Mathematical Methods Units 1 & 2

Practice Examinations — Units 1 & 2

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Contents

Exam 1 — Technology-free (Easy)

*Reading time: 15 minutes. Writing time: 1 hour. Total 40 marks. Students are **not** permitted to bring any technology (calculators or software), or notes of any kind, into the examination room. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown. Diagrams are not to scale.*

Question 1 (4 marks)

a. Solve $3x-7=11$. 1 mark

b. Solve the simultaneous equations $y=2x-1$ and $x+y=8$. 3 marks

Question 2 (4 marks)

Sketch the graph of $y=(x-1)^2-4$, labelling the turning point and all axis intercepts. 4 marks

Question 3 (4 marks)

Let $f(x)=\sqrt{x+3}$.

a. State the implied domain. 1 mark

b. Find $f(6)$. 1 mark

c. State the range. 1 mark

d. Is f a one-to-one function? 1 mark

Question 4 (5 marks)

Let $P(x)=x^3-7x+6$.

a. Show that $(x-1)$ is a factor. 1 mark

b. Factorise $P(x)$ fully. 3 marks

c. Hence solve $P(x)=0$. 1 mark

Question 5 (5 marks)

a. Simplify $(2x^3)^2\times x$. 2 marks

b. Evaluate $\log_3 81$. 1 mark

c. Solve $2^x=32$. 2 marks

Question 6 (5 marks)

a. Convert $135°$ to radians, in terms of π. 1 mark

b. Find the exact value of $\sin\frac{3\pi}{4}$. 2 marks

c. Solve $\cos x=\frac{1}{2}$ for $x\in[0,2\pi]$. 2 marks

Question 7 (5 marks)

A bag contains 5 red and 4 blue counters. Two counters are drawn at random without replacement. Find:

a. $Pr(\text{both red})$; 2 marks

b. $Pr(\text{one of each colour})$. 3 marks

Question 8 (3 marks)

Find the value(s) of k for which $x^2+kx+25=0$ has exactly one solution. 3 marks

Question 9 (5 marks)

Consider $y=x^3+3x^2+1$.

a. Find $\frac{dy}{dx}$. 2 marks

b. Find the gradient of the curve at $x=3$. 1 mark

c. Find the coordinates of the stationary points. 2 marks

End of examination questions

Exam 1 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a triangle	$\frac{1}{2}bh$	volume of a sphere	$\frac{4}{3}\pi r^3$
circumference of a circle	$2\pi r$	volume of a cylinder	$\pi r^2 h$
area of a circle	πr^2	volume of a cone	$\frac{1}{3}\pi r^2 h$
arc length	$r\theta$, θ in radians	volume of a pyramid	$\frac{1}{3}Ah$
area of a sector	$\frac{1}{2}r^2\theta$, θ in radians	surface area of a sphere	$4\pi r^2$

Coordinate geometry

distance between points	$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	gradient	$m=\frac{y_2-y_1}{x_2-x_1}$
midpoint	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$	straight line through (x_1,y_1)	$y-y_1=m(x-x_1)$

Algebra

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	discriminant	$\Delta=b^2-4ac$
logarithm laws	$\log_a(xy)=\log_a x+\log_a y$		$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
	$\log_a(x^n)=n\log_a x$	index–log relation	$a^x=y \Leftrightarrow x=\log_a y$

Circular functions

Pythagorean identity	$\sin^2(x)+\cos^2(x)=1$	tangent	$\tan(x)=\frac{\sin(x)}{\cos(x)}$
$y=a\sin(n(t\pm\varepsilon))\pm b$	period $\frac{2\pi}{n}$; amplitude a		

Calculus

derivative of x^n	nx^{n-1}	anti-derivative of x^n ($n\neq-1$)	$\frac{x^{n+1}}{n+1}+c$
average rate of change	$\frac{f(b)-f(a)}{b-a}$	central difference	$f'(x)\approx\frac{f(x+h)-f(x-h)}{2h}$
Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$		

Probability and counting

addition rule	$P(A\cup B)=P(A)+P(B)$	conditional probability	$P(A/B)=\frac{P(A\cap B)}{P(B)}$

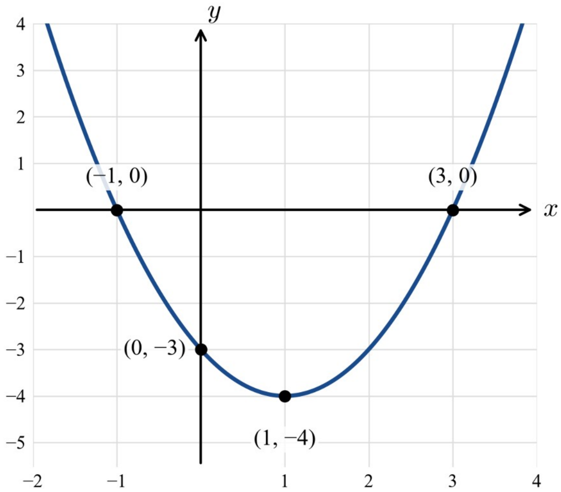
independence	$P(A \cap B) = P(A) \times P(B)$	combinations	${}^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$

End of Formula Sheet

Exam 1 — Answers and solutions

Q1. (a) $3x-7=11 \Rightarrow 3x=18 \Rightarrow x=6$. (b) Substitute $y=2x-1$ into $x+y=8$: $x+2x-1=8 \Rightarrow 3x=9 \Rightarrow x=3$, so $y=5$. $\therefore (x,y)=(3,5)$.

Q2. Turning point $(1,-4)$; y-intercept $x=0 \Rightarrow y=1-4=-3$, giving $(0,-3)$; x-intercepts $(x-1)^2=4 \Rightarrow x-1=\pm 2$, so $x=3$ or $x=-1$, giving $(3,0)$ and $(-1,0)$.



Q3. (a) Require $x+3\geq 0$, so the domain is $[-3,\infty)$. (b) $f(6)=\sqrt{9}=3$. (c) Range $[0,\infty)$. (d) Yes — f is increasing on its domain, so it is one-to-one.

Q4. (a) $P(1)=1-7+6=0$, so by the factor theorem $(x-1)$ is a factor. (b) Dividing, $x^3-7x+6=(x-1)(x^2+x-6)=(x-1)(x+3)(x-2)$. (c) $P(x)=0 \Rightarrow x=1, x=-3$ or $x=2$.

Q5. (a) $(2x^3)^2\times x=4x^6\times x=4x^7$. (b) $\log_3 81=4$ (since $3^4=81$). (c) $2^x=32=2^5 \Rightarrow x=5$.

Q6. (a) $135°=135\times\frac{\pi}{180}=\frac{3\pi}{4}$. (b) $\frac{3\pi}{4}$ is in the second quadrant with reference angle $\frac{\pi}{4}$, where sine is positive: $\sin\frac{3\pi}{4}=\frac{\sqrt{2}}{2}$. (c) $\cos x=\frac{1}{2}$ with base angle $\frac{\pi}{3}$; cosine is positive in the 1st and 4th quadrants, so $x=\frac{\pi}{3}$ or $x=\frac{5\pi}{3}$.

Q7. (a) $Pr(\text{both red})=\frac{5}{9}\times\frac{4}{8}=\frac{20}{72}=\frac{5}{18}$. (b) $Pr(\text{one of each})=Pr(RB)+Pr(BR)=\frac{5}{9}\times\frac{4}{8}+\frac{4}{9}\times\frac{5}{8}=\frac{20}{72}+\frac{20}{72}=\frac{40}{72}=\frac{5}{9}$.

Q8. One solution requires $\Delta=0$: $k^2-4(1)(25)=0 \Rightarrow k^2=100 \Rightarrow k=\pm 10$.

Q9. (a) $\frac{dy}{dx}=3x^2+6x$. (b) At $x=3$: $3(9)+6(3)=27+18=45$. (c) Stationary where $3x^2+6x=0 \Rightarrow 3x(x+2)=0 \Rightarrow x=0$ or $x=-2$; $y(0)=1$ and $y(-2)=-8+12+1=5$, so the stationary points are $(0,1)$ and $(-2,5)$.

Exam 2 — Technology-free (Medium)

Reading time: 15 minutes. Writing time: 1 hour. Total 40 marks. Students are ***not*** *permitted to bring any technology (calculators or software), or notes of any kind, into the examination room. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown. Diagrams are not to scale.*

Question 1 (4 marks)

a. Solve $x^2 - 5x + 6 = 0$. 2 marks

b. Solve $8^x = 4$. 2 marks

Question 2 (5 marks)

Let $f(x) = \sqrt{x} + 2$.

a. State the implied domain and the range of f. 2 marks

b. Find $f(4)$. 1 mark

c. Find the rule for $f^{-1}(x)$ and state its domain. 2 marks

Question 3 (5 marks)

A circle has equation $x^2 + y^2 - 4x + 6y - 12 = 0$.

a. By completing the square, find the centre and radius of the circle. 3 marks

b. Sketch the circle, showing the centre. 2 marks

Question 4 (6 marks)

a. Find the exact value of each of the following.

i. $\sin \frac{5\pi}{6}$ **ii.** $\cos \frac{5\pi}{4}$ **iii.** $\tan \frac{7\pi}{6}$ 3 marks

b. Solve $2\cos x = \sqrt{3}$ for $x \in [0, 2\pi]$. 3 marks

Question 5 (5 marks)

A bag contains 4 red and 3 blue marbles. Two marbles are drawn at random, one after the other, **without replacement**.

a. Draw a tree diagram showing the probabilities. 1 mark

b. Find the probability that both marbles are red. 2 marks

c. Find the probability that the two marbles are of different colours. 2 marks

Question 6 (5 marks)

a. Solve $\log_2(x-1) = 3$. 2 marks

b. Simplify $\log_{10} 40 + \log_{10} 25 - \log_{10} 10$. 3 marks

Question 7 (5 marks)

