

Mathematical Methods Units 3 & 4

Chapter 1 — Functions, relations and graphs

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Theory

Much of this course is about **sets** — collections of objects (usually numbers). This section fixes the notation used throughout to describe sets, intervals, and the domains and ranges of functions.

Sets of numbers. Four sets recur constantly:

- $\mathbb{N}$ — the **natural numbers** $1, 2, 3, \dots$
- $\mathbb{Z}$ — the **integers** $\dots, -2, -1, 0, 1, 2, \dots$
- $\mathbb{Q}$ — the **rational numbers**, those that can be written as a fraction $\frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$
- $\mathbb{R}$ — the **real numbers**.

These are nested: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. A real number that is **not** rational (such as $\sqrt{2}$ or π) is **irrational**; the set of irrationals is $\mathbb{R} \setminus \mathbb{Q}$.

Membership and subsets. $a \in A$ means “ a is an element of A ”; $a \notin A$ means it is not. $A \subseteq B$ means every element of A is also in B . The **empty set** $= \{\}$ has no elements.

Set-builder notation. A set is often described by a defining property, written with a **colon**:

$$\{x : P(x)\}$$

read “the set of all x such that $P(x)$ is true”. For example $\{x : x > 3\}$ and $\{x : 0 \leq x < 5\}$.

Interval notation. For real numbers $a < b$, a square bracket **includes** the endpoint and a round bracket **excludes** it:

Interval	Set-builder	Meaning
$[a, b]$	$\{x : a \leq x \leq b\}$	both endpoints included
(a, b)	$\{x : a < x < b\}$	both endpoints excluded
$[a, b)$	$\{x : a \leq x < b\}$	left included, right excluded
$(a, b]$	$\{x : a < x \leq b\}$	left excluded, right included
$[a, \infty)$	$\{x : x \geq a\}$	unbounded above
$(-\infty, b]$	$\{x : x \leq b\}$	unbounded below

Infinity is never included, so ∞ and $-\infty$ always take a round bracket. The whole real line is $(-\infty, \infty) = \mathbb{R}$.

Number-line representation. On a number line a **filled** dot marks an included endpoint and a **hollow** dot an excluded one; an arrow shows the set continues to infinity.

Set operations. For sets A and B :

- **Union** $A \cup B$ — everything in A **or** B (or both).
- **Intersection** $A \cap B$ — everything in A **and** B .
- **Set difference** $A \setminus B$ — everything in A but **not** in B .

These are used to write domains as **unions of intervals**. For instance a function undefined only at $x = 1$ has domain $\mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, \infty)$.

Worked examples

Example 1 — scaffolded

Express $\{x : -2 \leq x < 3\}$ in interval notation, show it on a number line, and state whether -2 , 0 and 3 belong to the set.

Working

Comment

The condition is $-2 \leq x < 3$: -2 is included ($\leq$) and 3 is excluded ($<$).

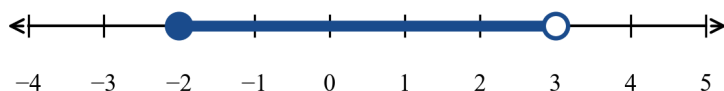
Match each inequality sign to a bracket.

Interval: $[-2, 3)$.

Square bracket for the included endpoint, round for the excluded one.

$-2 \in [-2, 3)$ (included); $0 \in [-2, 3)$; $3 \notin [-2, 3)$ (excluded).

Test each value against $-2 \leq x < 3$.



Example 2 — scaffolded

Let $A = [-1, 4)$ and $B = (2, 6]$. Find $A \cap B$ and $A \cup B$, giving each in interval notation, and illustrate on a number line.

Working

Comment

$A = \{x : -1 \leq x < 4\}$, $B = \{x : 2 < x \leq 6\}$.

Rewrite each interval as an inequality.

$A \cap B$: need x in **both**, so $2 < x < 4$.

Intersection keeps the overlap: the larger left endpoint and the smaller right endpoint.

$A \cap B = (2, 4)$.

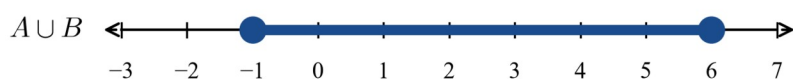
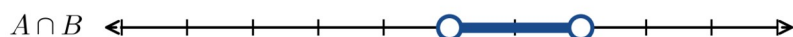
Both endpoints are excluded (one from each set).

$A \cup B$: everything in **either**, from -1 (included) to 6 (included).

Union spans from the smaller left endpoint to the larger right endpoint.

$A \cup B = [-1, 6]$.

The overlap means there is no gap.



Example 3 — unscaffolded

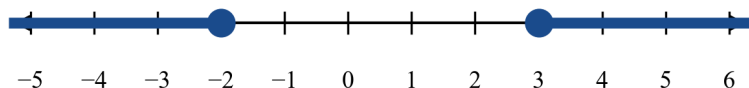
Solve $x^2 - x - 6 \geq 0$ and express the solution set in both set-builder and interval notation.

Factorise: $x^2 - x - 6 = (x - 3)(x + 2)$.

The parabola $y = (x - 3)(x + 2)$ opens upward with x -intercepts -2 and 3 , so it is **above or on** the x -axis for $x \leq -2$ or $x \geq 3$.

Set-builder: $\{x : x \leq -2\} \cup \{x : x \geq 3\}$; interval: $(-\infty, -2] \cup [3, \infty)$.

$\therefore$ the solution set is $(-\infty, -2] \cup [3, \infty)$.



Example 4 — unscaffolded

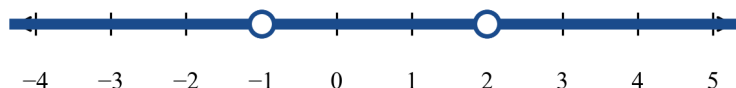
A function f is defined for all real numbers except $x = -1$ and $x = 2$. Write its domain, $\mathbb{R} \setminus \{-1, 2\}$, as a union of intervals.

Removing the two points -1 and 2 from the real line leaves three open pieces:

$$\mathbb{R} \setminus \{-1, 2\} = (-\infty, -1) \cup (-1, 2) \cup (2, \infty).$$

Each excluded point becomes a **hollow** dot, so all six endpoints here are excluded.

$\therefore$ the domain is $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.



Practice questions

Scaffolded

Q1. Rewrite each set in the other notation (set-builder $\leftrightarrow$ interval): (a) $\{x : x \geq -3\}$; (b) $\{x : -1 < x \leq 4\}$; (c) $(-\infty, 2)$; (d) $[0, 5)$.

Q2. Let $A = [-3, 2]$ and $B = [0, 5)$. (a) Find $A \cap B$; (b) find $A \cup B$; (c) find $A \setminus B$; (d) illustrate A , B and your answers on a number line.

Q3. A set is graphed as the closed interval from -1 to 4 (both endpoints filled). (a) Write it in interval notation; (b) write it in set-builder notation; (c) state whether -1 , 4 and 5 are members.

Q4. Solve $2x - 5 < 3$. (a) Give the solution in interval notation; (b) represent it on a number line.

Q5. State whether each statement is true or false: (a) $\sqrt{2} \in \mathbb{Q}$; (b) $-3 \in \mathbb{N}$; (c) $22/7 \in \mathbb{Q}$; (d) $0 \in \mathbb{Z}$; (e) $\mathbb{N} \subseteq \mathbb{Z}$.

Q6. Find the maximal (implied) domain of each function, giving your answer in interval notation: (a) $f(x) = \frac{1}{x-2}$; (b) $g(x) = \sqrt{x-1}$.

Unscaffolded

For **Q7–Q13**, evaluate the set operation or solve the inequality, giving the answer in interval notation (and on a number line where a diagram is shown in the solutions).

Q7. $[-2, 5) \cap (0, 7]$

Q8. $(-\infty, 3] \cup (1, \infty)$

Q9. Solve $x^2 < 9$.

Q10. Solve $x^2 - 4 \geq 0$.

Q11. Write $\mathbb{R} \setminus \{0, 3\}$ as a union of intervals.

Q12. Solve $3 - 2x \geq 1$.

Q13. Solve $|x| < 4$.

Q14. Find the maximal domain of $h(x) = \sqrt{4 - x}$, in interval notation.

Q15. Find the maximal domain of $p(x) = \frac{1}{(x-1)(x+3)}$, as a union of intervals.

Q16. Express $[-5, -1] \cup (2, 4)$ in set-builder notation, and state whether -1 and 2 are members.

Q17. Solve $x^2 - x - 2 \leq 0$.

Q18. Find $\{x : x > 0\} \cap \{x : x \leq 5\}$, in interval notation.

Q19. Find $\{x : x^2 - 5x + 6 > 0\}$, giving your answer in interval notation.

Q20. State the maximal domain of $f(x) = \frac{\sqrt{x+2}}{x-3}$, in interval notation.

Q21. Express $(-\infty, -2) \cup [0, 3)$ in set-builder notation, and list the integers that lie in $[0, 3)$.

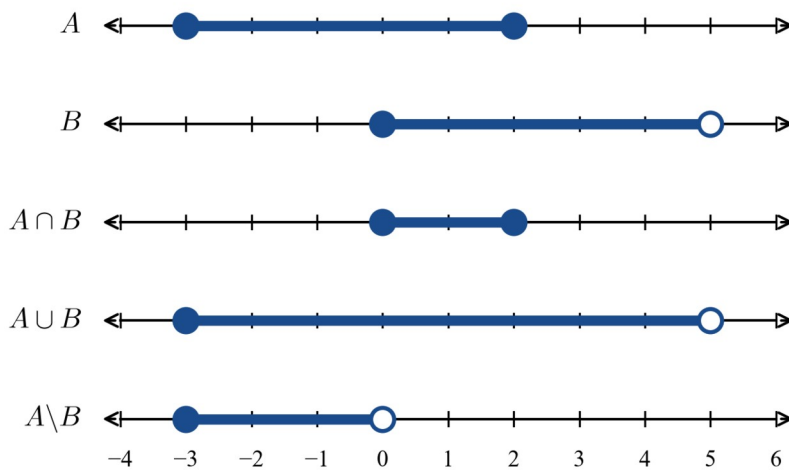
Answers

Q1. (a) $[-3, \infty)$; (b) $(-1, 4]$; (c) $\{x : x < 2\}$; (d) $\{x : 0 \leq x < 5\}$. **Q2.** (a) $[0, 2]$; (b) $[-3, 5]$; (c) $[-3, 0]$. **Q3.** (a) $[-1, 4]$; (b) $\{x : -1 \leq x \leq 4\}$; (c) -1 yes, 4 yes, 5 no. **Q4.** (a) $(-\infty, 4)$. **Q5.** (a) F; (b) F; (c) T; (d) T; (e) T. **Q6.** (a) $(-\infty, 2) \cup (2, \infty)$; (b) $[1, \infty)$. **Q7.** $(0, 5)$. **Q8.** $(-\infty, \infty) = \mathbb{R}$. **Q9.** $(-3, 3)$. **Q10.** $(-\infty, -2] \cup [2, \infty)$. **Q11.** $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. **Q12.** $(-\infty, 1]$. **Q13.** $(-4, 4)$. **Q14.** $(-\infty, 4]$. **Q15.** $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$. **Q16.** $\{x : -5 \leq x \leq -1\} \cup \{x : 2 < x < 4\}$; -1 is a member, 2 is not. **Q17.** $[-1, 2]$. **Q18.** $(0, 5]$. **Q19.** $(-\infty, 2) \cup (3, \infty)$. **Q20.** $[-2, 3) \cup (3, \infty)$. **Q21.** $\{x : x < -2\} \cup \{x : 0 \leq x < 3\}$; the integers in $[0, 3)$ are $0, 1, 2$.

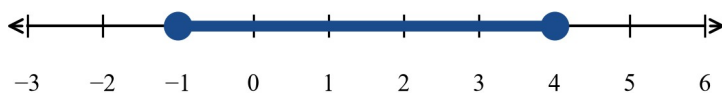
Fully-worked solutions

Q1. (a) $\{x : x \geq -3\} = [-3, \infty)$. (b) $\{x : -1 < x \leq 4\} = (-1, 4]$. (c) $(-\infty, 2) = \{x : x < 2\}$. (d) $[0, 5) = \{x : 0 \leq x < 5\}$.

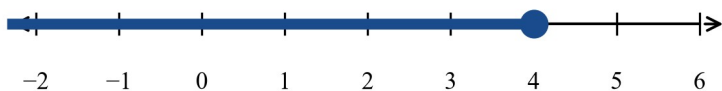
Q2. $A = [-3, 2]$, $B = [0, 5)$. (a) $A \cap B$ needs x in both: $0 \leq x \leq 2$, so $[0, 2]$. (b) $A \cup B$ spans -3 (included) to 5 (excluded) with no gap, so $[-3, 5)$. (c) $A \setminus B$ is A with the part in B removed: $-3 \leq x < 0$, so $[-3, 0)$.



Q3. (a) $[-1, 4]$. (b) $\{x : -1 \leq x \leq 4\}$. (c) $-1 \in [-1, 4]$ and $4 \in [-1, 4]$; $5 \notin [-1, 4]$.



Q4. $2x - 5 < 3 \Rightarrow 2x < 8 \Rightarrow x < 4$, so the solution is $(-\infty, 4)$.



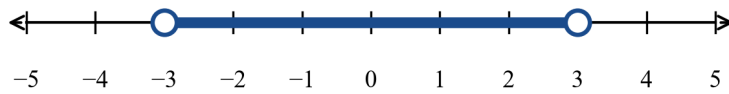
Q5. (a) False — $\sqrt{2}$ is irrational. (b) False — -3 is an integer but not a natural number. (c) True — $22/7$ is a ratio of integers. (d) True. (e) True — every natural number is an integer.

Q6. (a) $\frac{1}{x-2}$ is undefined at $x=2$, so the domain is $\mathbb{R} \setminus \{2\} = (-\infty, 2) \cup (2, \infty)$. (b) $\sqrt{x-1}$ requires $x-1 \geq 0$, i.e. $x \geq 1$, so $[1, \infty)$.

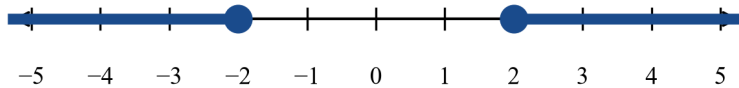
Q7. $[-2, 5) \cap (0, 7]$ keeps the overlap: $0 < x < 5$, so $(0, 5)$.

Q8. $(-\infty, 3]$ covers all $x \leq 3$ and $(1, \infty)$ covers all $x > 1$; together they cover every real number, so $(-\infty, \infty) = \mathbb{R}$.

Q9. $x^2 < 9 \Rightarrow -3 < x < 3$, so $(-3, 3)$.



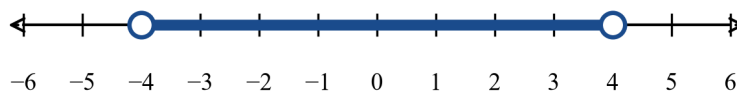
Q10. $x^2 - 4 \geq 0 \Rightarrow (x - 2)(x + 2) \geq 0$, which holds for $x \leq -2$ or $x \geq 2$, so $(-\infty, -2] \cup [2, \infty)$.



Q11. Removing 0 and 3 from $\mathbb{R}$ leaves $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$.

Q12. $3 - 2x \geq 1 \Rightarrow -2x \geq -2 \Rightarrow x \leq 1$ (the inequality reverses on dividing by -2), so $(-\infty, 1]$.

Q13. $|x| < 4 \Rightarrow -4 < x < 4$, so $(-4, 4)$.

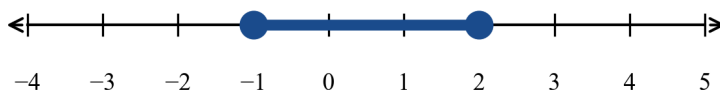


Q14. $\sqrt{4 - x}$ requires $4 - x \geq 0$, i.e. $x \leq 4$, so $(-\infty, 4]$.

Q15. $\frac{1}{(x - 1)(x + 3)}$ is undefined where the denominator is zero, at $x = 1$ and $x = -3$; the domain is $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$.

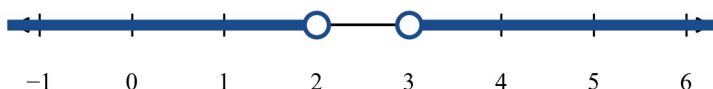
Q16. $[-5, -1] \cup (2, 4) = \{x : -5 \leq x \leq -1\} \cup \{x : 2 < x < 4\}$. Since -1 is a closed endpoint, -1 **is** a member; since 2 is an open endpoint, 2 is **not**.

Q17. $x^2 - x - 2 = (x - 2)(x + 1) \leq 0$ between the roots, so $-1 \leq x \leq 2$, giving $[-1, 2]$.



Q18. $\{x : x > 0\} \cap \{x : x \leq 5\}$ needs $0 < x \leq 5$, so $(0, 5]$.

Q19. $x^2 - 5x + 6 = (x - 2)(x - 3) > 0$ **outside** the roots, so $x < 2$ or $x > 3$, giving $(-\infty, 2) \cup (3, \infty)$.



Q20. $\frac{\sqrt{x+2}}{x-3}$ requires $x+2 \geq 0$ (so $x \geq -2$) **and** $x-3 \neq 0$ (so $x \neq 3$). Combining, the maximal domain is $[-2, 3) \cup (3, \infty)$.

Q21. $(-\infty, -2) \cup [0, 3) = \{x : x < -2\} \cup \{x : 0 \leq x < 3\}$. The integers in $[0, 3)$ satisfy $0 \leq x < 3$, namely 0, 1, 2.

Chapter 1 Functions, relations and graphs — 1B Relations and functions

Theory

Relations. A **relation** is a set of ordered pairs (x, y) . A relation can be described by listing its pairs, by a rule together with a domain, or by a graph.

- The **domain** of a relation is the set of all first coordinates (x -values).
- The **range** of a relation is the set of all second coordinates (y -values).

For example, the relation $\{(1, 2), (1, 4), (3, 5)\}$ has domain $\{1, 3\}$ and range $\{2, 4, 5\}$.

Functions. A **function** is a relation in which **each x -value is paired with exactly one y -value**. Equivalently, no two ordered pairs have the same first coordinate.

The vertical-line test. A graph represents a function if and only if **every vertical line meets the graph at most once**. If some vertical line meets the graph more than once, the relation is not a function.

Function notation. A function is often written $f: D \rightarrow \mathbb{R}, f(x) = \dots$, where D is the domain (the *maximal* domain unless a smaller one is stated) and the rule tells you how each x is mapped to its **image** $f(x)$. For example, if $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 - 4$, then $f(3) = 3^2 - 4 = 5$.

One-to-one and many-to-one. A function is **one-to-one** if different x -values always give different y -values — equivalently, every **horizontal** line meets the graph at most once. If some horizontal line meets the graph more than once (two or more x -values share a y -value), the function is **many-to-one**. For instance, $f(x) = x^2$ is many-to-one because $f(-2) = f(2) = 4$.

Finding the range from a restricted domain. When a function is given on a restricted domain, find the range from its graph: check the values at the endpoints and at any turning point, and note whether each endpoint is **included** (closed dot, square bracket) or **excluded** (open dot, round bracket).

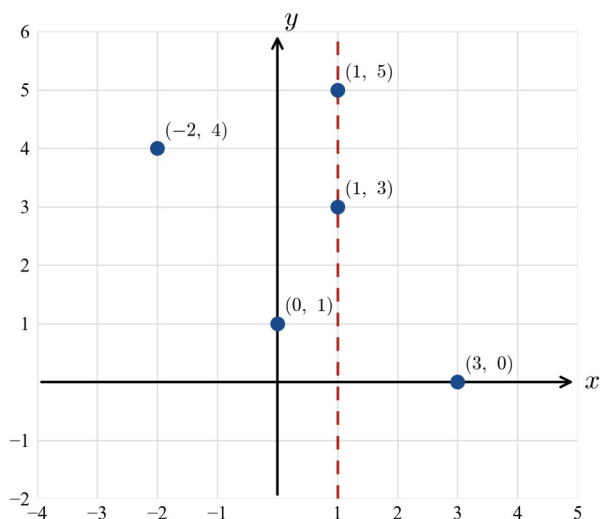
To describe a relation: state its rule and its domain, or read its domain and range from the graph; then decide whether it is a function (vertical-line test) and, if so, whether it is one-to-one (horizontal-line test).

Worked examples

Example 1 — scaffolded

For the relation $S = \{(-2, 4), (0, 1), (1, 3), (1, 5), (3, 0)\}$, state the domain and range, and determine whether S is a function.

Working	Comment
Domain = $\{-2, 0, 1, 3\}$.	The set of first coordinates (list each once).
Range = $\{0, 1, 3, 4, 5\}$.	The set of second coordinates.
The pairs $(1, 3)$ and $(1, 5)$ have the same first coordinate.	Look for a repeated x -value.
$\therefore S$ is not a function : $x = 1$ is paired with both 3 and 5.	One x -value gives two y -values, so a vertical line at $x = 1$ meets the relation twice.



Example 2 — scaffolded

Consider the relation with rule $y = 2x - 1$ and domain $[-1, 3)$. Sketch its graph, state the domain and range, and determine whether it is a function and whether it is one-to-one.

Working

At $x = -1$: $y = 2(-1) - 1 = -3$, giving the closed endpoint $(-1, -3)$.

As $x \rightarrow 3^-$: $y \rightarrow 2(3) - 1 = 5$, giving the open endpoint $(3, 5)$.

Domain $[-1, 3)$; since the line increases, range $[-3, 5)$.

Every vertical line meets the graph once, so it **is a function**; every horizontal line meets it once, so it **is one-to-one**.

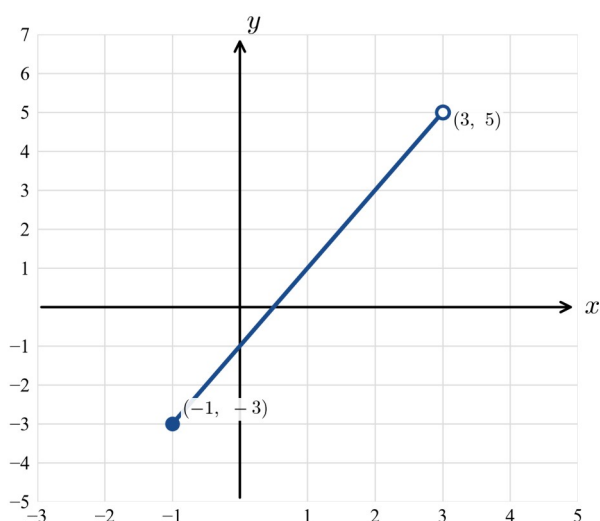
Comment

$x = -1$ **is** included, so a filled dot.

$x = 3$ is **excluded**, so an open dot.

Increasing line: smallest y at the left end, largest approached at the right.

A non-horizontal straight line is always a one-to-one function.



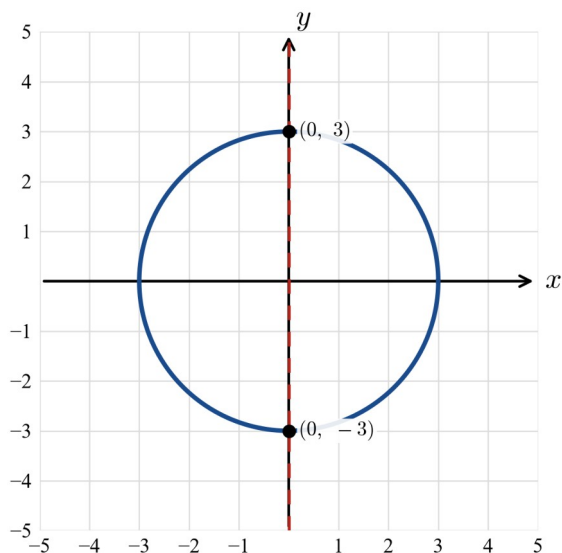
Example 3 — unscaffolded

Determine whether the relation $x^2 + y^2 = 9$ is a function, and state its domain and range.

The graph is a circle of radius 3 centred at the origin.

Substituting $x = 0$ gives $y^2 = 9$, so $y = 3$ or $y = -3$: the vertical line $x = 0$ meets the circle at $(0, 3)$ and $(0, -3)$.

$\therefore$ the relation is **not a function**. Its domain is $[-3, 3]$ and its range is $[-3, 3]$.



Example 4 — unscaffolded

Let $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 - 4$. Find $f(3)$, $f(-2)$ and $f(a+1)$, state whether f is one-to-one, and solve $f(x) = 0$.

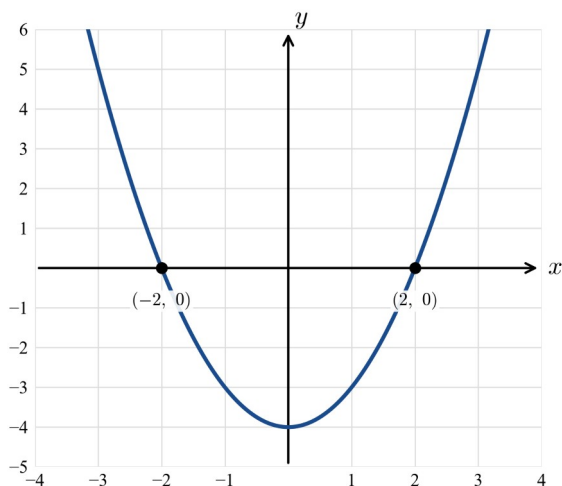
$$f(3) = 3^2 - 4 = 5 \text{ and } f(-2) = (-2)^2 - 4 = 0.$$

$$f(a+1) = (a+1)^2 - 4 = a^2 + 2a + 1 - 4 = a^2 + 2a - 3.$$

Since $f(-2) = f(2) = 0$, two different x -values share a y -value, so f is **many-to-one**.

$$f(x) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x^2 = 4, \text{ so } x = -2 \text{ or } x = 2.$$

$\therefore$ the horizontal line $y = 0$ meets the parabola twice, at $(-2, 0)$ and $(2, 0)$.



Practice questions

Scaffolded

Q1. For the relation $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$: (a) state the domain; (b) state the range; (c) is it a function? Justify your answer; (d) is it one-to-one?

Q2. For $y = 3 - x$ with domain $(-2, 2]$: (a) sketch the graph, showing open/closed endpoints; (b) state the domain; (c) state the range; (d) is it a function, and is it one-to-one?

Q3. For $y = x^2$ with domain $[-2, 3]$: (a) sketch the graph; (b) state the range; (c) is it a function? (d) is it one-to-one? Justify your answer.

Q4. For the relation $x^2 + y^2 = 4$: (a) sketch the graph; (b) state the domain and range; (c) determine whether it is a function, justifying your answer with the vertical-line test; (d) state a restriction of the range that would make it a function.

Q5. For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 5$: (a) find $f(0)$; (b) find $f(-3)$; (c) solve $f(x) = 1$; (d) find $f(a-1)$ in simplest form.

Q6. For $g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = 5 - x^2$: (a) find $g(2)$; (b) find $g(0)$; (c) solve $g(x) = 1$; (d) state the range and whether g is one-to-one.

Un scaffolded

Q7. State the domain and range of $\{(-3, 2), (-1, 2), (0, 5), (2, -1)\}$, and determine whether it is a function.

Q8. Determine whether $\{(1, 2), (1, 3), (2, 4)\}$ is a function, and state its domain and range.

Q9. For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 + 1$, find $f(0)$, $f(-3)$ and $f(2t)$.

Q10. For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x - 2$, find the value of x for which $f(x) = 10$, and the value for which $f(x) = 0$.

Q11. Sketch $y = 2x - 4$ with domain $[0, 4]$, state its range, and determine whether it is one-to-one.

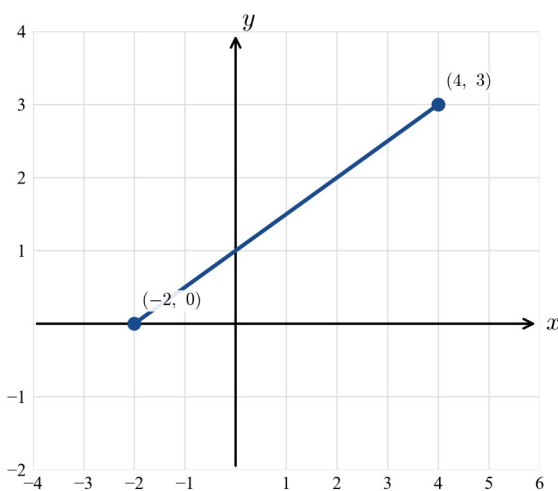
Q12. Sketch $y = -x^2 + 4$ with domain $[-1, 3]$, state its range, and determine whether it is a function and whether it is one-to-one.

Q13. State the domain and range of $x^2 + y^2 = 16$, and determine whether it is a function.

Q14. Sketch the relation $y^2 = x$, state its domain and range, and determine whether it is a function, justifying your answer.

Q15. For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 - 6x + 5$, find $f(0)$, solve $f(x) = 0$, solve $f(x) = -4$, and state whether f is one-to-one.

Q16. Write down a rule and domain describing the relation graphed below.



Q17. For $h: [2, \infty) \rightarrow \mathbb{R}, h(x) = (x - 2)^2$, determine whether h is one-to-one and state its range.

Q18. Determine, with justification, which of the following are functions: (i) $y = x^3$; (ii) $x = y^2$; (iii) $\{(1, 2), (2, 2), (3, 2)\}$; (iv) $x^2 + y^2 = 1$.

Q19. For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 4 - 2x$, find $f(a+1) - f(a)$ and interpret your answer.

Q20. For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$, find $\{x: f(x) = 9\}$ and $\{x: f(x) < 0\}$.

