

Mathematical Methods Units 3 & 4

Chapter 3 — Transformations of the plane

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Chapter 3 Transformations of the plane — 3A Translations

Theory

A **translation** slides a graph without changing its shape or orientation. For a function $y = f(x)$:

$$y = f(x - h) + k$$

is the graph of $y = f(x)$ translated h **units horizontally** and k **units vertically** — to the right when $h > 0$ (left when $h < 0$), and up when $k > 0$ (down when $k < 0$).

Image of a point. Every point (a, b) on $y = f(x)$ moves to $(a + h, b + k)$ on the translated graph. Conversely, a point (x, y) on the image came from $(x - h, y - k)$ on the original.

The standard graphs under a translation $y = f(x - h) + k$:

Base $f(x)$	Translated rule	Key feature moves to
x^2 (parabola)	$y = (x - h)^2 + k$	turning point (h, k)
x^3 (cubic)	$y = (x - h)^3 + k$	stationary point of inflection (h, k)
$\frac{1}{x}$ (hyperbola)	$y = \frac{1}{x - h} + k$	asymptotes $x = h, y = k$
$\frac{1}{x^2}$	$y = \frac{1}{(x - h)^2} + k$	asymptotes $x = h, y = k$
$\sqrt{x}$ (square root)	$y = \sqrt{x - h} + k$	endpoint (h, k) , domain $x \geq h$

Finding the intercepts (exact values). Substitute $x = 0$ for the y -intercept (if $x = 0$ is in the domain) and solve $y = 0$ for the x -intercepts.

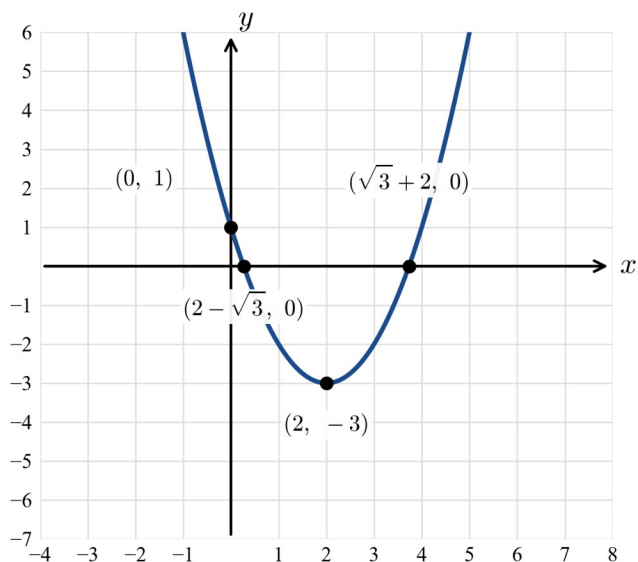
To sketch a translated standard graph: (1) identify the base graph; (2) locate its moved key feature (turning point / point of inflection / asymptotes / endpoint); (3) find the exact axis intercepts; (4) sketch the same-shaped curve through those features.

Worked examples

Example 1 — scaffolded

Sketch $y = (x - 2)^2 - 3$, labelling the turning point and the exact axis intercepts.

Working	Comment
Base graph $y = x^2$, with $h = 2, k = -3$.	Compare with $y = (x - h)^2 + k$: a translation 2 right and 3 down.
Turning point $(2, -3)$ (a minimum).	The vertex of $y = x^2$ at $(0, 0)$ moves to (h, k) .
y-intercept: $x = 0 \Rightarrow y = (0 - 2)^2 - 3 = 4 - 3 = 1$, giving $(0, 1)$.	Substitute $x = 0$.
x-intercepts: $(x - 2)^2 - 3 = 0 \Rightarrow (x - 2)^2 = 3 \Rightarrow x - 2 = \pm\sqrt{3}$, so $x = 2 \pm \sqrt{3}$, giving $(2 - \sqrt{3}, 0)$ and $(2 + \sqrt{3}, 0)$.	Keep the surds exact.



Example 2 — scaffolded

Sketch $y = \frac{1}{x+1} + 2$, stating the asymptotes and the exact axis intercepts.

Working

Base graph $y = \frac{1}{x}$, with $h = -1$, $k = 2$.

Asymptotes: $x = -1$ and $y = 2$.

y-intercept: $x = 0 \Rightarrow y = \frac{1}{1} + 2 = 3$, giving $(0, 3)$.

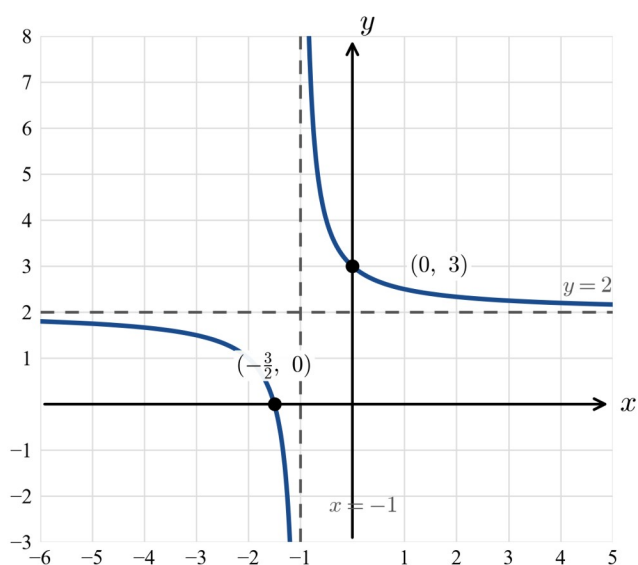
x-intercept: $\frac{1}{x+1} = -2 \Rightarrow x+1 = -\frac{1}{2}$, so $x = -\frac{3}{2}$, giving $(-\frac{3}{2}, 0)$.

Comment

Compare with $y = \frac{1}{x-h} + k$; here $x-h = x+1$: a translation 1 left and 2 up.

The asymptotes $x=0$, $y=0$ move to $x=h$, $y=k$.
Substitute $x=0$.

Isolate the fraction, then solve.



Example 3 — unscaffolded

Sketch $y = \sqrt{x+3} - 1$, stating the domain and the exact axis intercepts.

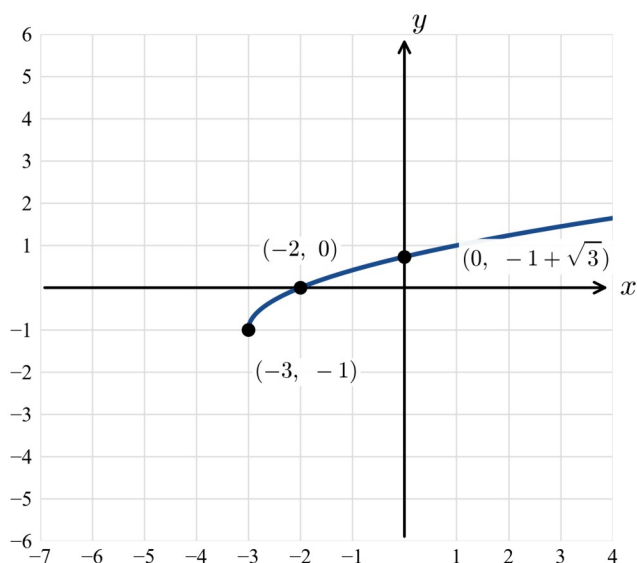
Base graph $y = \sqrt{x}$, with $h = -3$, $k = -1$: a translation 3 left and 1 down.

Endpoint $(-3, -1)$; domain $\{x : x \geq -3\}$.

y-intercept: $x=0 \Rightarrow y=\sqrt{3}-1$, giving $(0, \sqrt{3}-1)$.

x-intercept: $\sqrt{x+3}=1 \Rightarrow x+3=1$, so $x=-2$, giving $(-2, 0)$.

$\therefore$ the graph runs from the endpoint $(-3, -1)$ up through $(-2, 0)$ and $(0, \sqrt{3}-1)$.



Example 4 — unscaffolded

Sketch $y = (x-1)^3 + 2$, labelling the stationary point of inflection and the exact axis intercepts.

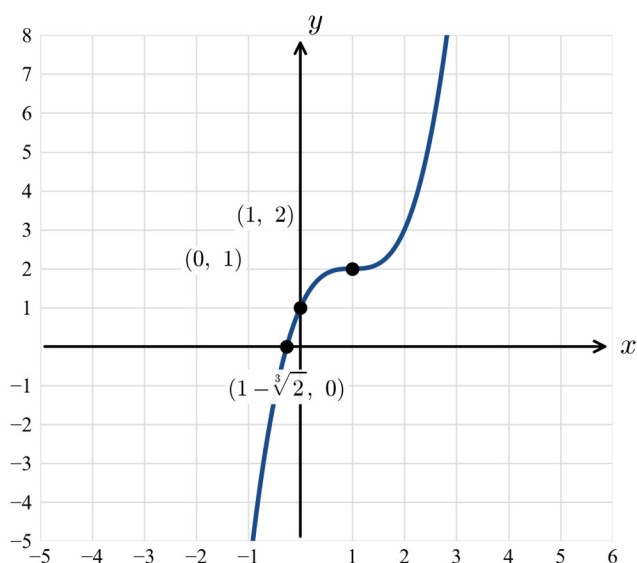
Base graph $y = x^3$, with $h=1$, $k=2$: a translation 1 right and 2 up.

Stationary point of inflection $(1, 2)$.

y-intercept: $x=0 \Rightarrow y = (-1)^3 + 2 = 1$, giving $(0, 1)$.

x-intercept: $(x-1)^3 = -2 \Rightarrow x-1 = -\sqrt[3]{2}$, so $x = 1 - \sqrt[3]{2}$, giving $(1 - \sqrt[3]{2}, 0)$.

$\therefore$ the cubic increases through $(1 - \sqrt[3]{2}, 0)$ and $(0, 1)$, flattening at $(1, 2)$.



Practice questions

Scaffolded

Q1. For $y = (x + 1)^2 - 4$: (a) state the translation from $y = x^2$ and the turning point; (b) find the y -intercept; (c) find the exact x -intercepts; (d) sketch the graph.

Q2. For $y = \frac{1}{x-2} - 1$: (a) state the asymptotes; (b) find the exact axis intercepts; (c) describe the translation from $y = \frac{1}{x}$; (d) sketch the graph.

Q3. For $y = \frac{1}{(x+2)^2} + 1$: (a) state the asymptotes; (b) explain why the graph has no x -intercept; (c) find the y -intercept; (d) sketch the graph.

Q4. For $y = \sqrt{x-2} + 1$: (a) state the endpoint and the domain; (b) explain why the graph has no axis intercepts; (c) sketch the graph.

Q5. For $y = (x+2)^3 - 1$: (a) state the stationary point of inflection; (b) find the axis intercepts; (c) sketch the graph.

Q6. For $y = (x-3)^2 + 2$: (a) state the turning point; (b) explain why the graph has no x -intercept; (c) find the y -intercept and sketch the graph.

Un scaffolded

For **Q7–Q16** and **Q20–Q21**, sketch the graph, labelling the turning point / stationary point of inflection / asymptotes / endpoint (as appropriate) and all axis intercepts with exact coordinates.

Q7. $y = (x - 1)^2 - 9$

Q8. $y = \frac{1}{x+3} - 2$

Q9. $y = \frac{1}{(x-2)^2} - 1$

Q10. $y = \sqrt{x+1} - 3$

Q11. $y = (x - 1)^3 - 1$

Q12. $y = (x+2)^2 - 1$

Q13. $y = \frac{1}{x-1} + 3$

Q14. $y = \frac{1}{(x+1)^2} + 2$

Q15. $y = \sqrt{x+2} - 1$

Q16. $y = (x+1)^3 - 8$

Q17. The point $(3, -2)$ lies on the graph of $y = f(x)$. State the coordinates of its image on the graph of $y = f(x+4) + 5$.

Q18. The graph of $y = \frac{1}{x^2}$ is translated so that its asymptotes become $x = 3$ and $y = -2$. Write down the rule of the translated graph and state the translation.

Q19. A parabola has the same shape as $y = x^2$ and its turning point is $(-2, 5)$. Write down its rule, and state whether it has any x -intercepts, giving a reason.

Q20. $y = (x + 1)^3 + 3$

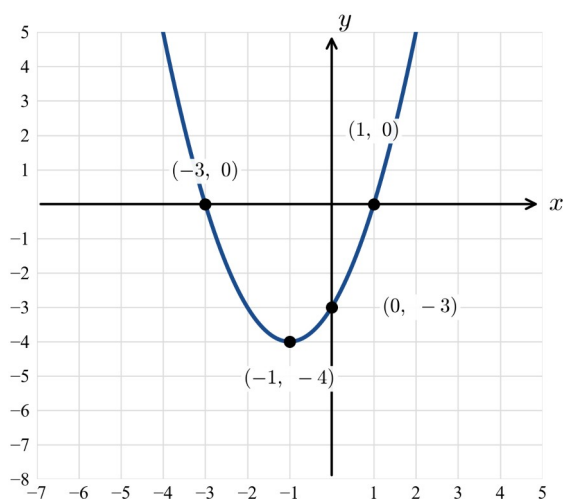
Q21. $y = \sqrt{x + 4} + 1$ — sketch the graph, and state its domain and range.

Answers

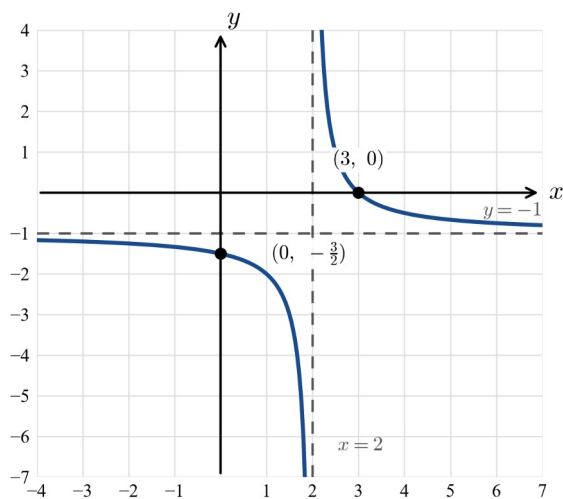
Q1. Translation 1 left, 4 down; turning point $(-1, -4)$; y -int $(0, -3)$; x -ints $(-3, 0), (1, 0)$. **Q2.** Asymptotes $x=2, y=-1$; intercepts $(3, 0), (0, -\frac{3}{2})$; translation 2 right, 1 down. **Q3.** Asymptotes $x=-2, y=1$; no x -intercept (since $\frac{1}{(x+2)^2} > 0$, so $y > 1$); y -int $(0, \frac{5}{4})$. **Q4.** Endpoint $(2, 1)$; domain $\{x: x \geq 2\}$; no intercepts ($x=0$ outside the domain and $y \geq 1$). **Q5.** Stationary point of inflection $(-2, -1)$; intercepts $(-1, 0), (0, 7)$. **Q6.** Turning point $(3, 2)$; no x -intercept (minimum value $2 > 0$); y -int $(0, 11)$. **Q7.** Turning point $(1, -9)$; y -int $(0, -8)$; x -ints $(-2, 0), (4, 0)$. **Q8.** Asymptotes $x=-3, y=-2$; intercepts $(-\frac{5}{2}, 0), (0, -\frac{5}{3})$. **Q9.** Asymptotes $x=2, y=-1$; intercepts $(1, 0), (3, 0), (0, -\frac{3}{4})$. **Q10.** Endpoint $(-1, -3)$; domain $\{x: x \geq -1\}$; intercepts $(8, 0), (0, -2)$. **Q11.** Stationary point of inflection $(1, -1)$; intercepts $(2, 0), (0, -2)$. **Q12.** Turning point $(-2, -1)$; x -ints $(-3, 0), (-1, 0)$; y -int $(0, 3)$. **Q13.** Asymptotes $x=1, y=3$; intercepts $(\frac{2}{3}, 0), (0, 2)$. **Q14.** Asymptotes $x=-1, y=2$; no x -intercept; y -int $(0, 3)$. **Q15.** Endpoint $(-2, -1)$; domain $\{x: x \geq -2\}$; intercepts $(-1, 0), (0, \sqrt{2}-1)$. **Q16.** Stationary point of inflection $(-1, -8)$; intercepts $(1, 0), (0, -7)$. **Q17.** $(-1, 3)$. **Q18.** $y = \frac{1}{(x-3)^2} - 2$; translation 3 right and 2 down. **Q19.** $y = (x+2)^2 + 5$; no x -intercepts (the minimum value is $5 > 0$). **Q20.** Stationary point of inflection $(-1, 3)$; x -int $(-1 - \sqrt[3]{3}, 0)$; y -int $(0, 4)$. **Q21.** Endpoint $(-4, 1)$; domain $\{x: x \geq -4\}$, range $\{y: y \geq 1\}$; y -int $(0, 3)$; no x -intercept.

Fully-worked solutions

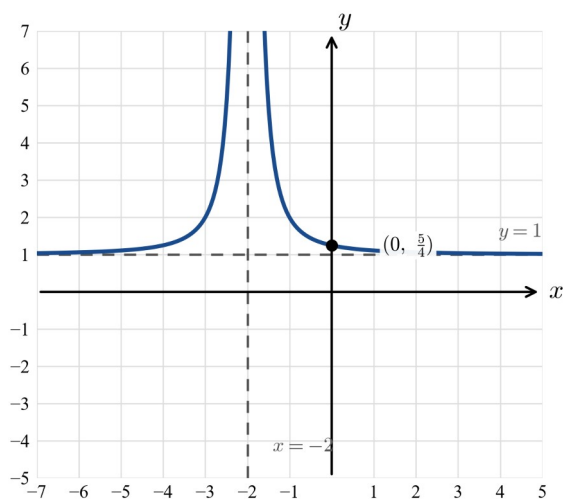
Q1. $y = (x+1)^2 - 4$: base $y = x^2$ translated 1 left, 4 down; turning point $(-1, -4)$. y -int $x=0 \Rightarrow y = 1 - 4 = -3$, giving $(0, -3)$. x -ints $(x+1)^2 = 4 \Rightarrow x+1 = \pm 2$, so $x = 1$ or $x = -3$, giving $(1, 0)$ and $(-3, 0)$.



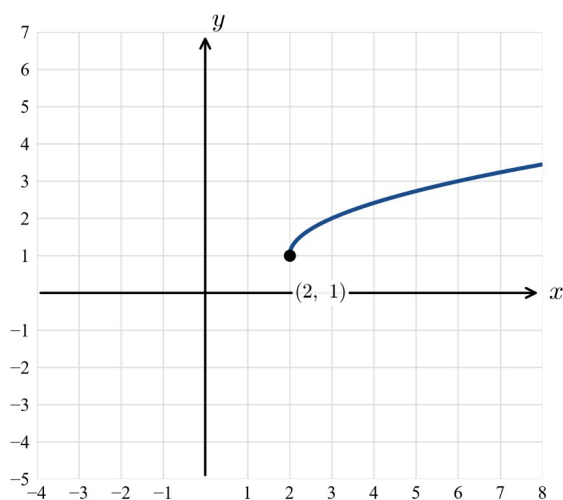
Q2. $y = \frac{1}{x-2} - 1$: asymptotes $x=2, y=-1$. y -int $x=0 \Rightarrow y = \frac{1}{-2} - 1 = -\frac{3}{2}$, giving $(0, -\frac{3}{2})$. x -int $\frac{1}{x-2} = 1 \Rightarrow x-2 = 1 \Rightarrow x=3$, giving $(3, 0)$. Translation 2 right and 1 down.



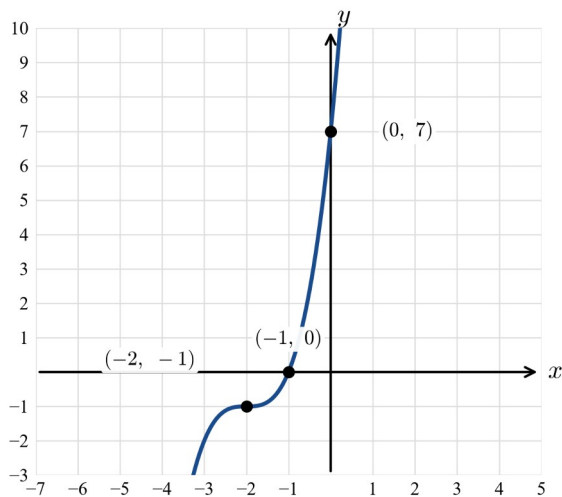
Q3. $y = \frac{1}{(x+2)^2} + 1$: asymptotes $x = -2$, $y = 1$. Since $\frac{1}{(x+2)^2} > 0$, we have $y > 1$ always, so there is no x -intercept. y -int $x = 0 \Rightarrow y = \frac{1}{4} + 1 = \frac{5}{4}$, giving $(0, \frac{5}{4})$.



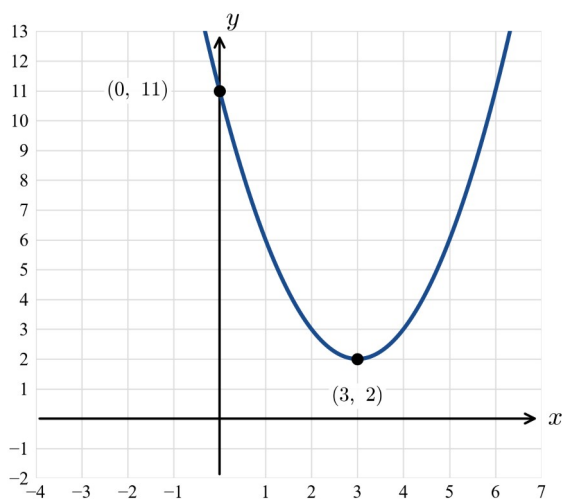
Q4. $y = \sqrt{x-2} + 1$: endpoint $(2, 1)$, domain $\{x : x \geq 2\}$. No y -intercept because $x = 0$ is outside the domain; no x -intercept because $y \geq 1 > 0$.



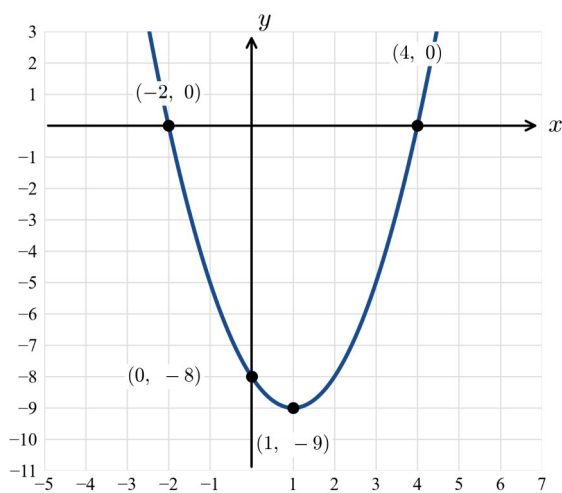
Q5. $y = (x+2)^3 - 1$: stationary point of inflection $(-2, -1)$. x -int $(x+2)^3 = 1 \Rightarrow x+2 = 1 \Rightarrow x = -1$, giving $(-1, 0)$. y -int $x = 0 \Rightarrow y = 8 - 1 = 7$, giving $(0, 7)$.



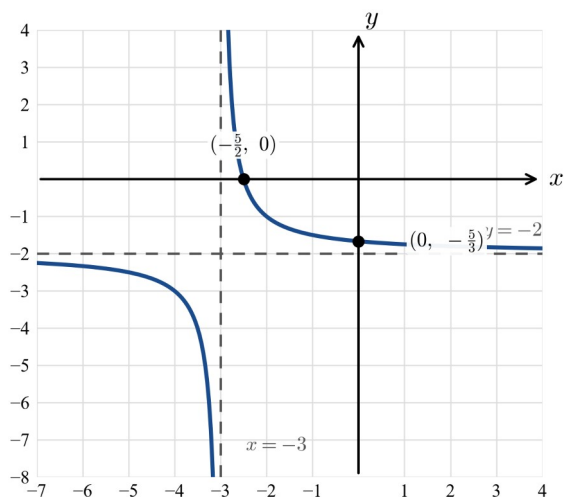
Q6. $y = (x - 3)^2 + 2$: turning point $(3, 2)$, a minimum. Since the minimum value is $2 > 0$, the parabola lies entirely above the x -axis, so there is no x -intercept. y -int $x = 0 \Rightarrow y = 9 + 2 = 11$, giving $(0, 11)$.



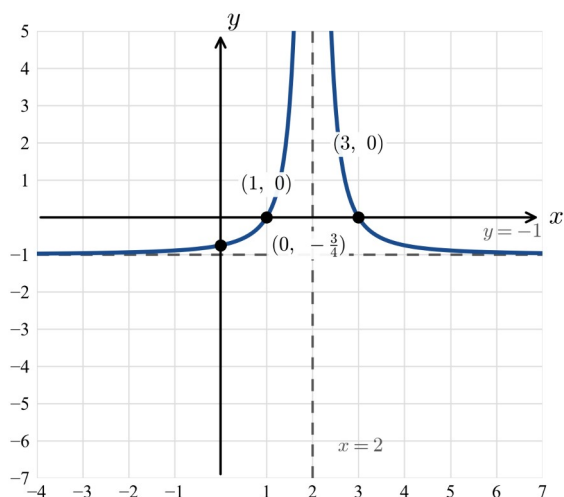
Q7. $y = (x - 1)^2 - 9$: turning point $(1, -9)$. y -int $(0, -8)$. x -ints $(x - 1)^2 = 9 \Rightarrow x - 1 = \pm 3$, so $x = 4$ or $x = -2$, giving $(4, 0)$ and $(-2, 0)$.



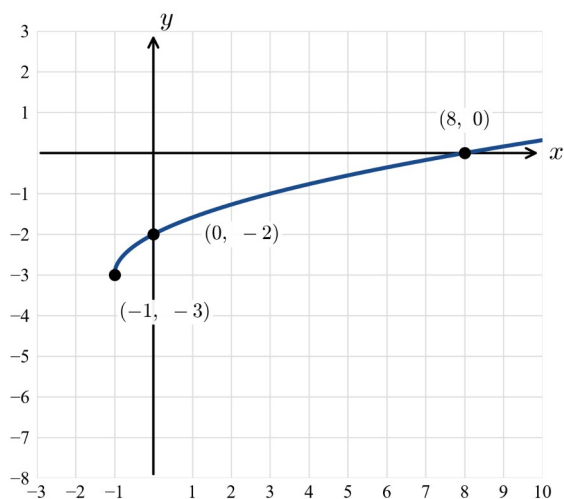
Q8. $y = \frac{1}{x+3} - 2$: asymptotes $x = -3$, $y = -2$. y -int $x = 0 \Rightarrow y = \frac{1}{3} - 2 = -\frac{5}{3}$, giving $(0, -\frac{5}{3})$. x -int $\frac{1}{x+3} = 2 \Rightarrow x + 3 = \frac{1}{2} \Rightarrow x = -\frac{5}{2}$, giving $(-\frac{5}{2}, 0)$.



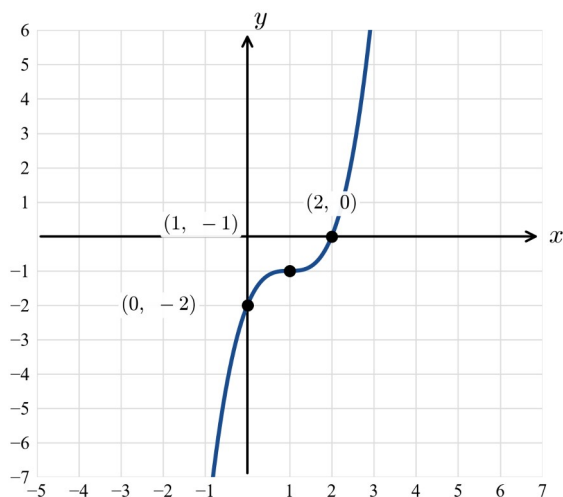
Q9. $y = \frac{1}{(x-2)^2} - 1$: asymptotes $x=2$, $y=-1$. x-ints $(x-2)^2=1 \Rightarrow x-2=\pm 1$, so $x=1$ or $x=3$, giving $(1,0)$ and $(3,0)$. y-int $x=0 \Rightarrow y = \frac{1}{4} - 1 = -\frac{3}{4}$, giving $(0, -\frac{3}{4})$.



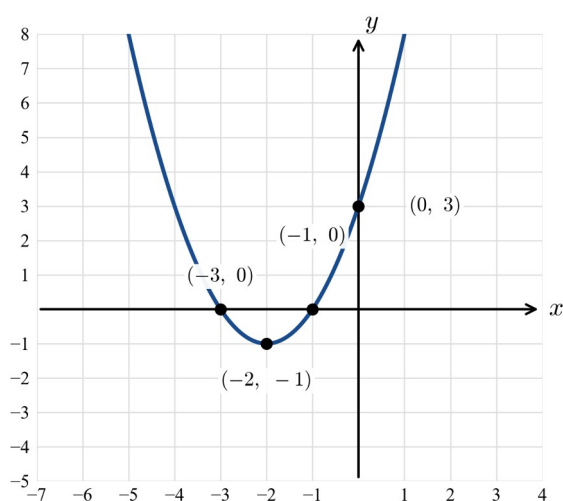
Q10. $y = \sqrt{x+1} - 3$: endpoint $(-1, -3)$, domain $\{x : x \geq -1\}$. x-int $\sqrt{x+1}=3 \Rightarrow x+1=9 \Rightarrow x=8$, giving $(8,0)$. y-int $x=0 \Rightarrow y = 1 - 3 = -2$, giving $(0, -2)$.



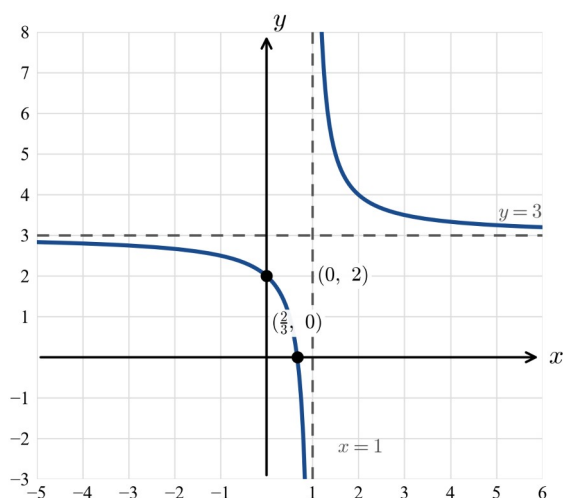
Q11. $y = (x-1)^3 - 1$: stationary point of inflection $(1, -1)$. x-int $(x-1)^3=1 \Rightarrow x-1=1 \Rightarrow x=2$, giving $(2,0)$. y-int $x=0 \Rightarrow y = -1 - 1 = -2$, giving $(0, -2)$.



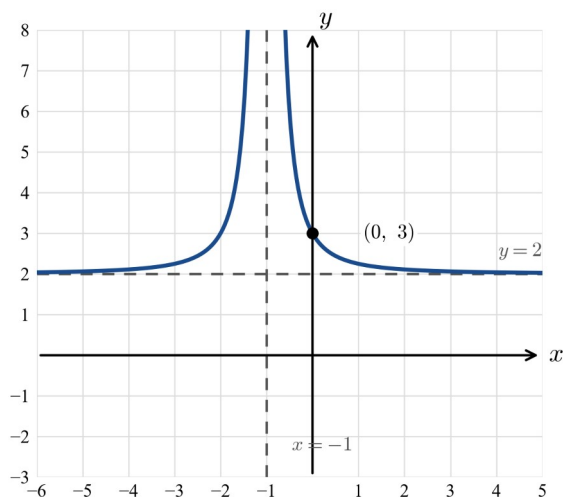
Q12. $y = (x+2)^2 - 1$: turning point $(-2, -1)$. x-ints $(x+2)^2 = 1 \Rightarrow x = -1$ or $x = -3$, giving $(-1, 0)$ and $(-3, 0)$. y-int $x=0 \Rightarrow y = 4 - 1 = 3$, giving $(0, 3)$.



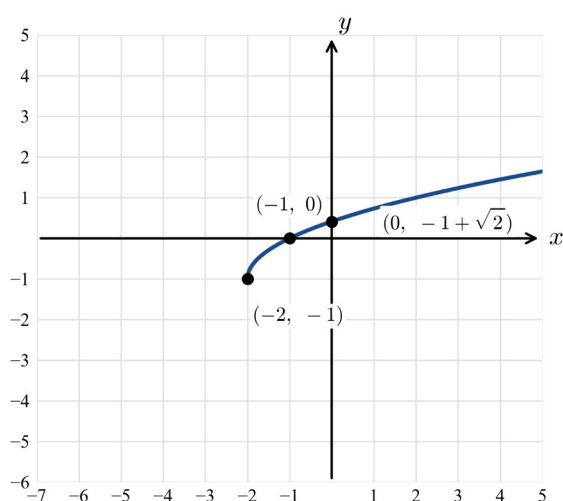
Q13. $y = \frac{1}{x-1} + 3$: asymptotes $x=1$, $y=3$. y-int $x=0 \Rightarrow y = -1 + 3 = 2$, giving $(0, 2)$. x-int $\frac{1}{x-1} = -3 \Rightarrow x-1 = -\frac{1}{3} \Rightarrow x = \frac{2}{3}$, giving $(\frac{2}{3}, 0)$.



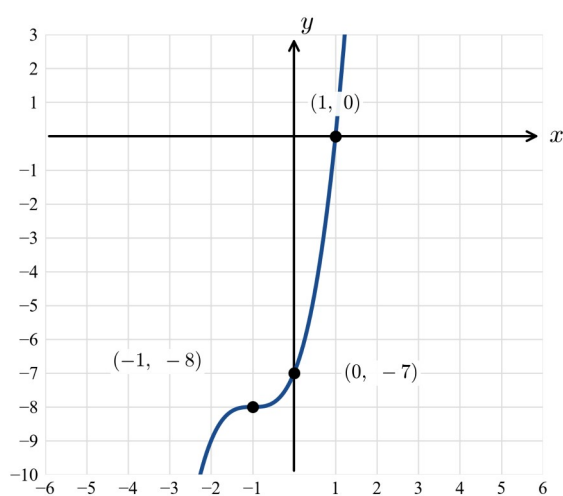
Q14. $y = \frac{1}{(x+1)^2} + 2$: asymptotes $x = -1$, $y = 2$; since $\frac{1}{(x+1)^2} > 0$, $y > 2$ and there is no x-intercept. y-int $x=0 \Rightarrow y = 1 + 2 = 3$, giving $(0, 3)$.



Q15. $y = \sqrt{x+2} - 1$: endpoint $(-2, -1)$, domain $\{x : x \geq -2\}$. x-int $\sqrt{x+2} = 1 \Rightarrow x+2=1 \Rightarrow x=-1$, giving $(-1, 0)$. y-int $x=0 \Rightarrow y = \sqrt{2} - 1$, giving $(0, \sqrt{2} - 1)$.



Q16. $y = (x+1)^3 - 8$: stationary point of inflection $(-1, -8)$. x-int $(x+1)^3 = 8 \Rightarrow x+1=2 \Rightarrow x=1$, giving $(1, 0)$. y-int $x=0 \Rightarrow y = 1 - 8 = -7$, giving $(0, -7)$.



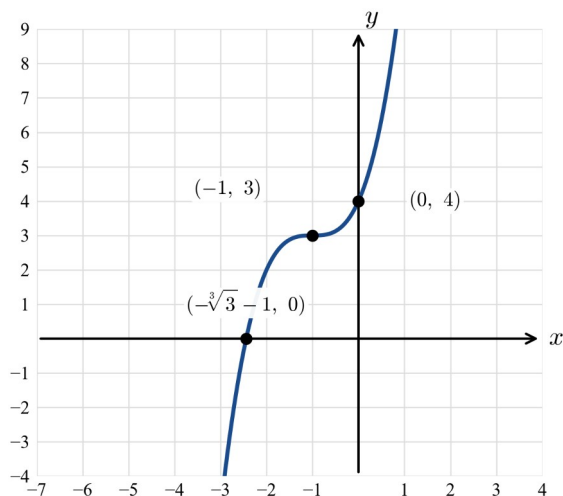
Q17. The map $x \mapsto x - 4$, $y \mapsto y + 5$ takes $y = f(x)$ to $y = f(x + 4) + 5$, translating every point 4 left and 5 up. So $(3, -2) \mapsto (3 - 4, -2 + 5) = (-1, 3)$.

Q18. Translating $y = \frac{1}{x^2}$ so the asymptotes $x=0$, $y=0$ become $x=3$, $y=-2$ requires $h=3$, $k=-2$: $y = \frac{1}{(x-3)^2} - 2$.

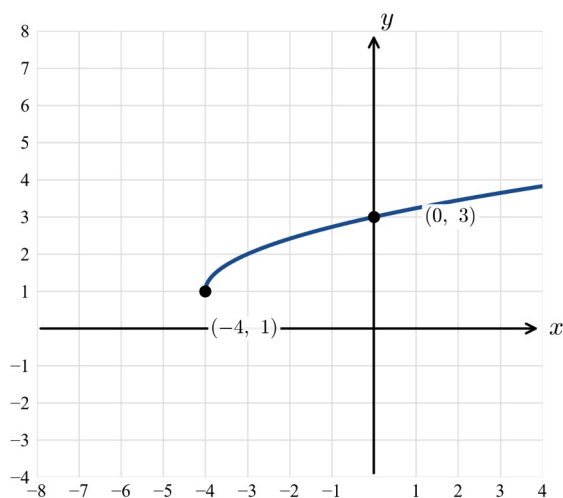
The translation is 3 right and 2 down.

Q19. Same shape as $y = x^2$ with turning point $(-2, 5)$ gives $y = (x + 2)^2 + 5$. The minimum value is $5 > 0$, so the graph lies entirely above the x -axis and has no x -intercepts.

Q20. $y = (x + 1)^3 + 3$: stationary point of inflection $(-1, 3)$. x -int $(x + 1)^3 = -3 \Rightarrow x + 1 = -\sqrt[3]{3} \Rightarrow x = -1 - \sqrt[3]{3}$, giving $(-1 - \sqrt[3]{3}, 0)$. y -int $x = 0 \Rightarrow y = 1 + 3 = 4$, giving $(0, 4)$.



Q21. $y = \sqrt{x + 4} + 1$: endpoint $(-4, 1)$, domain $\{x : x \geq -4\}$, range $\{y : y \geq 1\}$. Since $y \geq 1 > 0$ there is no x -intercept. y -int $x = 0 \Rightarrow y = \sqrt{4} + 1 = 3$, giving $(0, 3)$.



Chapter 3 Transformations of the plane — 3B Dilations

Theory

A **dilation** stretches or compresses a graph away from (or towards) one of the axes. There are two kinds, and the key is to be clear about **which axis** the dilation is measured from.

Dilation from the x -axis, factor a (a vertical stretch): the rule becomes

$$y = af(x).$$

- Every point (x, y) maps to (x, ay) — the y -coordinates are multiplied by a .
- If $a > 1$ the graph is stretched **away** from the x -axis (steeper); if $0 < a < 1$ it is compressed **towards** the x -axis.
- The **x -intercepts are unchanged** (a point with $y = 0$ still has $y = 0$).

Dilation from the y -axis, factor b (a horizontal stretch): the rule becomes

$$y = f(x/b).$$

- Every point (x, y) maps to (bx, y) — the x -coordinates are multiplied by b .
- If $b > 1$ the graph is stretched **away** from the y -axis (wider); if $0 < b < 1$ it is compressed towards it. The **y -intercept is unchanged**.
- Note that $y = f(nx)$ is a dilation of factor $\frac{1}{n}$ from the y -axis.

Effect on the standard graphs.

Base	From the x -axis, factor a	From the y -axis, factor b
$y = x^2$	$y = ax^2$	$y = \left(\frac{x}{b}\right)^2 = \frac{x^2}{b^2}$
$y = \sqrt{x}$	$y = a\sqrt{x}$	$y = \sqrt{\frac{x}{b}}$
$y = \frac{1}{x}$	$y = \frac{a}{x}$	$y = \frac{b}{x}$
$y = \frac{1}{x^2}$	$y = \frac{a}{x^2}$	$y = \frac{b^2}{x^2}$

A useful fact. For a **power function** $y = x^n$, a dilation from the y -axis produces the same family of graphs as a dilation from the x -axis — for example $y = \left(\frac{x}{2}\right)^2 = \frac{1}{4}x^2$ is also the dilation of $y = x^2$ by factor $\frac{1}{4}$ from the x -axis. The **image graph** is identical; only the description (and the point-mapping used to obtain it) differs. This is **not** true for graphs in general (for example the circular functions in Chapter 6).

Describing a dilation. Given a graph and its image, compare y -coordinates at a fixed x (for a dilation from the x -axis) or x -coordinates at a fixed y (from the y -axis) to read off the factor.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = 3\sqrt{x}$, describing the dilation applied to $y = \sqrt{x}$ and giving the images of the points $(1, 1)$ and $(4, 2)$.

Working

$$y = 3\sqrt{x} = 3f(x) \text{ where } f(x) = \sqrt{x}.$$

Dilation of factor 3 from the x -axis.

$$(1, 1) \rightarrow (1, 3 \times 1) = (1, 3) \text{ and}$$

$$(4, 2) \rightarrow (4, 3 \times 2) = (4, 6).$$

Endpoint $(0, 0)$ is unchanged; domain $\{x : x \geq 0\}$, range $\{y : y \geq 0\}$.

