Mathematical Methods Units 1 & 2

Chapter 17 — Integration (Units 3 & 4 preview)

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Chapter 17 Integration — 17A Approximating areas under curves

Units 3 & 4 preview. This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

Theory

For a function $y=f(x)$ with $f(x)\geq 0$, we often want the **area** of the region between the curve, the x-axis, and the lines $x=a$ and $x=b$. This section estimates that area; the next section finds it exactly.

Strips. Divide $[a,b]$ into n equal strips, each of width

$$\Delta x=\frac{b-a}{n},$$

with cut points $x_0=a, x_1, x_2, \ldots, x_n=b$. On each strip we replace the curve by a rectangle.

Left-endpoint estimate. Take the height of each rectangle at the **left** side of its strip:

$$L=\Delta x[f(x_0)+f(x_1)+\cdots+f(x_{n-1})].$$

Right-endpoint estimate. Take the height at the **right** side:

$$R=\Delta x[f(x_1)+f(x_2)+\cdots+f(x_n)].$$

For an **increasing** function, L **under-estimates** and R **over-estimates** the area (the true area lies between them); for a **decreasing** function it is the other way around.

Trapezium rule. Joining the tops of the strips by straight lines gives trapeziums, and usually a better estimate. It is the average of the left and right estimates:

$$T=1/2(L+R)=\Delta x\left[1/2f(x_0)+f(x_1)+\cdots+f(x_{n-1})+1/2f(x_n)\right].$$

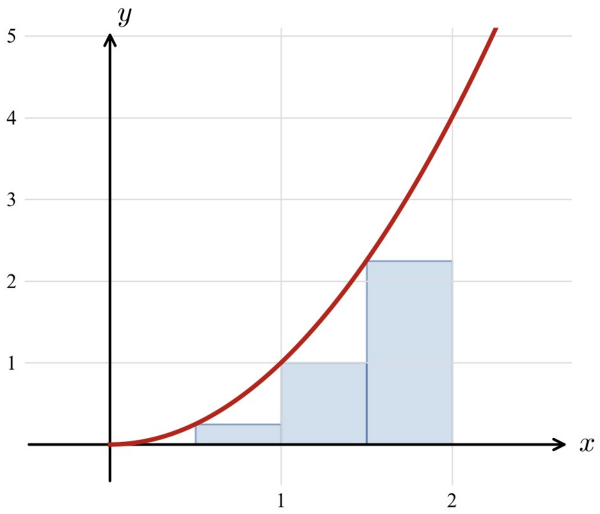
More strips, better estimate. Increasing n makes Δx smaller and the estimate more accurate. As $n\to\infty$ the estimate approaches the **exact** area — that limit is the *definite integral* (§17B).

Worked examples

Example 1 — scaffolded

Estimate the area under $y=x^2$ from $x=0$ to $x=2$ using **4 left-endpoint rectangles**.

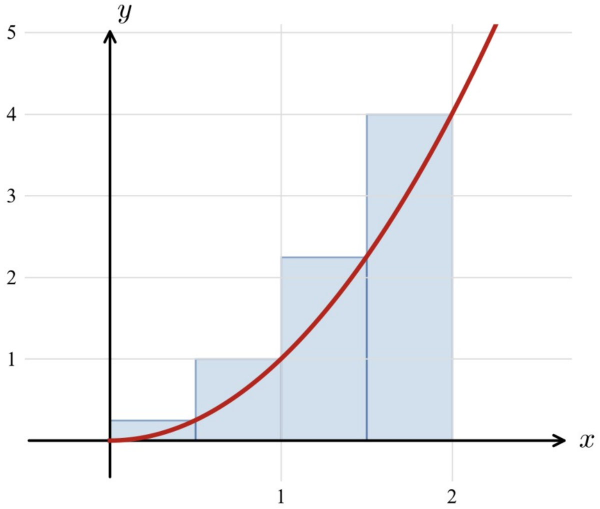
Working	Comment
$\Delta x=\frac{2-0}{4}=0.5$; cut points $0, 0.5, 1, 1.5, 2$.	Four equal strips.
Left heights: $f(0)=0$, $f(0.5)=0.25$, $f(1)=1$, $f(1.5)=2.25$.	Use the **left** end of each strip.
$L=0.5(0+0.25+1+2.25)=0.5\times 3.5=1.75$.	Add the heights, multiply by Δx.
$y=x^2$ is increasing, so this is an **under-estimate**.	The rectangles sit below the curve.



Example 2 — scaffolded

Estimate the same area using **4 right-endpoint rectangles**.

Working	Comment
Same $\Delta x = 0.5$ and cut points.	
Right heights: $f(0.5) = 0.25$, $f(1) = 1$, $f(1.5) = 2.25$, $f(2) = 4$.	Use the **right** end of each strip.
$R = 0.5(0.25 + 1 + 2.25 + 4) = 0.5 \times 7.5 = 3.75$.	An **over-estimate**.



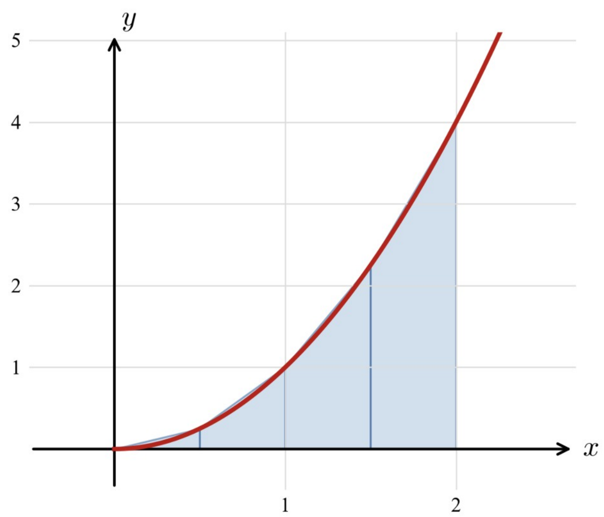
So the true area lies between 1.75 and 3.75.

Example 3 — unscaffolded

Use the **trapezium rule** with 4 strips to estimate the area under $y = x^2$ on $[0, 2]$.

$$T = 1/2(L + R) = 1/2(1.75 + 3.75) = 2.75.$$

$\therefore$ the area ≈ 2.75 (the exact area is $8/3 \approx 2.667$, so the trapezium estimate is much closer than either rectangle estimate).



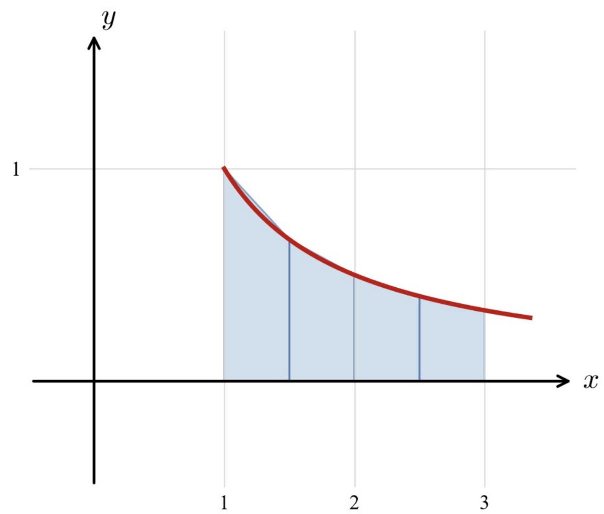
Example 4 — unscaffolded

Estimate the area under $y=\dfrac{1}{x}$ from $x=1$ to $x=3$ using the trapezium rule with 4 strips.

$\Delta x=0.5$; cut points $1, 1.5, 2, 2.5, 3$; heights $1, 2/3, 1/2, 2/5, 1/3$.

$$T=0.5\left[1/2(1)+2/3+1/2+2/5+1/2(1/3)\right]=0.5\times 67/30=67/60\approx 1.12.$$

$\therefore$ the area ≈ 1.12 (the exact area is $\log_e 3\approx 1.099$).



Practice questions

Scaffolded

Q1. Estimate the area under $y=x^2$ from $x=0$ to $x=2$ using **2 left-endpoint rectangles**. Is it an under- or over-estimate?

Q2. Repeat Q1 using **2 right-endpoint rectangles**.

Q3. Estimate the area under $y=x^2+1$ from $x=0$ to $x=3$ using **3 left-endpoint rectangles**.

Q4. Repeat Q3 using **3 right-endpoint rectangles**.

Q5. Use the **trapezium rule** with 3 strips to estimate the area under $y=x^2+1$ on $[0,3]$.

Q6. Use the trapezium rule with 4 strips to estimate the area under $y = 2x + 1$ on $[0, 4]$. Explain why this estimate is **exact**.

Unscaffolded

Estimate each area as directed. Give exact values where possible, otherwise a decimal.

Q7. $y = x^2$ on $[1, 3]$, 4 left-endpoint rectangles.

Q8. $y = x^2$ on $[1, 3]$, 4 right-endpoint rectangles.

Q9. $y = x^2$ on $[1, 3]$, trapezium rule with 4 strips.

Q10. $y = 4 - x^2$ on $[0, 2]$, 4 left-endpoint rectangles. Under- or over-estimate?

Q11. $y = 4 - x^2$ on $[0, 2]$, 4 right-endpoint rectangles.

Q12. $y = x^3$ on $[0, 2]$, 4 left-endpoint rectangles.

Q13. $y = x^3$ on $[0, 2]$, trapezium rule with 4 strips.

Q14. $y = \frac{1}{x}$ on $[1, 3]$, trapezium rule with 4 strips (you may reuse Example 4).

Q15. $y = 10 - x^2$ on $[0, 3]$, 3 left-endpoint rectangles.

Q16. $y = 2^x$ on $[0, 3]$, trapezium rule with 3 strips.

Q17. $y = x^2 + 2$ on $[0, 4]$, 4 left-endpoint rectangles. Compare with the right estimate (38) and the trapezium estimate (30); which is closest to the exact area $88/3$?

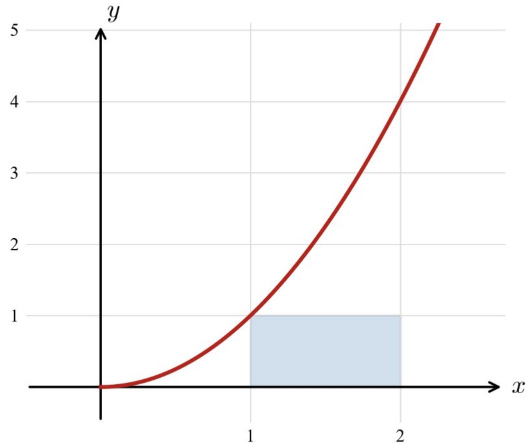
Q18. $y = x^2$ on $[0, 2]$, 8 left-endpoint rectangles. Compare with the 4-rectangle estimate of 1.75 from Example 1.

Answers

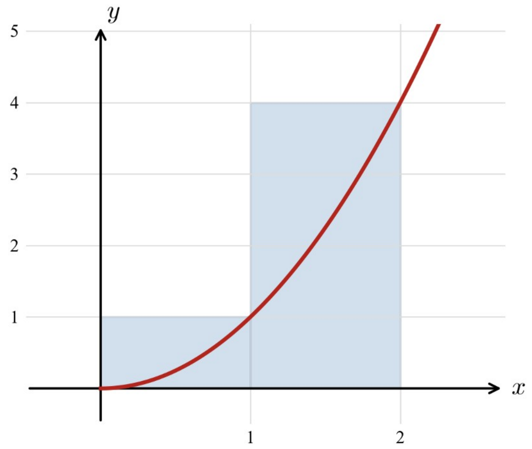
Q1. 1 (under-estimate). **Q2.** 5 (over-estimate). **Q3.** 8. **Q4.** 17. **Q5.** 12.5. **Q6.** 20 (exact — a straight line makes the trapeziums fit exactly). **Q7.** $27/4=6.75$. **Q8.** $43/4=10.75$. **Q9.** $35/4=8.75$. **Q10.** $25/4=6.25$ (over-estimate — the function is decreasing). **Q11.** $17/4=4.25$. **Q12.** $9/4=2.25$. **Q13.** $17/4=4.25$. **Q14.** $67/60\approx1.12$. **Q15.** 25. **Q16.** 10.5. **Q17.** 22; the trapezium estimate 30 is closest to $88/3\approx29.33$. **Q18.** 2.1875 — closer to the exact $8/3\approx2.667$ than the 4-rectangle estimate.