Q21. State the domain and range of the relation with rule $y = x^2 - 4$ and domain $[-1, 3]$, and determine whether it defines a function and whether it is one-to-one, justifying each answer.

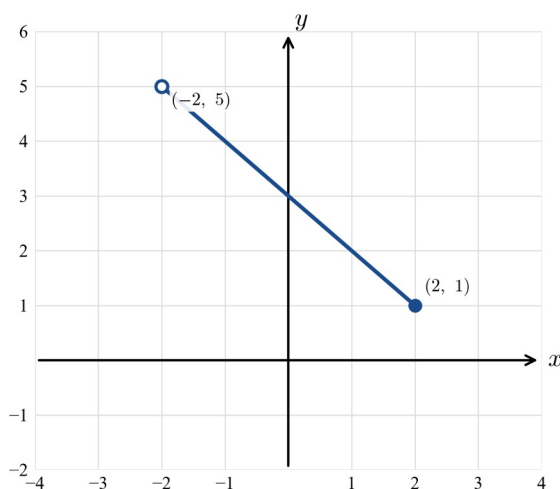
Answers

Q1. (a) $\{1, 2, 3, 4\}$; (b) $\{1, 4, 9, 16\}$; (c) yes — each x appears once; (d) yes. **Q2.** (b) $(-2, 2]$; (c) $[1, 5]$; (d) a function and one-to-one. **Q3.** (b) $[0, 9]$; (c) yes; (d) no — many-to-one (e.g. $x = -2$ and $x = 2$ both give $y = 4$). **Q4.** (b) domain $[-2, 2]$, range $[-2, 2]$; (c) not a function; (d) e.g. range $[0, 2]$ (upper semicircle). **Q5.** (a) 5; (b) -1 ; (c) $x = -2$; (d) $2a + 3$. **Q6.** (a) 1; (b) 5; (c) $x = -2$ or $x = 2$; (d) range $(-\infty, 5]$, not one-to-one. **Q7.** Domain $\{-3, -1, 0, 2\}$, range $\{-1, 2, 5\}$; a function. **Q8.** Not a function ($x = 1$ gives 2 and 3); domain $\{1, 2\}$, range $\{2, 3, 4\}$. **Q9.** $f(0) = 1$, $f(-3) = 10$, $f(2t) = 4t^2 + 1$. **Q10.** $f(x) = 10 \Rightarrow x = 4$; $f(x) = 0 \Rightarrow x = \frac{2}{3}$. **Q11.** Range $[-4, 4]$; one-to-one. **Q12.** Range $[-5, 4]$; a function; not one-to-one. **Q13.** Domain $[-4, 4]$, range $[-4, 4]$; not a function. **Q14.** Domain $[0, \infty)$, range $\mathbb{R}$; not a function ($x = 4$ gives $y = 2$ and $y = -2$). **Q15.** $f(0) = 5$; $f(x) = 0 \Rightarrow x = 1$ or $x = 5$; $f(x) = -4 \Rightarrow x = 3$; not one-to-one. **Q16.** $y = \frac{1}{2}x + 1$, domain $[-2, 4]$. **Q17.** One-to-one; range $[0, \infty)$. **Q18.** (i) function; (ii) not a function; (iii) function; (iv) not a function. **Q19.** $f(a+1) - f(a) = -2$; the function decreases by 2 for each unit increase in x (constant gradient -2). **Q20.** $\{x : f(x) = 9\} = \{-3, 3\}$; $\{x : f(x) < 0\} = \emptyset$ (no real x , since $x^2 \geq 0$). **Q21.** Domain $[-1, 3]$, range $[-4, 5]$; a function; not one-to-one.

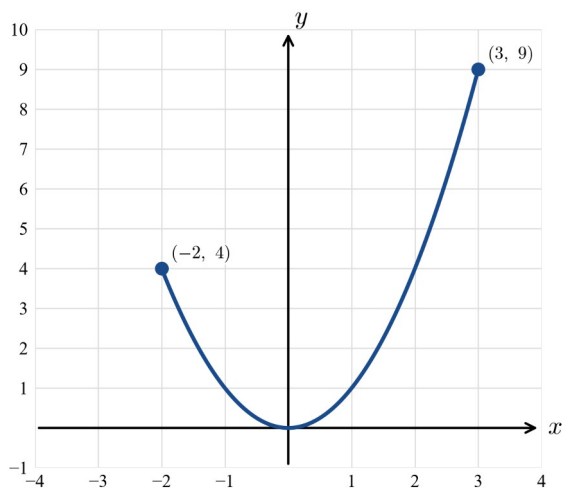
Fully-worked solutions

Q1. Domain $\{1, 2, 3, 4\}$, range $\{1, 4, 9, 16\}$. Each first coordinate appears exactly once, so it **is a function**. Different x -values give different y -values, so it **is one-to-one**.

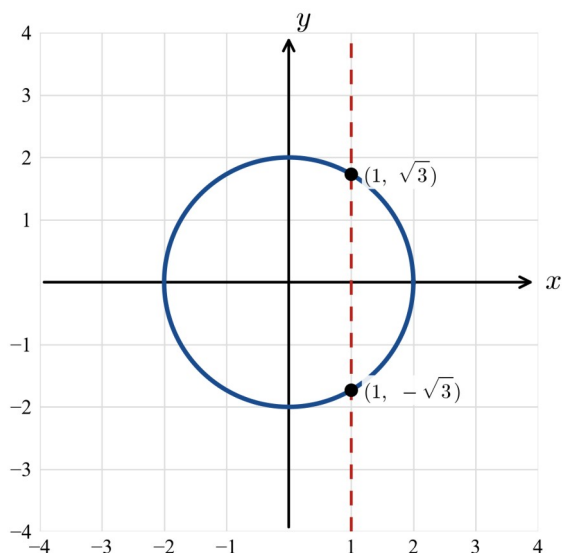
Q2. $y = 3 - x$ is a decreasing line. At $x = 2$ (included): $y = 1$, closed endpoint $(2, 1)$. As $x \rightarrow -2^+$ (excluded): $y \rightarrow 5$, open endpoint $(-2, 5)$. Domain $(-2, 2]$, range $[1, 5]$. It passes the vertical-line test (**a function**) and the horizontal-line test (**one-to-one**).



Q3. $y = x^2$ on $[-2, 3]$. Minimum at $x = 0$ gives $y = 0$; the larger endpoint $x = 3$ gives $y = 9$. Range $[0, 9]$. It **is a function** (vertical-line test). It is **not one-to-one**: $x = -2$ and $x = 2$ both give $y = 4$, so a horizontal line meets it twice — many-to-one.



Q4. $x^2 + y^2 = 4$ is a circle of radius 2 centred at the origin; domain $[-2, 2]$, range $[-2, 2]$. The vertical line $x = 1$ meets it at $(1, \sqrt{3})$ and $(1, -\sqrt{3})$, so it is **not a function**. Restricting to $y \geq 0$ (the upper semicircle, range $[0, 2]$) makes it a function.



Q5. $f(x) = 2x + 5$. (a) $f(0) = 5$. (b) $f(-3) = 2(-3) + 5 = -1$. (c) $2x + 5 = 1 \Rightarrow 2x = -4 \Rightarrow x = -2$. (d) $f(a-1) = 2(a-1) + 5 = 2a - 2 + 5 = 2a + 3$.

Q6. $g(x) = 5 - x^2$. (a) $g(2) = 5 - 4 = 1$. (b) $g(0) = 5$. (c) $5 - x^2 = 1 \Rightarrow x^2 = 4 \Rightarrow x = -2$ or $x = 2$. (d) The maximum value is 5 (at $x = 0$), so range $(-\infty, 5]$; since $g(-2) = g(2)$, g is not one-to-one.

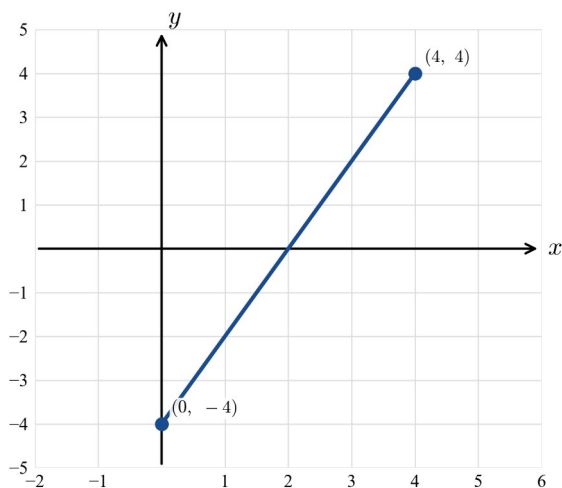
Q7. First coordinates $\{-3, -1, 0, 2\}$ (domain); second coordinates $\{2, 5, -1\}$, i.e. range $\{-1, 2, 5\}$. No first coordinate is repeated, so it is **a function** (two different x -values may share the y -value 2 — that is allowed).

Q8. The pairs $(1, 2)$ and $(1, 3)$ share the first coordinate 1, so it is **not a function**. Domain $\{1, 2\}$, range $\{2, 3, 4\}$.

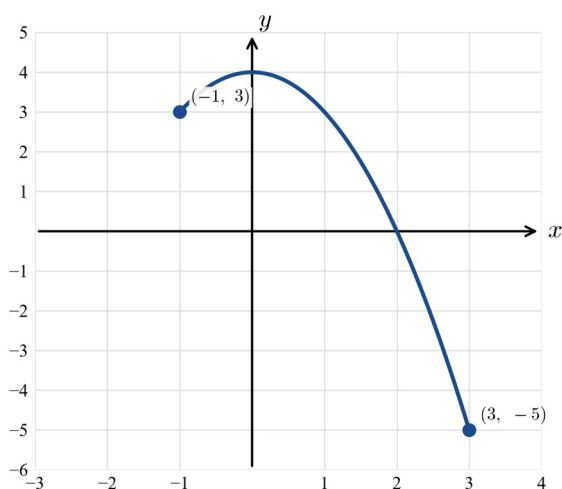
Q9. $f(x) = x^2 + 1$. $f(0) = 1$; $f(-3) = 9 + 1 = 10$; $f(2t) = (2t)^2 + 1 = 4t^2 + 1$.

Q10. $3x - 2 = 10 \Rightarrow 3x = 12 \Rightarrow x = 4$. $3x - 2 = 0 \Rightarrow x = \frac{2}{3}$.

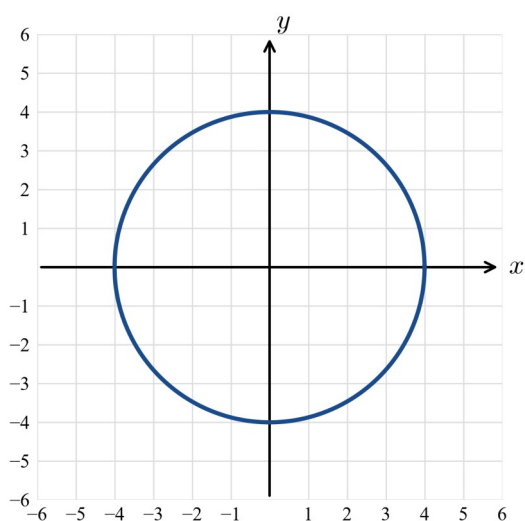
Q11. $y = 2x - 4$ on $[0, 4]$. Endpoints $(0, -4)$ and $(4, 4)$, both included. Range $[-4, 4]$. The line is increasing, so it is **one-to-one**.



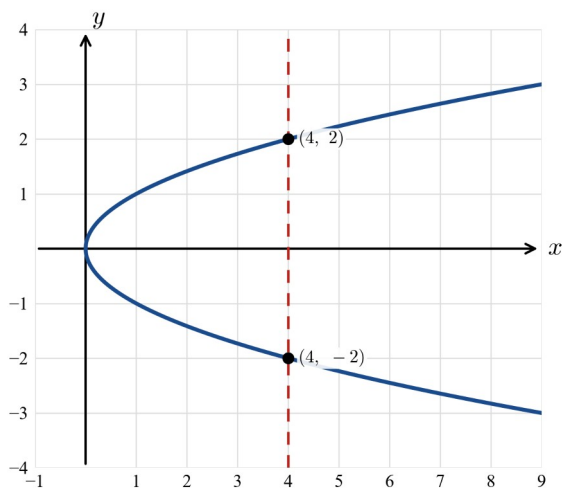
Q12. $y = -x^2 + 4$ on $[-1, 3]$. Maximum at $x=0$ gives $y=4$; the endpoint $x=3$ gives $y=-5$ (the lower value). Range $[-5, 4]$. It is a **function**; it is **not one-to-one** (e.g. $x=-1$ and $x=1$ both give $y=3$).



Q13. $x^2 + y^2 = 16$ is a circle of radius 4: domain $[-4, 4]$, range $[-4, 4]$. It fails the vertical-line test (e.g. $x=0$ gives $y=\pm 4$), so it is **not a function**.



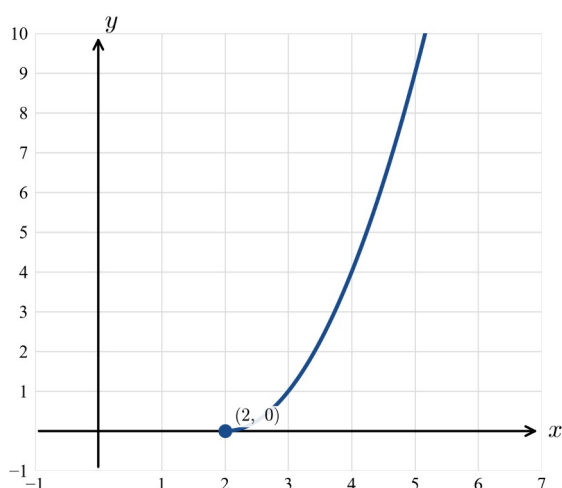
Q14. $y^2 = x$ is a parabola opening to the right. For any $x > 0$ there are two values $y = \pm\sqrt{x}$ (e.g. $x=4$ gives $y=2$ and $y=-2$), so it fails the vertical-line test and is **not a function**. Domain $[0, \infty)$, range $\mathbb{R}$.



Q15. $f(x) = x^2 - 6x + 5$. $f(0) = 5$. $f(x) = 0 \Rightarrow x^2 - 6x + 5 = (x-1)(x-5) = 0 \Rightarrow x = 1$ or $x = 5$.
 $f(x) = -4 \Rightarrow x^2 - 6x + 5 = -4 \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$. Since $f(1) = f(5) = 0$, f is **not one-to-one**.

Q16. The graph is a line segment from the closed endpoint $(-2, 0)$ to the closed endpoint $(4, 3)$. Its gradient is $\frac{3-0}{4-(-2)} = \frac{1}{2}$ and it passes through $(-2, 0)$, so $y = \frac{1}{2}(x+2) = \frac{1}{2}x + 1$. Rule $y = \frac{1}{2}x + 1$, domain $[-2, 4]$.

Q17. $h(x) = (x-2)^2$ on $[2, \infty)$. On this restricted domain the graph is the right-hand half of the parabola (increasing), so **it is one-to-one**. The minimum is at $x = 2$, giving $h(2) = 0$; range $[0, \infty)$.

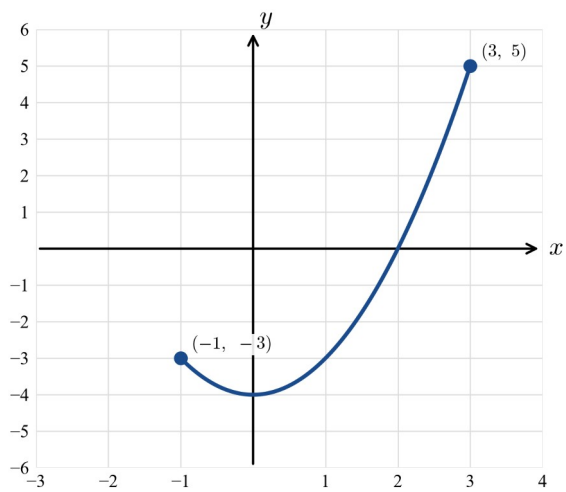


Q18. (i) $y = x^3$: each x gives one y — a **function**. (ii) $x = y^2$: each $x > 0$ gives $y = \pm\sqrt{x}$ — **not a function**. (iii) $\{(1, 2), (2, 2), (3, 2)\}$: no x repeated — a **function** (many-to-one). (iv) $x^2 + y^2 = 1$: a circle, fails the vertical-line test — **not a function**.

Q19. $f(a+1) - f(a) = [4 - 2(a+1)] - [4 - 2a] = (2 - 2a) - (4 - 2a) = -2$. The result is constant: each increase of 1 in x decreases f by 2, which is the constant gradient of the line.

Q20. $f(x) = x^2$. $f(x) = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$, so $\{x : f(x) = 9\} = \{-3, 3\}$. Since $x^2 \geq 0$ for all real x , $f(x) < 0$ has no solutions: $\{x : f(x) < 0\} = \emptyset$.

Q21. $y = x^2 - 4$ on $[-1, 3]$. Domain $[-1, 3]$. Minimum at $x = 0$ gives $y = -4$; the endpoint $x = 3$ gives $y = 5$. Range $[-4, 5]$. Every vertical line meets the graph once, so **it is a function**. It is **not one-to-one**: $f(-1) = -3 = f(1)$ with $-1, 1 \in [-1, 3]$, so a horizontal line meets it twice.



Chapter 1 Functions, relations and graphs — 1C Function types and maximal domains

Theory

One-to-one and many-to-one. A function is **one-to-one** if different x -values always give different y -values — equivalently, every horizontal line meets the graph at most once (the **horizontal-line test**). Otherwise it is **many-to-one**. For example $f(x) = 2x + 1$ is one-to-one, while $f(x) = x^2$ is many-to-one (since $f(-2) = f(2) = 4$).

Implied (maximal) domain. When a function is given by a rule with no stated domain, its **implied domain** is the largest subset of $\mathbb{R}$ for which the rule gives real values. The three restrictions to watch for:

Rule	Requires
$\sqrt{g(x)}$	$g(x) \geq 0$
$\frac{1}{g(x)}$	$g(x) \neq 0$
$\log_e(g(x))$	$g(x) > 0$

Polynomials are defined for all $x \in \mathbb{R}$. When a rule combines several of these, the implied domain is the **intersection** of the separate requirements.

Restrictions. We may deliberately restrict a function to a smaller domain, written $f: D \rightarrow \mathbb{R}, f(x) = \dots$. Restricting the domain of a many-to-one function can make it one-to-one — for example $f(x) = x^2$ becomes one-to-one on $[0, \infty)$. (This is needed later to form inverses.)

Piecewise-defined (hybrid) functions. A hybrid function uses different rules on different parts of its domain, for example

$$f(x) = \begin{cases} x+1 & x < 1 \\ 3-x & x \geq 1. \end{cases}$$

To evaluate $f(a)$, choose the branch whose condition a satisfies. To sketch, draw each branch only on its sub-domain, using a **closed dot** ($\bullet$) where an endpoint is included and an **open dot** ($\circ$) where it is excluded. The function is **continuous** at a join if the two branches meet there.

Even and odd functions.

- f is **even** if $f(-x) = f(x)$ for all x — its graph is symmetric about the y -axis (e.g. x^2, x^4).
- f is **odd** if $f(-x) = -f(x)$ for all x — its graph has point symmetry about the origin (e.g. x, x^3).
- Most functions are **neither**. Test algebraically by simplifying $f(-x)$ and comparing it with $f(x)$ and $-f(x)$.

Worked examples

Example 1 — scaffolded

Find the implied domain of $f(x) = \frac{\sqrt{x+1}}{x-2}$.

Working	Comment
The numerator $\sqrt{x+1}$ requires $x+1 \geq 0$, i.e. $x \geq -1$.	A square root needs a non-negative radicand.
The denominator requires $x-2 \neq 0$, i.e. $x \neq 2$.	Division by zero is undefined.
Combine (intersection): $x \geq -1$ and $x \neq 2$.	Both conditions must hold.

$\therefore$ implied domain $[-1, 2) \cup (2, \infty)$.

The point $x=2$ is cut out of $[-1, \infty)$.

Example 2 — scaffolded

Determine whether $f(x) = x^3 - 4x$ is even, odd or neither, and describe the symmetry of its graph.

$$f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x.$$

Substitute $-x$ and simplify.

$$= -(x^3 - 4x) = -f(x).$$

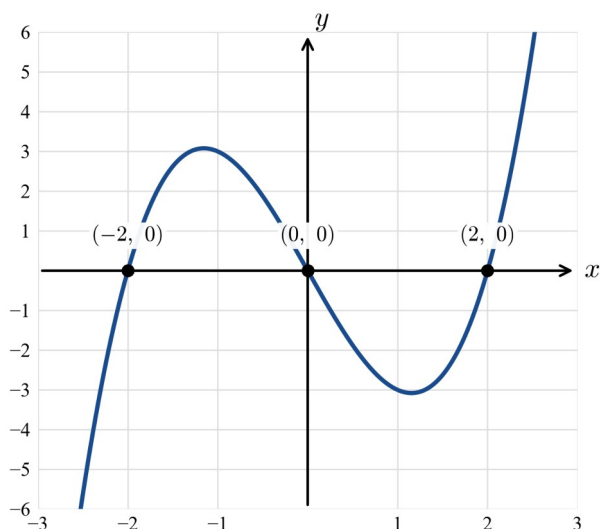
Factor out -1 and compare with $f(x)$.

Since $f(-x) = -f(x)$, f is **odd**.

Matches the definition of odd.

The graph has point symmetry about the origin.

Odd functions are symmetric under a 180° rotation about O .



Example 3 — unscaffolded

For the hybrid function $f(x) = \begin{cases} x+1 & x < 1 \\ 3-x & x \geq 1 \end{cases}$, find $f(-2)$, $f(0)$ and $f(3)$, state the domain, and sketch the graph.

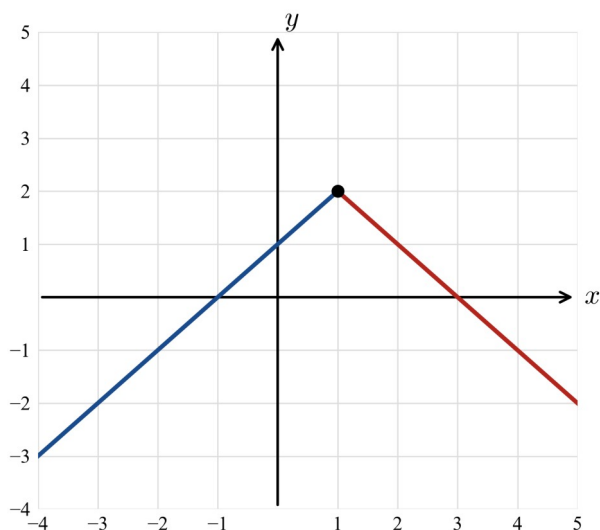
$f(-2)$: since $-2 < 1$, use $x+1$: $f(-2) = -2+1 = -1$.

$f(0)$: since $0 < 1$, use $x+1$: $f(0) = 0+1 = 1$.

$f(3)$: since $3 \geq 1$, use $3-x$: $f(3) = 3-3 = 0$.

The domain is $\mathbb{R}$. At the join $x=1$: the left branch approaches $1+1=2$ and the right branch gives $3-1=2$, so the graph is **continuous** there (a single closed point $(1, 2)$).

$\therefore$ the graph is a line of gradient 1 for $x < 1$, meeting a line of gradient -1 for $x \geq 1$ at $(1, 2)$.



Example 4 — unscaffolded

Find the implied domain of $f(x) = \sqrt{x^2 - 9}$ and state whether f is one-to-one or many-to-one.

The radicand must be non-negative: $x^2 - 9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow x \leq -3$ or $x \geq 3$.

$\therefore$ implied domain $(-\infty, -3] \cup [3, \infty)$.

Since $f(-3) = f(3) = 0$ (and, more generally, $f(-x) = f(x)$), different x -values can share a y -value, so f is **many-to-one**.

Practice questions

Scaffolded

Q1. For $f(x) = \sqrt{x - 5}$: (a) state the implied domain; (b) find the range; (c) state whether f is one-to-one or many-to-one.

Q2. For $f(x) = \frac{1}{x+3}$: (a) state the implied domain; (b) explain why $x = -3$ must be excluded; (c) state whether f is one-to-one or many-to-one.

Q3. For $f(x) = x^4 - 3x^2$: (a) find and simplify $f(-x)$; (b) compare $f(-x)$ with $f(x)$; (c) state whether f is even, odd or neither.

Q4. For $f(x) = x^3 + x$: (a) find and simplify $f(-x)$; (b) state whether f is even, odd or neither; (c) describe the symmetry of the graph.

Q5. For $f(x) = \begin{cases} -x & x < 0 \\ x^2 & x \geq 0 \end{cases}$: (a) find $f(-3)$, $f(-1)$ and $f(2)$; (b) state the domain; (c) sketch the graph.

Q6. For $f(x) = \sqrt{4 - x^2}$: (a) state the implied domain; (b) find the range; (c) state whether f is one-to-one or many-to-one.

Unscaffolded

State the implied (maximal) domain of each of the following (Q7–Q12).

Q7. $f(x) = \frac{1}{\sqrt{x-3}}$

Q8. $f(x) = \log_e(2x - 1)$

Q9. $f(x) = \frac{1}{x^2 - 4}$

Q10. $f(x) = \sqrt{x+2} + \frac{1}{x-1}$

Q11. $f(x) = \sqrt{(x-1)(x-4)}$

Q12. $f(x) = \log_e(x^2 - x - 6)$

Determine whether each is even, odd or neither (Q13–Q16).

Q13. Show that $f(x) = x^5 - x$ is odd.

Q14. $f(x) = 2x^2 + 1$

Q15. $f(x) = x^3 + x^2$

Q16. Show that $f(x) = \frac{x}{x^2 + 1}$ is odd.

Q17. For $f(x) = \begin{cases} x^2 & x \leq 1 \\ 2x - 1 & x > 1 \end{cases}$ find $f(-2)$, $f(1)$ and $f(3)$, state the domain, and sketch the graph.

Q18. For $f(x) = \begin{cases} 2x + 3 & x < -1 \\ 1 - x^2 & x \geq -1 \end{cases}$ find $f(-3)$, $f(-1)$ and $f(2)$, and sketch the graph, showing clearly whether it is continuous at $x = -1$.

Q19. The function $f(x) = x^2$ is many-to-one on $\mathbb{R}$. State a restricted domain on which f is one-to-one, and give the range of f for that restriction.

Q20. State the implied domain of $f(x) = \frac{1}{x^2 + 1}$ and explain why f is many-to-one.

Q21. A function is defined by $f(x) = \begin{cases} ax + 1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$. Find the value of a for which f is continuous at $x = 2$.

Answers

Q1. (a) $[5, \infty)$; (b) $[0, \infty)$; (c) one-to-one. **Q2.** (a) $\mathbb{R} \setminus \{-3\}$; (c) one-to-one. **Q3.** $f(-x) = x^4 - 3x^2 = f(x)$; **even**. **Q4.** $f(-x) = -x^3 - x = -f(x)$; **odd**; symmetric about the origin. **Q5.** (a) $f(-3) = 3$, $f(-1) = 1$, $f(2) = 4$; (b) $\mathbb{R}$. **Q6.** (a) $[-2, 2]$; (b) $[0, 2]$; (c) many-to-one. **Q7.** $(3, \infty)$. **Q8.** $(\frac{1}{2}, \infty)$. **Q9.** $\mathbb{R} \setminus \{-2, 2\}$. **Q10.** $[-2, 1) \cup (1, \infty)$. **Q11.** $(-\infty, 1] \cup [4, \infty)$. **Q12.** $(-\infty, -2) \cup (3, \infty)$. **Q13.** $f(-x) = -x^5 + x = -f(x)$; **odd**. **Q14.** $f(-x) = 2x^2 + 1 = f(x)$; **even**. **Q15.** $f(-x) = -x^3 + x^2$; neither $f(x)$ nor $-f(x)$; **neither**. **Q16.** $f(-x) = \frac{-x}{x^2 + 1} = -f(x)$; **odd**. **Q17.** $f(-2) = 4$, $f(1) = 1$, $f(3) = 5$; domain $\mathbb{R}$; continuous at $x = 1$. **Q18.** $f(-3) = -3$, $f(-1) = 0$, $f(2) = -3$; **not continuous** at $x = -1$ (jump from 1 to 0). **Q19.** For example $[0, \infty)$ (or $(-\infty, 0]$), with range $[0, \infty)$. **Q20.** Domain $\mathbb{R}$; many-to-one because $f(-x) = f(x)$ (e.g. $f(-1) = f(1) = 1/2$). **Q21.** $a = \frac{3}{2}$.

Fully-worked solutions

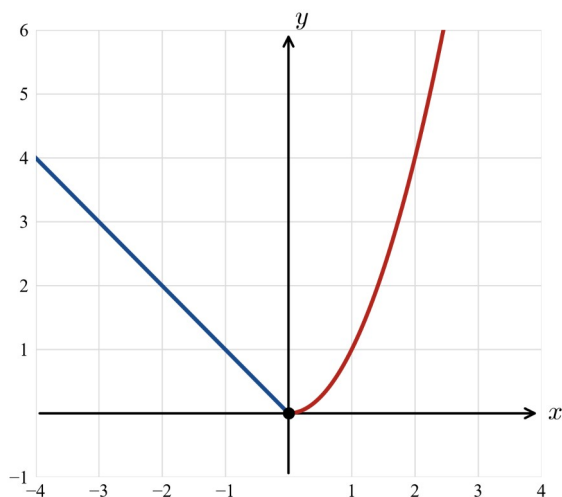
Q1. $f(x) = \sqrt{x-5}$. (a) Require $x-5 \geq 0 \Rightarrow x \geq 5$: implied domain $[5, \infty)$. (b) As x ranges over $[5, \infty)$, $\sqrt{x-5}$ takes every value ≥ 0 : range $[0, \infty)$. (c) $\sqrt{x-5}$ increases, so each y occurs once: **one-to-one**.