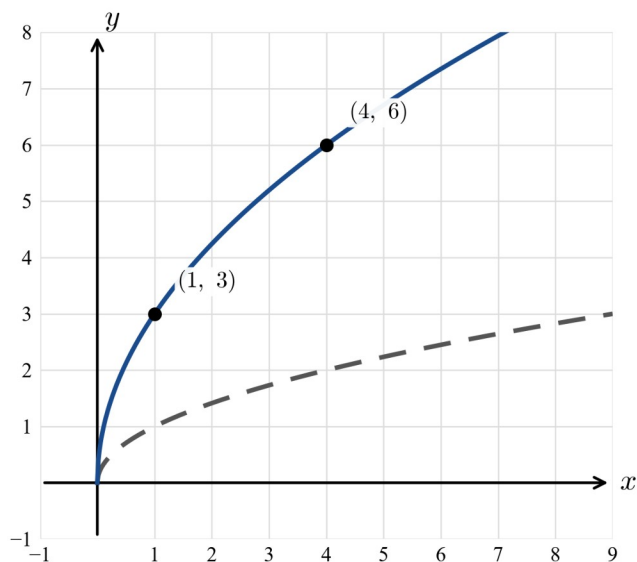
Comment

Compare with $y = af(x)$: this is a dilation from the x -axis.

$a = 3$; y -coordinates are tripled.

Map each point by $(x, y) \rightarrow (x, 3y)$.

A point with $y = 0$ is fixed by a dilation from the x -axis.



Example 2 — scaffolded

The graph of $y = x^2$ is dilated by a factor of 2 from the y -axis. Find the rule of the image, sketch it, and give the image of the point $(2, 4)$.

Working

$$\text{Replace } x \text{ by } \frac{x}{2}: y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}.$$

$$(2, 4) \rightarrow (2 \times 2, 4) = (4, 4).$$

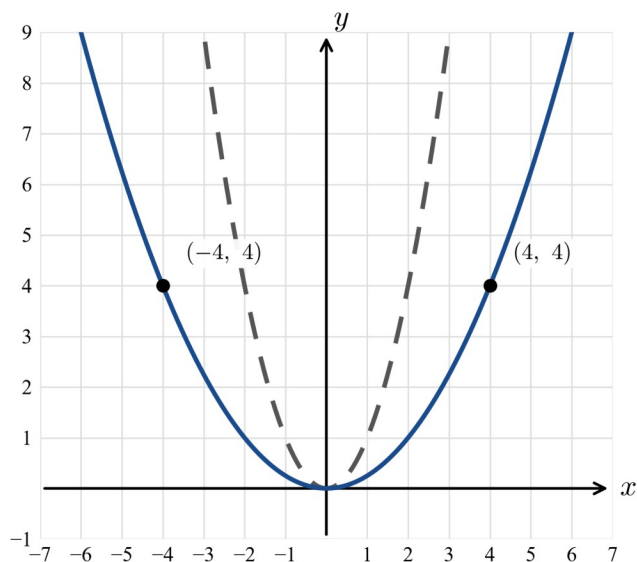
The parabola is **wider** than $y = x^2$ (a horizontal stretch), same turning point $(0, 0)$.

Comment

A dilation of factor b from the y -axis gives $y = f(x/b)$, here $b = 2$.

Map each point by $(x, y) \rightarrow (2x, y)$.

$b = 2 > 1$ stretches away from the y -axis.



Example 3 — unscaffolded

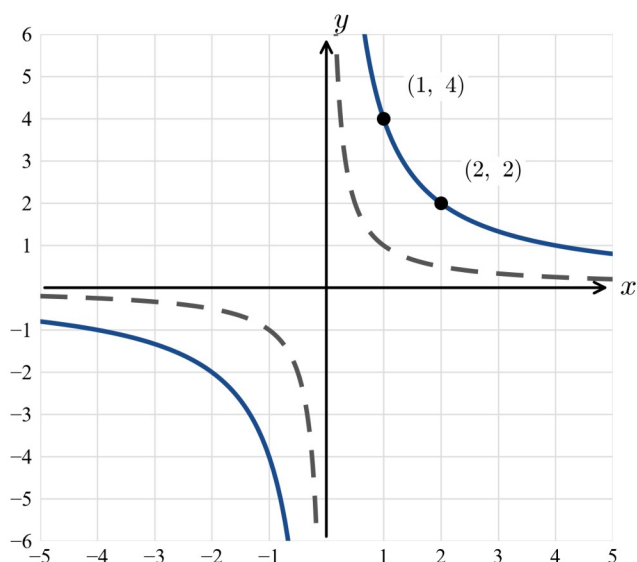
Sketch the graph of $y = \frac{4}{x}$, describing the dilation of $y = \frac{1}{x}$ that produces it.

$y = \frac{4}{x} = 4 \times \frac{1}{x}$, so this is a **dilation of factor 4 from the x -axis**.

Points map by $(x, y) \rightarrow (x, 4y)$: for example $\left(2, \frac{1}{2}\right) \rightarrow (2, 2)$ and $(1, 1) \rightarrow (1, 4)$.

The asymptotes $x = 0$ and $y = 0$ are unchanged; domain $\mathbb{R} \setminus \{0\}$, range $\mathbb{R} \setminus \{0\}$.

$\therefore$ the branches are pushed further from the origin than those of $y = \frac{1}{x}$.



Example 4 — unscaffolded

The graph of $y = x^2$ is mapped to the graph of $y = 5x^2$. Describe the dilation and find the image of the point $(2, 4)$.

$y = 5x^2 = 5f(x)$ with $f(x) = x^2$, so it is a **dilation of factor 5 from the x -axis**.

The point maps by $(x, y) \rightarrow (x, 5y)$: $(2, 4) \rightarrow (2, 20)$.

$\therefore$ the image of $(2, 4)$ is $(2, 20)$.

Practice questions

Scaffolded

Q1. The graph of $y = x^2$ is dilated by a factor of 3 from the x -axis. (a) Write the rule of the image. (b) Find the images of $(1, 1)$ and $(-2, 4)$. (c) Sketch the image, showing the base graph and two labelled points.

Q2. The graph of $y = \sqrt{x}$ is dilated by a factor of 4 from the y -axis. (a) Write the rule of the image. (b) Find the images of $(4, 2)$ and $(16, 4)$. (c) Sketch the image.

Q3. The graph of $y = \frac{1}{x^2}$ is dilated by a factor of 2 from the x -axis. (a) Write the rule of the image and state its asymptotes. (b) Find the images of $(1, 1)$ and $(2, \frac{1}{4})$. (c) Sketch the image.

Q4. The graph of $y = \frac{1}{x}$ is dilated by a factor of 3 from the y -axis. (a) Write the rule of the image. (b) Find the images of $(1, 1)$ and $(2, \frac{1}{2})$. (c) Explain why this image is also a dilation of $y = \frac{1}{x}$ from the x -axis, and state its factor.

Q5. The points $(2, 3)$ and $(-4, 6)$ lie on the graph of $y = f(x)$. Find their images under: (a) a dilation of factor $\frac{1}{2}$ from the x -axis; (b) a dilation of factor 2 from the y -axis.

Q6. The graph of $y = x^2$ is mapped to the graph of $y = \frac{1}{4}x^2$. (a) Describe the dilation from the x -axis that does this. (b) Find the image of the point $(2, 4)$.

Un scaffolded

For **Q7–Q12** and **Q17–Q18**, sketch the image (showing the base graph dashed and two labelled points), and state the rule of the image.

Q7. $y = x^2$ under a dilation of factor 4 from the x -axis.

Q8. $y = \frac{1}{x}$ under a dilation of factor $\frac{1}{2}$ from the x -axis.

Q9. $y = \frac{1}{x}$ under a dilation of factor 5 from the x -axis.

Q10. $y = \frac{1}{x^2}$ under a dilation of factor $\frac{1}{2}$ from the x -axis.

Q11. $y = x^2$ under a dilation of factor 3 from the y -axis.

Q12. $y = \sqrt{x}$ under a dilation of factor 9 from the y -axis.

Q13. The point $(6, -2)$ lies on $y = f(x)$. Find its image under (a) a dilation of factor 3 from the x -axis, and (b) a dilation of factor $\frac{1}{2}$ from the y -axis.

Q14. Describe the dilation from the x -axis that maps $y = x^2$ onto $y = \frac{1}{2}x^2$.

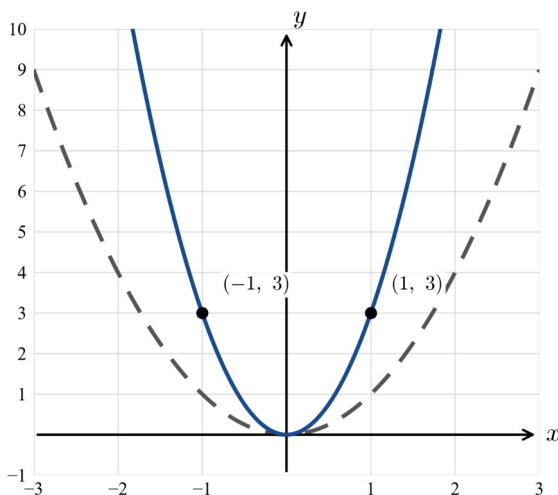
- Q15.** The graph of $y = a x^2$ passes through the point $(2, 20)$. Find the value of a and describe the dilation from the x -axis of $y = x^2$ it represents.
- Q16.** State the rule of the image of $y = \frac{1}{x}$ under a dilation of factor 2 from the y -axis, and explain why the same graph results from a dilation of factor 2 from the x -axis.
- Q17.** $y = \sqrt{x}$ under a dilation of factor 2 from the x -axis.
- Q18.** $y = \frac{1}{x^2}$ under a dilation of factor 3 from the x -axis.
- Q19.** The graph of $y = f(x)$ has x -intercepts $(-3, 0)$ and $(2, 0)$. State the x -intercepts of the image under (a) a dilation of factor 5 from the x -axis, and (b) a dilation of factor 2 from the y -axis.
- Q20.** The graph of $y = x^2$ is mapped to $y = 16 x^2$. Describe this transformation as a dilation from the x -axis, and also as a dilation from the y -axis.
- Q21.** The graph of $y = \sqrt{x}$ is mapped to $y = 2 \sqrt{x}$. Describe this as a dilation from the x -axis, and find the factor of the dilation from the y -axis that produces the same image.
-

Answers

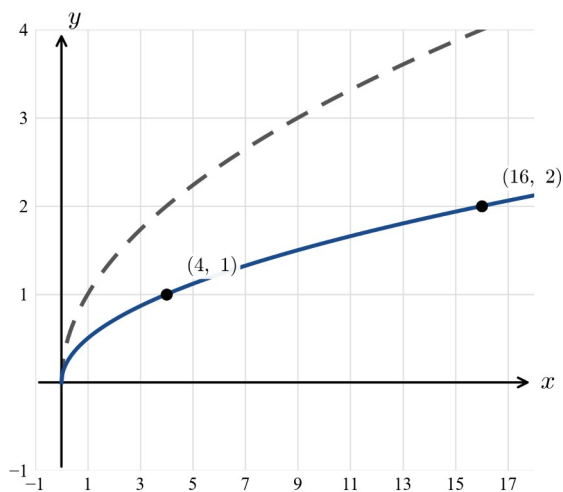
Q1. (a) $y = 3x^2$; (b) $(1, 3), (-2, 12)$. **Q2.** (a) $y = \sqrt{\frac{x}{4}} = \frac{\sqrt{x}}{2}$; (b) $(16, 2), (64, 4)$. **Q3.** (a) $y = \frac{2}{x^2}$, asymptotes $x = 0$, $y = 0$; (b) $(1, 2), \left(2, \frac{1}{2}\right)$. **Q4.** (a) $y = \frac{3}{x}$; (b) $(3, 1), (6, 1/2)$; (c) $\frac{3}{x} = 3 \times \frac{1}{x}$, a dilation of factor 3 from the x -axis. **Q5.** (a) $\left(2, \frac{3}{2}\right), (-4, 3)$; (b) $(4, 3), (-8, 6)$. **Q6.** (a) factor $\frac{1}{4}$ from the x -axis; (b) $(2, 1)$. **Q7.** $y = 4x^2$; points $(1, 4), (-1, 4)$. **Q8.** $y = \frac{1}{2x}$; points $\left(1, \frac{1}{2}\right), \left(2, \frac{1}{4}\right)$. **Q9.** $y = \frac{5}{x}$; points $(1, 5), (5, 1)$. **Q10.** $y = \frac{1}{2x^2}$; points $\left(1, \frac{1}{2}\right), \left(\frac{1}{2}, 2\right)$. **Q11.** $y = \frac{x^2}{9}$; points $(3, 1), (6, 4)$. **Q12.** $y = \frac{\sqrt{x}}{3}$; points $(9, 1), (36, 2)$. **Q13.** (a) $(6, -6)$; (b) $(3, -2)$. **Q14.** A dilation of factor $\frac{1}{2}$ from the x -axis. **Q15.** $a = 5$; a dilation of factor 5 from the x -axis. **Q16.** $y = \frac{2}{x}$; since $\frac{2}{x} = 2 \times \frac{1}{x}$, it is also a dilation of factor 2 from the x -axis (power-function equivalence). **Q17.** $y = 2\sqrt{x}$; points $(1, 2), (4, 4)$. **Q18.** $y = \frac{3}{x^2}$; points $(1, 3), \left(3, \frac{1}{3}\right)$. **Q19.** (a) $(-3, 0)$ and $(2, 0)$ (unchanged); (b) $(-6, 0)$ and $(4, 0)$. **Q20.** Factor 16 from the x -axis, or factor $\frac{1}{4}$ from the y -axis. **Q21.** Factor 2 from the x -axis; the equivalent dilation from the y -axis has factor $\frac{1}{4}$.

Fully-worked solutions

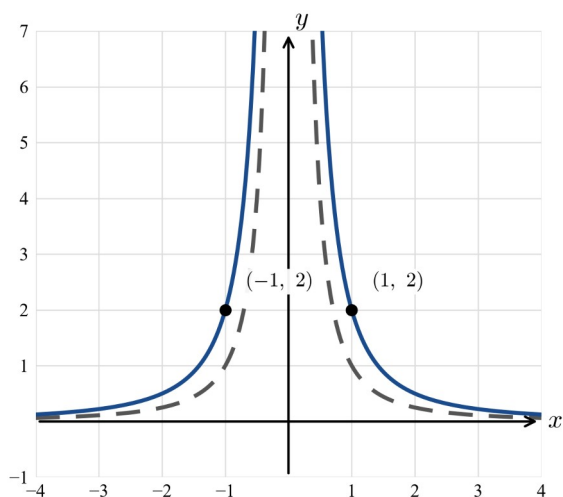
Q1. $y = 3f(x)$ with $f(x) = x^2$, so the image is $y = 3x^2$ (dilation of factor 3 from the x -axis). Points map by $(x, y) \rightarrow (x, 3y)$: $(1, 1) \rightarrow (1, 3)$ and $(-2, 4) \rightarrow (-2, 12)$.



Q2. Replace x by $\frac{x}{4}$: $y = \sqrt{\frac{x}{4}} = \frac{\sqrt{x}}{2}$. Points map by $(x, y) \rightarrow (4x, y)$: $(4, 2) \rightarrow (16, 2)$ and $(16, 4) \rightarrow (64, 4)$.

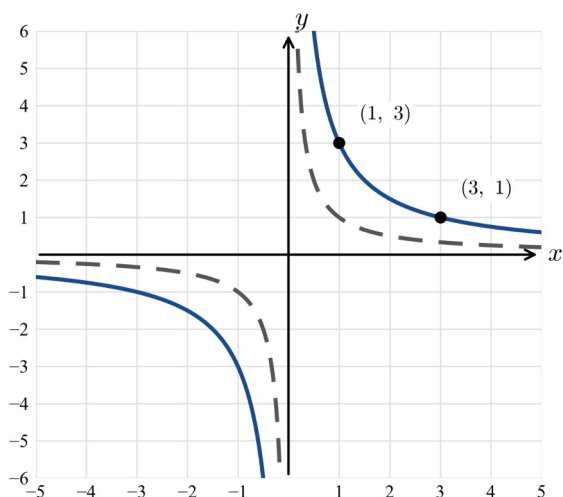


Q3. $y = 2 \times \frac{1}{x^2} = \frac{2}{x^2}$; asymptotes $x=0$, $y=0$. Points map by $(x, y) \rightarrow (x, 2y)$: $(1, 1) \rightarrow (1, 2)$ and $\left(2, \frac{1}{4}\right) \rightarrow \left(2, \frac{1}{2}\right)$.



Q4. Replace x by $\frac{x}{3}$: $y = \frac{1}{x/3} = \frac{3}{x}$. Points map by $(x, y) \rightarrow (3x, y)$: $(1, 1) \rightarrow (3, 1)$ and $\left(2, \frac{1}{2}\right) \rightarrow \left(6, \frac{1}{2}\right)$. (c)

Since $\frac{3}{x} = 3 \times \frac{1}{x}$, the same graph is a dilation of factor 3 from the x -axis.

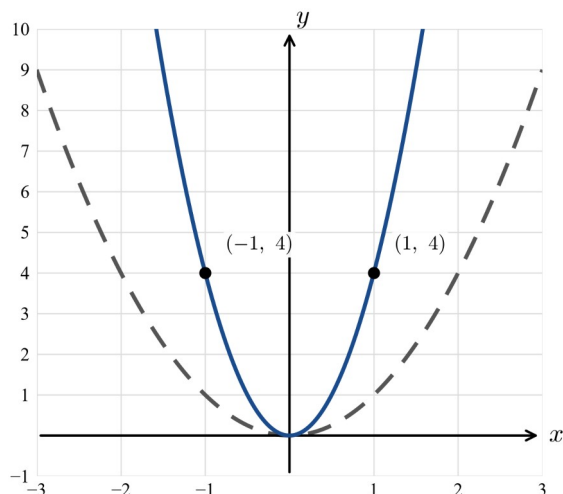


Q5. (a) $(x, y) \rightarrow \left(x, \frac{1}{2}y\right): (2, 3) \rightarrow \left(2, \frac{3}{2}\right)$ and $(-4, 6) \rightarrow (-4, 3)$. (b) $(x, y) \rightarrow (2x, y): (2, 3) \rightarrow (4, 3)$ and $(-4, 6) \rightarrow (-8, 6)$.

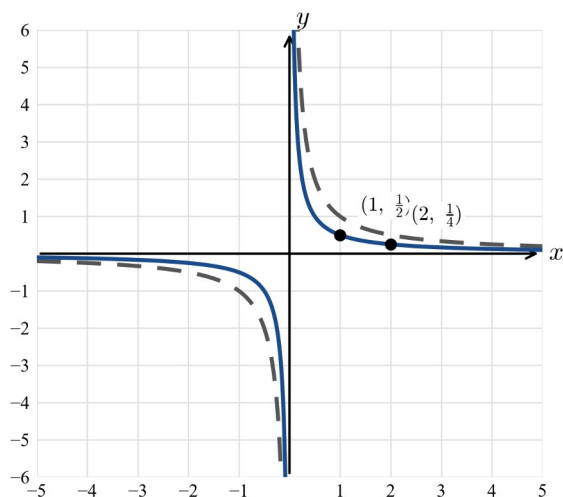
Q6. (a) Comparing $y = \frac{1}{4}x^2$ with $y = ax^2$ gives $a = \frac{1}{4}$: a dilation of factor $\frac{1}{4}$ from the x -axis. (b)

$$(2, 4) \rightarrow \left(2, \frac{1}{4} \times 4\right) = (2, 1).$$

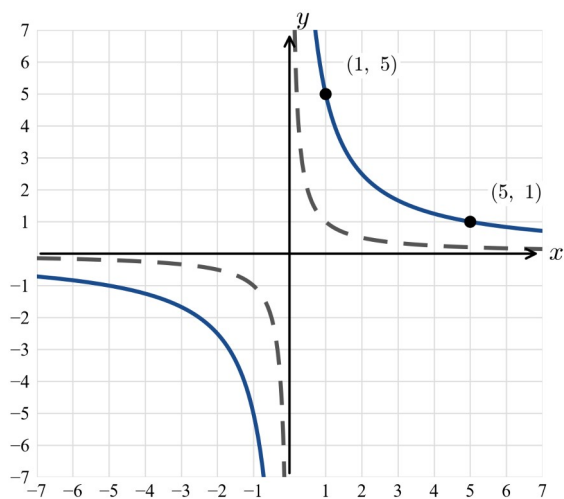
Q7. $y = 4x^2$ (factor 4 from the x -axis). $(1, 1) \rightarrow (1, 4), (-1, 1) \rightarrow (-1, 4)$.



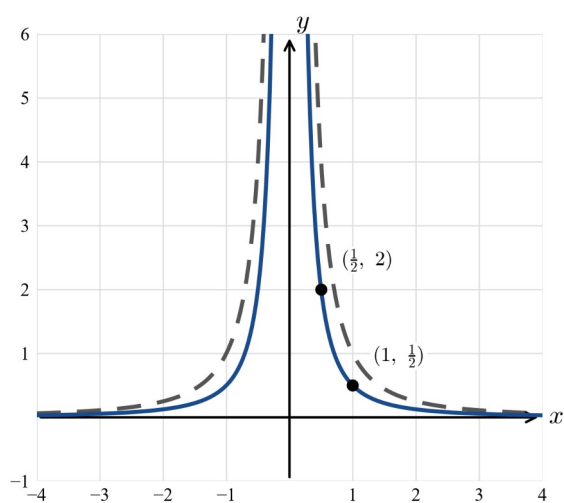
Q8. $y = \frac{1}{2} \times \frac{1}{x} = \frac{1}{2x}$ (factor $\frac{1}{2}$ from the x -axis). $(1, 1) \rightarrow \left(1, \frac{1}{2}\right), \left(2, \frac{1}{2}\right) \rightarrow \left(2, \frac{1}{4}\right)$.



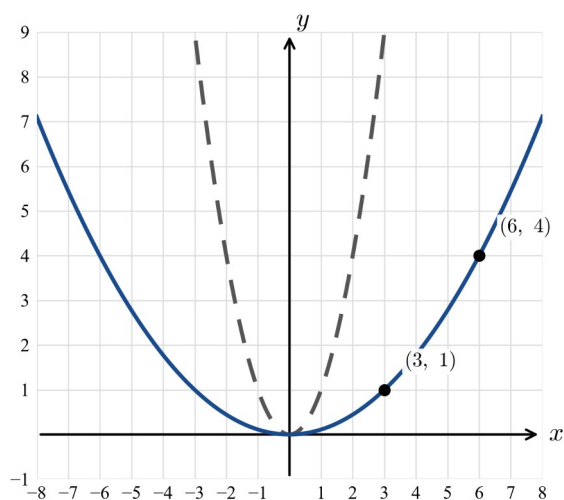
Q9. $y = \frac{5}{x}$ (factor 5 from the x -axis). $(1, 1) \rightarrow (1, 5), (5, 1/5) \rightarrow (5, 1)$.



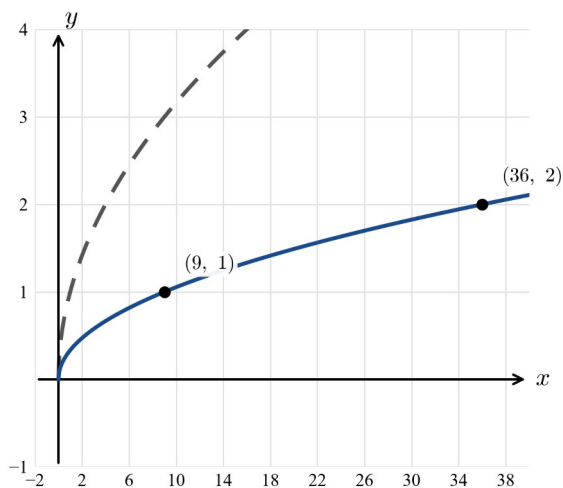
Q10. $y = \frac{1}{2} \times \frac{1}{x^2} = \frac{1}{2x^2}$ (factor $\frac{1}{2}$ from the x -axis). $(1, 1) \rightarrow \left(1, \frac{1}{2}\right), \left(\frac{1}{2}, 4\right) \rightarrow \left(\frac{1}{2}, 2\right)$.



Q11. Replace x by $\frac{x}{3}$: $y = \left(\frac{x}{3}\right)^2 = \frac{x^2}{9}$ (factor 3 from the y -axis). $(1, 1) \rightarrow (3, 1), (2, 4) \rightarrow (6, 4)$.



Q12. Replace x by $\frac{x}{9}$: $y = \sqrt{\frac{x}{9}} = \frac{\sqrt{x}}{3}$ (factor 9 from the y -axis). $(1, 1) \rightarrow (9, 1), (4, 2) \rightarrow (36, 2)$.



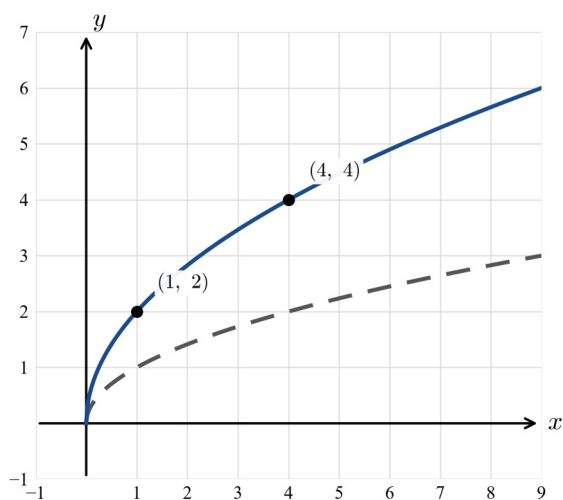
Q13. (a) $(x, y) \rightarrow (x, 3y): (6, -2) \rightarrow (6, -6)$. (b) $(x, y) \rightarrow \left(\frac{1}{2}x, y\right): (6, -2) \rightarrow (3, -2)$.

Q14. Comparing $y = \frac{1}{2}x^2$ with $y = ax^2$ gives $a = \frac{1}{2}$: a dilation of factor $\frac{1}{2}$ from the x -axis.

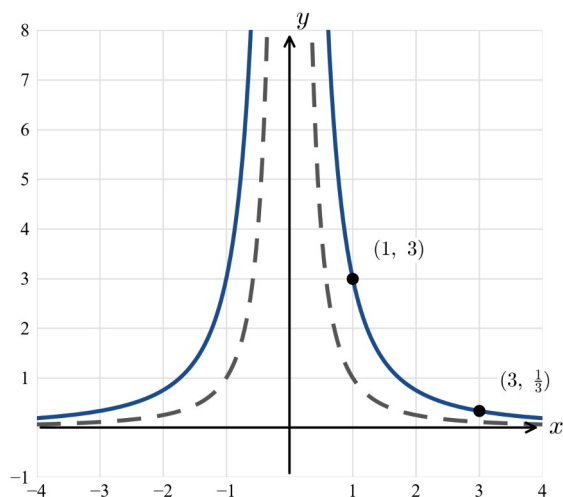
Q15. Substituting $(2, 20)$: $20 = a(2)^2 = 4a$, so $a = 5$. The graph $y = 5x^2$ is a dilation of factor 5 from the x -axis.

Q16. Replacing x by $\frac{x}{2}$ gives $y = \frac{1}{x/2} = \frac{2}{x}$. Because $\frac{2}{x} = 2 \times \frac{1}{x}$, the same graph is produced by a dilation of factor 2 from the x -axis — the two descriptions give the identical image for this power function.

Q17. $y = 2\sqrt{x}$ (factor 2 from the x -axis). $(1, 1) \rightarrow (1, 2), (4, 2) \rightarrow (4, 4)$.



Q18. $y = \frac{3}{x^2}$ (factor 3 from the x -axis). $(1, 1) \rightarrow (1, 3), \left(3, \frac{1}{9}\right) \rightarrow \left(3, \frac{1}{3}\right)$.



- Q19.** (a) A dilation from the x -axis fixes points with $y=0$, so the x -intercepts are **unchanged**: $(-3, 0)$ and $(2, 0)$.
 (b) A dilation of factor 2 from the y -axis maps $(x, 0) \rightarrow (2x, 0)$: $(-3, 0) \rightarrow (-6, 0)$ and $(2, 0) \rightarrow (4, 0)$.

Q20. $y = 16x^2 = 16f(x)$ is a dilation of factor 16 from the x -axis. Alternatively, $16x^2 = \left(\frac{x}{1/4}\right)^2$, i.e. replacing x by

$\frac{x}{1/4}$, so it is also a dilation of factor $\frac{1}{4}$ from the y -axis.

Q21. $y = 2\sqrt{x} = 2f(x)$ is a dilation of factor 2 from the x -axis. For a dilation of factor b from the y -axis,

$\sqrt{\frac{x}{b}} = \frac{1}{\sqrt{b}}\sqrt{x}$; setting $\frac{1}{\sqrt{b}} = 2$ gives $b = \frac{1}{4}$.

Chapter 3 Transformations of the plane — 3C Reflections

Theory

A **reflection** is a transformation that flips a graph across a line (a *mirror line*). In this chapter the two axis reflections are studied: reflection in the x -axis and reflection in the y -axis.

Reflection in the x -axis. Each point (x, y) maps to $(x, -y)$: the x -coordinate is unchanged and the y -coordinate changes sign. The image of $y = f(x)$ has the rule

$$y = -f(x).$$

Every function value is negated, so the graph is flipped top-to-bottom. Points on the x -axis are unchanged, so **x -intercepts stay fixed**; the y -intercept and any turning point have their y -coordinate negated (a minimum becomes a maximum, and vice versa).

Reflection in the y -axis. Each point (x, y) maps to $(-x, y)$: the y -coordinate is unchanged and the x -coordinate changes sign. The image of $y = f(x)$ has the rule

$$y = f(-x),$$

obtained by replacing x with $-x$ in the rule. The graph is flipped left-to-right. Points on the y -axis are unchanged, so the **y -intercept stays fixed**; x -intercepts and endpoints have their x -coordinate negated. For a square-root graph the domain is reversed (for example $x \geq h$ becomes $x \leq -h$).

Effect on the standard graphs.

Graph	Reflected in x -axis, $y = -f(x)$	Reflected in y -axis, $y = f(-x)$
$y = x^2$	$y = -x^2$ (opens down)	$y = x^2$ (unchanged — even)
$y = \sqrt{x}$	$y = -\sqrt{x}$ (below the axis)	$y = \sqrt{-x}$ (domain $x \leq 0$)
$y = \frac{1}{x}$	$y = -\frac{1}{x}$	$y = -\frac{1}{x}$ (same — odd)
$y = \frac{1}{x^2}$	$y = -\frac{1}{x^2}$ (below the axis)	$y = \frac{1}{x^2}$ (unchanged — even)
$y = x^3$	$y = -x^3$	$y = -x^3$ (same — odd)

- An **even** function satisfies $f(-x) = f(x)$: its graph is unchanged by reflection in the y -axis (it is symmetric about the y -axis).
- An **odd** function satisfies $f(-x) = -f(x)$: reflecting in the y -axis gives the *same* image as reflecting in the x -axis (the graph has point symmetry about the origin).

To sketch a reflection: (1) write the image rule — negate the whole rule (x -axis) or replace x by $-x$ (y -axis); (2) transform the key features accordingly (which intercepts stay fixed); (3) find any new intercepts exactly; (4) sketch, labelling the key features.

Worked examples

Example 1 — scaffolded

The graph of $y = (x - 1)^2 - 4$ is reflected in the x -axis. Find the rule of the image and sketch it.

Working

Image rule: $y = -((x - 1)^2 - 4) = 4 - (x - 1)^2$.

Comment

Reflection in the x -axis negates the whole rule, $y = -f(x)$.

Working

Original turning point $(1, -4)$ (minimum) $\rightarrow (1, 4)$ (maximum).

x -intercepts:

$4 - (x - 1)^2 = 0 \Rightarrow (x - 1)^2 = 4 \Rightarrow x - 1 = \pm 2$, so $x = -1$ or $x = 3$: $(-1, 0), (3, 0)$.

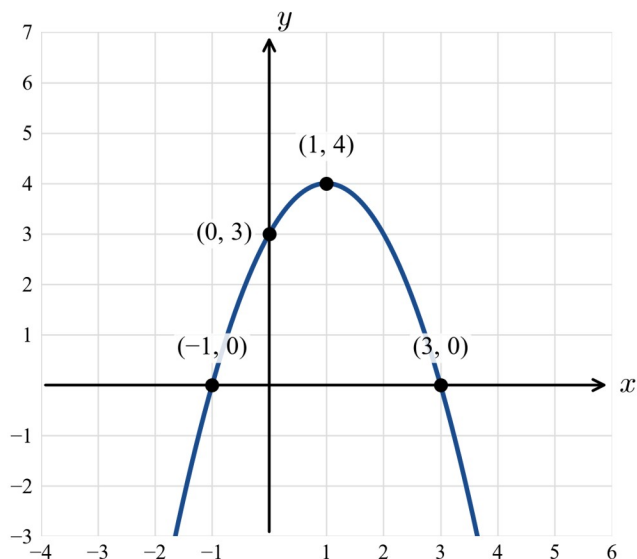
y -intercept: $x = 0 \Rightarrow y = 4 - (0 - 1)^2 = 4 - 1 = 3$, giving $(0, 3)$.

Comment

The y -coordinate is negated; a minimum becomes a maximum.

x -intercepts are **unchanged** by a reflection in the x -axis.

The original y -intercept $(0, -3)$ has its sign reversed.



Example 2 — scaffolded

The graph of $y = \sqrt{x - 1}$ is reflected in the y -axis. Find the rule of the image, state its domain, and sketch it.

Working

Image rule: replace x with $-x$:

$$y = \sqrt{-x - 1} = \sqrt{-(x + 1)}.$$

Original endpoint $(1, 0) \rightarrow (-1, 0)$.

Domain: need $-(x + 1) \geq 0 \Rightarrow x \leq -1$.

No y -intercept: $x = 0$ is outside the domain. Range $\{y : y \geq 0\}$ (unchanged).

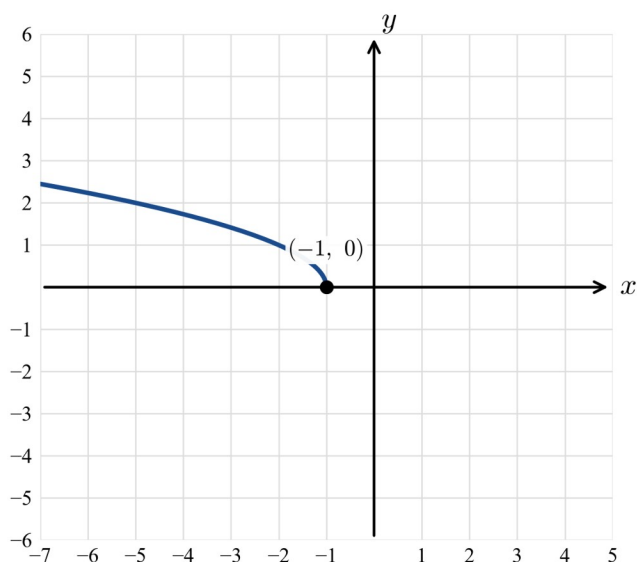
Comment

Reflection in the y -axis gives $y = f(-x)$.

The endpoint's x -coordinate is negated.

The original domain $x \geq 1$ becomes $x \leq -1$.

Reflection in the y -axis leaves y -values unchanged.



Example 3 — unscaffolded

The graph of $y = \frac{1}{x-2} + 1$ is reflected in the x -axis. Find the rule of the image, state the equations of its asymptotes, and sketch it.

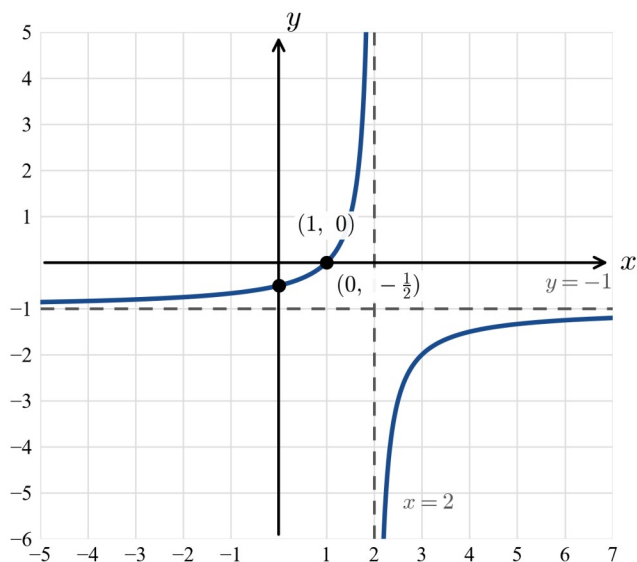
$$\text{Image rule: } y = -\left(\frac{1}{x-2} + 1\right) = -\frac{1}{x-2} - 1.$$

Reflection in the x -axis negates y : the horizontal asymptote $y = 1$ becomes $y = -1$; the vertical asymptote $x = 2$ is unchanged. Domain $\mathbb{R} \setminus \{2\}$, range $\mathbb{R} \setminus \{-1\}$.

$$x\text{-intercept: } -\frac{1}{x-2} - 1 = 0 \Rightarrow \frac{1}{x-2} = -1 \Rightarrow x-2 = -1, \text{ so } x=1, \text{ giving } (1, 0).$$

$$y\text{-intercept: } x=0 \Rightarrow y = -\frac{1}{-2} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}, \text{ giving } \left(0, -\frac{1}{2}\right).$$

$\therefore$ the branches are flipped to sit below/above the new asymptote $y = -1$.



Example 4 — unscaffolded

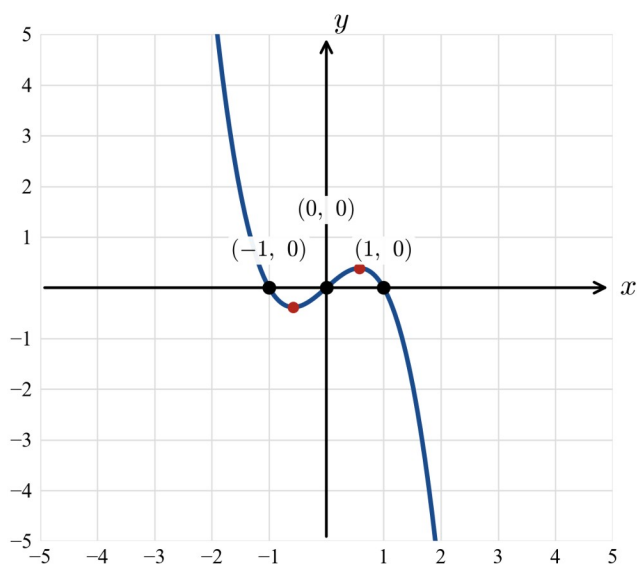
The graph of $y = x^3 - x$ is reflected in the y -axis. Find the rule of the image and sketch it. What do you notice?

$$\text{Image rule: replace } x \text{ with } -x: y = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x).$$

Since the image rule equals $-f(x)$, reflecting $y = x^3 - x$ in the y -axis gives the **same** result as reflecting it in the x -axis. This happens because $f(x) = x^3 - x$ is an **odd** function ($f(-x) = -f(x)$), so its graph has point symmetry about the origin.

$$x\text{-intercepts: } -x^3 + x = 0 \Rightarrow -x(x^2 - 1) = 0, \text{ so } x = 0, \pm 1: (-1, 0), (0, 0), (1, 0).$$

$\therefore$ the image passes through the same three intercepts, with the branches swapped end-for-end.



Practice questions

Scaffolded

Q1. The graph of $y = x^2 - 4$ is reflected in the x -axis. (a) Write the rule of the image; (b) state the coordinates and nature of the turning point; (c) find all axis intercepts; (d) sketch the graph.

Q2. The graph of $y = \sqrt{x} + 1$ is reflected in the y -axis. (a) Write the rule of the image; (b) state the domain and range; (c) write the coordinates of the endpoint; (d) sketch the graph.

Q3. The graph of $y = \frac{1}{x-1}$ is reflected in the x -axis. (a) Write the rule of the image; (b) state the equations of the asymptotes; (c) find any axis intercepts; (d) sketch the graph.

Q4. The graph of $y = (x+2)^2 - 1$ is reflected in the x -axis. (a) Write the rule of the image; (b) state the turning point; (c) find all axis intercepts; (d) sketch the graph.

Q5. The graph of $y = \frac{1}{x^2}$ is reflected in the x -axis. (a) Write the rule of the image; (b) state the asymptotes and the range; (c) sketch the graph.

Q6. The graph of $y = x^3$ is reflected in the y -axis. (a) Write the rule of the image; (b) explain why the image is the same as the reflection of $y = x^3$ in the x -axis; (c) sketch the graph.

Un scaffolded

For **Q7–Q19** and **Q21**, write the rule of the image, sketch it labelling the key features (with exact intercepts), and state the domain and range where relevant. For **Q20**, write the rules and state the domain and range of each image.

Q7. $y = (x-1)^2$, reflected in the x -axis.

Q8. $y = \sqrt{x-2}$, reflected in the x -axis.

Q9. $y = \sqrt{x+3}$, reflected in the y -axis.

Q10. $y = \frac{1}{x-2} + 1$, reflected in the y -axis.

Q11. $y = \frac{1}{(x+1)^2}$, reflected in the x -axis.

Q12. $y = x^3 - 4x$, reflected in the x -axis.

Q13. $y = (x - 1)^3$, reflected in the y -axis.

Q14. $y = \sqrt{x} - 2$, reflected in the x -axis.