Let $P(x) = x^4 - 10x^2 + 9$.

a. Show that $(x-1)$ and $(x+1)$ are both factors of $P(x)$. 1 mark

b. Hence factorise $P(x)$ fully, as a product of linear factors. 3 marks

c. Hence solve $P(x)=0$. 1 mark

Question 8 (5 marks)

A particle moves in a straight line. Its position, in metres, at time t seconds ($t \geq 0$) is $x(t)=t^3-9t^2+15t$.

a. Find the velocity $v(t)$. 1 mark

b. Find the time(s) at which the particle is momentarily at rest. 2 marks

c. Find the acceleration when $t=2$. 2 marks

End of examination questions

Exam 2 Formula Sheet

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Mensuration

area of a triangle	$\frac{1}{2}bh$	volume of a sphere	$\frac{4}{3}\pi r^3$
circumference of a circle	$2\pi r$	volume of a cylinder	$\pi r^2 h$
area of a circle	πr^2	volume of a cone	$\frac{1}{3}\pi r^2 h$
arc length	$r\theta$, θ in radians	volume of a pyramid	$\frac{1}{3}Ah$
area of a sector	$\frac{1}{2}r^2\theta$, θ in radians	surface area of a sphere	$4\pi r^2$

Coordinate geometry

distance between points	$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	gradient	$m=\frac{y_2-y_1}{x_2-x_1}$
midpoint	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$	straight line through (x_1, y_1)	$y-y_1=m(x-x_1)$

Algebra

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	discriminant	$\Delta=b^2-4ac$
logarithm laws	$\log_a(xy)=\log_a x+\log_a y$		$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
	$\log_a(x^n)=n\log_a x$	index–log relation	$a^x=y \Leftrightarrow x=\log_a y$

Circular functions

Pythagorean identity	$\sin^2(x)+\cos^2(x)=1$	tangent	$\tan(x)=\frac{\sin(x)}{\cos(x)}$
$y=a\sin(n(t\pm\varepsilon))\pm b$	period $\frac{2\pi}{n}$; amplitude a		

Calculus

derivative of x^n	nx^{n-1}	anti-derivative of x^n ($n\neq-1$)	$\frac{x^{n+1}}{n+1}+c$
average rate of change	$\frac{f(b)-f(a)}{b-a}$	central difference	$f'(x)\approx\frac{f(x+h)-f(x-h)}{2h}$
Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$		

Probability and counting

addition rule	$P(A\cup B)=P(A)+P(B)$	conditional probability	$P(A/B)=\frac{P(A\cap B)}{P(B)}$

independence	$P(A \cap B) = P(A) \times P(B)$	combinations	${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

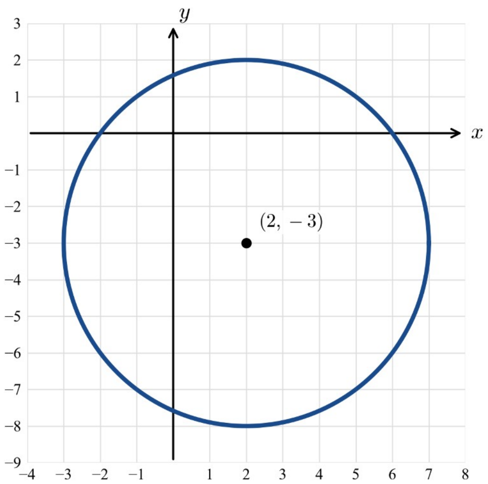
End of Formula Sheet

Exam 2 — Answers and solutions

Q1. (a) $x^2-5x+6=(x-2)(x-3)=0$, so $x=2$ or $x=3$. (b) $8^x=2^{3x}$ and $4=2^2$, so $3x=2$ and $\therefore x=\frac{2}{3}$.

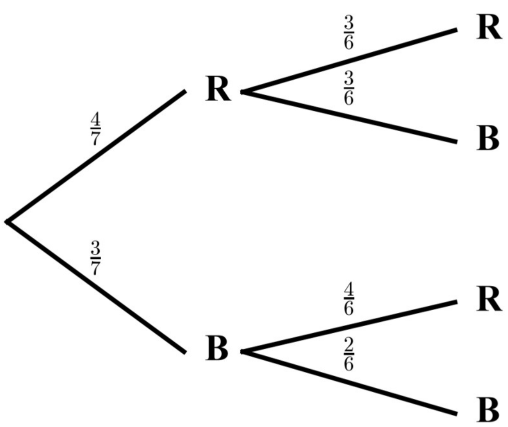
Q2. (a) The radicand requires $x\geq 0$, so the implied domain is $[0,\infty)$; as $\sqrt{x}\geq 0$, the range is $[2,\infty)$. (b) $f(4)=\sqrt{4}+2=2+2=4$. (c) Let $y=\sqrt{x}+2$; swap x and y: $x=\sqrt{y}+2$, so $\sqrt{y}=x-2$ and $y=(x-2)^2$. $\therefore f^{-1}(x)=(x-2)^2$, with domain $[2,\infty)$ (the range of f).

Q3. (a) $x^2-4x+y^2+6y=12\Rightarrow(x-2)^2-4+(y+3)^2-9=12$, so $(x-2)^2+(y+3)^2=25$. Centre $(2,-3)$, radius 5. (b)



Q4. (a) (i) $\sin\frac{5\pi}{6}=\frac{1}{2}$ (2nd quadrant, reference $\frac{\pi}{6}$). (ii) $\cos\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$ (3rd quadrant, reference $\frac{\pi}{4}$). (iii) $\tan\frac{7\pi}{6}=\frac{\sqrt{3}}{3}$ (3rd quadrant, reference $\frac{\pi}{6}$, tan positive). (b) $\cos x=\frac{\sqrt{3}}{2}$; reference angle $\frac{\pi}{6}$, cosine positive in quadrants 1 and 4: $\therefore x=\frac{\pi}{6}$ or $x=\frac{11\pi}{6}$.

Q5. (a)



(b) $Pr(\text{RR})=\frac{4}{7}\times\frac{3}{6}=\frac{12}{42}=\frac{2}{7}$.

(c) $Pr(\text{different})=Pr(\text{RB})+Pr(\text{BR})=\frac{4}{7}\cdot\frac{3}{6}+\frac{3}{7}\cdot\frac{4}{6}=\frac{12}{42}+\frac{12}{42}=\frac{24}{42}=\frac{4}{7}$.

Q6. (a) $\log_2(x-1)=3\Rightarrow x-1=2^3=8$, so $x=9$. (b) $\log_{10}40+\log_{10}25-\log_{10}10=\log_{10}\frac{40\times 25}{10}=\log_{10}100=2$.

Q7. (a) $P(1)=1-10+9=0$ and $P(-1)=1-10+9=0$, so by the factor theorem $(x-1)$ and $(x+1)$ are both factors. (b) Hence (x^2-1) is a factor; dividing, $P(x)=(x^2-1)(x^2-9)=(x-1)(x+1)(x-3)(x+3)$. (c) $P(x)=0 \Rightarrow x=\pm 1, \pm 3$.

Q8. (a) $v(t)=x'(t)=3t^2-18t+15$. (b) At rest when $v=0$: $3(t^2-6t+5)=3(t-1)(t-5)=0$, so $t=1$ s or $t=5$ s. (c) $a(t)=v'(t)=6t-18$; $a(2)=12-18=-6\,\text{m/s}^2$.

Exam 3 — Technology-free (Hard)

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Question 1 (5 marks)

Consider the equation $x^2+(m-1)x+(m+2)=0$, where m is a real constant.

a. Show that the discriminant is $\Delta=(m-7)(m+1)$. 2 marks

b. Find the values of m for which the equation has two distinct real solutions. 2 marks

c. State the values of m for which the equation has exactly one solution. 1 mark

Question 2 (6 marks)

Let $P(x)=2x^3-3x^2-11x+k$, where k is a constant. It is given that $(x-3)$ is a factor of $P(x)$.

a. Find the value of k. 1 mark

b. Hence factorise $P(x)$ fully. 3 marks

c. Solve $P(x)=0$. 1 mark

d. State the y-intercept of $y=P(x)$. 1 mark

Question 3 (5 marks)

a. Find the exact value of $\sin\frac{5\pi}{3}+\cos\frac{5\pi}{6}$. 2 marks

b. Solve $2\cos^2 x-1=0$ for $x\in[0,2\pi]$. 3 marks

Question 4 (4 marks)

Solve $\log_3(x)+\log_3(x+6)=3$ for x. 4 marks

Question 5 (5 marks)

Let $f(x)=\sqrt{3-2x}$, with maximal domain.

a. State the domain and range of f. 2 marks

b. Find the rule for $f^{-1}(x)$ and state its domain. 2 marks

c. Verify that $f\left(f^{-1}(x)\right)=x$ for x in the domain of f^{-1}. 1 mark

Question 6 (4 marks)

Two events A and B from the same sample space satisfy $P(A)=\frac{1}{3}$, $P(B/A)=\frac{3}{5}$ and $P(B/A')=\frac{3}{10}$, where A' denotes the complement of A.

a. Find $P(B)$. 2 marks

b. Find $P(A/B)$. 1 mark

c. Determine, with a reason, whether A and B are independent. 1 mark

Question 7 (6 marks)

For $f(x)=x^3-6x^2+9x+1$:

a. find $f'(x)$; 1 mark

b. find the coordinates of the stationary points and determine their nature; 3 marks

c. find the equation of the tangent to the curve at $x=0$. 2 marks

Question 8 (5 marks)

A particle moves in a straight line with position $x(t)=2t^3-9t^2+12t$ metres, $t\geq 0$ seconds.