Q2. $f(x) = \frac{1}{x+3}$. (a) Require $x+3 \neq 0 \Rightarrow x \neq -3$: domain $\mathbb{R} \setminus \{-3\}$. (b) At $x = -3$ the denominator is 0, so f is undefined there. (c) A hyperbola is **one-to-one** (each $y \neq 0$ comes from exactly one x).

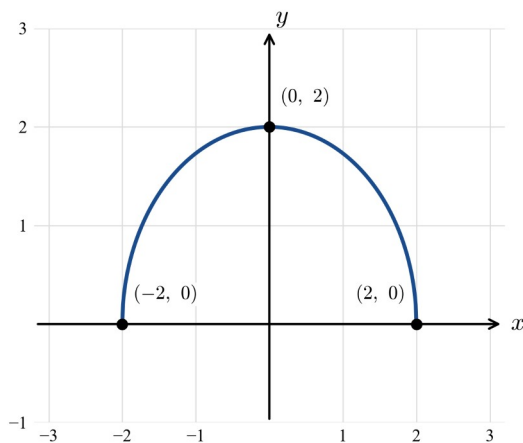
Q3. $f(x) = x^4 - 3x^2$. (a) $f(-x) = (-x)^4 - 3(-x)^2 = x^4 - 3x^2$. (b) $f(-x) = f(x)$. (c) $\therefore$ **even**.

Q4. $f(x) = x^3 + x$. (a) $f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$. (b) **odd**. (c) point symmetry about the origin.

Q5. $f(-3) = -(-3) = 3$; $f(-1) = -(-1) = 1$ (both use $-x$ since $x < 0$); $f(2) = 2^2 = 4$. Domain $\mathbb{R}$. The branches meet at $(0, 0)$ (continuous).



Q6. $f(x) = \sqrt{4-x^2}$. (a) Require $4-x^2 \geq 0 \Rightarrow x^2 \leq 4 \Rightarrow -2 \leq x \leq 2$: domain $[-2, 2]$. (b) The maximum of $\sqrt{4-x^2}$ is 2 (at $x = 0$) and the minimum is 0 (at $x = \pm 2$): range $[0, 2]$. (c) Since $f(-x) = f(x)$ (a semicircle), it is **many-to-one**.



Q7. $f(x) = \frac{1}{\sqrt{x-3}}$. The square root needs $x-3 \geq 0$ and the denominator needs $\sqrt{x-3} \neq 0$, so $x-3 > 0 \Rightarrow x > 3$: domain $(3, \infty)$.

Q8. $f(x) = \log_e(2x-1)$. Require $2x-1 > 0 \Rightarrow x > \frac{1}{2}$: domain $\left(\frac{1}{2}, \infty\right)$.

Q9. $f(x) = \frac{1}{x^2-4}$. Require $x^2-4 \neq 0 \Rightarrow x \neq \pm 2$: domain $\mathbb{R} \setminus \{-2, 2\}$.

Q10. $f(x) = \sqrt{x+2} + \frac{1}{x-1}$. Require $x+2 \geq 0$ ($x \geq -2$) and $x-1 \neq 0$ ($x \neq 1$): domain $[-2, 1) \cup (1, \infty)$.

Q11. $f(x) = \sqrt{(x-1)(x-4)}$. Require $(x-1)(x-4) \geq 0$. The product is non-negative outside the roots: $x \leq 1$ or $x \geq 4$: domain $(-\infty, 1] \cup [4, \infty)$.

Q12. $f(x) = \log_e(x^2-x-6) = \log_e((x-3)(x+2))$. Require $(x-3)(x+2) > 0 \Rightarrow x < -2$ or $x > 3$: domain $(-\infty, -2) \cup (3, \infty)$.

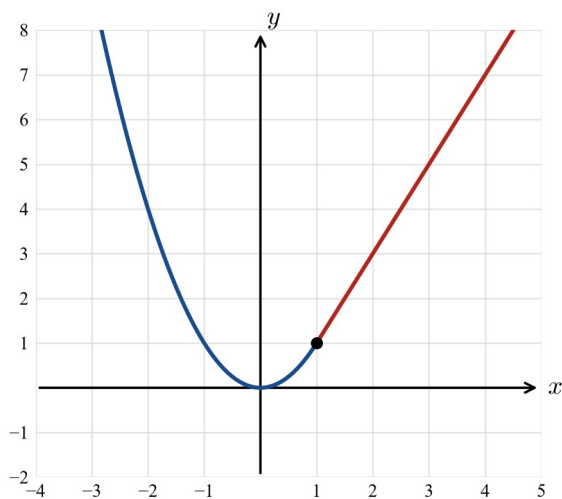
Q13. $f(x) = x^5 - x$. $f(-x) = (-x)^5 - (-x) = -x^5 + x = -(x^5 - x) = -f(x)$. $\therefore$ **odd**.

Q14. $f(x) = 2x^2 + 1$. $f(-x) = 2(-x)^2 + 1 = 2x^2 + 1 = f(x)$. $\therefore$ **even**.

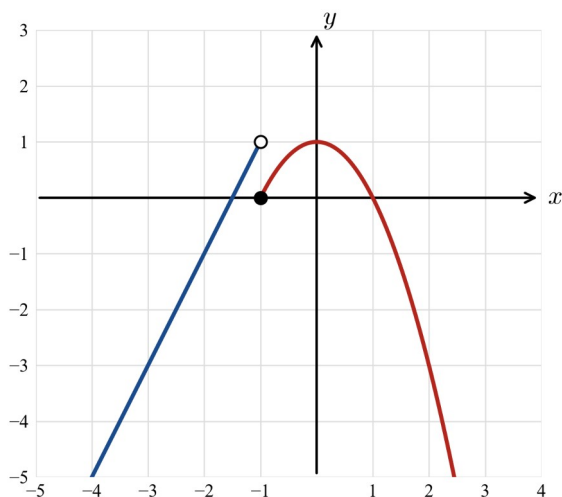
Q15. $f(x) = x^3 + x^2$. $f(-x) = -x^3 + x^2$. This equals neither $f(x) = x^3 + x^2$ nor $-f(x) = -x^3 - x^2$. $\therefore$ **neither**.

Q16. $f(x) = \frac{x}{x^2+1}$. $f(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -f(x)$. $\therefore$ **odd**.

Q17. $f(-2) = (-2)^2 = 4$ (since $-2 \leq 1$); $f(1) = 1^2 = 1$ (since $1 \leq 1$); $f(3) = 2(3) - 1 = 5$ (since $3 > 1$). Domain $\mathbb{R}$; at $x=1$ both branches give 1, so continuous.



Q18. $f(-3) = 2(-3) + 3 = -3$ (since $-3 < -1$); $f(-1) = 1 - (-1)^2 = 0$ (since $-1 \geq -1$); $f(2) = 1 - 2^2 = -3$. At $x = -1$ the left branch approaches $2(-1) + 3 = 1$ but the right branch gives 0, so there is a **jump** — the function is **not continuous** at $x = -1$ (open circle at $(-1, 1)$, closed circle at $(-1, 0)$).



Q19. $f(x) = x^2$ is many-to-one because $f(-x) = f(x)$. Restricting to $[0, \infty)$ (where f is increasing) makes it one-to-one; the range for this restriction is $[0, \infty)$. (The restriction $(-\infty, 0]$ works equally well.)

Q20. $x^2 + 1 > 0$ for all x , so the implied domain is $\mathbb{R}$. Since $f(-x) = \frac{1}{(-x)^2 + 1} = \frac{1}{x^2 + 1} = f(x)$, distinct inputs such as $x = 1$ and $x = -1$ share the value $1/2$, so f is **many-to-one**.

Q21. For continuity at $x = 2$ the two branches must agree there: $a(2) + 1 = 2^2 \Rightarrow 2a + 1 = 4 \Rightarrow 2a = 3. \therefore a = \frac{3}{2}$.

Chapter 1 Functions, relations and graphs — 1D Operations on functions

Theory

From two functions f and g we can build new functions by adding, subtracting or multiplying their outputs.

Definitions. For each x in a common domain,

$$(f+g)(x)=f(x)+g(x), (f-g)(x)=f(x)-g(x), (fg)(x)=f(x)g(x).$$

Domain of a combination. Each of $f+g$, $f-g$ and fg is only defined where **both** f and g are defined, so

$$\text{dom}(f \pm g) = \text{dom}(fg) = \text{dom } f \cap \text{dom } g.$$

This is the single most common place to lose marks: **find the intersection of the two domains first**, and keep it even after the rule simplifies. For example, if $f(x)=\sqrt{x-1}$ and $g(x)=\sqrt{x-1}$ then $(fg)(x)=x-1$, but the domain is **not** $\mathbb{R}$ — it is $[1, \infty)$, inherited from the square roots.

Addition of ordinates. To sketch $y=(f+g)(x)$ without first simplifying, add the y -coordinates (ordinates) of $y=f(x)$ and $y=g(x)$ at each value of x . Useful landmarks:

- where $f(x)=0$, the sum equals $g(x)$ (the sum meets the graph of g);
- where $g(x)=0$, the sum equals $f(x)$;
- where $f(x)=-g(x)$, the sum is 0 (an x -intercept of the sum);
- where one function is much larger than the other, the sum follows that function.

The sum is only drawn over $\text{dom } f \cap \text{dom } g$.

Worked examples

Example 1 — scaffolded

If $f(x)=x^2$ and $g(x)=2x+3$, find $(f+g)(x)$, $(f-g)(x)$ and $(fg)(x)$, state the domain of each, and evaluate $(f+g)(2)$.

Working	Comment
$\text{dom } f = \mathbb{R}$ and $\text{dom } g = \mathbb{R}$, so every combination has domain $\mathbb{R}$.	Both are polynomials, defined for all x ; the intersection is $\mathbb{R}$.
$(f+g)(x)=x^2+(2x+3)=x^2+2x+3.$	Add the rules.
$(f-g)(x)=x^2-(2x+3)=x^2-2x-3.$	Subtract; watch the sign on every term of g .
$(fg)(x)=x^2(2x+3)=2x^3+3x^2.$	Multiply the rules.
$(f+g)(2)=2^2+2(2)+3=4+4+3=11.$	Substitute $x=2$ into the sum.

Example 2 — scaffolded

If $f(x)=\sqrt{x+2}$ and $g(x)=\sqrt{3-x}$, find $(f+g)(x)$ and its maximal domain, evaluate $(f+g)(1)$ exactly, and find $(fg)(x)$.

Working	Comment
$\text{dom } f$: need $x+2 \geq 0$, so $x \geq -2$, i.e. $[-2, \infty)$.	A square root requires a non-negative radicand.
$\text{dom } g$: need $3-x \geq 0$, so $x \leq 3$, i.e. $(-\infty, 3]$.	Similarly for g .
$\text{dom}(f+g)=[-2, \infty) \cap (-\infty, 3]=[-2, 3]$.	Intersect the two domains.

$$(f+g)(x) = \sqrt{x+2} + \sqrt{3-x}.$$

The rules cannot be combined further.

$$(f+g)(1) = \sqrt{1+2} + \sqrt{3-1} = \sqrt{3} + \sqrt{2}.$$

$x=1$ lies in $[-2, 3]$; keep the surds exact.

$$(fg)(x) = \sqrt{x+2}\sqrt{3-x} = \sqrt{(x+2)(3-x)}, \text{ still with domain } [-2, 3].$$

Product of square roots; the domain is unchanged.

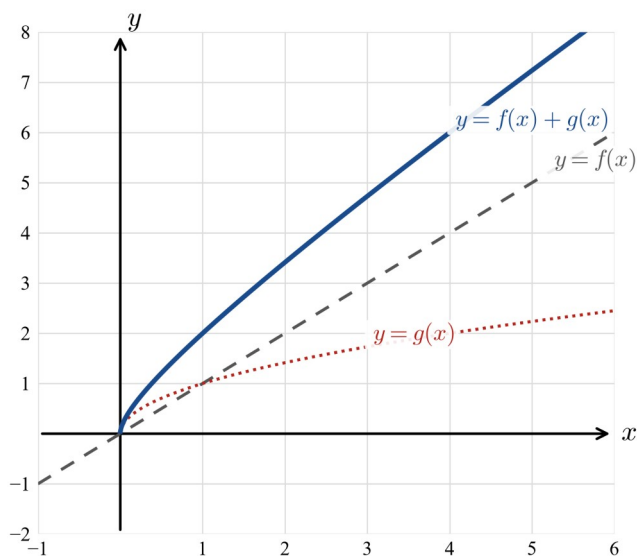
Example 3 — unscaffolded

Given $f(x) = x$ and $g(x) = \sqrt{x}$, sketch $y = (f+g)(x)$ using addition of ordinates, and state its domain.

$$\text{dom } f = \mathbb{R}, \text{ dom } g = [0, \infty), \text{ so } \text{dom } (f+g) = [0, \infty).$$

$(f+g)(x) = x + \sqrt{x}$. At $x=0$ both graphs meet the origin, so the sum starts at $(0, 0)$; for $x > 0$ the sum lies **above** the line $y = x$ by the height $\sqrt{x}$. At $x=4$: $f=4$, $g=2$, so the sum is 6.

$\therefore$ the sum increases from $(0, 0)$, always above $y = x$.



Example 4 — unscaffolded

If $f(x) = x^2$ and $g(x) = \frac{1}{x}$, find $(fg)(x)$ and state its domain.

$$\text{dom } f = \mathbb{R} \text{ and } \text{dom } g = \mathbb{R} \setminus \{0\}, \text{ so } \text{dom } (fg) = \mathbb{R} \cap (\mathbb{R} \setminus \{0\}) = \mathbb{R} \setminus \{0\}.$$

$$(fg)(x) = x^2 \times \frac{1}{x} = x \text{ for } x \neq 0.$$

The rule simplifies to the line $y = x$, but g is undefined at $x=0$, so that value stays excluded even after the cancellation.

$\therefore (fg)(x) = x$, with domain $\mathbb{R} \setminus \{0\}$ — **not** $\mathbb{R}$.

Practice questions

Scaffolded

Q1. For $f(x) = x^2 - 1$ and $g(x) = x + 1$: (a) find $(f+g)(x)$; (b) find $(f-g)(x)$; (c) find $(fg)(x)$; (d) state the domain of each combination.

Q2. For $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{5-x}$: (a) write down $(f+g)(x)$; (b) find the maximal domain of $f+g$; (c) evaluate $(f+g)(2)$, giving an exact answer; (d) find $(fg)(x)$ and state its domain.

Q3. For $f(x) = 2x$ and $g(x) = x^2$: (a) write $(f+g)(x)$ in simplest form; (b) complete a table of $f(x)$, $g(x)$ and $(f+g)(x)$ at $x = -2, -1, 0, 1$; (c) sketch $y = (f+g)(x)$ using addition of ordinates, labelling the turning point; (d) state the coordinates of the turning point.

Q4. For $f(x) = x$ and $g(x) = \frac{1}{x}$: (a) state the domain of $f+g$; (b) write $(f+g)(x)$; (c) sketch $y = (f+g)(x)$ using addition of ordinates, labelling the turning points; (d) describe the behaviour of the sum as $x \rightarrow \infty$.

Q5. For $f(x) = \sqrt{x+4}$ and $g(x) = \sqrt{x+4}$: (a) find $(f+g)(x)$; (b) find $(fg)(x)$; (c) state the domain of (fg) and explain why it is not $\mathbb{R}$.

Q6. For $f(x) = x^2$ and $g(x) = 3-x$: (a) find $(f+g)(x)$; (b) find $(f-g)(x)$; (c) find $(fg)(x)$; (d) evaluate each of the three combinations at $x = 2$.

Unscaffolded

For each pair, find $(f+g)(x)$, $(f-g)(x)$ and $(fg)(x)$ and state the maximal domain of the combined functions. Where a **sketch** is asked for, use addition of ordinates.

Q7. $f(x) = x^2 + 2$, $g(x) = x - 4$.

Q8. $f(x) = 3x$, $g(x) = x^2 + x$.

Q9. $f(x) = \sqrt{x}$, $g(x) = \sqrt{6-x}$; also evaluate $(f+g)(2)$ exactly.

Q10. $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{x+1}$; **show that** $(fg)(x) = x+1$ and state its domain.

Q11. $f(x) = \frac{1}{x-2}$, $g(x) = x$.

Q12. $f(x) = x$, $g(x) = -x^2$; sketch $y = (f+g)(x)$ using addition of ordinates, labelling the turning point.

Q13. $f(x) = x^2$, $g(x) = \frac{1}{x^2}$; **show that** $(fg)(x) = 1$ and state its domain.

Q14. $f(x) = \sqrt{x+3}$, $g(x) = \sqrt{1-x}$; state the maximal domain of $f+g$.

Q15. $f(x) = x+1$, $g(x) = x^2 - 4$.

Q16. $f(x) = \sqrt{x-2}$, $g(x) = x$; sketch $y = (f+g)(x)$ using addition of ordinates.

Q17. $f(x) = 2x+1$, $g(x) = 3-2x$. Comment on $(f+g)(x)$.

Q18. $f(x) = x^3$, $g(x) = x$.

Q19. $f(x) = \sqrt{4-x}$, $g(x) = \sqrt{4-x}$; state the domain of (fg) .

Q20. $f(x) = x$, $g(x) = \frac{2}{x}$; sketch $y = (f+g)(x)$ using addition of ordinates, giving the exact coordinates of the turning points.

Q21. $f(x) = x^2 - 2x$, $g(x) = 2x$. **Show that** $(f+g)(x) = x^2$, then sketch $y = (f+g)(x)$ using addition of ordinates.

Answers

Q1. (a) $x^2 + x$; (b) $x^2 - x - 2$; (c) $(x+1)(x^2-1) = x^3 + x^2 - x - 1$; (d) all $\mathbb{R}$. **Q2.** (a) $\sqrt{x-1} + \sqrt{5-x}$; (b) $[1, 5]$; (c) $1 + \sqrt{3}$; (d) $\sqrt{(x-1)(5-x)}$, domain $[1, 5]$. **Q3.** (a) $x^2 + 2x$; (d) turning point $(-1, -1)$. **Q4.** (a) $\mathbb{R} \setminus \{0\}$; (b) $x + \frac{1}{x}$; (c) turning points $(1, 2)$ and $(-1, -2)$; (d) the sum approaches the line $y = x$ from above. **Q5.** (a) $2\sqrt{x+4}$; (b) $x+4$; (c) domain $[-4, \infty)$ — inherited from the square roots. **Q6.** (a) $x^2 - x + 3$; (b) $x^2 + x - 3$; (c) $x^2(3-x) = 3x^2 - x^3$; (d) 5, 3, 4. **Q7.** $x^2 + x - 2$; $x^2 - x + 6$; $(x-4)(x^2+2)$; domain $\mathbb{R}$. **Q8.** $x^2 + 4x$; $2x - x^2$; $3x^2(x+1)$; domain $\mathbb{R}$. **Q9.** $\sqrt{x} + \sqrt{6-x}$; $\sqrt{x} - \sqrt{6-x}$; $\sqrt{x(6-x)}$; domain $[0, 6]$; $(f+g)(2) = \sqrt{2} + 2$. **Q10.** $2\sqrt{x+1}$; 0; $x+1$; domain $[-1, \infty)$. **Q11.** $\frac{1}{x-2} + x$; $\frac{1}{x-2} - x$; $\frac{x}{x-2}$; domain $\mathbb{R} \setminus \{2\}$. **Q12.** $(f+g)(x) = x - x^2$; $(f-g)(x) = x + x^2$; $(fg)(x) = -x^3$; turning point $(\frac{1}{2}, \frac{1}{4})$; domain $\mathbb{R}$. **Q13.** $(f+g)(x) = x^2 + \frac{1}{x^2}$; $(f-g)(x) = x^2 - \frac{1}{x^2}$; $(fg)(x) = 1$; domain $\mathbb{R} \setminus \{0\}$. **Q14.** $\sqrt{x+3} + \sqrt{1-x}$; $\sqrt{x+3} - \sqrt{1-x}$; $\sqrt{(x+3)(1-x)}$; domain $[-3, 1]$. **Q15.** $x^2 + x - 3$; $-x^2 + x + 5$; $(x+1)(x^2-4)$; domain $\mathbb{R}$. **Q16.** $(f+g)(x) = x + \sqrt{x-2}$; $(f-g)(x) = \sqrt{x-2} - x$; $(fg)(x) = x\sqrt{x-2}$; domain $[2, \infty)$. **Q17.** $(f+g)(x) = 4$ (a constant); $(f-g)(x) = 4x - 2$; $(fg)(x) = -4x^2 + 4x + 3$; domain $\mathbb{R}$. **Q18.** $x^3 + x$; $x^3 - x$; x^4 ; domain $\mathbb{R}$. **Q19.** $2\sqrt{4-x}$; 0; $4-x$; domain $(-\infty, 4]$. **Q20.** $(f+g)(x) = x + \frac{2}{x}$; $(f-g)(x) = x - \frac{2}{x}$; $(fg)(x) = 2$; turning points $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$; domain $\mathbb{R} \setminus \{0\}$. **Q21.** $(f+g)(x) = x^2$; $(f-g)(x) = x^2 - 4x$; $(fg)(x) = 2x^3 - 4x^2$; domain $\mathbb{R}$; turning point $(0, 0)$.

Fully-worked solutions

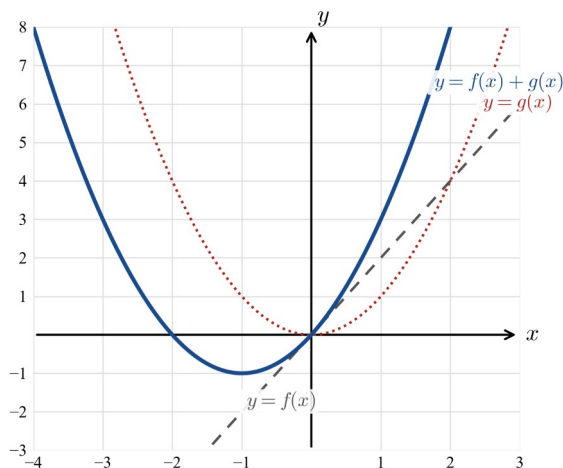
Q1. $\text{dom } f = \text{dom } g = \mathbb{R}$, so every combination has domain $\mathbb{R}$. (a) $(f+g)(x) = (x^2-1) + (x+1) = x^2 + x$. (b) $(f-g)(x) = (x^2-1) - (x+1) = x^2 - x - 2$. (c) $(fg)(x) = (x^2-1)(x+1) = x^3 + x^2 - x - 1$.

Q2. $\text{dom } f = [1, \infty)$ (need $x-1 \geq 0$); $\text{dom } g = (-\infty, 5]$ (need $5-x \geq 0$). (a) $(f+g)(x) = \sqrt{x-1} + \sqrt{5-x}$. (b) domain $= [1, \infty) \cap (-\infty, 5] = [1, 5]$. (c) $(f+g)(2) = \sqrt{1} + \sqrt{3} = 1 + \sqrt{3}$. (d) $(fg)(x) = \sqrt{(x-1)(5-x)}$, domain $[1, 5]$.

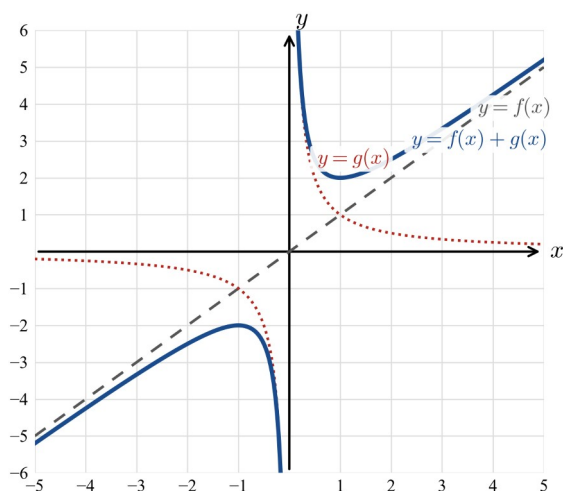
Q3. (a) $(f+g)(x) = 2x + x^2 = x^2 + 2x$. (b)

x	-2	-1	0	1
$f(x) = 2x$	-4	-2	0	2
$g(x) = x^2$	4	1	0	1
$(f+g)(x)$	0	-1	0	3

(c), (d) The sum $x^2 + 2x = x(x+2)$ is a parabola with x -intercepts 0 and -2 and turning point $(-1, -1)$.



Q4. (a) $\text{dom } f = \mathbb{R}$, $\text{dom } g = \mathbb{R} \setminus \{0\}$, so domain $= \mathbb{R} \setminus \{0\}$. (b) $(f+g)(x) = x + \frac{1}{x}$. (c) For large $|x|$ the term $\frac{1}{x}$ is small, so the sum hugs the line $y = x$; near $x = 0$ the term $\frac{1}{x}$ dominates. Setting the derivative $1 - \frac{1}{x^2} = 0$ gives $x^2 = 1$, so $x = \pm 1$: turning points $(1, 2)$ (minimum, $x > 0$) and $(-1, -2)$ (maximum, $x < 0$). (d) As $x \rightarrow \infty$, $(f+g)(x) \rightarrow x^+$: the graph approaches the line $y = x$ from above.



Q5. (a) $(f+g)(x) = 2\sqrt{x+4}$. (b) $(fg)(x) = \sqrt{x+4}\sqrt{x+4} = x+4$. (c) Each factor needs $x+4 \geq 0$, i.e. $x \geq -4$, so the domain is $[-4, \infty)$. Although the rule $x+4$ is defined for all reals, the product only exists where **both** square roots do.

Q6. $\text{dom} = \mathbb{R}$ for all. (a) $(f+g)(x) = x^2 + (3-x) = x^2 - x + 3$. (b) $(f-g)(x) = x^2 - (3-x) = x^2 + x - 3$. (c) $(fg)(x) = x^2(3-x) = 3x^2 - x^3$. (d) At $x = 2$: $(f+g)(2) = 4 - 2 + 3 = 5$; $(f-g)(2) = 4 + 2 - 3 = 3$; $(fg)(2) = 4(1) = 4$.

Q7. Domain $\mathbb{R}$. $(f+g)(x) = x^2 + x - 2$; $(f-g)(x) = x^2 - x + 6$; $(fg)(x) = (x-4)(x^2+2)$.

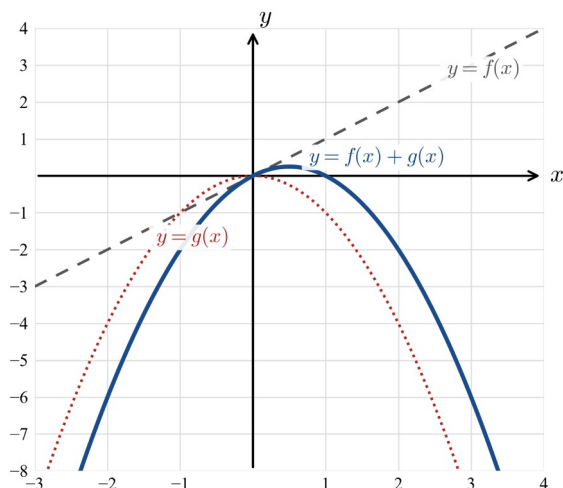
Q8. Domain $\mathbb{R}$. $(f+g)(x) = 3x + (x^2+x) = x^2 + 4x$; $(f-g)(x) = 3x - (x^2+x) = 2x - x^2$; $(fg)(x) = 3x(x^2+x) = 3x^2(x+1)$.

Q9. $\text{dom } f = [0, \infty)$, $\text{dom } g = (-\infty, 6]$, so domain $[0, 6]$. $(f+g)(x) = \sqrt{x} + \sqrt{6-x}$; $(f-g)(x) = \sqrt{x} - \sqrt{6-x}$; $(fg)(x) = \sqrt{x(6-x)}$. $(f+g)(2) = \sqrt{2} + \sqrt{4} = \sqrt{2} + 2$.

Q10. Each root needs $x+1 \geq 0$, so domain $[-1, \infty)$. $(f+g)(x) = 2\sqrt{x+1}$; $(f-g)(x) = 0$; $(fg)(x) = \sqrt{x+1}\sqrt{x+1} = x+1$; the domain remains $[-1, \infty)$, not $\mathbb{R}$.

Q11. $\text{dom } f = \mathbb{R} \setminus \{2\}$, $\text{dom } g = \mathbb{R}$, so domain $\mathbb{R} \setminus \{2\}$. $(f+g)(x) = \frac{1}{x-2} + x$; $(f-g)(x) = \frac{1}{x-2} - x$;
 $(fg)(x) = \frac{x}{x-2}$.

Q12. Domain $\mathbb{R}$. $(f+g)(x) = x - x^2 = x(1-x)$, a parabola with x -intercepts 0 and 1 and turning point $\left(\frac{1}{2}, \frac{1}{4}\right)$.
 $(f-g)(x) = x - (-x^2) = x + x^2$; $(fg)(x) = x(-x^2) = -x^3$.