Q15. $y = \frac{2}{x - 1}$, reflected in the x -axis.

Q16. $y = x^2 - 2x$, reflected in the x -axis.

Q17. $y = \frac{1}{(x - 3)^2}$, reflected in the x -axis.

Q18. $y = \sqrt{x + 1} + 2$, reflected in the y -axis.

Q19. $y = x^3 + 1$, reflected in the x -axis.

Q20. For $f(x) = \sqrt{x}$, write the rules of the reflections of f in (i) the x -axis and (ii) the y -axis, and state the domain and range of each image.

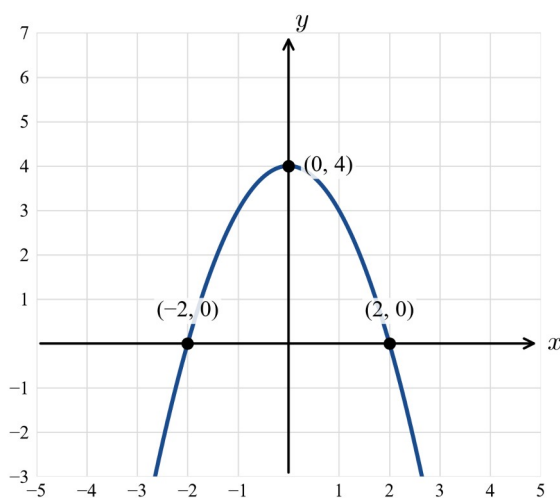
Q21. The graph of $y = \frac{1}{x - 2}$ is reflected in the y -axis. Find the rule of the image and the equations of its asymptotes, and hence sketch it.

Answers

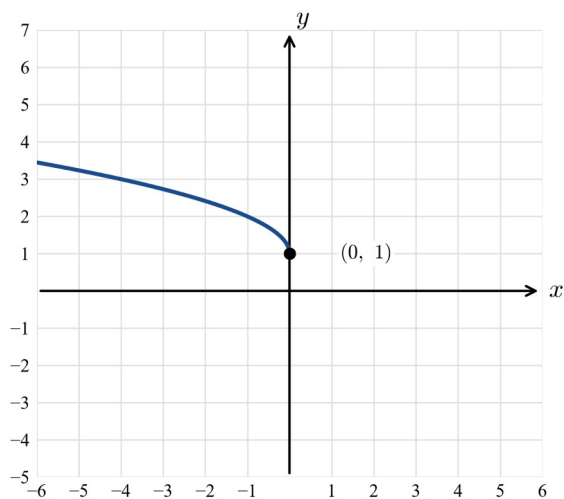
Q1. $y = 4 - x^2$; maximum TP $(0, 4)$; intercepts $(-2, 0), (2, 0), (0, 4)$. **Q2.** $y = \sqrt{-x} + 1$; domain $\{x : x \leq 0\}$, range $\{y : y \geq 1\}$; endpoint $(0, 1)$. **Q3.** $y = -\frac{1}{x-1}$; asymptotes $x = 1, y = 0$; intercept $(0, 1)$ (no x -intercept). **Q4.** $y = 1 - (x+2)^2$; maximum TP $(-2, 1)$; intercepts $(-3, 0), (-1, 0), (0, -3)$. **Q5.** $y = -\frac{1}{x^2}$; asymptotes $x = 0, y = 0$; range $\{y : y < 0\}$. **Q6.** $y = -x^3$; equal because x^3 is odd; passes through the origin as a stationary point of inflection. **Q7.** $y = -(x-1)^2$; maximum TP $(1, 0)$; intercepts $(1, 0), (0, -1)$. **Q8.** $y = -\sqrt{x-2}$; endpoint $(2, 0)$; domain $\{x : x \geq 2\}$, range $\{y : y \leq 0\}$; intercept $(2, 0)$. **Q9.** $y = \sqrt{3-x}$; endpoint $(3, 0)$; domain $\{x : x \leq 3\}$, range $\{y : y \geq 0\}$; intercepts $(3, 0), (0, \sqrt{3})$. **Q10.** $y = -\frac{1}{x+2} + 1$; asymptotes $x = -2, y = 1$; intercepts $(-1, 0), (0, \frac{1}{2})$. **Q11.** $y = -\frac{1}{(x+1)^2}$; asymptotes $x = -1, y = 0$; range $\{y : y < 0\}$; intercept $(0, -1)$ (no x -intercept). **Q12.** $y = -x^3 + 4x$; intercepts $(-2, 0), (0, 0), (2, 0)$. **Q13.** $y = -(x+1)^3$; stationary point of inflection $(-1, 0)$; intercepts $(-1, 0), (0, -1)$. **Q14.** $y = 2 - \sqrt{x}$; endpoint $(0, 2)$; domain $\{x : x \geq 0\}$, range $\{y : y \leq 2\}$; intercepts $(4, 0), (0, 2)$. **Q15.** $y = -\frac{2}{x-1}$; asymptotes $x = 1, y = 0$; intercept $(0, 2)$ (no x -intercept). **Q16.** $y = 2x - x^2$; maximum TP $(1, 1)$; intercepts $(0, 0), (2, 0)$. **Q17.** $y = -\frac{1}{(x-3)^2}$; asymptotes $x = 3, y = 0$; range $\{y : y < 0\}$; intercept $(0, -\frac{1}{9})$ (no x -intercept). **Q18.** $y = \sqrt{1-x} + 2$; endpoint $(1, 2)$; domain $\{x : x \leq 1\}$, range $\{y : y \geq 2\}$; intercept $(0, 3)$. **Q19.** $y = -x^3 - 1$; stationary point of inflection $(0, -1)$; intercepts $(-1, 0), (0, -1)$. **Q20.** (i) $y = -\sqrt{x}$, domain $\{x : x \geq 0\}$, range $\{y : y \leq 0\}$; (ii) $y = \sqrt{-x}$, domain $\{x : x \leq 0\}$, range $\{y : y \geq 0\}$. **Q21.** $y = -\frac{1}{x+2}$; asymptotes $x = -2, y = 0$; intercept $(0, -\frac{1}{2})$ (no x -intercept).

Fully-worked solutions

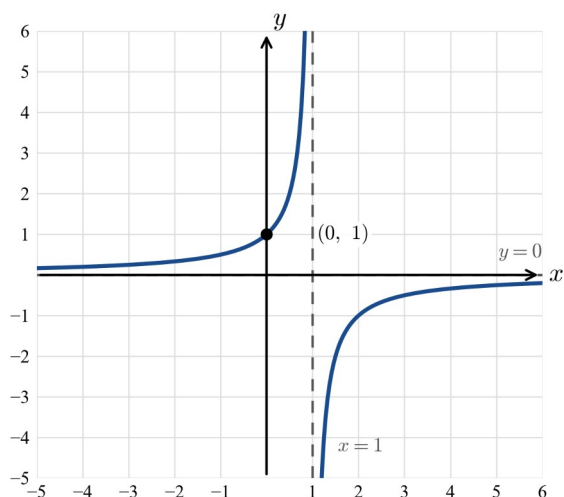
Q1. Image $y = -(x^2 - 4) = 4 - x^2$. Maximum TP $(0, 4)$. x -intercepts: $4 - x^2 = 0 \Rightarrow x = \pm 2$, giving $(-2, 0), (2, 0)$. y -intercept $(0, 4)$.



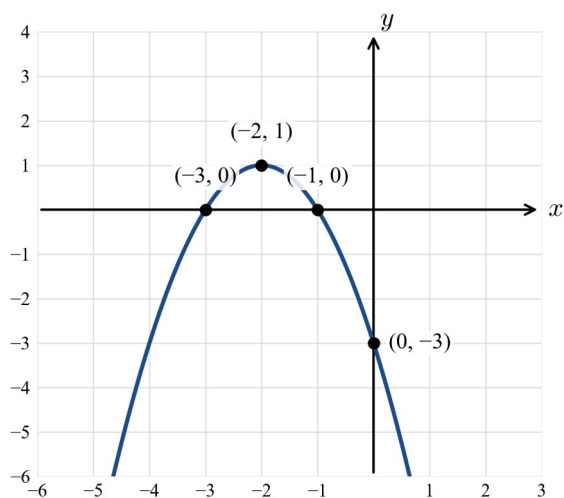
Q2. Image $y = \sqrt{-x} + 1$. Endpoint $(0, 1)$; domain $\{x : x \leq 0\}$; since $\sqrt{-x} \geq 0$, range $\{y : y \geq 1\}$. y -intercept $x = 0 \Rightarrow y = 1$, giving $(0, 1)$; no x -intercept ($y \geq 1$).



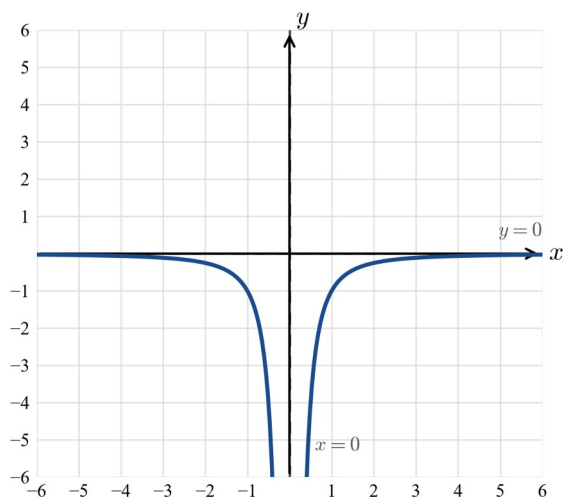
Q3. Image $y = -\frac{1}{x-1}$. Asymptotes $x=1$, $y=0$. No x -intercept ($y \neq 0$). y -intercept: $x=0 \Rightarrow y = -\frac{1}{-1} = 1$, giving $(0, 1)$.



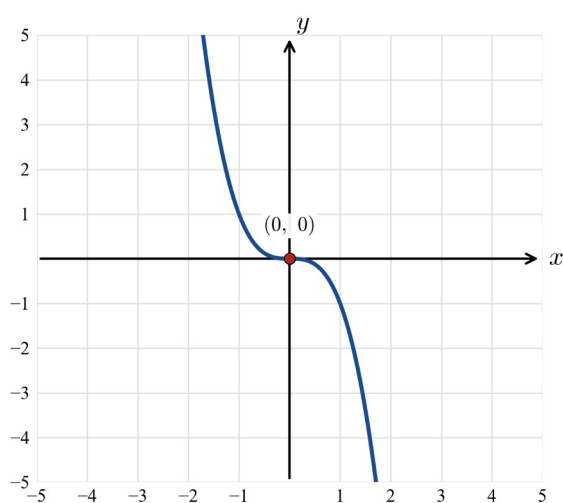
Q4. Image $y = -((x+2)^2 - 1) = 1 - (x+2)^2$. Maximum TP $(-2, 1)$. x -intercepts: $(x+2)^2 = 1 \Rightarrow x = -1$ or $x = -3$, giving $(-1, 0), (-3, 0)$. y -intercept: $x=0 \Rightarrow y = 1 - 4 = -3$, giving $(0, -3)$.



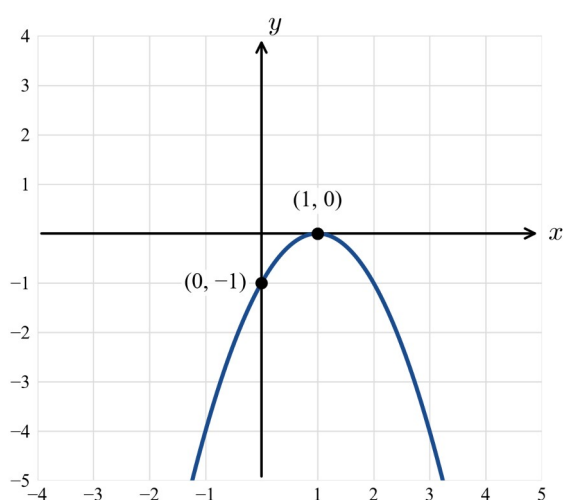
Q5. Image $y = -\frac{1}{x^2}$. Asymptotes $x=0$, $y=0$; since $-\frac{1}{x^2} < 0$, range $\{y : y < 0\}$; both branches lie below the x -axis. No intercepts.



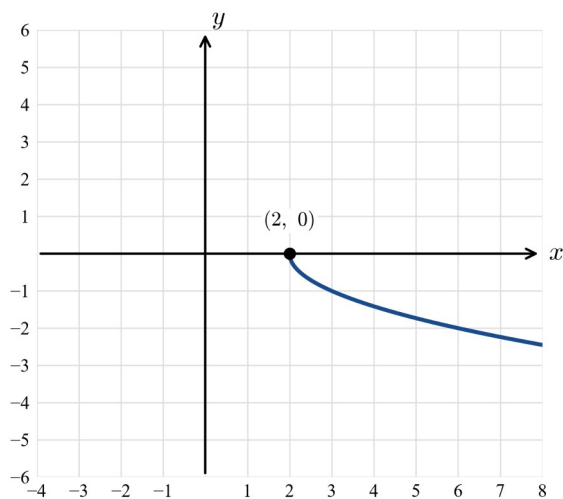
Q6. Image $y = (-x)^3 = -x^3$. Because x^3 is odd, $f(-x) = -f(x)$, so reflecting in the y -axis gives $y = -x^3$, the same as reflecting in the x -axis. The graph has a stationary point of inflection at the origin and is decreasing.



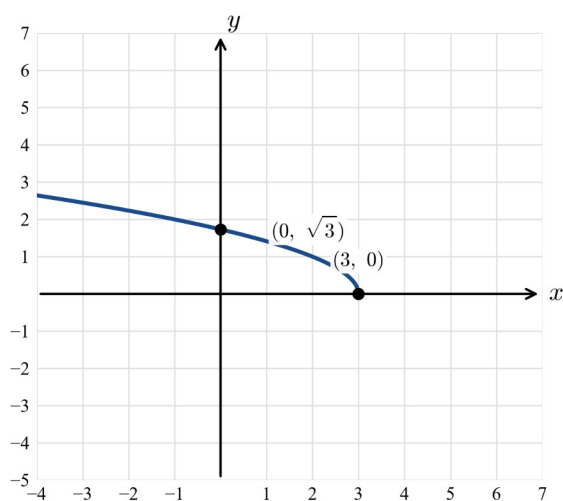
Q7. Image $y = -(x-1)^2$. Maximum TP $(1, 0)$. y -intercept: $x=0 \Rightarrow y=-1$, giving $(0, -1)$. Only x -intercept $(1, 0)$.



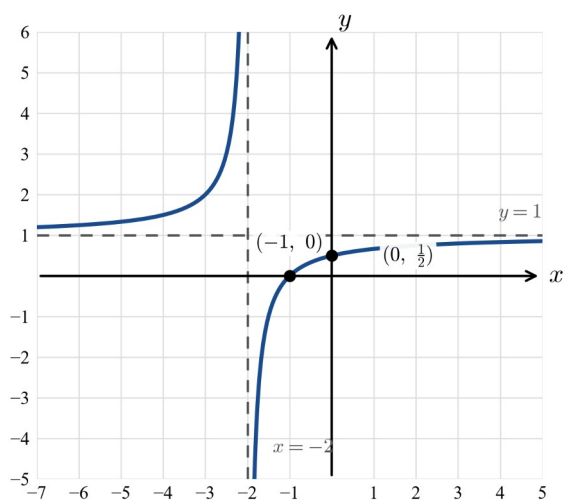
Q8. Image $y = -\sqrt{x-2}$. Endpoint $(2, 0)$; domain $\{x : x \geq 2\}$; since the negative root is ≤ 0 , range $\{y : y \leq 0\}$. Only intercept $(2, 0)$; no y -intercept.



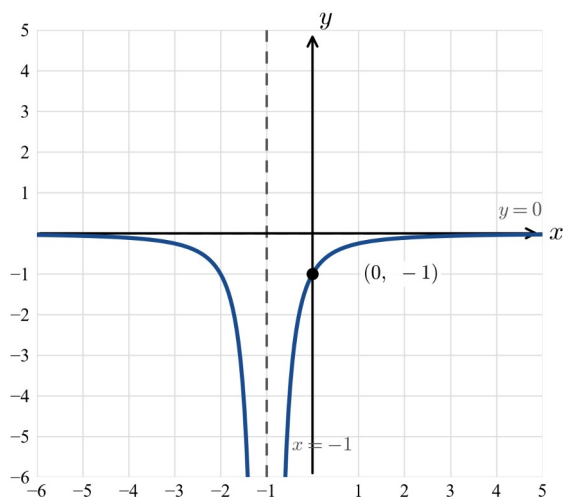
Q9. Image $y = \sqrt{-x+3} = \sqrt{3-x}$. Endpoint $(3, 0)$; domain $\{x : x \leq 3\}$, range $\{y : y \geq 0\}$. y-intercept: $x=0 \Rightarrow y = \sqrt{3}$, giving $(0, \sqrt{3})$. x-intercept $(3, 0)$.



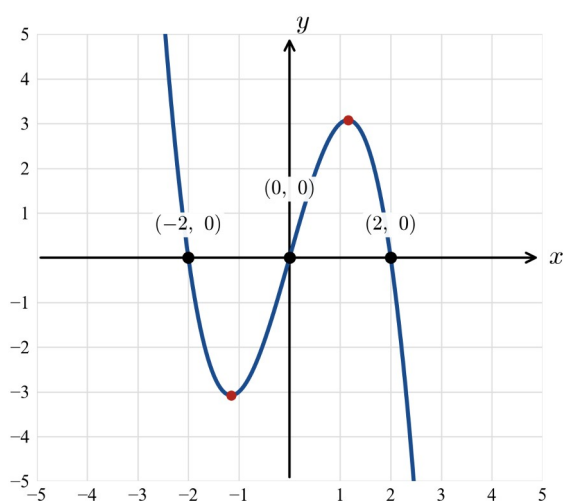
Q10. Image (replace x by $-x$): $y = \frac{1}{-x-2} + 1 = -\frac{1}{x+2} + 1$. Asymptotes $x = -2$, $y = 1$. x-intercept: $\frac{1}{x+2} = 1 \Rightarrow x+2=1 \Rightarrow x = -1$, giving $(-1, 0)$. y-intercept: $x=0 \Rightarrow y = -\frac{1}{2} + 1 = \frac{1}{2}$, giving $(0, \frac{1}{2})$.



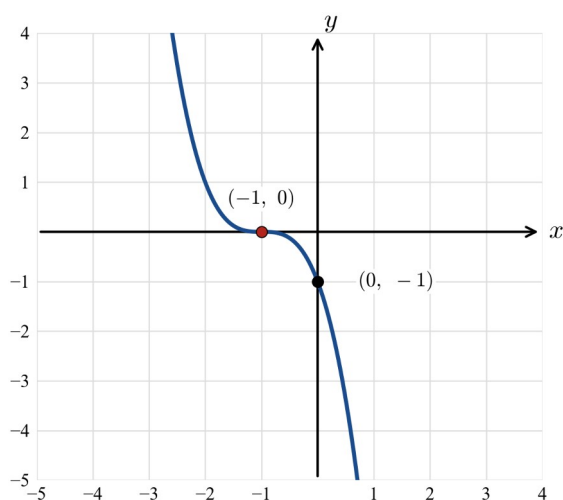
Q11. Image $y = -\frac{1}{(x+1)^2}$. Asymptotes $x = -1$, $y = 0$; range $\{y : y < 0\}$ (both branches below the axis). y-intercept: $x=0 \Rightarrow y = -1$, giving $(0, -1)$; no x-intercept.



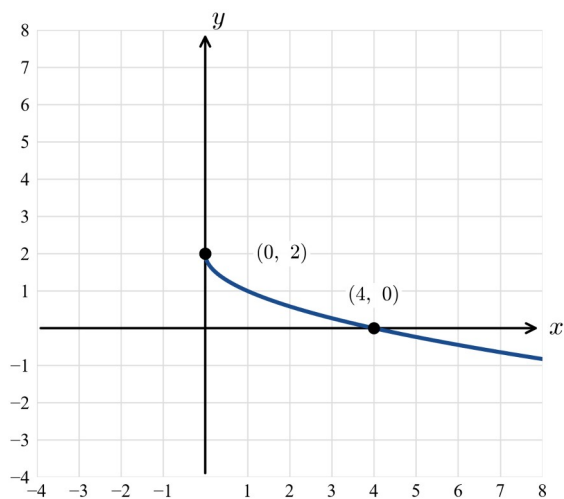
Q12. Image $y = -(x^3 - 4x) = -x^3 + 4x$. x -intercepts: $-x(x^2 - 4) = 0 \Rightarrow x = 0, \pm 2$, giving $(-2, 0), (0, 0), (2, 0)$. y -intercept $(0, 0)$. The original graph is flipped top-to-bottom.



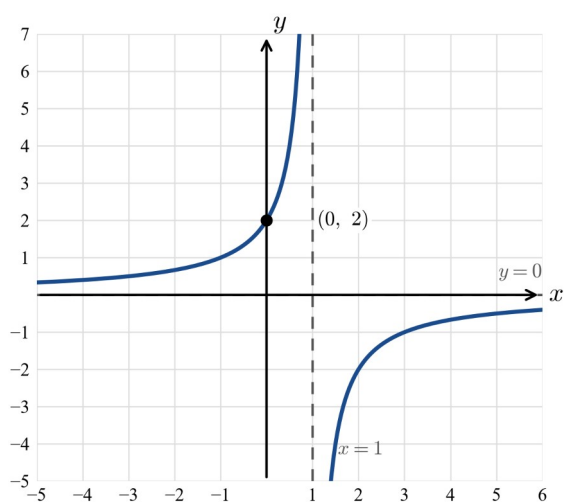
Q13. Image $y = (-x - 1)^3 = -(x + 1)^3$. Stationary point of inflection at $(-1, 0)$. y -intercept: $x = 0 \Rightarrow y = -(1)^3 = -1$, giving $(0, -1)$. Only x -intercept $(-1, 0)$.



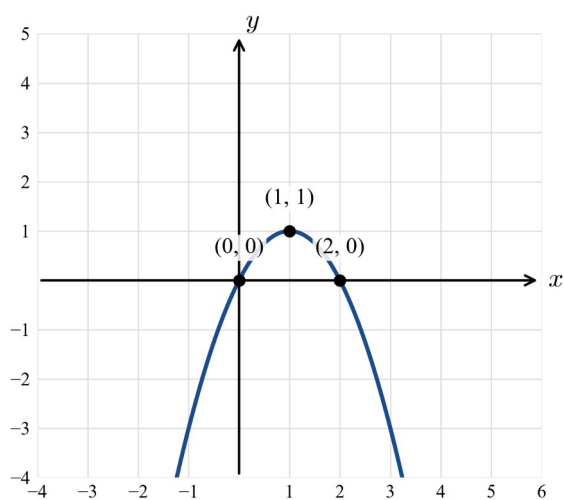
Q14. Image $y = -(\sqrt{x} - 2) = 2 - \sqrt{x}$. Endpoint $(0, 2)$; domain $\{x : x \geq 0\}$, range $\{y : y \leq 2\}$. x -intercept: $\sqrt{x} = 2 \Rightarrow x = 4$, giving $(4, 0)$. y -intercept $(0, 2)$.



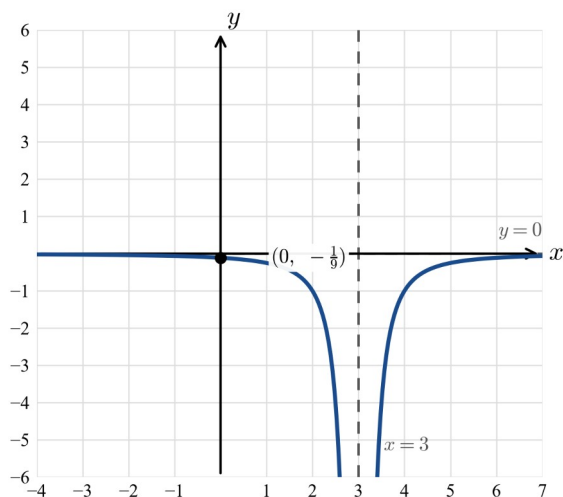
Q15. Image $y = -\frac{2}{x-1}$. Asymptotes $x=1$, $y=0$. No x -intercept. y -intercept: $x=0 \Rightarrow y = -\frac{2}{-1} = 2$, giving $(0, 2)$.



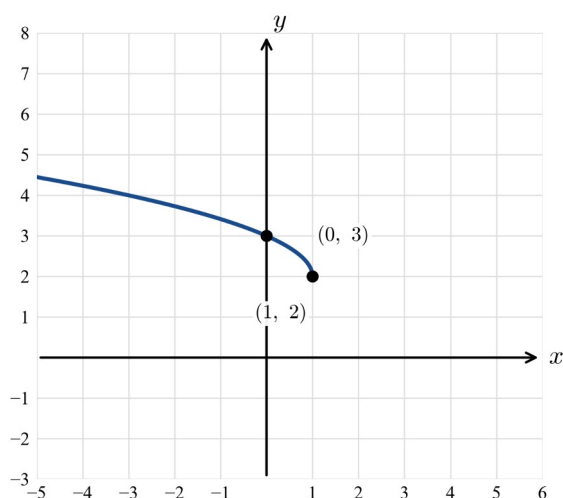
Q16. Image $y = -(x^2 - 2x) = 2x - x^2$. Maximum TP: axis $x=1$, $y(1) = 2 - 1 = 1$, giving $(1, 1)$. x -intercepts: $x(2-x) = 0 \Rightarrow x=0, 2$, giving $(0, 0), (2, 0)$.



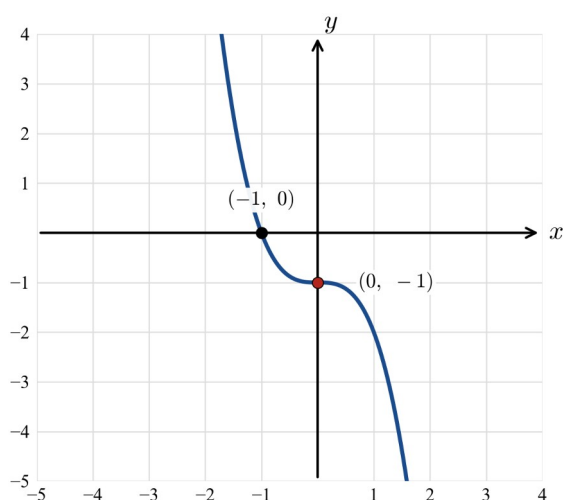
Q17. Image $y = -\frac{1}{(x-3)^2}$. Asymptotes $x=3$, $y=0$; range $\{y : y < 0\}$. y -intercept: $x=0 \Rightarrow y = -\frac{1}{9}$, giving $(0, -\frac{1}{9})$; no x -intercept.



Q18. Image $y = \sqrt{-x+1} + 2 = \sqrt{1-x} + 2$. Endpoint $(1, 2)$; domain $\{x : x \leq 1\}$, range $\{y : y \geq 2\}$. No x -intercept ($y \geq 2$). y -intercept: $x=0 \Rightarrow y = \sqrt{1} + 2 = 3$, giving $(0, 3)$.



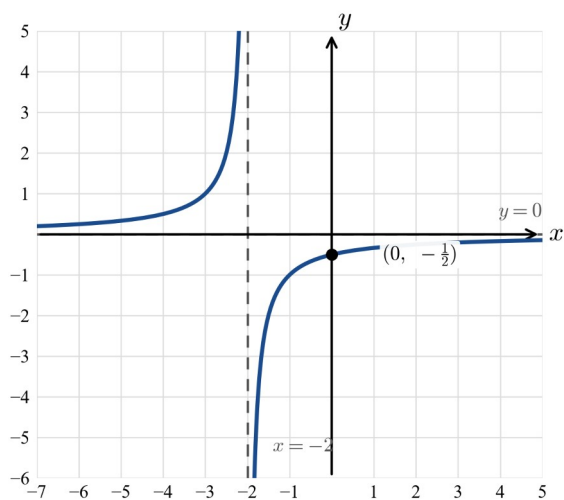
Q19. Image $y = -(x^3 + 1) = -x^3 - 1$. Stationary point of inflection at $(0, -1)$. x -intercept: $-x^3 - 1 = 0 \Rightarrow x^3 = -1 \Rightarrow x = -1$, giving $(-1, 0)$. y -intercept $(0, -1)$.



Q20. (i) Reflection in the x -axis: $y = -\sqrt{x}$, domain $\{x : x \geq 0\}$, range $\{y : y \leq 0\}$. (ii) Reflection in the y -axis: $y = \sqrt{-x}$, domain $\{x : x \leq 0\}$, range $\{y : y \geq 0\}$.

Q21. Image (replace x by $-x$): $y = \frac{1}{-x-2} = -\frac{1}{x+2}$. Asymptotes $x = -2$, $y = 0$. No x -intercept. y -intercept:

$$x=0 \Rightarrow y = -\frac{1}{2}, \text{ giving } \left(0, -\frac{1}{2}\right).$$



Chapter 3 Transformations of the plane — 3D Combinations of transformations

Theory

Sections 3A–3C dealt with translations, dilations and reflections one at a time. This section **combines** them. Every combination of a dilation, a reflection and a translation applied to a graph $y = f(x)$ can be written in the single form

$$y = af(b(x - h)) + k.$$

What each constant does (starting from $y = f(x)$):

- a — a **dilation of factor $|a|$ from the x -axis** (parallel to the y -axis), together with a **reflection in the x -axis** when $a < 0$. It multiplies every y -coordinate by a .
- b — a **dilation of factor $\frac{1}{|b|}$ from the y -axis** (parallel to the x -axis), together with a **reflection in the y -axis** when $b < 0$.
- h — a **translation of h units in the positive x -direction** (right).
- k — a **translation of k units in the positive y -direction** (up).

Order matters. Apply the **dilations and reflections first, then the translations**. Doing the translation first gives a different (wrong) graph, because a dilation also moves an already-translated point. Describing *both* transformations, in the right order and about the correct axis, is where marks are most often lost.

Effect on a point. Under $y = af(b(x - h)) + k$, a point (x, y) on $y = f(x)$ maps to

$$\left(\frac{x}{b} + h, ay + k\right).$$

So $y = f(x - 3)$ moves a point 3 right; $y = f(2x)$ halves each x -coordinate; $y = 2f(x)$ doubles each y -coordinate; $y = -f(x)$ reflects each point in the x -axis.

Effect on key features. Asymptotes, endpoints and turning points move with the graph: a vertical asymptote $x = 0$ becomes $x = h$; a horizontal asymptote $y = 0$ becomes $y = k$; the turning point or endpoint at $(0, 0)$ becomes (h, k) (for $b > 0$). Intercepts are then found from the image rule.

To find an image and sketch it: (1) read off a, b, h, k ; (2) name the transformations in order; (3) write the image rule; (4) find the new asymptotes/endpoint/turning point and the **exact** axis intercepts; (5) sketch.

Worked examples

Example 1 — scaffolded

The graph of $y = x^2$ is dilated by a factor of 2 from the x -axis, then translated 1 unit to the right and 3 units down. Find the rule of the image and sketch it, labelling the turning point and the exact axis intercepts.

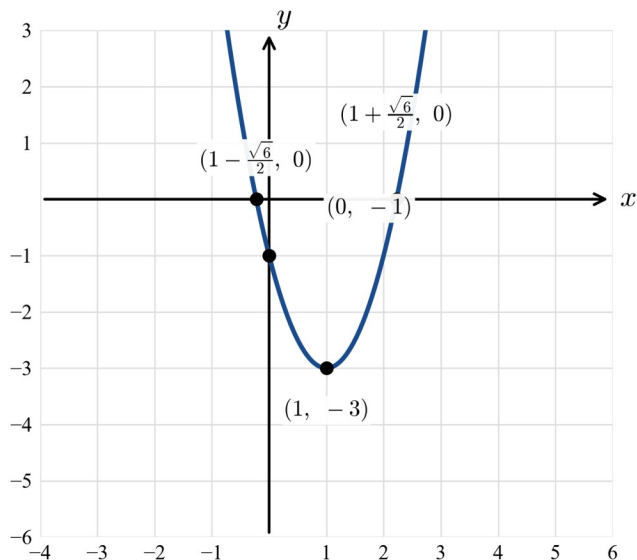
Working	Comment
Dilation factor 2 from the x -axis: $y = 2x^2$.	The dilation multiplies y by 2; apply it before the translation.
Then translate 1 right and 3 down: replace x with $x - 1$ and subtract 3: $y = 2(x - 1)^2 - 3$.	$h = 1, k = -3$.
Turning point $(1, -3)$ (a minimum, since $a = 2 > 0$).	The image of $(0, 0)$ is (h, k) .
y-intercept: $x = 0 \Rightarrow y = 2(0 - 1)^2 - 3 = 2 - 3 = -1$, giving $(0, -1)$.	Substitute $x = 0$.

x-intercepts:

$$2(x-1)^2 - 3 = 0 \Rightarrow (x-1)^2 = \frac{3}{2} \Rightarrow x-1 = \pm \frac{\sqrt{6}}{2}, \text{ so}$$

$$x = 1 \pm \frac{\sqrt{6}}{2}.$$

Keep the answer exact.

**Example 2 — scaffolded**

The graph of $y = \frac{1}{x^2}$ is reflected in the x -axis, dilated by a factor of 3 from the x -axis, then translated 2 units right and 1 unit up. Find the image rule and sketch it, stating the asymptotes, range, and exact intercepts.

Reflect in the x -axis and dilate by factor 3 from the x -axis: $y = -\frac{3}{x^2}$.

$a = -3$: reflection (negative) and dilation (magnitude 3) together.

Translate 2 right and 1 up: $y = -\frac{3}{(x-2)^2} + 1$.

$h = 2, k = 1$.

Asymptotes: $x = 2$ and $y = 1$; since $a < 0$, **range** $\{y : y < 1\}$.

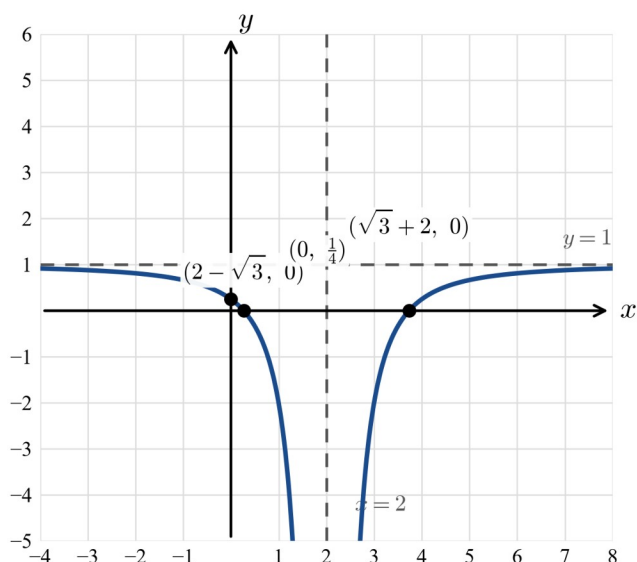
Vertical asymptote $x = h$; horizontal asymptote $y = k$.

y-intercept: $x = 0 \Rightarrow y = -\frac{3}{4} + 1 = \frac{1}{4}$, giving $\left(0, \frac{1}{4}\right)$.

Substitute $x = 0$.

x-intercepts: $\frac{3}{(x-2)^2} = 1 \Rightarrow (x-2)^2 = 3 \Rightarrow x = 2 \pm \sqrt{3}$.

Two intercepts.



Example 3 — unscaffolded

The point $(-2, 4)$ lies on the graph of $y = f(x)$. State the coordinates of the image of this point on each of the following graphs.

Using $(x, y) \rightarrow \left(\frac{x}{b} + h, a y + k\right)$:

(a) $y = f(x - 3) + 1$: here $h = 3, k = 1$, so $(-2, 4) \rightarrow (-2 + 3, 4 + 1) = (1, 5)$.

(b) $y = 2f(x)$: here $a = 2$, so $(-2, 4) \rightarrow (-2, 8)$.

(c) $y = f(2x)$: here $b = 2$, so $(-2, 4) \rightarrow (-1, 4)$.

(d) $y = -f(x) + 5$: here $a = -1, k = 5$, so $(-2, 4) \rightarrow (-2, -4 + 5) = (-2, 1)$.

$\therefore$ the images are $(1, 5)$, $(-2, 8)$, $(-1, 4)$ and $(-2, 1)$.

Example 4 — unscaffolded

Describe a sequence of transformations that maps $y = \frac{1}{x}$ onto $y = \frac{2}{x+3} - 1$, sketch the image, and state the image of the point $(1, 1)$ (which lies on $y = \frac{1}{x}$).

Writing the rule as $y = \frac{2}{x - (-3)} + (-1)$ gives $a = 2, h = -3, k = -1$. From $y = \frac{1}{x}$:

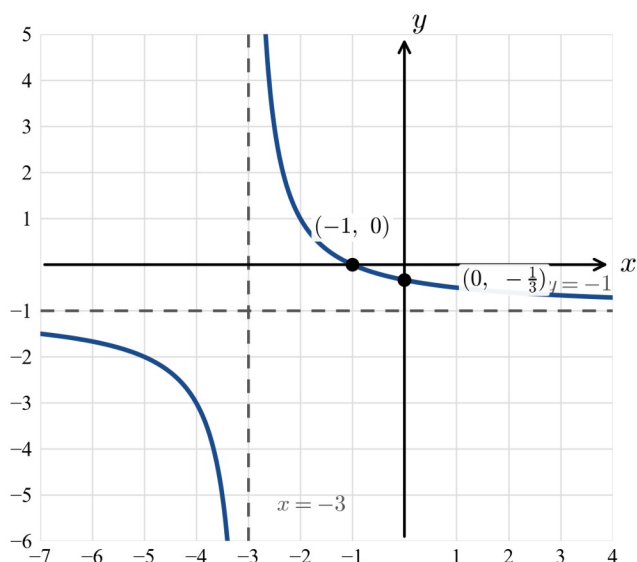
1. a **dilation of factor 2 from the x -axis**;
2. a **translation of 3 units left and 1 unit down**.

Asymptotes $x = -3, y = -1$. x -intercept: $\frac{2}{x+3} = 1 \Rightarrow x+3=2 \Rightarrow x=-1$, giving $(-1, 0)$. y -intercept:

$x=0 \Rightarrow y = \frac{2}{3} - 1 = -\frac{1}{3}$, giving $\left(0, -\frac{1}{3}\right)$.

Image of $(1, 1)$: $\left(\frac{1}{1} + (-3), 2(1) + (-1)\right) = (-2, 1)$.

$\therefore$ the image passes through $(-2, 1)$, with asymptotes $x = -3$ and $y = -1$.



Practice questions

Scaffolded

- Q1.** The graph of $y = x^2$ is dilated by a factor of 3 from the x -axis, then translated 2 units left and 1 unit down. (a) Write the image rule; (b) state the turning point; (c) find the exact axis intercepts; (d) sketch the graph.
- Q2.** Consider $y = \sqrt{x}$ reflected in the y -axis, then translated 4 units right, giving $y = \sqrt{4 - x}$. (a) State the endpoint and the domain; (b) find the axis intercepts; (c) sketch the graph.
- Q3.** The graph of $y = \frac{1}{x}$ is reflected in the x -axis, then translated 2 units right and 3 units up. (a) Write the image rule and state the asymptotes; (b) find the exact axis intercepts; (c) sketch the graph.
- Q4.** The point $(3, -2)$ lies on $y = f(x)$. State the image of this point on: (a) $y = f(x + 1)$; (b) $y = -2f(x)$; (c) $y = f(x) - 4$; (d) $y = f(3x)$.
- Q5.** Describe the sequence of transformations that maps $y = x^2$ onto $y = -(x - 1)^2 + 4$, then (a) state the turning point and (b) find the axis intercepts and (c) sketch the graph.
- Q6.** The graph of $y = \frac{1}{x^2}$ is dilated by a factor of 2 from the x -axis, then translated 1 unit left and 2 units down. (a) Write the image rule and state the asymptotes and range; (b) find the axis intercepts; (c) sketch the graph.

Un scaffolded

For **Q7–Q11** and **Q15, Q17–Q20**: write the image rule (where a sequence is described), find the exact axis intercepts and the turning point / asymptotes / endpoint, and sketch. For the point/description questions, give exact coordinates or a full description in the correct order.

- Q7.** $y = x^2$ is reflected in the x -axis, then translated 2 right and 5 up. Sketch the image.
- Q8.** $y = \frac{1}{x}$ is dilated by factor 4 from the x -axis, then translated 1 left and 2 down.
- Q9.** $y = \frac{1}{x^2}$ is reflected in the x -axis, then translated 3 left and 4 up.
- Q10.** $y = \sqrt{x}$ is dilated by factor 2 from the x -axis, reflected in the x -axis, then translated 1 right and 3 up.
- Q11.** $y = x^3$ is translated 1 right and 2 down.

Q12. The point $(-1, 3)$ lies on $y = f(x)$. State its image on (a) $y = f(x - 2) + 3$; (b) $y = 3f(x)$; (c) $y = f(-x)$; (d) $y = -f(x) - 1$.

Q13. Describe the sequence of transformations that maps $y = \frac{1}{x}$ onto $y = -\frac{2}{x-1} + 3$, and state the equations of the asymptotes of the image.

Q14. Describe the sequence of transformations that maps $y = \sqrt{x}$ onto $y = \sqrt{2-x} + 1$, and state the endpoint and domain of the image.