a. Find expressions for the velocity v and the acceleration a. 2 marks

b. Find the times when the particle is momentarily at rest. 1 mark

c. Find the total distance travelled in the first 2 seconds. 2 marks

End of examination questions

Exam 3 Formula Sheet

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Mensuration

area of a triangle	$\frac{1}{2}bh$	volume of a sphere	$\frac{4}{3}\pi r^3$
circumference of a circle	$2\pi r$	volume of a cylinder	$\pi r^2 h$
area of a circle	πr^2	volume of a cone	$\frac{1}{3}\pi r^2 h$
arc length	$r\theta$, θ in radians	volume of a pyramid	$\frac{1}{3}Ah$
area of a sector	$\frac{1}{2}r^2\theta$, θ in radians	surface area of a sphere	$4\pi r^2$

Coordinate geometry

distance between points	$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	gradient	$m=\frac{y_2-y_1}{x_2-x_1}$
midpoint	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$	straight line through (x_1, y_1)	$y-y_1=m(x-x_1)$

Algebra

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	discriminant	$\Delta=b^2-4ac$
logarithm laws	$\log_a(xy)=\log_a x+\log_a y$		$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
	$\log_a(x^n)=n\log_a x$	index–log relation	$a^x=y \Leftrightarrow x=\log_a y$

Circular functions

Pythagorean identity	$\sin^2(x)+\cos^2(x)=1$	tangent	$\tan(x)=\frac{\sin(x)}{\cos(x)}$
$y=a\sin(n(t\pm\varepsilon))\pm b$	period $\frac{2\pi}{n}$; amplitude a		

Calculus

derivative of x^n	nx^{n-1}	anti-derivative of x^n ($n\neq-1$)	$\frac{x^{n+1}}{n+1}+c$
average rate of change	$\frac{f(b)-f(a)}{b-a}$	central difference	$f'(x)\approx\frac{f(x+h)-f(x-h)}{2h}$
Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$		

Probability and counting

addition rule	$P(A\cup B)=P(A)+P(B)$	conditional probability	$P(A/B)=\frac{P(A\cap B)}{P(B)}$

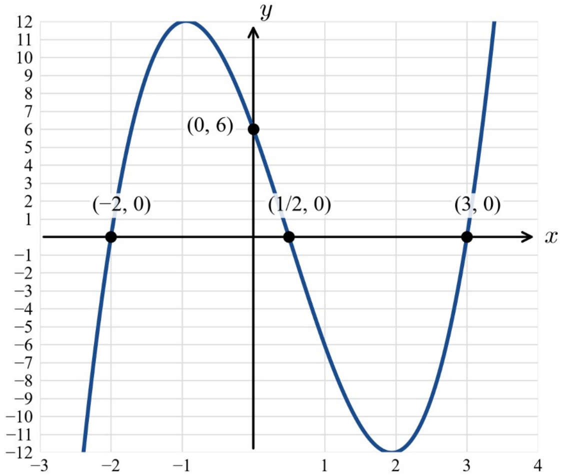
independence	$P(A \cap B) = P(A) \times P(B)$	combinations	${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

End of Formula Sheet

Exam 3 — Answers and solutions

Q1. (a) $\Delta=(m-1)^2-4(m+2)=m^2-2m+1-4m-8=m^2-6m-7=(m-7)(m+1)$. (b) Two distinct real solutions when $\Delta>0$: $(m-7)(m+1)>0 \Rightarrow m<-1$ or $m>7$. (c) Exactly one solution when $\Delta=0$: $m=-1$ or $m=7$.

Q2. (a) Since $(x-3)$ is a factor, the factor theorem gives $P(3)=0$: $2(27)-3(9)-11(3)+k=54-27-33+k=k-6=0$, so $k=6$. (b) With $k=6$, dividing $P(x)=2x^3-3x^2-11x+6$ by $(x-3)$ gives $P(x)=(x-3)(2x^2+3x-2)=(x-3)(2x-1)(x+2)$. (c) $P(x)=0 \Rightarrow x=3, x=1/2, x=-2$. (d) $P(0)=6$, so the y-intercept is $(0,6)$.



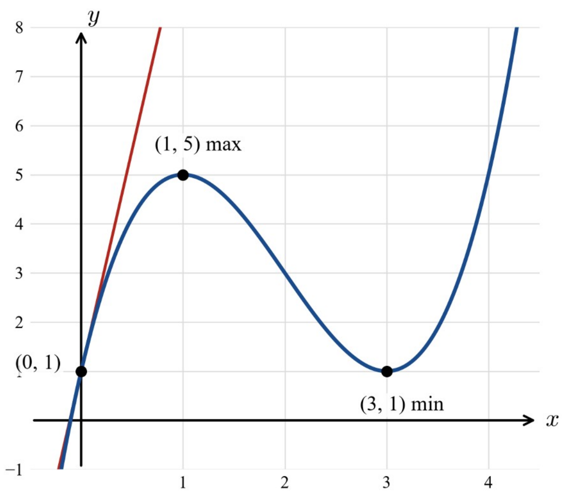
Q3. (a) $\sin\frac{5\pi}{3}=-\frac{\sqrt{3}}{2}$ and $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$, so the sum is $-\sqrt{3}$. (b) $2\cos^2 x-1=0 \Rightarrow \cos^2 x=1/2 \Rightarrow \cos x=\pm\frac{\sqrt{2}}{2}$. $\therefore x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$.

Q4. $\log_3(x(x+6))=3 \Rightarrow x^2+6x=27 \Rightarrow x^2+6x-27=0 \Rightarrow (x+9)(x-3)=0$. Since $x>0$ is required, $\therefore x=3$ (rejecting $x=-9$).

Q5. (a) Require $3-2x\geq 0 \Rightarrow x\leq 3/2$, so the domain is $(-\infty,3/2]$; the range is $[0,\infty)$. (b) Let $y=\sqrt{3-2x}$; swap: $x=\sqrt{3-2y} \Rightarrow x^2=3-2y \Rightarrow y=\frac{3-x^2}{2}$. $\therefore f^{-1}(x)=\frac{3-x^2}{2}$, domain $[0,\infty)$ (the range of f). (c) $f(f^{-1}(x))=\sqrt{3-2\left(\frac{3-x^2}{2}\right)}=\sqrt{x^2}=x$ for $x\geq 0$. ✓

Q6. (a) By the law of total probability, $P(B)=P(A)P(B/A)+P(A')P(B/A')=\frac{1}{3}\times\frac{3}{5}+\frac{2}{3}\times\frac{3}{10}=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$. (b) $P(A\cap B)=P(A)P(B/A)=\frac{1}{5}$, so $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{1/5}{2/5}=\frac{1}{2}$. (c) $P(A)\times P(B)=\frac{1}{3}\times\frac{2}{5}=\frac{2}{15}$, but $P(A\cap B)=\frac{1}{5}=\frac{3}{15}\neq\frac{2}{15}$. $\therefore$ A and B are **not** independent.

Q7. (a) $f'(x)=3x^2-12x+9$. (b) $f'(x)=3(x^2-4x+3)=3(x-1)(x-3)=0 \Rightarrow x=1$ or $x=3$. $f(1)=5$, $f(3)=1$. The gradient $f'(x)=3(x-1)(x-3)$ is positive for $x<1$, negative for $1<x<3$ and positive for $x>3$, so $(1,5)$ is a **local maximum** and $(3,1)$ is a **local minimum**. (c) $f(0)=1$ and $f'(0)=9$, so the tangent is $y=9x+1$.



Q8. (a) $v=\frac{dx}{dt}=6t^2-18t+12$; $a=\frac{dv}{dt}=12t-18$. (b) At rest: $v=0\Rightarrow 6(t^2-3t+2)=6(t-1)(t-2)=0\Rightarrow t=1$ or $t=2$. (c) $x(0)=0$, $x(1)=5$, $x(2)=4$. The particle moves forward 5 m (to $t=1$), then back 1 m (to $t=2$). $\therefore$ total distance $=5+1=6$ m.

Exam 4 — Technology-active (Easy)

Reading time: 15 minutes. Writing time: 2 hours. ***Section A:*** *20 multiple-choice questions (20 marks).* ***Section B:*** *written response (60 marks). Total 80 marks. Approved materials: protractors, set squares and aids for curve sketching; one bound reference; one approved CAS calculator or CAS software, and one scientific calculator. For Section A, choose the response that is* ***correct*** *for the question. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown. Diagrams are not to scale.*

Section A — Multiple-choice

Question 1

The gradient of the line through $(1,2)$ and $(3,8)$ is

A. 2

B. 3

C. 4

D. -3

Question 2

The solution of $3x-5=7$ is

A. 4

B. 2

C. 12

D. -4

Question 3

$(x-3)(x+5)$ expands to

A. $x^2-2x-15$

B. $x^2+2x-15$

C. $x^2+2x+15$

D. $x^2-8x-15$

Question 4

The turning point of $y=(x-2)^2+3$ is

A. $(-2,3)$

B. $(2,-3)$

C. $(2,3)$

D. $(3,2)$

Question 5

The discriminant of x^2+4x+5 is

A. -4

B. 4

C. -36

D. 36

Question 6

x^2-9 factorises to

A. $(x-3)^2$

B. $(x+3)^2$

C. $(x-3)(x+3)$

D. $(x-9)(x+1)$

Question 7

The (implied) domain of $y=\sqrt{x-4}$ is

A. $x>4$

B. $x\geq 4$

C. $x\leq 4$

D. $x\geq -4$

Question 8

If $f(x)=2x^2-1$, then $f(3)$ equals

A. 11

B. 35

C. 5

D. 17

Question 9

The equation of the circle with centre the origin and radius 5 is

A. $x^2+y^2=5$

B. $x^2+y^2=10$

C. $x^2+y^2=25$

D. $(x-5)^2+y^2=25$

Question 10

The vertical asymptote of $y = \frac{1}{x-3}$ is

A. $x = -3$

B. $x = 3$

C. $y = 3$

D. $x = 0$

Question 11

A fair die is rolled. The probability of an even number is

A. $\frac{1}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{6}$

D. $\frac{1}{2}$

Question 12

Two fair coins are tossed. The probability of two heads is

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. $\frac{1}{3}$

D. $\frac{3}{4}$

Question 13

A café menu offers 5 different sandwiches and 3 different drinks. The number of ways to choose one sandwich and one drink is