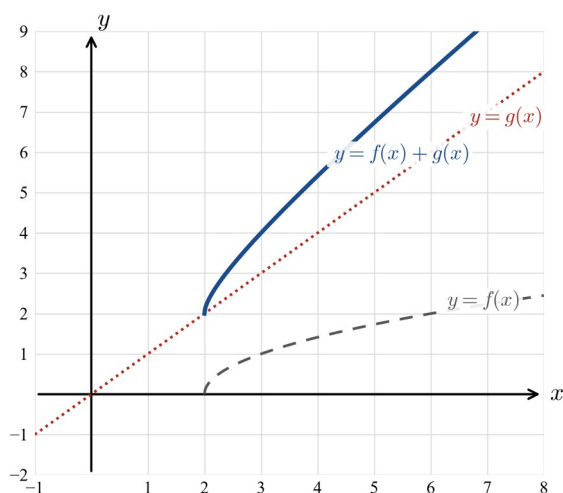


Q13. $\text{dom } g = \mathbb{R} \setminus \{0\}$, so domain $\mathbb{R} \setminus \{0\}$. $(f+g)(x) = x^2 + \frac{1}{x^2}$; $(f-g)(x) = x^2 - \frac{1}{x^2}$; $(fg)(x) = x^2 \cdot \frac{1}{x^2} = 1$ for every $x \neq 0$ (the value $x=0$ is excluded).

Q14. $\text{dom } f = [-3, \infty)$, $\text{dom } g = (-\infty, 1]$, so domain $[-3, 1]$. $(f+g)(x) = \sqrt{x+3} + \sqrt{1-x}$;
 $(f-g)(x) = \sqrt{x+3} - \sqrt{1-x}$; $(fg)(x) = \sqrt{(x+3)(1-x)}$.

Q15. Domain $\mathbb{R}$. $(f+g)(x) = x^2 + x - 3$; $(f-g)(x) = (x+1) - (x^2 - 4) = -x^2 + x + 5$; $(fg)(x) = (x+1)(x^2 - 4)$.

Q16. $\text{dom } f = [2, \infty)$, $\text{dom } g = \mathbb{R}$, so domain $[2, \infty)$. $(f+g)(x) = x + \sqrt{x-2}$: the sum starts at $(2, 2)$ (where $\sqrt{x-2} = 0$, so the sum equals g) and rises above the line $y = x$. $(f-g)(x) = \sqrt{x-2} - x$; $(fg)(x) = x\sqrt{x-2}$.



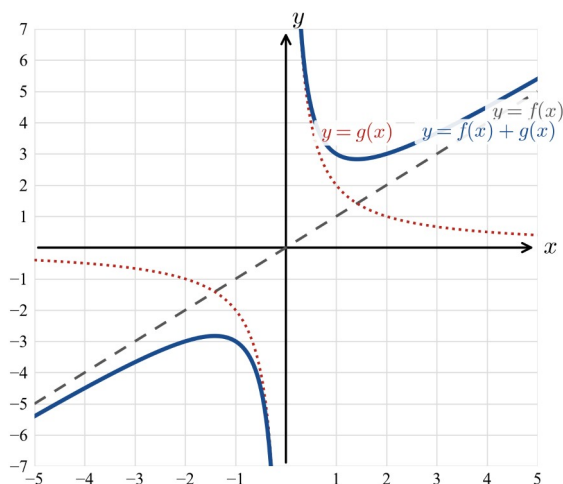
Q17. Domain $\mathbb{R}$. $(f+g)(x) = (2x+1) + (3-2x) = 4$: the two linear terms cancel, giving the **constant** function 4.
 $(f-g)(x) = 4x - 2$; $(fg)(x) = (2x+1)(3-2x) = -4x^2 + 4x + 3$.

Q18. Domain $\mathbb{R}$. $(f+g)(x) = x^3 + x$; $(f-g)(x) = x^3 - x$; $(fg)(x) = x^3 \cdot x = x^4$.

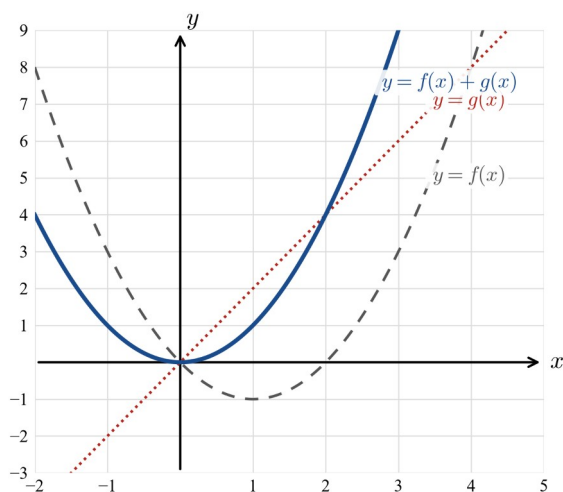
Q19. Each root needs $4 - x \geq 0$, so domain $(-\infty, 4]$. $(f + g)(x) = 2\sqrt{4 - x}$; $(f - g)(x) = 0$; $(fg)(x) = 4 - x$, with domain $(-\infty, 4]$.

Q20. Domain $\mathbb{R} \setminus \{0\}$. $(f + g)(x) = x + \frac{2}{x}$; $(f - g)(x) = x - \frac{2}{x}$; $(fg)(x) = x \cdot \frac{2}{x} = 2$ for $x \neq 0$. Setting the derivative

$1 - \frac{2}{x^2} = 0$ gives $x^2 = 2$, so the turning points are at $x = \pm\sqrt{2}$, giving $(\sqrt{2}, 2\sqrt{2})$ (minimum) and $(-\sqrt{2}, -2\sqrt{2})$ (maximum). The graph approaches $y = x$ for large $|x|$.



Q21. Domain $\mathbb{R}$. $(f + g)(x) = (x^2 - 2x) + 2x = x^2$: the $-2x$ and $+2x$ cancel, leaving the basic parabola $y = x^2$ with turning point $(0, 0)$. $(f - g)(x) = (x^2 - 2x) - 2x = x^2 - 4x$; $(fg)(x) = (x^2 - 2x)(2x) = 2x^3 - 4x^2$.



Chapter 1 Functions, relations and graphs — 1E Composite functions

Theory

Given two functions f and g , the **composite function** $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

To evaluate it, apply g **first**, then apply f to the result. In general $f \circ g \neq g \circ f$, so the **order matters**.

When does $f \circ g$ exist? The output $g(x)$ must be an allowable input for f ; that is,

$$\text{ran}(g) \subseteq \text{dom}(f).$$

If this fails, $f \circ g$ can still be formed on a **restricted domain** — keep only those x in $\text{dom}(g)$ for which $g(x) \in \text{dom}(f)$:

$$\text{dom}(f \circ g) = \{x \in \text{dom}(g) : g(x) \in \text{dom}(f)\}.$$

Finding the rule. Substitute the rule for g into f . For example, if $f(x) = \sqrt{x}$ and $g(x) = x - 4$, then $(f \circ g)(x) = f(x - 4) = \sqrt{x - 4}$.

Domain and range.

- The **domain** of $f \circ g$ is a subset of $\text{dom}(g)$ — never larger. Even when the rule *simplifies*, the domain is inherited from the inner function and any new restriction.
- The **range** of $f \circ g$ is found from the composite rule over its domain.

Graphing a composite. To sketch the graph of $y = (f \circ g)(x)$, first find the rule for $f(g(x))$, then sketch the resulting function. For example, if $f(x) = x^2$ and $g(x) = x + 1$, then $(f \circ g)(x) = (x + 1)^2$, whose graph is a parabola with vertex at $(-1, 0)$ and y -intercept $(0, 1)$.

Watch the domain. If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 - 1$ *simplifies* to $x - 1$, but the domain is still $\{x : x \geq 0\}$ (inherited from $\sqrt{x}$), **not** all of $\mathbb{R}$. This is a very common exam slip.

Decomposing a function. Any “function of a function” can be written as a composite. For $h(x) = (3x - 1)^4$, choose the inner function $g(x) = 3x - 1$ and the outer function $f(x) = x^4$, so that $h = f \circ g$.

Worked examples

Example 1 — scaffolded

For $f(x) = x^2 - 2$ and $g(x) = 3x + 1$ (both with domain $\mathbb{R}$), find $f(g(x))$ and $g(f(x))$, and evaluate $f(g(2))$.

Working

Comment

$$f(g(x)) = f(3x + 1) = (3x + 1)^2 - 2$$

Substitute $g(x)$ into f : replace the input of f by $3x + 1$.

$$= 9x^2 + 6x + 1 - 2 = 9x^2 + 6x - 1$$

Expand $(3x + 1)^2 = 9x^2 + 6x + 1$.

$$g(f(x)) = g(x^2 - 2) = 3(x^2 - 2) + 1 = 3x^2 - 5$$

Now substitute $f(x)$ into g . Note $g(f(x)) \neq f(g(x))$.

$$f(g(2)): \text{ first } g(2) = 3(2) + 1 = 7, \text{ then}$$

Work from the inside out: g first, then f .

$$f(7) = 7^2 - 2 = 47.$$

$$\therefore f(g(x)) = 9x^2 + 6x - 1, g(f(x)) = 3x^2 - 5,$$

Both composites have domain $\mathbb{R}$.

$$f(g(2)) = 47.$$

Example 2 — scaffolded

Let $f(x) = \sqrt{x}$, with domain $\{x : x \geq 0\}$, and $g(x) = x - 4$, with domain $\mathbb{R}$. Determine whether $f \circ g$ exists, and find its rule, domain and range.

Working	Comment
$\text{ran}(g) = \mathbb{R}$, but $\text{dom}(f) = \{x : x \geq 0\}$, so $\text{ran}(g) \not\subseteq \text{dom}(f)$.	$f \circ g$ does not exist for all real x ; restrict the domain.
$(f \circ g)(x) = f(x - 4) = \sqrt{x - 4}$.	Substitute g into f .
Need $x - 4 \geq 0$, i.e. $x \geq 4$. Domain $\{x : x \geq 4\}$.	Keep only the inputs that land in $\text{dom}(f)$.
Since $\sqrt{x - 4} \geq 0$, the range is $\{y : y \geq 0\}$.	Read the range from the rule over its domain.
$\therefore (f \circ g)(x) = \sqrt{x - 4}$, domain $\{x : x \geq 4\}$, range $\{y : y \geq 0\}$.	For comparison, $(g \circ f)(x) = \sqrt{x} - 4$ has domain $\{x : x \geq 0\}$.

Example 3 — unscaffolded

For $f(x) = \frac{1}{x}$, with domain $\mathbb{R} \setminus \{0\}$, and $g(x) = x + 2$, find $f(g(x))$ and $g(f(x))$, stating the domain and range of each.

$f(g(x)) = f(x + 2) = \frac{1}{x + 2}$. This is undefined when $x + 2 = 0$, so the domain is $\mathbb{R} \setminus \{-2\}$, and since $\frac{1}{x + 2} \neq 0$, the range is $\mathbb{R} \setminus \{0\}$.

$g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 2$. This is undefined when $x = 0$, so the domain is $\mathbb{R} \setminus \{0\}$, and since $\frac{1}{x} \neq 0$, we have $g(f(x)) \neq 2$, giving range $\mathbb{R} \setminus \{2\}$.

$\therefore$ the two composites have different rules, domains and ranges.

Example 4 — unscaffolded

Express each function as a composite $f \circ g$, stating suitable rules for f and g . (a) $h(x) = (5x - 2)^3$ (b) $k(x) = \sqrt{x^2 + 1}$.

(a) The “inside” is $5x - 2$ and the “outside” operation is cubing. Take $g(x) = 5x - 2$ and $f(x) = x^3$; then $f(g(x)) = (5x - 2)^3 = h(x)$.

(b) The “inside” is $x^2 + 1$ and the “outside” is the square root. Take $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$; then $f(g(x)) = \sqrt{x^2 + 1} = k(x)$.

$\therefore$ (a) $f(x) = x^3$, $g(x) = 5x - 2$; (b) $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$.

Practice questions

Scaffolded

Q1. For $f(x) = 2x + 1$ and $g(x) = x^2$: (a) find $f(g(x))$; (b) find $g(f(x))$; (c) evaluate $f(g(3))$ and $g(f(3))$; (d) state whether $f \circ g = g \circ f$.

Q2. For $f(x) = x^2$ and $g(x) = x - 3$: (a) find $f(g(x))$ and expand; (b) find $g(f(x))$; (c) evaluate $f(g(5))$; (d) state the domain and range of $f(g(x))$.

Q3. For $f(x) = \frac{1}{x}$ (domain $\mathbb{R} \setminus \{0\}$) and $g(x) = 3 - x$: (a) find $f(g(x))$; (b) state the domain of $f(g(x))$; (c) find $g(f(x))$; (d) state the domain of $g(f(x))$.

Q4. For $f(x) = \sqrt{x}$ (domain $\{x : x \geq 0\}$) and $g(x) = 9 - x$: (a) find $f(g(x))$; (b) find the domain and range of $f(g(x))$; (c) find $g(f(x))$; (d) find the domain and range of $g(f(x))$.

Q5. For $f(x) = x^2 + 2$ and $g(x) = \frac{1}{x}$ (domain $\mathbb{R} \setminus \{0\}$): (a) find $f(g(x))$ and state its range; (b) find $g(f(x))$; (c) explain why $g(f(x))$ is defined for all real x ; (d) state the range of $g(f(x))$.

Q6. For $f(x) = \sqrt{x}$ (domain $\{x : x \geq 0\}$) and $g(x) = x^2 - 9$: (a) find the rule for $f(g(x))$; (b) find the values of x for which $f(g(x))$ is defined; (c) find $g(f(x))$ and its domain; (d) evaluate $f(g(5))$, giving an exact value.

Unscaffolded

Q7. For $f(x) = 3x - 2$ and $g(x) = x + 5$, find $f(g(x))$ and $g(f(x))$.

Q8. For $f(x) = x^2$ and $g(x) = 2x - 1$, find $f(g(x))$ and $g(f(x))$.

Q9. For $f(x) = \frac{1}{x+1}$ (domain $\mathbb{R} \setminus \{-1\}$) and $g(x) = x^2$, find $f(g(x))$ and $g(f(x))$, stating the domain of each.

Q10. For $f(x) = \sqrt{x-1}$ (domain $\{x : x \geq 1\}$) and $g(x) = x + 3$, find $f(g(x))$ and $g(f(x))$, stating the domain of each.

Q11. Express $h(x) = (2x + 7)^5$ as a composite $f \circ g$, stating rules for f and g .

Q12. Express $h(x) = \frac{1}{x^2 - 4}$ as a composite $f \circ g$, and state the maximal domain of h .

Q13. Express $h(x) = \sqrt{5 - 2x}$ as a composite $f \circ g$, and state the maximal domain of h .

Q14. For $f(x) = x^2 - 4$ and $g(x) = \sqrt{x}$ (domain $\{x : x \geq 0\}$): (a) find and **simplify** $f(g(x))$, and state its domain and range; (b) find $g(f(x))$ and state its domain.

Q15. For $f(x) = \frac{2}{x}$ (domain $\mathbb{R} \setminus \{0\}$) and $g(x) = x - 3$, find $f(g(x))$ and $g(f(x))$, stating the domain and range of each.

Q16. If $f(x) = x^2$ and $g(x) = x + 2$, solve $f(g(x)) = 9$ for x .

Q17. If $f(x) = 2x - 1$ and $g(x) = x^2 + 1$, solve $f(g(x)) = 19$ for x .

Q18. For $f(x) = \sqrt{x}$ (domain $\{x : x \geq 0\}$) and $g(x) = x + 4$: (a) find $f(g(x))$ and state its domain; (b) sketch the graph of $y = f(g(x))$, labelling the endpoint and the y -intercept; (c) state the range of $f(g(x))$.

Q19. State the maximal domain of $h(x) = \frac{1}{x^2 - x - 6}$, and express h as a composite $f \circ g$.

Q20. Given that $f(x) = 3x + 1$ and $f(g(x)) = 6x + 7$, find the rule for $g(x)$.

Q21. Given that $g(x) = x - 4$ and $f(g(x)) = x^2 - 8x + 16$, find the rule for $f(x)$.

Answers

Q1. (a) $2x^2 + 1$; (b) $4x^2 + 4x + 1$; (c) $f(g(3)) = 19$, $g(f(3)) = 49$; (d) no, $f \circ g \neq g \circ f$. **Q2.** (a) $x^2 - 6x + 9$; (b) $x^2 - 3$; (c) 4; (d) domain $\mathbb{R}$, range $\{y: y \geq 0\}$. **Q3.** (a) $\frac{1}{3-x}$; (b) $\mathbb{R} \setminus \{3\}$; (c) $3 - \frac{1}{x}$; (d) $\mathbb{R} \setminus \{0\}$. **Q4.** (a) $\sqrt{9-x}$; (b) domain $\{x: x \leq 9\}$, range $\{y: y \geq 0\}$; (c) $9 - \sqrt{x}$; (d) domain $\{x: x \geq 0\}$, range $\{y: y \leq 9\}$. **Q5.** (a) $\frac{1}{x^2} + 2$, range $\{y: y > 2\}$; (b) $\frac{1}{x^2 + 2}$; (c) $x^2 + 2 \geq 2 > 0$ for all x , so the denominator is never 0; (d) $\left\{y: 0 < y \leq \frac{1}{2}\right\}$. **Q6.** (a) $\sqrt{x^2 - 9}$; (b) $x \leq -3$ or $x \geq 3$; (c) $g(f(x)) = x - 9$, domain $\{x: x \geq 0\}$; (d) $f(g(5)) = \sqrt{16} = 4$. **Q7.** $f(g(x)) = 3x + 13$; $g(f(x)) = 3x + 3$. **Q8.** $f(g(x)) = 4x^2 - 4x + 1$; $g(f(x)) = 2x^2 - 1$. **Q9.** $f(g(x)) = \frac{1}{x^2 + 1}$, domain $\mathbb{R}$; $g(f(x)) = \frac{1}{(x+1)^2}$, domain $\mathbb{R} \setminus \{-1\}$. **Q10.** $f(g(x)) = \sqrt{x+2}$, domain $\{x: x \geq -2\}$; $g(f(x)) = \sqrt{x-1} + 3$, domain $\{x: x \geq 1\}$. **Q11.** $f(x) = x^5$, $g(x) = 2x + 7$. **Q12.** $f(x) = \frac{1}{x}$, $g(x) = x^2 - 4$; maximal domain $\mathbb{R} \setminus \{-2, 2\}$. **Q13.** $f(x) = \sqrt{x}$, $g(x) = 5 - 2x$; maximal domain $\left\{x: x \leq \frac{5}{2}\right\}$. **Q14.** (a) $x - 4$, domain $\{x: x \geq 0\}$, range $\{y: y \geq -4\}$; (b) $\sqrt{x^2 - 4}$, domain $\{x: x \leq -2 \text{ or } x \geq 2\}$. **Q15.** $f(g(x)) = \frac{2}{x-3}$, domain $\mathbb{R} \setminus \{3\}$, range $\mathbb{R} \setminus \{0\}$; $g(f(x)) = \frac{2}{x} - 3$, domain $\mathbb{R} \setminus \{0\}$, range $\mathbb{R} \setminus \{-3\}$. **Q16.** $x = 1$ or $x = -5$. **Q17.** $x = 3$ or $x = -3$. **Q18.** (a) $\sqrt{x+4}$, domain $\{x: x \geq -4\}$; (b) square-root curve starting at $(-4, 0)$, increasing, through $(0, 2)$; (c) $\{y: y \geq 0\}$. **Q19.** Maximal domain $\mathbb{R} \setminus \{-2, 3\}$; $f(x) = \frac{1}{x}$, $g(x) = x^2 - x - 6$. **Q20.** $g(x) = 2x + 2$. **Q21.** $f(x) = x^2$.

Fully-worked solutions

Q1. (a) $f(g(x)) = 2(x^2) + 1 = 2x^2 + 1$. (b) $g(f(x)) = (2x + 1)^2 = 4x^2 + 4x + 1$. (c) $f(g(3)) = 2(9) + 1 = 19$; $g(f(3)) = (2 \cdot 3 + 1)^2 = 7^2 = 49$. (d) Since $2x^2 + 1 \neq 4x^2 + 4x + 1$, $f \circ g \neq g \circ f$.

Q2. (a) $f(g(x)) = (x - 3)^2 = x^2 - 6x + 9$. (b) $g(f(x)) = x^2 - 3$. (c) $f(g(5)) = (5 - 3)^2 = 4$. (d) $(x - 3)^2 \geq 0$ for all real x , so domain $\mathbb{R}$, range $\{y: y \geq 0\}$.

Q3. (a) $f(g(x)) = \frac{1}{3-x}$. (b) Undefined at $x = 3$: domain $\mathbb{R} \setminus \{3\}$. (c) $g(f(x)) = 3 - \frac{1}{x}$. (d) Undefined at $x = 0$: domain $\mathbb{R} \setminus \{0\}$.

Q4. (a) $f(g(x)) = \sqrt{9-x}$. (b) Need $9 - x \geq 0 \Rightarrow x \leq 9$: domain $\{x: x \leq 9\}$; since $\sqrt{9-x} \geq 0$, range $\{y: y \geq 0\}$. (c) $g(f(x)) = 9 - \sqrt{x}$. (d) Domain $\{x: x \geq 0\}$; since $\sqrt{x} \geq 0$, $9 - \sqrt{x} \leq 9$, so range $\{y: y \leq 9\}$.

Q5. (a) $f(g(x)) = \left(\frac{1}{x}\right)^2 + 2 = \frac{1}{x^2} + 2$; since $\frac{1}{x^2} > 0$, range $\{y: y > 2\}$. (b) $g(f(x)) = \frac{1}{x^2 + 2}$. (c) $x^2 + 2 \geq 2 > 0$ for all x , so the denominator is never zero — defined on $\mathbb{R}$. (d) As $x^2 + 2$ ranges over $[2, \infty)$, $\frac{1}{x^2 + 2}$ ranges over $\left(0, \frac{1}{2}\right]$.

Q6. (a) $f(g(x)) = \sqrt{x^2 - 9}$. (b) Need $x^2 - 9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow x \leq -3$ or $x \geq 3$. (c) $g(f(x)) = (\sqrt{x})^2 - 9 = x - 9$, with domain $\{x: x \geq 0\}$ inherited from $\sqrt{x}$. (d) $f(g(5)) = \sqrt{25 - 9} = \sqrt{16} = 4$.

Q7. $f(g(x)) = 3(x + 5) - 2 = 3x + 13$; $g(f(x)) = (3x - 2) + 5 = 3x + 3$.

Q8. $f(g(x)) = (2x - 1)^2 = 4x^2 - 4x + 1$; $g(f(x)) = 2(x^2) - 1 = 2x^2 - 1$.

Q9. $f(g(x)) = \frac{1}{x^2+1}$; since $x^2+1 \geq 1 > 0$, domain $\mathbb{R}$. $g(f(x)) = \left(\frac{1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$; undefined at $x = -1$, domain $\mathbb{R} \setminus \{-1\}$.

Q10. $f(g(x)) = \sqrt{(x+3)-1} = \sqrt{x+2}$; need $x+2 \geq 0$, domain $\{x: x \geq -2\}$. $g(f(x)) = \sqrt{x-1}+3$; the inner $\sqrt{x-1}$ needs $x \geq 1$, domain $\{x: x \geq 1\}$.

Q11. Let $g(x) = 2x+7$ and $f(x) = x^5$; then $f(g(x)) = (2x+7)^5 = h(x)$.

Q12. Let $g(x) = x^2 - 4$ and $f(x) = \frac{1}{x}$; then $f(g(x)) = \frac{1}{x^2-4}$. This is undefined when $x^2 - 4 = 0 \Rightarrow x = \pm 2$, so the maximal domain is $\mathbb{R} \setminus \{-2, 2\}$.

Q13. Let $g(x) = 5 - 2x$ and $f(x) = \sqrt{x}$; then $f(g(x)) = \sqrt{5-2x}$. Need $5 - 2x \geq 0 \Rightarrow x \leq \frac{5}{2}$, so the maximal domain is $\left\{x: x \leq \frac{5}{2}\right\}$.

Q14. (a) $f(g(x)) = (\sqrt{x})^2 - 4 = x - 4$. The domain is inherited from $g(x) = \sqrt{x}: \{x: x \geq 0\}$. Over this domain $x - 4 \geq -4$, so range $\{y: y \geq -4\}$. (The rule simplifies, but the domain is **not** all of $\mathbb{R}$.) (b) $g(f(x)) = \sqrt{x^2 - 4}$; need $x^2 - 4 \geq 0 \Rightarrow x \leq -2$ or $x \geq 2$.

Q15. $f(g(x)) = \frac{2}{x-3}$; undefined at $x = 3$, domain $\mathbb{R} \setminus \{3\}$, and since the value is never 0, range $\mathbb{R} \setminus \{0\}$.

$g(f(x)) = \frac{2}{x} - 3$; undefined at $x = 0$, domain $\mathbb{R} \setminus \{0\}$, and since $\frac{2}{x} \neq 0$, range $\mathbb{R} \setminus \{-3\}$.

Q16. $f(g(x)) = (x+2)^2$. Solve $(x+2)^2 = 9 \Rightarrow x+2 = \pm 3 \Rightarrow x = 1$ or $x = -5$.

Q17. $f(g(x)) = 2(x^2+1) - 1 = 2x^2+1$. Solve $2x^2+1 = 19 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.

Q18. (a) $f(g(x)) = f(x+4) = \sqrt{x+4}$. Need $g(x) = x+4 \geq 0$, i.e. $x \geq -4$: domain $\{x: x \geq -4\}$. (b) The graph has the shape of $y = \sqrt{x}$ shifted 4 units left: it begins at the endpoint $(-4, 0)$ and increases, crossing the y -axis at $(0, 2)$ since $\sqrt{0+4} = 2$. (c) Since $\sqrt{x+4} \geq 0$ on the domain, the range is $\{y: y \geq 0\}$.

Q19. $h(x) = \frac{1}{x^2-x-6}$ is undefined where $x^2-x-6 = (x-3)(x+2) = 0$, i.e. $x = 3$ or $x = -2$: maximal domain

$\mathbb{R} \setminus \{-2, 3\}$. With $g(x) = x^2 - x - 6$ and $f(x) = \frac{1}{x}$, $h = f \circ g$.

Q20. $f(g(x)) = 3g(x) + 1$. Setting $3g(x) + 1 = 6x + 7$ gives $3g(x) = 6x + 6$, so $g(x) = 2x + 2$.

Q21. $f(g(x)) = x^2 - 8x + 16 = (x-4)^2$. Since $g(x) = x - 4$, we have $f(x-4) = (x-4)^2$, so $f(u) = u^2$; that is, $f(x) = x^2$.

Chapter 1 Functions, relations and graphs — 1F Inverse functions

Theory

The **inverse** of a function reverses its effect: if f maps a to b , then its inverse maps b back to a . Formally, the inverse relation of f is obtained by interchanging the coordinates in every ordered pair, so the point (a, b) on $y = f(x)$ corresponds to the point (b, a) on the inverse.

When does an inverse function exist? The inverse relation is itself a *function* only when f is **one-to-one** — that is, no two different x -values give the same y -value (the graph passes a **horizontal-line test**). When f is one-to-one we write its inverse function as f^{-1} .

Finding the rule of f^{-1} .

1. Write $y = f(x)$.
2. **Interchange x and y .**
3. Solve for y ; this expression is $f^{-1}(x)$.

Domain and range swap. Because the coordinates are interchanged,

$$\text{dom } f^{-1} = \text{ran } f \text{ and } \text{ran } f^{-1} = \text{dom } f.$$

Always state the domain of f^{-1} — it is the range of f , and is an essential part of the answer.

The graph of f^{-1} is the reflection of the graph of f in the line $y = x$. Reflecting swaps the axes, so intercepts, endpoints and asymptotes of f become the corresponding features of f^{-1} with their coordinates interchanged.

Composition. For every x in the appropriate domain,

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$

Many-to-one functions. A many-to-one function (for example $f(x) = x^2$) has no inverse function over its full domain. **Restricting the domain** to an interval on which f is one-to-one produces a function that does have an inverse. Different restrictions give different inverses (for $f(x) = x^2$, the restriction $x \geq 0$ gives $f^{-1}(x) = \sqrt{x}$, while $x \leq 0$ gives $f^{-1}(x) = -\sqrt{x}$).

Points of intersection. Because f^{-1} is the reflection of f in $y = x$, any point where the two graphs meet on the line $y = x$ satisfies $f(x) = x$. (Graphs can also meet off the line, but solving $f(x) = x$ finds the common points on it.)

Worked examples

Example 1 — scaffolded

For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 3$, find the rule and domain of f^{-1} , and sketch f , f^{-1} and the line $y = x$ on one set of axes.

Working

Comment

Let $y = 2x + 3$.

Start from $y = f(x)$.

Interchange x and y : $x = 2y + 3$.

The defining step for an inverse.

Solve for y : $2y = x - 3$, so $y = \frac{x-3}{2}$. Thus

Make y the subject.

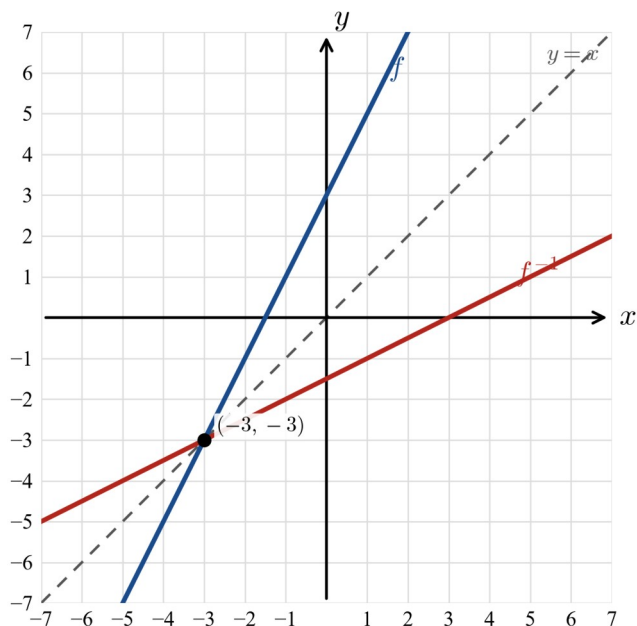
$$f^{-1}(x) = \frac{x-3}{2}.$$

f is linear (one-to-one) with domain and range $\mathbb{R}$, so

$$\text{dom } f^{-1} = \text{ran } f = \mathbb{R}.$$

$\text{dom } f^{-1} = \mathbb{R}$.