Q15. $y = x^2$ is dilated by a factor of $\frac{1}{2}$ from the y -axis, then translated 3 up. (Recall a dilation of factor $\frac{1}{2}$ from the y -axis replaces x with $2x$.)

Q16. The point $(4, 7)$ lies on $y = f(x)$. State its image on (a) $y = f\left(\frac{x}{2}\right)$; (b) $y = f(2x)$; (c) $y = f(x) + 3$; (d) $y = -f(x)$.

Q17. $y = \frac{1}{x}$ is reflected in the y -axis, then translated 2 up.

Q18. $y = \frac{1}{x^2}$ is dilated by factor 3 from the x -axis, then translated 2 right and 1 down.

Q19. $y = \sqrt{x}$ is dilated by a factor of $\frac{1}{2}$ from the y -axis, then translated 1 down.

Q20. The graph of $y = x^2$ is mapped to $y = 2(x + 3)^2 - 5$. Describe the sequence of transformations in the correct order, sketch the image, and state the image of the turning point $(0, 0)$.

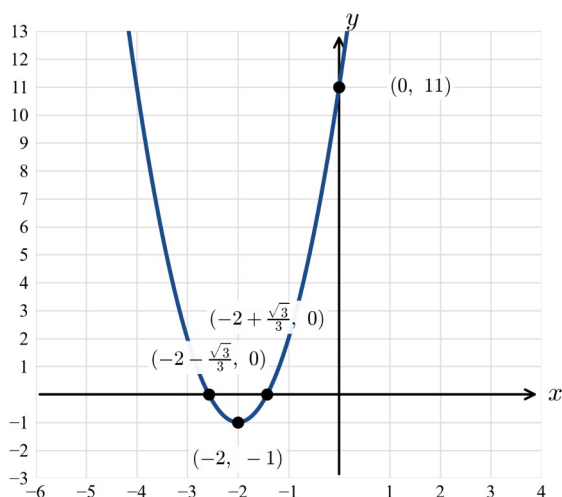
Q21. The graph of $y = f(x)$ has a local maximum at $(2, 5)$. State the coordinates of the corresponding point on (a) $y = 3f(x - 1) - 4$ and (b) $y = f(2x)$.

Answers

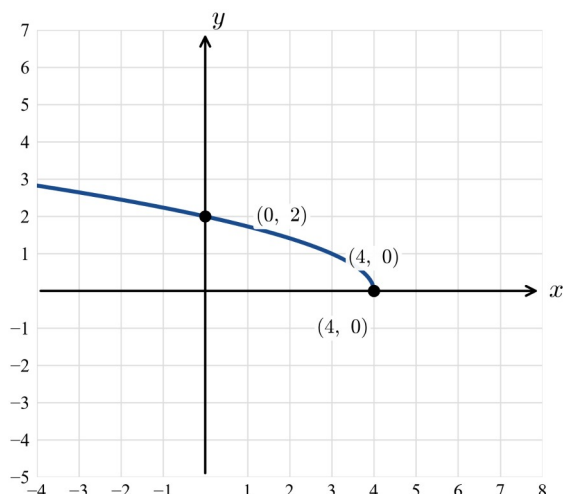
Q1. $y = 3(x+2)^2 - 1$; TP $(-2, -1)$; intercepts $\left(-2 \pm \frac{\sqrt{3}}{3}, 0\right)$, $(0, 11)$. **Q2.** Endpoint $(4, 0)$, domain $\{x : x \leq 4\}$; intercepts $(4, 0)$ and $(0, 2)$. **Q3.** $y = -\frac{1}{x-2} + 3$; asymptotes $x = 2$, $y = 3$; intercepts $\left(\frac{7}{3}, 0\right)$, $\left(0, \frac{7}{2}\right)$. **Q4.** (a) $(2, -2)$; (b) $(3, 4)$; (c) $(3, -6)$; (d) $(1, -2)$. **Q5.** Reflection in the x -axis, then translation 1 right and 4 up; TP $(1, 4)$; intercepts $(-1, 0)$, $(3, 0)$, $(0, 3)$. **Q6.** $y = \frac{2}{(x+1)^2} - 2$; asymptotes $x = -1$, $y = -2$, range $\{y : y > -2\}$; intercepts $(-2, 0)$, $(0, 0)$. **Q7.** $y = -(x-2)^2 + 5$; TP $(2, 5)$; intercepts $(2 \pm \sqrt{5}, 0)$, $(0, 1)$. **Q8.** $y = \frac{4}{x+1} - 2$; asymptotes $x = -1$, $y = -2$; intercepts $(1, 0)$, $(0, 2)$. **Q9.** $y = -\frac{1}{(x+3)^2} + 4$; asymptotes $x = -3$, $y = 4$; intercepts $\left(-\frac{7}{2}, 0\right)$, $\left(-\frac{5}{2}, 0\right)$, $\left(0, \frac{35}{9}\right)$. **Q10.** $y = -2\sqrt{x-1} + 3$; endpoint $(1, 3)$, domain $\{x : x \geq 1\}$, range $\{y : y \leq 3\}$; intercept $\left(\frac{13}{4}, 0\right)$. **Q11.** $y = (x-1)^3 - 2$; stationary point of inflection $(1, -2)$; intercepts $(1 + \sqrt[3]{2}, 0)$, $(0, -3)$. **Q12.** (a) $(1, 6)$; (b) $(-1, 9)$; (c) $(1, 3)$; (d) $(-1, -4)$. **Q13.** Dilation of factor 2 from the x -axis, reflection in the x -axis, then translation 1 right and 3 up; asymptotes $x = 1$, $y = 3$. **Q14.** Reflection in the y -axis, then translation 2 right and 1 up; endpoint $(2, 1)$, domain $\{x : x \leq 2\}$. **Q15.** $y = 4x^2 + 3$; TP $(0, 3)$; no x -intercepts; $(0, 3)$. **Q16.** (a) $(8, 7)$; (b) $(2, 7)$; (c) $(4, 10)$; (d) $(4, -7)$. **Q17.** $y = -\frac{1}{x} + 2$; asymptotes $x = 0$, $y = 2$; intercept $\left(\frac{1}{2}, 0\right)$ (no y -intercept). **Q18.** $y = \frac{3}{(x-2)^2} - 1$; asymptotes $x = 2$, $y = -1$; intercepts $(2 \pm \sqrt{3}, 0)$, $\left(0, -\frac{1}{4}\right)$. **Q19.** $y = \sqrt{2x} - 1$; endpoint $(0, -1)$, domain $\{x : x \geq 0\}$; intercepts $\left(\frac{1}{2}, 0\right)$, $(0, -1)$. **Q20.** Dilation of factor 2 from the x -axis, then translation 3 left and 5 down; TP $(-3, -5)$; intercepts $\left(-3 \pm \frac{\sqrt{10}}{2}, 0\right)$, $(0, 13)$; image of $(0, 0)$ is $(-3, -5)$. **Q21.** (a) $(3, 11)$; (b) $(1, 5)$.

Fully-worked solutions

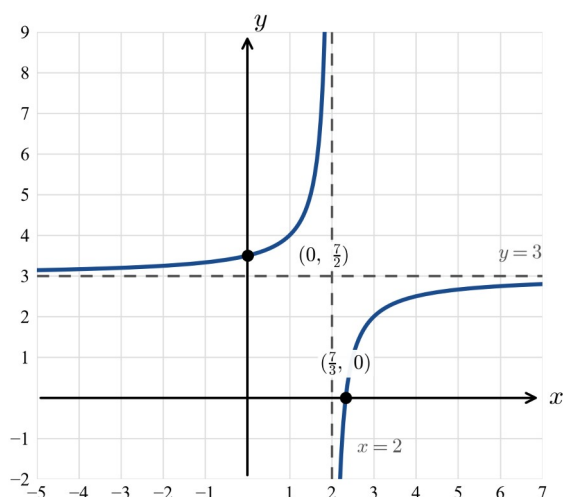
Q1. Dilation factor 3: $y = 3x^2$; translate 2 left, 1 down: $y = 3(x+2)^2 - 1$. TP $(-2, -1)$. y -intercept $x = 0$: $y = 3(4) - 1 = 11$, so $(0, 11)$. x -intercepts: $3(x+2)^2 = 1 \Rightarrow (x+2)^2 = \frac{1}{3} \Rightarrow x+2 = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}$, so $x = -2 \pm \frac{\sqrt{3}}{3}$.



Q2. $y = \sqrt{4-x} = \sqrt{-(x-4)}$. The inside is $4-x$, so the domain is $\{x : x \leq 4\}$ with endpoint $(4, 0)$. x -intercept $(4, 0)$; y -intercept $x=0$: $y = \sqrt{4} = 2$, so $(0, 2)$.

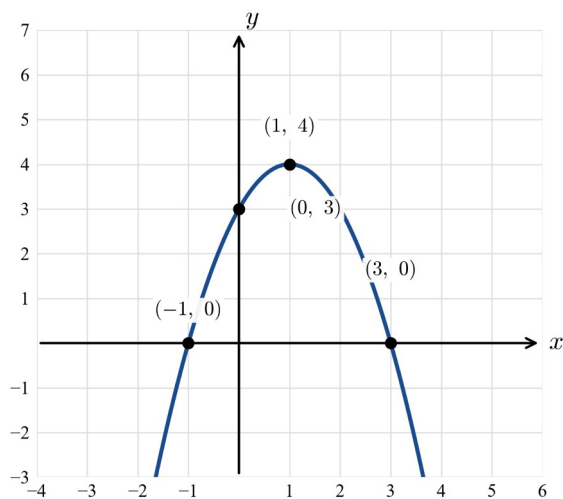


Q3. Reflection in the x -axis: $y = -\frac{1}{x}$; translate 2 right, 3 up: $y = -\frac{1}{x-2} + 3$. Asymptotes $x=2$, $y=3$. x -intercept: $\frac{1}{x-2} = 3 \Rightarrow x-2 = \frac{1}{3} \Rightarrow x = \frac{7}{3}$, so $(\frac{7}{3}, 0)$. y -intercept $x=0$: $y = -\frac{1}{-2} + 3 = \frac{7}{2}$, so $(0, \frac{7}{2})$.

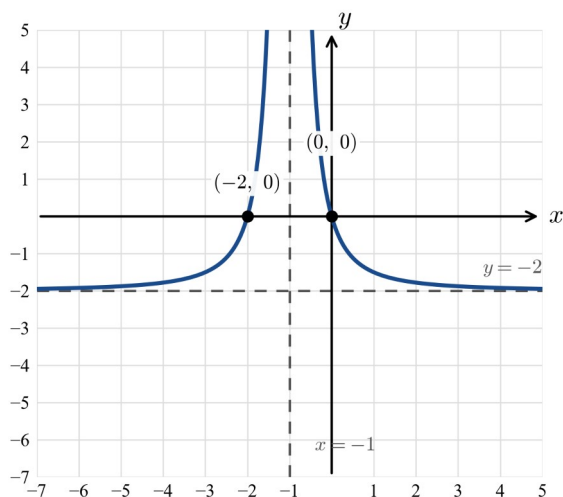


Q4. $(x, y) \rightarrow (\frac{x}{b} + h, ay + k)$. (a) $h = -1$: $(3-1, -2) = (2, -2)$. (b) $a = -2$: $(3, 4)$. (c) $k = -4$: $(3, -6)$. (d) $b = 3$: $(\frac{3}{3}, -2) = (1, -2)$.

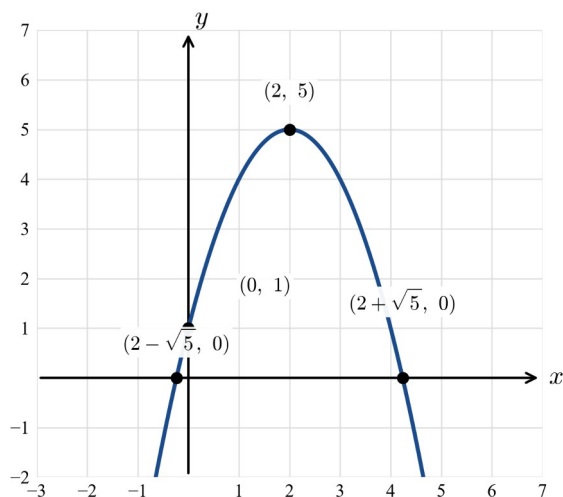
Q5. $y = -(x-1)^2 + 4$: $a = -1$ (reflection in the x -axis), $h = 1$, $k = 4$ (translate 1 right, 4 up). TP $(1, 4)$, a maximum. y -intercept $x=0$: $y = -1 + 4 = 3$, so $(0, 3)$. x -intercepts: $(x-1)^2 = 4 \Rightarrow x = -1$ or $x = 3$.



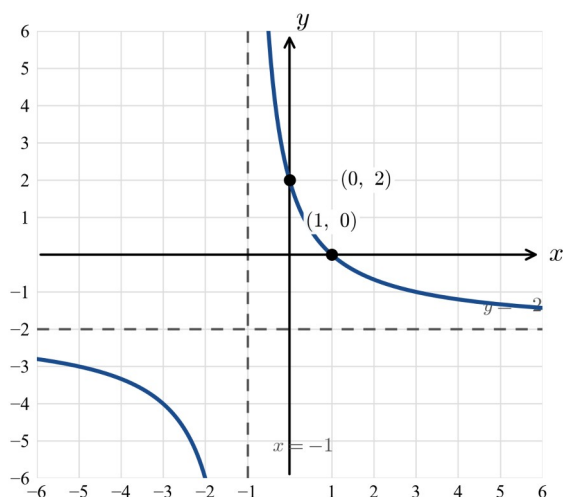
Q6. Dilation factor 2: $y = \frac{2}{x^2}$; translate 1 left, 2 down: $y = \frac{2}{(x+1)^2} - 2$. Asymptotes $x = -1$, $y = -2$; since $a = 2 > 0$, range $\{y : y > -2\}$. y-intercept $x = 0$: $y = 2 - 2 = 0$, so $(0, 0)$. x-intercepts: $\frac{2}{(x+1)^2} = 2 \Rightarrow (x+1)^2 = 1 \Rightarrow x = 0$ or $x = -2$.



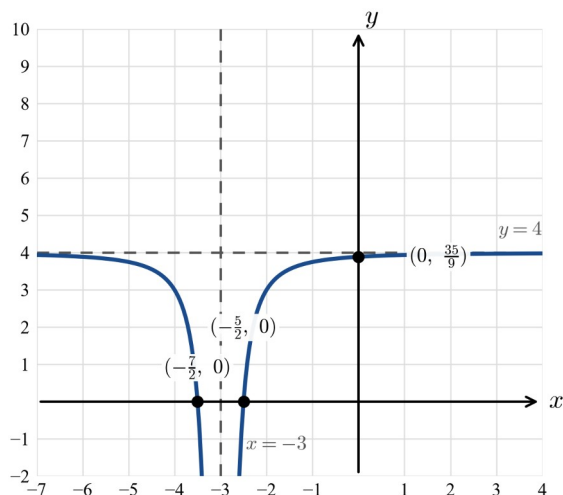
Q7. $y = -(x-2)^2 + 5$. TP $(2, 5)$ (maximum). y-intercept $x = 0$: $y = -4 + 5 = 1$, so $(0, 1)$. x-intercepts: $(x-2)^2 = 5 \Rightarrow x = 2 \pm \sqrt{5}$.



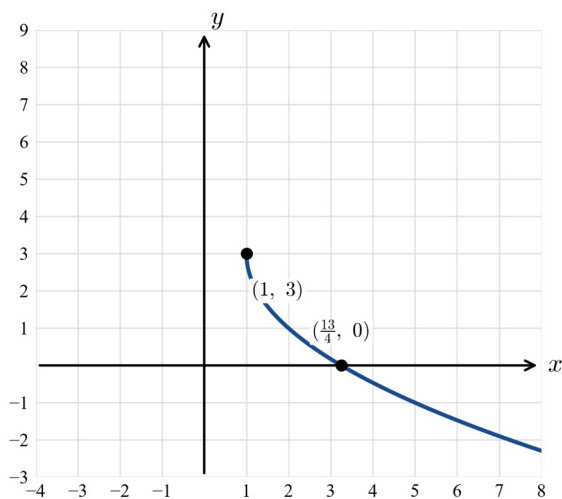
Q8. $y = \frac{4}{x+1} - 2$. Asymptotes $x = -1$, $y = -2$. x -intercept: $\frac{4}{x+1} = 2 \Rightarrow x+1=2 \Rightarrow x=1$, so $(1, 0)$. y -intercept $x=0$: $y = 4 - 2 = 2$, so $(0, 2)$.



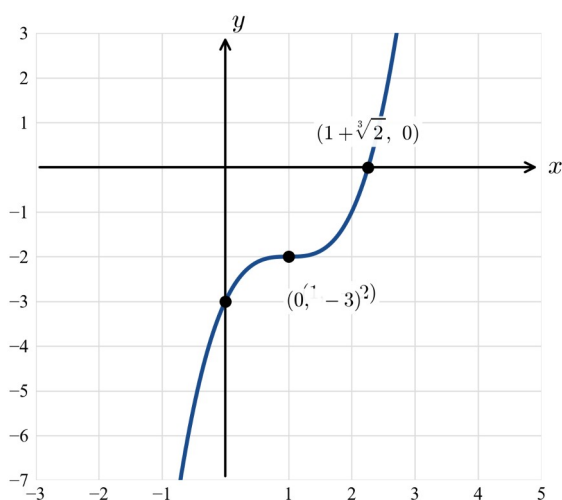
Q9. $y = -\frac{1}{(x+3)^2} + 4$. Asymptotes $x = -3$, $y = 4$; range $\{y : y < 4\}$. y -intercept $x=0$: $y = -\frac{1}{9} + 4 = \frac{35}{9}$, so $(0, \frac{35}{9})$.
 x -intercepts: $(x+3)^2 = \frac{1}{4} \Rightarrow x+3 = \pm \frac{1}{2}$, so $x = -\frac{5}{2}$ or $x = -\frac{7}{2}$.



Q10. $y = -2\sqrt{x-1} + 3$. Endpoint $(1, 3)$; $a = -2 < 0$ so the branch decreases: domain $\{x : x \geq 1\}$, range $\{y : y \leq 3\}$. x -intercept: $2\sqrt{x-1} = 3 \Rightarrow \sqrt{x-1} = \frac{3}{2} \Rightarrow x-1 = \frac{9}{4} \Rightarrow x = \frac{13}{4}$. No y -intercept ($x=0$ outside the domain).



Q11. $y = (x - 1)^3 - 2$. Stationary point of inflection at $(1, -2)$. y -intercept $x = 0$: $y = (-1)^3 - 2 = -3$, so $(0, -3)$. x -intercept: $(x - 1)^3 = 2 \Rightarrow x - 1 = \sqrt[3]{2} \Rightarrow x = 1 + \sqrt[3]{2}$.

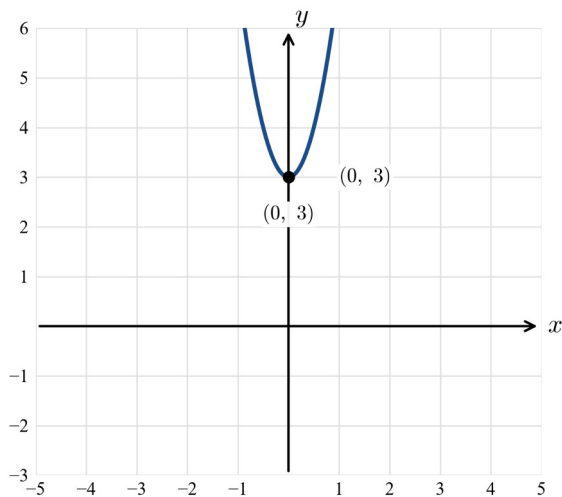


Q12. (a) $h = 2, k = 3$: $(-1 + 2, 3 + 3) = (1, 6)$. (b) $a = 3$: $(-1, 9)$. (c) $b = -1$: $\left(\frac{-1}{-1}, 3\right) = (1, 3)$. (d) $a = -1, k = -1$: $(-1, -4)$.

Q13. $y = -\frac{2}{x-1} + 3$: $a = -2$ (dilation factor 2 from the x -axis and reflection in the x -axis), $h = 1, k = 3$ (translate 1 right, 3 up). Asymptotes $x = 1, y = 3$.

Q14. $y = \sqrt{2-x} + 1 = \sqrt{-(x-2)} + 1$: reflection in the y -axis, then translation 2 right and 1 up. Endpoint $(2, 1)$, domain $\{x : x \leq 2\}$.

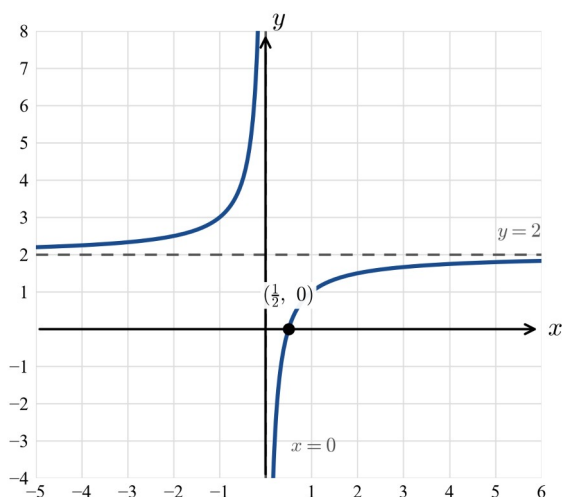
Q15. A dilation of factor $\frac{1}{2}$ from the y -axis replaces x with $2x$: $y = (2x)^2 = 4x^2$; then translate 3 up: $y = 4x^2 + 3$. TP $(0, 3)$; since the minimum value is $3 > 0$ there are no x -intercepts; y -intercept $(0, 3)$.



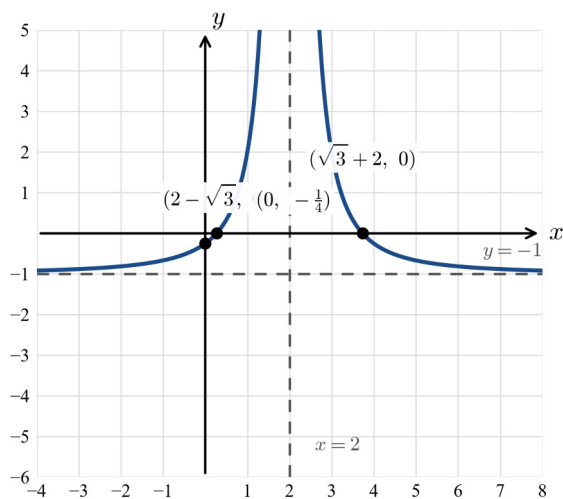
Q16. (a) $b = \frac{1}{2}: \left(\frac{4}{1/2}, 7\right) = (8, 7)$. (b) $b = 2: (2, 7)$. (c) $k = 3: (4, 10)$. (d) $a = -1: (4, -7)$.

Q17. Reflection in the y -axis replaces x with $-x$: $y = \frac{1}{-x} = -\frac{1}{x}$; translate 2 up: $y = -\frac{1}{x} + 2$. Asymptotes $x = 0$, $y = 2$

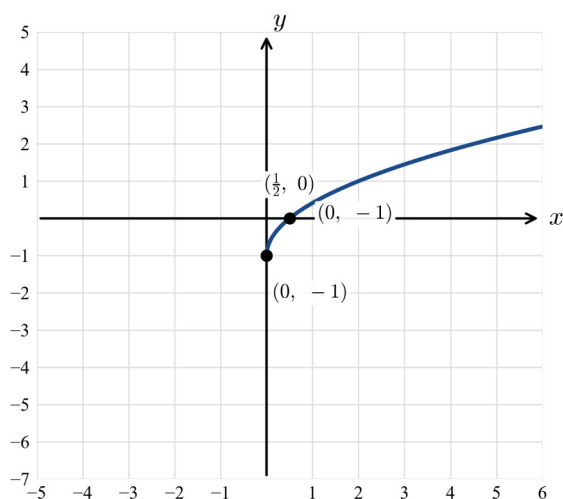
. x -intercept: $\frac{1}{x} = 2 \Rightarrow x = \frac{1}{2}$, so $\left(\frac{1}{2}, 0\right)$. No y -intercept ($x = 0$ is the asymptote).



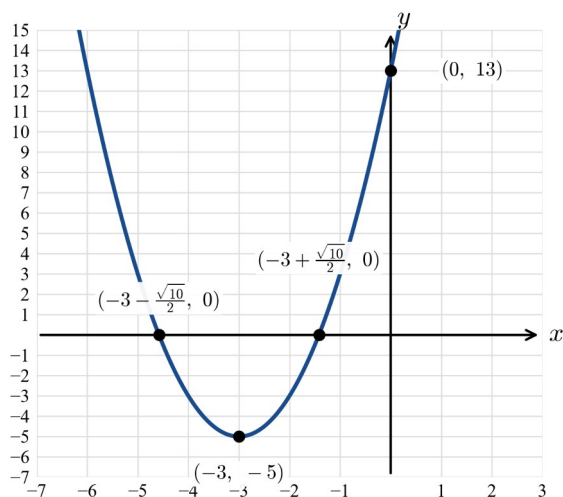
Q18. $y = \frac{3}{(x-2)^2} - 1$. Asymptotes $x = 2$, $y = -1$; range $\{y: y > -1\}$. y -intercept $x = 0$: $y = \frac{3}{4} - 1 = -\frac{1}{4}$, so $\left(0, -\frac{1}{4}\right)$. x -intercepts: $\frac{3}{(x-2)^2} = 1 \Rightarrow (x-2)^2 = 3 \Rightarrow x = 2 \pm \sqrt{3}$.



Q19. A dilation of factor $\frac{1}{2}$ from the y -axis replaces x with $2x$: $y = \sqrt{2x}$; translate 1 down: $y = \sqrt{2x} - 1$. Endpoint $(0, -1)$, domain $\{x : x \geq 0\}$. x -intercept: $\sqrt{2x} = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$. y -intercept $(0, -1)$.



Q20. $y = 2(x + 3)^2 - 5$: $a = 2$ (dilation factor 2 from the x -axis), $h = -3$, $k = -5$ (translate 3 left, 5 down). TP $(-3, -5)$; image of $(0, 0)$ is $(-3, -5)$. y -intercept $x = 0$: $y = 2(9) - 5 = 13$, so $(0, 13)$. x -intercepts: $2(x + 3)^2 = 5 \Rightarrow (x + 3)^2 = \frac{5}{2} \Rightarrow x + 3 = \pm \frac{\sqrt{10}}{2}$, so $x = -3 \pm \frac{\sqrt{10}}{2}$.



Q21. $(x, y) \rightarrow \left(\frac{x}{b} + h, a y + k\right)$ applied to $(2, 5)$. (a) $a = 3, h = 1, k = -4$: $(2 + 1, 3(5) - 4) = (3, 11)$. (b) $b = 2$:
 $\left(\frac{2}{2}, 5\right) = (1, 5)$.

Chapter 3 Transformations of the plane — 3E Identifying transformations

Theory

Sections 3A–3D took a rule and produced a graph. This section works **backwards**: given a graph (or its rule) and its image, we **determine the transformation** — that is, we identify the dilation(s), reflection(s) and translation(s) that map one onto the other.

The general form. Every image of a base graph $y = f(x)$ can be written

$$y = af(b(x - h)) + k,$$

where, reading the parameters off the rule:

- a — a **dilation of factor $|a|$ from the x -axis**, together with a **reflection in the x -axis** when $a < 0$;
- b — a **dilation of factor $\frac{1}{|b|}$ from the y -axis**, with a **reflection in the y -axis** when $b < 0$ (in this section $b = 1$ unless stated);
- h — a **horizontal translation** of h units (right if $h > 0$);
- k — a **vertical translation** of k units (up if $k > 0$).

Describing a sequence. State each transformation with its factor, axis and direction, and give them in the order **dilations and reflections first, then translations**. Naming *both* the dilation and the translation, with the correct axis and order, is exactly where marks are most often lost.

Determining a from a point. When the *type* of image and its key feature are known — the vertex of a parabola, or the asymptotes of a $\frac{1}{x}$ - or $\frac{1}{x^2}$ -form graph, or the endpoint of a square-root graph — the rule has the form (for example) $y = a(x - h)^2 + k$ with h, k known. Substitute a second known point and solve for a .

The reverse transformation. The transformation that maps the image *back* onto the base is the inverse: undo the translations first (opposite direction), then undo the dilation (reciprocal factor).

To determine a transformation: (1) identify the base graph and the image form; (2) match a, b, h, k (using a known point to find a if needed); (3) describe the sequence in the correct order, or write the image rule, as asked.

Worked examples

Example 1 — scaffolded

Describe the sequence of transformations that maps the graph of $y = x^2$ onto $y = 3(x + 2)^2 - 4$, and state the coordinates of the vertex of the image.

Working

Comment

Compare $y = 3(x + 2)^2 - 4$ with $y = a(x - h)^2 + k$: $a = 3$, $h = -2$, $k = -4$.

Read the parameters straight off the rule.

$a = 3 > 0$: a **dilation of factor 3 from the x -axis**.

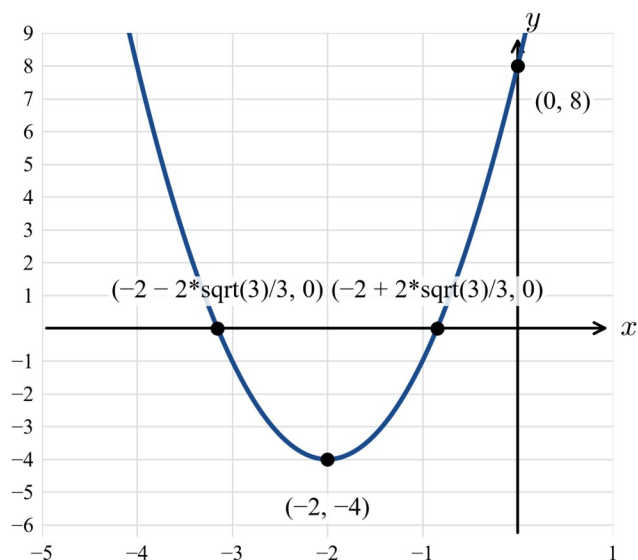
$|a|$ is the dilation factor; $a > 0$ so there is no reflection.

Then a **translation of 2 units to the left and 4 units down**.

$h = -2$ (left), $k = -4$ (down). Translations come after the dilation.

Vertex: the vertex of $y = x^2$ is $(0, 0)$, which maps to $(-2, -4)$.

The turning point moves with the translation.



Example 2 — scaffolded

The graph of $y = \frac{1}{x}$ undergoes a dilation of factor 2 from the x -axis, then a reflection in the x -axis, then a translation of 1 unit right and 3 units up. Find the rule of the image and state the equations of its asymptotes.

Working

Comment

Dilation of factor 2 from the x -axis: $y = \frac{2}{x}$.

Multiply the function by $a = 2$.

Reflection in the x -axis: $y = -\frac{2}{x}$.

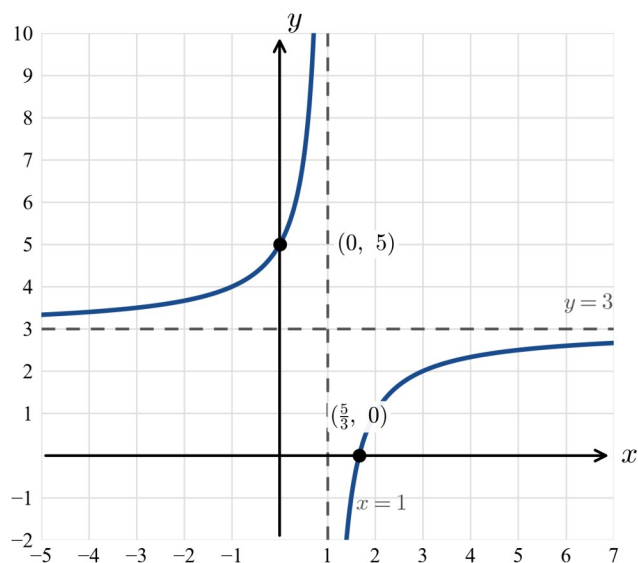
Replace y by $-y$ (i.e. a becomes -2).

Translation 1 right and 3 up: replace x by $x - 1$ and add 3:
3: $y = -\frac{2}{x-1} + 3$.

Apply the translations last.

Asymptotes: $x = 1$ and $y = 3$.

Vertical asymptote $x = h$; horizontal asymptote $y = k$.



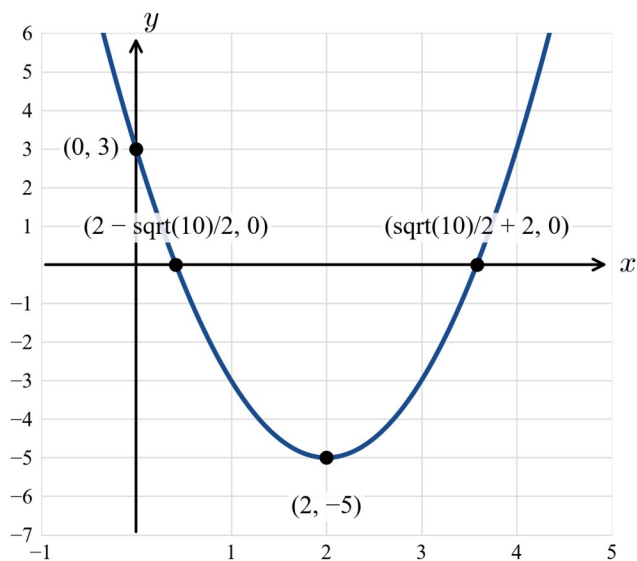
Example 3 — unscaffolded

A parabola is obtained from $y = x^2$ by a dilation from the x -axis and a translation. Its image has vertex $(2, -5)$ and passes through $(0, 3)$. Find the value of the dilation factor and the rule of the image.

The vertex gives the form $y = a(x - 2)^2 - 5$.

Substituting $(0, 3)$: $3 = a(0 - 2)^2 - 5 = 4a - 5 \Rightarrow 4a = 8 \Rightarrow a = 2$.

$\therefore$ the dilation factor is 2 and the rule is $y = 2(x - 2)^2 - 5$.



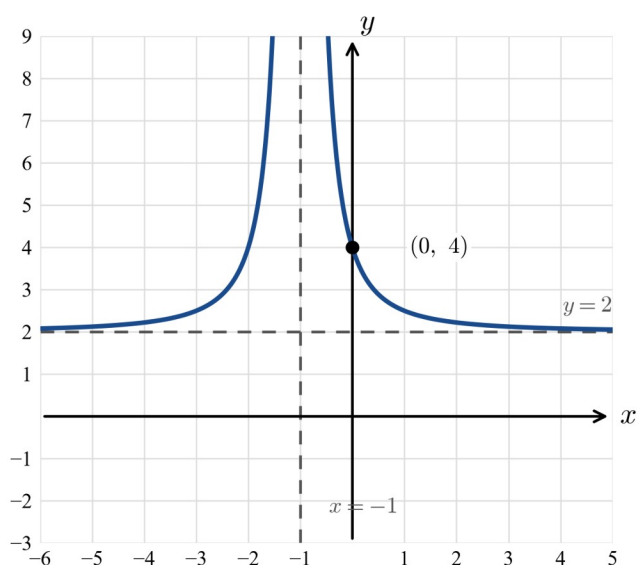
Example 4 — unscaffolded

A graph is obtained from $y = \frac{1}{x^2}$ by a dilation from the x -axis and a translation. Its image has asymptotes $x = -1$ and $y = 2$ and passes through $(0, 4)$. Find its rule.

The asymptotes give the form $y = \frac{a}{(x + 1)^2} + 2$.

Substituting $(0, 4)$: $4 = \frac{a}{(0 + 1)^2} + 2 = a + 2 \Rightarrow a = 2$.

$\therefore y = \frac{2}{(x + 1)^2} + 2$ (a dilation of factor 2 from the x -axis, then a translation 1 left and 2 up).



Practice questions

Scaffolded

Q1. The graph of $y = x^2$ is mapped onto $y = -(x - 1)^2 + 3$. (a) describe the sequence of transformations; (b) state the coordinates of the vertex of the image; (c) find the y -intercept of the image.

Q2. The graph of $y = \sqrt{x}$ undergoes a dilation of factor 2 from the x -axis, then a translation of 3 units right and 1 unit down. (a) find the rule of the image; (b) state the coordinates of the endpoint; (c) find the exact x -intercept.

Q3. The graph of $y = \frac{1}{x}$ is mapped onto $y = \frac{1}{x - 2} - 1$. (a) describe the sequence of transformations; (b) state the equations of the asymptotes; (c) find the axis intercepts.

Q4. A parabola with rule $y = a(x - h)^2 + k$ is obtained from $y = x^2$. Its vertex is $(-1, 4)$ and it passes through $(1, 0)$. (a) state h and k ; (b) find a ; (c) write the rule and describe the transformations.

Q5. The graph of $y = x^3$ is mapped onto $y = 2(x - 1)^3 + 3$. (a) describe the sequence of transformations; (b) state the coordinates of the point of inflection of the image.

Q6. The graph of $y = \frac{1}{x^2}$ is transformed. Its image has asymptotes $x = 3$ and $y = -2$ and passes through $(2, 0)$. (a) write the form of the rule using the asymptotes; (b) find a ; (c) write the rule and state the range.

Un scaffolded

For **Q7–Q11**, describe the sequence of transformations mapping the stated base graph onto the given image, and state the vertex / asymptotes / endpoint / point of inflection as appropriate.

Q7. $y = x^2 \rightarrow y = 4(x - 2)^2 - 1$

Q8. $y = \frac{1}{x} \rightarrow y = -\frac{3}{x + 2} - 1$

Q9. $y = \sqrt{x} \rightarrow y = -\sqrt{x - 1} + 2$

Q10. $y = \frac{1}{x^2} \rightarrow y = \frac{3}{(x + 1)^2} + 1$

Q11. $y = x^3 \rightarrow y = -(x + 2)^3 - 1$

For **Q12–Q15**, use the given feature and point to find the dilation factor and the rule.

Q12. A parabola from $y = x^2$ with vertex $(3, -4)$ passing through $(5, 4)$.

Q13. A hyperbola from $y = \frac{1}{x}$ with asymptotes $x = -1$, $y = 3$ passing through $(0, 1)$.

Q14. A square-root graph from $y = \sqrt{x}$ with endpoint $(-2, 3)$ passing through $(2, 1)$.

Q15. An image of $y = \frac{1}{x^2}$ with asymptotes $x = 2$, $y = 1$ passing through $(0, \frac{3}{2})$.

Q16. The graph of $y = x^2$ is mapped onto $y = 2(x - 1)^2 + 3$. (a) describe the forward transformation; (b) describe the reverse transformation that maps the image back onto $y = x^2$.

Q17. Describe the sequence of transformations mapping $y = \frac{1}{x}$ onto $y = \frac{1}{2(x - 2)} + 3$.

Q18. Describe the sequence of transformations mapping $y = x^2$ onto $y = (x + 3)^2 + 2$, and explain why no dilation is involved.

Q19. Describe a sequence of transformations that maps the graph of $y = \sqrt{x}$ onto $y = 3\sqrt{x - 2}$.

Q20. A transformation T maps the graph of $y = \frac{1}{x^2}$ onto $y = -\frac{1}{(x - 4)^2} + 2$. State T as a sequence of transformations, and find the coordinates of the point onto which $(1, 1)$ is mapped.

Q21. Find the values of a and k so that $y = af(x) + k$ maps the graph of $f(x) = x^2$ onto a parabola with range $[3, \infty)$ whose graph passes through $(2, 11)$.

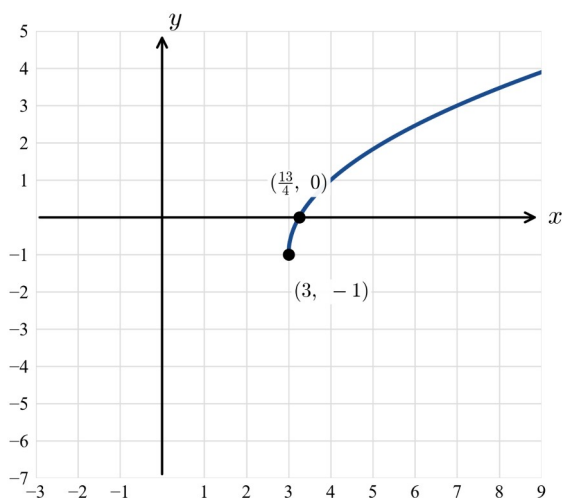
Answers

Q1. (a) reflection in the x -axis, then translation 1 right and 3 up; (b) $(1, 3)$; (c) $(0, 2)$. **Q2.** (a) $y = 2\sqrt{x-3} - 1$; (b) $(3, -1)$; (c) $\left(\frac{13}{4}, 0\right)$. **Q3.** (a) translation 2 right and 1 down; (b) $x = 2$, $y = -1$; (c) $(3, 0)$, $\left(0, -\frac{3}{2}\right)$. **Q4.** (a) $h = -1$, $k = 4$; (b) $a = -1$; (c) $y = -(x+1)^2 + 4$ — reflection in the x -axis, then translation 1 left and 4 up. **Q5.** (a) dilation of factor 2 from the x -axis, then translation 1 right and 3 up; (b) $(1, 3)$. **Q6.** (a) $y = \frac{a}{(x-3)^2} - 2$; (b) $a = 2$; (c) $y = \frac{2}{(x-3)^2} - 2$, range $\{y : y > -2\}$. **Q7.** Dilation of factor 4 from the x -axis, then translation 2 right and 1 down; vertex $(2, -1)$. **Q8.** Dilation of factor 3 from the x -axis, reflection in the x -axis, then translation 2 left and 1 down; asymptotes $x = -2$, $y = -1$. **Q9.** Reflection in the x -axis, then translation 1 right and 2 up; endpoint $(1, 2)$. **Q10.** Dilation of factor 3 from the x -axis, then translation 1 left and 1 up; asymptotes $x = -1$, $y = 1$. **Q11.** Reflection in the x -axis, then translation 2 left and 1 down; point of inflection $(-2, -1)$. **Q12.** $a = 2$; $y = 2(x-3)^2 - 4$. **Q13.** $a = -2$; $y = -\frac{2}{x+1} + 3$. **Q14.** $a = -1$; $y = -\sqrt{x+2} + 3$. **Q15.** $a = 2$; $y = \frac{2}{(x-2)^2} + 1$. **Q16.** (a) dilation of factor 2 from the x -axis, then translation 1 right and 3 up; (b) translation 1 left and 3 down, then a dilation of factor $\frac{1}{2}$ from the x -axis. **Q17.** Dilation of factor $\frac{1}{2}$ from the x -axis, then translation 2 right and 3 up. **Q18.** Translation 3 left and 2 up; $a = 1$, so the shape is unchanged (no dilation). **Q19.** Dilation of factor 3 from the x -axis, then translation 2 right. **Q20.** Reflection in the x -axis, then translation 4 right and 2 up; $(1, 1) \rightarrow (5, 1)$. **Q21.** $a = 2$, $k = 3$.