Fully-worked solutions

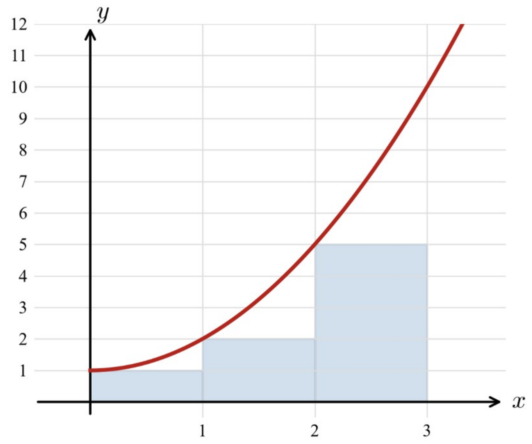
Q1. $\Delta x=1$; cut points $0,1,2$. Left heights $f(0)=0$, $f(1)=1$. $L=1(0+1)=1$. Under-estimate (increasing function).



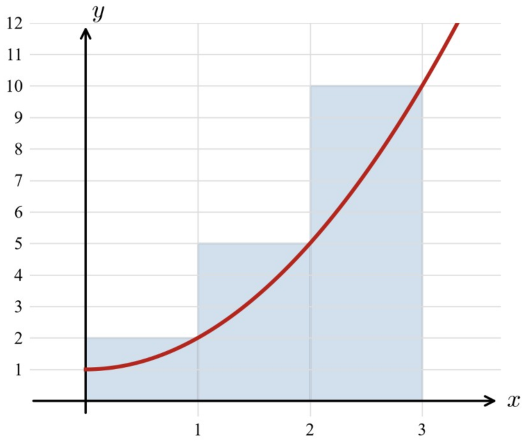
Q2. Right heights $f(1)=1$, $f(2)=4$. $R=1(1+4)=5$. Over-estimate.



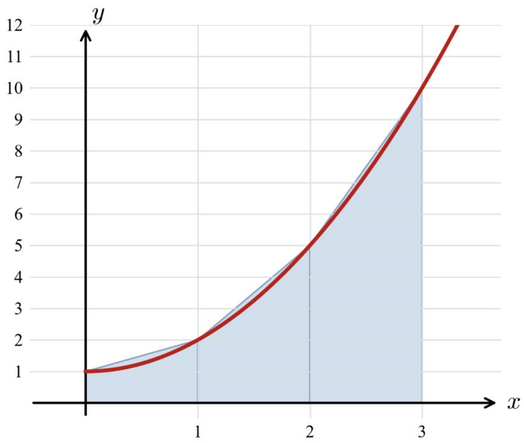
Q3. $\Delta x=1$; cut points $0,1,2,3$. Left heights $f(0)=1$, $f(1)=2$, $f(2)=5$. $L=1(1+2+5)=8$.



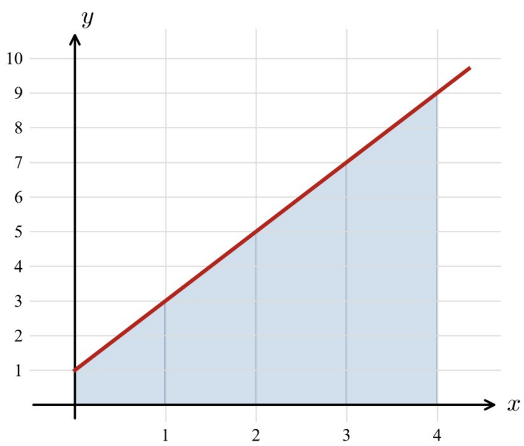
Q4. Right heights $f(1)=2$, $f(2)=5$, $f(3)=10$. $R=1(2+5+10)=17$.



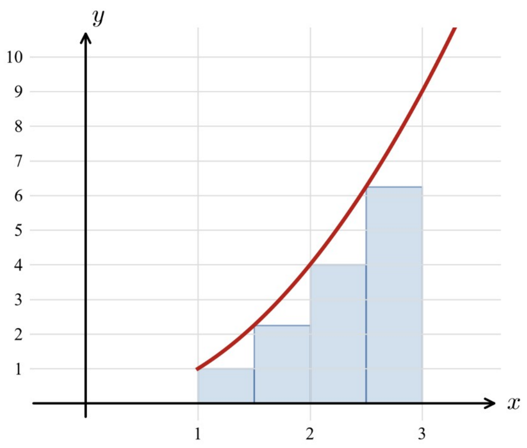
Q5. $T=1/2(L+R)=1/2(8+17)=12.5$.



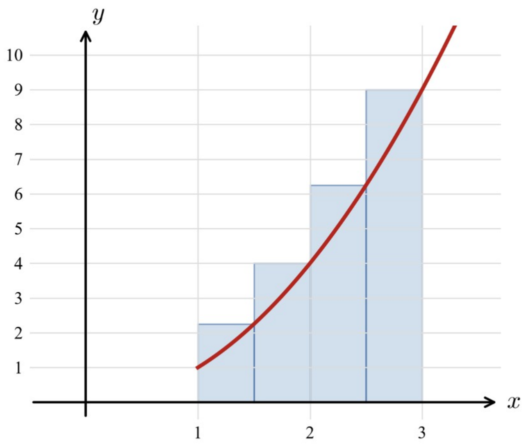
Q6. $\Delta x=1$; heights $1,3,5,7,9$. $T=1[1/2(1)+3+5+7+1/2(9)]=20$. Because $y=2x+1$ is a straight line, the tops of the trapeziums lie exactly on the graph, so there is no error — the estimate equals the exact area.



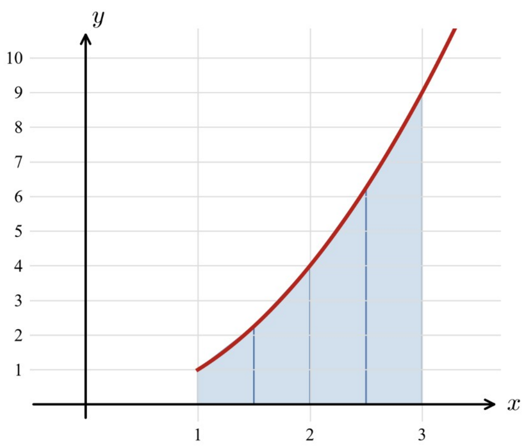
Q7. $\Delta x=0.5$; cut points $1,1.5,2,2.5,3$. Left heights $1,2.25,4,6.25$.
$L=0.5(1+2.25+4+6.25)=0.5\times13.5=27/4=6.75$.



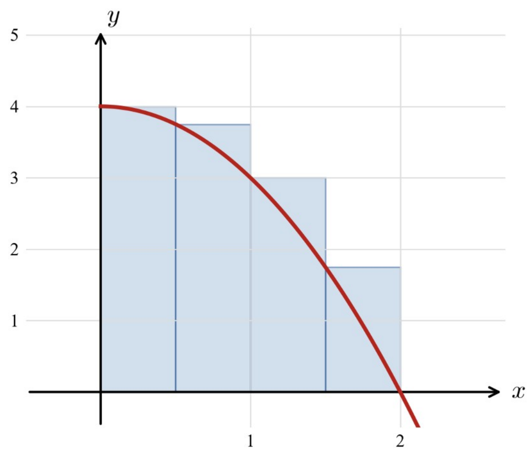
Q8. Right heights $2.25, 4, 6.25, 9$. $R = 0.5(2.25+4+6.25+9) = 0.5 \times 21.5 = 43/4 = 10.75$.



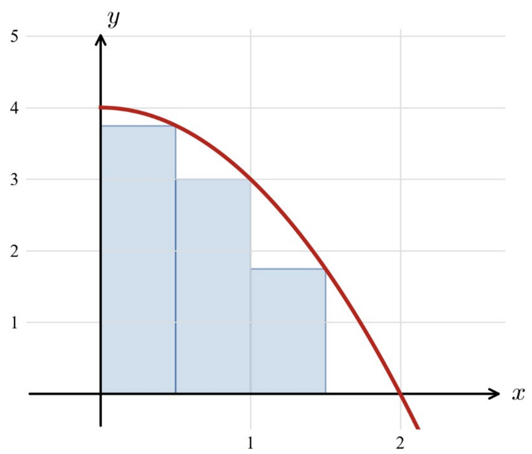
Q9. $T = 1/2(6.75+10.75) = 35/4 = 8.75$ (exact area $26/3 \approx 8.67$).



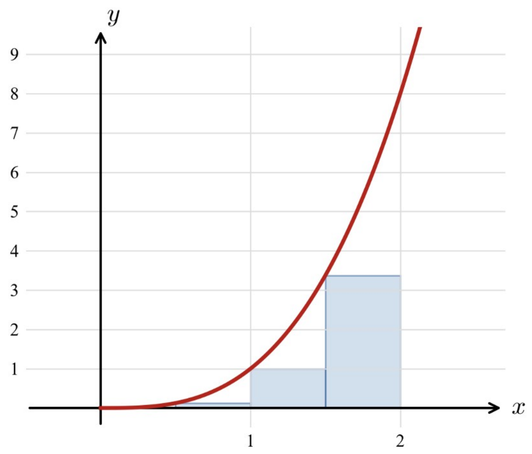
Q10. $\Delta x = 0.5$; cut points $0, 0.5, 1, 1.5, 2$. Left heights $f(0)=4$, $f(0.5)=3.75$, $f(1)=3$, $f(1.5)=1.75$. $L = 0.5(4+3.75+3+1.75) = 0.5 \times 12.5 = 25/4 = 6.25$. The function is **decreasing**, so the left estimate **over-**estimates.



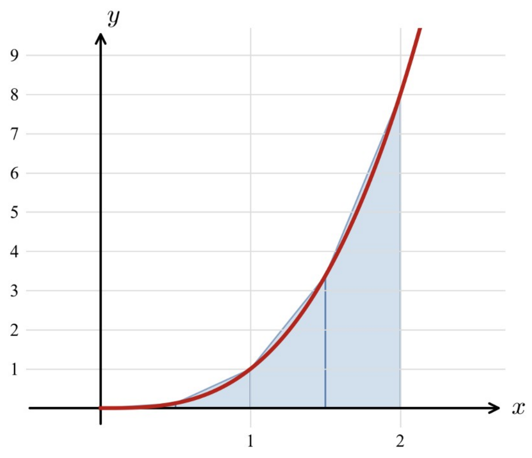
Q11. Right heights $3.75, 3, 1.75, 0$. $R = 0.5(3.75 + 3 + 1.75 + 0) = 17/4 = 4.25$ (under-estimate).



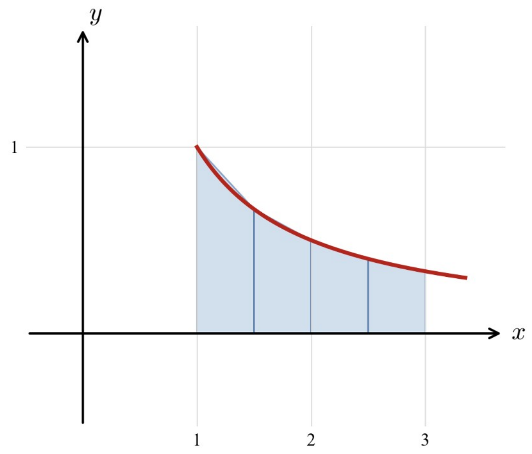
Q12. $\Delta x = 0.5$; cut points $0, 0.5, 1, 1.5, 2$. Left heights $f(0) = 0$, $f(0.5) = 0.125$, $f(1) = 1$, $f(1.5) = 3.375$. $L = 0.5(0 + 0.125 + 1 + 3.375) = 0.5 \times 4.5 = 9/4 = 2.25$.