A. 8

B. 15

C. 125

D. 243

Question 14

$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ equals

A. 10

B. 20

C. 15

D. 5

Question 15

$2^3 \times 2^4$ equals

A. 2^{12}

B. 2^7

C. 4^7

D. 2^1

Question 16

$\log_2 8$ equals

A. 2

B. 4

C. 8

D. 3

Question 17

180° in radians is

A. π

B. 2π

C. $\frac{\pi}{2}$

D. $\frac{\pi}{3}$

Question 18

The period of $y = \sin x$ is

A. π

B. 2π

C. $\frac{\pi}{2}$

D. 4π

Question 19

The derivative of x^3 is

A. $3x^3$

B. x^2

C. $3x^2$

D. $3x$

Question 20

An antiderivative of $2x$ is

A. $2+c$

B. x^2+c

C. $2x^2+c$

D. $\frac{x^2}{2}+c$

End of Section A

Section B — Written response

Question 1 (12 marks)

Consider $f(x)=x^2-4x+3$.

a. Find the coordinates of the axis intercepts. 2 marks

b. Find the coordinates of the turning point. 2 marks

c. Sketch the graph, labelling the turning point and intercepts. 2 marks

d. State the range of f. 1 mark

e. Solve $f(x)=8$. 3 marks

f. Find the values of x for which $f(x)<0$. 2 marks

Question 2 (12 marks)

A jar contains 6 green and 3 yellow lollies. Two lollies are drawn at random **without replacement**.

a. Draw a tree diagram showing the probabilities. 2 marks

b. Find the probability that both lollies are green. 2 marks

c. Find the probability that the two lollies are of different colours. 3 marks

d. Find the probability that at least one lolly is yellow. 3 marks

e. Given that the first lolly drawn is yellow, find the probability that the second lolly drawn is green. 2 marks

Question 3 (12 marks)

A population is modelled by $P=500\times 2^{t/10}$, where t is the time in years.

a. State the initial population. 1 mark

b. Find the population after 20 years. 2 marks

c. Find the time taken for the population to reach 4000. 3 marks

d. Sketch the graph of P against t for $0 \le t \le 40$. 2 marks

e. State whether the model represents growth or decay, giving a reason. 2 marks

f. Find the average rate of change of the population between $t=0$ and $t=20$. 2 marks

Question 4 (12 marks)

a. Convert 150° to radians. 1 mark

b. Find the exact value of $\sin\frac{2\pi}{3}$. 2 marks

c. Solve $\sin x=\frac{1}{2}$ for $x\in[0,2\pi]$. 3 marks

d. State the amplitude and period of $y=3\sin(2x)$. 2 marks

e. State the maximum value of $y=3\sin(2x)+1$. 2 marks

f. Solve $3\sin(2x)+1=1$ for $x\in[0,\pi]$. 2 marks

Question 5 (12 marks)

Consider $f(x)=x^3-3x^2+2$.

a. Find $f'(x)$. 2 marks

b. Find the coordinates of the stationary points. 3 marks

c. Classify each stationary point. 2 marks

d. Find the equation of the tangent to the curve at $x=1$. 3 marks

e. Taking $x_0=3$ as a first approximation to the largest solution of $f(x)=0$, use one application of Newton's method to find x_1. 2 marks

End of examination questions

Exam 4 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a triangle	$\frac{1}{2}bh$	volume of a sphere	$\frac{4}{3}\pi r^3$
circumference of a circle	$2\pi r$	volume of a cylinder	$\pi r^2 h$
area of a circle	πr^2	volume of a cone	$\frac{1}{3}\pi r^2 h$
arc length	$r\theta$, θ in radians	volume of a pyramid	$\frac{1}{3}Ah$
area of a sector	$\frac{1}{2}r^2\theta$, θ in radians	surface area of a sphere	$4\pi r^2$

Coordinate geometry

distance between points	$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	gradient	$m=\frac{y_2-y_1}{x_2-x_1}$
midpoint	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$	straight line through (x_1,y_1)	$y-y_1=m(x-x_1)$

Algebra

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	discriminant	$\Delta=b^2-4ac$
logarithm laws	$\log_a(xy)=\log_a x+\log_a y$		$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
	$\log_a(x^n)=n\log_a x$	index–log relation	$a^x=y \Leftrightarrow x=\log_a y$

Circular functions

Pythagorean identity	$\sin^2(x)+\cos^2(x)=1$	tangent	$\tan(x)=\frac{\sin(x)}{\cos(x)}$
$y=a\sin(n(t\pm\varepsilon))\pm b$	period $\frac{2\pi}{n}$; amplitude a		

Calculus

derivative of x^n	nx^{n-1}	anti-derivative of x^n ($n\neq-1$)	$\frac{x^{n+1}}{n+1}+c$
average rate of change	$\frac{f(b)-f(a)}{b-a}$	central difference	$f'(x)\approx\frac{f(x+h)-f(x-h)}{2h}$
Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$		

Probability and counting

addition rule	$P(A\cup B)=P(A)+P(B)$	conditional probability	$P(A/B)=\frac{P(A\cap B)}{P(B)}$

independence	$P(A \cap B) = P(A) \times P(B)$	combinations	${}^{n}C_r = \dfrac{n!}{r!(n-r)!}$

End of Formula Sheet

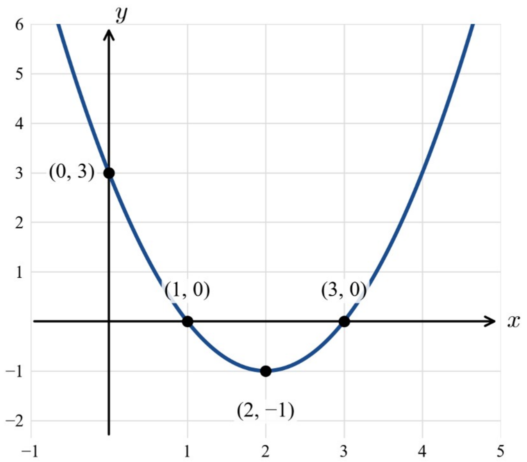
Exam 4 — Answers and solutions

Section A answer key

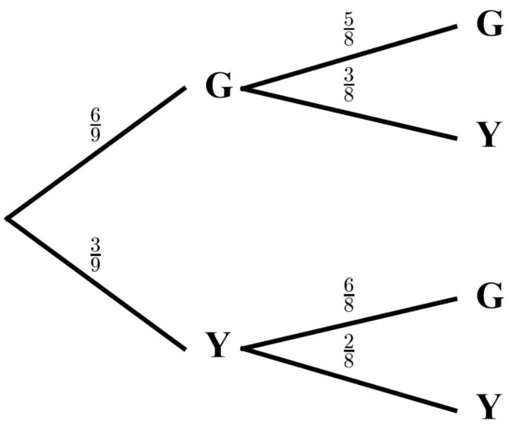
Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	6	C	11	D	16	D
2	A	7	B	12	A	17	A
3	B	8	D	13	B	18	B
4	C	9	C	14	A	19	C
5	A	10	B	15	B	20	B

Section B solutions

Q1. (a) $x^2-4x+3=(x-1)(x-3)=0$, so x-intercepts $(1,0)$ and $(3,0)$; y-intercept $(0,3)$. (b) Axis $x=2$; $f(2)=4-8+3=-1$: turning point $(2,-1)$. (c) See graph. (d) Range $[-1,\infty)$. (e) $x^2-4x+3=8 \Rightarrow x^2-4x-5=0 \Rightarrow (x-5)(x+1)=0$, so $x=5$ or $x=-1$. (f) $f(x)=(x-1)(x-3)<0$ between the x-intercepts, $\therefore 1<x<3$.



Q2. (a) Tree:



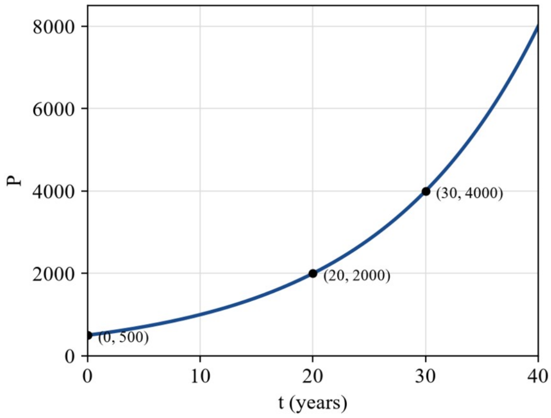
(b) $Pr(GG)=\frac{6}{9}\times\frac{5}{8}=\frac{30}{72}=\frac{5}{12}$.

(c) $Pr(\text{one of each})=\frac{6}{9}\times\frac{3}{8}+\frac{3}{9}\times\frac{6}{8}=\frac{18}{72}+\frac{18}{72}=\frac{36}{72}=\frac{1}{2}$.

(d) $Pr(\text{at least one yellow})=1-Pr(GG)=1-\frac{5}{12}=\frac{7}{12}$.

