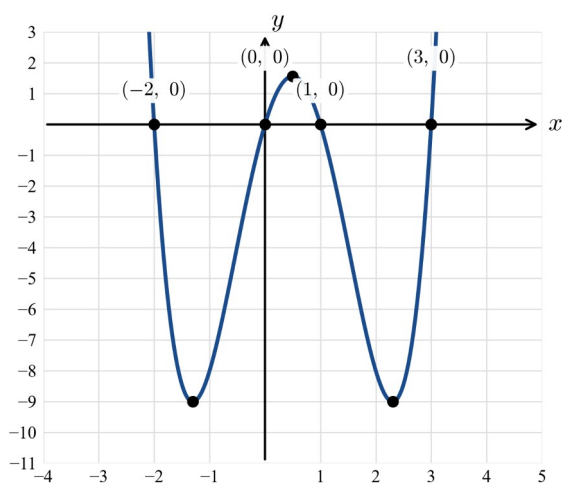
Q14. $y = x(x-2)^2(x+2)$. Crosses at $x=0$ and $x=-2$; touches at $x=2$ (double). Both arms up. y -intercept $(0, 0)$.



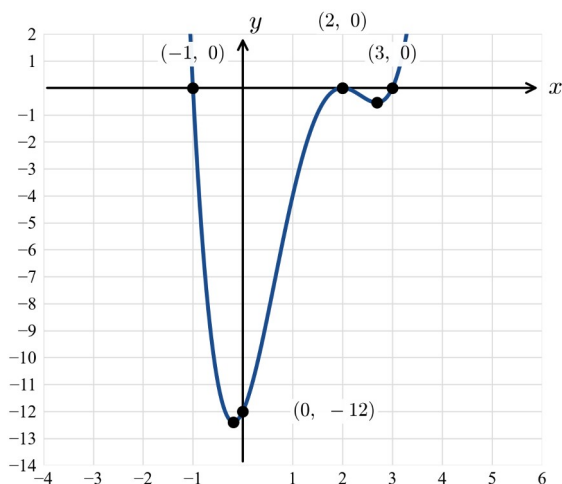
Q15. $y = (x+2)(x+1)(x-1)(x-2)(x-3)$. Degree 5, leading coefficient $+1$: left arm down, right arm up. Five simple roots $-2, -1, 1, 2, 3$. y -intercept $x=0 \Rightarrow (2)(1)(-1)(-2)(-3) = -12$, giving $(0, -12)$.



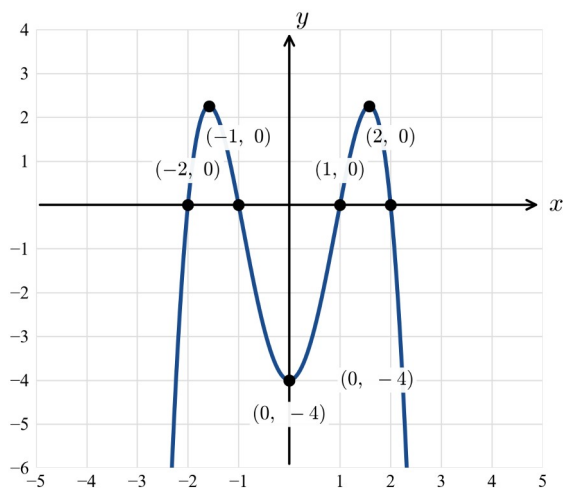
Q16. $y = x(x-1)(x-3)(x+2)$ has simple roots $-2, 0, 1, 3$ and both arms up. Reading where the curve is **above** the axis: $x < -2$, then $0 < x < 1$, then $x > 3$. $\therefore$ solution $(-\infty, -2) \cup (0, 1) \cup (3, \infty)$.



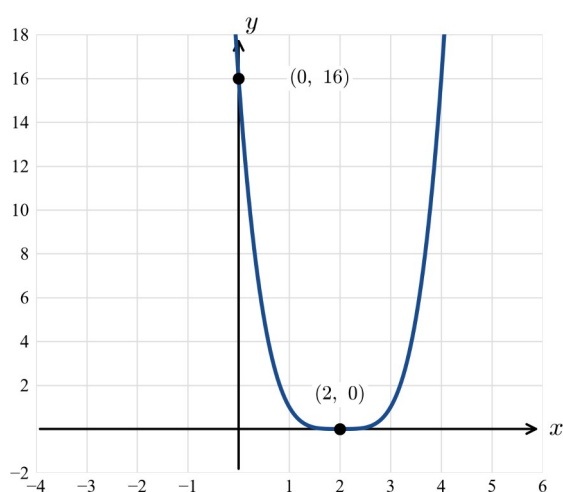
Q17. $y = (x+1)(x-2)^2(x-3)$ crosses at -1 and 3 and touches at 2 ; both arms up. The curve is on or above the axis for $x \leq -1$, only at the touch $x = 2$, and for $x \geq 3$. $\therefore$ solution $(-\infty, -1] \cup \{2\} \cup [3, \infty)$.



Q18. $y = -(x^2-1)(x^2-4) = -(x-1)(x+1)(x-2)(x+2)$. Four simple roots $\pm 1, \pm 2$; degree 4, leading coefficient -1 : both arms down; even function. y -intercept $x = 0 \Rightarrow -(-1)(-4) = -4$, giving $(0, -4)$.

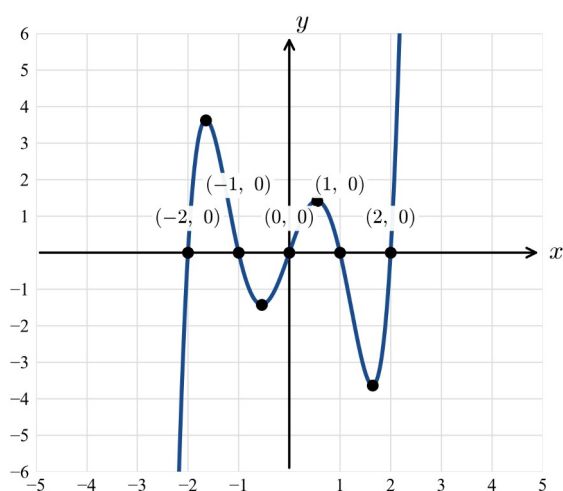


Q19. $y = (x - 2)^4$. A single root $x = 2$ with multiplicity 4 (even), so the graph **touches** the axis there, very flat. Degree 4, leading coefficient $+1$: both arms up; $y \geq 0$ everywhere. y -intercept $(0, 16)$.



Q20. $y = -2x^5 + 3x^2 - 1$. The leading term is $-2x^5$: degree 5 (odd) and leading coefficient $-2 < 0$. So as $x \rightarrow -\infty$, $y \rightarrow +\infty$ (left arm up), and as $x \rightarrow +\infty$, $y \rightarrow -\infty$ (right arm down).

Q21. $y = x(x - 2)(x + 2)(x - 1)(x + 1)$. Degree 5, leading coefficient $+1$: left arm down, right arm up. Five simple roots $-2, -1, 0, 1, 2$; odd function. y -intercept $(0, 0)$.



Chapter 4 Polynomial functions — 4G Finding polynomial rules from graphs

Theory

Given the graph of a polynomial, we can reconstruct its rule by reading off the **x -intercepts** and the **way the curve meets the x -axis** at each one, then using one more point (usually the **y -intercept**) to find the leading coefficient a .

Multiplicity — how the curve meets the x -axis. If $x = p$ is a root of a polynomial, the *multiplicity* of the factor $(x - p)$ controls the local behaviour:

- **Single factor** $(x - p)$ (multiplicity 1): the graph **crosses** the axis straight through.
- **Double factor** $(x - p)^2$ (multiplicity 2): the graph **touches** the axis (a turning point sits on the axis) and turns back.
- **Triple factor** $(x - p)^3$ (multiplicity 3): the graph has a **stationary point of inflection** on the axis — it flattens, crosses, and continues.