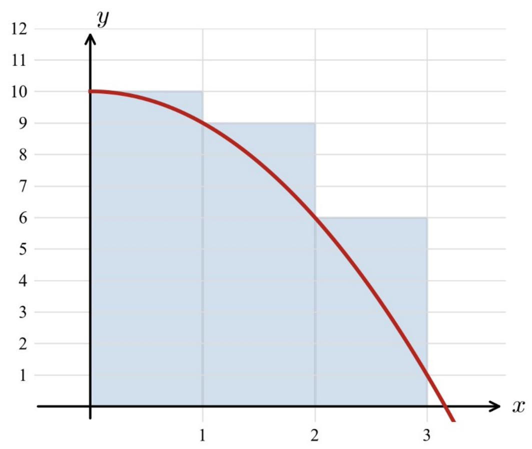
Q13. Right heights $0.125, 1, 3.375, 8$, so $R = 0.5 \times 12.5 = 6.25$; then $T = 1/2(2.25 + 6.25) = 17/4 = 4.25$ (exact area 4).



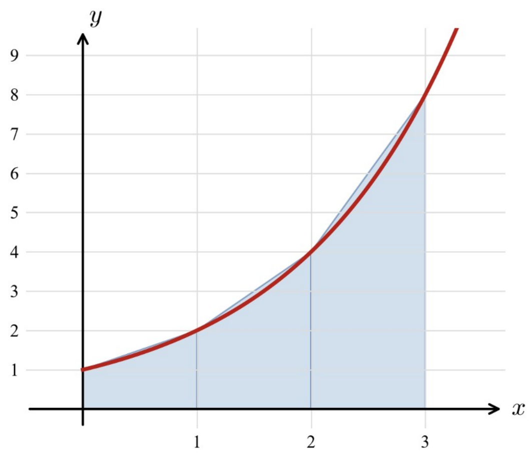
Q14. As in Example 4, $T = 67/60 \approx 1.12$ (exact area $\log_e 3 \approx 1.099$).



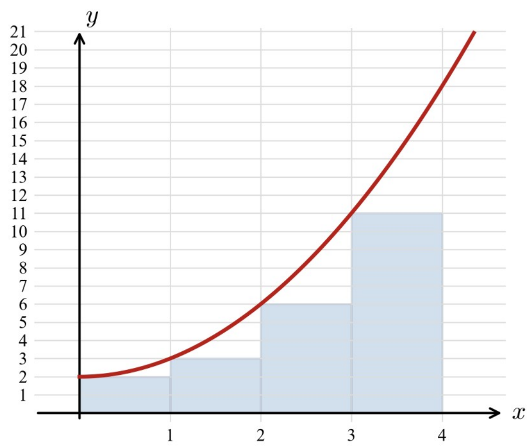
Q15. $\Delta x = 1$; cut points $0, 1, 2, 3$. Left heights $f(0) = 10$, $f(1) = 9$, $f(2) = 6$. $L = 1(10 + 9 + 6) = 25$ (over-estimate — decreasing).



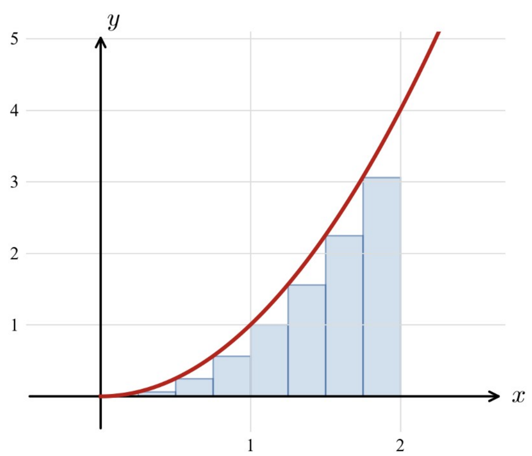
Q16. $\Delta x = 1$; heights $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$. $T = 1[1/2(1) + 2 + 4 + 1/2(8)] = 10.5$.



Q17. $\Delta x=1$; cut points $0,1,2,3,4$. Left heights $f(0)=2$, $f(1)=3$, $f(2)=6$, $f(3)=11$. $L=1(2+3+6+11)=22$. Comparing $L=22$, $R=38$ and $T=30$ with the exact area $88/3\approx 29.33$, the **trapezium estimate** 30 **is closest**.



Q18. $\Delta x=0.25$; cut points $0,0.25,0.5,\ldots,2$. The eight left heights are $0,0.0625,0.25,0.5625,1,1.5625,2.25,3.0625$, summing to 8.75, so $L=0.25\times 8.75=2.1875$. This is closer to the exact area $8/3\approx 2.667$ than Example 1's 1.75 — more strips give a better estimate.



Chapter 17 Integration — 17B The definite integral

Units 3 & 4 preview. This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

Theory

The **definite integral** of f from $x=a$ to $x=b$ is written $\int_a^b f(x)\,dx$ and is evaluated using the **fundamental theorem of calculus**:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where F is any antiderivative of f. The arbitrary constant $+c$ cancels ($F(b)+c-(F(a)+c)$), so we may use the simplest antiderivative.

Area under a curve. If $f(x)\geq 0$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx = \text{the area of the region between the curve } y=f(x) \text{ and the } x\text{-axis, from } x=a \text{ to } x=b.$$

Finding the limits. To find the area bounded by a curve and the x-axis, first find where the curve **crosses** the x-axis (solve $f(x)=0$); these give the limits a and b.

Useful properties.

$$\int_a^a f(x)\,dx = 0, \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

and $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$, $\int_a^b (f\pm g)\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx$.

*(This section deals with regions **above** the x-axis. Regions below the axis give a negative integral — signed area — which is treated in 17C.)*

Worked examples

Example 1 — scaffolded

Evaluate $\int_1^3 2x\,dx$.

Working	Comment
An antiderivative of $2x$ is x^2.	Antidifferentiate; the $+c$ is not needed.
$\int_1^3 2x\,dx = [x^2]_1^3$	Write the antiderivative in square brackets with the limits.
$=(3)^2-(1)^2=9-1=8.$	Substitute the upper limit, then subtract the lower: $F(b)-F(a)$.

Example 2 — scaffolded

Evaluate $\int_0^2 (3x^2+1)\,dx$.

Working	Comment
Antiderivative: x^3+x.	Antidifferentiate term by term.
$\int_0^2 (3x^2+1)\,dx=[x^3+x]_0^2$	Insert the limits.
$=(8+2)-(0+0)=10$.	$F(2)-F(0)$.

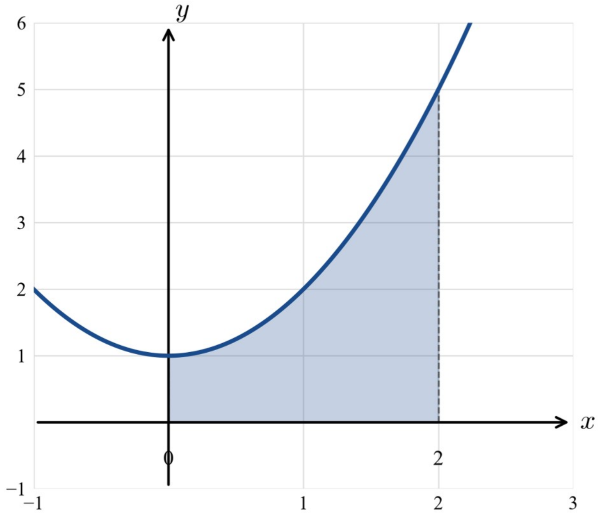
Example 3 — unscaffolded

Find the exact area of the region bounded by $y=x^2+1$, the x-axis, and the lines $x=0$ and $x=2$.

Since $x^2+1>0$, the area equals the definite integral:

$$\text{Area}=\int_0^2 (x^2+1)\,dx=[x^3/3+x]_0^2=(8/3+2)-0=\frac{14}{3}.$$

$\therefore$ the area is $\frac{14}{3}$ square units.



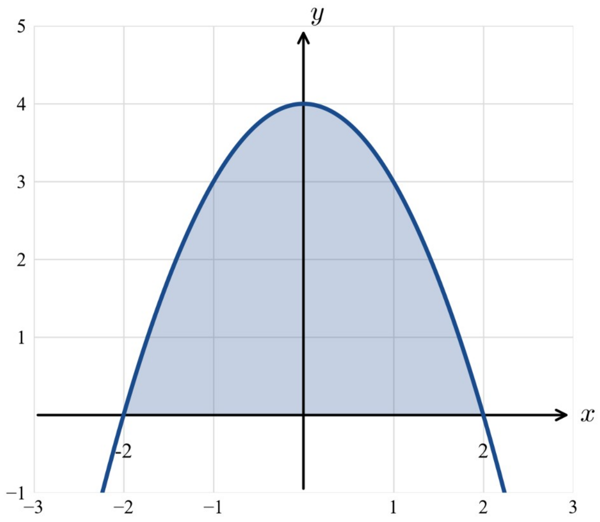
Example 4 — unscaffolded

Find the exact area of the region bounded by $y=4-x^2$ and the x-axis.

The curve cuts the x-axis where $4-x^2=0$, i.e. $x=-2$ and $x=2$, and $y\ge 0$ between them. So

$$\text{Area}=\int_{-2}^2 (4-x^2)\,dx=[4x-x^3/3]_{-2}^2=(8-8/3)-(-8+8/3)=\frac{32}{3}.$$

$\therefore$ the area is $\frac{32}{3}$ square units.



Practice questions

Scaffolded

Q1. Evaluate $\int_{0}^{1} 4x^3 \, dx$.

Q2. Evaluate $\int_{1}^{2} 6x^2 \, dx$.

Q3. Evaluate $\int_{-1}^{1} 3x^2 \, dx$.

Q4. Evaluate $\int_{0}^{3} (2x+1) \, dx$.

Q5. Find the area under $y=x^2$ from $x=0$ to $x=3$.

Q6. Find the area of the region bounded by $y=4x-x^2$ and the x-axis.

Unscaffolded

Evaluate each definite integral, or find the exact area, as required.

Q7. $\int_{0}^{2} x^3 \, dx$

Q8. $\int_{1}^{3} (x^2-1) \, dx$

Q9. $\int_{-2}^{1} (x+3) \, dx$

Q10. $\int_{0}^{1} (x^2+2x) \, dx$

Q11. $\int_{1}^{4} (2x-3) \, dx$

Q12. Find the area under $y=x^2+2$ from $x=1$ to $x=3$.

Q13. Find the area under $y=3x^2$ from $x=0$ to $x=2$.

Q14. Find the area of the region bounded by $y=9-x^2$ and the x-axis.

Q15. Find the area under $y=(x-2)^2$ from $x=0$ to $x=2$.

Q16. Given that $\int_0^3 f(x)\,dx=10$ and $\int_0^1 f(x)\,dx=3$, find $\int_1^3 f(x)\,dx$.

Q17. Evaluate $\int_2^6 x\,dx$.

Q18. Find the area of the region bounded by $y=3x-x^2$ and the x-axis.

Answers

Q1. 1. **Q2.** 14. **Q3.** 2. **Q4.** 12. **Q5.** 9. **Q6.** $\frac{32}{3}$. **Q7.** 4. **Q8.** $\frac{20}{3}$. **Q9.** $\frac{15}{2}$. **Q10.** $\frac{4}{3}$. **Q11.** 6. **Q12.** $\frac{38}{3}$. **Q13.** 8. **Q14.** 36. **Q15.** $\frac{8}{3}$. **Q16.** 7. **Q17.** 16. **Q18.** $\frac{9}{2}$.

Fully-worked solutions

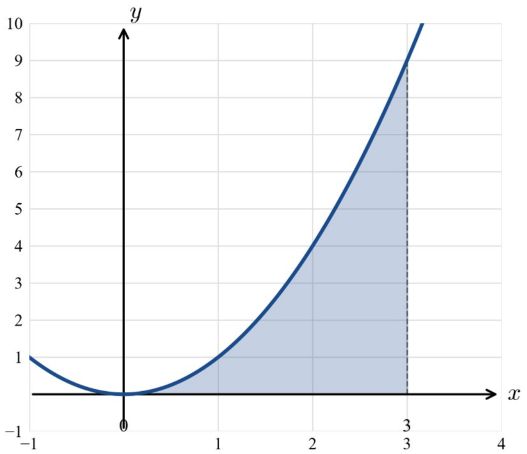
Q1. $\int_0^1 4x^3\,dx=[x^4]_0^1=1-0=1$.