The graphs meet on $y = x$ where $2x + 3 = x$, i.e. $x = -3$: Solve $f(x) = x$ for the point on the line.
the point $(-3, -3)$.



Example 2 — scaffolded

For $f(x) = x^2 - 4$ with domain $\{x : x \geq 0\}$, find f^{-1} , stating its domain, and sketch both graphs with $y = x$.

Working

On $x \geq 0$, f is one-to-one, so f^{-1} exists.

Range of f : the minimum is $f(0) = -4$, so
 $\text{ran } f = [-4, \infty)$.

Let $y = x^2 - 4$; interchange: $x = y^2 - 4$, so $y^2 = x + 4$.

Since $y = f^{-1}(x)$ takes values in $\text{dom } f = [0, \infty)$, take the **positive** root: $y = \sqrt{x + 4}$.

$\therefore f^{-1}(x) = \sqrt{x + 4}$, $\text{dom } f^{-1} = [-4, \infty)$.

Comment

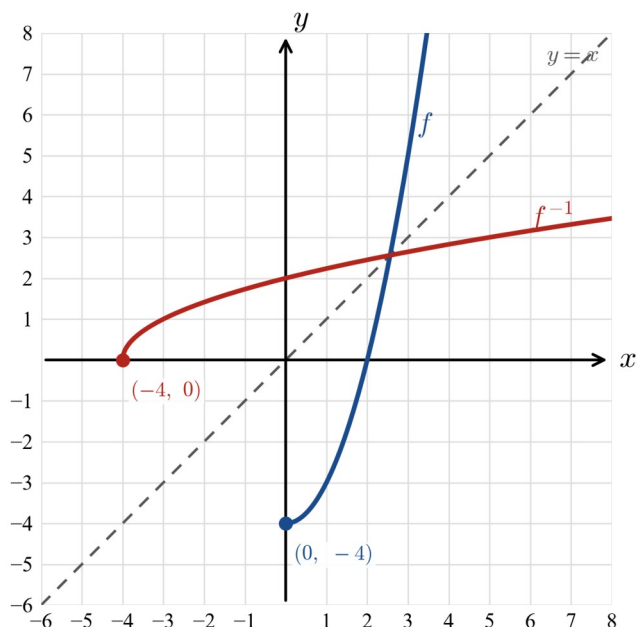
Restricting to $x \geq 0$ makes the parabola one-to-one.

This will be the domain of f^{-1} .

Swap x and y , then solve.

The range of f^{-1} is $[0, \infty)$, so choose the positive root.

State the domain — it is essential.



Example 3 — unscaffolded

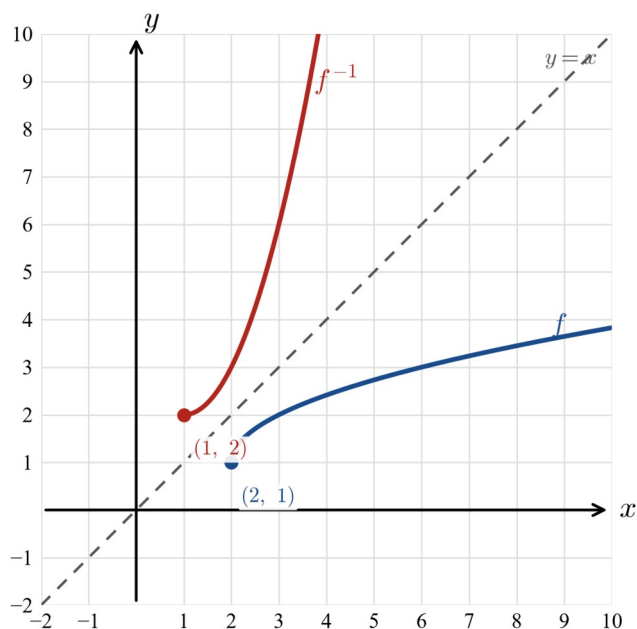
For $f(x) = \sqrt{x-2} + 1$ with domain $\{x : x \geq 2\}$, find f^{-1} and its domain.

f is one-to-one on $x \geq 2$. Its endpoint is $(2, 1)$ and it increases, so $\text{ran } f = [1, \infty)$.

Let $y = \sqrt{x-2} + 1$. Interchange: $x = \sqrt{y-2} + 1$, so $x-1 = \sqrt{y-2}$.

Square (valid since $x-1 \geq 0$ on the range): $(x-1)^2 = y-2$, giving $y = (x-1)^2 + 2$.

$\therefore f^{-1}(x) = (x-1)^2 + 2$, with $\text{dom } f^{-1} = [1, \infty)$.



Example 4 — unscaffolded

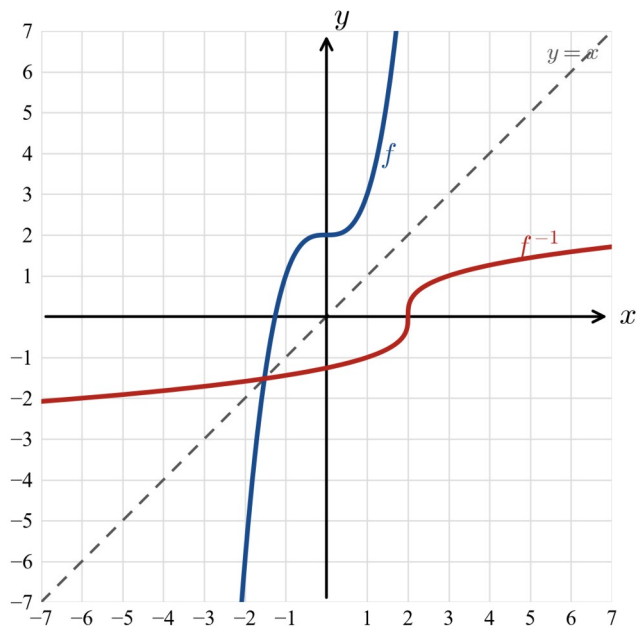
For $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3 + 2$, find f^{-1} and describe how its graph is related to the graph of f .

f is a one-to-one cubic with domain and range $\mathbb{R}$.

Let $y = x^3 + 2$. Interchange: $x = y^3 + 2$, so $y^3 = x-2$ and $y = \sqrt[3]{x-2}$.

$\therefore f^{-1}(x) = \sqrt[3]{x-2}$, with $\text{dom } f^{-1} = \mathbb{R}$.

The graph of f^{-1} is the reflection of the graph of f in the line $y = x$; for instance the point $(0, 2)$ on f corresponds to $(2, 0)$ on f^{-1} .



Practice questions

Scaffolded

Q1. For $f(x) = 3x - 6$, $x \in \mathbb{R}$: (a) find the rule of f^{-1} ; (b) state the domain of f^{-1} ; (c) find where the graphs of f and f^{-1} meet on the line $y = x$; (d) sketch f , f^{-1} and $y = x$.

Q2. For $f(x) = x^2 + 1$, $x \geq 0$: (a) state the range of f ; (b) find $f^{-1}(x)$ and its domain; (c) sketch f , f^{-1} and $y = x$.

Q3. For $f(x) = \sqrt{x+3}$, $x \geq -3$: (a) state the range of f ; (b) find $f^{-1}(x)$ and its domain; (c) sketch both graphs with $y = x$.

Q4. For $f(x) = \frac{1}{x}$, $x \neq 0$: (a) find $f^{-1}(x)$; (b) what do you notice about f and f^{-1} ? (c) state the domain and range of f^{-1} .

Q5. For $f(x) = (x-1)^2$, $x \geq 1$: (a) explain why the domain restriction is needed for an inverse to exist; (b) find $f^{-1}(x)$ and its domain; (c) sketch f , f^{-1} and $y = x$.

Q6. For $f(x) = x^3 - 1$, $x \in \mathbb{R}$: (a) find $f^{-1}(x)$; (b) verify that $f(f^{-1}(x)) = x$; (c) sketch both graphs with $y = x$.

Un scaffolded

For each function, find the rule and domain of f^{-1} (state the range of f where needed), and sketch f , f^{-1} and the line $y = x$.

Q7. $f(x) = 4x + 1$, $x \in \mathbb{R}$

Q8. $f(x) = -2x + 5$, $x \in \mathbb{R}$ (also find where the graphs meet on $y = x$)

Q9. $f(x) = x^2 - 9$, $x \geq 0$

Q10. $f(x) = x^2 + 2$, $x \leq 0$

Q11. $f(x) = \sqrt{x-4}$, $x \geq 4$

Q12. $f(x) = \sqrt{2-x}, x \leq 2$

Q13. $f(x) = 2x^3, x \in \mathbb{R}$

Q14. $f(x) = (x+2)^3, x \in \mathbb{R}$

Q15. $f(x) = \frac{1}{x-2}, x \neq 2$

Q16. $f(x) = \frac{2}{x} + 1, x \neq 0$

Q17. $f(x) = (x-3)^2 + 1, x \geq 3$

Q18. $f(x) = 4 - x^2, x \geq 0$

Q19. $f(x) = \sqrt{x} - 2, x \geq 0$

Q20. The function $g(x) = (x-2)^2 - 3$ is many-to-one. State the largest domain of the form $[a, \infty)$ on which g has an inverse, then find $g^{-1}(x)$ and its domain. (VCAA-style.)

Q21. For $f(x) = \sqrt{x+2}, x \geq -2$, find f^{-1} and hence find the exact coordinates of any point of intersection of the graphs of f and f^{-1} . (VCAA-style.)

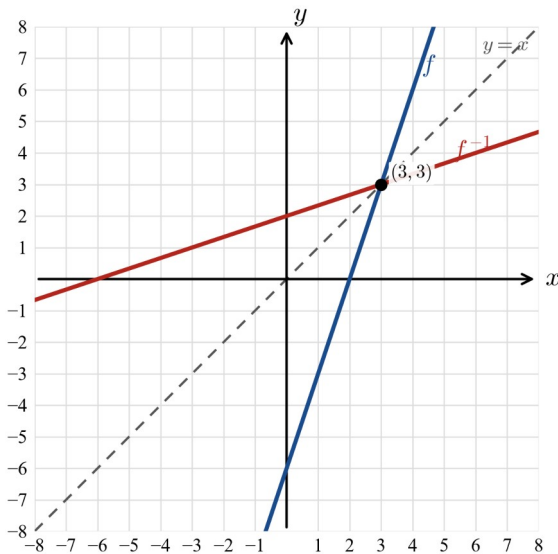
Answers

Q1. $f^{-1}(x) = \frac{x+6}{3}$; dom $\mathbb{R}$; meet at $(3, 3)$. **Q2.** $\text{ran } f = [1, \infty)$; $f^{-1}(x) = \sqrt{x-1}$, dom $[1, \infty)$. **Q3.** $\text{ran } f = [0, \infty)$; $f^{-1}(x) = x^2 - 3$, dom $[0, \infty)$. **Q4.** $f^{-1}(x) = \frac{1}{x}$; f is its own inverse (self-inverse); dom and $\text{ran } \mathbb{R} \setminus \{0\}$. **Q5.** $f^{-1}(x) = \sqrt{x+1}$, dom $[0, \infty)$. **Q6.** $f^{-1}(x) = \sqrt[3]{x+1}$. **Q7.** $f^{-1}(x) = \frac{x-1}{4}$, dom $\mathbb{R}$. **Q8.** $f^{-1}(x) = \frac{5-x}{2}$, dom $\mathbb{R}$; meet at $\left(\frac{5}{3}, \frac{5}{3}\right)$. **Q9.** $\text{ran } f = [-9, \infty)$; $f^{-1}(x) = \sqrt{x+9}$, dom $[-9, \infty)$. **Q10.** $\text{ran } f = [2, \infty)$; $f^{-1}(x) = -\sqrt{x-2}$, dom $[2, \infty)$ (range ≤ 0). **Q11.** $\text{ran } f = [0, \infty)$; $f^{-1}(x) = x^2 + 4$, dom $[0, \infty)$. **Q12.** $\text{ran } f = [0, \infty)$; $f^{-1}(x) = 2 - x^2$, dom $[0, \infty)$. **Q13.** $f^{-1}(x) = \sqrt[3]{\frac{x}{2}}$, dom $\mathbb{R}$. **Q14.** $f^{-1}(x) = \sqrt[3]{x} - 2$, dom $\mathbb{R}$. **Q15.** $f^{-1}(x) = \frac{1}{x} + 2$, dom $\mathbb{R} \setminus \{0\}$. **Q16.** $f^{-1}(x) = \frac{2}{x-1}$, dom $\mathbb{R} \setminus \{1\}$. **Q17.** $\text{ran } f = [1, \infty)$; $f^{-1}(x) = \sqrt{x-1} + 3$, dom $[1, \infty)$. **Q18.** $\text{ran } f = (-\infty, 4]$; $f^{-1}(x) = \sqrt{4-x}$, dom $(-\infty, 4]$. **Q19.** $\text{ran } f = [-2, \infty)$; $f^{-1}(x) = (x+2)^2$, dom $[-2, \infty)$. **Q20.** Largest domain $[2, \infty)$; $g^{-1}(x) = \sqrt{x+3} + 2$, dom $[-3, \infty)$. **Q21.** $f^{-1}(x) = x^2 - 2$, dom $[0, \infty)$; graphs meet at $(2, 2)$.

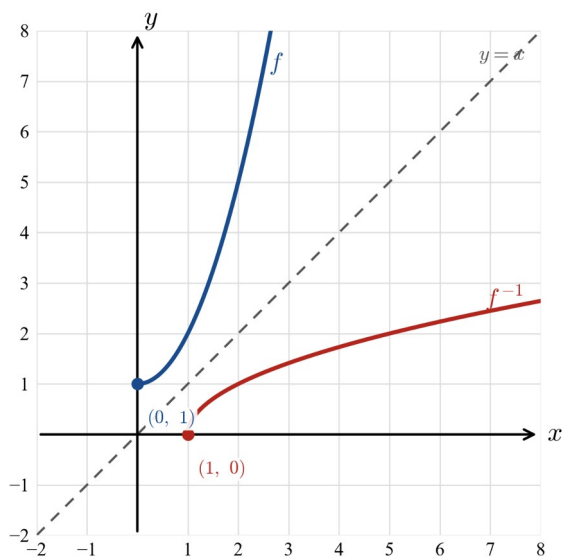
Fully-worked solutions

Q1. $y = 3x - 6$; swap: $x = 3y - 6 \Rightarrow 3y = x + 6 \Rightarrow y = \frac{x+6}{3}$. So $f^{-1}(x) = \frac{x+6}{3}$, dom $\mathbb{R}$. Meet on $y = x$:

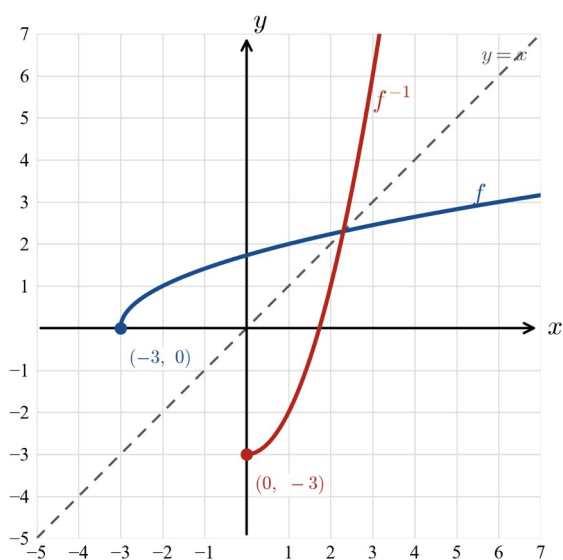
$3x - 6 = x \Rightarrow 2x = 6 \Rightarrow x = 3$, giving $(3, 3)$.



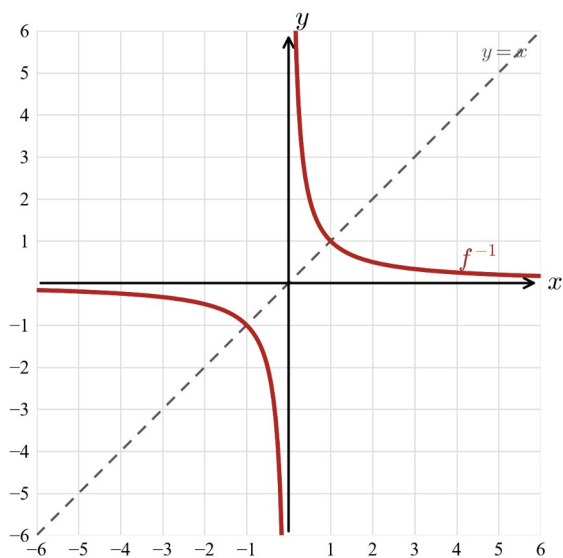
Q2. On $x \geq 0$, $\text{ran } f = [1, \infty)$. Swap $y = x^2 + 1$: $x = y^2 + 1 \Rightarrow y^2 = x - 1$; take the positive root (range of f^{-1} is $[0, \infty)$): $f^{-1}(x) = \sqrt{x-1}$, dom $[1, \infty)$.



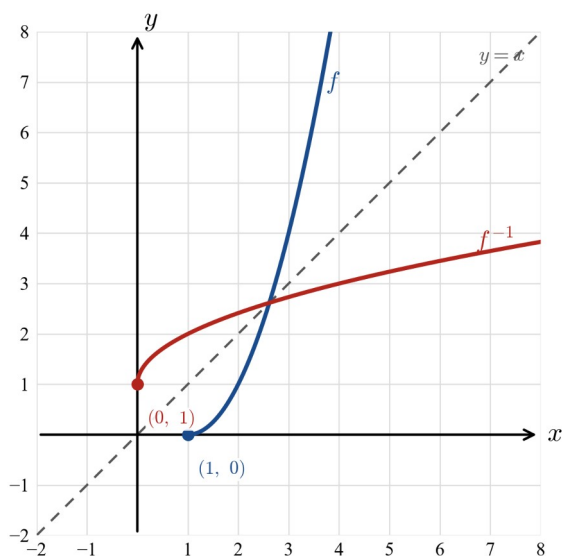
Q3. $\text{ran } f = [0, \infty)$. Swap $y = \sqrt{x+3}$: $x = \sqrt{y+3} \Rightarrow x^2 = y+3 \Rightarrow y = x^2 - 3$. So $f^{-1}(x) = x^2 - 3$, $\text{dom } [0, \infty)$ (the range of f).



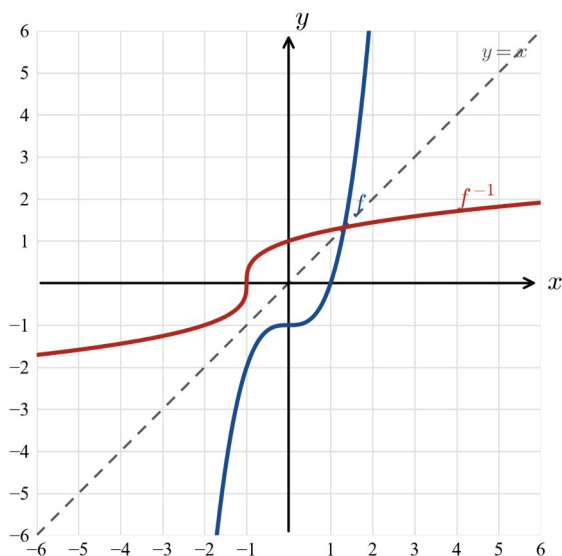
Q4. Swap $y = \frac{1}{x}$: $x = \frac{1}{y} \Rightarrow y = \frac{1}{x}$. So $f^{-1}(x) = \frac{1}{x} = f(x)$ — the function is its **own inverse** (self-inverse), which is why its graph is symmetric in $y = x$. dom and $\text{ran } \mathbb{R} \setminus \{0\}$.



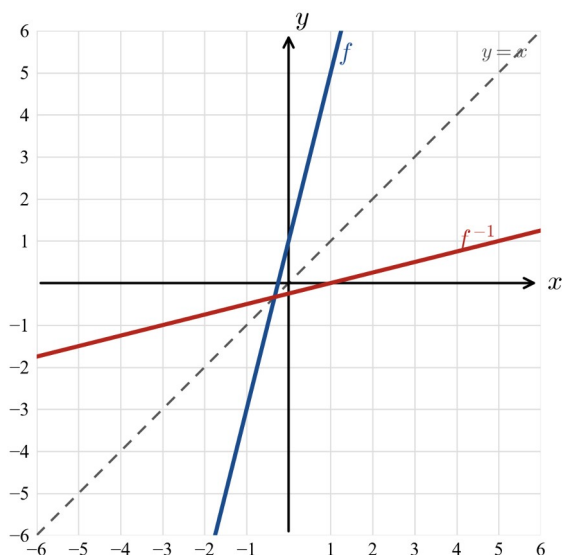
Q5. (a) On its full domain x^2 is many-to-one, so no inverse function exists; restricting to $x \geq 1$ makes f one-to-one. (b) $\text{ran } f = [0, \infty)$. Swap $y = (x - 1)^2$: $x = (y - 1)^2 \Rightarrow y - 1 = \sqrt{x}$ (positive root, since $y \geq 1$) $\Rightarrow y = \sqrt{x} + 1$. So $f^{-1}(x) = \sqrt{x} + 1$, $\text{dom } [0, \infty)$.



Q6. Swap $y = x^3 - 1$: $x = y^3 - 1 \Rightarrow y^3 = x + 1 \Rightarrow y = \sqrt[3]{x + 1}$. So $f^{-1}(x) = \sqrt[3]{x + 1}$. Check: $f(f^{-1}(x)) = (\sqrt[3]{x + 1})^3 - 1 = (x + 1) - 1 = x$. ✓

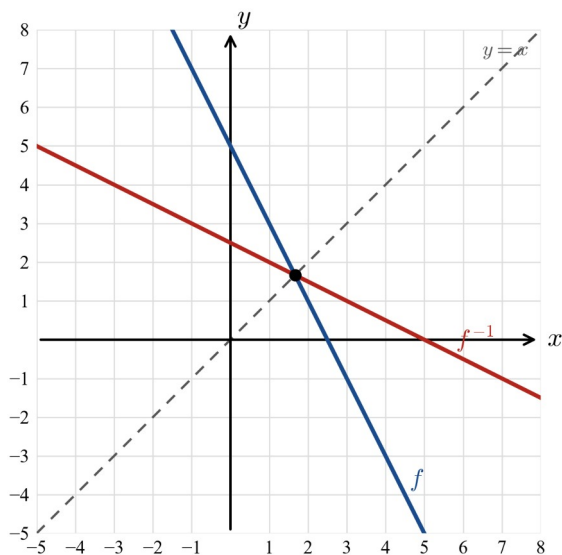


Q7. $y = 4x + 1$; swap: $x = 4y + 1 \Rightarrow y = \frac{x - 1}{4}$. So $f^{-1}(x) = \frac{x - 1}{4}$, $\text{dom } \mathbb{R}$.

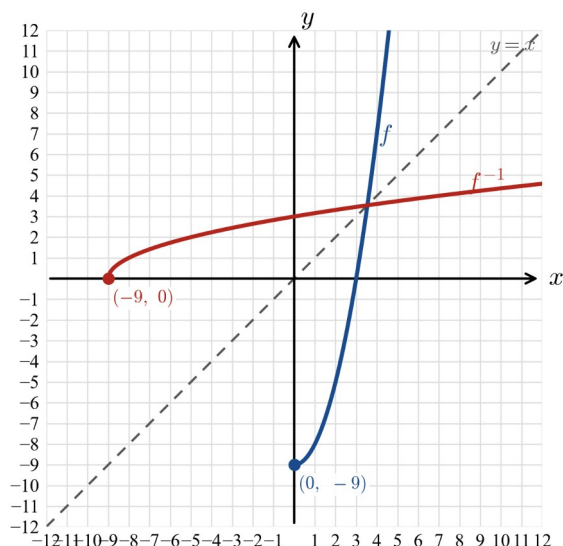


Q8. $y = -2x + 5$; swap: $x = -2y + 5 \Rightarrow 2y = 5 - x \Rightarrow y = \frac{5-x}{2}$. So $f^{-1}(x) = \frac{5-x}{2}$, $\text{dom } \mathbb{R}$. Meet on $y = x$:

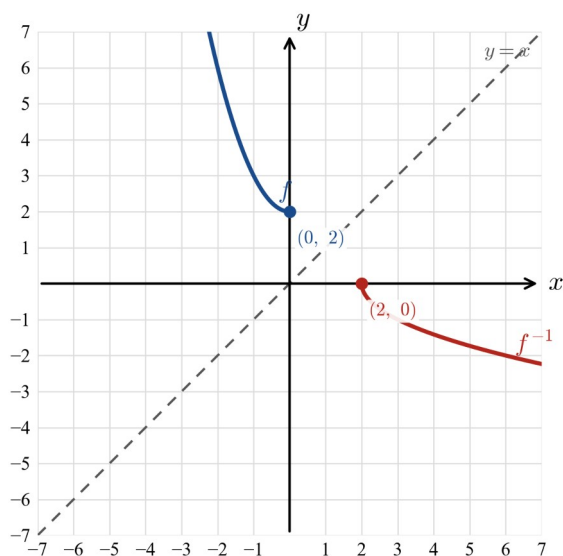
$$-2x + 5 = x \Rightarrow 3x = 5 \Rightarrow x = \frac{5}{3}, \text{ giving } \left(\frac{5}{3}, \frac{5}{3}\right).$$



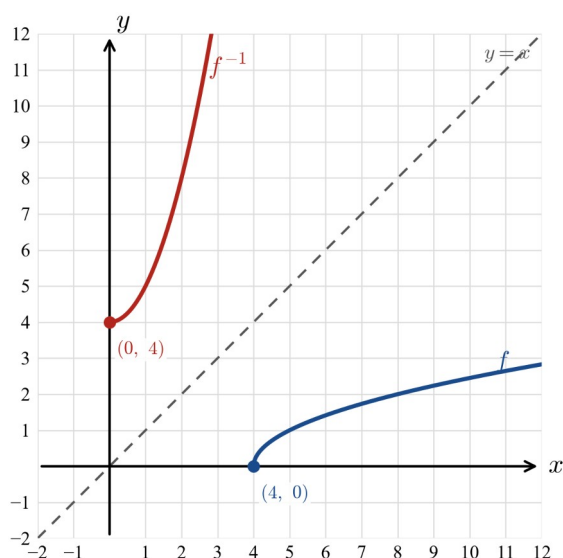
Q9. On $x \geq 0$, $\text{ran } f = [-9, \infty)$. Swap $y = x^2 - 9$: $x = y^2 - 9 \Rightarrow y^2 = x + 9 \Rightarrow y = \sqrt{x+9}$ (positive root). So $f^{-1}(x) = \sqrt{x+9}$, $\text{dom } [-9, \infty)$.



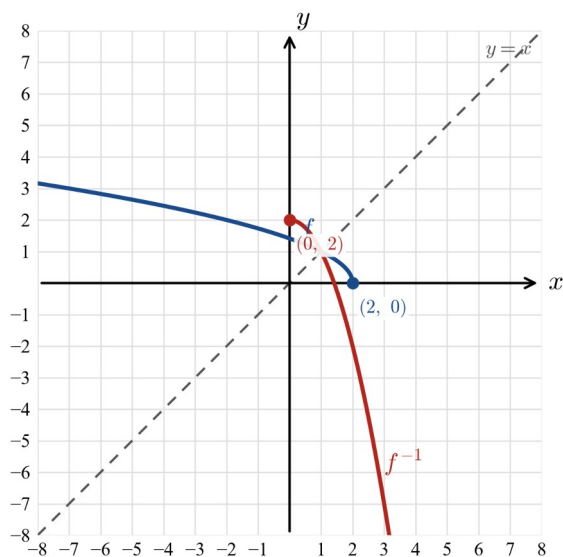
Q10. On $x \leq 0$, f is decreasing and one-to-one; $\text{ran } f = [2, \infty)$. Swap $y = x^2 + 2$: $x = y^2 + 2 \Rightarrow y^2 = x - 2$. Since $\text{ran } f^{-1} = \text{dom } f = (-\infty, 0]$, take the **negative** root: $f^{-1}(x) = -\sqrt{x-2}$, $\text{dom } [2, \infty)$.



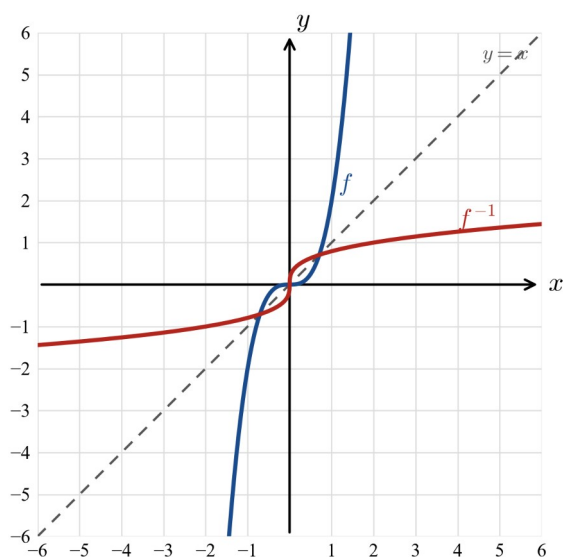
Q11. $\text{ran } f = [0, \infty)$. Swap $y = \sqrt{x-4}$: $x = \sqrt{y-4} \Rightarrow x^2 = y-4 \Rightarrow y = x^2+4$. So $f^{-1}(x) = x^2+4$, $\text{dom } [0, \infty)$.