Reading the degree and sign. The number of factors (counted with multiplicity) is the **degree**. The **ends** of the graph give the sign of a : for an odd degree, up-to-the-right means $a > 0$; for an even degree, both ends up means $a > 0$ (and down means $a < 0$).

Method.

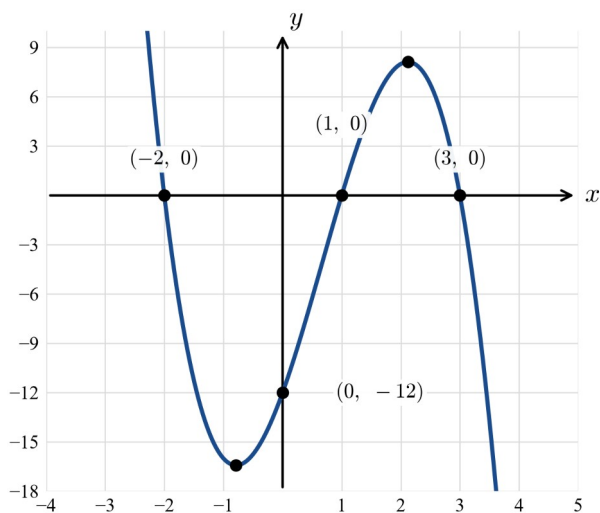
1. Write a factor for each x -intercept, raised to the power of its multiplicity.
2. Write $y = a \times (\text{product of factors})$.
3. Substitute the coordinates of one more known point (the y -intercept is easiest) and solve for a .

Stationary point of inflection form. If the graph is a cubic with a stationary point of inflection at (h, k) , use $y = a(x - h)^3 + k$, then substitute another point to find a .

Worked examples

Example 1 — scaffolded

The cubic graph shown crosses the x -axis at $x = -2$, $x = 1$ and $x = 3$, and has y -intercept $(0, -12)$. Determine its rule.



Working

Each intercept is a simple crossing, so the factors are $(x + 2)$, $(x - 1)$, $(x - 3)$.

Comment

Multiplicity 1 at each root — single factors.

Working

$$y = a(x+2)(x-1)(x-3).$$

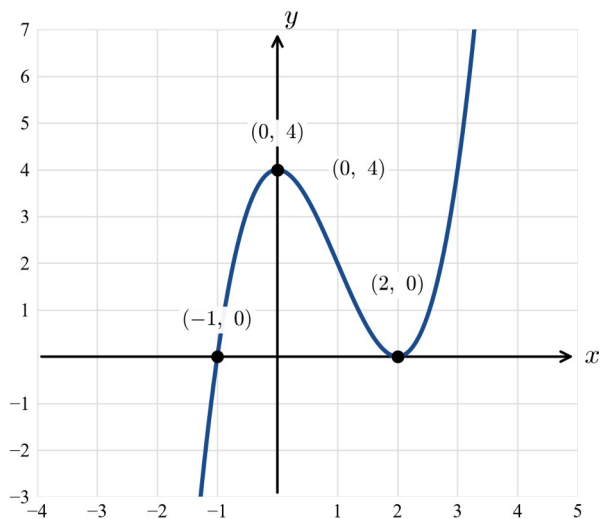
$$\text{Substitute } (0, -12): -12 = a(2)(-1)(-3) = 6a.$$

$$a = -2.$$

$$\therefore y = -2(x+2)(x-1)(x-3).$$

Example 2 — scaffolded

The cubic graph shown touches the x -axis at $x=2$, crosses it at $x=-1$, and has y -intercept $(0, 4)$. Determine its rule.



Comment

Three factors $\Rightarrow$ a cubic.

Use the y -intercept as the extra point.

Solve for the leading coefficient.

The negative a agrees with the down-to-the-right end.

Working

$$\text{Touch at } x=2 \Rightarrow (x-2)^2; \text{ cross at } x=-1 \Rightarrow (x+1).$$

$$y = a(x-2)^2(x+1).$$

$$\text{Substitute } (0, 4): 4 = a(-2)^2(1) = 4a, \text{ so } a = 1.$$

$$\therefore y = (x-2)^2(x+1).$$

Comment

A touch is a double factor; a cross is a single factor.

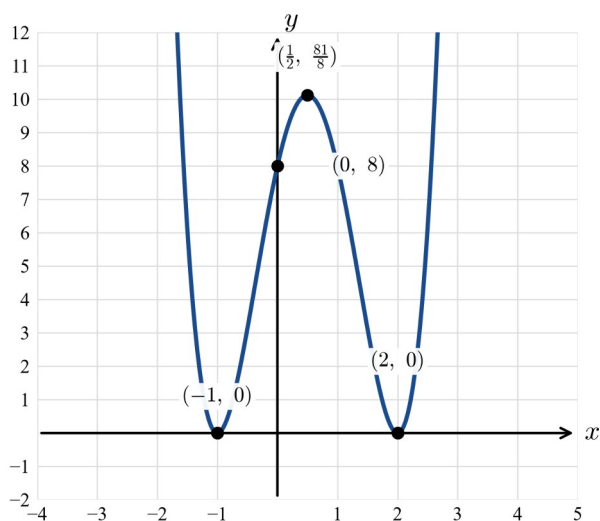
Multiplicities sum to 3 $\Rightarrow$ a cubic.

Find a from the y -intercept.

Monic cubic.

Example 3 — unscaffolded

Determine the rule of the quartic shown, which touches the x -axis at $x=-1$ and $x=2$ and has y -intercept $(0, 8)$.



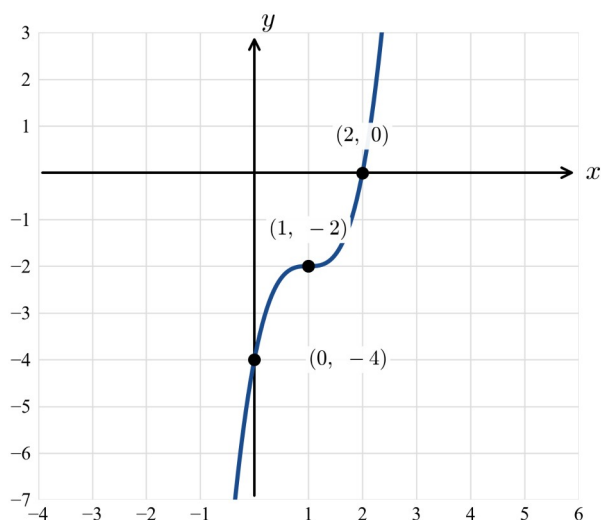
Both intercepts are touches, so each factor is squared: $y = a(x+1)^2(x-2)^2$.

Substitute $(0, 8)$: $8 = a(1)(4) = 4a$, so $a = 2$.

$$\therefore y = 2(x+1)^2(x-2)^2.$$

Example 4 — unscaffolded

The cubic shown has a stationary point of inflection at $(1, -2)$ and passes through $(0, -4)$. Determine its rule.



A stationary point of inflection at $(1, -2)$ gives the form $y = a(x-1)^3 - 2$.