Q2. $\int_1^2 6x^2\,dx=[2x^3]_1^2=16-2=14$.

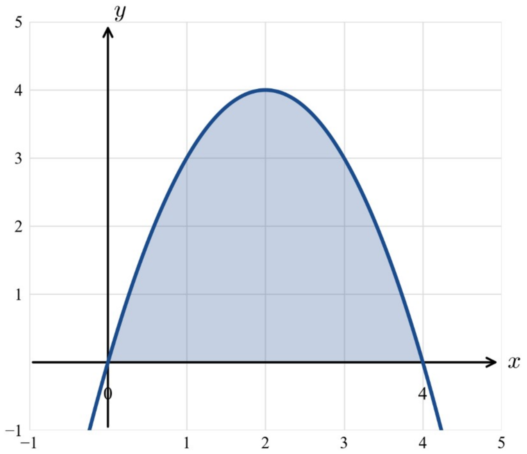
Q3. $\int_{-1}^1 3x^2\,dx=[x^3]_{-1}^1=1-(-1)=2$.

Q4. $\int_0^3 (2x+1)\,dx=[x^2+x]_0^3=(9+3)-0=12$.

Q5. Area = $\int_0^3 x^2\,dx=[x^3/3]_0^3=9-0=9$ square units.



Q6. $4x-x^2=x(4-x)=0$ at $x=0$ and $x=4$, with $y\geq 0$ between. Area $=\int_0^4 (4x-x^2)\,dx=[2x^2-x^3/3]_0^4=32-64/3=\frac{32}{3}$ square units.



Q7. $\int_0^2 x^3\,dx = [x^4/4]_0^2 = 16/4 - 0 = 4.$

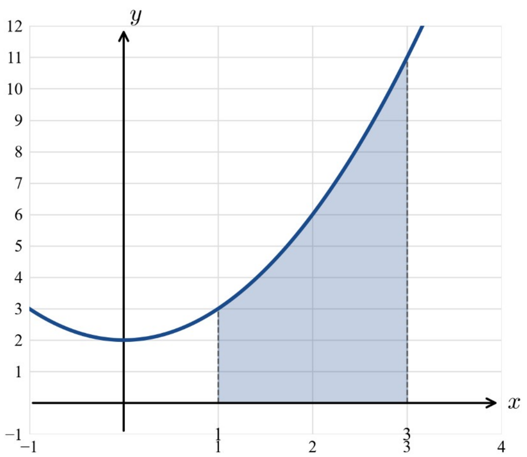
Q8. $\int_1^3 (x^2-1)\,dx = [x^3/3 - x]_1^3 = (9-3)-(1/3-1) = 6+2/3 = \dfrac{20}{3}.$

Q9. $\int_{-2}^1 (x+3)\,dx = [x^2/2+3x]_{-2}^1 = (1/2+3)-(2-6) = 7/2+4 = \dfrac{15}{2}.$

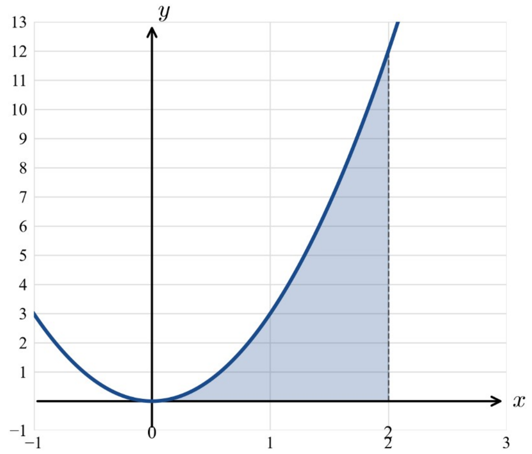
Q10. $\int_0^1 (x^2+2x)\,dx = [x^3/3+x^2]_0^1 = 1/3+1 = \dfrac{4}{3}.$

Q11. $\int_1^4 (2x-3)\,dx = [x^2-3x]_1^4 = (16-12)-(1-3) = 4+2 = 6.$

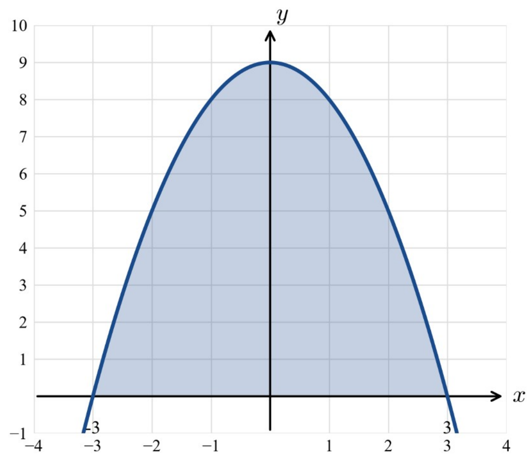
Q12. Area = $\int_1^3 (x^2+2)\,dx = [x^3/3+2x]_1^3 = (9+6)-(1/3+2) = 15-7/3 = \dfrac{38}{3}$ square units.



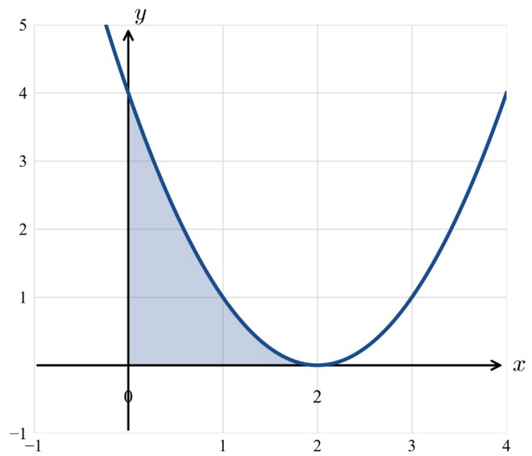
Q13. Area = $\int_0^2 3x^2\,dx = [x^3]_0^2 = 8$ square units.



Q14. $9-x^2=0$ at $x=\pm3$, with $y\ge0$ between. Area $=\int_{-3}^{3}(9-x^2)\,dx=[9x-x^3/3]_{-3}^{3}=(27-9)-(-27+9)=18+18=36$ square units.



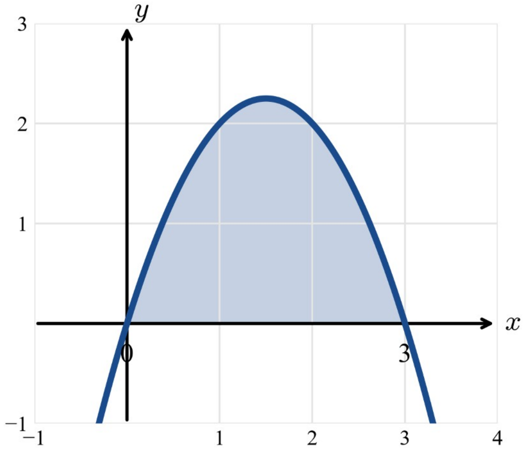
Q15. Area $=\int_{0}^{2}(x-2)^2\,dx=[(x-2)^3/3]_0^2=0-(-2)^3/3=\frac{8}{3}$ square units.



Q16. By additivity, $\int_0^3 f=\int_0^1 f+\int_1^3 f$, so $\int_1^3 f(x)\,dx=10-3=7$.

Q17. $\int_2^6 x\,dx=[x^2/2]_2^6=18-2=16$.

Q18. $3x - x^2 = x(3 - x) = 0$ at $x = 0$ and $x = 3$, with $y \geq 0$ between. Area $= \int_0^3 (3x - x^2)\,dx = [3x^2/2 - x^3/3]_0^3 = 27/2 - 9 = \frac{9}{2}$ square units.



Chapter 17 Integration — 17C Signed area

Units 3 & 4 preview. This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

Theory

The definite integral $\int_a^b f(x)\,dx$ measures **signed area**: a region **above** the x-axis counts as **positive** area, and a region **below** the x-axis counts as **negative** area. So the value of the integral is the **net signed area** between the curve and the x-axis.

Net signed area versus total area. Because areas below the axis are subtracted, the integral can be small — even zero — while there is still plenty of area between the curve and the axis. The **total area** treats every region as positive.

Finding the total area. To find the total area between $y=f(x)$ and the x-axis over $[a,b]$:

1. Find the x-intercepts of f **inside** the interval (solve $f(x)=0$).
2. **Split** the interval at those intercepts.
3. Integrate over each piece separately.
4. Add the **absolute values** of the pieces:

$$\text{total area}=\sum\left|\int_{\text{piece}} f(x)\,dx\right|.$$

A region entirely below the axis gives a **negative** integral; its **area** is the absolute value. If the curve only **touches** the axis without crossing, it stays on the one side and no splitting is needed there. For a polynomial this happens at a repeated factor of **even** multiplicity, such as $(x-2)^2$; a repeated factor of **odd** multiplicity, such as x^3 or $(x-2)^3$, still **crosses** the axis (a stationary point of inflection) and must be split there.

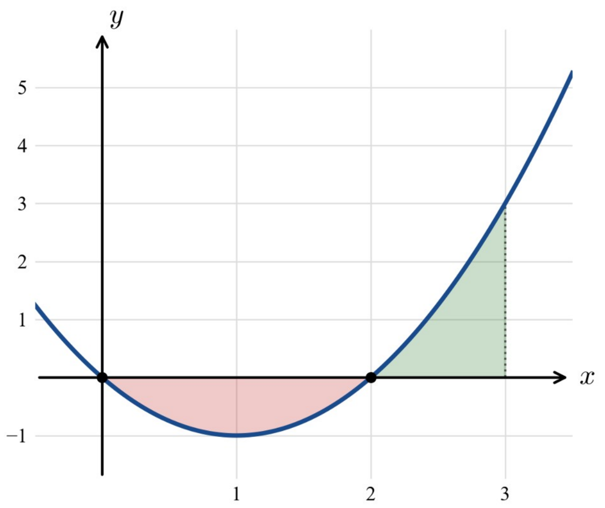
Worked examples

Example 1 — scaffolded

For $y=x^2-2x$: (a) evaluate $\int_0^3 (x^2-2x)\,dx$; (b) find the total area between the curve and the x-axis from $x=0$ to $x=3$.

Working	Comment
$x^2-2x=x(x-2)$, so the x-intercepts are $x=0$ and $x=2$. The curve is **below** the axis on $(0,2)$ and **above** on $(2,3)$.	Find where the curve crosses the axis inside the interval.
(a) $\int_0^3 (x^2-2x)\,dx=\left[\frac{x^3}{3}-x^2\right]_0^3=(9-9)-0=0.$	This is the **net signed area** — the area below cancels the area above.
(b) $\int_0^2 (x^2-2x)\,dx=\left[\frac{x^3}{3}-x^2\right]_0^2=\frac{8}{3}-4=-\frac{4}{3}$, so this area is $\frac{4}{3}$.	Below the axis → negative integral; take the absolute value.
$\int_2^3 (x^2-2x)\,dx=\left[\frac{x^3}{3}-x^2\right]_2^3=0-\left(-\frac{4}{3}\right)=\frac{4}{3}.$	Above the axis → positive.

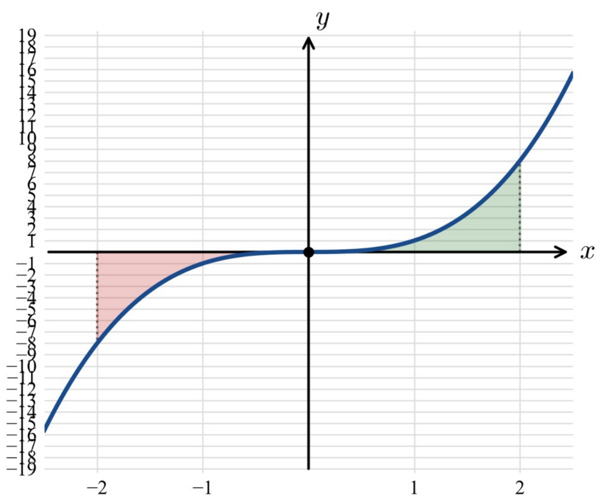
Working	Comment
Total area $=\frac{4}{3}+\frac{4}{3}=\frac{8}{3}$.	Add the absolute values.