Fully-worked solutions

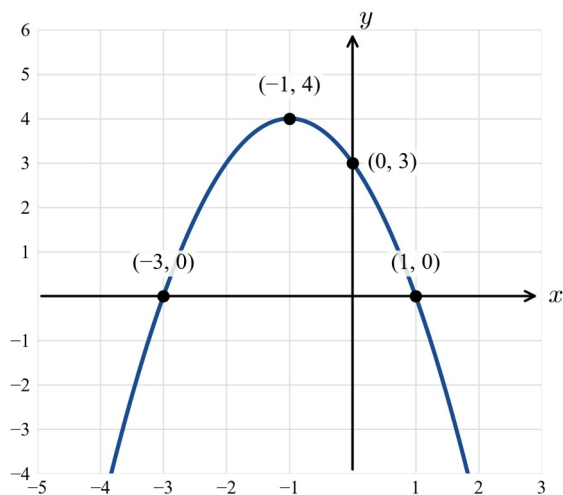
Q1. $y = -(x-1)^2 + 3$; $a = -1$, $h = 1$, $k = 3$. (a) $a = -1$: a reflection in the x -axis (the dilation factor is 1), then a translation 1 right and 3 up. (b) Vertex $(1, 3)$. (c) $x = 0 \Rightarrow y = -(0-1)^2 + 3 = -1 + 3 = 2$, so $(0, 2)$.

Q2. Dilation of factor 2 from the x -axis: $y = 2\sqrt{x}$. Translation 3 right and 1 down: $y = 2\sqrt{x-3} - 1$. (b) Endpoint where $x-3=0$: $(3, -1)$. (c) $2\sqrt{x-3} - 1 = 0 \Rightarrow \sqrt{x-3} = \frac{1}{2} \Rightarrow x-3 = \frac{1}{4}$, so $x = \frac{13}{4}$, giving $\left(\frac{13}{4}, 0\right)$.



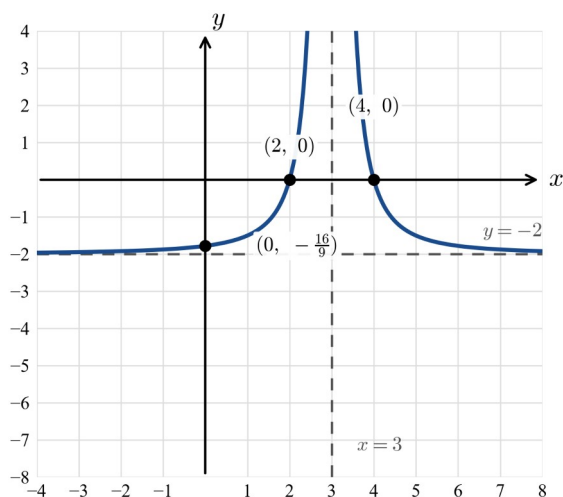
Q3. $y = \frac{1}{x-2} - 1$: $h=2$, $k=-1$, $a=1$. (a) A translation 2 right and 1 down (no dilation or reflection). (b) Asymptotes $x=2$, $y=-1$. (c) x -intercept: $\frac{1}{x-2} = 1 \Rightarrow x=3$, giving $(3, 0)$; y -intercept: $x=0 \Rightarrow y = -\frac{1}{2} - 1 = -\frac{3}{2}$, giving $(0, -\frac{3}{2})$.

Q4. (a) $h=-1$, $k=4$, so $y = a(x+1)^2 + 4$. (b) $(1, 0)$: $0 = a(1+1)^2 + 4 = 4a + 4 \Rightarrow a = -1$. (c) $y = -(x+1)^2 + 4$ — a reflection in the x -axis, then a translation 1 left and 4 up.



Q5. $y = 2(x-1)^3 + 3$: $a=2$, $h=1$, $k=3$. (a) A dilation of factor 2 from the x -axis, then a translation 1 right and 3 up. (b) The point of inflection of $y = x^3$ is $(0, 0)$, which maps to $(1, 3)$.

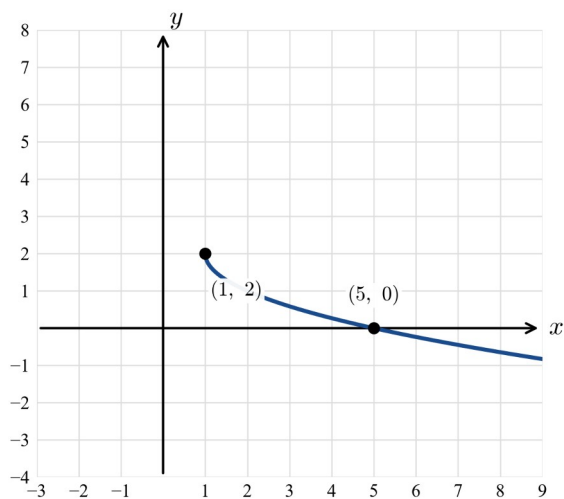
Q6. (a) $y = \frac{a}{(x-3)^2} - 2$. (b) $(2, 0)$: $0 = \frac{a}{(2-3)^2} - 2 = a - 2 \Rightarrow a = 2$. (c) $y = \frac{2}{(x-3)^2} - 2$; since $a=2 > 0$, range $\{y : y > -2\}$.



Q7. $y = 4(x-2)^2 - 1$: $a=4$, $h=2$, $k=-1$. Dilation of factor 4 from the x -axis, then translation 2 right and 1 down. Vertex $(2, -1)$.

Q8. $y = -\frac{3}{x+2} - 1$: $a=-3$, $h=-2$, $k=-1$. Dilation of factor 3 from the x -axis, reflection in the x -axis, then translation 2 left and 1 down. Asymptotes $x=-2$, $y=-1$.

Q9. $y = -\sqrt{x-1} + 2$: $a=-1$, $h=1$, $k=2$. Reflection in the x -axis, then translation 1 right and 2 up. Endpoint $(1, 2)$.

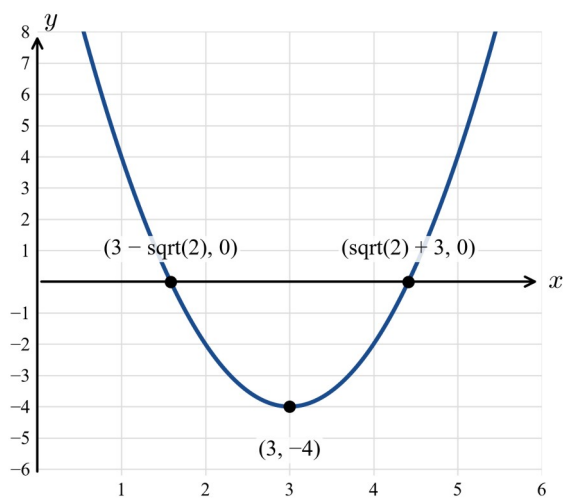


Q10. $y = \frac{3}{(x+1)^2} + 1$: $a=3$, $h=-1$, $k=1$. Dilation of factor 3 from the x -axis, then translation 1 left and 1 up.

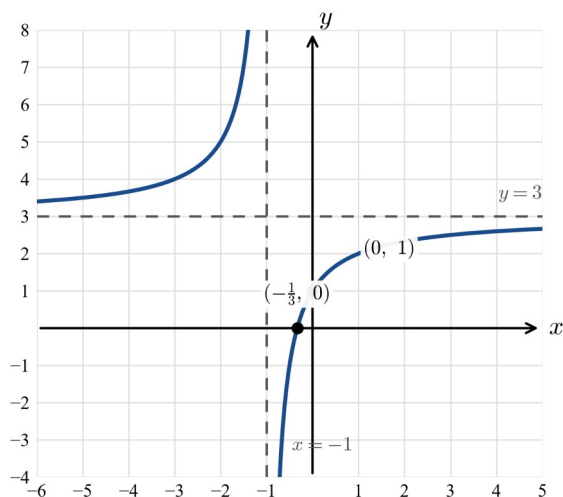
Asymptotes $x=-1$, $y=1$.

Q11. $y = -(x+2)^3 - 1$: $a=-1$, $h=-2$, $k=-1$. Reflection in the x -axis, then translation 2 left and 1 down. Point of inflection $(-2, -1)$.

Q12. Vertex $(3, -4)$ gives $y = a(x-3)^2 - 4$. Through $(5, 4)$: $4 = a(5-3)^2 - 4 = 4a - 4 \Rightarrow 4a = 8 \Rightarrow a = 2$.
 $\therefore y = 2(x-3)^2 - 4$.

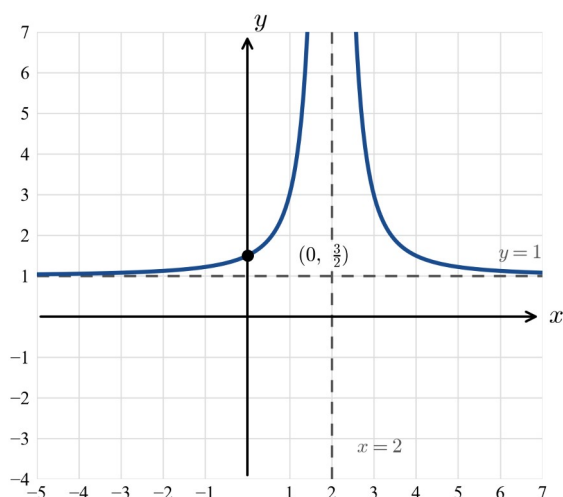


Q13. Asymptotes $x=-1$, $y=3$ give $y = \frac{a}{x+1} + 3$. Through $(0, 1)$: $1 = \frac{a}{1} + 3 \Rightarrow a = -2$. $\therefore y = -\frac{2}{x+1} + 3$.



Q14. Endpoint $(-2, 3)$ gives $y = a\sqrt{x+2} + 3$. Through $(2, 1)$: $1 = a\sqrt{2+2} + 3 = 2a + 3 \Rightarrow 2a = -2 \Rightarrow a = -1$.
 $\therefore y = -\sqrt{x+2} + 3$.

Q15. Asymptotes $x = 2$, $y = 1$ give $y = \frac{a}{(x-2)^2} + 1$. Through $(0, \frac{3}{2})$: $\frac{3}{2} = \frac{a}{4} + 1 \Rightarrow \frac{a}{4} = \frac{1}{2} \Rightarrow a = 2$. $\therefore y = \frac{2}{(x-2)^2} + 1$.



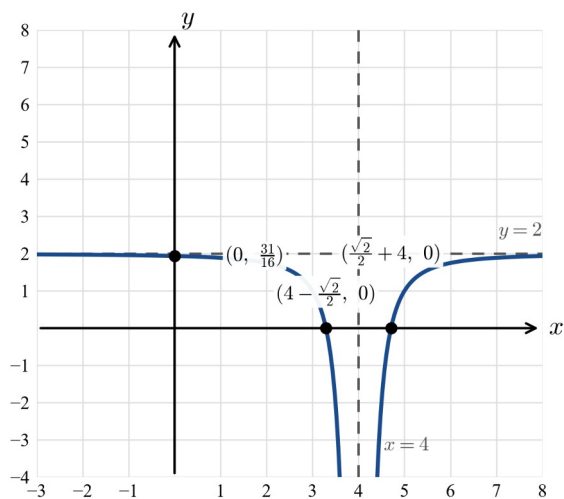
Q16. (a) $y = 2(x-1)^2 + 3$: a dilation of factor 2 from the x -axis, then a translation 1 right and 3 up. (b) Reversing: undo the translation (1 left and 3 down), then undo the dilation (a dilation of factor $\frac{1}{2}$ from the x -axis).

Q17. $y = \frac{1}{2(x-2)} + 3 = \frac{\frac{1}{2}}{x-2} + 3$: $a = \frac{1}{2}$, $h = 2$, $k = 3$. A dilation of factor $\frac{1}{2}$ from the x -axis, then a translation 2 right and 3 up.

Q18. $y = (x+3)^2 + 2$: $a = 1$, $h = -3$, $k = 2$. A translation 3 left and 2 up. Since $a = 1$, the graph is neither stretched nor reflected, so no dilation is involved.

Q19. $y = 3\sqrt{x-2}$: $a = 3$, $h = 2$, $k = 0$. A dilation of factor 3 from the x -axis, then a translation 2 units right.

Q20. $y = -\frac{1}{(x-4)^2} + 2$: $a = -1$, $h = 4$, $k = 2$. T is a reflection in the x -axis, then a translation 4 right and 2 up. The point $(1, 1)$ (which lies on $y = \frac{1}{x^2}$) maps to $(1+4, -1+2) = (5, 1)$. Check: at $x = 5$, $-\frac{1}{(5-4)^2} + 2 = -1 + 2 = 1$. $\checkmark$



Q21. $y = af(x) + k = ax^2 + k$ has vertex $(0, k)$ and, for $a > 0$, range $[k, \infty)$. Range $[3, \infty)$ gives $k = 3$. Through $(2, 11)$: $11 = a(2)^2 + 3 = 4a + 3 \Rightarrow 4a = 8 \Rightarrow a = 2$. $\therefore a = 2, k = 3$ (image $y = 2x^2 + 3$).

Theory

In Sections 3A–3E you learned to translate, dilate and reflect graphs. This section applies those transformations to **sketch** the graph of a function that is built from one of the three *standard* graphs below — showing its key features (asymptotes or endpoint, and axis intercepts) accurately, though not to scale.

The three standard graphs.

- **Rectangular hyperbola** $y = \frac{1}{x}$: two branches (in the first and third quadrants), with a **vertical asymptote** $x = 0$ and a **horizontal asymptote** $y = 0$. Domain $\mathbb{R} \setminus \{0\}$, range $\mathbb{R} \setminus \{0\}$.
- **Reciprocal-square graph** $y = \frac{1}{x^2}$: two branches, **both above** the x -axis, with asymptotes $x = 0$ and $y = 0$. Domain $\mathbb{R} \setminus \{0\}$, range $\{y : y > 0\}$.
- **Square-root function** $y = \sqrt{x}$: a single branch starting at the **endpoint** $(0, 0)$ and increasing. Domain $\{x : x \geq 0\}$, range $\{y : y \geq 0\}$.

The transformed forms. Applying a dilation of factor a from the x -axis (a reflection in the x -axis as well when $a < 0$), then a translation of h units horizontally and k units vertically, gives:

Base	Transformed rule	Asymptotes / endpoint	Domain	Range
$\frac{1}{x}$	$y = \frac{a}{x-h} + k$	asymptotes $x = h$, $y = k$	$\mathbb{R} \setminus \{h\}$	$\mathbb{R} \setminus \{k\}$
$\frac{1}{x^2}$	$y = \frac{a}{(x-h)^2} + k$	asymptotes $x = h$, $y = k$	$\mathbb{R} \setminus \{h\}$	$y > k$ if $a > 0$; $y < k$ if $a < 0$
$\sqrt{x}$	$y = a\sqrt{x-h} + k$	endpoint (h, k)	$x \geq h$	$y \geq k$ if $a > 0$; $y \leq k$ if $a < 0$

- The **vertical translation** k moves the horizontal asymptote to $y = k$ (or the endpoint up to height k); the **horizontal translation** h moves the vertical asymptote to $x = h$ (or the endpoint across to $x = h$).
- A negative coefficient **inside** the function — for example $y = a\sqrt{-(x-h)} + k$ or $y = \frac{a}{-(x-h)} + k$ — is a **reflection in the y -axis**; for the square-root graph this reverses the domain to $x \leq h$.
- Always apply **dilations and reflections before translations**. Naming *both* the dilation and the translation, in the correct order and about the correct axis, is exactly where marks are most often lost.

Finding the intercepts (give exact values).

- **y -intercept:** substitute $x = 0$ — but only if $x = 0$ is in the domain.
- **x -intercepts:** solve $y = 0$. A transformed hyperbola has one when $k \neq 0$ and none when $k = 0$; a transformed $\frac{1}{x^2}$ graph has two or none; a square-root graph has one or none.

To sketch: (1) identify the base graph; (2) write down the asymptotes (or the endpoint) and the domain and range; (3) find the exact axis intercepts; (4) draw the asymptotes as dashed lines, plot the intercepts and endpoint, then draw each branch approaching its asymptotes.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = \frac{2}{x-1} + 3$, stating the equations of the asymptotes and the exact coordinates of the axis intercepts.

Working

Comment

Base graph $y = \frac{1}{x}$ (a hyperbola), with $a = 2$, $h = 1$, $k = 3$.

Compare with $y = \frac{a}{x-h} + k$.

Asymptotes: $x = 1$ and $y = 3$.

The vertical asymptote is $x = h$; the horizontal asymptote is $y = k$.

y-intercept: $x = 0 \Rightarrow y = \frac{2}{0-1} + 3 = -2 + 3 = 1$, giving $(0, 1)$.

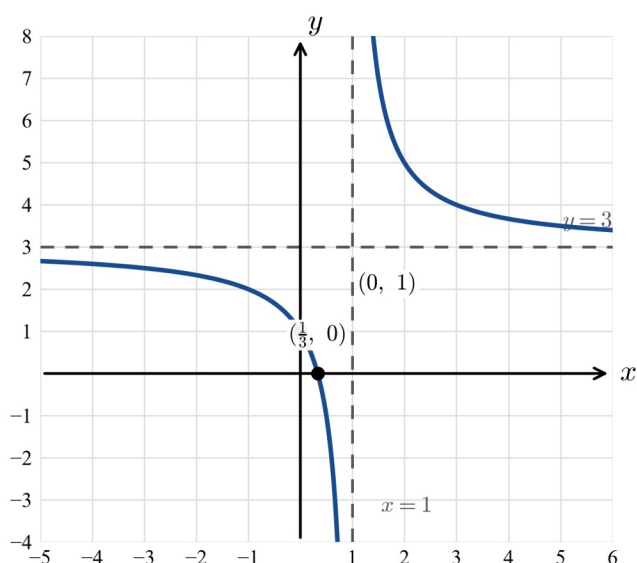
Substitute $x = 0$ (which is in the domain).

x-intercept: $\frac{2}{x-1} + 3 = 0 \Rightarrow \frac{2}{x-1} = -3 \Rightarrow x-1 = -\frac{2}{3}$,
so $x = \frac{1}{3}$, giving $(\frac{1}{3}, 0)$.

Isolate the fraction, then solve. Keep the answer exact.

Since $a = 2 > 0$, the branches lie in the *upper-right* and *lower-left* regions formed by the asymptotes. Domain $\mathbb{R} \setminus \{1\}$, range $\mathbb{R} \setminus \{3\}$.

The sign of a fixes which regions the branches occupy.



Example 2 — scaffolded

Sketch the graph of $y = -\frac{1}{(x+2)^2} + 1$, labelling the asymptotes and all axis intercepts, and state the range.

Working

Comment

Base graph $y = \frac{1}{x^2}$, with $a = -1$, $h = -2$, $k = 1$.

Compare with $y = \frac{a}{(x-h)^2} + k$; here $x-h = x+2$.

Asymptotes: $x = -2$ and $y = 1$.

Vertical asymptote $x = h = -2$; horizontal asymptote $y = k = 1$.

Because $a = -1 < 0$, the graph is **reflected in the x-axis**, so both branches lie **below** $y = 1$. Range $\{y : y < 1\}$.

For this form, $a < 0$ gives range $y < k$.

$$\text{y-intercept: } x=0 \Rightarrow y = -\frac{1}{(0+2)^2} + 1 = -\frac{1}{4} + 1 = \frac{3}{4},$$

Substitute $x=0$.

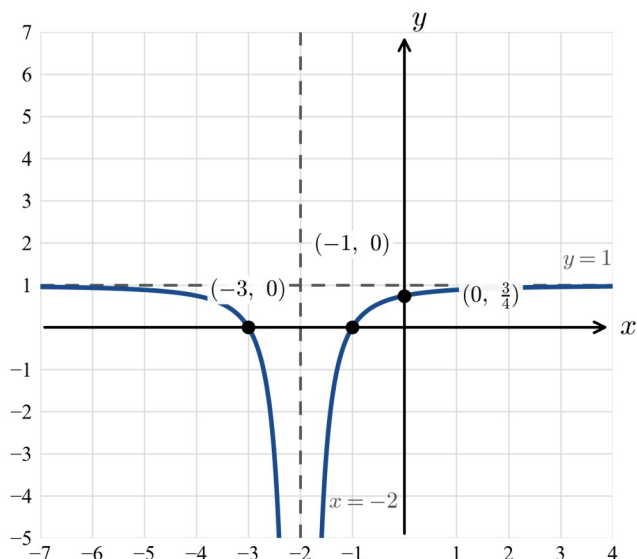
$$\text{giving } \left(0, \frac{3}{4}\right).$$

x-intercepts:

Keep the $\pm$: a square gives two solutions.

$$-\frac{1}{(x+2)^2} + 1 = 0 \Rightarrow (x+2)^2 = 1 \Rightarrow x+2 = \pm 1, \text{ so } x = -1$$

or $x = -3$, giving $(-1, 0)$ and $(-3, 0)$.



Example 3 — unscaffolded

Sketch the graph of $y = \sqrt{x+1} - 2$, stating the domain and range and the exact axis intercepts.

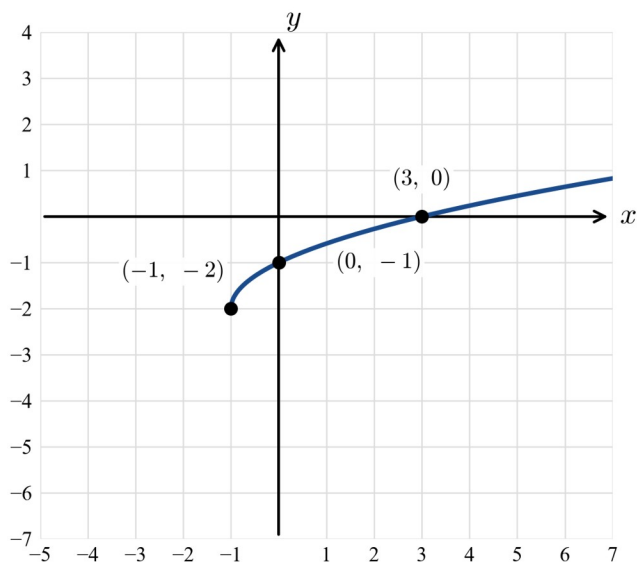
This is the square-root graph $y = \sqrt{x}$ translated 1 unit left and 2 units down; since $a=1 > 0$ the branch increases from its endpoint.

Endpoint $(-1, -2)$; domain $\{x : x \geq -1\}$, range $\{y : y \geq -2\}$.

$$\text{y-intercept: } x=0 \Rightarrow y = \sqrt{0+1} - 2 = 1 - 2 = -1, \text{ giving } (0, -1).$$

$$\text{x-intercept: } \sqrt{x+1} - 2 = 0 \Rightarrow \sqrt{x+1} = 2 \Rightarrow x+1 = 4, \text{ so } x = 3, \text{ giving } (3, 0).$$

$\therefore$ the graph runs from $(-1, -2)$ up through $(0, -1)$ and $(3, 0)$.



Example 4 — unscaffolded

The graph of $y = \frac{1}{x}$ is transformed to produce $y = -\frac{3}{x+1} + 2$. Describe the sequence of transformations, then sketch the graph, labelling the asymptotes and axis intercepts.

Writing the rule as $y = \frac{-3}{x - (-1)} + 2$, we read $a = -3$, $h = -1$, $k = 2$. From $y = \frac{1}{x}$:

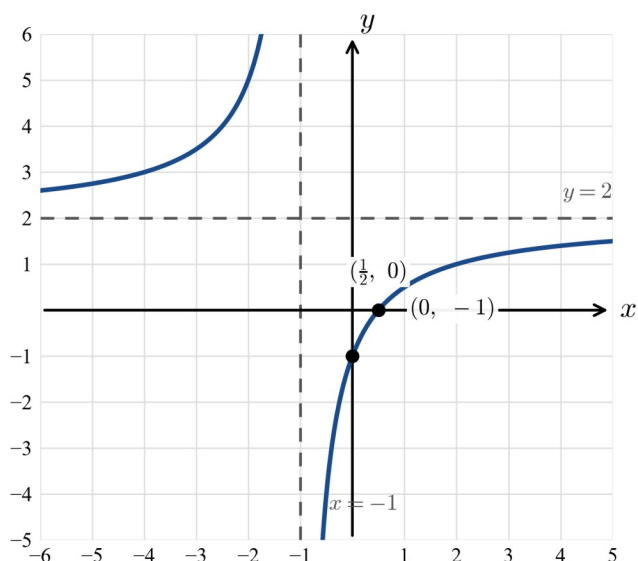
1. a **dilation of factor 3 from the x -axis** ($|a| = 3$);
2. a **reflection in the x -axis** (since $a < 0$);
3. a **translation of 1 unit to the left and 2 units up**.

Asymptotes $x = -1$ and $y = 2$; domain $\mathbb{R} \setminus \{-1\}$, range $\mathbb{R} \setminus \{2\}$.

y -intercept: $x = 0 \Rightarrow y = -\frac{3}{1} + 2 = -1$, giving $(0, -1)$.

x -intercept: $-\frac{3}{x+1} + 2 = 0 \Rightarrow \frac{3}{x+1} = 2 \Rightarrow x+1 = \frac{3}{2}$, so $x = \frac{1}{2}$, giving $(\frac{1}{2}, 0)$.

$\therefore$ the reflection places the branches in the *upper-left* and *lower-right* regions of the asymptotes.



Practice questions

Scaffolded

Q1. For $y = \frac{1}{x-2} + 1$: (a) state the equations of the two asymptotes; (b) find the exact coordinates of the axis intercepts; (c) state the domain and range; (d) sketch the graph, labelling the asymptotes and intercepts.

Q2. For $y = \frac{1}{(x-1)^2} - 1$: (a) name the base graph and state the asymptotes; (b) find all axis intercepts; (c) state the range; (d) sketch the graph.

Q3. For $y = \sqrt{x-1} + 2$: (a) write the coordinates of the endpoint; (b) state the domain and range; (c) explain why the graph has no axis intercepts; (d) sketch the graph.

Q4. For $y = -\frac{2}{x-2} + 1$: (a) state the asymptotes; (b) find the exact axis intercepts; (c) describe the transformations that map $y = \frac{1}{x}$ onto this graph; (d) sketch the graph.

Q5. For $y = -\frac{2}{(x-1)^2} + 2$: (a) state the asymptotes and the range; (b) find all axis intercepts; (c) sketch the graph; (d) how does the graph differ from that of $y = \frac{1}{(x-1)^2} + 2$?

Q6. For $y = -\sqrt{x+2} + 1$: (a) state the endpoint, domain and range; (b) find the exact axis intercepts; (c) sketch the graph.

Un scaffolded

Sketch each of the following, labelling the asymptotes (or endpoint) and the exact coordinates of all axis intercepts, and state the domain and range.

Q7. $y = \frac{3}{x-2} - 1$

Q8. $y = -\frac{1}{x+1} - 2$

Q9. $y = \frac{1}{(x+2)^2} - 1$

Q10. $y = -\frac{1}{(x-1)^2} + 4$

Q11. $y = \sqrt{x-3} + 1$

Q12. $y = \sqrt{x+4} - 2$

Q13. $y = -\sqrt{x-1} + 4$

Q14. $y = \sqrt{2-x} + 1$

Q15. $y = \frac{1}{2(x-2)} + 1$

Q16. $y = -\frac{4}{x-1} + 2$

Q17. $y = \frac{3}{(x-2)^2} + 1$

Q18. $y = -\frac{1}{(x+1)^2} - 1$

Q19. $y = \sqrt{2(x+4)} - 2$

Q20. Describe the sequence of transformations that maps $y = \frac{1}{x^2}$ onto $y = -\frac{2}{(x-3)^2} + 4$, and hence sketch the graph.

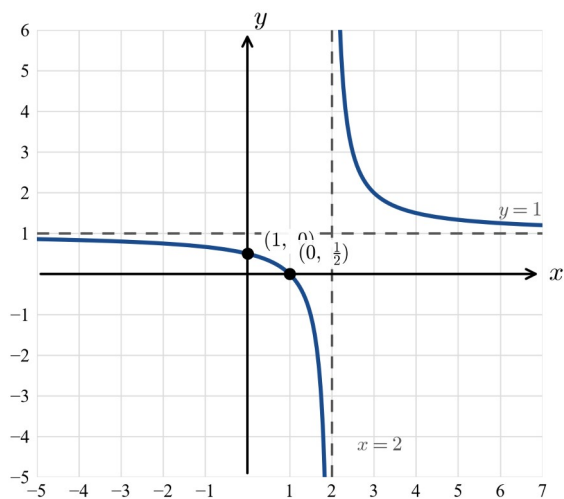
Q21. For $y = -\frac{2}{x-3} + 1$, state the equations of the asymptotes, find the exact axis intercepts, and hence sketch the graph.

Answers

Q1. Asymptotes $x=2$, $y=1$; intercepts $(1,0)$, $(0, \frac{1}{2})$; domain $\mathbb{R} \setminus \{2\}$, range $\mathbb{R} \setminus \{1\}$. **Q2.** Reciprocal-square graph; asymptotes $x=1$, $y=-1$; intercepts $(0,0)$, $(2,0)$; range $\{y: y > -1\}$. **Q3.** Endpoint $(1,2)$; domain $\{x: x \geq 1\}$, range $\{y: y \geq 2\}$; no intercepts (the graph never reaches $x=0$ or $y=0$). **Q4.** Asymptotes $x=2$, $y=1$; intercepts $(4,0)$, $(0,2)$; (c) dilation of factor 2 from the x -axis, reflection in the x -axis, then translation 2 right and 1 up. **Q5.** Asymptotes $x=1$, $y=2$, range $\{y: y < 2\}$; intercepts $(0,0)$, $(2,0)$; (d) the factor 2 makes it twice as steep and the reflection in the x -axis turns the branches to point down from $y=2$ instead of up. **Q6.** Endpoint $(-2,1)$; domain $\{x: x \geq -2\}$, range $\{y: y \leq 1\}$; intercepts $(-1,0)$, $(0, 1-\sqrt{2})$. **Q7.** Asymptotes $x=2$, $y=-1$; intercepts $(5,0)$, $(0, -\frac{5}{2})$; domain $\mathbb{R} \setminus \{2\}$, range $\mathbb{R} \setminus \{-1\}$. **Q8.** Asymptotes $x=-1$, $y=-2$; intercepts $(-\frac{3}{2}, 0)$, $(0, -3)$; domain $\mathbb{R} \setminus \{-1\}$, range $\mathbb{R} \setminus \{-2\}$. **Q9.** Asymptotes $x=-2$, $y=-1$; intercepts $(-3,0)$, $(-1,0)$, $(0, -\frac{3}{4})$; domain $\mathbb{R} \setminus \{-2\}$, range $\{y: y > -1\}$. **Q10.** Asymptotes $x=1$, $y=4$; intercepts $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$, $(0, 3)$; domain $\mathbb{R} \setminus \{1\}$, range $\{y: y < 4\}$. **Q11.** Endpoint $(3,1)$; domain $\{x: x \geq 3\}$, range $\{y: y \geq 1\}$; no intercepts. **Q12.** Endpoint $(-4, -2)$; domain $\{x: x \geq -4\}$, range $\{y: y \geq -2\}$; intercept $(0,0)$. **Q13.** Endpoint $(1,4)$; domain $\{x: x \geq 1\}$, range $\{y: y \leq 4\}$; intercept $(17,0)$. **Q14.** Endpoint $(2,1)$; domain $\{x: x \leq 2\}$, range $\{y: y \geq 1\}$; intercept $(0, 1+\sqrt{2})$. **Q15.** Asymptotes $x=2$, $y=1$; intercepts $(\frac{3}{2}, 0)$, $(0, \frac{3}{4})$; domain $\mathbb{R} \setminus \{2\}$, range $\mathbb{R} \setminus \{1\}$. **Q16.** Asymptotes $x=1$, $y=2$; intercepts $(3,0)$, $(0,6)$; domain $\mathbb{R} \setminus \{1\}$, range $\mathbb{R} \setminus \{2\}$. **Q17.** Asymptotes $x=2$, $y=1$; intercept $(0, \frac{7}{4})$ (no x -intercept); domain $\mathbb{R} \setminus \{2\}$, range $\{y: y > 1\}$. **Q18.** Asymptotes $x=-1$, $y=-1$; intercept $(0, -2)$ (no x -intercept); domain $\mathbb{R} \setminus \{-1\}$, range $\{y: y < -1\}$. **Q19.** Endpoint $(-4, -2)$; domain $\{x: x \geq -4\}$, range $\{y: y \geq -2\}$; intercepts $(-2,0)$, $(0, -2+2\sqrt{2})$. **Q20.** Dilation of factor 2 from the x -axis, reflection in the x -axis, then translation 3 right and 4 up; asymptotes $x=3$, $y=4$; intercepts $(3-\frac{\sqrt{2}}{2}, 0)$, $(3+\frac{\sqrt{2}}{2}, 0)$, $(0, \frac{34}{9})$; range $\{y: y < 4\}$. **Q21.** Asymptotes $x=3$, $y=1$; intercepts $(5,0)$, $(0, \frac{5}{3})$; range $\mathbb{R} \setminus \{1\}$.

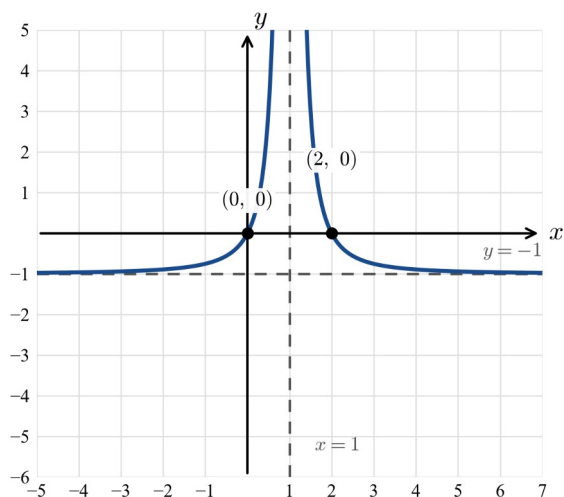
Fully-worked solutions

Q1. $y = \frac{1}{x-2} + 1$. Asymptotes $x=2$, $y=1$. y -intercept: $x=0 \Rightarrow y = \frac{1}{-2} + 1 = \frac{1}{2}$, giving $(0, \frac{1}{2})$. x -intercept: $\frac{1}{x-2} = -1 \Rightarrow x-2 = -1 \Rightarrow x=1$, giving $(1,0)$. As $a=1 > 0$, branches lie upper-right and lower-left.

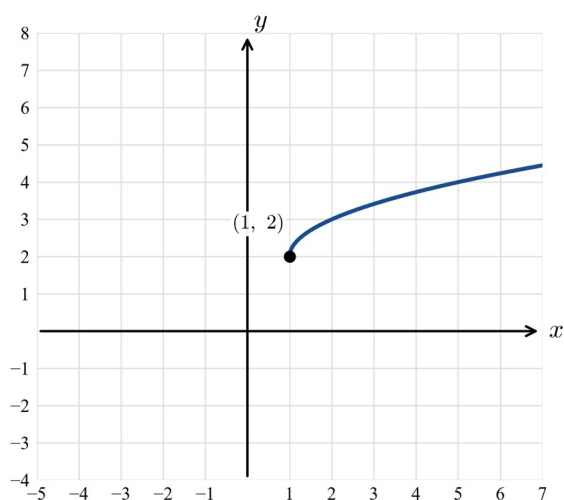


Q2. $y = \frac{1}{(x-1)^2} - 1$. Asymptotes $x = 1$, $y = -1$; since $a = 1 > 0$, range $\{y : y > -1\}$. y -intercept:

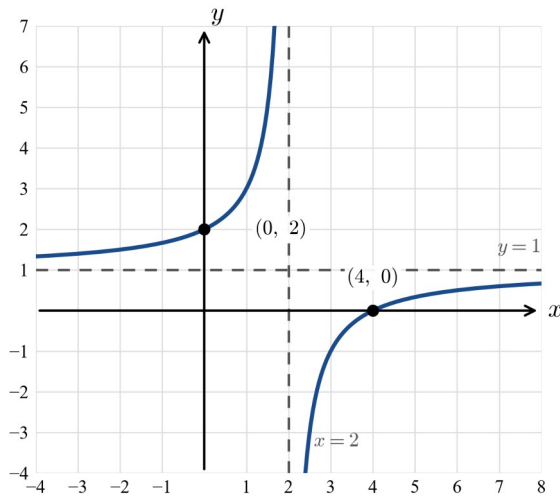
$x = 0 \Rightarrow y = \frac{1}{1} - 1 = 0$, giving $(0, 0)$. x -intercepts: $(x-1)^2 = 1 \Rightarrow x = 0$ or $x = 2$, giving $(0, 0)$ and $(2, 0)$.



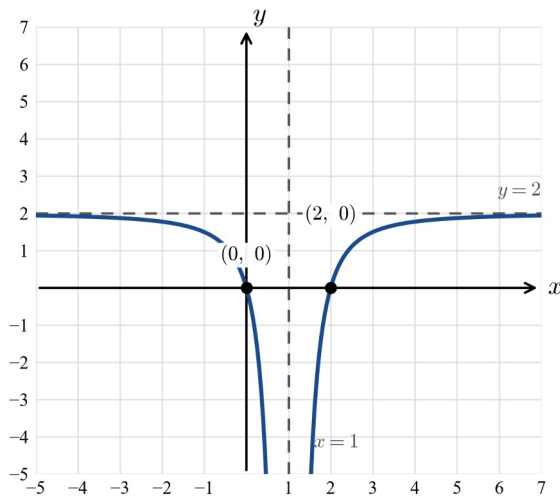
Q3. $y = \sqrt{x-1} + 2$. Endpoint $(1, 2)$; domain $\{x : x \geq 1\}$, range $\{y : y \geq 2\}$. There is no y -intercept because $x = 0$ is outside the domain, and no x -intercept because $y \geq 2 > 0$ for all x in the domain.



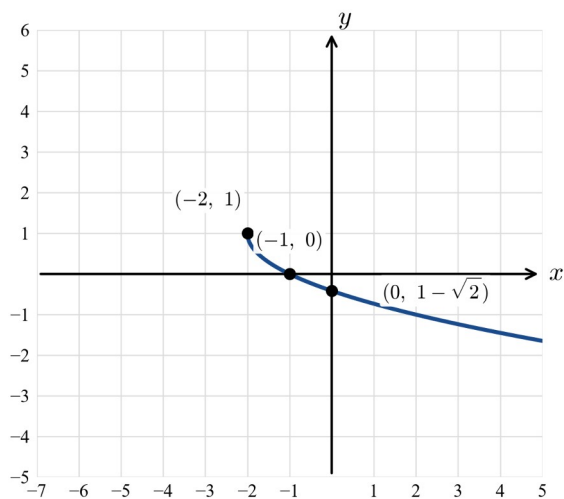
Q4. $y = -\frac{2}{x-2} + 1$. Asymptotes $x=2$, $y=1$. y -intercept: $x=0 \Rightarrow y = -\frac{2}{-2} + 1 = 2$, giving $(0, 2)$. x -intercept: $\frac{2}{x-2} = 1 \Rightarrow x-2=2 \Rightarrow x=4$, giving $(4, 0)$. (c) From $y = \frac{1}{x}$: dilation of factor 2 from the x -axis, reflection in the x -axis, then translation 2 right and 1 up.



Q5. $y = -\frac{2}{(x-1)^2} + 2$. Asymptotes $x=1$, $y=2$; $a = -2 < 0$, so range $\{y : y < 2\}$. y -intercept: $x=0 \Rightarrow y = -\frac{2}{1} + 2 = 0$, giving $(0, 0)$. x -intercepts: $\frac{2}{(x-1)^2} = 2 \Rightarrow (x-1)^2 = 1 \Rightarrow x=0$ or $x=2$, giving $(0, 0)$ and $(2, 0)$. (d) The factor 2 makes it twice as steep and the negative sign reflects it in the x -axis, so the branches point **down** from $y=2$ instead of up.