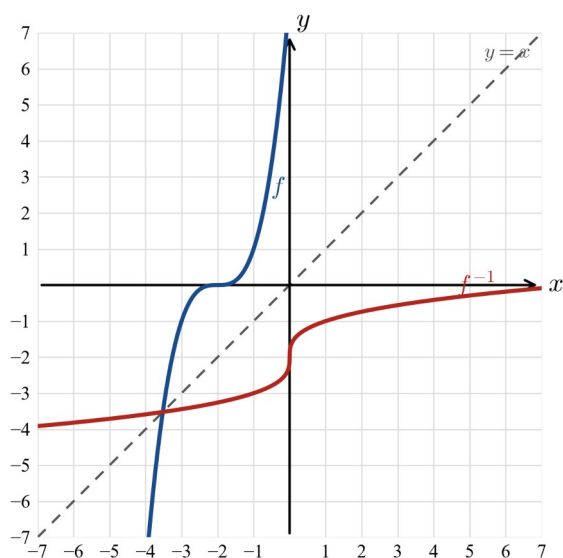
Q12. $\text{ran } f = [0, \infty)$. Swap $y = \sqrt{2-x}$: $x = \sqrt{2-y} \Rightarrow x^2 = 2-y \Rightarrow y = 2-x^2$. So $f^{-1}(x) = 2-x^2$, $\text{dom } [0, \infty)$.



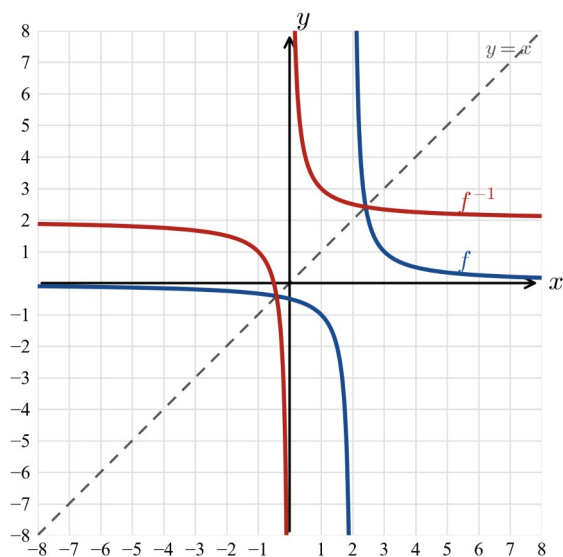
Q13. Swap $y = 2x^3$: $x = 2y^3 \Rightarrow y^3 = \frac{x}{2} \Rightarrow y = \sqrt[3]{\frac{x}{2}}$. So $f^{-1}(x) = \sqrt[3]{\frac{x}{2}}$, $\text{dom } \mathbb{R}$.



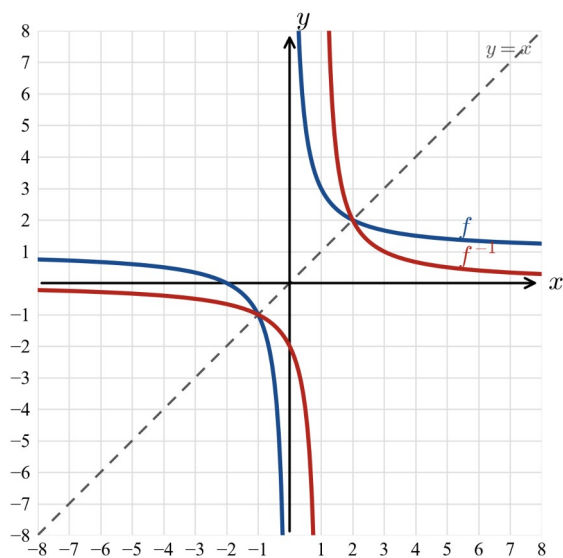
Q14. Swap $y = (x+2)^3$: $x = (y+2)^3 \Rightarrow y+2 = \sqrt[3]{x} \Rightarrow y = \sqrt[3]{x} - 2$. So $f^{-1}(x) = \sqrt[3]{x} - 2$, $\text{dom } \mathbb{R}$.



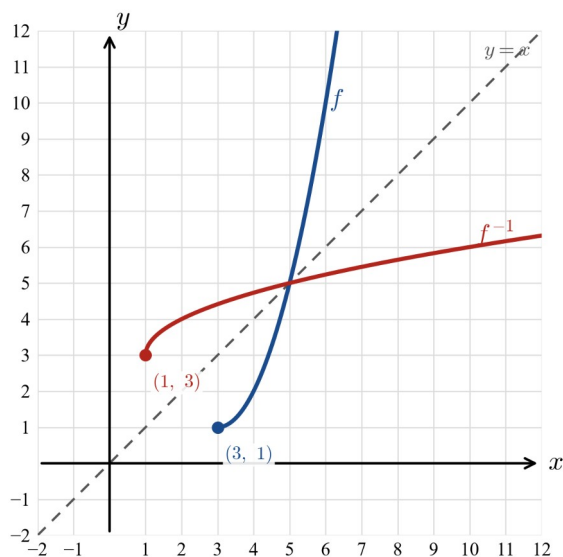
Q15. Swap $y = \frac{1}{x-2}$: $x = \frac{1}{y-2} \Rightarrow y-2 = \frac{1}{x} \Rightarrow y = \frac{1}{x} + 2$. So $f^{-1}(x) = \frac{1}{x} + 2$, $\text{dom } \mathbb{R} \setminus \{0\}$ ($= \text{ran } f$).



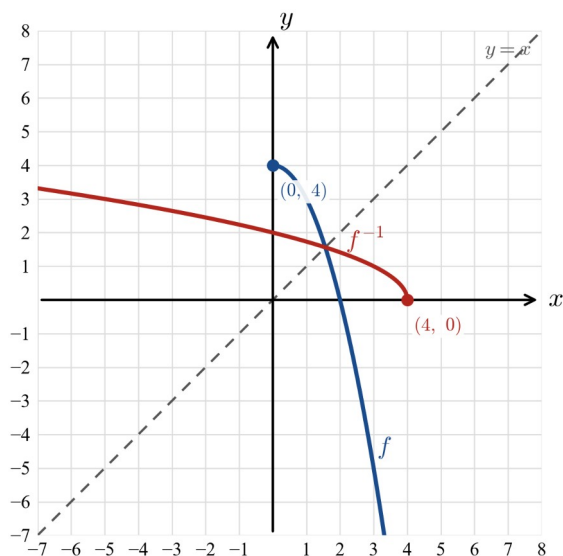
Q16. Swap $y = \frac{2}{x} + 1$: $x = \frac{2}{y} + 1 \Rightarrow x - 1 = \frac{2}{y} \Rightarrow y = \frac{2}{x-1}$. So $f^{-1}(x) = \frac{2}{x-1}$, $\text{dom } \mathbb{R} \setminus \{1\}$.



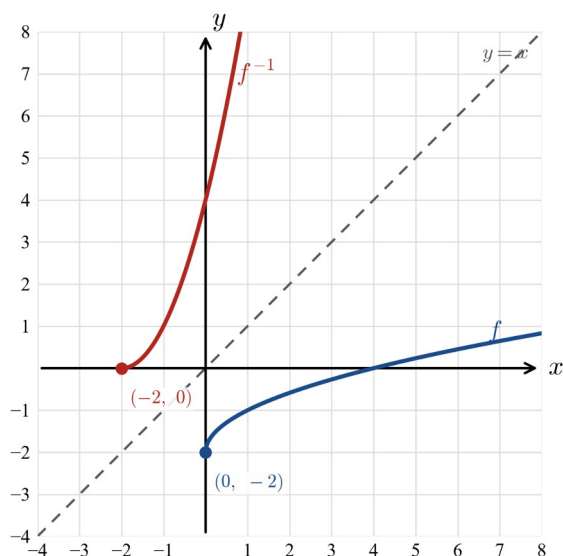
Q17. On $x \geq 3$, $\text{ran } f = [1, \infty)$. Swap $y = (x-3)^2 + 1$: $x = (y-3)^2 + 1 \Rightarrow (y-3)^2 = x-1 \Rightarrow y-3 = \sqrt{x-1}$ (positive root, $y \geq 3$) $\Rightarrow y = \sqrt{x-1} + 3$. So $f^{-1}(x) = \sqrt{x-1} + 3$, $\text{dom } [1, \infty)$.



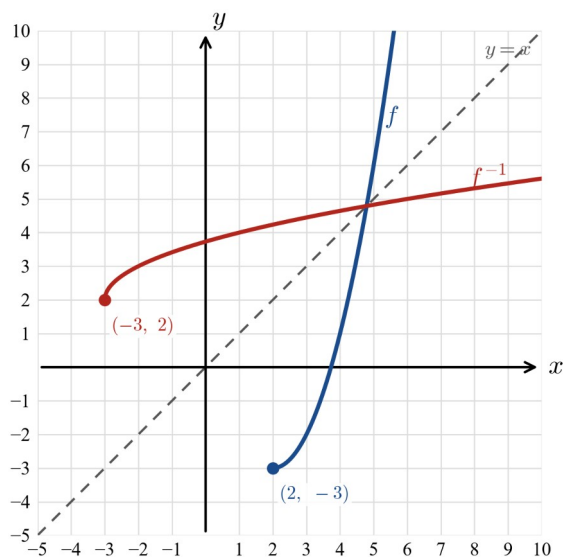
Q18. On $x \geq 0$, f is decreasing; $\text{ran } f = (-\infty, 4]$. Swap $y = 4 - x^2$: $x = 4 - y^2 \Rightarrow y^2 = 4 - x \Rightarrow y = \sqrt{4 - x}$ (positive root, since $\text{ran } f^{-1} = [0, \infty)$). So $f^{-1}(x) = \sqrt{4 - x}$, $\text{dom } (-\infty, 4]$.



Q19. $\text{ran } f = [-2, \infty)$. Swap $y = \sqrt{x} - 2$: $x = \sqrt{y} - 2 \Rightarrow x + 2 = \sqrt{y} \Rightarrow y = (x + 2)^2$. So $f^{-1}(x) = (x + 2)^2$, $\text{dom } [-2, \infty)$.

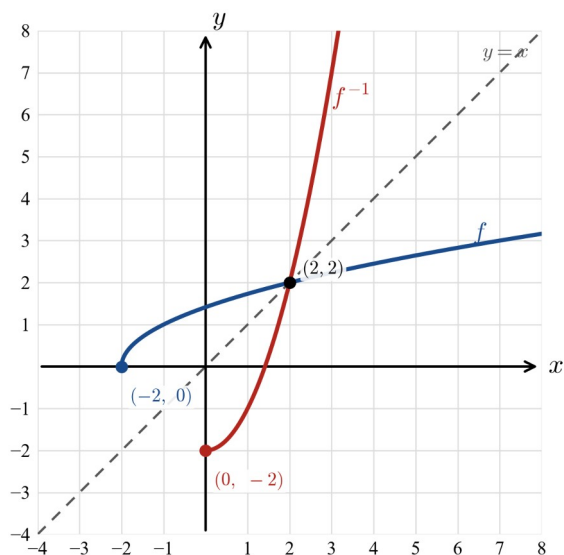


Q20. $g(x) = (x - 2)^2 - 3$ has turning point $(2, -3)$, so it is one-to-one on $[2, \infty)$ — the largest domain of the form $[a, \infty)$ is $a = 2$. Then $\text{ran } g = [-3, \infty)$. Swap $y = (x - 2)^2 - 3$: $x = (y - 2)^2 - 3 \Rightarrow (y - 2)^2 = x + 3 \Rightarrow y - 2 = \sqrt{x + 3}$ (positive root, $y \geq 2$) $\Rightarrow y = \sqrt{x + 3} + 2$. So $g^{-1}(x) = \sqrt{x + 3} + 2$, $\text{dom } [-3, \infty)$.



Q21. $\text{ran } f = [0, \infty)$. Swap $y = \sqrt{x+2}$: $x = \sqrt{y+2} \Rightarrow x^2 = y+2 \Rightarrow y = x^2 - 2$, so $f^{-1}(x) = x^2 - 2$, $\text{dom } [0, \infty)$.

Intersection: solve $f(x) = x$, i.e. $\sqrt{x+2} = x \Rightarrow x+2 = x^2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$. Only $x=2$ lies in the domain (and ≥ 0), giving the point $(2, 2)$.



Chapter 1 Functions, relations and graphs — 1G Power functions

Theory

A **power function** is a function of the form

$$y = x^n,$$

where the index n is a constant. The shape of the graph, and its domain and range, depend on the **type** of the index. In Methods you must know the graphs of x^n for n a positive integer, a negative integer, and the fractional indices $\frac{1}{2}$ and $\frac{1}{3}$.

Positive integer index. For $y = x^n$ with $n = 1, 2, 3, \dots$ the domain is $\mathbb{R}$.

- **Even n ($x^2, x^4, x^6, \dots$):** the function is **even**, since $(-x)^n = x^n$, so the graph is **symmetric about the y-axis**. Range $[0, \infty)$. The graph has a minimum at the origin and becomes flatter near $x = 0$ and steeper beyond $x = \pm 1$ as n increases.
- **Odd n ($x^3, x^5, \dots$):** the function is **odd**, since $(-x)^n = -x^n$, so the graph is **symmetric about the origin**. Range $\mathbb{R}$. For $n \geq 3$ there is a **stationary point of inflection** at the origin.

Every power function passes through $(1, 1)$; every positive-index one also passes through $(0, 0)$; even ones also through $(-1, 1)$ and odd ones through $(-1, -1)$.

Negative integer index. These are the reciprocal power functions, with domain $\mathbb{R} \setminus \{0\}$.

- $y = x^{-1} = \frac{1}{x}$ — the **rectangular hyperbola**: **odd**, asymptotes $x = 0$ and $y = 0$, range $\mathbb{R} \setminus \{0\}$; passes through $(1, 1)$ and $(-1, -1)$.
- $y = x^{-2} = \frac{1}{x^2}$ (sometimes called a *truncus*): **even**, asymptotes $x = 0$ and $y = 0$, range $(0, \infty)$; passes through $(1, 1)$ and $(-1, 1)$.

Fractional index.

- $y = x^{1/2} = \sqrt{x}$: domain $[0, \infty)$, range $[0, \infty)$; increasing from the endpoint $(0, 0)$, through $(1, 1)$ and $(4, 2)$. Its domain is not symmetric, so it is **neither** even nor odd.
- $y = x^{1/3} = \sqrt[3]{x}$: domain $\mathbb{R}$, range $\mathbb{R}$; **odd**, through $(0, 0)$, $(1, 1)$, $(-1, -1)$ and $(8, 2)$.

Summary.

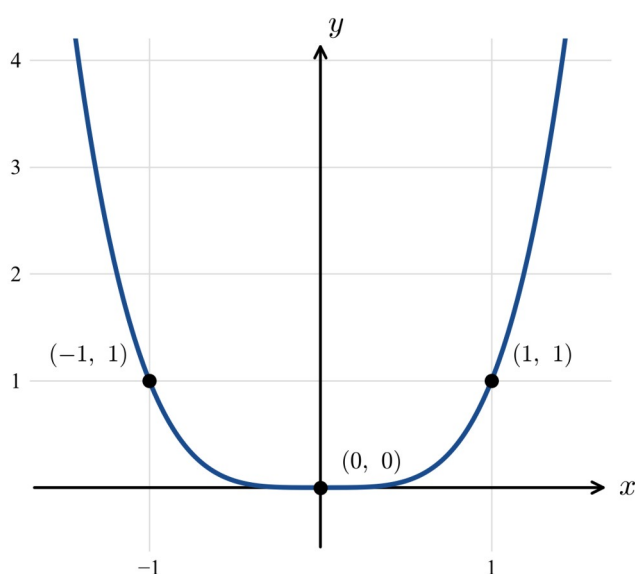
Type	Example	Domain	Range	Even/odd
even positive integer	x^2, x^4	$\mathbb{R}$	$[0, \infty)$	even
odd positive integer	x^3, x^5	$\mathbb{R}$	$\mathbb{R}$	odd
x^{-1}	$\frac{1}{x}$	$\mathbb{R} \setminus \{0\}$	$\mathbb{R} \setminus \{0\}$	odd
x^{-2}	$\frac{1}{x^2}$	$\mathbb{R} \setminus \{0\}$	$(0, \infty)$	even
$x^{1/2}$	$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$	neither
$x^{1/3}$	$\sqrt[3]{x}$	$\mathbb{R}$	$\mathbb{R}$	odd

Worked examples

Example 1 — scaffolded

For $y = x^4$, state the domain, range and axis of symmetry, determine whether the function is even or odd, and sketch the graph, labelling the points where $x = -1, 0, 1$.

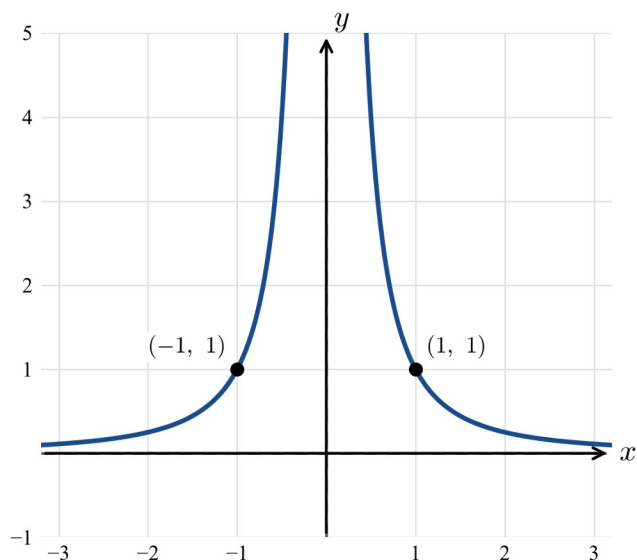
Working	Comment
Domain $\mathbb{R}$ (any real number can be raised to the power 4).	Positive integer index.
$(-x)^4 = x^4$, so $f(-x) = f(x)$: the function is even , symmetric about the y -axis.	Test $f(-x)$ against $f(x)$.
Since $x^4 \geq 0$ for all x , and $x^4 = 0$ only at $x = 0$, the range is $[0, \infty)$.	The minimum value 0 occurs at the origin.
Key points: $(0, 0)$, $(1, 1)$ and $(-1, 1)$.	Substitute $x = 0, 1, -1$.



Example 2 — scaffolded

For $y = \frac{1}{x^2}$, state the domain, range and equations of the asymptotes, determine whether the function is even or odd, and sketch the graph.

Working	Comment
$y = x^{-2} = \frac{1}{x^2}$. Domain $\mathbb{R} \setminus \{0\}$ (we cannot divide by 0).	The rule is undefined at $x = 0$.
$(-x)^{-2} = x^{-2}$, so $f(-x) = f(x)$: the function is even .	Symmetric about the y -axis.
Asymptotes $x = 0$ and $y = 0$. Since $\frac{1}{x^2} > 0$ for all $x \neq 0$, the range is $(0, \infty)$.	Both branches lie above the x -axis.
Key points $(1, 1)$ and $(-1, 1)$.	Substitute $x = \pm 1$.



Example 3 — unscaffolded

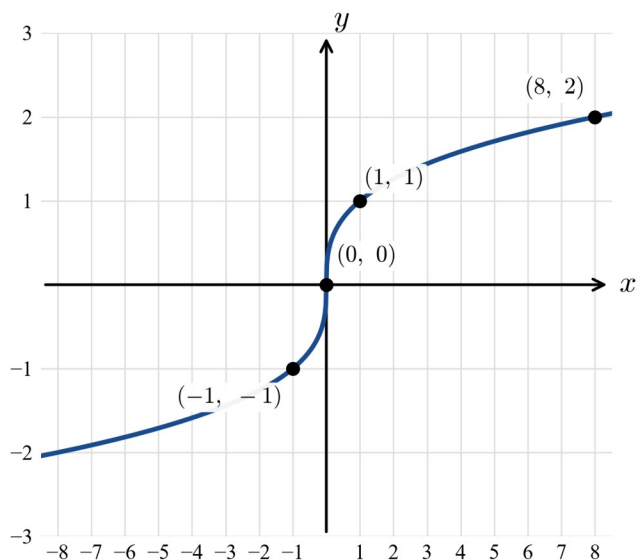
For $y = \sqrt[3]{x}$, state the domain and range, determine whether the function is even or odd, and sketch the graph.

$y = x^{1/3}$ is the cube-root function. Every real number has exactly one real cube root, so the domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

$\sqrt[3]{-x} = -\sqrt[3]{x}$, so $f(-x) = -f(x)$: the function is **odd**, symmetric about the origin.

Key points: $(0, 0)$, $(1, 1)$, $(-1, -1)$ and $(8, 2)$.

$\therefore$ the graph increases through the origin with a vertical tangent there.



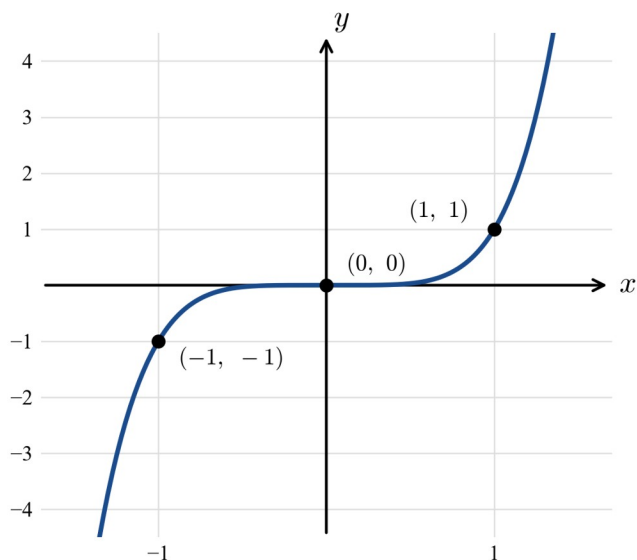
Example 4 — unscaffolded

For $y = x^5$, state the domain and range, describe the symmetry, and sketch the graph, labelling the behaviour at the origin.

Domain $\mathbb{R}$, range $\mathbb{R}$. Since $(-x)^5 = -x^5$, the function is **odd** — symmetric about the origin.

Because the index is $5 \geq 3$, the graph has a **stationary point of inflection** at $(0, 0)$; it is flat near the origin and steep beyond $x = \pm 1$. Key points $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

$\therefore$ the curve rises from the third quadrant, flattens at the origin, then rises into the first.



Practice questions

Scaffolded

- Q1.** For $y = x^2$: (a) state the domain and range; (b) is the function even or odd? Justify your answer; (c) write the coordinates of the points where $x = -1, 0, 1$; (d) sketch the graph.
- Q2.** For $y = x^3$: (a) state the domain and range; (b) describe the symmetry of the graph; (c) state the nature of the point at the origin; (d) sketch the graph, labelling $x = -1, 0, 1$.
- Q3.** For $y = x$ (that is, x^1): (a) state the domain and range; (b) is the function even or odd? (c) sketch the graph.
- Q4.** For $y = x^6$: (a) state the domain and range; (b) explain why the function is even; (c) sketch the graph, labelling $x = -1, 0, 1$.
- Q5.** For $y = \frac{1}{x}$: (a) state the domain, range and asymptotes; (b) is the function even or odd? (c) sketch the graph.
- Q6.** For $y = \sqrt{x}$: (a) state the domain and range; (b) explain why the function is neither even nor odd; (c) write the coordinates where $x = 0, 1, 4$; (d) sketch the graph.

Un scaffolded

- Q7.** Sketch $y = 2x^2$, labelling the points where $x = -1, 0, 1$, and state the range.
- Q8.** Sketch $y = -x^3$, state the domain and range, and describe its symmetry.
- Q9.** Sketch $y = -\frac{1}{x}$, stating the asymptotes and the points where $x = \pm 1$.
- Q10.** Sketch $y = x^4$ and state the domain and range.
- Q11.** On the one set of axes sketch $y = x^2$ and $y = x^4$. State, with reasons, which graph is higher for $0 < x < 1$ and which is higher for $x > 1$.
- Q12.** On the one set of axes sketch $y = x^3$ and $y = x^5$, and describe how they compare near the origin and for $x > 1$.
- Q13.** Sketch $y = \sqrt[3]{x}$, labelling the points where $x = -8, -1, 0, 1, 8$, and state the domain and range.
- Q14.** Sketch $y = \frac{1}{x^2}$ and state the domain, range and asymptotes.

Q15. The graph of a power function $y = x^n$ passes through the point $(2, 8)$. Find the value of n , and hence state whether the function is even or odd.

Q16. Solve each equation, and explain why the numbers of real solutions differ: (a) $x^4 = 16$; (b) $x^3 = -27$.

Q17. For the power functions $y = x^2$ and $y = x^4$, state which is larger when $0 < x < 1$ and which is larger when $x > 1$. Justify your answers.

Q18. Solve $x^3 = x^5$ for x .

Q19. State the domain, range and whether the function is even, odd or neither, for: (a) $y = x^6$; (b) $y = \frac{1}{x}$; (c) $y = \sqrt{x}$; (d) $y = \sqrt[3]{x}$.

Q20. Solve each equation exactly: (a) $\frac{1}{x} = \frac{1}{4}$; (b) $\frac{1}{x^2} = \frac{1}{9}$. Comment on why part (b) has two solutions but part (a) has one.

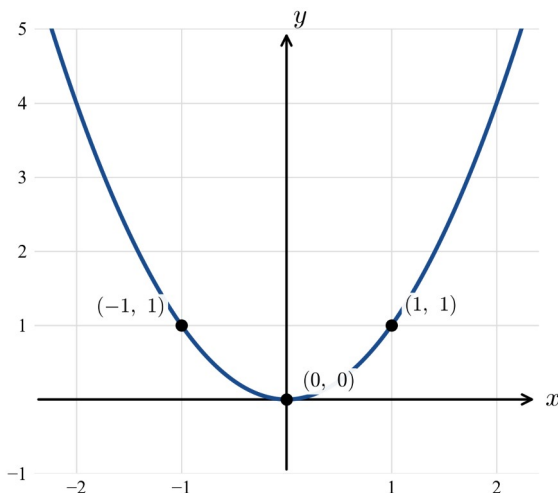
Q21. The graph of a power function $y = x^n$ passes through the point $(4, 2)$. (a) Find the value of n . (b) State the domain and range of the function. (c) Hence find the exact value of x for which $x^n = 3$.

Answers

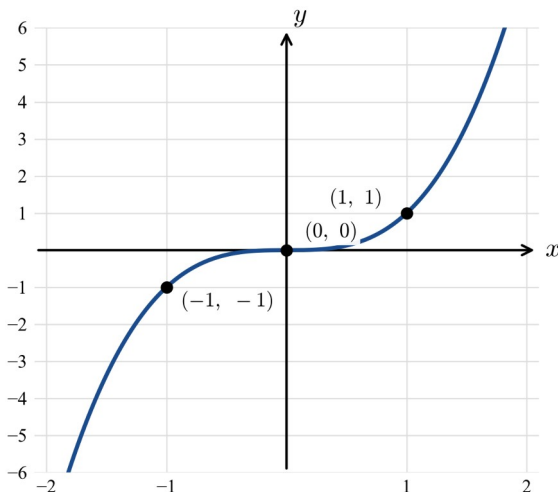
Q1. Domain $\mathbb{R}$, range $[0, \infty)$; even; $(-1, 1), (0, 0), (1, 1)$. **Q2.** Domain $\mathbb{R}$, range $\mathbb{R}$; symmetric about the origin (odd); stationary point of inflection at $(0, 0)$. **Q3.** Domain $\mathbb{R}$, range $\mathbb{R}$; odd. **Q4.** Domain $\mathbb{R}$, range $[0, \infty)$; even because $(-x)^6 = x^6$. **Q5.** Domain $\mathbb{R} \setminus \{0\}$, range $\mathbb{R} \setminus \{0\}$, asymptotes $x=0, y=0$; odd. **Q6.** Domain $[0, \infty)$, range $[0, \infty)$; neither (its domain is not symmetric); $(0, 0), (1, 1), (4, 2)$. **Q7.** Range $[0, \infty)$; points $(-1, 2), (0, 0), (1, 2)$. **Q8.** Domain $\mathbb{R}$, range $\mathbb{R}$; odd (symmetric about the origin), the graph of $y = x^3$ reflected in the x -axis. **Q9.** Asymptotes $x=0, y=0$; through $(1, -1)$ and $(-1, 1)$. **Q10.** Domain $\mathbb{R}$, range $[0, \infty)$. **Q11.** For $0 < x < 1$, $y = x^2$ is higher; for $x > 1$, $y = x^4$ is higher (they meet at $(0, 0)$ and $(1, 1)$). **Q12.** Both are odd and pass through $(0, 0), (1, 1), (-1, -1)$; for $x > 1$, $y = x^5$ is higher; near the origin $y = x^5$ is flatter. **Q13.** Domain $\mathbb{R}$, range $\mathbb{R}$; $(-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2)$. **Q14.** Domain $\mathbb{R} \setminus \{0\}$, range $(0, \infty)$, asymptotes $x=0, y=0$. **Q15.** $n=3$; odd. **Q16.** (a) $x = \pm 2$; (b) $x = -3$. **Q17.** $0 < x < 1$: $x^2 > x^4$. $x > 1$: $x^4 > x^2$. **Q18.** $x = -1, 0, 1$. **Q19.** (a) $\mathbb{R}, [0, \infty)$, even; (b) $\mathbb{R} \setminus \{0\}, \mathbb{R} \setminus \{0\}$, odd; (c) $[0, \infty), [0, \infty)$, neither; (d) $\mathbb{R}, \mathbb{R}$, odd. **Q20.** (a) $x = 4$; (b) $x = \pm 3$. **Q21.** (a) $n = \frac{1}{2}$; (b) domain $[0, \infty)$, range $[0, \infty)$; (c) $x = 9$.