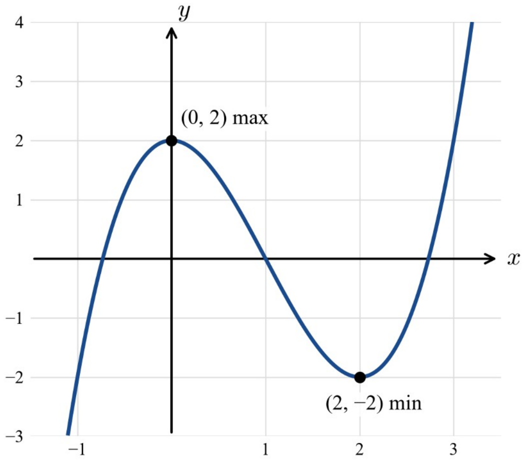
(e) After a yellow is drawn, 6 of the remaining 8 lollies are green, $\therefore$ $Pr(\text{second green / first yellow})=\frac{6}{8}=\frac{3}{4}$.

Q3. (a) $P(0)=500$. (b) $P(20)=500\times 2^2=2000$. (c) $500\times 2^{t/10}=4000 \Rightarrow 2^{t/10}=8=2^3 \Rightarrow \frac{t}{10}=3 \Rightarrow t=30$ years. (d) See graph. (e) Growth, because the base $2>1$ so P increases as t increases. (f) Average rate of change $=\frac{P(20)-P(0)}{20-0}=\frac{2000-500}{20}=75$ individuals per year.



Q4. (a) $150^{\circ}=150\times\frac{\pi}{180}=\frac{5\pi}{6}$. (b) $\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$. (c) $\sin x=\frac{1}{2}$: base angle $\frac{\pi}{6}$; in $[0,2\pi]$, $x=\frac{\pi}{6}$ or $x=\frac{5\pi}{6}$. (d) Amplitude 3, period $\frac{2\pi}{2}=\pi$. (e) Maximum $=3+1=4$. (f) $3\sin(2x)+1=1 \Rightarrow \sin(2x)=0$; for $x\in[0,\pi]$ we have $2x\in[0,2\pi]$, so $2x=0,\pi,2\pi$, $\therefore\ x=0,\frac{\pi}{2},\pi$.

Q5. (a) $f'(x)=3x^2-6x$. (b) $3x^2-6x=3x(x-2)=0 \Rightarrow x=0$ or $x=2$; $f(0)=2$, $f(2)=8-12+2=-2$: stationary points $(0,2)$ and $(2,-2)$. (c) Sign of $f'(x)=3x(x-2)$: at $x=-1$, $f'(-1)=9>0$; at $x=1$, $f'(1)=-3<0$; at $x=3$, $f'(3)=9>0$. Gradient changes $+\to-$ at $x=0$ and $-\to+$ at $x=2$, $\therefore$ $(0,2)$ is a local **maximum** and $(2,-2)$ is a local **minimum**. (d) $f(1)=1-3+2=0$, $f'(1)=3-6=-3$; tangent $y-0=-3(x-1)$, i.e. $y=-3x+3$. (e) $f(3)=27-27+2=2$, $f'(3)=27-18=9$; $x_1=x_0-\frac{f(x_0)}{f'(x_0)}=3-\frac{2}{9}=\frac{25}{9}\approx 2.78$.



Exam 5 — Technology-active (Medium)

Reading time: 15 minutes. Writing time: 2 hours. ***Section A:*** *20 multiple-choice questions (20 marks).* ***Section B:*** *written response (60 marks). Total 80 marks. Approved materials: protractors, set squares and aids for curve sketching; one bound reference; one approved CAS calculator or CAS software, and one scientific calculator. For Section A, choose the response that is* ***correct*** *for the question. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown. Diagrams are not to scale.*

Section A — Multiple-choice

Answer all questions. Each question is worth 1 mark.

Question 1

The maximal domain of $f(x)=\frac{1}{\sqrt{x-2}}$ is

A. $x \geq 2$

B. $x > 2$

C. $x \neq 2$

D. $x \in \mathbb{R}$

Question 2

If $f(x)=x^2-2x$, then $f(-3)$ equals

A. 3

B. 15

C. -3

D. -15

Question 3

The gradient of a line perpendicular to $y=2x-5$ is

A. 2

B. -2

C. $1/2$

D. $-1/2$

Question 4

The turning point of $y=2(x-3)^2+1$ is

A. $(3,1)$

B. $(-3,1)$

C. $(3,-1)$

D. $(-3,-1)$

Question 5

x^2+2x-8 factorises as

A. $(x+2)(x-4)$

B. $(x-2)(x+4)$

C. $(x+2)(x+4)$

D. $(x-2)(x-4)$

Question 6

The solutions of $2x^2-5x-3=0$ are

A. $x=3,-1/2$

B. $x=-3,1/2$

C. $x=3,1/2$

D. $x=-3,-1/2$

Question 7

The discriminant of x^2-3x+4 is

A. -7

B. 7

C. 25

D. -25

Question 8

If $y=x^4-2x^2$, then $\frac{dy}{dx}$ equals

A. $4x^3-4x$

B. $4x^3-2x$

C. $x^5/5-2x^3/3$

D. $4x^3-4$

Question 9

If $f'(x)=3x^2-2$ and $f(1)=4$, then $f(x)$ equals

A. x^3-2x+4

B. $6x$

C. x^3-2x+5

D. x^3-2x

Question 10

The period of $y = 3\cos(2x)$ is

A. 3

B. π

C. 2π

D. $\pi/2$

Question 11

$\cos\frac{5\pi}{6}$ equals

A. $\sqrt{3}/2$

B. $-\sqrt{3}/2$

C. $1/2$

D. $-1/2$

Question 12

$\log_3\frac{1}{9}$ equals

A. 2

B. -2

C. $1/2$

D. 3

Question 13

$16^{3/4}$ equals

A. 8

B. 12

C. 64

D. 4

Question 14

A committee of 3 students is to be chosen from a group of 7 students. The number of different committees that can be chosen is

A. 21

B. 210

C. 35

D. 343

Question 15

A discrete random variable X has probability distribution given by $Pr(X=x)=k\,x$ for $x=1,2,3,4$. The value of k is

A. $1/4$

B. $1/6$

C. $1/9$

D. $1/10$

Question 16

A fair die is rolled. $Pr(\text{even}\,/\,\text{greater than}\,2)$ equals

A. $1/2$

B. $2/3$

C. $1/3$

D. $1/4$

Question 17

The tangent to $y=x^2-x$ at $x=2$ has gradient

A. 2

B. 3

C. 4

D. 1

Question 18

The graph of $y=\frac{1}{x}$ is transformed to $y=\frac{1}{x+2}-1$. Its asymptotes are

A. $x=2,\,y=-1$

B. $x=-2,\,y=-1$

C. $x=-2,\,y=1$

D. $x=2,\,y=1$

Question 19

A zero of $f(x)=x^3-10$ is approximated using Newton's method. Starting from $x_0=2$, the next approximation x_1 is

A. $11/6$

B. $13/6$

C. $7/3$

D. $5/2$

Question 20

The average rate of change of $f(x)=x^2$ between $x=1$ and $x=3$ is

A. 2

B. 3

C. 4

D. 8

End of Section A

Section B — Written response

Answer all questions. A total of 60 marks is available.

Question 1 (13 marks)

Consider the function $f(x)=x^3-9x^2+24x-16$.

a. Find the coordinates of all axis intercepts. 3 marks

b. Find the coordinates of the stationary points and determine their nature. 5 marks

c. Sketch the graph of $y=f(x)$, labelling the intercepts and stationary points. 3 marks

d. Find the equation of the tangent to the curve at the point where $x=5$. 2 marks

Question 2 (12 marks)

A box contains 10 light globes: 3 are defective and 7 are in working order. Two globes are drawn at random from the box, one at a time, **without replacement**.

a. Draw a tree diagram showing the probabilities. 2 marks

b. Find the probability that both globes drawn are defective. 2 marks

c. Find the probability that exactly one of the two globes drawn is defective. 3 marks

d. Let X be the number of defective globes drawn. Construct the probability distribution table of X, and hence find the probability that at least one of the globes drawn is defective.
5 marks

Question 3 (11 marks)

The temperature T °C in a rural town is modelled by $T(t)=22+10\sin\left(\frac{\pi(t-8)}{12}\right)$ for $0\le t\le 24$, where t is the number of hours after midnight.

a. State the maximum and minimum temperatures, and the times at which they occur. 3 marks

b. Find the exact temperature at $t=11$. 2 marks

c. Find the times during the day when the temperature is 27 °C. 4 marks

d. Sketch the graph of $T(t)$ for $0\le t\le 24$. 2 marks

Question 4 (12 marks)

A farmer builds a rectangular enclosure against a straight river. No fence is needed along the river. She has 120 m of fencing for the other three sides. Let the two sides perpendicular to the river each be x m.

a. Show that the area enclosed is $A = x(120 - 2x)$, and state the domain of A. 2 marks

b. Find the value of x that maximises the area. Justify that this gives a maximum, and state the maximum area. 6 marks

c. State the dimensions of the enclosure of maximum area. 2 marks

d. Sketch the graph of A against x, marking the maximum. 2 marks

Question 5 (12 marks)

The value of a car is modelled by $V = 25\,000(0.85)^t$, where V is in dollars and t is the number of years after purchase.

a. State the value of the car when it is purchased. 1 mark

b. Find the value of the car after 3 years, to the nearest dollar. 2 marks

c. Find, to one decimal place, the number of years until the car is worth $\$10\,000$. 4 marks

d. By what percentage does the value decrease each year? 1 mark

e. A second model gives the value as $V = 25\,000 - 3000t$. Using a CAS calculator, find the value of $t > 0$ (to one decimal place) at which the two models give the same value. 4 marks

End of examination questions

Exam 5 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a triangle	$\frac{1}{2}bh$	volume of a sphere	$\frac{4}{3}\pi r^3$
circumference of a circle	$2\pi r$	volume of a cylinder	$\pi r^2 h$
area of a circle	πr^2	volume of a cone	$\frac{1}{3}\pi r^2 h$
arc length	$r\theta$, θ in radians	volume of a pyramid	$\frac{1}{3}Ah$
area of a sector	$\frac{1}{2}r^2\theta$, θ in radians	surface area of a sphere	$4\pi r^2$