Example 2 — scaffolded

For $y=x^3$: (a) evaluate $\int_{-2}^{2} x^3\,dx$; (b) find the total area between the curve and the x-axis from $x=-2$ to $x=2$.

Working	Comment
$x^3=0$ at $x=0$. The curve is **below** the axis on $(-2,0)$ and **above** on $(0,2)$.	Locate the intercept.
(a) $\int_{-2}^{2} x^3\,dx=\left[\frac{x^4}{4}\right]_{-2}^{2}=4-4=0$.	x^3 is an odd function, so the signed areas cancel.
(b) $\int_{-2}^{0} x^3\,dx=0-4=-4$ (area 4); $\int_{0}^{2} x^3\,dx=4-0=4$ (area 4).	Integrate each side separately.
Total area $=4+4=8$.	Add the absolute values.



Example 3 — unscaffolded

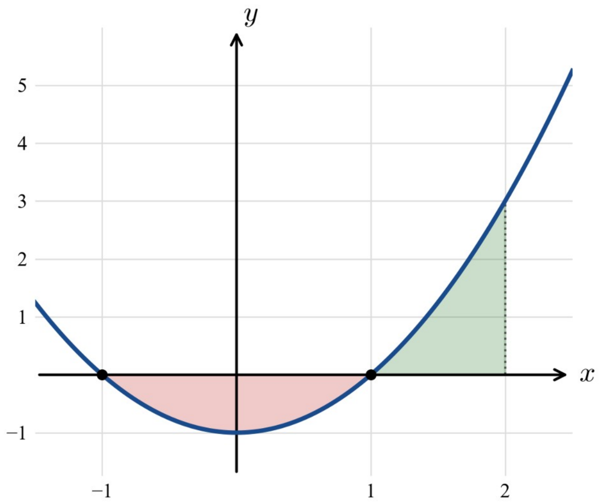
Evaluate $\int_{-1}^{2} (x^2-1)\,dx$ and find the total area between $y=x^2-1$ and the x-axis over $[-1,2]$.

Intercepts: $x^2-1=0 \Rightarrow x=\pm 1$; inside the interval the crossing is at $x=1$. Below the axis on $(-1,1)$, above on $(1,2)$.

$$\int_{-1}^{2}(x^2-1)\,dx=\left[\frac{x^3}{3}-x\right]_{-1}^{2}=\left(\frac{8}{3}-2\right)-\left(-\frac{1}{3}+1\right)=\frac{2}{3}-\frac{2}{3}=0.$$

$\int_{-1}^{1}(x^2-1)\,dx=-\frac{4}{3}$ (area $4/3$) and $\int_{1}^{2}(x^2-1)\,dx=\frac{4}{3}$.

$\therefore$ the net signed area is 0 and the total area is $\frac{4}{3}+\frac{4}{3}=\frac{8}{3}$.



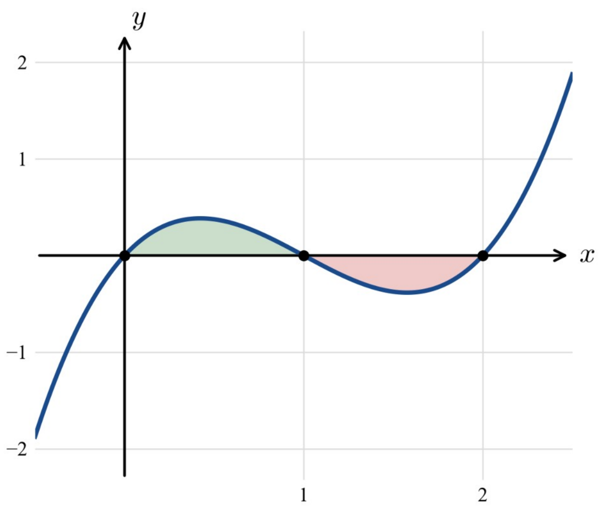
Example 4 — unscaffolded

Find the total area between $y=x(x-1)(x-2)$ and the x-axis from $x=0$ to $x=2$.

Expanding, $y=x^3-3x^2+2x$, with intercepts $x=0,1,2$. The curve is **above** the axis on $(0,1)$ and **below** on $(1,2)$.

$$\int_{0}^{1}(x^3-3x^2+2x)\,dx=\left[\frac{x^4}{4}-x^3+x^2\right]_{0}^{1}=\frac{1}{4},\ \int_{1}^{2}(x^3-3x^2+2x)\,dx=-\frac{1}{4}(\text{area } 1/4).$$

$\therefore$ total area $=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. (The net signed area $\int_{0}^{2}=0$.)



Practice questions

Scaffolded

Q1. For $y=x^2-4x$: (a) evaluate $\int_0^4 (x^2-4x)\,dx$; (b) find the total area between the curve and the x-axis from $x=0$ to $x=4$.

Q2. For $y=x^3-x$: (a) evaluate $\int_{-1}^{1} (x^3-x)\,dx$; (b) find the total area between the curve and the x-axis over $[-1,1]$.

Q3. For $y=4-x^2$: (a) evaluate $\int_0^3 (4-x^2)\,dx$; (b) find the total area between the curve and the x-axis from $x=0$ to $x=3$.

Q4. Find the total area between $y=x^2-x-2$ and the x-axis from $x=-1$ to $x=2$.

Q5. Find the total area between $y=x^2-3x$ and the x-axis from $x=0$ to $x=3$.

Q6. For $y=x^3-4x$: (a) evaluate $\int_{-2}^{2} (x^3-4x)\,dx$; (b) find the total area between the curve and the x-axis over $[-2,2]$.

Unscaffolded

For each of the following, evaluate the definite integral (the net signed area) **and** find the total area between the curve and the x-axis over the given interval.

Q7. $y=x^2-9$, from $x=0$ to $x=3$

Q8. $y=x^2-2x-3$, from $x=-1$ to $x=3$

Q9. $y=x^3$, from $x=-1$ to $x=2$

Q10. $y=1-x^2$, from $x=0$ to $x=2$

Q11. $y=x^2-5x+6$, from $x=0$ to $x=3$

Q12. $y=x^3-4x^2+4x$, from $x=0$ to $x=3$

Q13. $y=3-x$, from $x=0$ to $x=5$

Q14. $y=x^2-1$, from $x=0$ to $x=2$

Q15. $y=-x^2+4$, from $x=-3$ to $x=2$

Q16. $y=x^3-9x$, from $x=0$ to $x=3$

Q17. $y=(x-1)(x-2)(x-3)$, from $x=1$ to $x=3$

Q18. A particle moves in a straight line with velocity $v(t)=t^2-4$ metres per second for $0\le t\le 3$. (a) Find its displacement over this time ($\int_0^3 v\,dt$). (b) Find the total distance travelled.

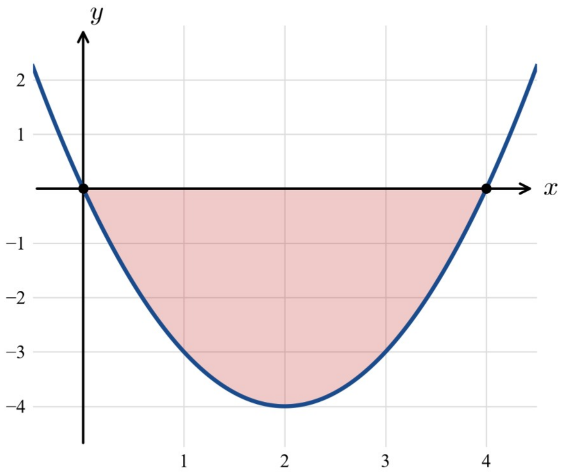
Answers

Each answer gives the definite integral (net signed area) then the total area.

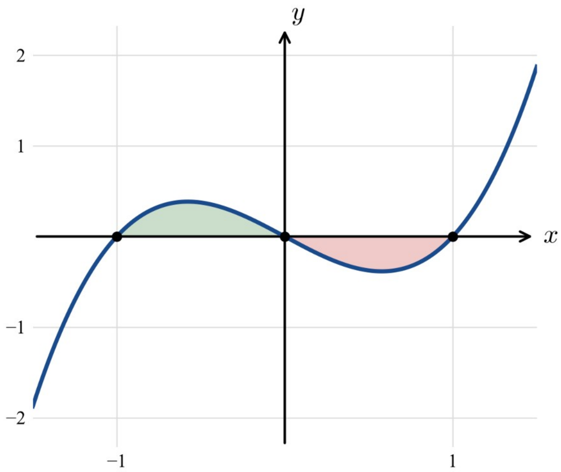
Q1. $-\frac{32}{3}$; total $\frac{32}{3}$. **Q2.** 0; total $\frac{1}{2}$. **Q3.** 3; total $\frac{23}{3}$. **Q4.** integral $-\frac{9}{2}$; total area $\frac{9}{2}$. **Q5.** integral $-\frac{9}{2}$; total area $\frac{9}{2}$. **Q6.** 0; total 8. **Q7.** -18; total 18. **Q8.** $-\frac{32}{3}$; total $\frac{32}{3}$. **Q9.** $\frac{15}{4}$; total $\frac{17}{4}$. **Q10.** $-\frac{2}{3}$; total 2. **Q11.** $\frac{9}{2}$; total $\frac{29}{6}$. **Q12.** $\frac{9}{4}$; total $\frac{9}{4}$ (the curve touches the axis at $x=2$ and stays above it). **Q13.** $\frac{5}{2}$; total $\frac{13}{2}$. **Q14.** $\frac{2}{3}$; total 2. **Q15.** $\frac{25}{3}$; total 13. **Q16.** $-\frac{81}{4}$; total $\frac{81}{4}$. **Q17.** 0; total $\frac{1}{2}$. **Q18.** displacement -3 m; total distance $\frac{23}{3}$ m.

Fully-worked solutions

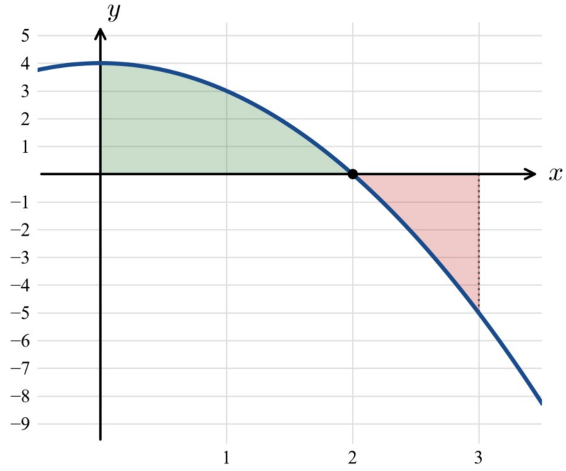
Q1. $y=x^2-4x=x(x-4)$; intercepts $x=0,4$ — the whole interval lies **below** the axis.
$\int_0^4 (x^2-4x)\,dx=\left[\frac{x^3}{3}-2x^2\right]_0^4=\frac{64}{3}-32=-\frac{32}{3}$. ∴ total area $=\frac{32}{3}$.



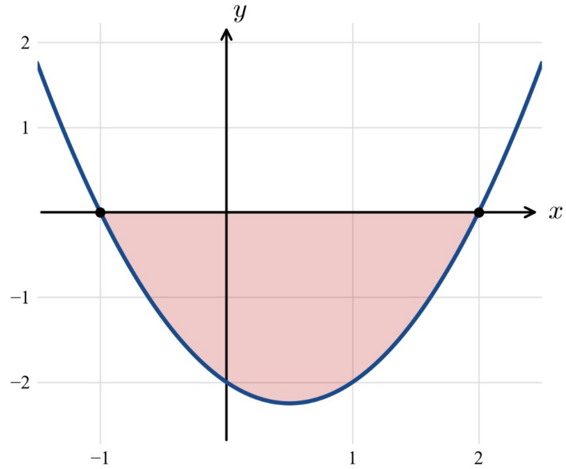
Q2. $y=x^3-x=x(x-1)(x+1)$; crossing at $x=0$. Above on $(-1,0)$, below on $(0,1)$. $\int_{-1}^{1} (x^3-x)\,dx=0$; $\int_{-1}^{0}=\frac{1}{4}$, $\int_0^1=-\frac{1}{4}$. ∴ total area $=\frac{1}{2}$.