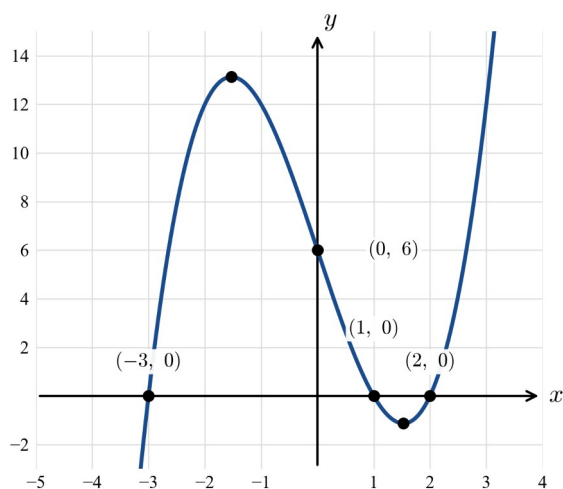
Substitute $(0, -4)$: $-4 = a(-1)^3 - 2 = -a - 2$, so $a = 2$.

$$\therefore y = 2(x-1)^3 - 2.$$

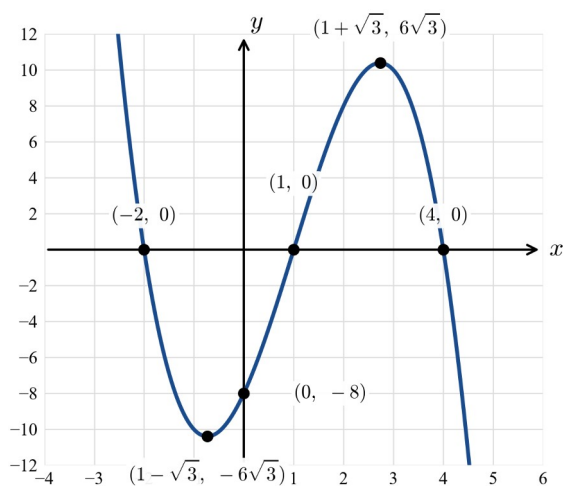
Practice questions

Scaffolded

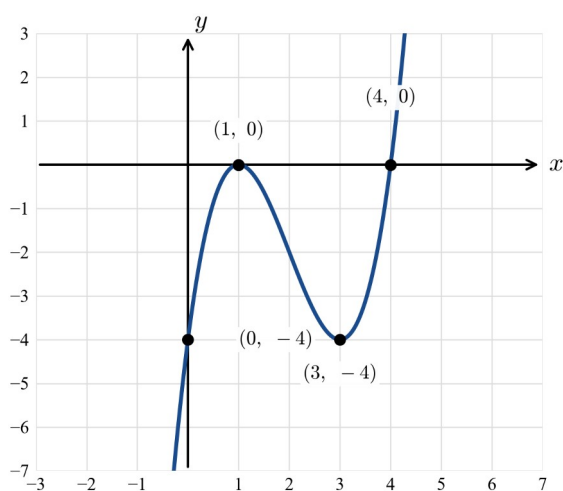
Q1. The cubic shown crosses the x -axis at $x = -3, 1$ and 2 , with y -intercept $(0, 6)$. (a) write a factor for each intercept; (b) write $y = a(\dots)$ and substitute $(0, 6)$; (c) state the rule.



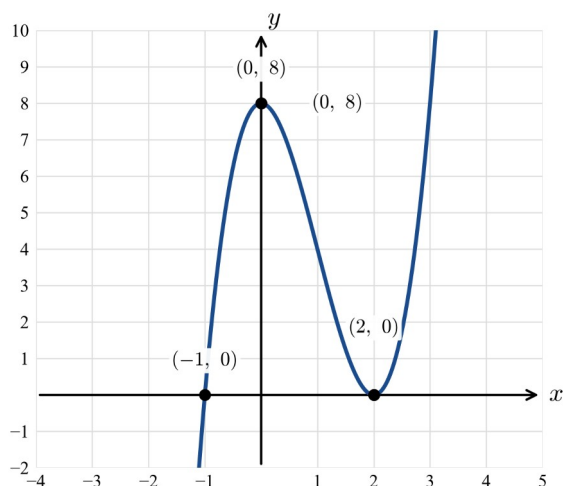
Q2. The cubic shown crosses at $x = -2, 1$ and 4 , with y -intercept $(0, -8)$. (a) form the factors; (b) find a ; (c) state the rule.



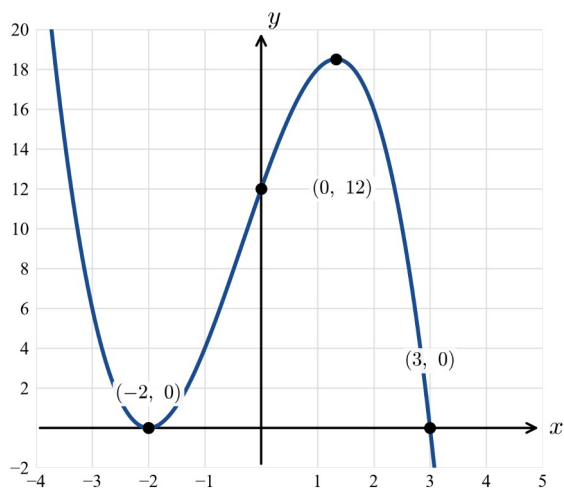
Q3. The cubic shown touches the axis at $x=1$ and crosses at $x=4$, with y -intercept $(0, -4)$. (a) which factor is squared, and why? (b) find a ; (c) state the rule.



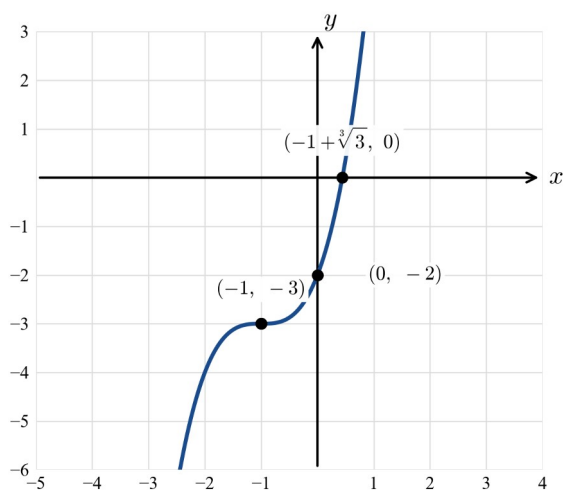
Q4. The cubic shown crosses at $x=-1$ and touches at $x=2$, with y -intercept $(0, 8)$. (a) form the factors; (b) find a ; (c) state the rule.



Q5. The cubic shown touches at $x=-2$ and crosses at $x=3$, with y -intercept $(0, 12)$. (a) form the factors; (b) find a ; (c) state the rule.



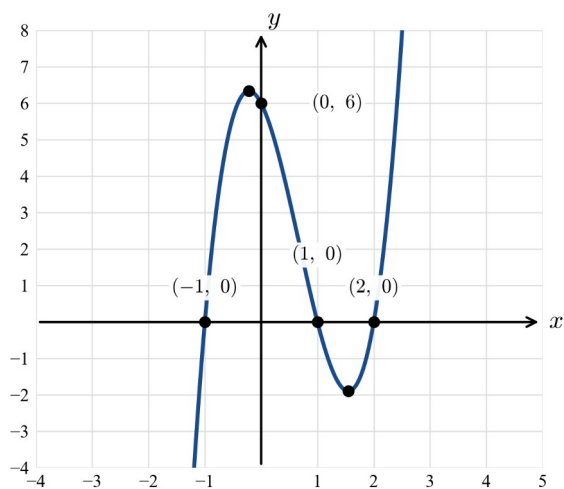
Q6. The cubic shown has a stationary point of inflection at $(-1, -3)$ and passes through $(0, -2)$. (a) write the form $y = a(x - h)^3 + k$; (b) find a ; (c) state the rule.



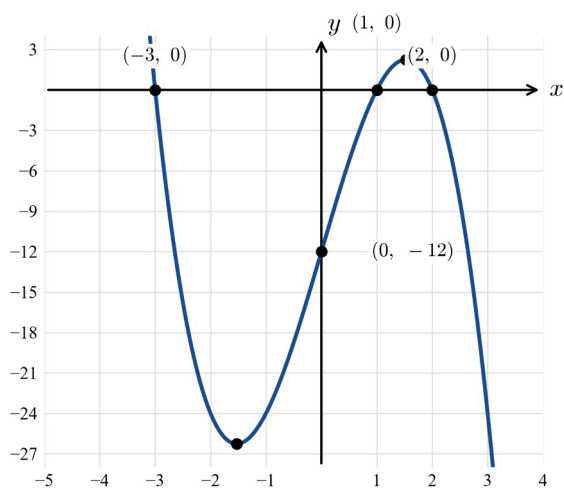
Un scaffolded

Determine the rule of each graph shown.