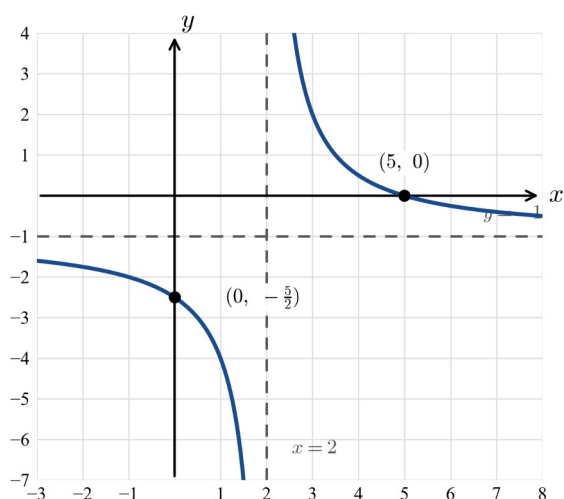


Q6. $y = -\sqrt{x+2} + 1$. Endpoint $(-2, 1)$; $a = -1 < 0$, so the branch **decreases**: domain $\{x : x \geq -2\}$, range $\{y : y \leq 1\}$. x -intercept: $\sqrt{x+2} = 1 \Rightarrow x+2=1 \Rightarrow x=-1$, giving $(-1, 0)$. y -intercept: $x=0 \Rightarrow y = -\sqrt{2} + 1 = 1 - \sqrt{2}$, giving $(0, 1 - \sqrt{2})$.



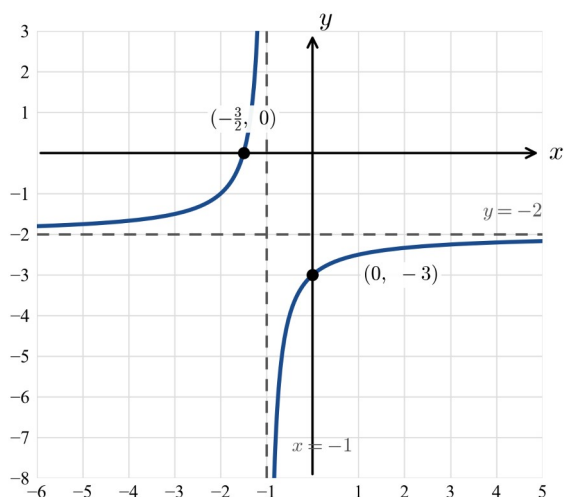
Q7. $y = \frac{3}{x-2} - 1$. Asymptotes $x=2$, $y=-1$; domain $\mathbb{R} \setminus \{2\}$, range $\mathbb{R} \setminus \{-1\}$. y -intercept:

$x=0 \Rightarrow y = \frac{3}{-2} - 1 = -\frac{5}{2}$, giving $(0, -\frac{5}{2})$. x -intercept: $\frac{3}{x-2} = 1 \Rightarrow x-2=3 \Rightarrow x=5$, giving $(5, 0)$.



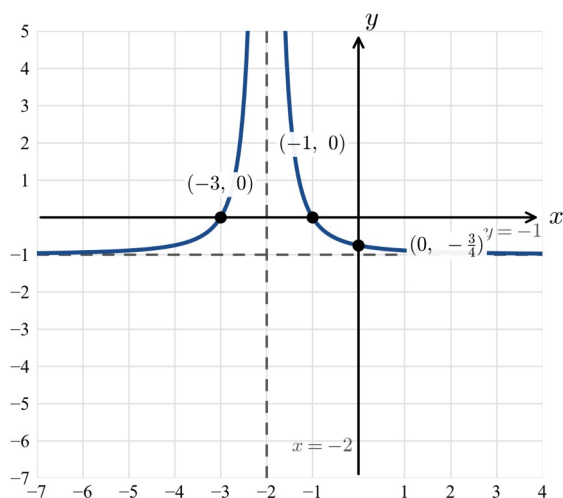
Q8. $y = -\frac{1}{x+1} - 2$. Asymptotes $x=-1$, $y=-2$; domain $\mathbb{R} \setminus \{-1\}$, range $\mathbb{R} \setminus \{-2\}$. y -intercept:

$x=0 \Rightarrow y = -\frac{1}{1} - 2 = -3$, giving $(0, -3)$. x -intercept: $-\frac{1}{x+1} = 2 \Rightarrow x+1 = -\frac{1}{2} \Rightarrow x = -\frac{3}{2}$, giving $(-\frac{3}{2}, 0)$.



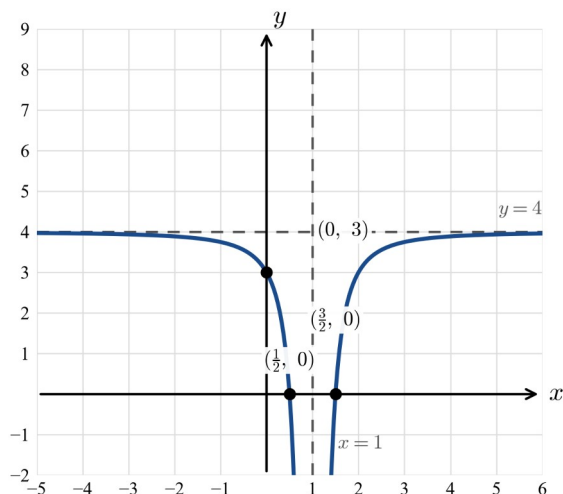
Q9. $y = \frac{1}{(x+2)^2} - 1$. Asymptotes $x = -2$, $y = -1$; domain $\mathbb{R} \setminus \{-2\}$, range $\{y : y > -1\}$. y-intercept:

$x=0 \Rightarrow y = \frac{1}{4} - 1 = -\frac{3}{4}$, giving $(0, -\frac{3}{4})$. x-intercepts: $(x+2)^2 = 1 \Rightarrow x = -1$ or $x = -3$, giving $(-1, 0)$ and $(-3, 0)$.

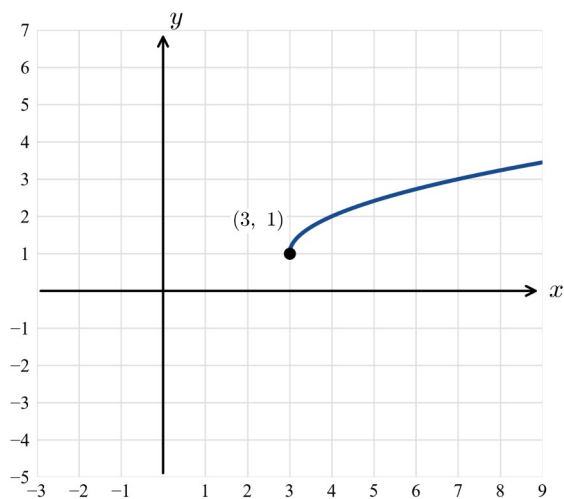


Q10. $y = -\frac{1}{(x-1)^2} + 4$. Asymptotes $x = 1$, $y = 4$; domain $\mathbb{R} \setminus \{1\}$, range $\{y : y < 4\}$. y-intercept:

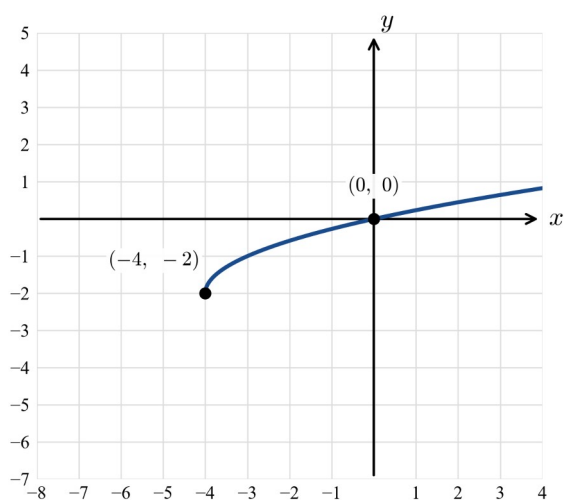
$x=0 \Rightarrow y = -\frac{1}{1} + 4 = 3$, giving $(0, 3)$. x-intercepts: $(x-1)^2 = \frac{1}{4} \Rightarrow x-1 = \pm \frac{1}{2}$, so $x = \frac{1}{2}$ or $x = \frac{3}{2}$, giving $(\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$.



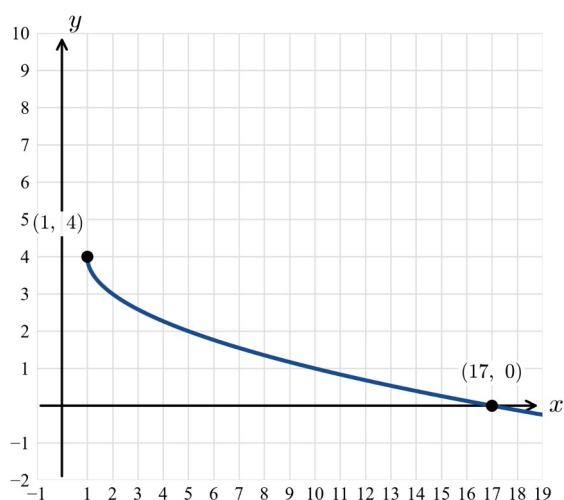
Q11. $y = \sqrt{x-3} + 1$. Endpoint $(3, 1)$; domain $\{x : x \geq 3\}$, range $\{y : y \geq 1\}$. No intercepts: $x=0$ is outside the domain and $y \geq 1 > 0$.



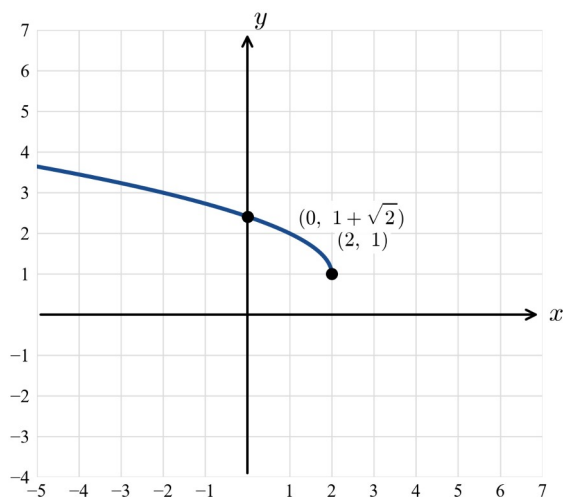
Q12. $y = \sqrt{x+4} - 2$. Endpoint $(-4, -2)$; domain $\{x : x \geq -4\}$, range $\{y : y \geq -2\}$. x -intercept:
 $\sqrt{x+4} = 2 \Rightarrow x+4 = 4 \Rightarrow x = 0$, giving $(0, 0)$, which is also the y -intercept.



Q13. $y = -\sqrt{x-1} + 4$. Endpoint $(1, 4)$; $a = -1 < 0$: domain $\{x : x \geq 1\}$, range $\{y : y \leq 4\}$. x -intercept:
 $\sqrt{x-1} = 4 \Rightarrow x-1 = 16 \Rightarrow x = 17$, giving $(17, 0)$. No y -intercept ($x = 0$ outside the domain).



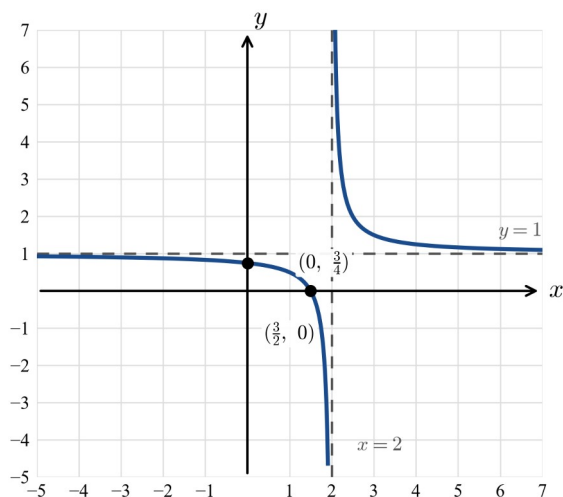
Q14. $y = \sqrt{2-x} + 1 = \sqrt{-(x-2)} + 1$. The inside is $2-x$, so the graph is reflected in the y -axis: endpoint $(2, 1)$, domain $\{x : x \leq 2\}$, range $\{y : y \geq 1\}$. No x -intercept ($y \geq 1$). y -intercept: $x = 0 \Rightarrow y = \sqrt{2} + 1$, giving $(0, 1 + \sqrt{2})$.



Q15. $y = \frac{1}{2(x-2)} + 1$. Asymptotes $x=2$, $y=1$; domain $\mathbb{R} \setminus \{2\}$, range $\mathbb{R} \setminus \{1\}$. y -intercept:

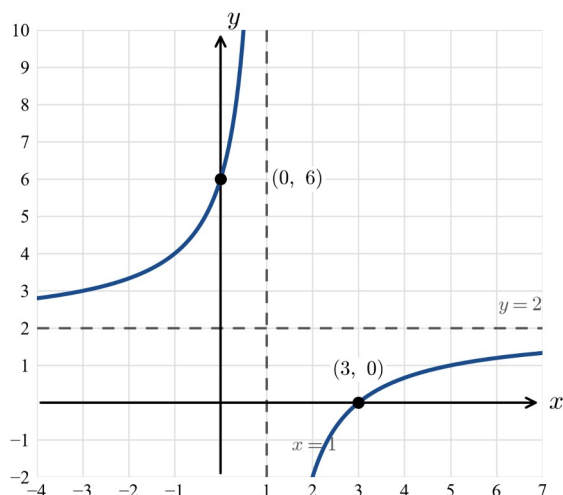
$x=0 \Rightarrow y = \frac{1}{2(-2)} + 1 = -\frac{1}{4} + 1 = \frac{3}{4}$, giving $(0, \frac{3}{4})$. x -intercept:

$\frac{1}{2(x-2)} = -1 \Rightarrow 2(x-2) = -1 \Rightarrow x-2 = -\frac{1}{2} \Rightarrow x = \frac{3}{2}$, giving $(\frac{3}{2}, 0)$.

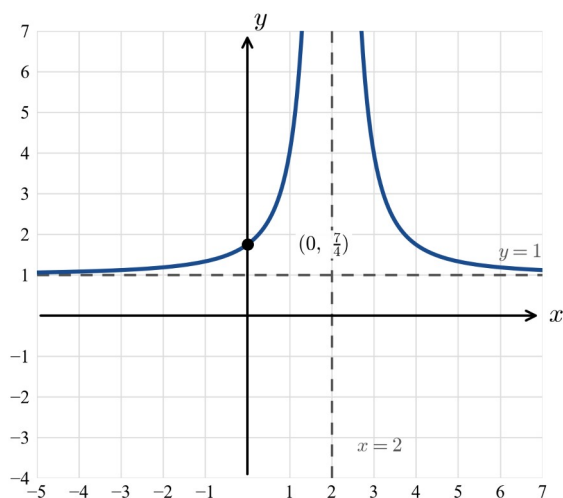


Q16. $y = -\frac{4}{x-1} + 2$. Asymptotes $x=1$, $y=2$; domain $\mathbb{R} \setminus \{1\}$, range $\mathbb{R} \setminus \{2\}$. y -intercept: $x=0 \Rightarrow y = -\frac{4}{-1} + 2 = 6$,

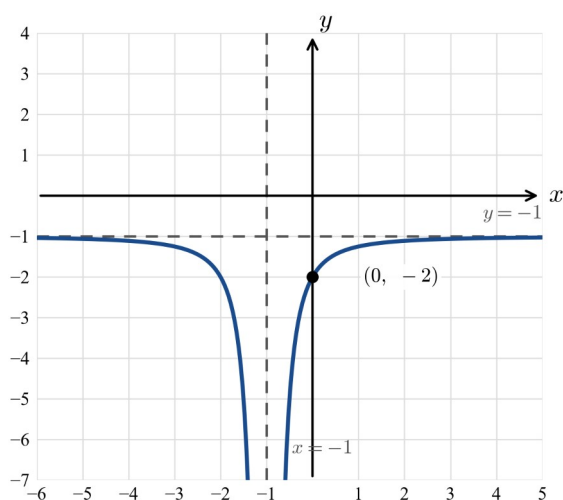
giving $(0, 6)$. x -intercept: $\frac{4}{x-1} = 2 \Rightarrow x-1 = 2 \Rightarrow x = 3$, giving $(3, 0)$.



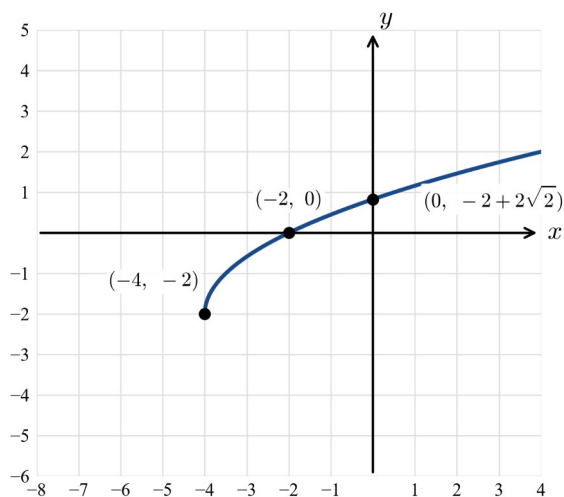
Q17. $y = \frac{3}{(x-2)^2} + 1$. Asymptotes $x=2$, $y=1$; domain $\mathbb{R} \setminus \{2\}$; $a=3>0$, range $\{y: y>1\}$, so there is **no** x -intercept. y -intercept: $x=0 \Rightarrow y = \frac{3}{4} + 1 = \frac{7}{4}$, giving $(0, \frac{7}{4})$.



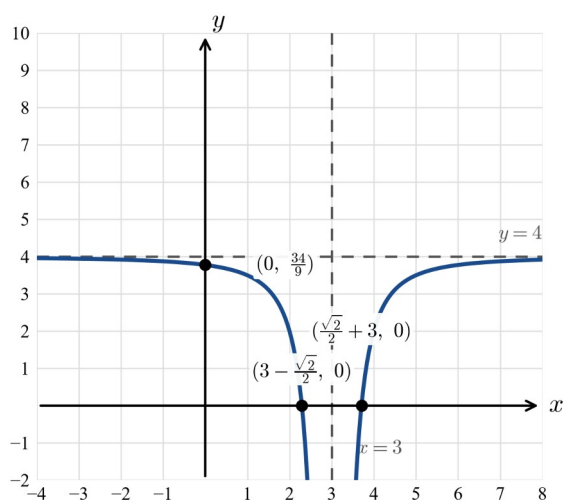
Q18. $y = -\frac{1}{(x+1)^2} - 1$. Asymptotes $x=-1$, $y=-1$; domain $\mathbb{R} \setminus \{-1\}$; $a=-1<0$, range $\{y: y<-1\}$, so there is **no** x -intercept. y -intercept: $x=0 \Rightarrow y = -\frac{1}{1} - 1 = -2$, giving $(0, -2)$.



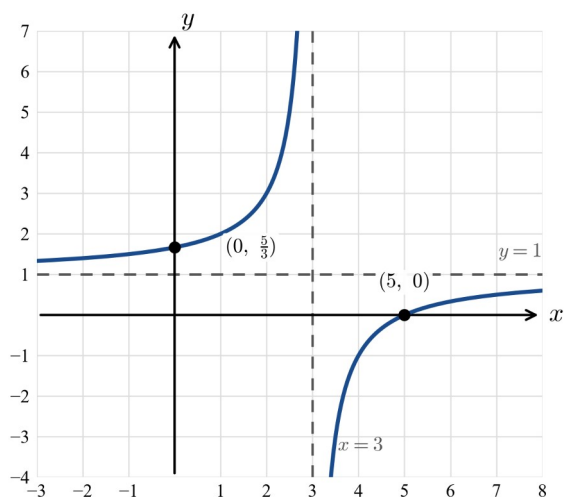
Q19. $y = \sqrt{2(x+4)} - 2$. Endpoint where $2(x+4)=0$, i.e. $(-4, -2)$; domain $\{x: x \geq -4\}$, range $\{y: y \geq -2\}$. x -intercept: $\sqrt{2(x+4)} = 2 \Rightarrow 2(x+4) = 4 \Rightarrow x+4 = 2 \Rightarrow x = -2$, giving $(-2, 0)$. y -intercept: $x=0 \Rightarrow y = \sqrt{8} - 2 = 2\sqrt{2} - 2$, giving $(0, -2 + 2\sqrt{2})$.



Q20. $y = -\frac{2}{(x-3)^2} + 4$. From $y = \frac{1}{x^2}$: a dilation of factor 2 from the x -axis, a reflection in the x -axis, then a translation 3 right and 4 up. Asymptotes $x=3$, $y=4$; range $\{y : y < 4\}$. y -intercept: $x=0 \Rightarrow y = -\frac{2}{9} + 4 = \frac{34}{9}$, giving $(0, \frac{34}{9})$. x -intercepts: $\frac{2}{(x-3)^2} = 4 \Rightarrow (x-3)^2 = \frac{1}{2} \Rightarrow x-3 = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$, giving $(3 - \frac{\sqrt{2}}{2}, 0)$ and $(3 + \frac{\sqrt{2}}{2}, 0)$.



Q21. $y = -\frac{2}{x-3} + 1$. Asymptotes $x=3$, $y=1$. x -intercept: $\frac{2}{x-3} = 1 \Rightarrow x-3=2 \Rightarrow x=5$, giving $(5, 0)$. y -intercept: $x=0 \Rightarrow y = -\frac{2}{-3} + 1 = \frac{2}{3} + 1 = \frac{5}{3}$, giving $(0, \frac{5}{3})$.



Chapter 3 Transformations of the plane — 3G Transformations of power functions

Theory

A **power function with a positive integer index** is $y = x^n$ where $n = 1, 2, 3, \dots$. This section applies the transformations of Sections 3A–3D to sketch graphs of the form

$$y = a(x - h)^n + k.$$

The two shapes. The value of n decides the basic shape.

- n **even** ($y = x^2, x^4, x^6, \dots$): a $\cup$ -shaped graph, symmetric about the y -axis, with a **turning point** at the origin. The transformed graph $y = a(x - h)^n + k$ has a turning point at (h, k) — a **minimum** if $a > 0$ and a **maximum** if $a < 0$.
- n **odd** ($y = x^3, x^5, \dots$, with $n \geq 3$): an increasing graph with a **stationary point of inflection** at the origin (it flattens out but does not turn back). The transformed graph $y = a(x - h)^n + k$ has a **stationary point of inflection** at (h, k) .

Reading the transformed rule $y = a(x - h)^n + k$.

Feature	Even n	Odd n ($n \geq 3$)
Key point (h, k)	turning point (min if $a > 0$, max if $a < 0$)	stationary point of inflection
Range	$\{y : y \geq k\}$ if $a > 0$; $\{y : y \leq k\}$ if $a < 0$	$\mathbb{R}$
Domain	$\mathbb{R}$	$\mathbb{R}$
a is a dilation of factor a from the x-axis (with a reflection in the x-axis when $a < 0$); h and k give a translation of h horizontally and k vertically. Apply the dilation/reflection before the translation. y-intercept: substitute $x = 0$, giving $(0, a(-h)^n + k)$. x-intercepts: solve $a(x - h)^n + k = 0$, i.e. $(x - h)^n = -\frac{k}{a}$. For odd n there is always exactly one real solution; for even n there are two if $-\frac{k}{a} > 0$, one (a touch, at $(h, 0)$) if $-\frac{k}{a} = 0$, and none if $-\frac{k}{a} < 0$. Give exact values.		

To sketch: identify n (shape) and the sign of a (orientation); plot the key point (h, k) ; find the exact axis intercepts; then draw the curve with the correct shape through those points.

Worked examples

Example 1 — scaffolded

Sketch $y = (x - 2)^3 + 1$, stating the coordinates of the stationary point of inflection and the exact axis intercepts.

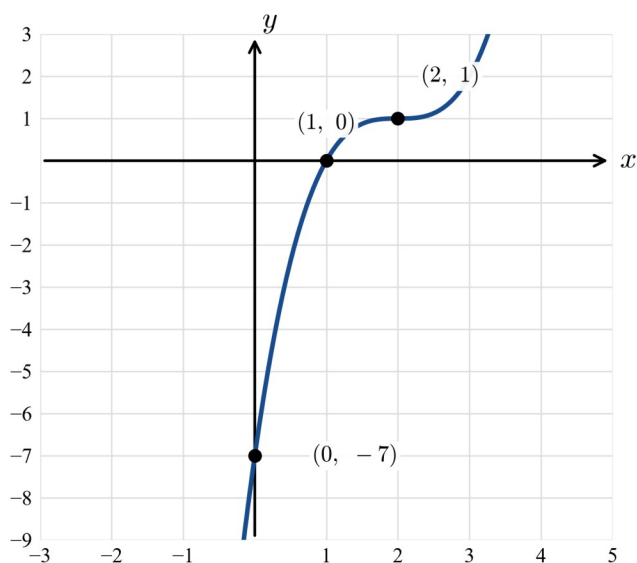
Working	Comment
Base graph $y = x^3$ ($n = 3$, odd), with $a = 1$, $h = 2$, $k = 1$.	Compare with $y = a(x - h)^n + k$.
n is odd, so there is a stationary point of inflection at $(2, 1)$.	For odd n the key point (h, k) is a stationary point of inflection.
y-intercept: $x = 0 \Rightarrow y = (0 - 2)^3 + 1 = -8 + 1 = -7$, giving $(0, -7)$.	Substitute $x = 0$.
x-intercept:	Take the cube root — an odd power gives exactly one

$(x-2)^3 + 1 = 0 \Rightarrow (x-2)^3 = -1 \Rightarrow x-2 = -1$, so $x = 1$, giving $(1, 0)$.

real solution.

$a = 1 > 0$, so the graph increases through the inflection. Range $\mathbb{R}$.

An odd-index power function has range $\mathbb{R}$.



Example 2 — scaffolded

Sketch $y = (x+1)^4 - 16$, labelling the turning point and all axis intercepts, and state the range.

Base graph $y = x^4$ ($n = 4$, even), with $a = 1$, $h = -1$, $k = -16$.

Compare with $y = a(x-h)^n + k$; here $x-h = x+1$.

n even and $a = 1 > 0$: a **minimum** turning point at $(-1, -16)$; range $\{y: y \geq -16\}$.

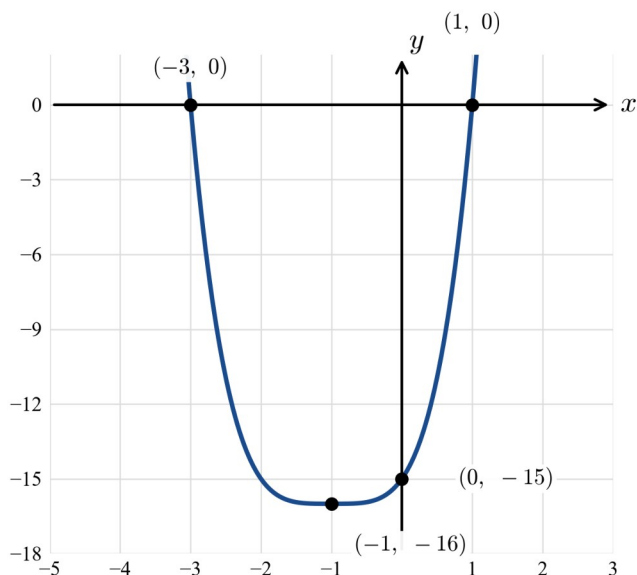
Even n with $a > 0$ gives a minimum and range $y \geq k$.

y-intercept: $x = 0 \Rightarrow y = (0+1)^4 - 16 = 1 - 16 = -15$, giving $(0, -15)$.

Substitute $x = 0$.

x-intercepts: $(x+1)^4 = 16 \Rightarrow x+1 = \pm 2$, so $x = 1$ or $x = -3$, giving $(1, 0)$ and $(-3, 0)$.

An even root gives $\pm$; here $\sqrt[4]{16} = 2$.



Example 3 — unscaffolded

The graph of $y = x^3$ is transformed to give $y = -2(x+1)^3 + 4$. Describe the sequence of transformations, then sketch the graph.

Writing the rule as $y = -2(x - (-1))^3 + 4$: $a = -2$, $h = -1$, $k = 4$. From $y = x^3$:

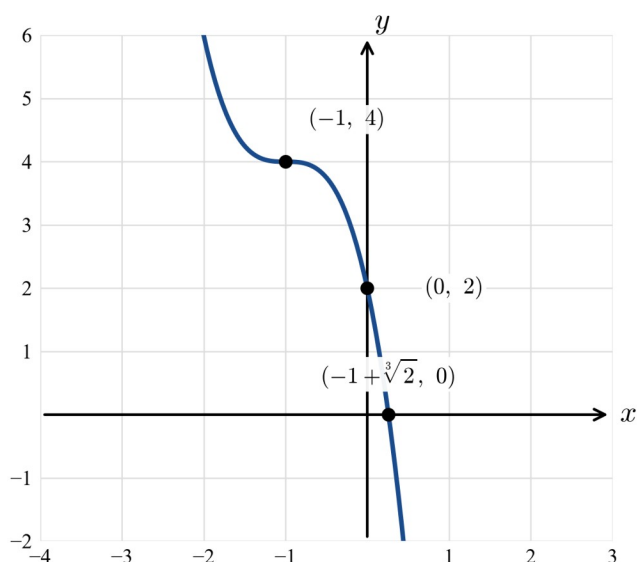
1. a **dilation of factor 2 from the x -axis**;
2. a **reflection in the x -axis** (since $a < 0$);
3. a **translation of 1 unit left and 4 units up**.

Stationary point of inflection $(-1, 4)$; range $\mathbb{R}$.

y -intercept: $x = 0 \Rightarrow y = -2(1)^3 + 4 = 2$, giving $(0, 2)$.

x -intercept: $-2(x+1)^3 + 4 = 0 \Rightarrow (x+1)^3 = 2 \Rightarrow x+1 = \sqrt[3]{2}$, so $x = -1 + \sqrt[3]{2}$, giving $(-1 + \sqrt[3]{2}, 0)$.

$\therefore$ the reflection makes the graph **decrease** through the inflection at $(-1, 4)$.



Example 4 — unscaffolded

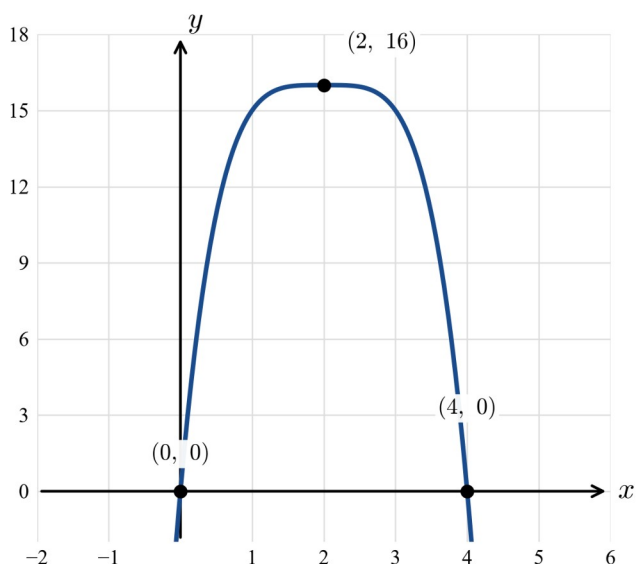
Sketch $y = -(x-2)^4 + 16$, labelling the turning point and axis intercepts, and state the range.

$n = 4$ (even) with $a = -1 < 0$: a **maximum** turning point at $(2, 16)$; range $\{y : y \leq 16\}$.

y -intercept: $x = 0 \Rightarrow y = -(0-2)^4 + 16 = -16 + 16 = 0$, giving $(0, 0)$.

x -intercepts: $-(x-2)^4 + 16 = 0 \Rightarrow (x-2)^4 = 16 \Rightarrow x-2 = \pm 2$, so $x = 0$ or $x = 4$, giving $(0, 0)$ and $(4, 0)$.

$\therefore$ the graph rises to the maximum $(2, 16)$ and cuts the axis at $(0, 0)$ and $(4, 0)$.



Practice questions

Scaffolded

Q1. For $y = (x - 1)^3 - 8$: (a) state the coordinates and type of the key point; (b) find the exact axis intercepts; (c) state the range; (d) sketch the graph.

Q2. For $y = (x + 2)^4 - 1$: (a) state the coordinates and type of the key point; (b) find all axis intercepts; (c) state the range; (d) sketch the graph.

Q3. For $y = 2(x - 1)^3$: (a) state the stationary point of inflection; (b) find the axis intercepts; (c) describe the transformations that map $y = x^3$ onto this graph; (d) sketch the graph.

Q4. For $y = -(x + 1)^4 + 16$: (a) state the turning point and the range; (b) find all axis intercepts; (c) sketch the graph.

Q5. For $y = (x - 1)^5 + 1$: (a) state the stationary point of inflection; (b) find the axis intercepts; (c) sketch the graph.

Q6. For $y = -2(x - 1)^4 + 8$: (a) state the turning point and the range; (b) find the exact axis intercepts; (c) sketch the graph.

Un scaffolded

Sketch each of the following, labelling the key point (turning point or stationary point of inflection) and the exact coordinates of all axis intercepts, and state the range.

Q7. $y = x^3 - 1$

Q8. $y = (x - 2)^3 - 8$

Q9. $y = 2(x + 1)^3 - 2$

Q10. $y = x^4 - 16$

Q11. $y = -(x - 2)^4 + 1$

Q12. $y = (x + 1)^5 - 32$

Q13. $y = (x - 1)^4 - 1$

Q14. Describe the sequence of transformations that maps $y = x^3$ onto $y = -2x^3 + 2$, and hence sketch the graph.

Q15. $y = 3(x + 1)^3$

Q16. $y = (x - 2)^4 - 16$

Q17. $y = 2(x + 1)^4 - 2$

Q18. $y = (x + 1)^3 + 1$

Q19. The graph of $y = x^3$ is translated so that its stationary point of inflection is at $(2, -3)$. Find the rule of the new graph, then sketch it, showing the exact x -intercept.

Q20. Describe the sequence of transformations that maps $y = x^3$ onto $y = -(x - 1)^3 + 1$, and hence sketch the graph.

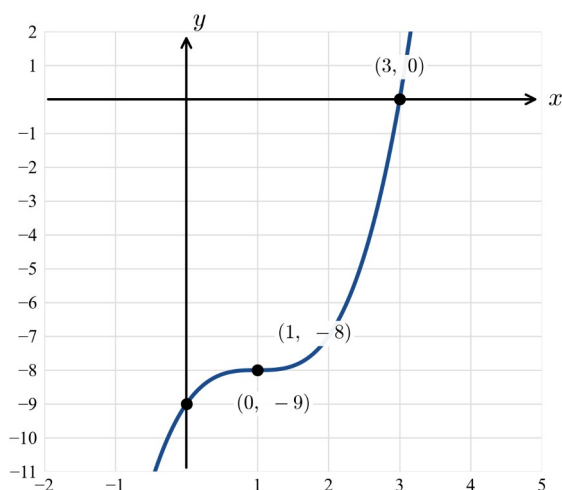
Q21. $y = -2(x + 1)^4 + 2$

Answers

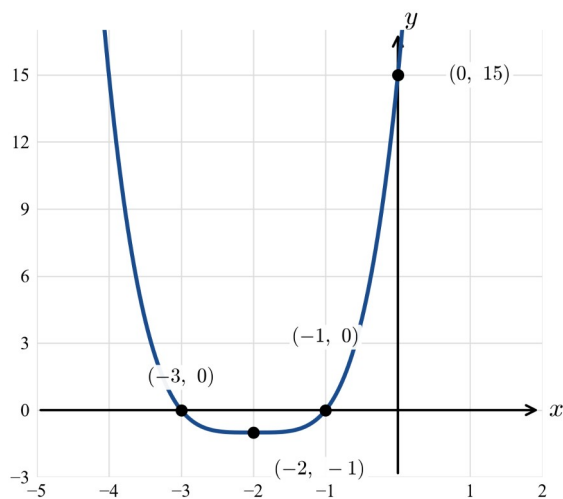
Q1. SPI $(1, -8)$; intercepts $(3, 0), (0, -9)$; range $\mathbb{R}$. **Q2.** Minimum $(-2, -1)$; intercepts $(-3, 0), (-1, 0), (0, 15)$; range $\{y: y \geq -1\}$. **Q3.** SPI $(1, 0)$; intercepts $(1, 0), (0, -2)$; (c) dilation factor 2 from the x -axis, then translation 1 right. **Q4.** Maximum $(-1, 16)$, range $\{y: y \leq 16\}$; intercepts $(-3, 0), (1, 0), (0, 15)$. **Q5.** SPI $(1, 1)$; intercept $(0, 0)$ (both axes); range $\mathbb{R}$. **Q6.** Maximum $(1, 8)$, range $\{y: y \leq 8\}$; intercepts $(1 - \sqrt{2}, 0), (1 + \sqrt{2}, 0), (0, 6)$. **Q7.** SPI $(0, -1)$; intercepts $(1, 0), (0, -1)$; range $\mathbb{R}$. **Q8.** SPI $(2, -8)$; intercepts $(4, 0), (0, -16)$; range $\mathbb{R}$. **Q9.** SPI $(-1, -2)$; intercept $(0, 0)$ (both axes); range $\mathbb{R}$. **Q10.** Minimum $(0, -16)$; intercepts $(-2, 0), (2, 0), (0, -16)$; range $\{y: y \geq -16\}$. **Q11.** Maximum $(2, 1)$; intercepts $(1, 0), (3, 0), (0, -15)$; range $\{y: y \leq 1\}$. **Q12.** SPI $(-1, -32)$; intercepts $(1, 0), (0, -31)$; range $\mathbb{R}$. **Q13.** Minimum $(1, -1)$; intercepts $(0, 0), (2, 0)$; range $\{y: y \geq -1\}$. **Q14.** Dilation factor 2 from the x -axis, reflection in the x -axis, translation 2 up; SPI $(0, 2)$; intercepts $(1, 0), (0, 2)$; range $\mathbb{R}$. **Q15.** SPI $(-1, 0)$; intercepts $(-1, 0), (0, 3)$; range $\mathbb{R}$. **Q16.** Minimum $(2, -16)$; intercepts $(0, 0), (4, 0)$; range $\{y: y \geq -16\}$. **Q17.** Minimum $(-1, -2)$; intercepts $(-2, 0), (0, 0)$; range $\{y: y \geq -2\}$. **Q18.** SPI $(-1, 1)$; intercepts $(-2, 0), (0, 2)$; range $\mathbb{R}$. **Q19.** $y = (x - 2)^3 - 3$; intercepts $(2 + \sqrt[3]{3}, 0), (0, -11)$; range $\mathbb{R}$. **Q20.** Reflection in the x -axis, translation 1 right and 1 up; SPI $(1, 1)$; intercepts $(2, 0), (0, 2)$; range $\mathbb{R}$. **Q21.** Maximum $(-1, 2)$; intercepts $(-2, 0), (0, 0)$; range $\{y: y \leq 2\}$.

Fully-worked solutions

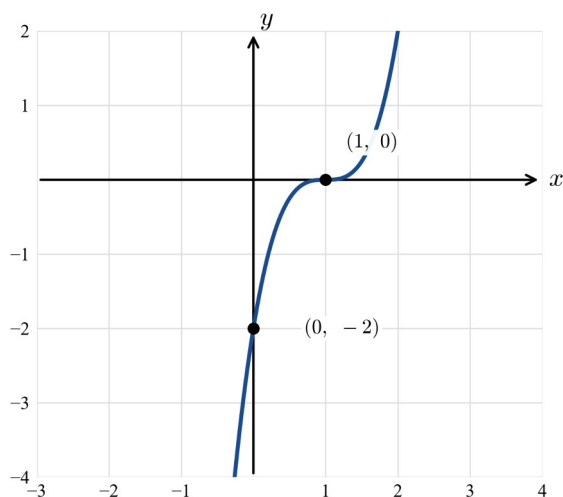
Q1. $y = (x - 1)^3 - 8$ (odd n). SPI $(1, -8)$; range $\mathbb{R}$. x -intercept: $(x - 1)^3 = 8 \Rightarrow x - 1 = 2 \Rightarrow x = 3$, giving $(3, 0)$. y -intercept: $x = 0 \Rightarrow y = (-1)^3 - 8 = -9$, giving $(0, -9)$.



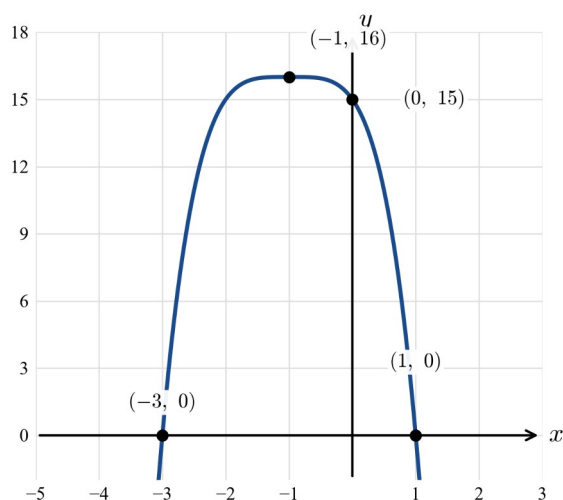
Q2. $y = (x + 2)^4 - 1$ (even $n, a > 0$). Minimum $(-2, -1)$; range $\{y: y \geq -1\}$. x -intercepts: $(x + 2)^4 = 1 \Rightarrow x + 2 = \pm 1$, so $x = -1$ or $x = -3$, giving $(-1, 0)$ and $(-3, 0)$. y -intercept: $x = 0 \Rightarrow y = 2^4 - 1 = 15$, giving $(0, 15)$.