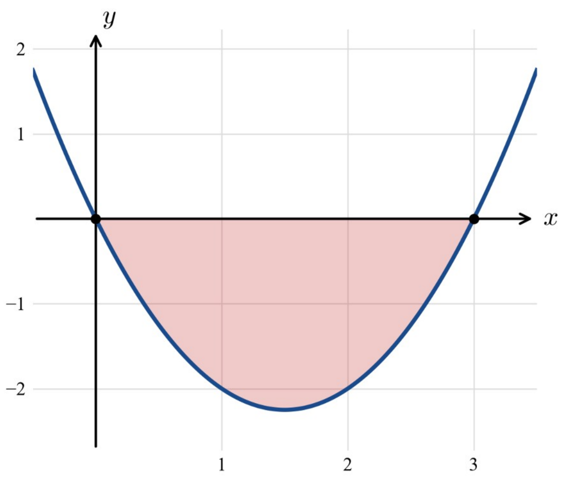
Q3. $y=4-x^2$; crossing at $x=2$. Above on $(0,2)$, below on $(2,3)$. $\int_0^3 (4-x^2)\,dx=\left[4x-\frac{x^3}{3}\right]_0^3=12-9=3$. $\int_0^2=\frac{16}{3}$, $\int_2^3=-\frac{7}{3}$. $\therefore$ total area $=\frac{16}{3}+\frac{7}{3}=\frac{23}{3}$.



Q4. $y=x^2-x-2=(x-2)(x+1)$; the interval $[-1,2]$ runs between the intercepts, so the curve is **below** the axis throughout. $\int_{-1}^{2} (x^2-x-2)\,dx=\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2}=-\frac{10}{3}-\frac{7}{6}=-\frac{9}{2}$. $\therefore$ total area $=\frac{9}{2}$.

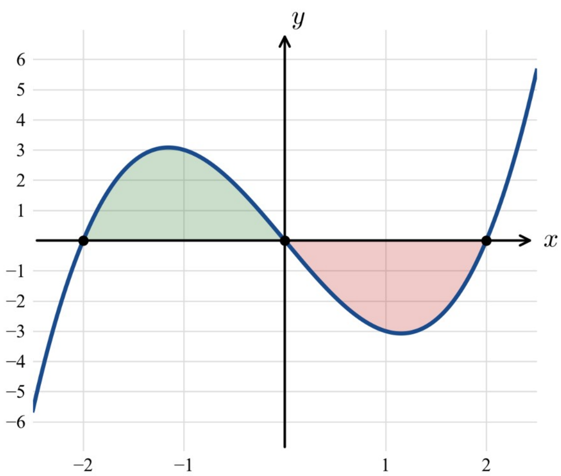


Q5. $y=x^2-3x=x(x-3)$; below the axis on all of $[0,3]$. $\int_0^3 (x^2-3x)\,dx=\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^3=9-\frac{27}{2}=-\frac{9}{2}$. $\therefore$ total area $=\frac{9}{2}$.



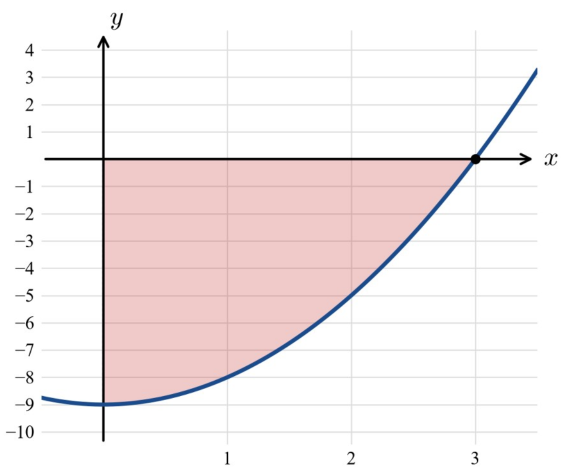
Q6. $y=x^3-4x=x(x-2)(x+2)$; crossing at $x=0$. Above on $(-2,0)$, below on $(0,2)$. $\int_{-2}^{2}(x^3-4x)\,dx=0$;

$\int_{-2}^{0}=4$, $\int_{0}^{2}=-4$. $\therefore$ total area $=4+4=8$.



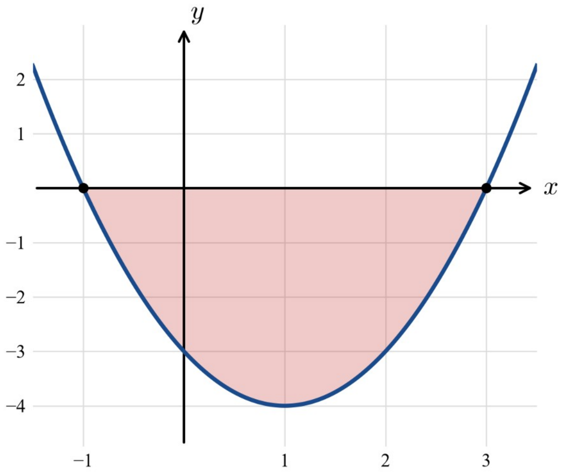
Q7. $y=x^2-9$; below the axis on all of $[0,3]$ (crossing at the endpoint $x=3$).

$\int_{0}^{3}(x^2-9)\,dx=\left[\frac{x^3}{3}-9x\right]_0^3=9-27=-18$. $\therefore$ total area $=18$.

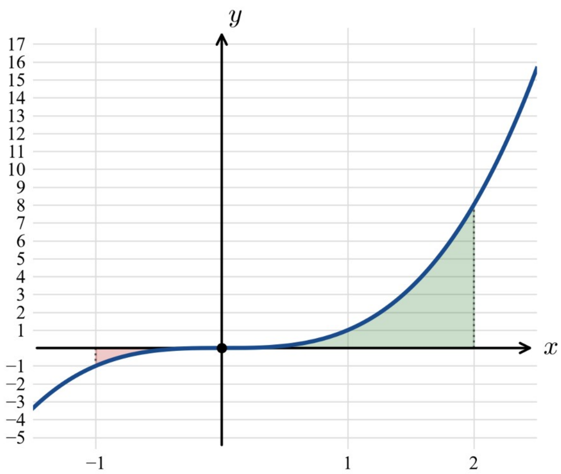


Q8. $y=x^2-2x-3=(x-3)(x+1)$; below the axis on all of $[-1,3]$ (crossings at the endpoints).

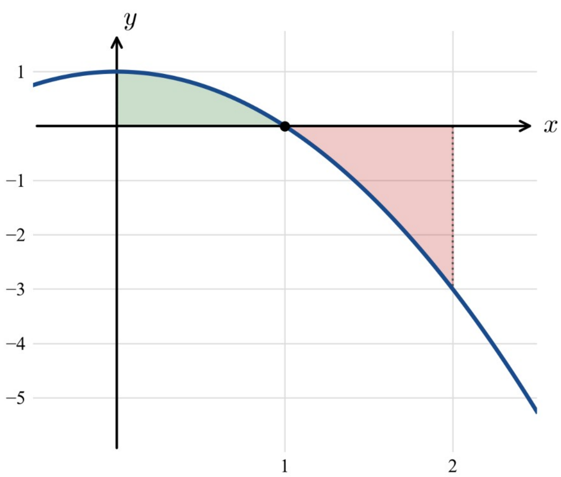
$\int_{-1}^{3}(x^2-2x-3)\,dx=\left[\frac{x^3}{3}-x^2-3x\right]_{-1}^{3}=-9-\frac{5}{3}=-\frac{32}{3}$. $\therefore$ total area $=\frac{32}{3}$.



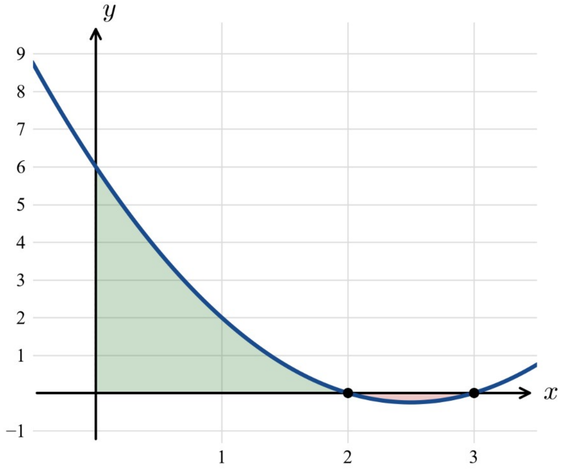
Q9. $y=x^3$; crossing at $x=0$. Below on $(-1,0)$, above on $(0,2)$. $\int_{-1}^{2} x^3\,dx=\left[\frac{x^4}{4}\right]_{-1}^{2}=4-\frac{1}{4}=\frac{15}{4}$. $\int_{-1}^{0}=-\frac{1}{4}$ (area 1/4), $\int_{0}^{2}=4$. $\therefore$ total area $=\frac{1}{4}+4=\frac{17}{4}$.



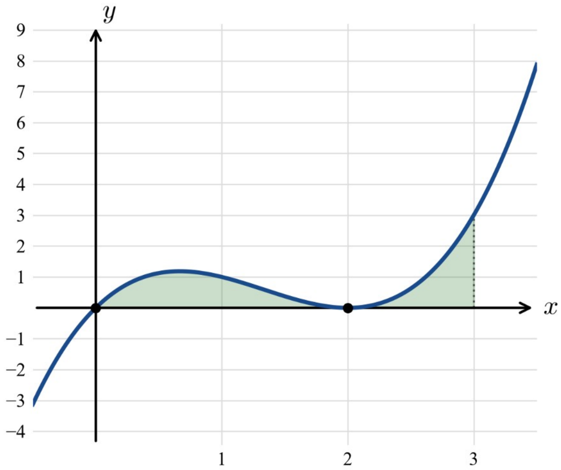
Q10. $y=1-x^2$; crossing at $x=1$. Above on $(0,1)$, below on $(1,2)$. $\int_{0}^{2}(1-x^2)\,dx=\left[x-\frac{x^3}{3}\right]_{0}^{2}=2-\frac{8}{3}=-\frac{2}{3}$. $\int_{0}^{1}=\frac{2}{3}$, $\int_{1}^{2}=-\frac{4}{3}$. $\therefore$ total area $=\frac{2}{3}+\frac{4}{3}=2$.



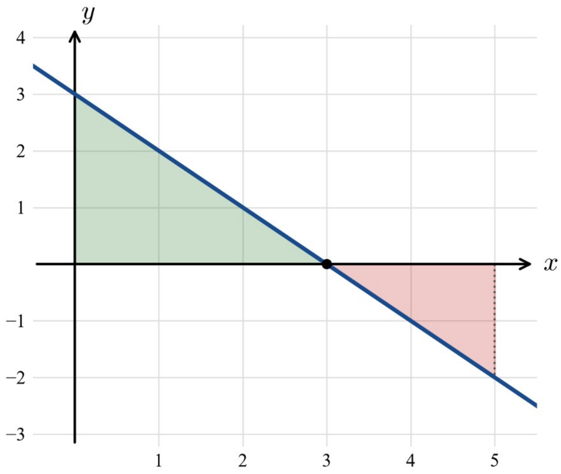
Q11. $y=x^2-5x+6=(x-2)(x-3)$; crossing at $x=2$ inside the interval. Above on $(0,2)$, below on $(2,3)$. $\int_0^3 (x^2-5x+6)\,dx=\left[\frac{x^3}{3}-\frac{5x^2}{2}+6x\right]_0^3=\frac{9}{2}$. $\int_0^2=\frac{14}{3}$, $\int_2^3=-\frac{1}{6}$. $\therefore$ total area $=\frac{14}{3}+\frac{1}{6}=\frac{29}{6}$.