Fully-worked solutions

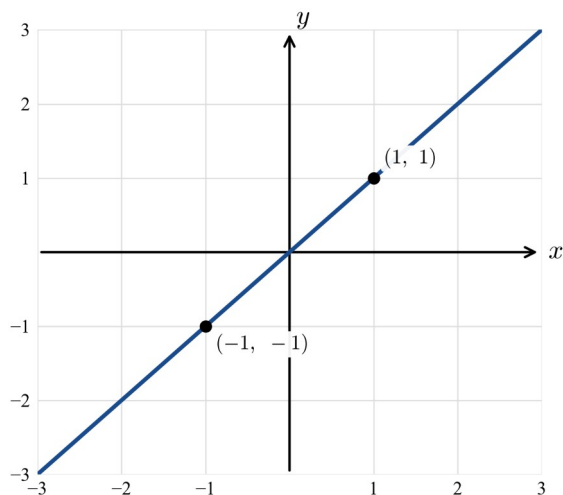
Q1. $y = x^2$. Domain $\mathbb{R}$; since $x^2 \geq 0$, range $[0, \infty)$. $(-x)^2 = x^2$, so it is **even** — symmetric about the y -axis. Points $(-1, 1), (0, 0), (1, 1)$.



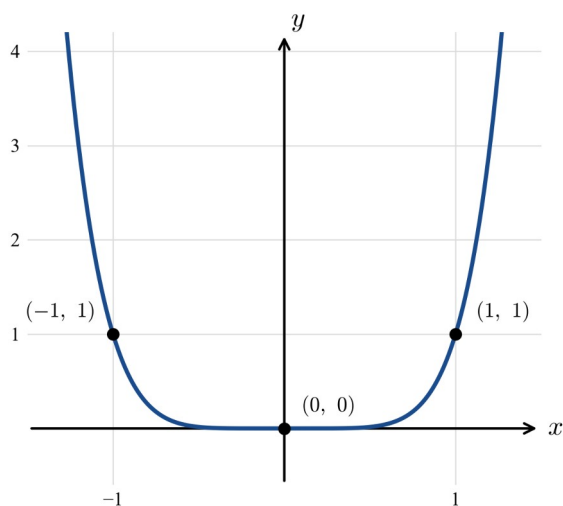
Q2. $y = x^3$. Domain $\mathbb{R}$, range $\mathbb{R}$. $(-x)^3 = -x^3$, so it is **odd** — symmetric about the origin. As the index is 3, there is a stationary point of inflection at $(0, 0)$.



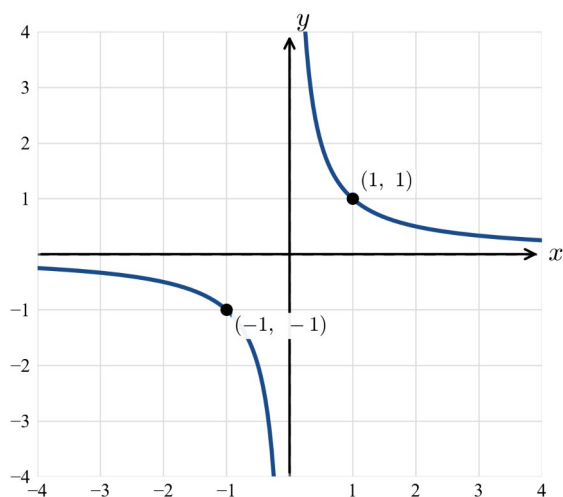
Q3. $y = x$. Domain $\mathbb{R}$, range $\mathbb{R}$. $(-x) = -x$, so it is **odd**. The graph is the straight line through the origin with gradient 1.



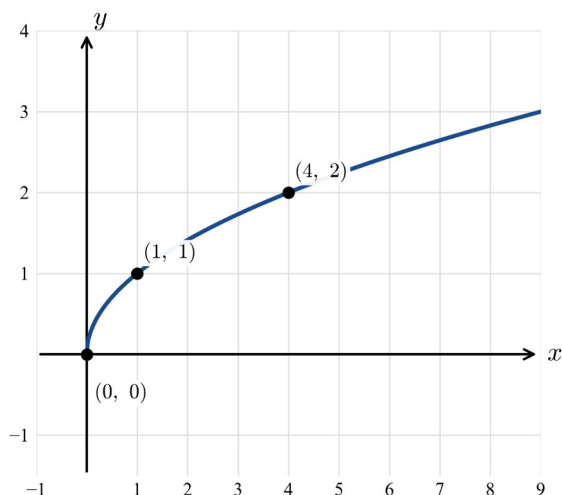
Q4. $y = x^6$. Domain $\mathbb{R}$; $x^6 \geq 0$, so range $[0, \infty)$. $(-x)^6 = x^6$, so it is **even**. Points $(-1, 1), (0, 0), (1, 1)$; the graph is flatter than $y = x^2$ near the origin and steeper beyond $x = \pm 1$.



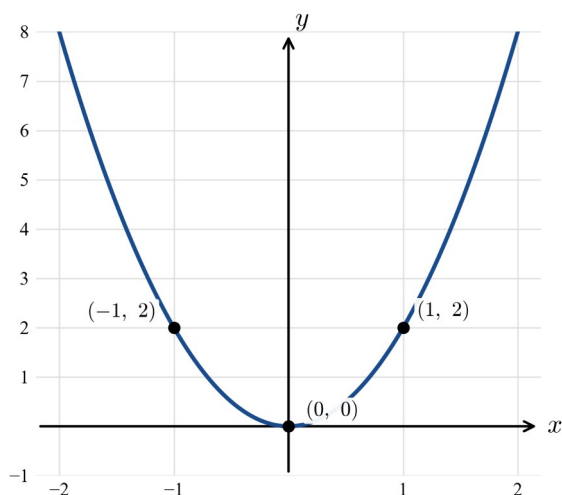
Q5. $y = \frac{1}{x}$. Domain $\mathbb{R} \setminus \{0\}$; the value 0 is never attained, so range $\mathbb{R} \setminus \{0\}$. Asymptotes $x = 0$ and $y = 0$. $\frac{1}{-x} = -\frac{1}{x}$, so it is **odd**. Through $(1, 1)$ and $(-1, -1)$.



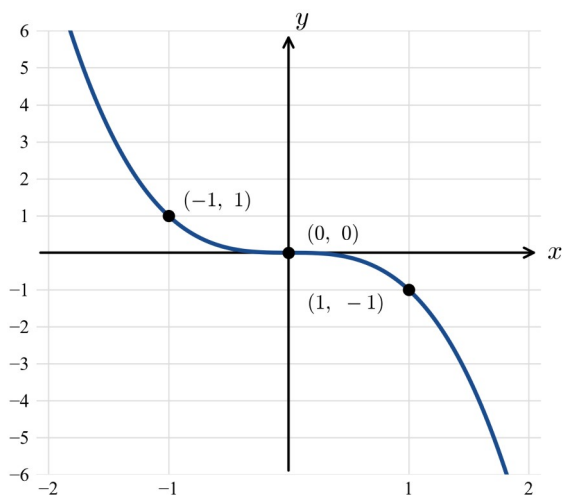
Q6. $y = \sqrt{x}$. We need $x \geq 0$, so domain $[0, \infty)$; the outputs are ≥ 0 , so range $[0, \infty)$. It is **neither** even nor odd because its domain is not symmetric about 0 (e.g. $\sqrt{-1}$ is undefined). Points $(0, 0), (1, 1), (4, 2)$.



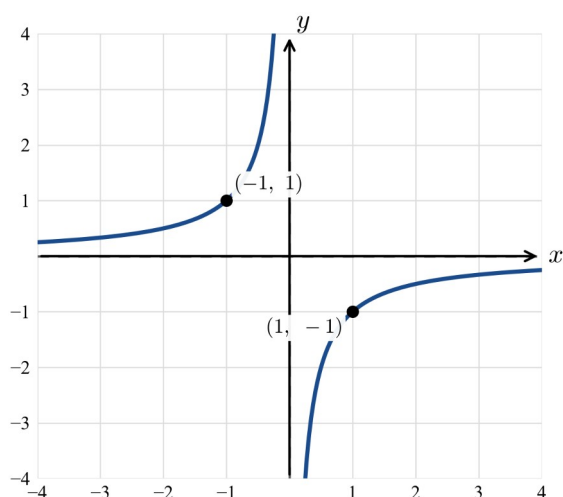
Q7. $y = 2x^2$. This is $y = x^2$ dilated by factor 2 from the x -axis: domain $\mathbb{R}$, range $[0, \infty)$, still even. Points $(-1, 2), (0, 0), (1, 2)$.



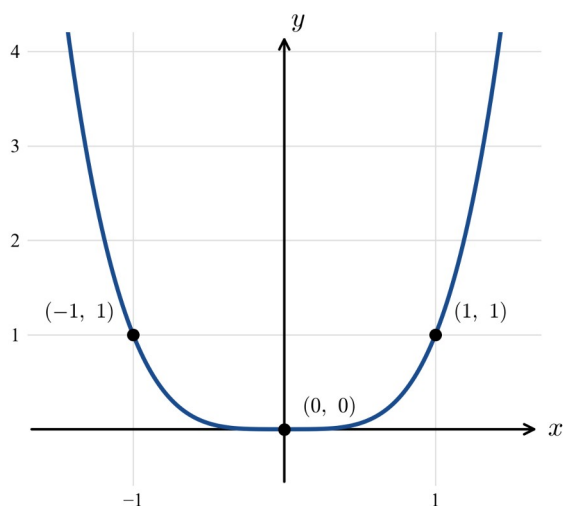
Q8. $y = -x^3$. Domain $\mathbb{R}$, range $\mathbb{R}$. $-(-x)^3 = -(-x^3) = x^3 = -(-x^3) \dots$ more simply $f(-x) = -(-x)^3 = x^3 = -f(x)$, so it is **odd**. It is $y = x^3$ reflected in the x -axis, so it falls from the second quadrant to the fourth.



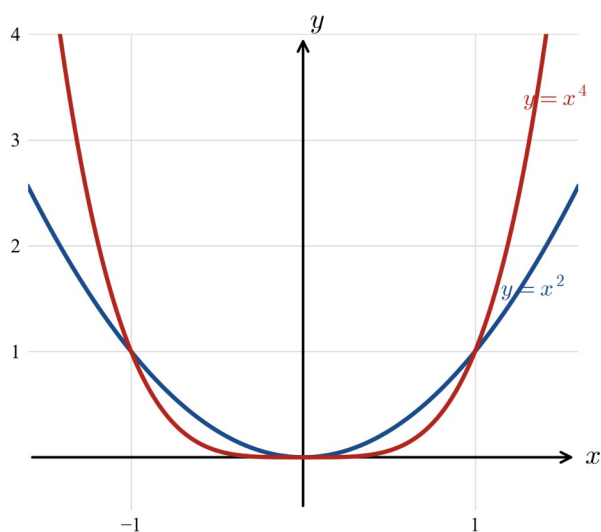
Q9. $y = -\frac{1}{x}$. Asymptotes $x=0$ and $y=0$; domain and range $\mathbb{R} \setminus \{0\}$. It is $y = \frac{1}{x}$ reflected in the x -axis, so the branches lie in the second and fourth quadrants, through $(-1, 1)$ and $(1, -1)$.



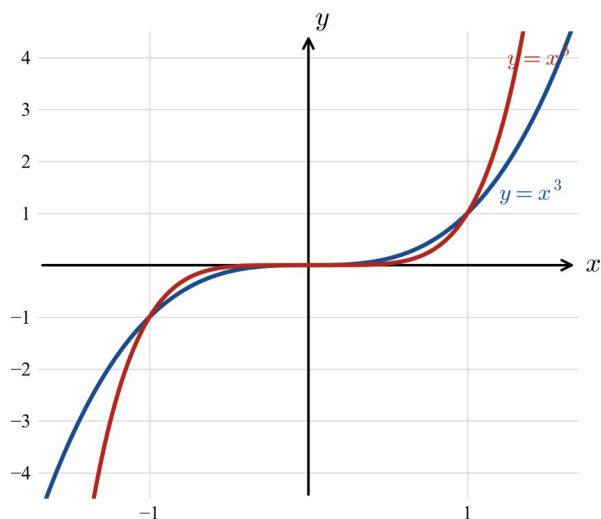
Q10. $y = x^4$. Domain $\mathbb{R}$, range $[0, \infty)$ (even; minimum 0 at the origin).



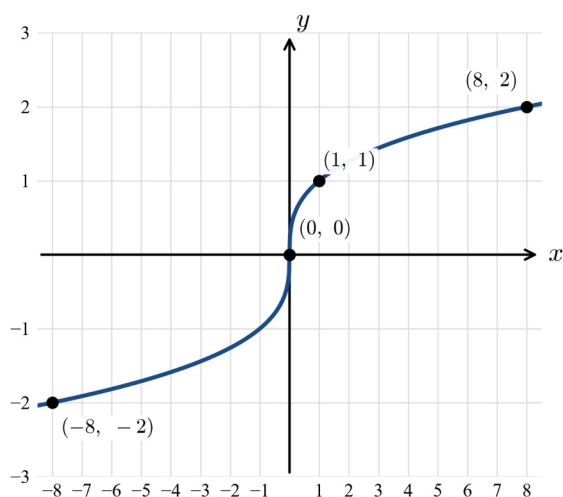
Q11. Both graphs pass through $(0, 0)$ and $(1, 1)$. For $0 < x < 1$, raising to a higher power makes the value **smaller**, so $x^2 > x^4$ (e.g. at $x = \frac{1}{2}$, $x^2 = \frac{1}{4} > \frac{1}{16} = x^4$). For $x > 1$, a higher power makes the value **larger**, so $x^4 > x^2$ (e.g. at $x = 2$, $16 > 4$).



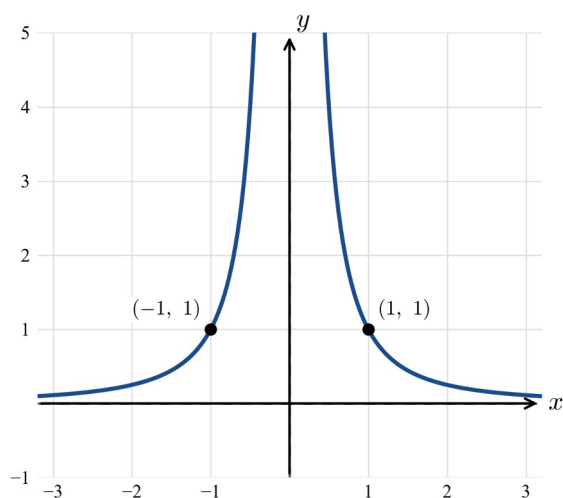
Q12. Both are odd and pass through $(0, 0), (1, 1), (-1, -1)$. Near the origin $y = x^5$ is flatter than $y = x^3$; for $x > 1$, $y = x^5$ is higher (e.g. at $x = 2$, $32 > 8$). Both have a stationary point of inflection at the origin.



Q13. $y = \sqrt[3]{x}$. Domain $\mathbb{R}$, range $\mathbb{R}$; odd. Points $(-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2)$.



Q14. $y = \frac{1}{x^2}$. Domain $\mathbb{R} \setminus \{0\}$, range $(0, \infty)$, asymptotes $x = 0$ and $y = 0$; even, so both branches are above the x -axis.



Q15. x^n through $(2, 8)$: $2^n = 8 = 2^3$, so $n = 3$. As 3 is odd, the function $y = x^3$ is **odd**.

Q16. (a) $x^4 = 16 \Rightarrow x = \pm \sqrt[4]{16} = \pm 2$. (b) $x^3 = -27 \Rightarrow x = \sqrt[3]{-27} = -3$. An even power hides the sign, so both $x = 2$ and $x = -2$ give 16; an odd power keeps the sign, so only $x = -3$ gives -27 .

Q17. For $0 < x < 1$: $x^2 > x^4$ (higher powers of a proper fraction are smaller). For $x > 1$: $x^4 > x^2$ (higher powers of a number greater than 1 are larger). The graphs cross at $(0, 0)$ and $(1, 1)$.

Q18. $x^3 = x^5 \Rightarrow x^5 - x^3 = 0 \Rightarrow x^3(x^2 - 1) = 0 \Rightarrow x^3(x - 1)(x + 1) = 0$, so $x = -1, 0, 1$.

Q19. (a) $y = x^6$: domain $\mathbb{R}$, range $[0, \infty)$, **even**. (b) $y = \frac{1}{x}$: domain $\mathbb{R} \setminus \{0\}$, range $\mathbb{R} \setminus \{0\}$, **odd**. (c) $y = \sqrt{x}$: domain $[0, \infty)$, range $[0, \infty)$, **neither**. (d) $y = \sqrt[3]{x}$: domain $\mathbb{R}$, range $\mathbb{R}$, **odd**.

Q20. (a) $\frac{1}{x} = \frac{1}{4} \Rightarrow x = 4$. (b) $\frac{1}{x^2} = \frac{1}{9} \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. Part (b) involves x^2 , which is even, so both $x = 3$ and $x = -3$ work; part (a) involves x^1 , which is odd (one-to-one), so there is a single solution.

Q21. (a) x^n through $(4, 2)$: $4^n = 2$, i.e. $2^{2n} = 2^1$, so $2n = 1$ and $n = \frac{1}{2}$. (b) $y = x^{1/2} = \sqrt{x}$: domain $[0, \infty)$, range $[0, \infty)$. (c) $\sqrt{x} = 3 \Rightarrow x = 9$.

Theory

A **mathematical model** uses a function to describe a real situation. The variable and the function value each have a real-world meaning and units, so every answer must be **interpreted in context** and checked for sense (a length or a time cannot be negative).

Building and using a model.

1. **Define the variables** (with units) and form the **rule**.
2. State the **domain that makes sense in the context** — often the natural domain further restricted by the situation (e.g. time only until a ball lands; a side length that keeps every dimension positive).
3. Answer the question using function notation, then **interpret** the result in context.

Key features and what they mean.

- **Axis intercepts:** the y -intercept is usually the *initial* value ($t=0$); an x -intercept is *when the quantity is zero* (a ball landing, a break-even point).
- **Turning point:** for a quadratic model the turning point is the **maximum or minimum** in the context (greatest height, largest area, maximum profit). Its coordinate $x = -\frac{b}{2a}$ is found by the axis-of-symmetry formula or by completing the square — **no calculus is needed**.
- **Endpoints:** the ends of the context domain; always check whether they are included.

Common model types.

- **Quadratic** — projectile height, area for a fixed perimeter, revenue and profit.
- **Hybrid (piecewise)** — different rules on different intervals (tiered pricing, a journey in stages, a tank that fills then drains). Evaluate with the **correct branch** for the given input; a value may be reached on **more than one branch**, and a candidate solution must lie in the branch's interval to be valid.
- **Power models** — inverse proportion $y = \frac{k}{x}$ (a hyperbola: as x increases, y decreases) and square-root models $y = k\sqrt{x}$.

Always reject any solution that falls outside the context domain.

Worked examples

Example 1 — scaffolded

A ball is thrown from a balcony. Its height above the ground, h metres, after t seconds is $h(t) = -5t^2 + 20t + 25$, until it lands.

Working

Comment

Initial height: $h(0) = -5(0)^2 + 20(0) + 25 = 25$, so the ball starts at 25 m.

Substitute $t=0$: the y -intercept is the starting height.

Greatest height: axis $t = -\frac{20}{2 \times (-5)} = 2$;

The maximum of a quadratic is at its turning point

$h(2) = -5(4) + 40 + 25 = 45$.

$$t = -\frac{b}{2a}.$$

So the ball reaches a **maximum height of 45 m after 2 seconds**.

Interpret both coordinates in context.

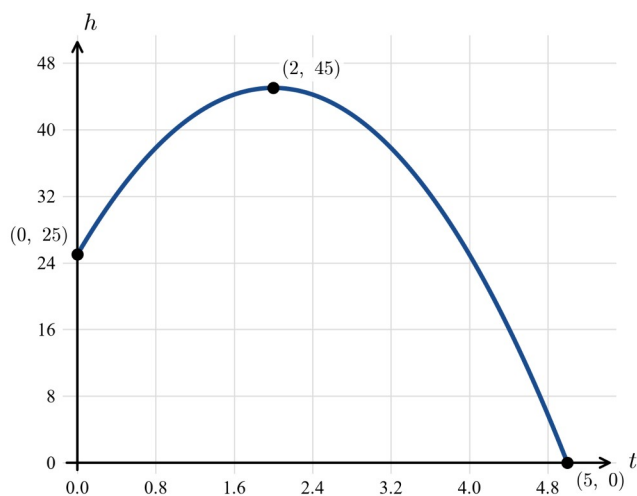
Landing:

Set $h=0$; a negative time is outside the context.

$h(t) = 0 \Rightarrow -5t^2 + 20t + 25 = 0 \Rightarrow t^2 - 4t - 5 = 0 \Rightarrow (t - 5)$
, so $t = 5$ (reject $t = -1$).

Domain $[0, 5]$, **range** $[0, 45]$. The ball lands after 5 seconds.

Time runs from launch to landing; height runs from 0 to the maximum.



Example 2 — scaffolded

A tank fills and then drains. Its volume V litres after t minutes is

$$V(t) = \begin{cases} 10t, & 0 \leq t \leq 3 \\ 30 - 6(t - 3), & 3 < t \leq 8. \end{cases}$$

At the change-over $t = 3$: filling gives
 $V(3) = 10(3) = 30$; draining gives $30 - 6(0) = 30$.

The two branches agree at $t = 3$, so the model is continuous.

Maximum volume is 30 litres, reached at $t = 3$ minutes.

The peak of the “tent” shape is the junction.

When is $V = 20$? Filling: $10t = 20 \Rightarrow t = 2$ (in $[0, 3]$ ✓). Draining:

Solve on **each** branch and keep only solutions inside that branch’s interval.

$$30 - 6(t - 3) = 20 \Rightarrow 6(t - 3) = 10 \Rightarrow t = \frac{14}{3} \text{ (in } (3, 8])$$

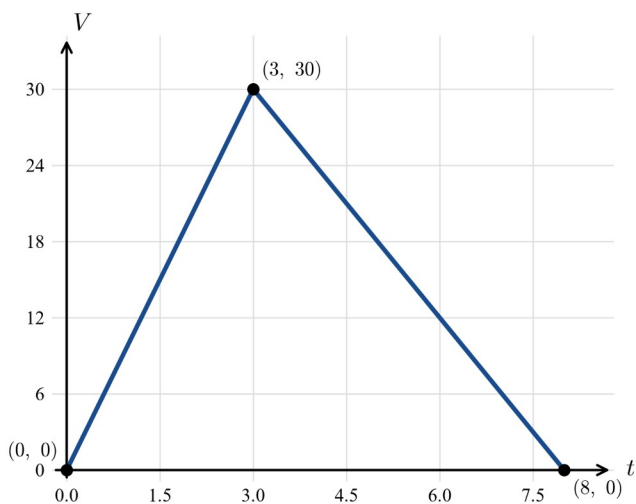
✓).

A value can occur on more than one branch.

So $V = 20$ litres at $t = 2$ min (filling) and $t = \frac{14}{3}$ min (draining).

Empty: $30 - 6(t - 3) = 0 \Rightarrow t = 8$. Domain $[0, 8]$, range $[0, 30]$.

The tank is empty again after 8 minutes.



Example 3 — unscaffolded

A rectangular enclosure is built against a straight wall, so only **three** sides need fencing. There are 60 m of fencing. If the two sides perpendicular to the wall each have length x m, find the dimensions giving the greatest area.

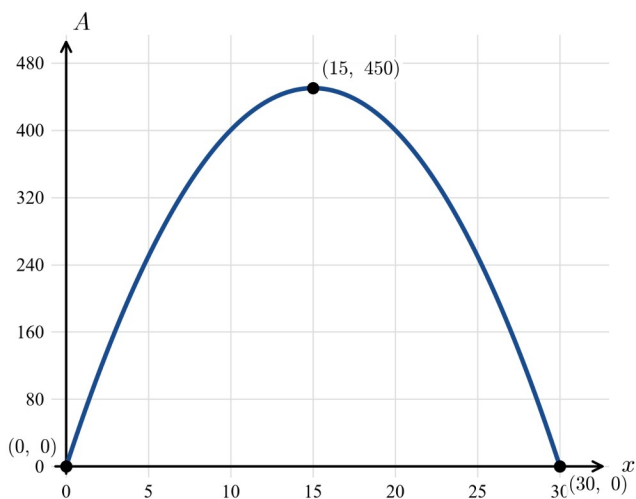
The side parallel to the wall is $60 - 2x$, so the area is

$$A(x) = x(60 - 2x) = 60x - 2x^2.$$

For all lengths to be positive we need $x > 0$ and $60 - 2x > 0$, i.e. the domain is $0 < x < 30$.

The maximum is at the turning point: $x = -\frac{60}{2 \times (-2)} = 15$, giving $A(15) = 15(60 - 30) = 450$.

$\therefore$ the greatest area is 450 m², when the enclosure is 15 m by 30 m.



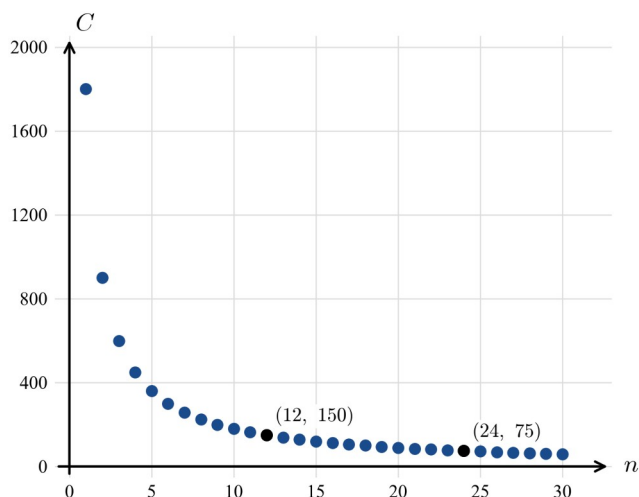
Example 4 — unscaffolded

A tour costing \$1800 is shared equally among n people. The cost per person is $C(n) = \frac{1800}{n}$, where n is a positive integer.

$C(12) = \frac{1800}{12} = 150$, so with 12 people each pays \$150.

For the cost per person to be \$75: $\frac{1800}{n} = 75 \Rightarrow n = \frac{1800}{75} = 24$.

∴ the cost per person is inversely proportional to n ; as more people join, each pays less, and 24 people are needed to bring the cost down to \$75 each.



Practice questions

Scaffolded

Q1. A stone is thrown upward from ground level; its height is $h(t) = -5t^2 + 30t$ metres after t seconds. (a) State the initial height. (b) Find when the stone lands, and hence state the domain. (c) Find the maximum height and when it occurs. (d) Find the time(s) when the height is 25 m.

Q2. A taxi fare (dollars) for a trip of d km is $F(d) = \begin{cases} 4 + 2d, & 0 \leq d \leq 10 \\ 24 + 3(d - 10), & 10 < d \leq 20. \end{cases}$ (a) Find the fare for a 5 km trip. (b) Find the fare for a 10 km trip. (c) Find the distance for which the fare is \$30. (d) Sketch the graph of F .

Q3. A rectangle has perimeter 40 m and one side of length x m. (a) Show that the area is $A(x) = 20x - x^2$. (b) State the domain in this context. (c) Find the maximum area and the value of x that gives it. (d) Find the value(s) of x for which the area is 75 m^2 .

Q4. For a 300 km journey at an average speed of v km/h, the time taken is $T(v) = \frac{300}{v}$ hours. (a) Find the time at 60 km/h. (b) Find the time at 75 km/h. (c) Find the speed needed for the trip to take 3 hours. (d) Sketch the graph of T for $40 \leq v \leq 120$.

Q5. An open box is made from a 10 cm by 6 cm card by cutting squares of side x cm from the corners and folding up the sides. (a) Show that the volume is $V(x) = 4x^3 - 32x^2 + 60x$. (b) State the domain in this context. (c) Find the volume when $x = 1$. (d) Sketch the graph of V .

Q6. A drone rises, hovers, then descends: $H(t) = \begin{cases} 15t, & 0 \leq t \leq 4 \\ 60, & 4 < t \leq 10 \\ 60 - 12(t - 10), & 10 < t \leq 15, \end{cases}$ where H is the height in metres

after t seconds. (a) State the maximum height. (b) State the time interval during which the drone hovers. (c) Find when the drone lands. (d) Sketch the graph of H .

Un scaffolded

Q7. A ball's height is $h(t) = -5t^2 + 20t$ metres after t seconds. Find the maximum height, when the ball lands, and the time(s) at which the height is 15 m. Sketch the graph.

Q8. The revenue from selling x items is $R(x) = 50x - x^2$ dollars, $0 \leq x \leq 50$. Find the number of items giving the maximum revenue and that maximum, and the value(s) of x for which $R = 600$.

Q9. A rectangular yard is fenced on three sides using 80 m of fencing, with the two equal sides of length x m. Find the rule for the area $A(x)$, its domain, and the maximum area.

Q10. A job requires 48 worker-days. With w workers it takes $D(w) = \frac{48}{w}$ days. Find $D(6)$, and the number of workers needed to finish in 4 days. Sketch the graph.