Coordinate geometry

distance between points	$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	gradient	$m=\frac{y_2-y_1}{x_2-x_1}$
midpoint	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$	straight line through (x_1,y_1)	$y-y_1=m(x-x_1)$

Algebra

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	discriminant	$\Delta=b^2-4ac$
logarithm laws	$\log_a(xy)=\log_a x+\log_a y$		$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
	$\log_a(x^n)=n\log_a x$	index–log relation	$a^x=y \Leftrightarrow x=\log_a y$

Circular functions

Pythagorean identity	$\sin^2(x)+\cos^2(x)=1$	tangent	$\tan(x)=\frac{\sin(x)}{\cos(x)}$
$y=a\sin(n(t\pm\varepsilon))\pm b$	period $\frac{2\pi}{n}$; amplitude a		

Calculus

derivative of x^n	nx^{n-1}	anti-derivative of x^n ($n\neq-1$)	$\frac{x^{n+1}}{n+1}+c$
average rate of change	$\frac{f(b)-f(a)}{b-a}$	central difference	$f'(x)\approx\frac{f(x+h)-f(x-h)}{2h}$
Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$		

Probability and counting

addition rule	$P(A\cup B)=P(A)+P(B)$	conditional probability	$P(A/B)=\frac{P(A\cap B)}{P(B)}$

independence	$P(A \cap B) = P(A) \times P(B)$	combinations	${}^{n}C_r = \dfrac{n!}{r!(n-r)!}$

End of Formula Sheet

Exam 5 — Answers and solutions

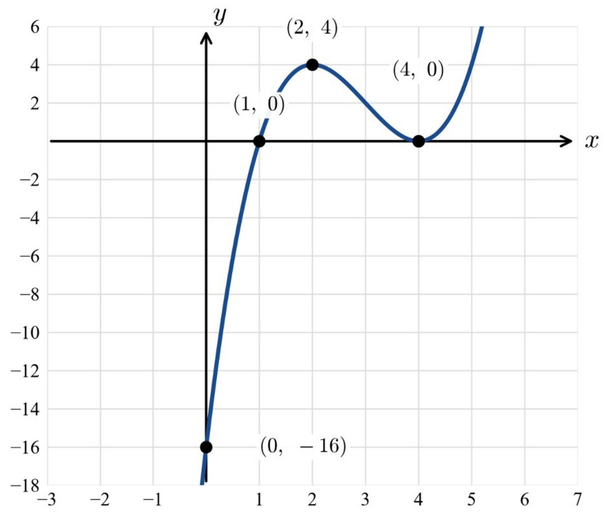
Section A — answer key

Q	1	2	3	4	5	6	7	8	9	10
Ans	B	B	D	A	B	A	A	A	C	B

Q	11	12	13	14	15	16	17	18	19	20
Ans	B	B	A	C	D	A	B	B	B	C

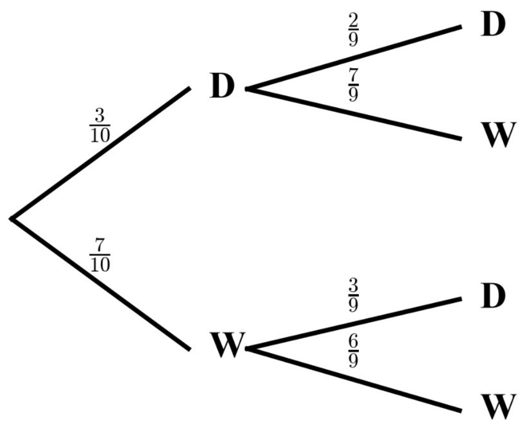
Section B — worked solutions

Question 1. (a) $f(1)=1-9+24-16=0$, so $(x-1)$ is a factor and $f(x)=x^3-9x^2+24x-16=(x-1)(x-4)^2$. x-intercepts: $x=1$ and $x=4$ (a repeated factor, so the graph touches at $(4,0)$); y-intercept $(0,-16)$. (b) $f'(x)=3x^2-18x+24=3(x-2)(x-4)$, so $f'(x)=0$ at $x=2$ and $x=4$. $f(2)=8-36+48-16=4$ and $f(4)=0$. The gradient $f'(x)=3(x-2)(x-4)$ is positive for $x<2$, negative for $2<x<4$ and positive for $x>4$, so $(2,4)$ is a **local maximum** and $(4,0)$ is a **local minimum**. (c)



(d) $f(5)=125-225+120-16=4$ and $f'(5)=3(25)-90+24=9$. Tangent: $y-4=9(x-5)$, so $\therefore y=9x-41$.

Question 2. (a)



(b) $Pr(DD)=\dfrac{3}{10}\times\dfrac{2}{9}=\dfrac{6}{90}=\dfrac{1}{15}$.

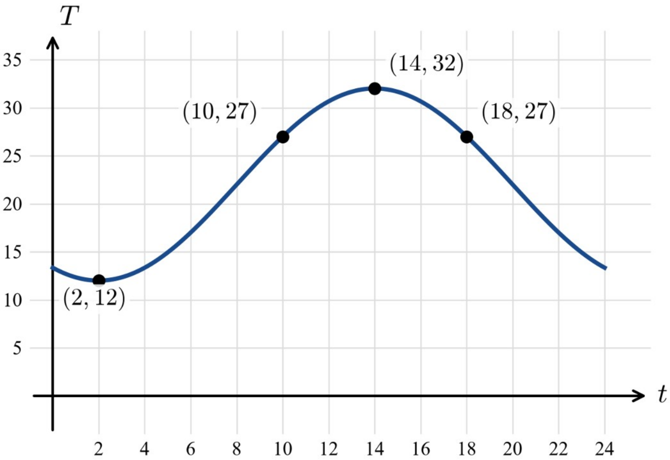
(c) $Pr(\text{exactly one defective})=Pr(DW)+Pr(WD)=\dfrac{3}{10}\cdot\dfrac{7}{9}+\dfrac{7}{10}\cdot\dfrac{3}{9}=\dfrac{21}{90}+\dfrac{21}{90}=\dfrac{42}{90}=\dfrac{7}{15}$.

(d) $Pr(X=0)=Pr(WW)=\frac{7}{10}\cdot\frac{6}{9}=\frac{42}{90}=\frac{7}{15}$; $Pr(X=1)=\frac{7}{15}$ (from part **c**); $Pr(X=2)=\frac{1}{15}$ (from part **b**).

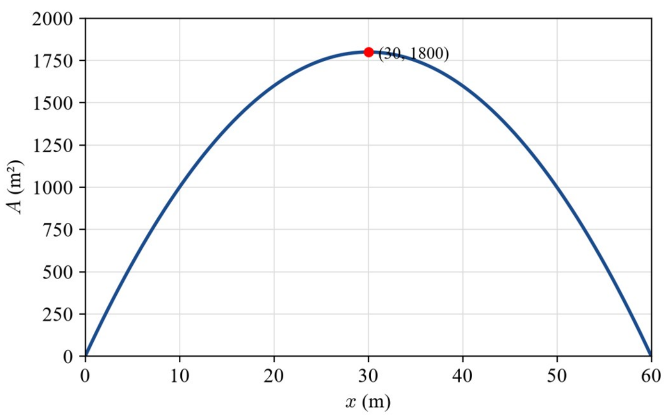
x	0	1	2
$Pr(X=x)$	$7/15$	$7/15$	$1/15$

The probabilities sum to 1. The probability that at least one globe drawn is defective is $Pr(X\geq 1)=1-Pr(X=0)=1-7/15=\frac{8}{15}$ (equivalently $7/15+1/15=8/15$).

Question 3. (a) The maximum of sin is 1, giving $T_{max}=32$ °C when $\frac{\pi(t-8)}{12}=\frac{\pi}{2}$, i.e. $t=14$ (2 p.m.). The minimum is $T_{min}=12$ °C when $\frac{\pi(t-8)}{12}=-\frac{\pi}{2}$, i.e. $t=2$ (2 a.m.). (b) $T(11)=22+10\sin\frac{\pi}{4}=22+10\cdot\frac{\sqrt{2}}{2}=22+5\sqrt{2}$ °C. (c) $22+10\sin\left(\frac{\pi(t-8)}{12}\right)=27\Rightarrow\sin\left(\frac{\pi(t-8)}{12}\right)=\frac{1}{2}$. For $0\leq t\leq 24$, $\frac{\pi(t-8)}{12}$ lies in $\left[-\frac{2\pi}{3},\frac{4\pi}{3}\right]$, so $\frac{\pi(t-8)}{12}=\frac{\pi}{6}$ or $\frac{5\pi}{6}$, giving $t-8=2$ or $t-8=10$, hence $t=10$ and $t=18$. The temperature is 27 °C at $t=10$ (10 a.m.) and $t=18$ (6 p.m.). (d)



Question 4. (a) The three fenced sides are the two of length x and the one parallel to the river, of length $120-2x$. Area $A=x(120-2x)$. Domain: $0<x<60$ (both dimensions positive). (b) $A=120x-2x^2$, so $\frac{dA}{dx}=120-4x=0\Rightarrow x=30$. The gradient $\frac{dA}{dx}$ is positive for $x<30$ and negative for $x>30$, so this is a maximum. $A(30)=30(120-60)=30\times 60=1800$ m². (c) The perpendicular sides are 30 m and the side parallel to the river is $120-2(30)=60$ m, so the enclosure is 30 m × 60 m. (d)



Question 5. (a) At $t=0$, $V=25\,000(0.85)^0=\$25\,000$. (b) $V(3)=25\,000(0.85)^3=25\,000\times 0.614125=15\,353.125\approx\$15\,353$. (c)

$25\,000(0.85)^t = 10\,000 \Rightarrow (0.85)^t = 0.4 \Rightarrow t = \dfrac{\log_{10} 0.4}{\log_{10} 0.85} \approx 5.6$ years. (d) Each year the value is multiplied by 0.85, a decrease of $15\,\%$ per year. (e) Solving $25\,000(0.85)^t = 25\,000 - 3000\,t$ with a CAS gives (for $t > 0$) $t \approx 3.9$ years.