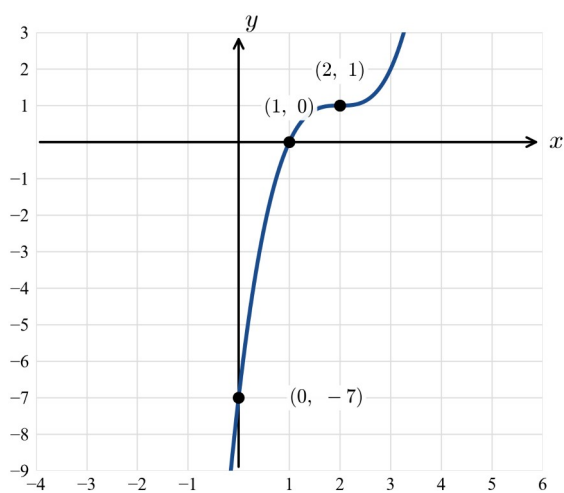
Q7. Crosses at $x = -1, 1, 2$; y -intercept $(0, 6)$.



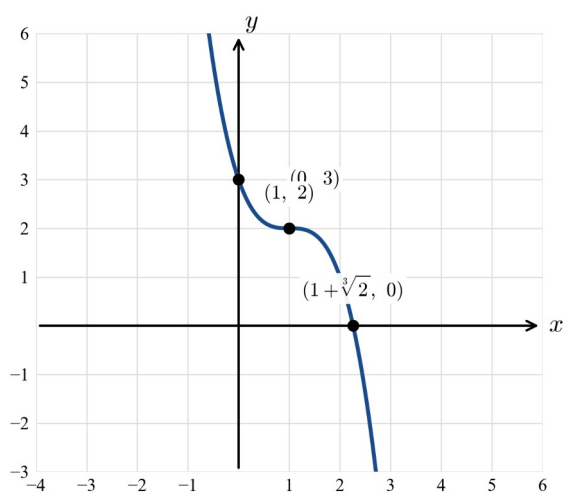
Q8. Crosses at $x = -3, 1, 2$; y -intercept $(0, -12)$.



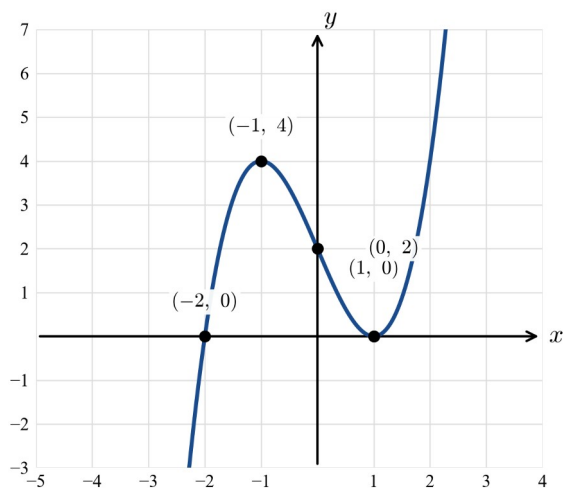
Q9. Stationary point of inflection at $(2, 1)$; y-intercept $(0, -7)$.



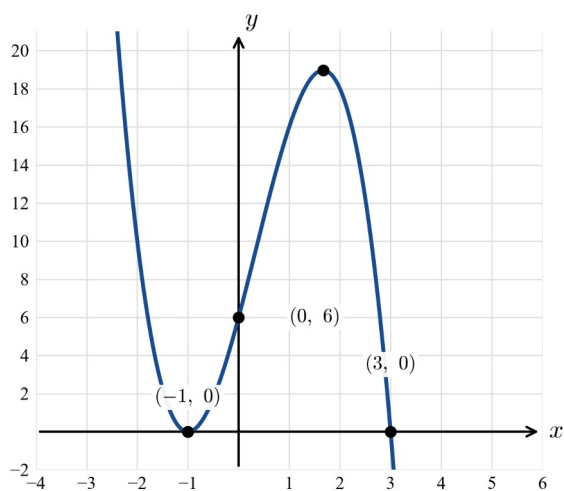
Q10. Stationary point of inflection at $(1, 2)$; y-intercept $(0, 3)$.



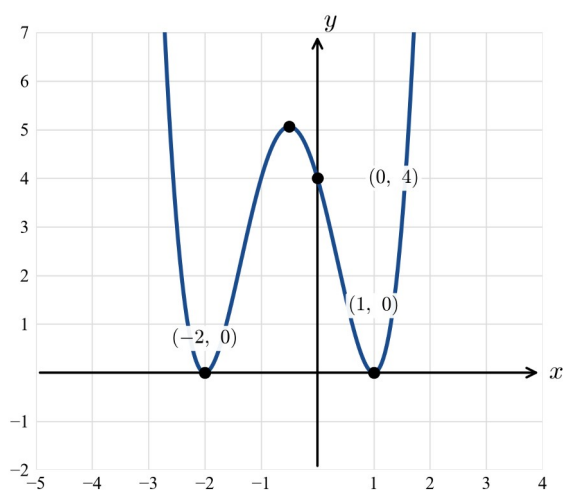
Q11. Crosses at $x = -2$, touches at $x = 1$; y-intercept $(0, 2)$.



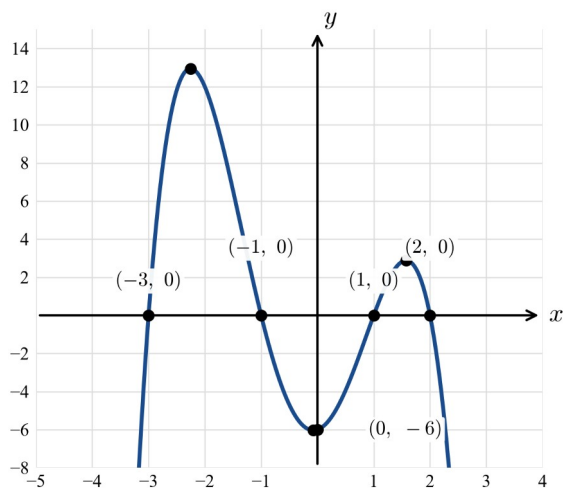
Q12. Touches at $x = -1$, crosses at $x = 3$; y-intercept $(0, 6)$.



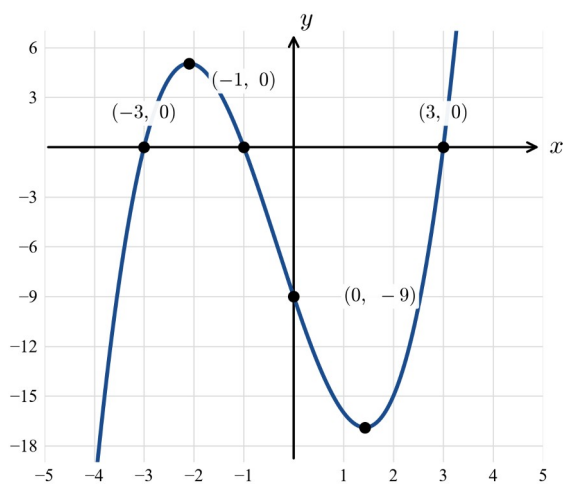
Q13. Quartic touching at $x = -2$ and $x = 1$; y-intercept $(0, 4)$.