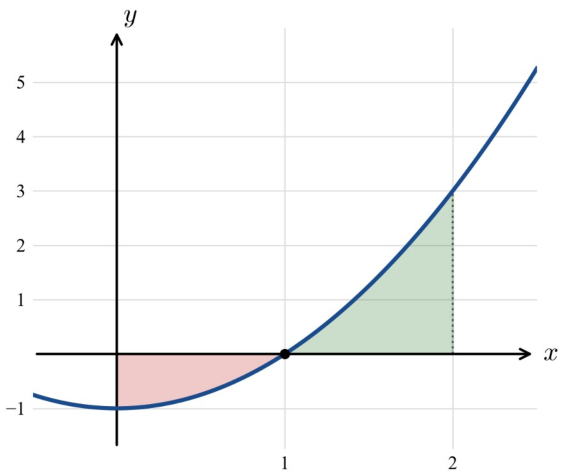
Q12. $y=x^3-4x^2+4x=x(x-2)^2$; the repeated factor $(x-2)^2$ has **even** multiplicity, so the curve **touches** the axis at $x=2$ without crossing, staying **above** the axis on $(0,3)$. $\int_0^3 (x^3-4x^2+4x)\,dx=\left[\frac{x^4}{4}-\frac{4x^3}{3}+2x^2\right]_0^3=\frac{9}{4}$. Since the curve does not cross, the signed area **equals** the total area $=\frac{9}{4}$.



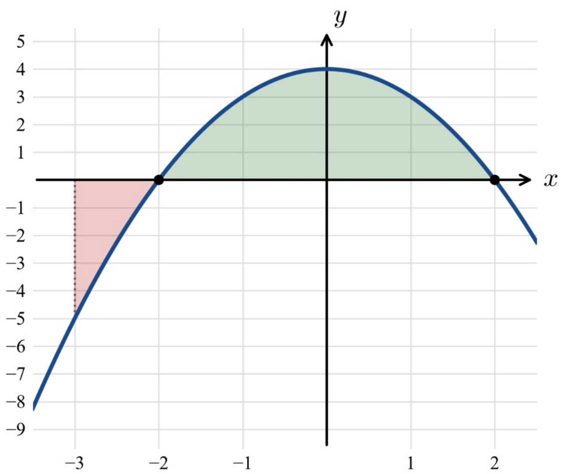
Q13. $y=3-x$; crossing at $x=3$. Above on $(0,3)$, below on $(3,5)$. $\int_0^5 (3-x)\,dx=\left[3x-\frac{x^2}{2}\right]_0^5=15-\frac{25}{2}=\frac{5}{2}$. $\int_0^3=\frac{9}{2}$, $\int_3^5=-2$. $\therefore$ total area $=\frac{9}{2}+2=\frac{13}{2}$.



Q14. $y=x^2-1$; crossing at $x=1$. Below on $(0,1)$, above on $(1,2)$. $\int_0^2 (x^2-1)\,dx=\left[\frac{x^3}{3}-x\right]_0^2=\frac{8}{3}-2=\frac{2}{3}$. $\int_0^1=-\frac{2}{3}$, $\int_1^2=\frac{4}{3}$. $\therefore$ total area $=\frac{2}{3}+\frac{4}{3}=2$.

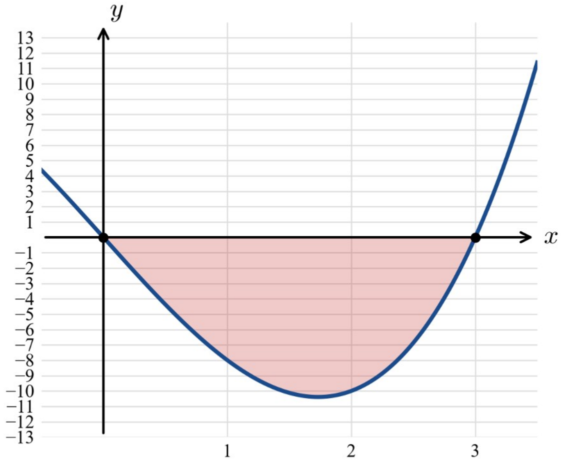


Q15. $y=-x^2+4$; crossing at $x=-2$ inside the interval. Below on $(-3,-2)$, above on $(-2,2)$. $\int_{-3}^{2} (-x^2+4)\,dx=\left[-\frac{x^3}{3}+4x\right]_{-3}^{2}=\frac{16}{3}-(-3)=\frac{25}{3}$. $\int_{-3}^{-2}=-\frac{7}{3}$, $\int_{-2}^{2}=\frac{32}{3}$. $\therefore$ total area $=\frac{7}{3}+\frac{32}{3}=13$.



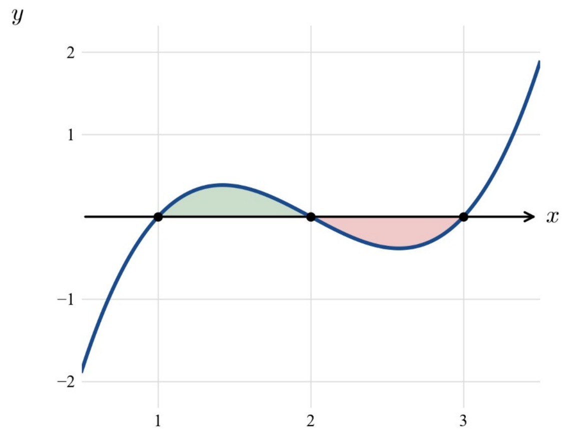
Q16. $y=x^3-9x=x(x-3)(x+3)$; below the axis on all of $[0,3]$ (crossings at the endpoints).

$\int_0^3 (x^3-9x)\,dx=\left[\frac{x^4}{4}-\frac{9x^2}{2}\right]_0^3=\frac{81}{4}-\frac{81}{2}=-\frac{81}{4}$. $\therefore$ total area $=\frac{81}{4}$.

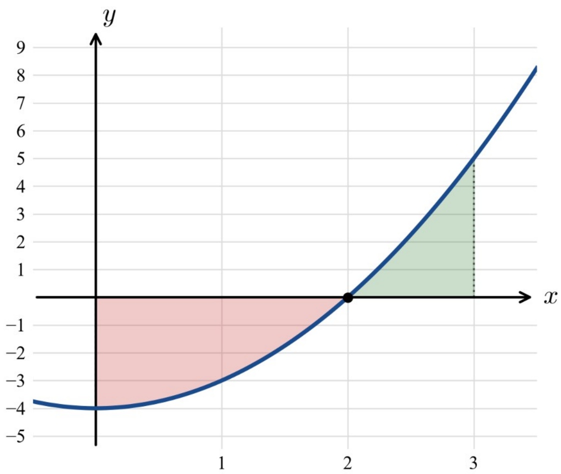


Q17. $y=(x-1)(x-2)(x-3)$; crossing at $x=2$ inside the interval. Above on $(1,2)$, below on $(2,3)$.

$\int_1^3 (x-1)(x-2)(x-3)\,dx=0$; $\int_1^2 =\frac{1}{4}$, $\int_2^3 =-\frac{1}{4}$. $\therefore$ total area $=\frac{1}{2}$.



Q18. $v(t)=t^2-4=(t-2)(t+2)$; $v=0$ at $t=2$. The velocity is **negative** on $(0,2)$ (moving backwards) and **positive** on $(2,3)$. (a) Displacement $=\int_0^3 (t^2-4)\,dt=\left[\frac{t^3}{3}-4t\right]_0^3=9-12=-3$ m. (b) $\int_0^2 (t^2-4)\,dt=-\frac{16}{3}$ (distance $16/3$), $\int_2^3 (t^2-4)\,dt=\frac{7}{3}$. $\therefore$ total distance $=\frac{16}{3}+\frac{7}{3}=\frac{23}{3}$ m.



Topic 5 Test A — Technology-active

Topic 5 covers Chapters 16–17. Reading time: 5 minutes. Writing time: 50 minutes. Total: 50 marks. One bound reference, one approved CAS calculator or CAS software, and one scientific calculator are permitted. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown. This topic is a Units 3 & 4 preview — the test is an ungraded extension. Some items are within the Units 1 & 2 study design (differentiation and anti-differentiation of polynomial functions, and curve sketching with derivatives); the chain rule, rational and negative powers, and definite integrals with their applications are beyond it.

Question 1 (3 marks)

Differentiate $y=(2x^2-3)^4$ and find the gradient of the curve at $x=1$.

Question 2 (3 marks)

Differentiate $y=3\sqrt{x}-\frac{2}{\sqrt{x}}$, giving your answer with positive indices or in surd form.

Question 3 (4 marks)

A function satisfies $f'(x)=3\sqrt{x}-\frac{4}{x^3}$ and $f(1)=0$. Find $f(x)$.

Question 4 (4 marks)

For $y=x^3+6x^2-1$, find $\frac{d^2y}{dx^2}$ and hence the coordinates of the point of inflection. State where the curve is concave up.

Question 5 (6 marks)

Sketch the graph of $y=x^3-75x$, labelling the axis intercepts, the stationary points (with their nature) and the point of inflection.

Question 6 (3 marks)

Estimate the area under $y=x^2+1$ from $x=0$ to $x=4$ using four rectangles of width 1 and the right endpoint of each strip. State whether this is an over- or under-estimate.

Question 7 (3 marks)

Evaluate $\int_0^3 (x^2-2x+2)\,dx$.

Question 8 (4 marks)

Find the area bounded by the curve $y=5x-x^2$ and the x-axis.

Question 9 (5 marks)

For $y=x^2-4$ over $0\le x\le 3$:

a. evaluate $\int_0^3 (x^2-4)\,dx$; 2 marks

b. find the total area between the curve and the x-axis. 3 marks

Question 10 (8 marks)

A particle moves in a straight line with velocity $v(t) = 3t^2 - 24t + 21$ m/s, for $t \geq 0$.

a. Find the acceleration $a(t)$. 1 mark

b. Find the times at which the particle is at rest. 2 marks

c. Find the displacement of the particle over the first 3 seconds. 2 marks

d. Find the total distance travelled in the first 3 seconds. 3 marks

Question 11 (7 marks)

The cross-section of a landscaped embankment is modelled by $h(x) = 3\sqrt{x} - 3/4\,x$, for $0 \leq x \leq 16$, where h metres is the height of the embankment at a horizontal distance of x metres from its left edge.

a. Find $h'(x)$. 1 mark

b. Find the value of x at which the embankment is highest, and state the maximum height. 2 marks

c. Estimate the area of the cross-section using four rectangles of width 4 and the left endpoint of each strip. Give your answer in square metres, correct to two decimal places. 2 marks

d. Use a definite integral to find the exact area of the cross-section. 2 marks

End of Test A

Topic 5 Test A — solutions

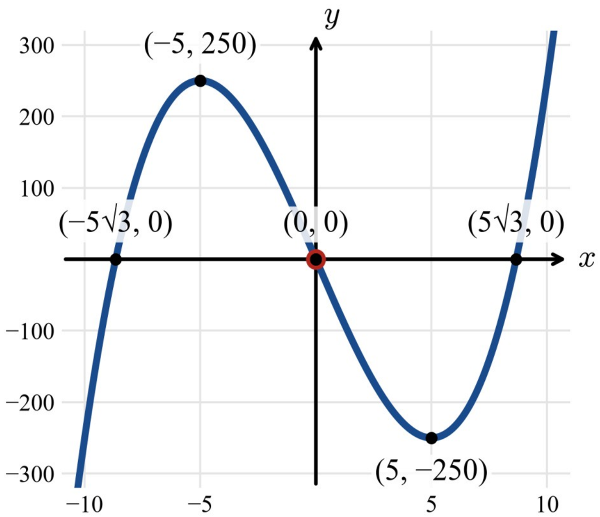
Q1. $\frac{dy}{dx}=4(2x^2-3)^3\cdot 4x=16x(2x^2-3)^3$. At $x=1$: $16(1)(2-3)^3=16(-1)=-16$.

Q2. $y=3x^{1/2}-2x^{-1/2}$, so $\frac{dy}{dx}=3/2x^{-1/2}+x^{-3/2}=\frac{3}{2\sqrt{x}}+\frac{1}{x\sqrt{x}}$.

Q3. $f(x)=\int(3x^{1/2}-4x^{-3})dx=2x^{3/2}+2x^{-2}+c$. Using $f(1)=0$: $2+2+c=0\Rightarrow c=-4$. $\therefore f(x)=2x^{3/2}+\frac{2}{x^2}-4$.

Q4. $\frac{dy}{dx}=3x^2+12x$, $\frac{d^2y}{dx^2}=6x+12=0\Rightarrow x=-2$; $y(-2)=-8+24-1=15$. Point of inflection $(-2,15)$. Concave up for $x>-2$ (where $y''>0$).