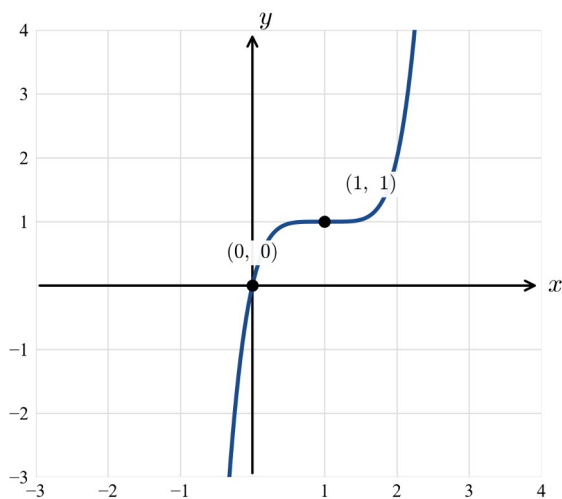
Q3. $y = 2(x - 1)^3$. SPI $(1, 0)$. x -intercept: $2(x - 1)^3 = 0 \Rightarrow x = 1$, giving $(1, 0)$. y -intercept: $x = 0 \Rightarrow y = 2(-1)^3 = -2$, giving $(0, -2)$. (c) From $y = x^3$: a dilation of factor 2 from the x -axis, then a translation of 1 unit right.



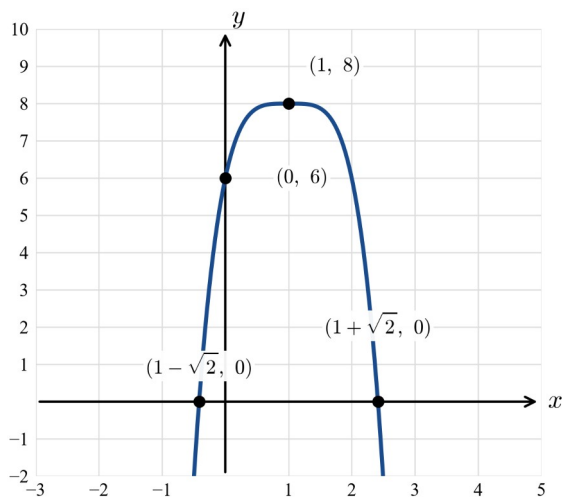
Q4. $y = -(x + 1)^4 + 16$ (even n , $a < 0$). Maximum $(-1, 16)$; range $\{y : y \leq 16\}$. x -intercepts: $(x + 1)^4 = 16 \Rightarrow x + 1 = \pm 2$, so $x = 1$ or $x = -3$, giving $(1, 0)$ and $(-3, 0)$. y -intercept: $x = 0 \Rightarrow y = -1 + 16 = 15$, giving $(0, 15)$.



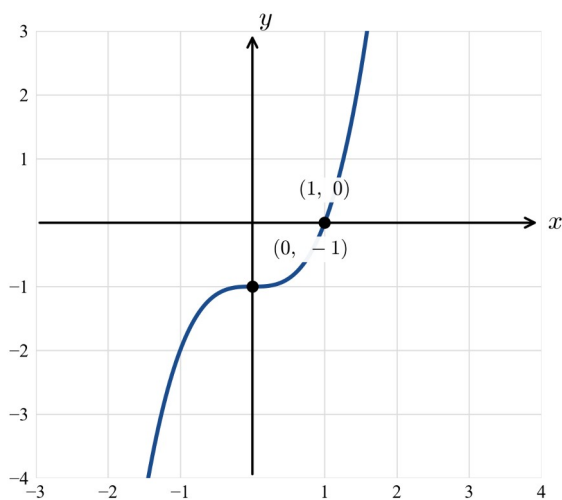
Q5. $y = (x - 1)^5 + 1$ (odd n). SPI $(1, 1)$; range $\mathbb{R}$. x -intercept: $(x - 1)^5 = -1 \Rightarrow x - 1 = -1 \Rightarrow x = 0$, giving $(0, 0)$, which is also the y -intercept.



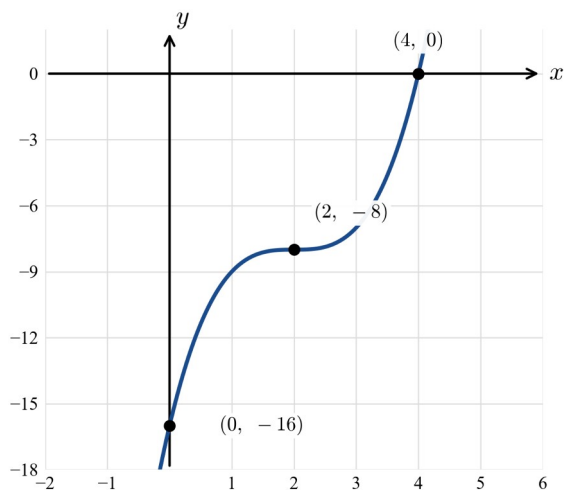
Q6. $y = -2(x-1)^4 + 8$ (even n , $a < 0$). Maximum $(1, 8)$; range $\{y : y \leq 8\}$. x -intercepts:
 $2(x-1)^4 = 8 \Rightarrow (x-1)^4 = 4 \Rightarrow x-1 = \pm\sqrt[4]{4} = \pm\sqrt{2}$ (since $\sqrt[4]{4} = \sqrt{2}$), giving $(1-\sqrt{2}, 0)$ and $(1+\sqrt{2}, 0)$. y -intercept:
 $x=0 \Rightarrow y = -2(1)^4 + 8 = 6$, giving $(0, 6)$.



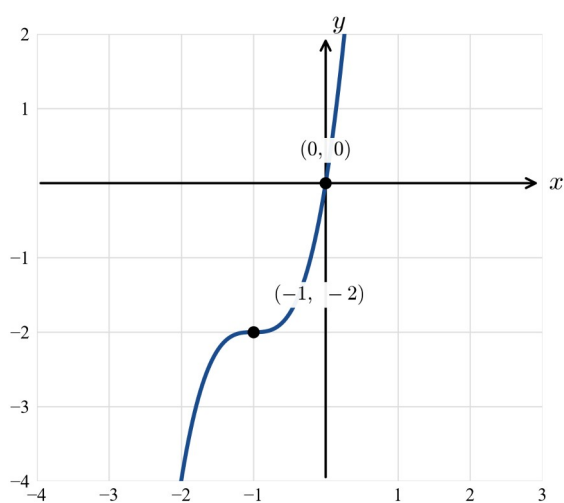
Q7. $y = x^3 - 1$. SPI $(0, -1)$; range $\mathbb{R}$. x -intercept: $x^3 = 1 \Rightarrow x = 1$, giving $(1, 0)$. y -intercept $(0, -1)$.



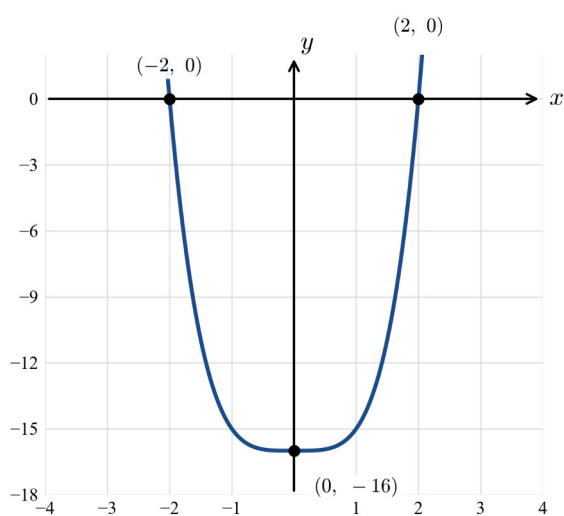
Q8. $y = (x-2)^3 - 8$. SPI $(2, -8)$; range $\mathbb{R}$. x -intercept: $(x-2)^3 = 8 \Rightarrow x-2 = 2 \Rightarrow x = 4$, giving $(4, 0)$. y -intercept:
 $x=0 \Rightarrow y = (-2)^3 - 8 = -16$, giving $(0, -16)$.



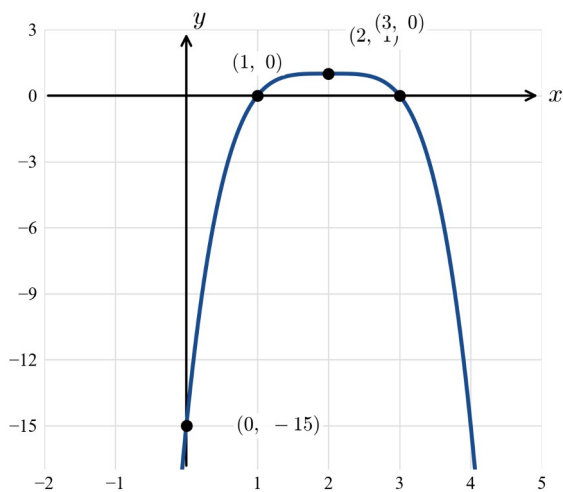
Q9. $y = 2(x+1)^3 - 2$. SPI $(-1, -2)$; range $\mathbb{R}$. x -intercept: $2(x+1)^3 = 2 \Rightarrow (x+1)^3 = 1 \Rightarrow x = 0$, giving $(0, 0)$, also the y -intercept.



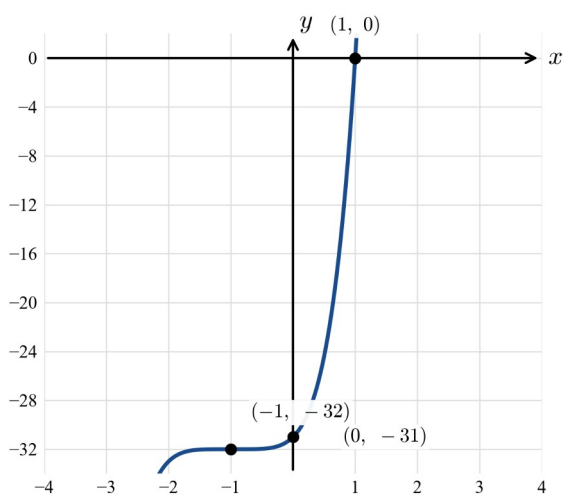
Q10. $y = x^4 - 16$. Minimum $(0, -16)$; range $\{y : y \geq -16\}$. x -intercepts: $x^4 = 16 \Rightarrow x = \pm 2$, giving $(-2, 0)$ and $(2, 0)$. y -intercept $(0, -16)$.



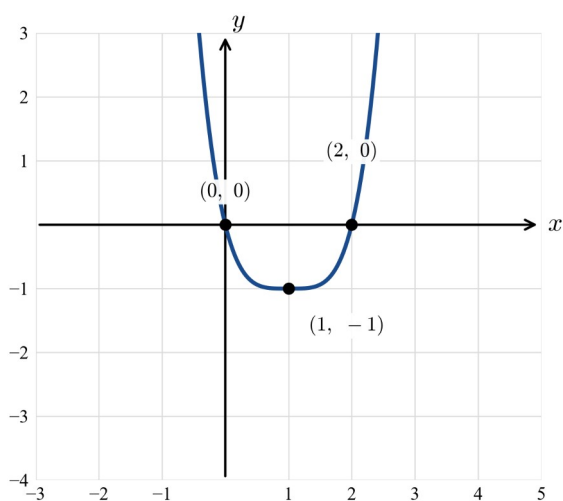
Q11. $y = -(x-2)^4 + 1$. Maximum $(2, 1)$; range $\{y : y \leq 1\}$. x -intercepts: $(x-2)^4 = 1 \Rightarrow x-2 = \pm 1$, so $x = 1$ or $x = 3$, giving $(1, 0)$ and $(3, 0)$. y -intercept: $x = 0 \Rightarrow y = -16 + 1 = -15$, giving $(0, -15)$.



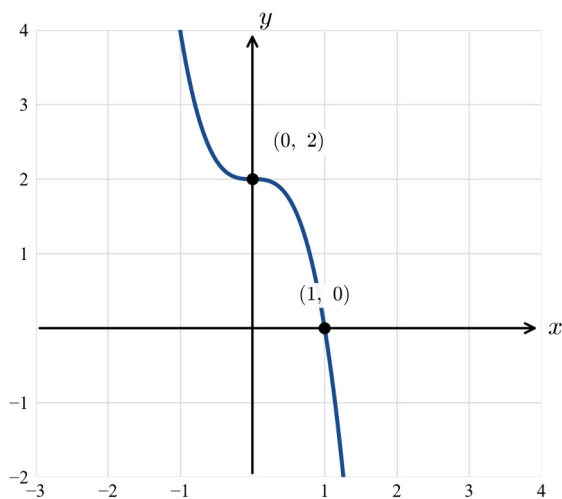
Q12. $y = (x+1)^5 - 32$. SPI $(-1, -32)$; range $\mathbb{R}$. x -intercept: $(x+1)^5 = 32 \Rightarrow x+1=2 \Rightarrow x=1$, giving $(1, 0)$. y -intercept: $x=0 \Rightarrow y=1-32=-31$, giving $(0, -31)$.



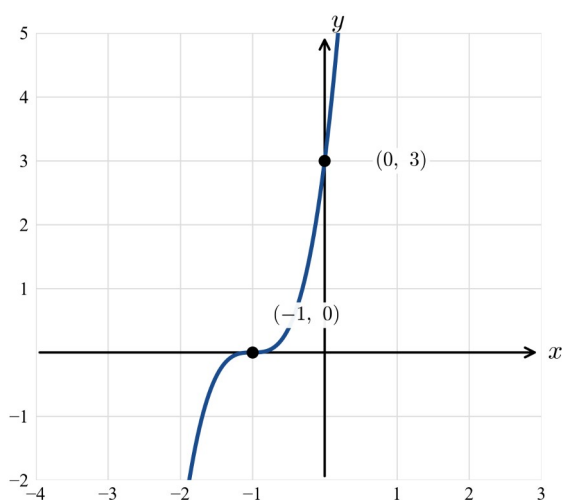
Q13. $y = (x-1)^4 - 1$. Minimum $(1, -1)$; range $\{y : y \geq -1\}$. x -intercepts: $(x-1)^4 = 1 \Rightarrow x-1 = \pm 1$, so $x=0$ or $x=2$, giving $(0, 0)$ and $(2, 0)$. y -intercept $(0, 0)$.



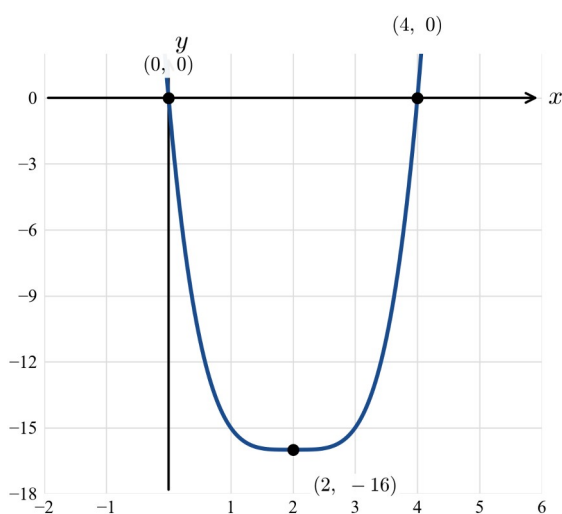
Q14. $y = -2x^3 + 2$. From $y = x^3$: a dilation of factor 2 from the x -axis, a reflection in the x -axis, then a translation of 2 units up. SPI $(0, 2)$; range $\mathbb{R}$. x -intercept: $-2x^3 + 2 = 0 \Rightarrow x^3 = 1 \Rightarrow x=1$, giving $(1, 0)$. y -intercept $(0, 2)$.



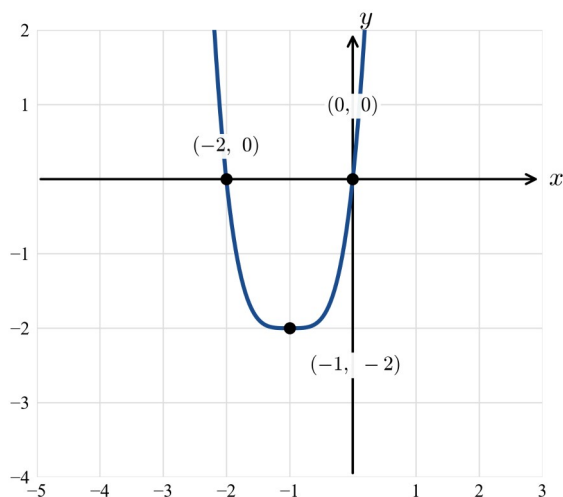
Q15. $y = 3(x+1)^3$. SPI $(-1, 0)$; range $\mathbb{R}$. x -intercept: $3(x+1)^3 = 0 \Rightarrow x = -1$, giving $(-1, 0)$. y -intercept: $x = 0 \Rightarrow y = 3(1)^3 = 3$, giving $(0, 3)$.



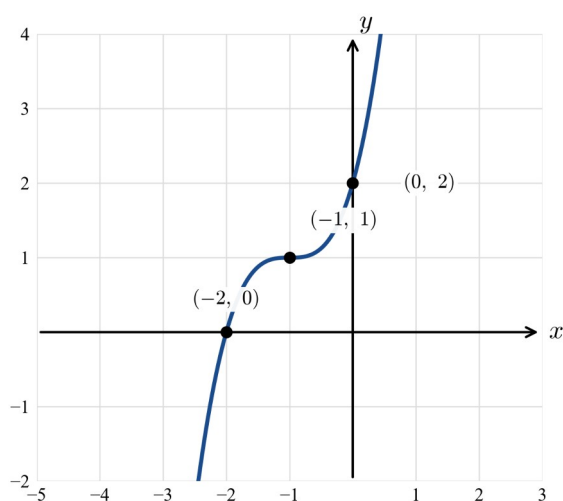
Q16. $y = (x-2)^4 - 16$. Minimum $(2, -16)$; range $\{y : y \geq -16\}$. x -intercepts: $(x-2)^4 = 16 \Rightarrow x-2 = \pm 2$, so $x = 0$ or $x = 4$, giving $(0, 0)$ and $(4, 0)$. y -intercept $(0, 0)$.



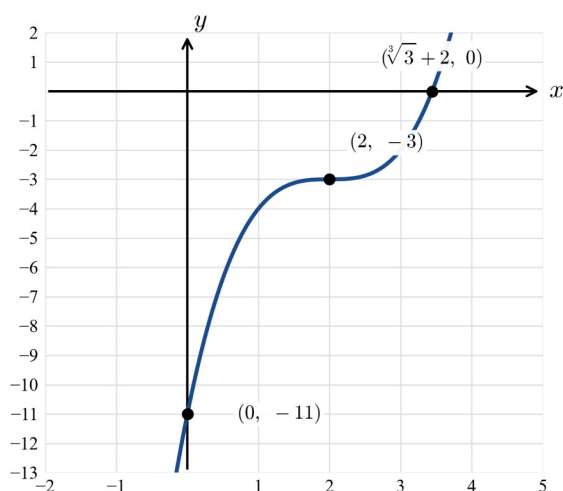
Q17. $y = 2(x+1)^4 - 2$. Minimum $(-1, -2)$; range $\{y : y \geq -2\}$. x -intercepts: $2(x+1)^4 = 2 \Rightarrow (x+1)^4 = 1 \Rightarrow x+1 = \pm 1$, so $x = 0$ or $x = -2$, giving $(0, 0)$ and $(-2, 0)$. y -intercept $(0, 0)$.



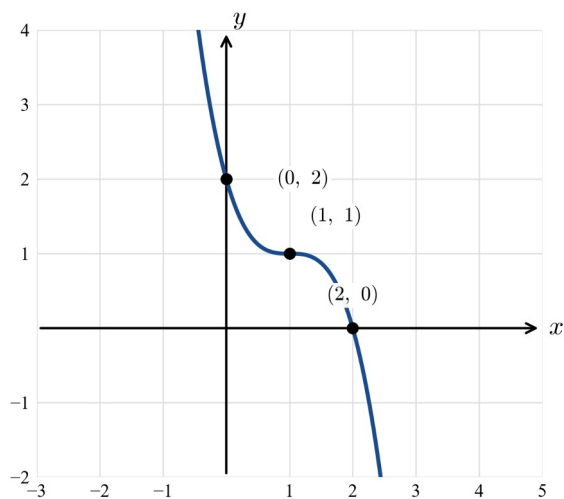
Q18. $y = (x+1)^3 + 1$. SPI $(-1, 1)$; range $\mathbb{R}$. x -intercept: $(x+1)^3 = -1 \Rightarrow x+1 = -1 \Rightarrow x = -2$, giving $(-2, 0)$. y -intercept: $x = 0 \Rightarrow y = 1 + 1 = 2$, giving $(0, 2)$.



Q19. Translating $y = x^3$ so its stationary point of inflection moves from $(0, 0)$ to $(2, -3)$ gives $y = (x-2)^3 - 3$. x -intercept: $(x-2)^3 = 3 \Rightarrow x-2 = \sqrt[3]{3}$, so $x = 2 + \sqrt[3]{3}$, giving $(2 + \sqrt[3]{3}, 0)$. y -intercept: $x = 0 \Rightarrow y = (-2)^3 - 3 = -11$, giving $(0, -11)$.

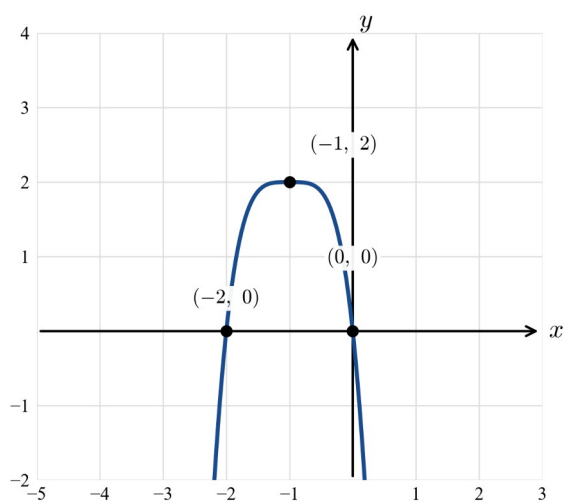


Q20. $y = -(x-1)^3 + 1$. From $y = x^3$: a reflection in the x -axis, then a translation of 1 unit right and 1 unit up. SPI $(1, 1)$; range $\mathbb{R}$. x -intercept: $-(x-1)^3 + 1 = 0 \Rightarrow (x-1)^3 = 1 \Rightarrow x = 2$, giving $(2, 0)$. y -intercept: $x = 0 \Rightarrow y = -(-1)^3 + 1 = 2$, giving $(0, 2)$.



Q21. $y = -2(x+1)^4 + 2$. Maximum $(-1, 2)$; range $\{y : y \leq 2\}$. x -intercepts:

$2(x+1)^4 = 2 \Rightarrow (x+1)^4 = 1 \Rightarrow x+1 = \pm 1$, so $x = 0$ or $x = -2$, giving $(0, 0)$ and $(-2, 0)$. y -intercept $(0, 0)$.



Chapter 3 Transformations of the plane — 3H Finding rules from graphs

Theory

The previous sections started from a rule and produced a graph. This section reverses the process: given the **graph** of a transformed standard function, we recover its **rule**.

The standard graphs and their tell-tale features.

Base graph	General transformed rule	Read directly from the graph
Parabola $y = x^2$	$y = a(x - h)^2 + k$	turning point (h, k)
Cubic $y = x^3$	$y = a(x - h)^3 + k$	stationary point of inflection (h, k)
Hyperbola $y = \frac{1}{x}$	$y = \frac{a}{x - h} + k$	asymptotes $x = h, y = k$
Reciprocal-square $y = \frac{1}{x^2}$	$y = \frac{a}{(x - h)^2} + k$	asymptotes $x = h, y = k$
Square root $y = \sqrt{x}$	$y = a\sqrt{x - h} + k$	endpoint (h, k)

The method.

1. **Identify the base graph** from its shape — a parabola turns once, a cubic has a point of inflection, a $\frac{1}{x}$ - or $\frac{1}{x^2}$ -form graph has two asymptotes (the $\frac{1}{x^2}$ form has both branches on the same side of the horizontal asymptote), a square-root graph starts at an endpoint.
2. **Read h and k** from the feature in the table above (turning point, point of inflection, asymptotes or endpoint).
3. **Find a** by substituting **one more known point** on the graph into the rule and solving.
4. **Write the rule**, and check it against a second feature if one is available.

A negative value of a means the graph is **reflected in the x -axis** (a parabola opens down, a square-root graph falls, a $\frac{1}{(x - h)^2}$ -form graph sits below its horizontal asymptote). A square-root graph that opens to the **left** has the form $y = a\sqrt{-(x - h)} + k$ (reflected in the y -axis), with domain $x \leq h$.

Worked examples

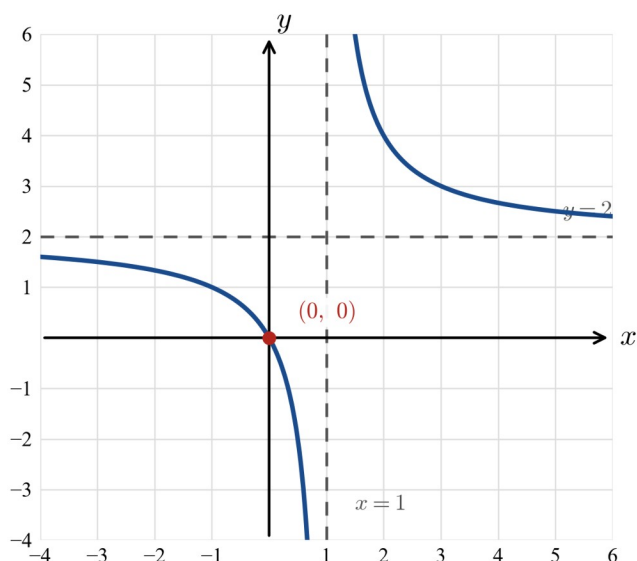
Example 1 — scaffolded

The graph shown is a hyperbola with asymptotes $x = 1$ and $y = 2$, passing through the origin. Determine its rule.

Working	Comment
The two asymptotes identify a hyperbola : $y = \frac{a}{x - h} + k$.	Two straight asymptotes $\Rightarrow$ base graph $y = \frac{1}{x}$.
Asymptotes $x = 1, y = 2$ give $h = 1, k = 2$: $y = \frac{a}{x - 1} + 2$.	Read h and k straight from the asymptotes.
Substitute the point $(0, 0)$: $0 = \frac{a}{0 - 1} + 2 = -a + 2$, so $a = 2$.	Use the extra known point to find a .

$$\therefore y = \frac{2}{x-1} + 2.$$

State the rule.



Example 2 — scaffolded

The parabola shown has turning point $(2, -1)$ and passes through $(0, 3)$. Determine its rule.

A parabola in turning-point form: $y = a(x - h)^2 + k$.

One turning point $\Rightarrow$ base graph $y = x^2$.

Turning point $(2, -1)$ gives $h = 2$, $k = -1$:

Read h and k from the turning point.

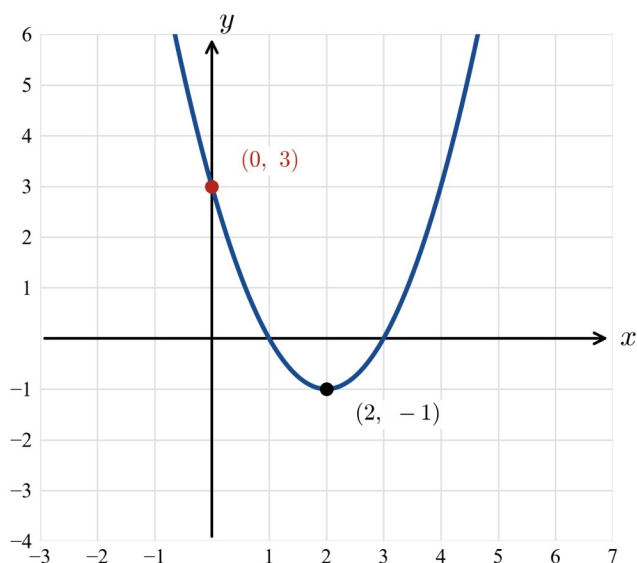
$$y = a(x - 2)^2 - 1.$$

Substitute $(0, 3)$: $3 = a(0 - 2)^2 - 1 = 4a - 1$, so $4a = 4$ and $a = 1$.

Use the extra point to find a .

$$\therefore y = (x - 2)^2 - 1.$$

State the rule.



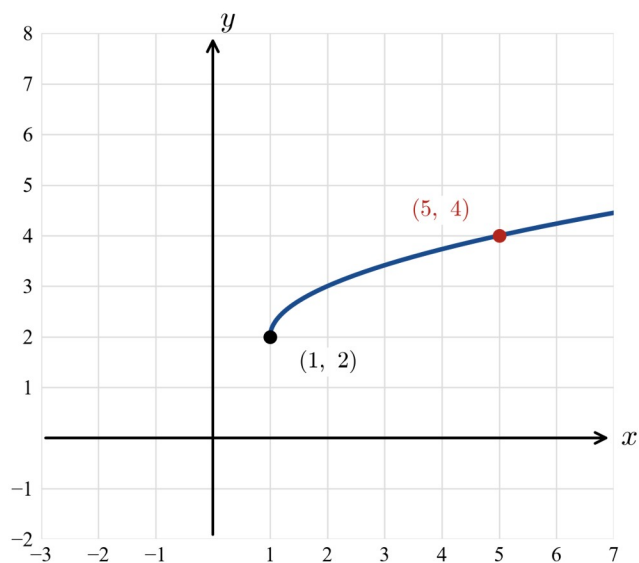
Example 3 — unscaffolded

The square-root graph shown has endpoint $(1, 2)$ and passes through $(5, 4)$. Determine its rule.

The endpoint gives the base graph $y = \sqrt{x}$ translated to $(h, k) = (1, 2)$: $y = a\sqrt{x - 1} + 2$.

Substitute $(5, 4)$: $4 = a\sqrt{5-1} + 2 = a\sqrt{4} + 2 = 2a + 2$, so $2a = 2$ and $a = 1$.

$$\therefore y = \sqrt{x-1} + 2.$$



Example 4 — unscaffolded

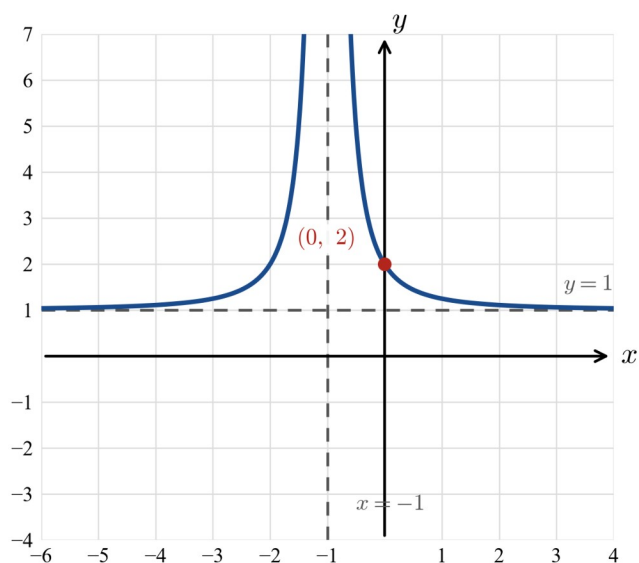
The graph shown is an image of $y = \frac{1}{x^2}$ under transformation, with asymptotes $x = -1$ and $y = 1$, passing through $(0, 2)$. Determine its rule.

Two asymptotes with both branches on the same side identify the form $y = \frac{a}{(x-h)^2} + k$: with $h = -1$, $k = 1$,

$$\text{i.e. } y = \frac{a}{(x+1)^2} + 1.$$

Substitute $(0, 2)$: $2 = \frac{a}{(0+1)^2} + 1 = a + 1$, so $a = 1$.

$$\therefore y = \frac{1}{(x+1)^2} + 1.$$



Practice questions

Scaffolded

Q1. The parabola shown has turning point $(-1, 4)$ and passes through $(1, 0)$. (a) Write the rule in the form $y = a(x - h)^2 + k$ with h and k filled in; (b) use the point $(1, 0)$ to find a ; (c) state the rule; (d) is the parabola upright or inverted, and why?

Q2. The hyperbola shown has asymptotes $x = -2$ and $y = 1$ and passes through $(0, 3)$. (a) State h and k ; (b) find a ; (c) write the rule.

Q3. The square-root graph shown has endpoint $(-4, -2)$ and passes through $(0, 0)$. (a) State the endpoint coordinates (h, k) ; (b) find a ; (c) write the rule and state the domain.

Q4. The cubic shown has a stationary point of inflection at $(1, 2)$ and passes through $(0, 1)$. (a) Write the rule in the form $y = a(x - h)^3 + k$ with h and k filled in; (b) find a ; (c) state the rule.

Q5. The graph shown, an image of $y = \frac{1}{x^2}$, has asymptotes $x = 2$ and $y = -1$ and passes through the origin. (a) State h and k ; (b) find a ; (c) write the rule and state the range.

Q6. The parabola shown has turning point $(3, -2)$ and passes through $(1, 6)$. (a) State h and k ; (b) find a ; (c) write the rule.

Un scaffolded

Each graph shows a transformed standard function with its key feature(s) and one further point labelled. Determine the rule of each.

Q7. Cubic: stationary point of inflection $(-1, 0)$, through $(1, 4)$.

Q8. Hyperbola: asymptotes $x = 2$, $y = -1$, through $(0, -2)$.

Q9. Square root: endpoint $(2, 1)$, through $(6, -1)$.

Q10. Reciprocal-square graph: asymptotes $x = 1$, $y = 3$, through $(0, 1)$.

Q11. Parabola: turning point $(0, 5)$, through $(2, -3)$.

Q12. Hyperbola: asymptotes $x = -1$, $y = -2$, through $(0, -5)$.

Q13. Cubic: stationary point of inflection $(0, -3)$, through $(1, -1)$.

Q14. Square root: endpoint $(0, 3)$, through $(4, 7)$.

Q15. Reciprocal-square graph: asymptotes $x = -2$, $y = 2$, through $(0, 3)$.

Q16. Parabola: turning point $(-2, -1)$, through $(0, 3)$.

Q17. Hyperbola: asymptotes $x = 3$, $y = 1$, through $(0, 3)$.

Q18. Cubic: stationary point of inflection $(2, 1)$, through $(0, 9)$.

Q19. Square-root graph opening to the **left**: endpoint $(4, 1)$, through $(0, 3)$. Give the rule in the form $y = a\sqrt{-(x - h)} + k$ and state the domain.

Q20. Parabola: turning point $(1, 2)$, through $(3, 10)$.

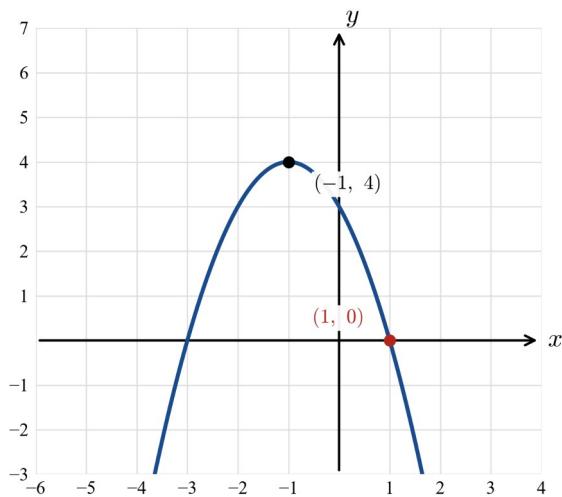
Q21. Hyperbola whose horizontal asymptote is the x -axis: asymptotes $x = 1$, $y = 0$, through $(2, 3)$.

Answers

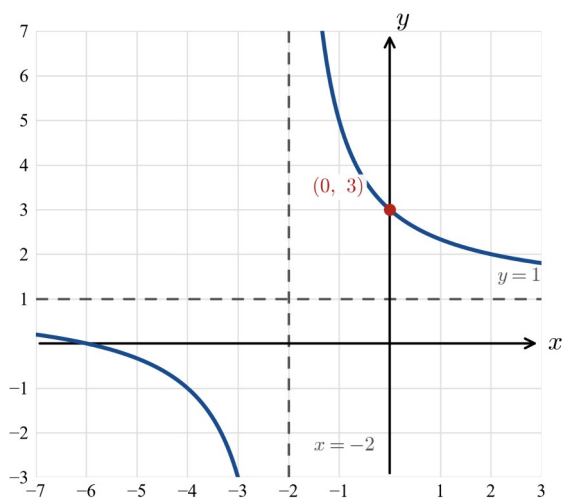
Q1. $y = -(x+1)^2 + 4$ (inverted, since $a = -1 < 0$). **Q2.** $y = \frac{4}{x+2} + 1$. **Q3.** $y = \sqrt{x+4} - 2$, domain $\{x : x \geq -4\}$. **Q4.** $y = (x-1)^3 + 2$. **Q5.** $y = \frac{4}{(x-2)^2} - 1$, range $\{y : y > -1\}$. **Q6.** $y = 2(x-3)^2 - 2$. **Q7.** $y = \frac{1}{2}(x+1)^3$. **Q8.** $y = \frac{2}{x-2} - 1$. **Q9.** $y = -\sqrt{x-2} + 1$. **Q10.** $y = -\frac{2}{(x-1)^2} + 3$. **Q11.** $y = -2x^2 + 5$. **Q12.** $y = -\frac{3}{x+1} - 2$. **Q13.** $y = 2x^3 - 3$. **Q14.** $y = 2\sqrt{x} + 3$. **Q15.** $y = \frac{4}{(x+2)^2} + 2$. **Q16.** $y = (x+2)^2 - 1$. **Q17.** $y = -\frac{6}{x-3} + 1$. **Q18.** $y = -(x-2)^3 + 1$. **Q19.** $y = \sqrt{-(x-4)} + 1 = \sqrt{4-x} + 1$, domain $\{x : x \leq 4\}$. **Q20.** $y = 2(x-1)^2 + 2$. **Q21.** $y = \frac{3}{x-1}$.

Fully-worked solutions

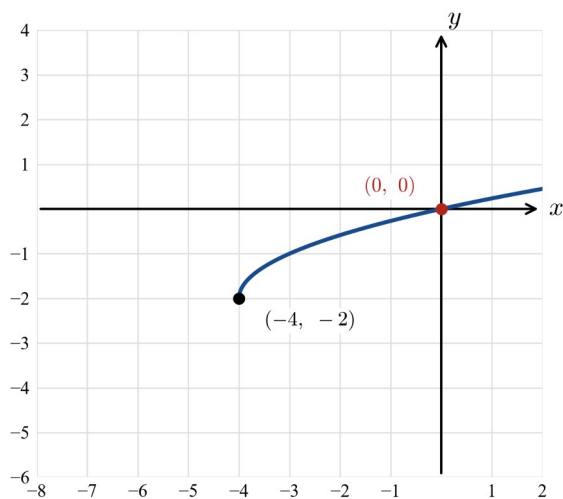
Q1. Turning point $(-1, 4)$: $y = a(x+1)^2 + 4$. Substitute $(1, 0)$: $0 = a(2)^2 + 4 = 4a + 4 \Rightarrow a = -1$.
 $\therefore y = -(x+1)^2 + 4$; as $a < 0$ the parabola is inverted.



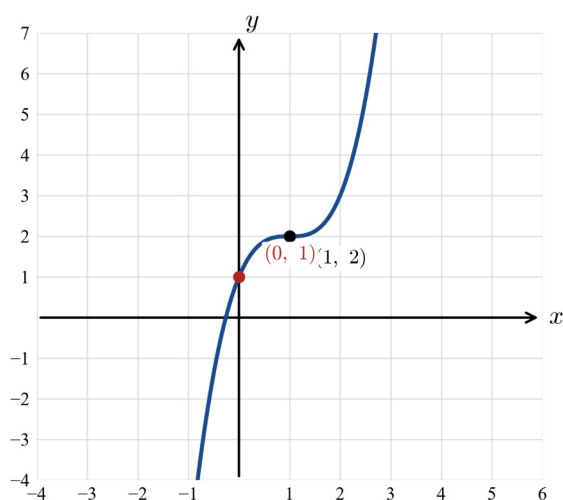
Q2. Asymptotes $x = -2$, $y = 1$: $y = \frac{a}{x+2} + 1$. Substitute $(0, 3)$: $3 = \frac{a}{2} + 1 \Rightarrow \frac{a}{2} = 2 \Rightarrow a = 4$. $\therefore y = \frac{4}{x+2} + 1$.