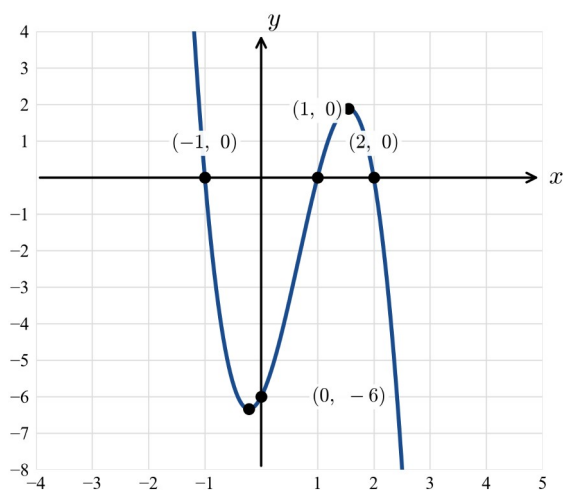
Q14. Quartic crossing at $x = -3, -1, 1, 2$; y-intercept $(0, -6)$.



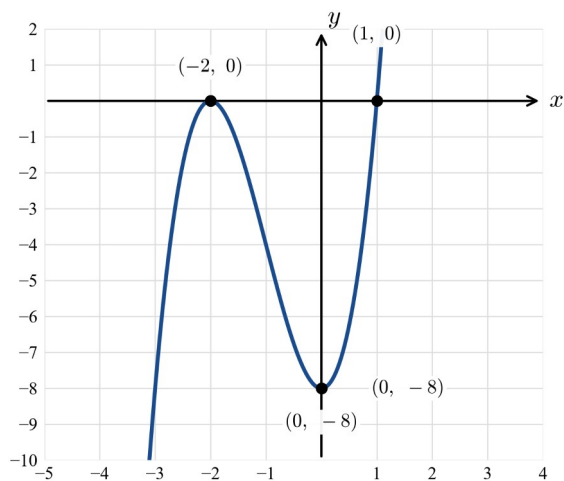
Q15. Crosses at $x = -3, -1, 3$; y-intercept $(0, -9)$.



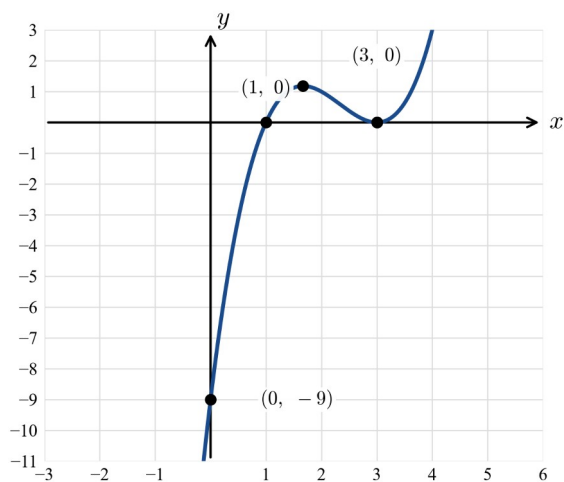
Q16. Crosses at $x = -1, 1, 2$; y-intercept $(0, -6)$.



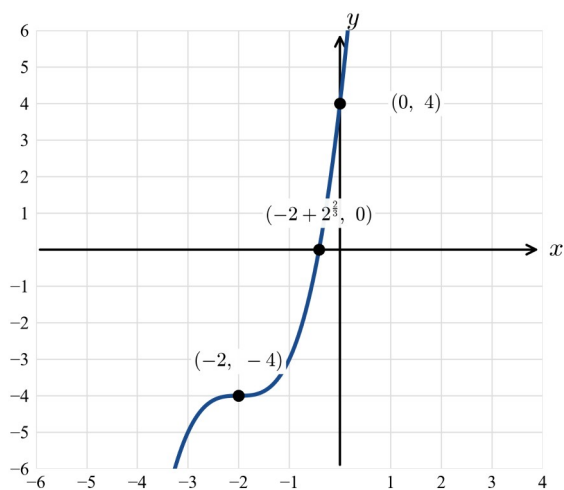
Q17. Touches at $x = -2$, crosses at $x = 1$; y-intercept $(0, -8)$.



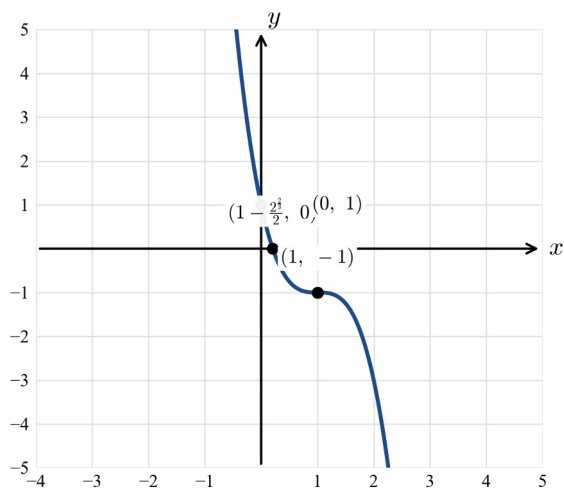
Q18. Crosses at $x=1$, touches at $x=3$; y -intercept $(0, -9)$.



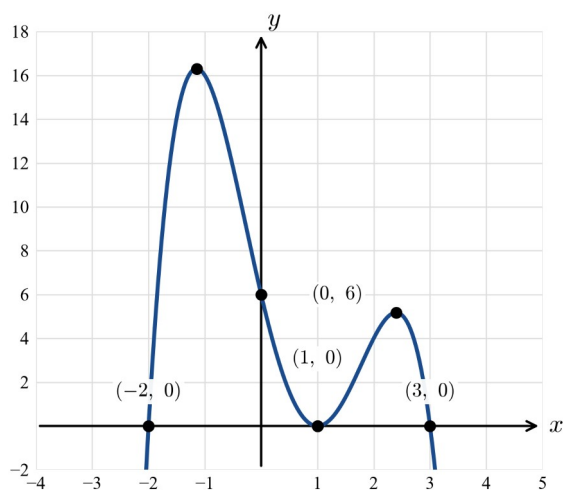
Q19. Stationary point of inflection at $(-2, -4)$; y -intercept $(0, 4)$.



Q20. Stationary point of inflection at $(1, -1)$; y -intercept $(0, 1)$.



Q21. Quartic crossing at $x = -2$ and $x = 3$ and touching at $x = 1$; y -intercept $(0, 6)$.



Answers

Q1. $y = (x+3)(x-1)(x-2)$ **Q2.** $y = -(x+2)(x-1)(x-4)$ **Q3.** $y = (x-1)^2(x-4)$ **Q4.** $y = 2(x+1)(x-2)^2$
Q5. $y = -(x+2)^2(x-3)$ **Q6.** $y = (x+1)^3 - 3$ **Q7.** $y = 3(x-1)(x+1)(x-2)$ **Q8.** $y = -2(x+3)(x-1)(x-2)$
Q9. $y = (x-2)^3 + 1$ **Q10.** $y = -(x-1)^3 + 2$ **Q11.** $y = (x+2)(x-1)^2$ **Q12.** $y = -2(x+1)^2(x-3)$ **Q13.**
 $y = (x+2)^2(x-1)^2$ **Q14.** $y = -(x+3)(x+1)(x-1)(x-2)$ **Q15.** $y = (x-3)(x+1)(x+3)$ **Q16.**
 $y = -3(x-1)(x-2)(x+1)$ **Q17.** $y = 2(x+2)^2(x-1)$ **Q18.** $y = (x-1)(x-3)^2$ **Q19.** $y = (x+2)^3 - 4$ **Q20.**
 $y = -2(x-1)^3 - 1$ **Q21.** $y = -(x+2)(x-1)^2(x-3)$

Fully-worked solutions

Q1. Simple crossings $\Rightarrow y = a(x+3)(x-1)(x-2)$. At $(0, 6)$: $6 = a(3)(-1)(-2) = 6a$, so $a = 1$.
 $\therefore y = (x+3)(x-1)(x-2)$.