Exam 6 — Technology-active (Hard)

Reading time: 15 minutes. Writing time: 2 hours. ***Section A:*** *20 multiple-choice questions (20 marks).* ***Section B:*** *written response (60 marks). Total 80 marks. Approved materials: protractors, set squares and aids for curve sketching; one bound reference; one approved CAS calculator or CAS software, and one scientific calculator. For Section A, choose the response that is* ***correct*** *for the question. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown. Diagrams are not to scale.*

Section A — Multiple-choice

Question 1

The maximal domain of $f(x)=\frac{\sqrt{x-2}}{x-5}$ is

A. $[2,\infty)$

B. $[2,5)\cup(5,\infty)$

C. $(2,5)\cup(5,\infty)$

D. $\mathbb{R}\setminus\{5\}$

Question 2

If $f(x)=2x^2-3x+1$, then $f(a+1)$ equals

A. $2a^2-a$

B. $2a^2+a$

C. $2a^2+7a+6$

D. $2a^2-3a+1$

Question 3

The number of real solutions of the equation $x^4-5x^2-36=0$ is

A. 0

B. 4

C. 3

D. 2

Question 4

The equation $x^2+kx+9=0$ has no real solutions when

A. $k<-6$ or $k>6$

B. $-6<k<6$

C. $k=\pm 6$

D. $-6\le k\le 6$

Question 5

The graph of $y=x^2$ is transformed to $y=-2(x+1)^2+3$ by

A. a reflection in the y-axis, dilation factor 2, translation 1 right and 3 up

B. a reflection in the x-axis, dilation factor 2 from the x-axis, translation 1 left and 3 up

C. a dilation factor 2, translation 1 left and 3 down

D. a reflection in the x-axis, dilation factor $1/2$, translation 1 left and 3 up

Question 6

$\cos\frac{2\pi}{3}$ equals

A. $\frac{\sqrt{3}}{2}$

B. $-\frac{\sqrt{3}}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

Question 7

The solutions of $2\sin x=\sqrt{3}$ over $[0,2\pi]$ are

A. $\frac{\pi}{6},\frac{5\pi}{6}$

B. $\frac{\pi}{3},\frac{2\pi}{3}$

C. $\frac{\pi}{3},\frac{5\pi}{3}$

D. $\frac{2\pi}{3},\frac{4\pi}{3}$

Question 8

The period of $y=3\tan(2x)$ is

A. 2π

B. π

C. $\frac{\pi}{2}$

D. $\frac{\pi}{4}$

Question 9

$\log_2 32 - \log_2 4$ equals

A. 8

B. $\log_2 28$

C. 3

D. 28

Question 10

The solution of $3^{2x} = 27$ is

A. 3

B. $\frac{3}{2}$

C. 2

D. $\frac{9}{2}$

Question 11

The range of $y = 2^x - 4$ is

A. $(0, \infty)$

B. $[-4, \infty)$

C. $(-4, \infty)$

D. $\mathbb{R}$

Question 12

Two fair dice are rolled. The probability that the sum is 7 is

A. $\frac{7}{36}$

B. $\frac{1}{9}$

C. $\frac{1}{6}$

D. $\frac{1}{12}$

Question 13

A committee of 4 people is to be chosen from a group of 7 women and 5 men. The number of different committees that contain exactly 2 women is

A. 210

B. 495

C. 31

D. 420

Question 14

A discrete random variable X has $Pr(X=x)=kx^2$ for $x \in \{1,2,3\}$, where k is a constant. $Pr(X \geq 2)$ equals

A. $\frac{13}{14}$

B. $\frac{1}{14}$

C. $\frac{9}{14}$

D. $\frac{5}{6}$

Question 15

For events A and B, $Pr(A)=0.5$, $Pr(B)=0.4$ and $Pr(A \cup B)=0.7$. Then $Pr(A/B)$ equals

A. $\frac{2}{5}$

B. $\frac{1}{4}$

C. $\frac{7}{10}$

D. $\frac{1}{2}$

Question 16

Newton's method is applied to $f(x)=x^3+2x-1$ with initial estimate $x_0=1$. The next estimate x_1 equals

A. $\frac{3}{5}$

B. $\frac{7}{5}$

C. $\frac{1}{3}$

D. 1

Question 17

The gradient of the tangent to $y=x^3-3x$ at $x=2$ is

A. 3

B. 6

C. 2

D. 9

Question 18

The local maximum of $y=x^3-3x^2-9x+5$ is at

A. $(-1,10)$

B. $(3,-22)$

C. $(0,5)$

D. $(1,-6)$

Question 19

A curve has $\frac{dy}{dx}=3x^2-4x+1$ and passes through the point $(2,5)$. The value of y when $x=-1$ is

A. 3

B. 1

C. -3

D. -1

Question 20

A particle has position $x(t)=t^3-12t^2+36t$ (metres, $t\geq 0$). It is momentarily at rest when

A. $t=2$ and $t=6$

B. $t=0$ and $t=6$

C. $t=6$ only

D. $t=2$ and $t=3$

End of Section A

Section B — Written response

Question 1 (12 marks)

Consider $f(x)=x^3+4x^2+4x$.

a. Show that $f(x)=x(x+2)^2$. 1 mark

b. Find the coordinates of the axis intercepts. 2 marks

c. Find and classify all stationary points. 4 marks

d. Sketch the graph of $y=f(x)$, labelling the key features. 2 marks

e. Find the equation of the tangent to the curve at the point where $x=-1$. 1 mark

f. Find all values of c for which the equation $f(x)=c$ has exactly three distinct real solutions. 2 marks

Question 2 (12 marks)

The depth of water at a pier is modelled by $h(t)=3\sin\left(\frac{\pi t}{6}\right)+5$, where h is in metres and t is the number of hours after midnight, for $0\le t\le 12$.

a. State the amplitude and the period. 2 marks

b. Find the maximum and minimum depths, and the first time each occurs. 3 marks

c. Sketch $y=h(t)$ for $0\le t\le 12$. 2 marks

d. Find the times during this interval at which the depth is 6.5 metres. 3 marks

e. Find the total length of time, in hours, during this interval for which the depth is at least 6.5 metres. 2 marks

Question 3 (12 marks)

An investment of $\$5000$ earns 8% per annum, compounded annually, so its value after t years is $V=5000(1.08)^t$.

a. Find the value after 10 years, to the nearest cent. 2 marks

b. Show that the time for the investment to double is $t=\log_{1.08}2$, and evaluate it to two decimal places. 4 marks

c. Find the least whole number of years after which the value first exceeds $\$8000$. 4 marks

d. A second investment of $\$7000$ is made at the same time, earning 5% per annum compounded annually. Find, correct to two decimal places, the number of years after which the two investments have equal value. 2 marks

Question 4 (12 marks)

A bag contains 5 red tokens and 3 blue tokens. Two tokens are drawn at random from the bag, one after the other, without replacement.

a. Find the probability that both tokens drawn are red. 2 marks

b. Find the probability that the two tokens drawn are of different colours. 2 marks

c. Given that the two tokens drawn are of different colours, find the probability that the first token drawn was red. 2 marks

d. Let X be the number of red tokens among the two drawn. Construct the probability distribution table for X and hence find $Pr(X\ge 1)$. 3 marks

e. All eight tokens are returned to the bag, and three of the eight tokens are then selected to be placed on display. Find the number of different selections that contain at least two red tokens. 3 marks

Question 5 (12 marks)

An open box is made from a square sheet of card of side 12 cm by cutting a square of side x cm from each corner and folding up the sides.

a. Show that the volume is $V(x)=x(12-2x)^2$. 2 marks

b. State the domain of V. 1 mark

c. Find the value of x that maximises the volume, justifying that it gives a maximum. 5 marks

d. State the maximum volume. 2 marks

e. Find the exact values of x in the domain of V for which $V(x) = 100$. 2 marks

End of examination questions

Exam 6 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Mensuration

area of a triangle	$\frac{1}{2}bh$	volume of a sphere	$\frac{4}{3}\pi r^3$
circumference of a circle	$2\pi r$	volume of a cylinder	$\pi r^2 h$
area of a circle	πr^2	volume of a cone	$\frac{1}{3}\pi r^2 h$
arc length	$r\theta$, θ in radians	volume of a pyramid	$\frac{1}{3}Ah$
area of a sector	$\frac{1}{2}r^2\theta$, θ in radians	surface area of a sphere	$4\pi r^2$

Coordinate geometry

distance between points	$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	gradient	$m=\frac{y_2-y_1}{x_2-x_1}$
midpoint	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$	straight line through (x_1,y_1)	$y-y_1=m(x-x_1)$

Algebra

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	discriminant	$\Delta=b^2-4ac$
logarithm laws	$\log_a(xy)=\log_a x+\log_a y$		$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
	$\log_a(x^n)=n\log_a x$	index–log relation	$a^x=y \Leftrightarrow x=\log_a y$

Circular functions

Pythagorean identity	$\sin^2(x)+\cos^2(x)=1$	tangent	$\tan(x)=\frac{\sin(x)}{\cos(x)}$
$y=a\sin(n(t\pm\varepsilon))\pm b$	period $\frac{2\pi}{n}$; amplitude a		