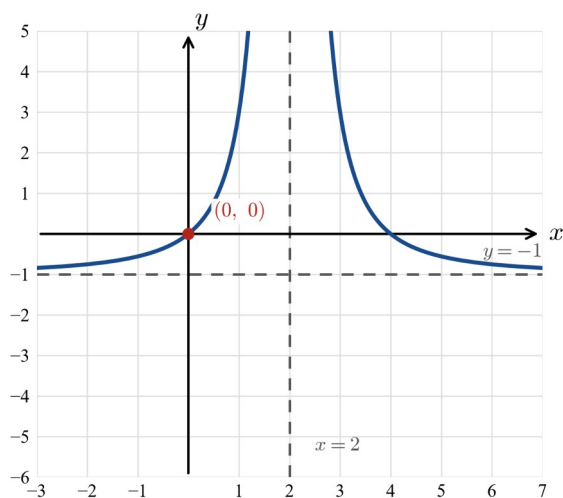
Q3. Endpoint $(-4, -2)$: $y = a\sqrt{x+4} - 2$. Substitute $(0, 0)$: $0 = a\sqrt{4} - 2 = 2a - 2 \Rightarrow a = 1$. $\therefore y = \sqrt{x+4} - 2$, domain $\{x : x \geq -4\}$.



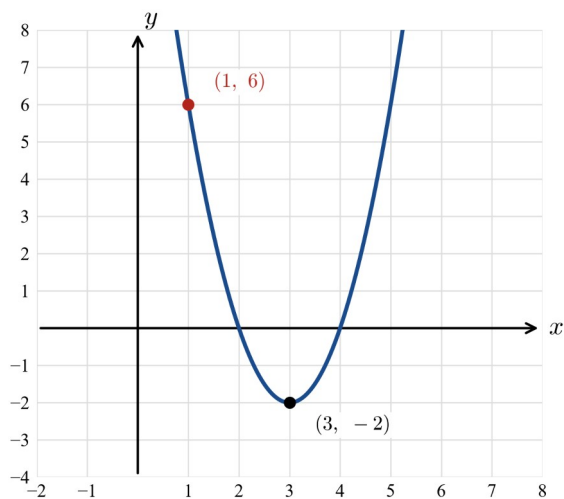
Q4. Stationary point of inflection $(1, 2)$: $y = a(x - 1)^3 + 2$. Substitute $(0, 1)$: $1 = a(-1)^3 + 2 = -a + 2 \Rightarrow a = 1$.
 $\therefore y = (x - 1)^3 + 2$.



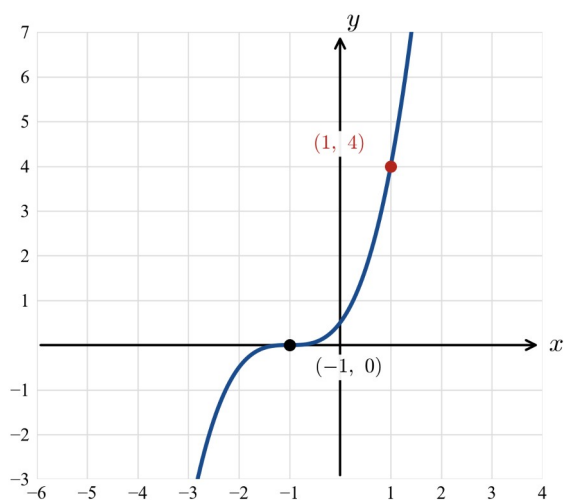
Q5. Asymptotes $x = 2$, $y = -1$: $y = \frac{a}{(x - 2)^2} - 1$. Substitute $(0, 0)$: $0 = \frac{a}{4} - 1 \Rightarrow a = 4$. $\therefore y = \frac{4}{(x - 2)^2} - 1$, range $\{y : y > -1\}$.



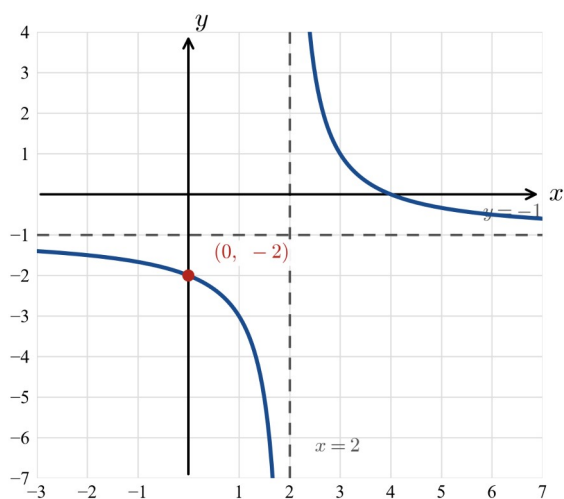
Q6. Turning point $(3, -2)$: $y = a(x - 3)^2 - 2$. Substitute $(1, 6)$: $6 = a(-2)^2 - 2 = 4a - 2 \Rightarrow 4a = 8 \Rightarrow a = 2$.
 $\therefore y = 2(x - 3)^2 - 2$.



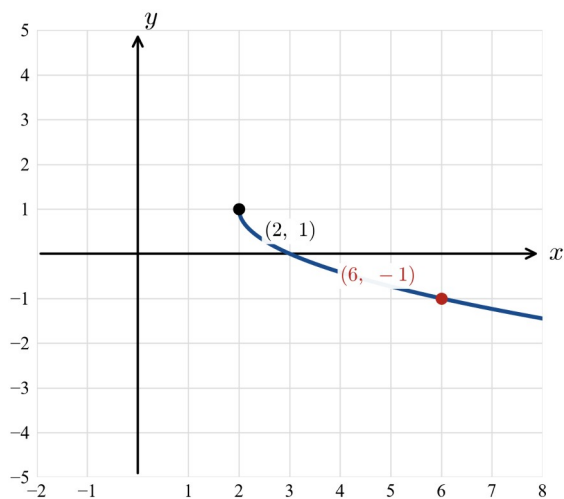
Q7. SPI $(-1, 0)$: $y = a(x+1)^3$. Substitute $(1, 4)$: $4 = a(2)^3 = 8a \Rightarrow a = \frac{1}{2}$. $\therefore y = \frac{1}{2}(x+1)^3$.



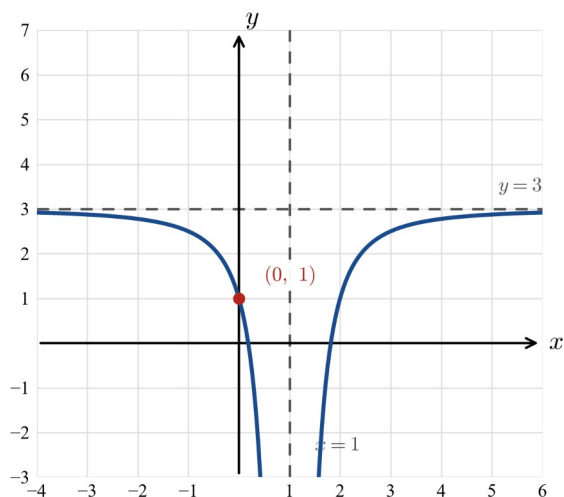
Q8. Asymptotes $x=2$, $y=-1$: $y = \frac{a}{x-2} - 1$. Substitute $(0, -2)$: $-2 = \frac{a}{-2} - 1 \Rightarrow -\frac{a}{2} = -1 \Rightarrow a = 2$. $\therefore y = \frac{2}{x-2} - 1$.



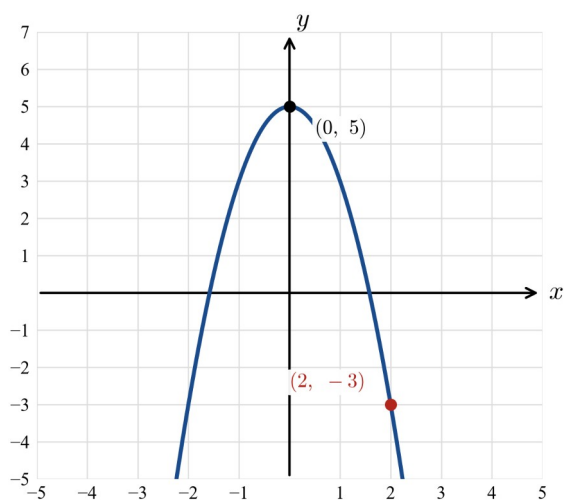
Q9. Endpoint $(2, 1)$: $y = a\sqrt{x-2} + 1$. Substitute $(6, -1)$: $-1 = a\sqrt{4} + 1 = 2a + 1 \Rightarrow a = -1$. $\therefore y = -\sqrt{x-2} + 1$ (the graph falls, since $a < 0$).



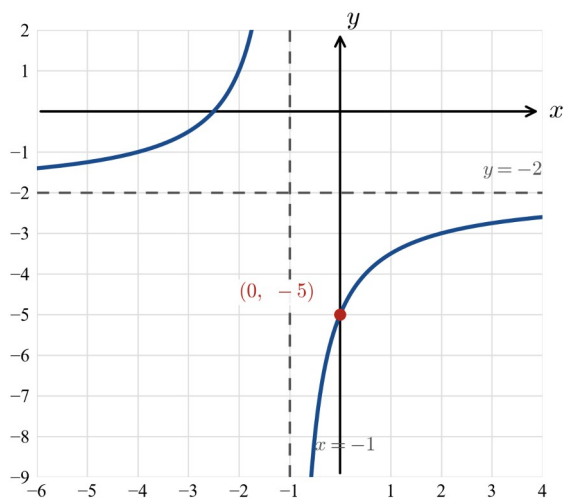
Q10. Asymptotes $x=1$, $y=3$: $y = \frac{a}{(x-1)^2} + 3$. Substitute $(0, 1)$: $1 = \frac{a}{1} + 3 \Rightarrow a = -2$. $\therefore y = -\frac{2}{(x-1)^2} + 3$ (branches below $y=3$).



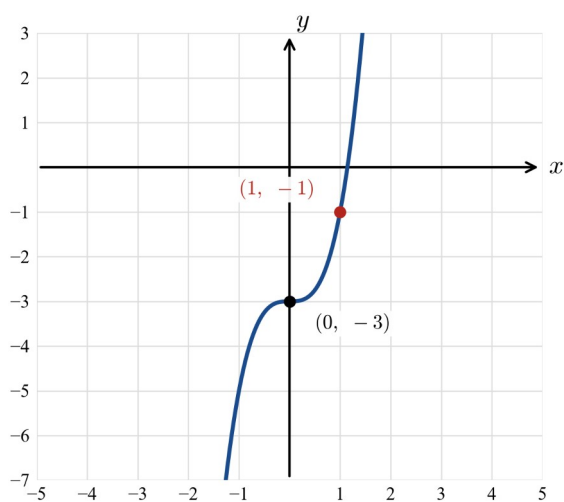
Q11. Turning point $(0, 5)$: $y = ax^2 + 5$. Substitute $(2, -3)$: $-3 = a(4) + 5 \Rightarrow 4a = -8 \Rightarrow a = -2$. $\therefore y = -2x^2 + 5$.



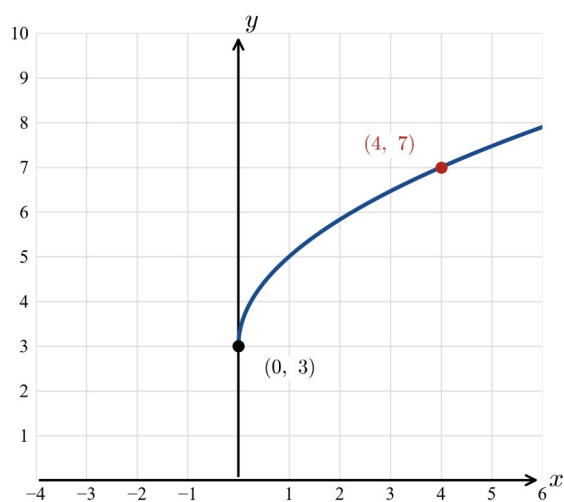
Q12. Asymptotes $x=-1$, $y=-2$: $y = \frac{a}{x+1} - 2$. Substitute $(0, -5)$: $-5 = \frac{a}{1} - 2 \Rightarrow a = -3$. $\therefore y = -\frac{3}{x+1} - 2$.



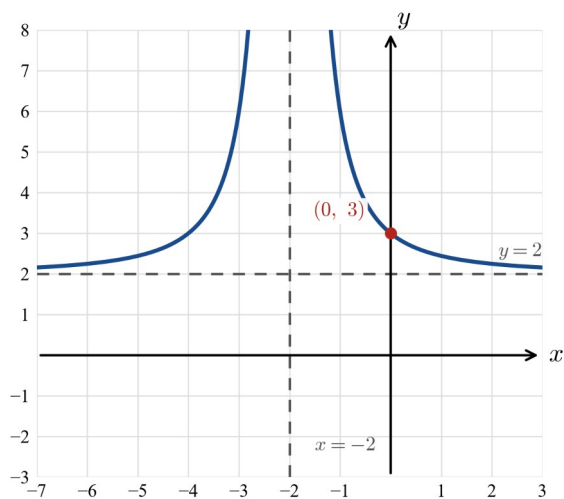
Q13. SPI $(0, -3)$: $y = ax^3 - 3$. Substitute $(1, -1)$: $-1 = a(1) - 3 \Rightarrow a = 2$. $\therefore y = 2x^3 - 3$.



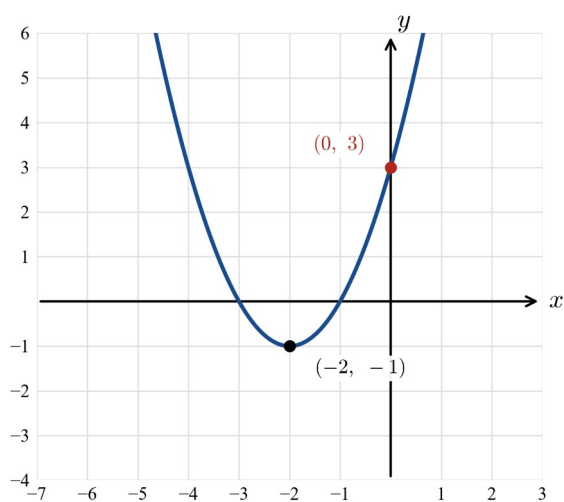
Q14. Endpoint $(0, 3)$: $y = a\sqrt{x} + 3$. Substitute $(4, 7)$: $7 = a\sqrt{4} + 3 = 2a + 3 \Rightarrow a = 2$. $\therefore y = 2\sqrt{x} + 3$.



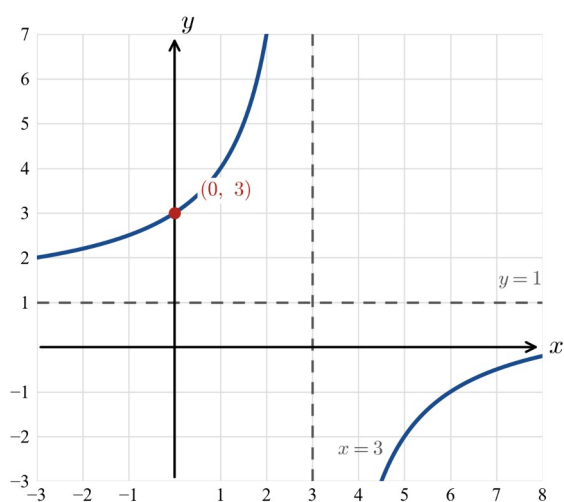
Q15. Asymptotes $x = -2$, $y = 2$: $y = \frac{a}{(x+2)^2} + 2$. Substitute $(0, 3)$: $3 = \frac{a}{4} + 2 \Rightarrow \frac{a}{4} = 1 \Rightarrow a = 4$. $\therefore y = \frac{4}{(x+2)^2} + 2$.



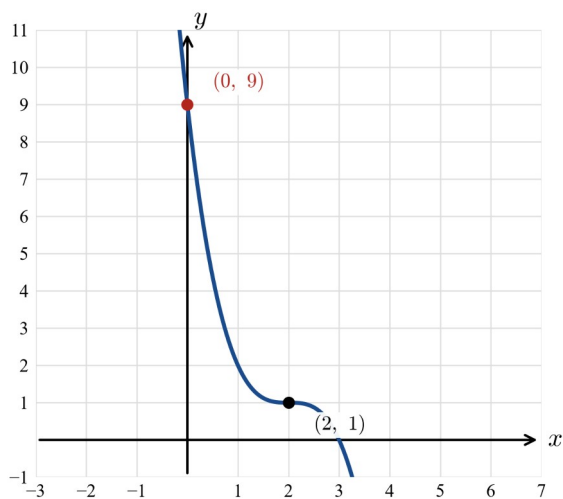
Q16. Turning point $(-2, -1)$: $y = a(x+2)^2 - 1$. Substitute $(0, 3)$: $3 = a(4) - 1 \Rightarrow 4a = 4 \Rightarrow a = 1$.
 $\therefore y = (x+2)^2 - 1$.



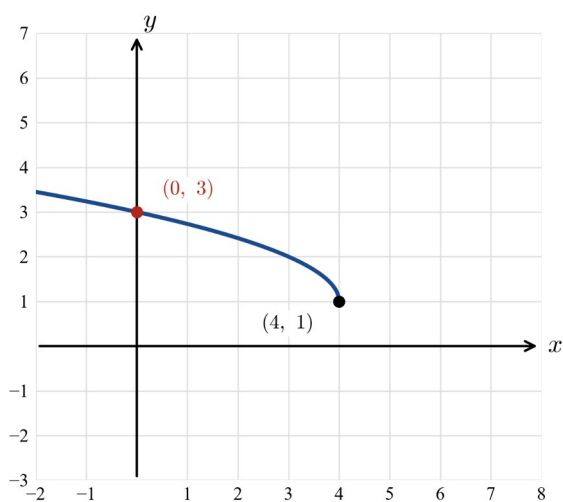
Q17. Asymptotes $x = 3$, $y = 1$: $y = \frac{a}{x-3} + 1$. Substitute $(0, 3)$: $3 = \frac{a}{-3} + 1 \Rightarrow -\frac{a}{3} = 2 \Rightarrow a = -6$. $\therefore y = -\frac{6}{x-3} + 1$.



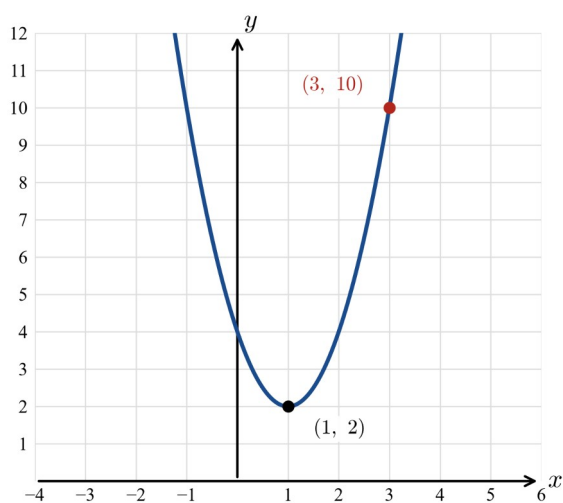
Q18. SPI $(2, 1)$: $y = a(x-2)^3 + 1$. Substitute $(0, 9)$: $9 = a(-2)^3 + 1 = -8a + 1 \Rightarrow -8a = 8 \Rightarrow a = -1$.
 $\therefore y = -(x-2)^3 + 1$.



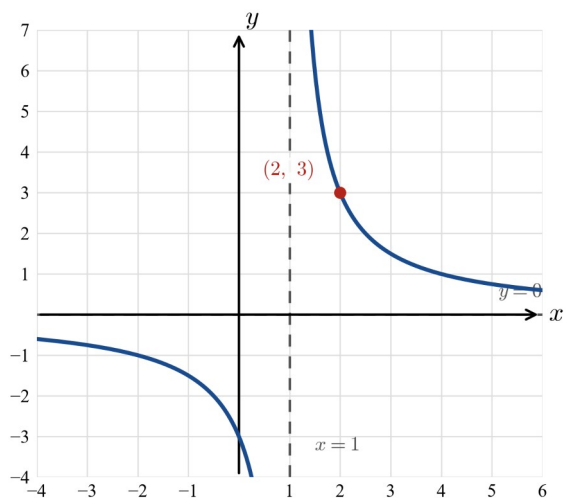
Q19. The graph opens to the left, so $y = a\sqrt{-(x-4)} + 1$ with endpoint $(4, 1)$. Substitute $(0, 3)$:
 $3 = a\sqrt{-(0-4)} + 1 = a\sqrt{4} + 1 = 2a + 1 \Rightarrow a = 1$. $\therefore y = \sqrt{-(x-4)} + 1 = \sqrt{4-x} + 1$, domain $\{x : x \leq 4\}$.



Q20. Turning point $(1, 2)$: $y = a(x-1)^2 + 2$. Substitute $(3, 10)$: $10 = a(2)^2 + 2 = 4a + 2 \Rightarrow 4a = 8 \Rightarrow a = 2$.
 $\therefore y = 2(x-1)^2 + 2$.



Q21. Horizontal asymptote $y = 0$ and vertical asymptote $x = 1$: $y = \frac{a}{x-1}$. Substitute $(2, 3)$: $3 = \frac{a}{1} \Rightarrow a = 3$.
 $\therefore y = \frac{3}{x-1}$.



Chapter 3 Transformations of the plane — 3I Transformation notation

Theory

A **transformation** of the plane can be written as a **mapping** $(x, y) \rightarrow (x', y')$ that sends each point (x, y) to its image (x', y') . This notation records a transformation compactly and makes it easy to find the image of a point or of a whole curve.

The basic transformations as mappings.

Transformation	Mapping
Translation h units right and k units up	$(x, y) \rightarrow (x + h, y + k)$
Dilation of factor a from the x -axis	$(x, y) \rightarrow (x, ay)$
Dilation of factor b from the y -axis	$(x, y) \rightarrow (bx, y)$
Reflection in the x -axis	$(x, y) \rightarrow (x, -y)$
Reflection in the y -axis	$(x, y) \rightarrow (-x, y)$

A **combination** of these is a single mapping $(x, y) \rightarrow (x', y')$ in which x' is a linear expression in x and y' is a linear expression in y , for example $(x, y) \rightarrow (bx + h, ay + k)$.

Image of a point. Substitute the coordinates directly. Under $(x, y) \rightarrow (x + 2, 3y)$ the image of $(1, 1)$ is $(1 + 2, 3 \times 1) = (3, 3)$.

Image of a curve. If the mapping is $x' = X(x)$ and $y' = Y(y)$:

1. make x and y the subjects: $x = X^{-1}(x')$ and $y = Y^{-1}(y')$;
2. substitute into the rule $y = f(x)$ of the original curve;
3. relabel $x' \rightarrow x$ and $y' \rightarrow y$ to write the rule of the image.

Example. Apply $(x, y) \rightarrow (x + 2, 3y)$ to $y = x^2$. Here $x' = x + 2$ and $y' = 3y$, so $x = x' - 2$ and $y = \frac{y'}{3}$. Substituting into $y = x^2$ gives $\frac{y'}{3} = (x' - 2)^2$, so $y' = 3(x' - 2)^2$. The image has rule $y = 3(x - 2)^2$.

Composition. Two transformations can be applied in sequence: apply the first mapping, then apply the second to the result. The single combined mapping is found by substitution.

Worked examples

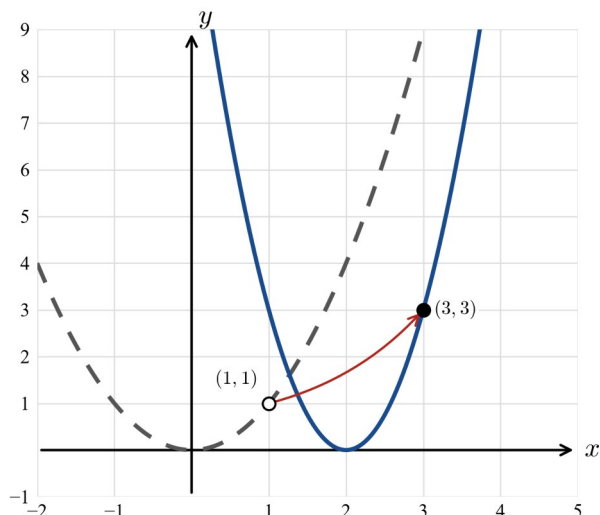
Example 1 — scaffolded

Find the image of the curve $y = x^2$ under the mapping $(x, y) \rightarrow (x + 2, 3y)$, and the image of the point $(1, 1)$.

Working	Comment
The mapping gives $x' = x + 2$ and $y' = 3y$.	Read off the two coordinate rules.
Make x, y the subjects: $x = x' - 2$ and $y = \frac{y'}{3}$.	Invert each coordinate rule.
Substitute into $y = x^2$: $\frac{y'}{3} = (x' - 2)^2$, so $y' = 3(x' - 2)^2$.	The pre-image lies on $y = x^2$.
Relabel: the image is $y = 3(x - 2)^2$.	Drop the dashes for the final rule.

Image of the point: $(1+2, 3 \times 1) = (3, 3)$.

Substitute the point directly into the mapping.



Example 2 — scaffolded

The mapping $(x, y) \rightarrow (2x, -y)$ is applied to $y = \frac{1}{x}$. Find the rule of the image and the image of the point $(1, 1)$.

Working

Comment

$x' = 2x$ and $y' = -y$.

A dilation of factor 2 from the y -axis and a reflection in the x -axis.

Make x, y the subjects: $x = \frac{x'}{2}$ and $y = -y'$.

Invert each rule.

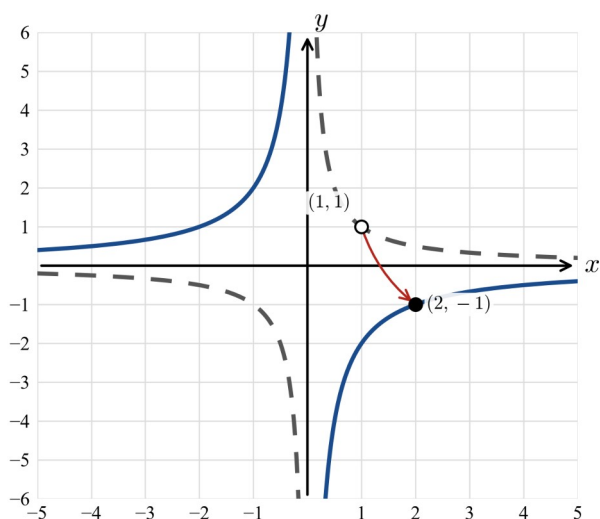
Substitute into $y = \frac{1}{x}$: $-y' = \frac{1}{x'/2} = \frac{2}{x'}$, so $y' = -\frac{2}{x'}$.

The pre-image lies on $y = 1/x$.

Relabel: the image is $y = -\frac{2}{x}$.

Image of the point: $(2 \times 1, -1) = (2, -1)$.

Substitute directly.



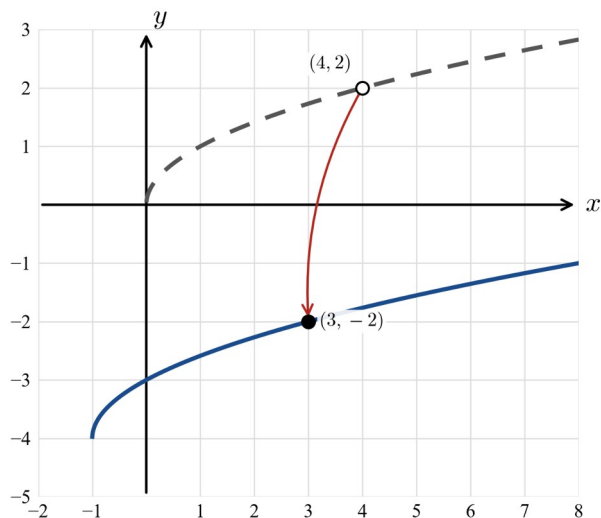
Example 3 — unscaffolded

Find the rule of the image of $y = \sqrt{x}$ under the mapping $(x, y) \rightarrow (x-1, y-4)$, and the image of the point $(4, 2)$.

$x' = x - 1$ and $y' = y - 4$, so $x = x' + 1$ and $y = y' + 4$.

Substituting into $y = \sqrt{x}$: $y' + 4 = \sqrt{x' + 1}$, so $y' = \sqrt{x' + 1} - 4$.

$\therefore$ the image is $y = \sqrt{x + 1} - 4$, and the image of $(4, 2)$ is $(4 - 1, 2 - 4) = (3, -2)$.



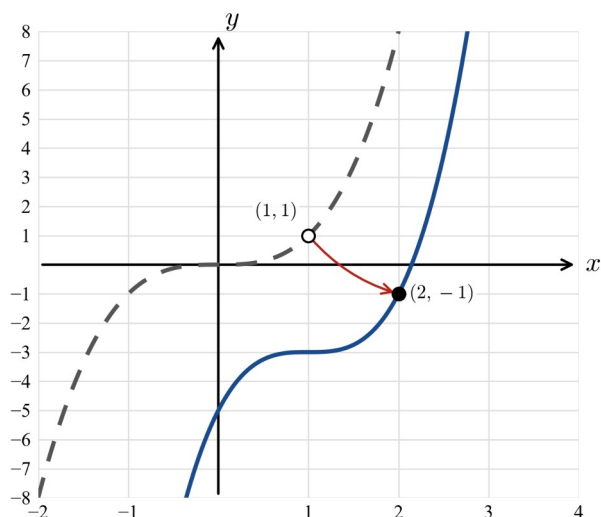
Example 4 — unscaffolded

The transformation $T_1: (x, y) \rightarrow (x, 2y)$ is applied to $y = x^3$, followed by $T_2: (x, y) \rightarrow (x + 1, y - 3)$. Find the single mapping and the rule of the final image.

Applying T_1 then T_2 to a point (x, y) gives $(x, 2y) \rightarrow (x + 1, 2y - 3)$, so the combined mapping is $(x, y) \rightarrow (x + 1, 2y - 3)$.

Then $x' = x + 1$ and $y' = 2y - 3$, so $x = x' - 1$ and $y = \frac{y' + 3}{2}$. Substituting into $y = x^3$: $\frac{y' + 3}{2} = (x' - 1)^3$, so $y' = 2(x' - 1)^3 - 3$.

$\therefore$ the image is $y = 2(x - 1)^3 - 3$. (Check: the point $(1, 1)$ maps to $(2, -1)$.)



Practice questions

Scaffolded

Q1. The mapping $(x, y) \rightarrow (x - 3, y + 1)$ is applied to $y = x^2$. (a) Write down x' and y' ; (b) make x and y the subjects; (c) find the rule of the image; (d) find the image of the point $(2, 4)$.

Q2. For the mapping $(x, y) \rightarrow (x + 1, 2y)$ applied to $y = \frac{1}{x^2}$: (a) state the transformation in words; (b) find the rule of the image; (c) find the image of $(1, 1)$.

Q3. The mapping $(x, y) \rightarrow (-x, y)$ is applied to $y = \sqrt{x}$. (a) name the transformation; (b) find the rule of the image; (c) state its domain; (d) find the image of $(4, 2)$.

Q4. For the mapping $(x, y) \rightarrow (x, -y)$ applied to $y = x^3$: (a) name the transformation; (b) find the rule of the image; (c) find the image of $(2, 8)$.

Q5. The mapping $(x, y) \rightarrow (x - 2, y + 1)$ is applied to $y = \frac{1}{x}$. (a) find the rule of the image; (b) state the equations of its asymptotes; (c) find the image of $(1, 1)$.

Q6. For the mapping $(x, y) \rightarrow (4x, y)$ applied to $y = \sqrt{x}$: (a) name the transformation; (b) find the rule of the image; (c) find the image of $(4, 2)$.

Un scaffolded

Q7. Find the image of the point $(2, -1)$ under $(x, y) \rightarrow (x + 5, 3y - 2)$.

Q8. Find the rule of the image of $y = x^2$ under $(x, y) \rightarrow (2x, y)$.

Q9. Find the rule of the image of $y = \frac{1}{x}$ under $(x, y) \rightarrow (x, y - 3)$.

Q10. Write the mapping for “a reflection in the x -axis followed by a translation of 2 units up”, and hence find the rule of the image of $y = x^2$.

Q11. Find the rule of the image of $y = \sqrt{x}$ under $(x, y) \rightarrow (x + 2, -y)$.

Q12. Find the rule of the image of $y = x^2$ under $(x, y) \rightarrow (x - 1, 2y - 3)$, and the image of $(1, 1)$.

Q13. The mapping $(x, y) \rightarrow (-x, y)$ is followed by $(x, y) \rightarrow (x + 3, y)$. Find the single mapping and the rule of the image of $y = \sqrt{x}$.

Q14. Find the rule of the image of $y = x^3$ under $(x, y) \rightarrow (2x, y - 1)$, and the image of $(1, 1)$.

Q15. Write the mapping for “a dilation of factor 2 from the y -axis and a translation of 1 unit down”, and hence find the rule of the image of $y = \frac{1}{x^2}$.

Q16. Find the rule of the image of $y = x^3$ under $(x, y) \rightarrow (x, -2y)$.

Q17. Find the rule of the image of $y = \frac{1}{x}$ under $(x, y) \rightarrow (-x, -y)$. Comment on your answer.

Q18. Find the rule of the image of $y = x^2$ under $(x, y) \rightarrow (-x, y)$. Comment on your answer.

Q19. Find the rule of the image of $y = x^2$ under $(x, y) \rightarrow (3x - 1, y + 4)$.

Q20. Find the rule of the image of $y = \frac{1}{x^2}$ under $(x, y) \rightarrow (x + 2, -y + 1)$, and state the equations of the asymptotes of the image.

Q21. Describe the mapping $(x, y) \rightarrow (x - 4, 2y)$ in words, and hence find the rule of the image of $y = \sqrt{x}$.

Answers

Q1. $x' = x - 3$, $y' = y + 1$; $x = x' + 3$, $y = y' - 1$; image $y = (x + 3)^2 + 1$; point $(-1, 5)$. **Q2.** Translation 1 right and dilation factor 2 from the x -axis; $y = \frac{2}{(x - 1)^2}$; $(2, 2)$. **Q3.** Reflection in the y -axis; $y = \sqrt{-x}$; domain $\{x : x \leq 0\}$; $(-4, 2)$. **Q4.** Reflection in the x -axis; $y = -x^3$; $(2, -8)$. **Q5.** $y = \frac{1}{x + 2} + 1$; asymptotes $x = -2$, $y = 1$; $(-1, 2)$. **Q6.** Dilation of factor 4 from the y -axis; $y = \frac{\sqrt{x}}{2}$; $(16, 2)$. **Q7.** $(7, -5)$. **Q8.** $y = \frac{x^2}{4}$. **Q9.** $y = \frac{1}{x} - 3$. **Q10.** $(x, y) \rightarrow (x, -y + 2)$; image $y = 2 - x^2$. **Q11.** $y = -\sqrt{x - 2}$. **Q12.** $y = 2(x + 1)^2 - 3$; point $(0, -1)$. **Q13.** $(x, y) \rightarrow (-x + 3, y)$; image $y = \sqrt{3 - x}$. **Q14.** $y = \frac{x^3}{8} - 1$; point $(2, 0)$. **Q15.** $(x, y) \rightarrow (2x, y - 1)$; image $y = \frac{4}{x^2} - 1$. **Q16.** $y = -2x^3$. **Q17.** $y = \frac{1}{x}$ — the curve is **unchanged**; $y = 1/x$ has point symmetry about the origin (it is an odd function). **Q18.** $y = x^2$ — the curve is **unchanged**; $y = x^2$ is symmetric about the y -axis (it is an even function). **Q19.** $y = \frac{(x + 1)^2}{9} + 4$. **Q20.** $y = -\frac{1}{(x - 2)^2} + 1$; asymptotes $x = 2$, $y = 1$. **Q21.** Translation 4 units left and dilation of factor 2 from the x -axis; image $y = 2\sqrt{x + 4}$.

Fully-worked solutions

Q1. $x' = x - 3$, $y' = y + 1 \Rightarrow x = x' + 3$, $y = y' - 1$. Substitute into $y = x^2$: $y' - 1 = (x' + 3)^2$, so $y' = (x' + 3)^2 + 1$. Image $y = (x + 3)^2 + 1$. Point: $(2 - 3, 4 + 1) = (-1, 5)$.

Q2. $x' = x + 1$, $y' = 2y$: a translation 1 unit right and a dilation of factor 2 from the x -axis. Then $x = x' - 1$, $y = \frac{y'}{2}$, so $\frac{y'}{2} = \frac{1}{(x' - 1)^2}$, giving $y = \frac{2}{(x - 1)^2}$. Point: $(1 + 1, 2 \times 1) = (2, 2)$.

Q3. $(x, y) \rightarrow (-x, y)$ is a reflection in the y -axis. $x = -x'$, $y = y'$, so $y' = \sqrt{-x'}$, giving $y = \sqrt{-x}$ with domain $\{x : x \leq 0\}$. Point: $(-4, 2)$.

Q4. $(x, y) \rightarrow (x, -y)$ is a reflection in the x -axis. $x = x'$, $y = -y'$, so $-y' = x'^3$, giving $y = -x^3$. Point: $(2, -8)$.

Q5. $x = x' + 2$, $y = y' - 1$, so $y' - 1 = \frac{1}{x' + 2}$, giving $y = \frac{1}{x + 2} + 1$. Asymptotes $x = -2$ and $y = 1$. Point: $(1 - 2, 1 + 1) = (-1, 2)$.

Q6. $(x, y) \rightarrow (4x, y)$ is a dilation of factor 4 from the y -axis. $x = \frac{x'}{4}$, $y = y'$, so $y' = \sqrt{\frac{x'}{4}} = \frac{\sqrt{x'}}{2}$, giving $y = \frac{\sqrt{x}}{2}$. Point: $(16, 2)$.

Q7. $(2 + 5, 3(-1) - 2) = (7, -5)$.

Q8. $x = \frac{x'}{2}$, $y = y'$, so $y' = \left(\frac{x'}{2}\right)^2 = \frac{x'^2}{4}$, giving $y = \frac{x^2}{4}$.

Q9. $x = x'$, $y = y' + 3$, so $y' + 3 = \frac{1}{x'}$, giving $y = \frac{1}{x} - 3$.

Q10. Reflection in the x -axis is $(x, y) \rightarrow (x, -y)$; then translating 2 up gives $(x, y) \rightarrow (x, -y + 2)$. So $y = -y' + 2$... i.e. $x = x'$, $y = 2 - y'$, and $2 - y' = x'^2$, giving $y = 2 - x^2$.

Q11. $x = x' - 2$, $y = -y'$, so $-y' = \sqrt{x' - 2}$, giving $y = -\sqrt{x - 2}$.

Q12. $x = x' + 1$, $y = \frac{y' + 3}{2}$, so $\frac{y' + 3}{2} = (x' + 1)^2$, giving $y = 2(x + 1)^2 - 3$. Point: $(1 - 1, 2 \times 1 - 3) = (0, -1)$.

Q13. Applying $(x, y) \rightarrow (-x, y)$ then $(x, y) \rightarrow (x + 3, y)$ gives $(-x, y) \rightarrow (-x + 3, y)$, so the single mapping is $(x, y) \rightarrow (-x + 3, y)$. Then $x = 3 - x'$, $y = y'$, so $y' = \sqrt{3 - x'}$, giving $y = \sqrt{3 - x}$.

Q14. $x = \frac{x'}{2}$, $y = y' + 1$, so $y' + 1 = \left(\frac{x'}{2}\right)^3 = \frac{x'^3}{8}$, giving $y = \frac{x^3}{8} - 1$. Point: $(2 \times 1, 1 - 1) = (2, 0)$.

Q15. A dilation of factor 2 from the y -axis is $(x, y) \rightarrow (2x, y)$; a translation 1 down is $(x, y) \rightarrow (x, y - 1)$; combined, $(x, y) \rightarrow (2x, y - 1)$. Then $x = \frac{x'}{2}$, $y = y' + 1$, so $y' + 1 = \frac{1}{(x'/2)^2} = \frac{4}{x'^2}$, giving $y = \frac{4}{x^2} - 1$.

Q16. $x = x'$, $y = -\frac{y'}{2}$, so $-\frac{y'}{2} = x'^3$, giving $y = -2x^3$.

Q17. $x = -x'$, $y = -y'$, so $-y' = \frac{1}{-x'} = -\frac{1}{x'}$, giving $y = \frac{1}{x}$. The image is **identical** to the original: $y = \frac{1}{x}$ has point symmetry about the origin (an odd function), so a rotation of 180° about the origin — which is exactly the mapping $(x, y) \rightarrow (-x, -y)$ — leaves it unchanged.

Q18. $x = -x'$, $y = y'$, so $y' = (-x')^2 = x'^2$, giving $y = x^2$. The image is **identical** to the original: $y = x^2$ is symmetric about the y -axis (an even function), so reflecting in the y -axis leaves it unchanged.

Q19. $x = \frac{x' + 1}{3}$, $y = y' - 4$, so $y' - 4 = \left(\frac{x' + 1}{3}\right)^2 = \frac{(x' + 1)^2}{9}$, giving $y = \frac{(x + 1)^2}{9} + 4$.

Q20. $x = x' - 2$, and $y' = -y + 1 \Rightarrow y = 1 - y'$, so $1 - y' = \frac{1}{(x' - 2)^2}$, giving $y' = 1 - \frac{1}{(x' - 2)^2}$, i.e. $y = -\frac{1}{(x - 2)^2} + 1$.

Asymptotes $x = 2$ and $y = 1$.

Q21. $(x, y) \rightarrow (x - 4, 2y)$: a translation of 4 units to the left and a dilation of factor 2 from the x -axis. Then $x = x' + 4$, $y = \frac{y'}{2}$, so $\frac{y'}{2} = \sqrt{x' + 4}$, giving $y = 2\sqrt{x + 4}$.