Q2. $y = a(x+2)(x-1)(x-4)$. At $(0, -8)$: $-8 = a(2)(-1)(-4) = 8a$, so $a = -1$.
 $\therefore y = -(x+2)(x-1)(x-4)$.

Q3. Touch at $1 \Rightarrow (x-1)^2$; cross at $4 \Rightarrow (x-4)$. $y = a(x-1)^2(x-4)$. At $(0, -4)$: $-4 = a(1)(-4) = -4a$, so $a = 1$. $\therefore y = (x-1)^2(x-4)$.

Q4. $y = a(x+1)(x-2)^2$. At $(0, 8)$: $8 = a(1)(4) = 4a$, so $a = 2$. $\therefore y = 2(x+1)(x-2)^2$.

Q5. Touch at $-2 \Rightarrow (x+2)^2$; cross at $3 \Rightarrow (x-3)$. $y = a(x+2)^2(x-3)$. At $(0, 12)$: $12 = a(4)(-3) = -12a$, so $a = -1$. $\therefore y = -(x+2)^2(x-3)$.

Q6. SPI at $(-1, -3) \Rightarrow y = a(x+1)^3 - 3$. At $(0, -2)$: $-2 = a(1) - 3$, so $a = 1$. $\therefore y = (x+1)^3 - 3$.

Q7. $y = a(x+1)(x-1)(x-2)$. At $(0, 6)$: $6 = a(1)(-1)(-2) = 2a$, so $a = 3$. $\therefore y = 3(x-1)(x+1)(x-2)$.

Q8. $y = a(x+3)(x-1)(x-2)$. At $(0, -12)$: $-12 = a(3)(-1)(-2) = 6a$, so $a = -2$.
 $\therefore y = -2(x+3)(x-1)(x-2)$.

Q9. SPI at $(2, 1) \Rightarrow y = a(x-2)^3 + 1$. At $(0, -7)$: $-7 = a(-8) + 1$, so $-8a = -8$, $a = 1$. $\therefore y = (x-2)^3 + 1$.

Q10. SPI at $(1, 2) \Rightarrow y = a(x-1)^3 + 2$. At $(0, 3)$: $3 = a(-1) + 2$, so $a = -1$. $\therefore y = -(x-1)^3 + 2$.

Q11. Cross at -2 , touch at 1 : $y = a(x+2)(x-1)^2$. At $(0, 2)$: $2 = a(2)(1) = 2a$, so $a = 1$. $\therefore y = (x+2)(x-1)^2$.

Q12. Touch at -1 , cross at 3 : $y = a(x+1)^2(x-3)$. At $(0, 6)$: $6 = a(1)(-3) = -3a$, so $a = -2$.
 $\therefore y = -2(x+1)^2(x-3)$.

Q13. Two touches $\Rightarrow y = a(x+2)^2(x-1)^2$. At $(0, 4)$: $4 = a(4)(1) = 4a$, so $a = 1$. $\therefore y = (x+2)^2(x-1)^2$.

Q14. Four crossings $\Rightarrow y = a(x+3)(x+1)(x-1)(x-2)$. At $(0, -6)$: $-6 = a(3)(1)(-1)(-2) = 6a$, so $a = -1$.
 $\therefore y = -(x+3)(x+1)(x-1)(x-2)$.

Q15. $y = a(x+3)(x+1)(x-3)$. At $(0, -9)$: $-9 = a(3)(1)(-3) = -9a$, so $a = 1$. $\therefore y = (x-3)(x+1)(x+3)$.

Q16. $y = a(x+1)(x-1)(x-2)$. At $(0, -6)$: $-6 = a(1)(-1)(-2) = 2a$, so $a = -3$.
 $\therefore y = -3(x-1)(x-2)(x+1)$.

Q17. Touch at -2 , cross at 1 : $y = a(x+2)^2(x-1)$. At $(0, -8)$: $-8 = a(4)(-1) = -4a$, so $a = 2$.
 $\therefore y = 2(x+2)^2(x-1)$.

Q18. Cross at 1, touch at 3: $y = a(x-1)(x-3)^2$. At $(0, -9)$: $-9 = a(-1)(9) = -9a$, so $a = 1$.
 $\therefore y = (x-1)(x-3)^2$.

Q19. SPI at $(-2, -4) \Rightarrow y = a(x+2)^3 - 4$. At $(0, 4)$: $4 = a(8) - 4$, so $8a = 8$, $a = 1$. $\therefore y = (x+2)^3 - 4$.

Q20. SPI at $(1, -1) \Rightarrow y = a(x-1)^3 - 1$. At $(0, 1)$: $1 = a(-1) - 1$, so $-a = 2$, $a = -2$. $\therefore y = -2(x-1)^3 - 1$.

Q21. Crosses at -2 and 3 , touches at 1 : $y = a(x+2)(x-1)^2(x-3)$. At $(0, 6)$: $6 = a(2)(1)(-3) = -6a$, so $a = -1$. $\therefore y = -(x+2)(x-1)^2(x-3)$.

Chapter 4 Polynomial functions — 4H Literal equations and systems of equations

Theory

This section brings together two ideas: solving equations whose coefficients contain **parameters** (pronumerals other than x), and deciding **how many times** a line meets a curve.

Literal equations. A *literal* quadratic such as $x^2 - 2px + p^2 - 9 = 0$ is solved for x exactly as a numerical quadratic — by factorising, completing the square, or the quadratic formula — but the answer is expressed **in terms of the parameters**. Here $x^2 - 2px + p^2 - 9 = (x - p)^2 - 9 = 0$, so $(x - p)^2 = 9$ and $x = p \pm 3$.

The discriminant as a condition on a parameter. For $ax^2 + bx + c = 0$ (with $a \neq 0$) the discriminant $\Delta = b^2 - 4ac$ decides the number of real solutions:

- $\Delta > 0$: **two** distinct solutions;
- $\Delta = 0$: **one** (repeated) solution;
- $\Delta < 0$: **no** real solutions.

When a coefficient contains a parameter k , Δ becomes an expression in k ; solving the inequality $\Delta > 0$ (or $\Delta = 0$, $\Delta < 0$) gives the values of k for each case. **Watch the leading coefficient:** if it contains k , the case that makes it zero reduces the equation to a linear one (a separate, single solution).

A line meeting a curve. To find where the line $y = mx + c$ meets the parabola $y = ax^2 + bx + d$, solve the two equations simultaneously by **substitution**: $ax^2 + bx + d = mx + c$, which rearranges to a quadratic $ax^2 + (b - m)x + (d - c) = 0$. The number of real solutions of this quadratic is the number of intersection points, so its discriminant tells us:

- $\Delta > 0$: the line **cuts** the curve at two points;
- $\Delta = 0$: the line is a **tangent** (touches at one point);
- $\Delta < 0$: the line **does not meet** the curve.

To find a tangency condition: form the quadratic, set its discriminant to 0, and solve for the parameter.

Worked examples

Example 1 — scaffolded

Solve $x^2 - 2px + p^2 - 9 = 0$ for x , giving your answer in terms of p .

Working	Comment
$x^2 - 2px + p^2 - 9 = 0$	The first three terms are a perfect square.
$(x - p)^2 - 9 = 0$	$x^2 - 2px + p^2 = (x - p)^2$.
$(x - p)^2 = 9$	Isolate the square.
$x - p = \pm 3$	Take the square root — keep the $\pm$.
$\therefore x = p + 3$ or $x = p - 3$	Solutions in terms of the parameter p .

Example 2 — scaffolded

For what values of k does $kx^2 + 4x + 2 = 0$ have (i) two solutions, (ii) one solution, (iii) no real solutions?