Q11. The speed of a falling object after dropping through h m is $v(h) = \sqrt{20h}$ m/s. Find $v(9)$, and the height fallen when the speed is 20 m/s. Sketch the graph.

Q12. An electricity charge (dollars) for x kWh is $C(x) = \begin{cases} 0.3x, & 0 \leq x \leq 100 \\ 30 + 0.5(x - 100), & x > 100. \end{cases}$ Find the charge for 80 kWh and for 200 kWh, and sketch the graph for $0 \leq x \leq 300$.

Q13. A parabolic bridge arch has height $h(x) = -\frac{1}{5}x^2 + 4x$ metres at a horizontal distance x m from one base, $0 \leq x \leq 20$. Find the span, the maximum height, and the height at $x = 5$. Sketch the graph.

Q14. An open box is made from a 12 cm by 8 cm card by cutting squares of side x cm from the corners. Show that $V(x) = 4x^3 - 40x^2 + 96x$, state the domain, and find the volume when $x = 2$.

Q15. The temperature is $T(t) = 25 - (t - 3)^2$ °C, t hours after midday, for $0 \leq t \leq 8$. Find the maximum temperature and when it occurs, the temperature at midday, and when the temperature is 21 °C. Sketch the graph.

Q16. For a fixed mass of gas, the pressure is $P(V) = \frac{600}{V}$ (in suitable units) at volume V . Find $P(4)$, and the volume when the pressure is 100. Sketch the graph for $2 \leq V \leq 12$.

Q17. The daily profit (dollars) from making x items is $P(x) = -2x^2 + 40x - 150$. Find the break-even values of x (where $P = 0$), and the number of items giving the maximum profit and that maximum. Sketch the graph.

Q18. The speed of a wave is $v(d) = \sqrt{10d}$ m/s in water of depth d m. Find $v(10)$, and the depth at which the speed is 20 m/s. Sketch the graph.

Q19. A car's distance from home is $d(t) = \begin{cases} 60t, & 0 \leq t \leq 2 \\ 120, & 2 < t \leq 3 \\ 120 + 80(t - 3), & 3 < t \leq 5, \end{cases}$ where d is in km and t in hours. Describe the motion in each stage, and find $d(4)$ and $d(5)$. Sketch the graph.

Q20. A rectangular paddock is fenced on three sides (the fourth is a river) using 100 m of fencing. The two sides perpendicular to the river each have length x m. **Show that** the area is $A(x) = 100x - 2x^2$, state the domain, and **hence** find the maximum area and the value of x that gives it.

Q21. The depth of water in a channel, f metres, at a distance x metres across is $f(x) = \begin{cases} -(x - 2)^2 + 4, & 0 \leq x \leq 2 \\ 4, & 2 < x \leq 5 \\ -2x + 14, & 5 < x \leq 7. \end{cases}$

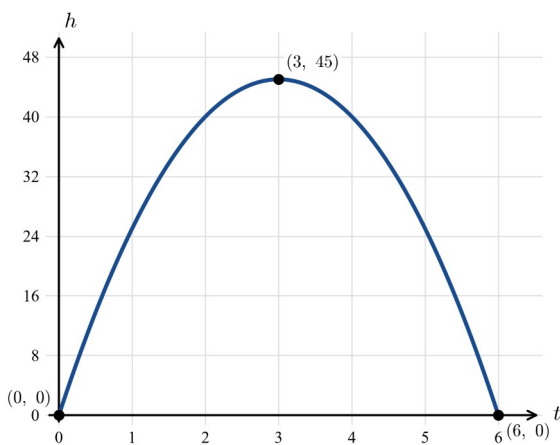
Sketch the graph, state the range, and find the interval on which the depth is 4 m.

Answers

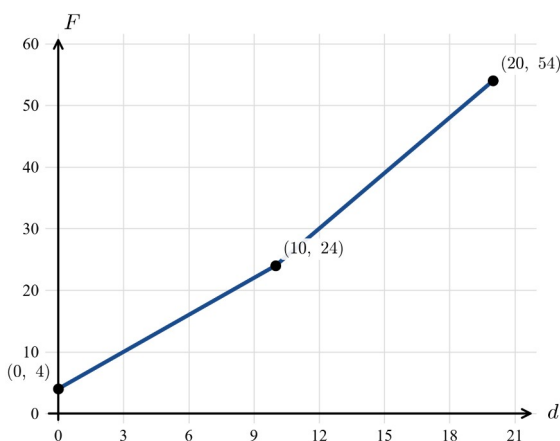
Q1. (a) 0 m; (b) lands at $t=6$ s, domain $[0, 6]$; (c) max 45 m at $t=3$ s; (d) $t=1$ s and $t=5$ s. **Q2.** (a) \$14; (b) \$24; (c) 12 km; (d) see solution. **Q3.** (a) shown; (b) $0 < x < 20$; (c) max 100 m² at $x=10$; (d) $x=5$ or $x=15$. **Q4.** (a) 5 h; (b) 4 h; (c) 100 km/h; (d) see solution. **Q5.** (a) shown; (b) $0 < x < 3$; (c) 32 cm³; (d) see solution. **Q6.** (a) 60 m; (b) $4 < t \leq 10$ s; (c) $t=15$ s; (d) see solution. **Q7.** Max 20 m at $t=2$ s; lands at $t=4$ s; $h=15$ m at $t=1$ s and $t=3$ s. **Q8.** $x=25$ items, max revenue \$625; $R=600$ at $x=20$ and $x=30$. **Q9.** $A(x)=80x-2x^2$, domain $0 < x < 40$; max area 800 m² at $x=20$. **Q10.** $D(6)=8$ days; 12 workers. **Q11.** $v(9)=6\sqrt{5}$ m/s; height 20 m. **Q12.** $C(80)=\$24$; $C(200)=\$80$. **Q13.** Span 20 m; max height 20 m at $x=10$; $h(5)=15$ m. **Q14.** Domain $0 < x < 4$; $V(2)=64$ cm³. **Q15.** Max 25 °C at $t=3$; $T(0)=16$ °C; $T=21$ °C at $t=1$ and $t=5$. **Q16.** $P(4)=150$; volume 6. **Q17.** Break-even $x=5$ and $x=15$; max profit \$50 at $x=10$ items. **Q18.** $v(10)=10$ m/s; depth 40 m. **Q19.** $d(4)=200$ km; $d(5)=280$ km. **Q20.** Domain $0 < x < 50$; max area 1250 m² at $x=25$. **Q21.** Range $[0, 4]$; depth is 4 m for $2 \leq x \leq 5$.

Fully-worked solutions

Q1. $h(t) = -5t^2 + 30t$. (a) $h(0) = 0$ m. (b) $-5t^2 + 30t = 0 \Rightarrow -5t(t-6) = 0 \Rightarrow t=0$ or $t=6$; the stone lands at $t=6$ s, so the domain is $[0, 6]$. (c) Axis $t = -\frac{30}{2(-5)} = 3$; $h(3) = -45 + 90 = 45$, a maximum height of 45 m at $t=3$ s. (d) $-5t^2 + 30t = 25 \Rightarrow t^2 - 6t + 5 = 0 \Rightarrow (t-1)(t-5) = 0$, so $t=1$ s and $t=5$ s.



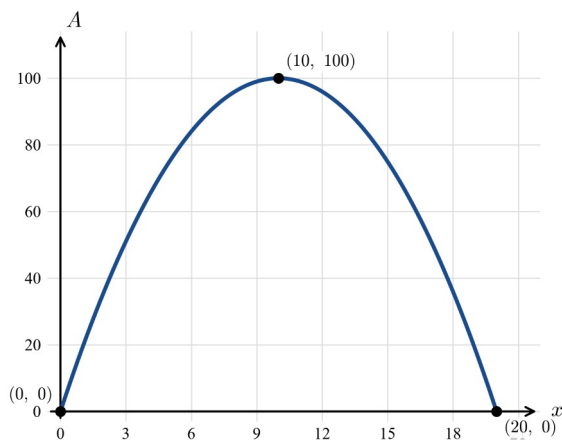
Q2. (a) $5 \leq 10$, so $F(5) = 4 + 2(5) = 14$, i.e. \$14. (b) $F(10) = 4 + 2(10) = 24$, i.e. \$24. (c) On $0 \leq d \leq 10$: $4 + 2d = 30 \Rightarrow d = 13$, which is **not** in $[0, 10]$, so reject. On $10 < d \leq 20$: $24 + 3(d-10) = 30 \Rightarrow 3(d-10) = 6 \Rightarrow d = 12$ ✓. So the fare is \$30 for a 12 km trip. (d) Two line segments meeting at $(10, 24)$.



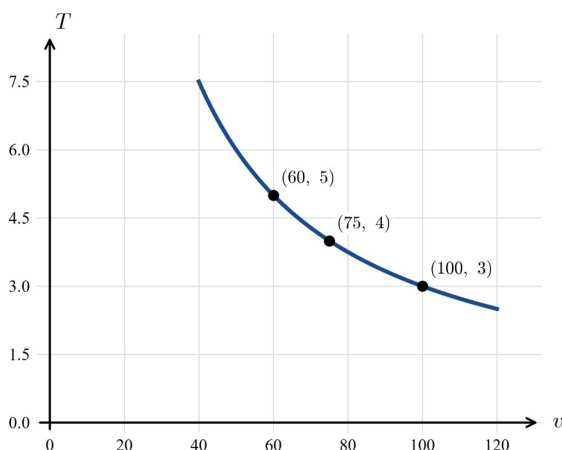
Q3. (a) The other side is $\frac{40-2x}{2} = 20-x$, so $A = x(20-x) = 20x - x^2$. (b) Need $x > 0$ and $20-x > 0$, so $0 < x < 20$.

(c) Axis $x = -\frac{20}{2(-1)} = 10$; $A(10) = 20(10) - 100 = 100$, a maximum area of 100 m^2 (a 10 m by 10 m square). (d)

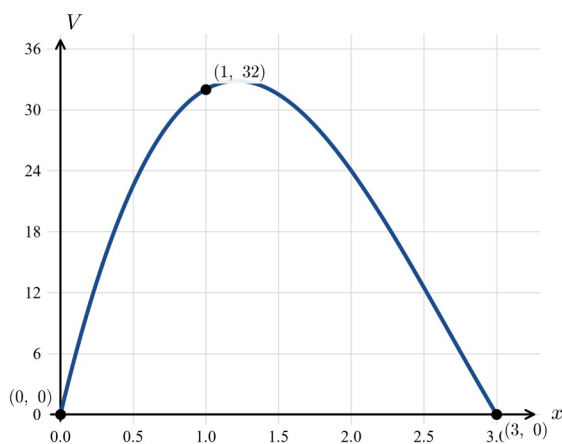
$20x - x^2 = 75 \Rightarrow x^2 - 20x + 75 = 0 \Rightarrow (x-5)(x-15) = 0$, so $x = 5$ or $x = 15$.



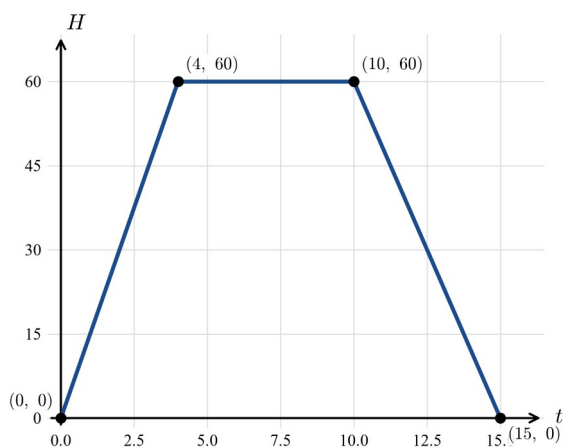
Q4. (a) $T(60) = \frac{300}{60} = 5 \text{ h}$. (b) $T(75) = \frac{300}{75} = 4 \text{ h}$. (c) $\frac{300}{v} = 3 \Rightarrow v = 100 \text{ km/h}$. (d) A decreasing hyperbola (inverse proportion) over $40 \leq v \leq 120$.



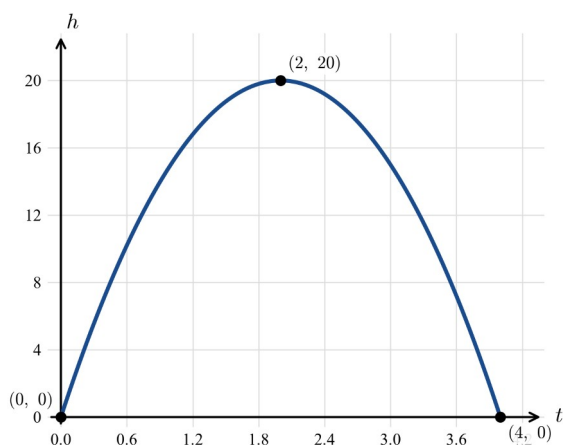
Q5. (a) The base measures $(10-2x)$ by $(6-2x)$ and the height is x , so $V = x(10-2x)(6-2x)$. Expanding: $(10-2x)(6-2x) = 60 - 32x + 4x^2$, so $V = 4x^3 - 32x^2 + 60x$. (b) Need $x > 0$ and $6-2x > 0$, so $0 < x < 3$. (c) $V(1) = 4 - 32 + 60 = 32 \text{ cm}^3$. (d) A cubic on $(0, 3)$, zero at each end.



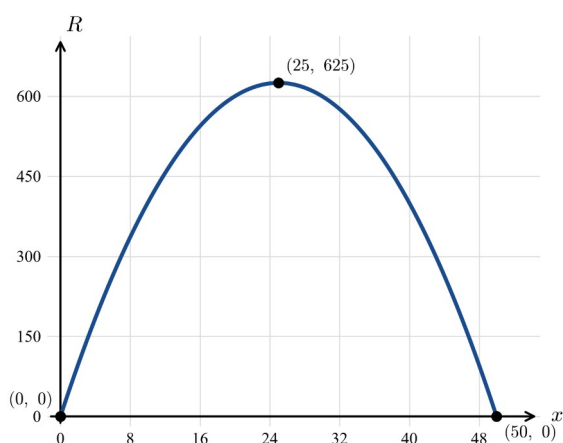
Q6. (a) Maximum height 60 m (the hover height). (b) The drone hovers while $H = 60$, i.e. for $4 < t \leq 10 \text{ s}$. (c) Descent: $60 - 12(t-10) = 0 \Rightarrow t - 10 = 5 \Rightarrow t = 15 \text{ s}$. (d) A rise, a flat hover, then a descent to the ground.



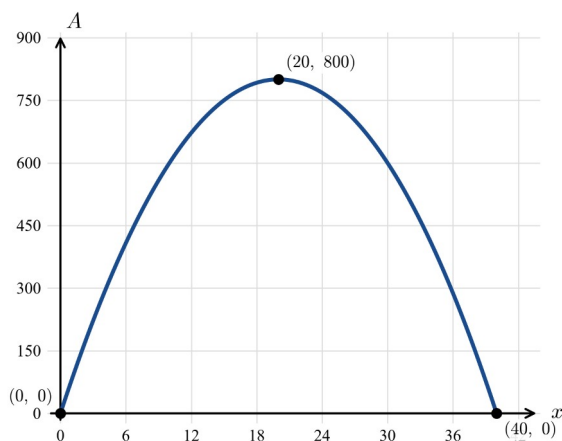
Q7. $h(t) = -5t^2 + 20t$. Axis $t = -\frac{20}{2(-5)} = 2$; $h(2) = -20 + 40 = 20$, a maximum height of 20 m at $t = 2$ s. Landing: $-5t(t - 4) = 0 \Rightarrow t = 4$ s (domain $[0, 4]$). $h = 15$: $-5t^2 + 20t = 15 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow (t - 1)(t - 3) = 0$, so $t = 1$ s and $t = 3$ s.



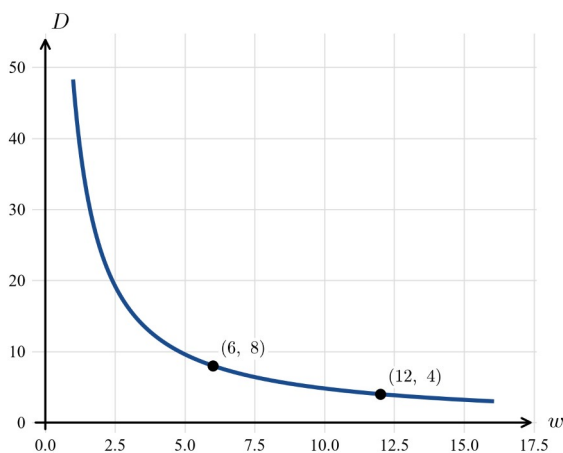
Q8. $R(x) = 50x - x^2$. Axis $x = -\frac{50}{2(-1)} = 25$; $R(25) = 1250 - 625 = 625$, so 25 items give the maximum revenue of \$625. $R = 600$: $50x - x^2 = 600 \Rightarrow x^2 - 50x + 600 = 0 \Rightarrow (x - 20)(x - 30) = 0$, so $x = 20$ or $x = 30$.



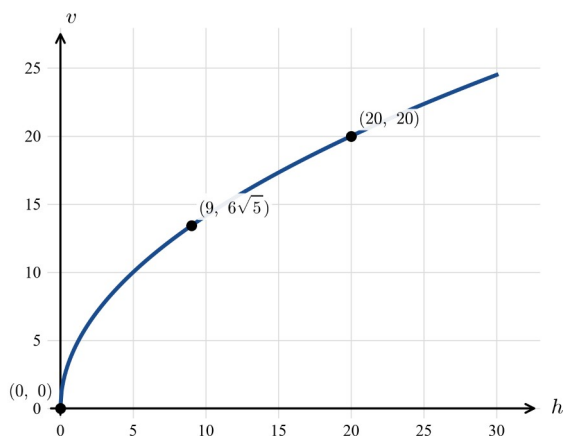
Q9. The side parallel to the missing side is $80 - 2x$, so $A(x) = x(80 - 2x) = 80x - 2x^2$; need $x > 0$ and $80 - 2x > 0$, so $0 < x < 40$. Axis $x = -\frac{80}{2(-2)} = 20$; $A(20) = 80(20) - 2(400) = 800$, a maximum area of 800 m².



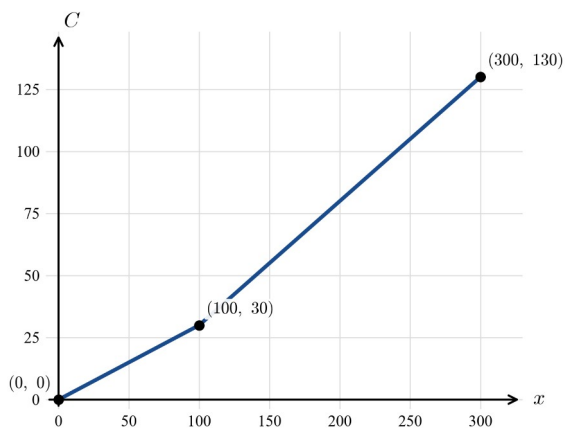
Q10. $D(6) = \frac{48}{6} = 8$ days. $\frac{48}{w} = 4 \Rightarrow w = 12$ workers. The graph is a decreasing hyperbola.



Q11. $v(9) = \sqrt{20 \times 9} = \sqrt{180} = 6\sqrt{5}$ m/s. $\sqrt{20h} = 20 \Rightarrow 20h = 400 \Rightarrow h = 20$ m. The graph is the square-root curve starting at $(0, 0)$.

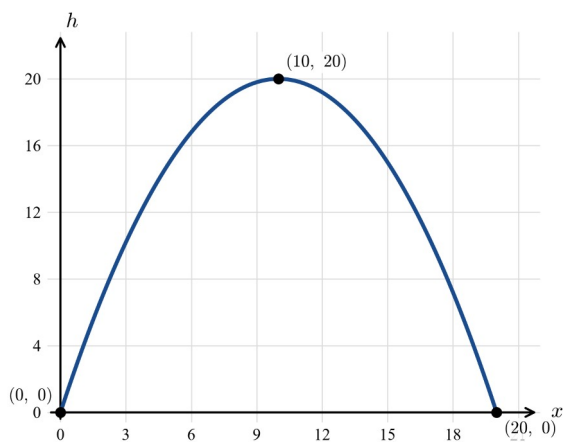


Q12. $80 \leq 100$, so $C(80) = 0.3 \times 80 = 24$, i.e. \$24. $200 > 100$, so $C(200) = 30 + 0.5(200 - 100) = 30 + 50 = 80$, i.e. \$80. The graph is two line segments meeting at $(100, 30)$, steeper after 100 kWh.



Q13. $h(x) = -\frac{1}{5}x^2 + 4x = -\frac{1}{5}x(x - 20)$, zero at $x=0$ and $x=20$, so the **span is 20 m**. Axis $x=10$;

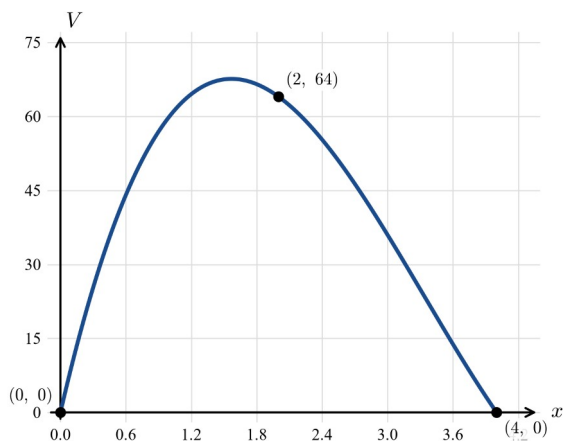
$h(10) = -\frac{1}{5}(100) + 40 = 20$, a **maximum height of 20 m**. $h(5) = -\frac{1}{5}(25) + 20 = -5 + 20 = 15$ m.



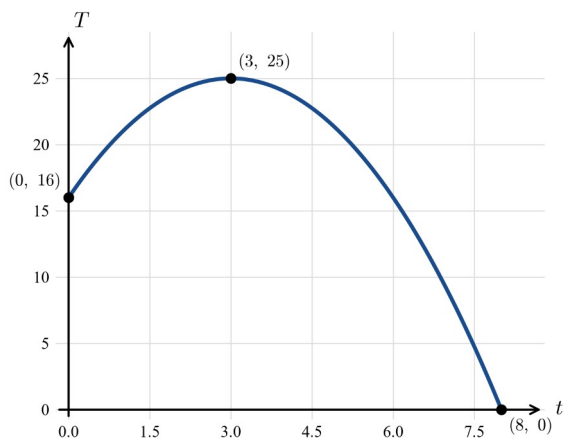
Q14. Base $(12 - 2x)$ by $(8 - 2x)$, height x : $V = x(12 - 2x)(8 - 2x)$. Expanding:

$(12 - 2x)(8 - 2x) = 96 - 40x + 4x^2$, so $V(x) = 4x^3 - 40x^2 + 96x$. Need $8 - 2x > 0$ and $x > 0$, so $0 < x < 4$.

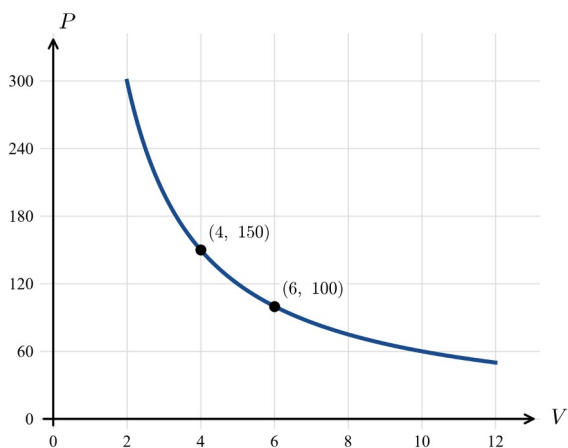
$V(2) = 32 - 160 + 192 = 64 \text{ cm}^3$.



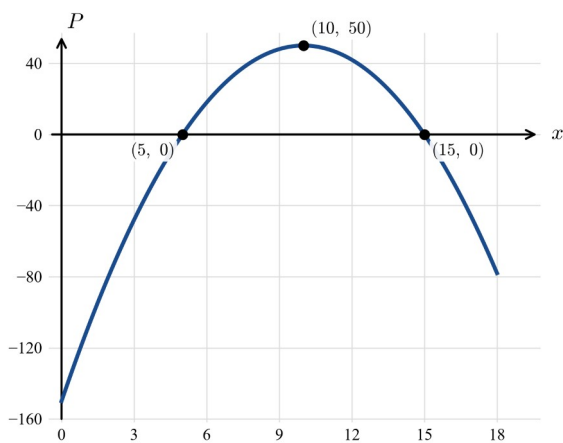
Q15. $T(t) = 25 - (t - 3)^2$ is already in turning-point form: maximum 25°C at $t=3$ (i.e. 3 pm). $T(0) = 25 - 9 = 16^\circ\text{C}$ at midday. $T=21$: $25 - (t - 3)^2 = 21 \Rightarrow (t - 3)^2 = 4 \Rightarrow t - 3 = \pm 2$, so $t=1$ and $t=5$.



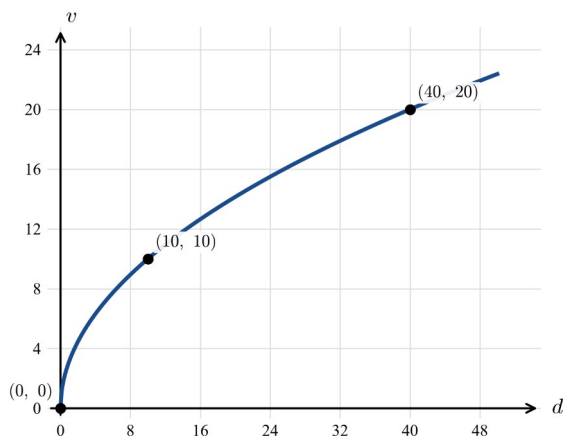
Q16. $P(4) = \frac{600}{4} = 150$. $\frac{600}{V} = 100 \Rightarrow V = 6$. The graph is a decreasing hyperbola (pressure inversely proportional to volume).



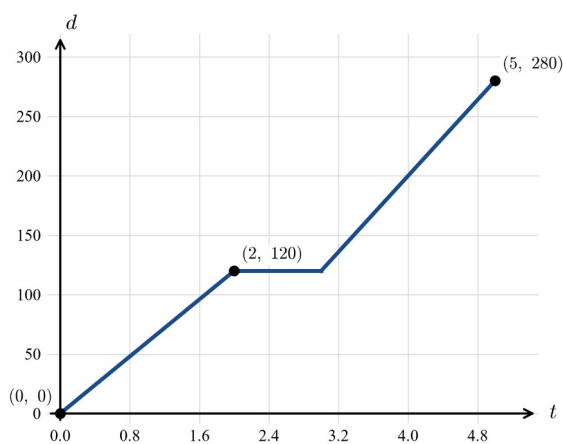
Q17. Break-even: $-2x^2 + 40x - 150 = 0 \Rightarrow x^2 - 20x + 75 = 0 \Rightarrow (x - 5)(x - 15) = 0$, so $x = 5$ and $x = 15$. Axis
 $x = -\frac{40}{2(-2)} = 10$; $P(10) = -200 + 400 - 150 = 50$, so the maximum profit is \$50 at $x = 10$ items.



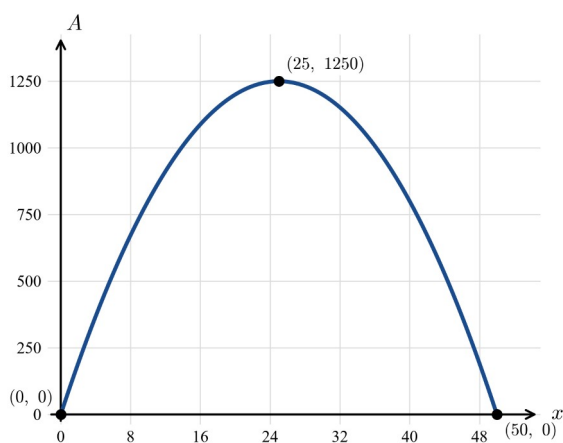
Q18. $v(10) = \sqrt{10 \times 10} = \sqrt{100} = 10$ m/s. $\sqrt{10d} = 20 \Rightarrow 10d = 400 \Rightarrow d = 40$ m. The graph is a square-root curve from $(0, 0)$.



Q19. Stage 1 ($0 \leq t \leq 2$): $d = 60t$ — driving away at 60 km/h. Stage 2 ($2 < t \leq 3$): $d = 120$ — stopped 120 km away. Stage 3 ($3 < t \leq 5$): $d = 120 + 80(t - 3)$ — driving further at 80 km/h. $d(4) = 120 + 80(1) = 200$ km; $d(5) = 120 + 80(2) = 280$ km.



Q20. The side parallel to the river is $100 - 2x$, so $A(x) = x(100 - 2x) = 100x - 2x^2$, as required. Domain: $x > 0$ and $100 - 2x > 0$, so $0 < x < 50$. Axis $x = -\frac{100}{2(-2)} = 25$; $A(25) = 100(25) - 2(625) = 2500 - 1250 = 1250$, so the maximum area is 1250 m^2 at $x = 25 \text{ m}$ (a 25 m by 50 m paddock).



Q21. On $[0, 2]$ the graph is the parabola $-(x - 2)^2 + 4$ from $(0, 0)$ up to the peak $(2, 4)$; on $(2, 5]$ it is constant at 4; on $(5, 7]$ it is the line $-2x + 14$ falling from $(5, 4)$ to $(7, 0)$. The greatest depth is 4 m and the least is 0 m, so the **range** is $[0, 4]$. The depth equals 4 m at the peak $x = 2$ and along the flat section, i.e. on the interval $2 \leq x \leq 5$.