Calculus

derivative of x^n	nx^{n-1}	anti-derivative of x^n ($n\neq-1$)	$\frac{x^{n+1}}{n+1}+c$
average rate of change	$\frac{f(b)-f(a)}{b-a}$	central difference	$f'(x)\approx\frac{f(x+h)-f(x-h)}{2h}$
Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$		

Probability and counting

addition rule	$P(A\cup B)=P(A)+P(B)$	conditional probability	$P(A/B)=\frac{P(A\cap B)}{P(B)}$

independence	$P(A \cap B) = P(A) \times P(B)$	combinations	${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

End of Formula Sheet

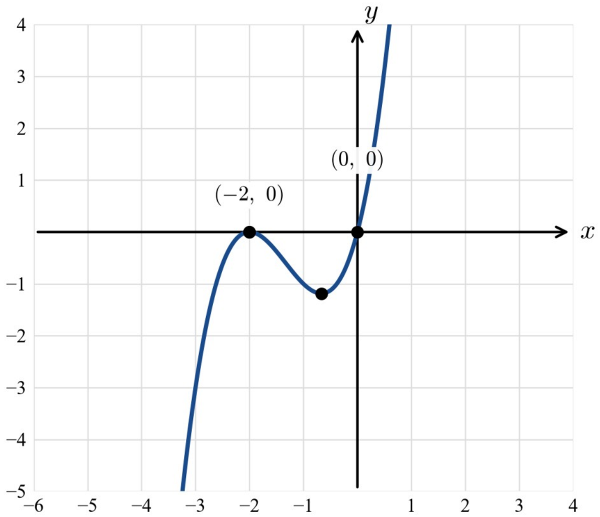
Exam 6 — Answers and solutions

Section A key

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans	B	B	D	B	B	C	B	C	C	B	C	C	A	A	D	A	D	A	D	A

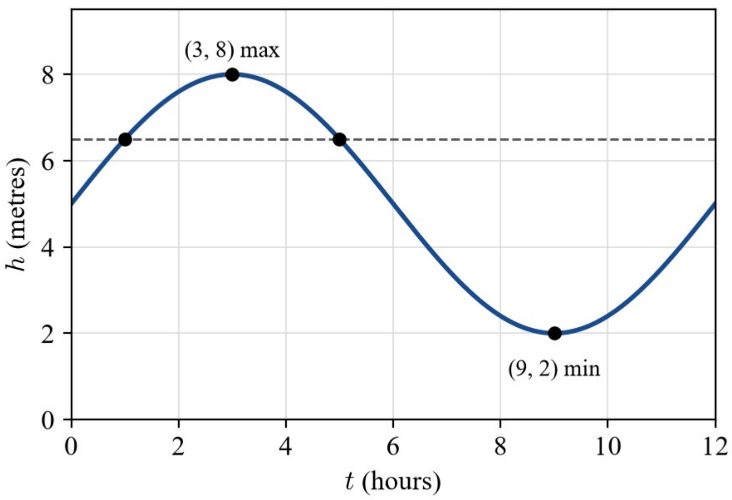
Selected reasoning: **Q3** $x^4-5x^2-36=(x^2-9)(x^2+4)=0 \Rightarrow x=\pm 3$ (two). **Q4** no real $\Leftrightarrow \Delta=k^2-36<0$. **Q13** ${}^7C_2 \times {}^5C_2=21\times 10=210$. **Q14** $k(1+4+9)=1 \Rightarrow k=1/14$, so $Pr(X\geq 2)=13k=13/14$. **Q15** $Pr(A\cap B)=0.5+0.4-0.7=0.2$, so $Pr(A/B)=0.2/0.4=1/2$. **Q16** $f'(x)=3x^2+2$, so $x_1=1-\dfrac{f(1)}{f'(1)}=1-\dfrac{2}{5}=\dfrac{3}{5}$. **Q19** $y=x^3-2x^2+x+c$; $y(2)=5 \Rightarrow c=3$, so $y(-1)=-1-2-1+3=-1$. **Q20** $v=3t^2-24t+36=3(t-2)(t-6)=0$.

Question 1. (a) $x(x+2)^2=x(x^2+4x+4)=x^3+4x^2+4x=f(x)$. ✓ (b) x-intercepts: $x(x+2)^2=0 \Rightarrow (0,0)$ and $(-2,0)$; y-intercept $(0,0)$. (c) $f'(x)=3x^2+8x+4=(x+2)(3x+2)$, so $f'=0$ at $x=-2,-\dfrac{2}{3}$. The gradient $f'(x)=(x+2)(3x+2)$ is positive for $x<-2$, negative for $-2<x<-\dfrac{2}{3}$ and positive for $x>-\dfrac{2}{3}$, so $(-2,0)$ is a **local maximum** and $\left(-\dfrac{2}{3},-\dfrac{32}{27}\right)$ is a **local minimum**. (d)



(e) $f(-1)=-1+4-4=-1$ and $f'(-1)=3-8+4=-1$, so $y+1=-(x+1)$, i.e. $\therefore y=-x-2$. (f) The horizontal line $y=c$ meets the graph in exactly three distinct points when c lies strictly between the local minimum value $-\dfrac{32}{27}$ and the local maximum value 0. $\therefore -\dfrac{32}{27}<c<0$.

Question 2. (a) Amplitude 3; period $\dfrac{2\pi}{\pi/6}=12$ hours. (b) Maximum depth $5+3=8$ m, first when $\sin(\pi t/6)=1$, i.e. $\pi t/6=\pi/2$, $t=3$ h. Minimum depth $5-3=2$ m, first at $\pi t/6=3\pi/2$, $t=9$ h. (c)



(d) $3\sin(\pi t/6)+5=6.5 \Rightarrow \sin(\pi t/6)=1/2 \Rightarrow \pi t/6=\pi/6$ or $5\pi/6$, giving $\therefore t=1$ and $t=5$ hours. (e) $h(t)\geq 6.5 \Leftrightarrow \sin(\pi t/6)\geq 1/2 \Leftrightarrow \pi/6\leq \pi t/6\leq 5\pi/6$, i.e. $1\leq t\leq 5$. $\therefore$ the depth is at least 6.5 m for $5-1=4$ hours.

Question 3. (a) $V(10)=5000(1.08)^{10}=\$10\,794.62$ (to the nearest cent). (b) Doubling: $5000(1.08)^t=10\,000 \Rightarrow 1.08^t=2 \Rightarrow t=\log_{1.08}2=\dfrac{\log 2}{\log 1.08}\approx 9.01$ years. (c) $5000(1.08)^t>8000 \Rightarrow 1.08^t>1.6 \Rightarrow t>\dfrac{\log 1.6}{\log 1.08}\approx 6.11$. Since $1.08^6\times 5000=\$7934.37$ and $1.08^7\times 5000=\$8569.12$, the value first exceeds $\$8000$ after $\therefore 7$ years. (d) $5000(1.08)^t=7000(1.05)^t \Rightarrow \left(\dfrac{1.08}{1.05}\right)^t=\dfrac{7}{5} \Rightarrow \left(\dfrac{36}{35}\right)^t=1.4 \Rightarrow t=\dfrac{\log_e 1.4}{\log_e\left(\dfrac{36}{35}\right)}\approx \therefore 11.94$ years (two decimal places).

Question 4. (a) $Pr(RR)=\dfrac{5}{8}\times\dfrac{4}{7}=\dfrac{20}{56}=\therefore \dfrac{5}{14}$. (b) $Pr(\text{different})=Pr(RB)+Pr(BR)=\dfrac{5}{8}\times\dfrac{3}{7}+\dfrac{3}{8}\times\dfrac{5}{7}=\dfrac{30}{56}=\therefore \dfrac{15}{28}$. (c) $Pr(\text{first red} / \text{different})=\dfrac{Pr(RB)}{Pr(\text{different})}=\dfrac{15/56}{15/28}=\therefore \dfrac{1}{2}$. (d) $Pr(X=0)=\dfrac{3}{8}\times\dfrac{2}{7}=\dfrac{3}{28}$, $Pr(X=1)=\dfrac{15}{28}$ (from part **b**), $Pr(X=2)=\dfrac{5}{14}=\dfrac{10}{28}$; the probabilities sum to 1.

x	0	1	2
$Pr(X=x)$	$\dfrac{3}{28}$	$\dfrac{15}{28}$	$\dfrac{10}{28}$

$Pr(X\geq 1)=1-Pr(X=0)=1-\dfrac{3}{28}=\therefore \dfrac{25}{28}$. (e) Exactly two red: ${}^5C_2\times{}^3C_1=10\times 3=30$; three red: ${}^5C_3=10$. $\therefore 30+10=40$ selections.

Question 5. (a) After cutting corners of side x, the base is a square of side $12-2x$ and the height is x, so $V=x(12-2x)^2$. ✓ (b) Need $x>0$ and $12-2x>0$, so the domain is $0<x<6$. (c) $V=4x^3-48x^2+144x$, so $V'(x)=12x^2-96x+144=12(x-2)(x-6)$. In the domain $V'=0$ at $x=2$ (since $x=6$ is excluded). $V'(x)=12(x-2)(x-6)$ is positive for $0<x<2$ and negative for $2<x<6$, so $x=2$ gives a **maximum**. (d) $V(2)=2(12-4)^2=2\times 64=\therefore 128\,\text{cm}^3$. (e) $x(12-2x)^2=100 \Rightarrow 4x^3-48x^2+144x-100=0 \Rightarrow x^3-12x^2+36x-25=0$. Since $x=1$ is a solution,

$(x-1)(x^2-11x+25)=0$, so $x=1$ or $x=\dfrac{11\pm\sqrt{21}}{2}$. As $\dfrac{11+\sqrt{21}}{2}\approx 7.79>6$, the solutions in the domain are $\therefore x=1$ and $x=\dfrac{11-\sqrt{21}}{2}$ (≈ 3.21).