Working	Comment
$a = k, b = 4, c = 2$, so $\Delta = 4^2 - 4(k)(2) = 16 - 8k$.	Provided $k \neq 0$ the equation is quadratic.
(i) Two solutions: $\Delta > 0 \Rightarrow 16 - 8k > 0 \Rightarrow k < 2$ (and	$\Delta > 0$.

$k \neq 0$).

(ii) One solution: $\Delta = 0 \Rightarrow k = 2$. (Also $k = 0$ gives the linear $4x + 2 = 0$, one solution.) $\Delta = 0$, plus the degenerate case.

(iii) No real solutions: $\Delta < 0 \Rightarrow 16 - 8k < 0 \Rightarrow k > 2$. $\Delta < 0$.

Example 3 — unscaffolded

Find the value(s) of m for which the line $y = mx + 1$ is a tangent to the parabola $y = x^2 + 3x + 4$.

Substituting the line into the parabola: $x^2 + 3x + 4 = mx + 1$, so

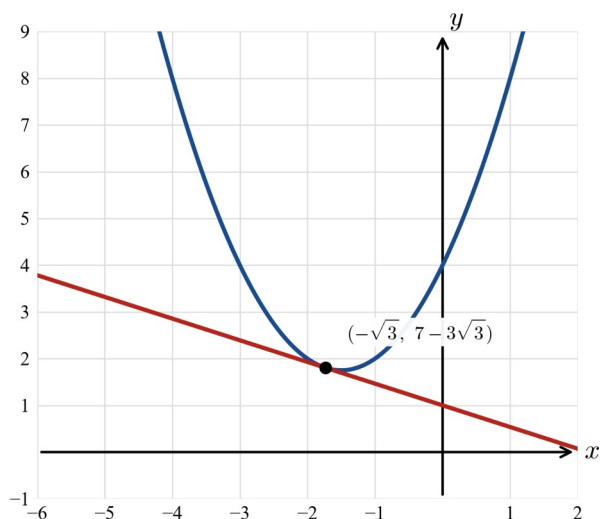
$$x^2 + (3 - m)x + 3 = 0.$$

For a tangent this quadratic has one solution, i.e. $\Delta = 0$:

$$\Delta = (3 - m)^2 - 4(1)(3) = m^2 - 6m - 3 = 0.$$

By the quadratic formula, $m = \frac{6 \pm \sqrt{36 + 12}}{2} = \frac{6 \pm 4\sqrt{3}}{2} = 3 \pm 2\sqrt{3}$.

$\therefore m = 3 - 2\sqrt{3}$ or $m = 3 + 2\sqrt{3}$.



Example 4 — unscaffolded

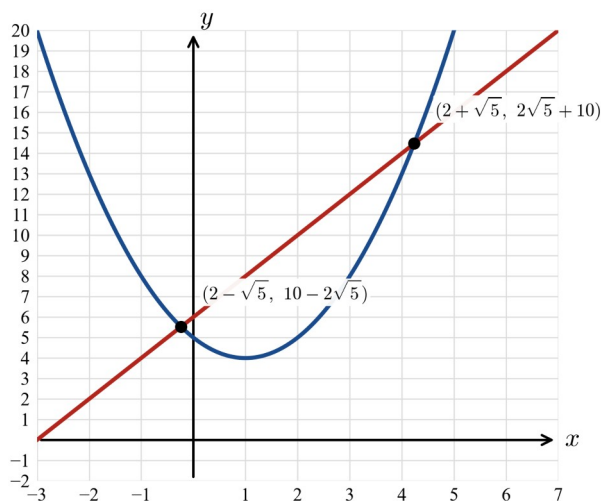
For what values of k does the line $y = 2x + k$ meet the parabola $y = x^2 - 2x + 5$ at two distinct points?

Substituting: $x^2 - 2x + 5 = 2x + k$, so $x^2 - 4x + (5 - k) = 0$.

$$\Delta = (-4)^2 - 4(1)(5 - k) = 16 - 20 + 4k = 4k - 4.$$

Two distinct points require $\Delta > 0$: $4k - 4 > 0 \Rightarrow k > 1$.

$\therefore$ the line meets the parabola twice when $k > 1$ (it is a tangent when $k = 1$, and misses when $k < 1$).



Practice questions

Scaffolded

Q1. Solve $x^2 - 2ax + a^2 - 16 = 0$ for x : (a) write $x^2 - 2ax + a^2$ as a perfect square; (b) hence solve for x in terms of a .

Q2. Solve $x^2 + (a - b)x - ab = 0$ for x : (a) factorise the left-hand side; (b) hence state the two solutions.

Q3. For $x^2 + 6x + k = 0$: (a) write down the discriminant in terms of k ; (b) find the value of k for which the equation has exactly one solution; (c) state the values of k for which it has two solutions.

Q4. For $kx^2 + 2x + 1 = 0$ (with $k \neq 0$): (a) find the discriminant in terms of k ; (b) find the value of k for which the line of solutions is a repeated root; (c) state the values of k for which there are no real solutions.

Q5. The line $y = mx + 2$ is a tangent to $y = x^2 - x + 3$. (a) show that $x^2 - (1 + m)x + 1 = 0$; (b) hence find the two possible values of m .

Q6. The line $y = x + k$ meets $y = x^2 + 2x - 1$ at two distinct points. (a) show that $x^2 + x - (1 + k) = 0$; (b) hence find the values of k .

Un scaffolded

Q7–Q11. Solve each equation for x , giving your answer in terms of the parameters.

Q7. $x^2 + 2ax + a^2 - b^2 = 0$

Q8. $x^2 - (a + b)x + ab = 0$

Q9. $x^2 - 5kx + 6k^2 = 0$

Q10. $px^2 - (p^2 + 1)x + p = 0$ (with $p \neq 0$)

Q11. $x^2 - 2ax + a^2 - 4b^2 = 0$

Q12–Q16. Find the value(s) of the parameter for the stated number of real solutions.

Q12. For $x^2 + kx + 9 = 0$, find the values of k for which there are (a) two, (b) one, (c) no real solutions.

Q13. For $x^2 - 4x + k = 0$, find the values of k for each of the three cases.

Q14. For $2x^2 + 3x + k = 0$, find the values of k for each of the three cases.

Q15. Show that $x^2 + (k+1)x + k = 0$ has real solutions for all values of k , and state when the solution is repeated.

Q16. For $kx^2 - 4x + k = 0$ (with $k \neq 0$), find the values of k for which there are (a) two, (b) one, (c) no real solutions.

Q17–Q21. Line-and-curve problems.

Q17. Find the value of c for which $y = x + c$ is a tangent to $y = x^2 - 3x + 4$.

Q18. Find the value(s) of k for which the line $y = kx$ is a tangent to $y = x^2 + 1$.

Q19. Find the value of k for which $y = 3x + k$ is a tangent to $y = x^2 + x + 3$, and the values of k for which the line cuts the parabola twice.

Q20. Find the value(s) of m for which $y = mx - 4$ is a tangent to $y = x^2$.

Q21. Find the value(s) of k for which the line $y = kx + 1$ is a tangent to $y = x^2 - x + 2$, and hence the values of k for which it does not meet the curve.