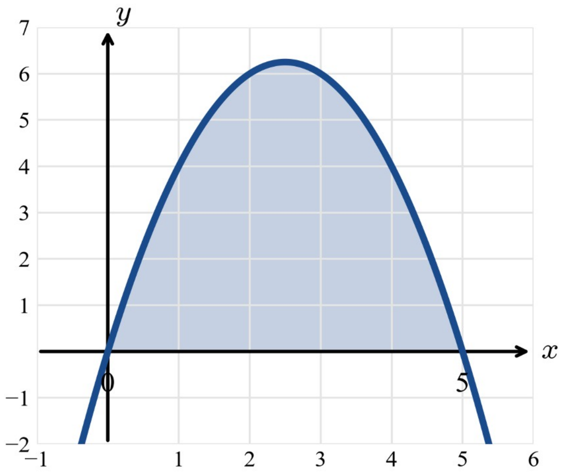
Q5. Intercepts: $x(x^2-75)=0\Rightarrow(0,0)$ and $(\pm5\sqrt{3},0)$. $y'=3x^2-75=3(x+5)(x-5)$, so stationary points at $x=-5$ and $x=5$: local **maximum** $(-5,250)$ and local **minimum** $(5,-250)$. $y''=6x=0\Rightarrow x=0$: inflection $(0,0)$.



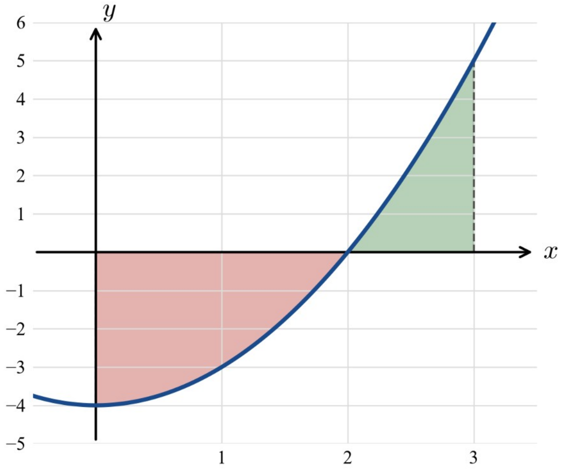
Q6. $\Delta x=1$; right endpoints $x=1,2,3,4$ give $f=2,5,10,17$. Estimate $=1(2+5+10+17)=34$. Since f is increasing, this is an **over-estimate**.

Q7. $\left[\frac{x^3}{3}-x^2+2x\right]_0^3=(9-9+6)-0=6$.

Q8. x-intercepts $5x-x^2=x(5-x)=0\Rightarrow x=0,5$; the curve is above the axis between them. Area $=\int_0^5(5x-x^2)dx=\left[5x^2/2-\frac{x^3}{3}\right]_0^5=125/2-\frac{125}{3}=\frac{125}{6}$ square units.



Q9. (a) $\left[\frac{x^3}{3}-4x\right]_0^3=(9-12)-0=-3$. (b) The curve crosses the axis at $x=2$. $\int_0^2 (x^2-4)\,dx=-\frac{16}{3}$ (below) and $\int_2^3 (x^2-4)\,dx=\frac{7}{3}$ (above). Total area $=\frac{16}{3}+\frac{7}{3}=\frac{23}{3}$ square units.



Q10. (a) $a(t)=\frac{dv}{dt}=6t-24$ m/s². (b) At rest: $3t^2-24t+21=3(t-1)(t-7)=0 \Rightarrow t=1$ or $t=7$ s. (c) Displacement $=\int_0^3 (3t^2-24t+21)\,dt=\left[t^3-12t^2+21t\right]_0^3=27-108+63=-18$ m. (d) $v>0$ on $(0,1)$ and $v<0$ on $(1,3)$: $\int_0^1 v\,dt=10$, $\int_1^3 v\,dt=-28$. ∴ total distance $=|10|+|-28|=38$ m.

Q11. (a) $h(x)=3x^{1/2}-3/4\,x$, so $h'(x)=3/2\,x^{-1/2}-3/4=\frac{3}{2\sqrt{x}}-\frac{3}{4}$. (b) $h'(x)=0 \Rightarrow \frac{3}{2\sqrt{x}}=\frac{3}{4} \Rightarrow \sqrt{x}=2 \Rightarrow x=4$. Since $h'(x)>0$ for $0<x<4$ and $h'(x)<0$ for $x>4$, this is a maximum. $h(4)=3(2)-3=3$. ∴ the embankment is highest at $x=4$, with maximum height 3 m. (c) $\Delta x=4$; left endpoints $x=0,4,8,12$ give $h=0,3,6\sqrt{2}-6,6\sqrt{3}-9$. Estimate $=4(0+3+6\sqrt{2}-6+6\sqrt{3}-9)=24\sqrt{2}+24\sqrt{3}-48\approx 27.51$ m². (d) Area $=\int_0^{16} (3x^{1/2}-3/4\,x)\,dx=\left[2x^{3/2}-3/8\,x^2\right]_0^{16}=2(64)-3/8(256)=128-96=32$ m².

Topic 5 Test B — Technology-free

Topic 5 covers Chapters 16–17. Reading time: 5 minutes. Writing time: 45 minutes. Total: 40 marks. Students are not permitted to bring any technology (calculators or software), or notes of any kind. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown. This topic is a Units 3 & 4 preview — the test is an ungraded extension. Some items are within the Units 1 & 2 study design (differentiation and anti-differentiation of polynomial functions, and curve sketching with derivatives); the chain rule, rational and negative powers, and definite integrals with their applications are beyond it.

Question 1 (2 marks)

Differentiate $y=(3x-1)^5$.

Question 2 (2 marks)

Differentiate $y=\sqrt{2x+1}$.

Question 3 (2 marks)

Find $\int (3x^2-8x+5)\,dx$.

Question 4 (2 marks)

Find $\int \frac{4}{\sqrt{x}}\,dx$.

Question 5 (3 marks)

For $y=2x^3-6x^2+1$, find the coordinates of the point of inflection.

Question 6 (5 marks)

For $y=x^3-48x$, find and classify the stationary points.

Question 7 (5 marks)

Hence sketch $y=x^3-48x$, labelling the stationary points and the x-intercepts.

Question 8 (3 marks)

Evaluate $\int_1^2 (3x^2-2x)\,dx$.

Question 9 (4 marks)

Find the area between the curve $y=x^2-16$ and the x-axis from $x=-4$ to $x=4$.

Question 10 (5 marks)

For $y=x^3-2x$ over $-1\le x\le 1$:

a. evaluate $\int_{-1}^{1} (x^3-2x)\,dx$; 2 marks

b. find the total area between the curve and the x-axis. 3 marks

Question 11 (3 marks)

Given $f'(x)=4x-6$ and $f(2)=1$, find $f(x)$.

Question 12 (4 marks)

A particle has velocity $v(t)=2t-8$ m/s for $t\geq 0$. Find the total distance it travels in the first 5 seconds.

End of Test B

Topic 5 Test B — solutions

Q1. $\dfrac{dy}{dx}=5(3x-1)^4\cdot 3=15(3x-1)^4$.

Q2. $y=(2x+1)^{1/2}$, so $\dfrac{dy}{dx}=1/2(2x+1)^{-1/2}\cdot 2=\dfrac{1}{\sqrt{2x+1}}$.

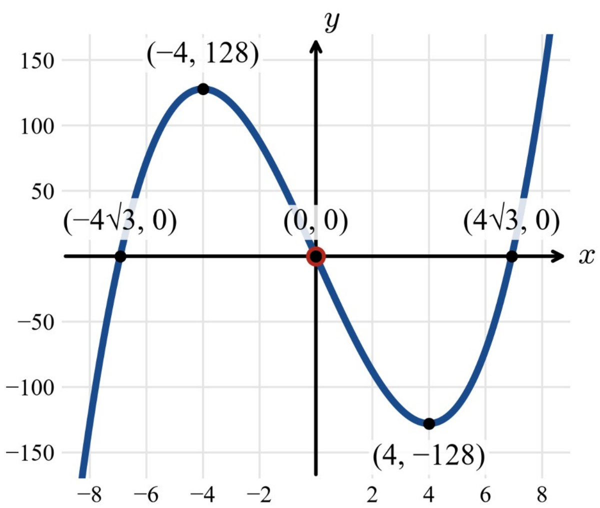
Q3. $\dfrac{3x^3}{3}-\dfrac{8x^2}{2}+5x+c=x^3-4x^2+5x+c$.

Q4. $\int 4x^{-1/2}\,dx=\dfrac{4x^{1/2}}{1/2}+c=8\sqrt{x}+c$.

Q5. $y'=6x^2-12x$, $y''=12x-12=0\Rightarrow x=1$; $y(1)=2-6+1=-3$. Point of inflection $(1,-3)$.

Q6. $y'=3x^2-48=0\Rightarrow x=\pm4$. $y''=6x$: $y''(4)=24>0$ so $(4,-128)$ is a local **minimum**; $y''(-4)=-24<0$ so $(-4,128)$ is a local **maximum**.

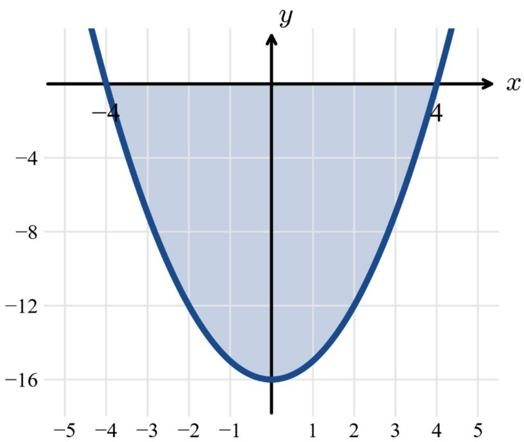
Q7. x-intercepts: $x(x^2-48)=0\Rightarrow x=0,\pm4\sqrt{3}$. With the stationary points from Q6:



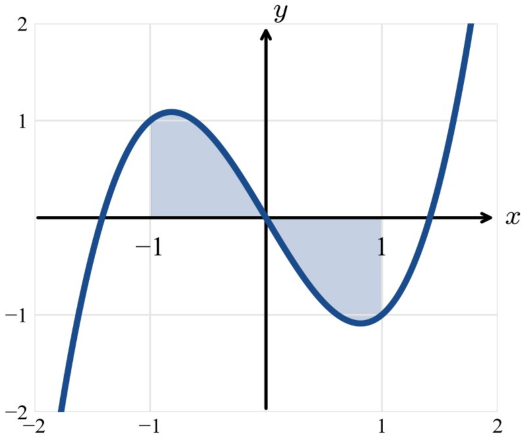
Q8. $\left[x^3-x^2\right]_1^2=(8-4)-(1-1)=4$.

Q9. On $(-4,4)$, $x^2-16<0$ (the curve is below the axis).

$$\int_{-4}^{4}(x^2-16)\,dx=\left[\frac{x^3}{3}-16x\right]_{-4}^{4}=(64/3-64)-(-64/3+64)=-\frac{256}{3}.\ \therefore\ \text{area}=\frac{256}{3}\ \text{square units.}$$



Q10. (a) $y = x^3 - 2x$ is an odd function, so $\int_{-1}^{1} (x^3 - 2x)\,dx = 0$. (b) The curve crosses the axis at $x = 0$. $\int_{-1}^{0} (x^3 - 2x)\,dx = 3/4$ (above) and $\int_{0}^{1} (x^3 - 2x)\,dx = -3/4$ (below). Total area $= \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$ square units.



Q11. $f(x) = 2x^2 - 6x + c$. Using $f(2) = 1$: $8 - 12 + c = 1 \Rightarrow c = 5$. $\therefore f(x) = 2x^2 - 6x + 5$.

Q12. $v = 0$ at $t = 4$. $\int_{0}^{4} (2t - 8)\,dt = [t^2 - 8t]_0^4 = -16$ (moving backwards) and $\int_{4}^{5} (2t - 8)\,dt = (25 - 40) - (16 - 32) = 1$. $\therefore$ total distance $= |-16| + |1| = 17$ m.