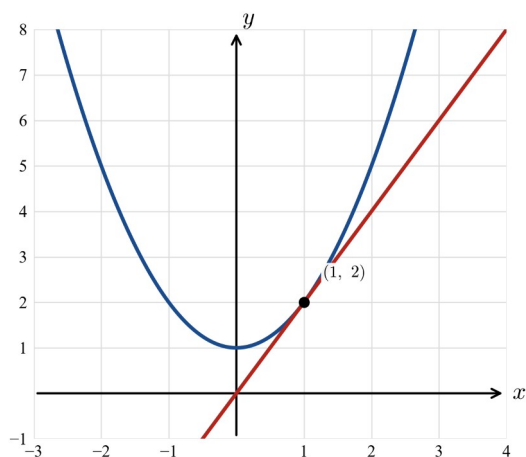
Answers

Q1. (a) $(x-a)^2$; (b) $x=a\pm 4$. **Q2.** (a) $(x+a)(x-b)$; (b) $x=-a$ or $x=b$. **Q3.** (a) $\Delta=36-4k$; (b) $k=9$; (c) $k<9$.
Q4. (a) $\Delta=4-4k$; (b) $k=1$; (c) $k>1$. **Q5.** (b) $m=1$ or $m=-3$. **Q6.** (b) $k>-\frac{5}{4}$. **Q7.** $x=-a\pm b$. **Q8.** $x=a$ or $x=b$.
Q9. $x=2k$ or $x=3k$. **Q10.** $x=p$ or $x=\frac{1}{p}$. **Q11.** $x=a\pm 2b$. **Q12.** $\Delta=k^2-36$: (a) $k<-6$ or $k>6$; (b) $k=\pm 6$; (c) $-6<k<6$. **Q13.** $\Delta=16-4k$: (a) $k<4$; (b) $k=4$; (c) $k>4$. **Q14.** $\Delta=9-8k$: (a) $k<\frac{9}{8}$; (b) $k=\frac{9}{8}$; (c) $k>\frac{9}{8}$. **Q15.** $\Delta=(k-1)^2\geq 0$ for all k ; repeated when $k=1$. **Q16.** $\Delta=16-4k^2$: (a) $-2<k<2$ (and $k\neq 0$); (b) $k=\pm 2$; (c) $k<-2$ or $k>2$. **Q17.** $c=0$. **Q18.** $k=\pm 2$. **Q19.** tangent $k=2$; cuts twice for $k>2$. **Q20.** $m=\pm 4$. **Q21.** tangent $k=1$ or $k=-3$; does not meet for $-3<k<1$.

Fully-worked solutions

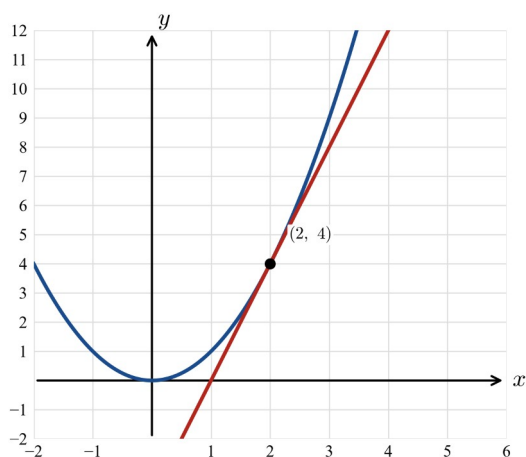
Q1. (a) $x^2-2ax+a^2=(x-a)^2$. (b) $(x-a)^2-16=0\Rightarrow(x-a)^2=16\Rightarrow x-a=\pm 4$, so $x=a+4$ or $x=a-4$.
Q2. (a) $x^2+(a-b)x-ab=(x+a)(x-b)$ (check: $-bx+ax-ab=(a-b)x-ab$ ✓). (b) $x=-a$ or $x=b$.
Q3. (a) $\Delta=6^2-4(1)(k)=36-4k$. (b) One solution: $36-4k=0\Rightarrow k=9$. (c) Two solutions: $36-4k>0\Rightarrow k<9$.
Q4. (a) $\Delta=2^2-4(k)(1)=4-4k$. (b) Repeated root: $4-4k=0\Rightarrow k=1$. (c) No real solutions: $4-4k<0\Rightarrow k>1$.
Q5. (a) $x^2-x+3=mx+2\Rightarrow x^2-x-mx+1=0\Rightarrow x^2-(1+m)x+1=0$. (b) Tangent $\Rightarrow \Delta=0$:
 $(1+m)^2-4=0\Rightarrow(1+m)^2=4\Rightarrow 1+m=\pm 2$, so $m=1$ or $m=-3$.
Q6. (a) $x^2+2x-1=x+k\Rightarrow x^2+x-1-k=0$, i.e. $x^2+x-(1+k)=0$. (b) Two points $\Rightarrow \Delta>0$:
 $1+4(1+k)>0\Rightarrow 4k+5>0\Rightarrow k>-\frac{5}{4}$.
Q7. $x^2+2ax+a^2-b^2=(x+a)^2-b^2=0\Rightarrow(x+a)^2=b^2\Rightarrow x+a=\pm b$, so $x=-a\pm b$.
Q8. $x^2-(a+b)x+ab=(x-a)(x-b)=0$, so $x=a$ or $x=b$.
Q9. $x^2-5kx+6k^2=(x-2k)(x-3k)=0$, so $x=2k$ or $x=3k$.
Q10. $px^2-(p^2+1)x+p=(px-1)(x-p)=0$ (check: px^2-p^2x-x+p ✓), so $x=\frac{1}{p}$ or $x=p$.
Q11. $x^2-2ax+a^2-4b^2=(x-a)^2-(2b)^2=0\Rightarrow(x-a)^2=4b^2\Rightarrow x-a=\pm 2b$, so $x=a\pm 2b$.
Q12. $\Delta=k^2-4(1)(9)=k^2-36$. (a) $\Delta>0\Rightarrow k^2>36\Rightarrow k<-6$ or $k>6$. (b) $\Delta=0\Rightarrow k=\pm 6$. (c) $\Delta<0\Rightarrow -6<k<6$.
Q13. $\Delta=(-4)^2-4k=16-4k$. (a) $16-4k>0\Rightarrow k<4$. (b) $k=4$. (c) $k>4$.
Q14. $\Delta=3^2-4(2)(k)=9-8k$. (a) $9-8k>0\Rightarrow k<\frac{9}{8}$. (b) $k=\frac{9}{8}$. (c) $k>\frac{9}{8}$.
Q15. $\Delta=(k+1)^2-4(1)(k)=k^2+2k+1-4k=k^2-2k+1=(k-1)^2$. Since $(k-1)^2\geq 0$ for all real k , there are always real solutions; the solution is repeated exactly when $(k-1)^2=0$, i.e. $k=1$.
Q16. $\Delta=(-4)^2-4(k)(k)=16-4k^2$. (a) $16-4k^2>0\Rightarrow k^2<4\Rightarrow -2<k<2$ (with $k\neq 0$ so it stays quadratic). (b) $\Delta=0\Rightarrow k=\pm 2$. (c) $16-4k^2<0\Rightarrow k<-2$ or $k>2$.
Q17. $x^2-3x+4=x+c\Rightarrow x^2-4x+(4-c)=0$. Tangent $\Rightarrow \Delta=0$: $16-4(4-c)=4c=0\Rightarrow c=0$.

Q18. $x^2 + 1 = kx \Rightarrow x^2 - kx + 1 = 0$. Tangent $\Rightarrow \Delta = 0: k^2 - 4 = 0 \Rightarrow k = \pm 2$.



Q19. $x^2 + x + 3 = 3x + k \Rightarrow x^2 - 2x + (3 - k) = 0$. $\Delta = 4 - 4(3 - k) = 4k - 8$. Tangent: $4k - 8 = 0 \Rightarrow k = 2$. Cuts twice: $4k - 8 > 0 \Rightarrow k > 2$.

Q20. $x^2 = mx - 4 \Rightarrow x^2 - mx + 4 = 0$. Tangent $\Rightarrow \Delta = 0: m^2 - 16 = 0 \Rightarrow m = \pm 4$.



Q21. $x^2 - x + 2 = kx + 1 \Rightarrow x^2 - (1 + k)x + 1 = 0$. $\Delta = (1 + k)^2 - 4 = k^2 + 2k - 3 = (k + 3)(k - 1)$. Tangent: $\Delta = 0 \Rightarrow k = 1$ or $k = -3$. Does not meet: $\Delta < 0 \Rightarrow (k + 3)(k - 1) < 0 \Rightarrow -3 < k < 1$.