# Mathematical Methods Units 1 & 2

## Chapter 16 — Calculus beyond polynomials (Units 3 & 4 preview)



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## Chapter 16 Calculus beyond polynomials — 16A The chain rule

**Units 3 & 4 preview.** This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

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### Theory

Some functions are **composite** — one function applied to another, such as $y=(3x+2)^4$ (the "outer" function is the fourth power; the "inner" function is $3x+2$). To differentiate a composite function we use the **chain rule**.

**The chain rule.** If $y$ is a function of $u$, and $u$ is a function of $x$, then

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

Equivalently, in function notation, if $y=f(g(x))$ then

$$\frac{dy}{dx}=f'(g(x))\times g'(x)$$

— "differentiate the outer function (keeping the inside unchanged), then multiply by the derivative of the inside".

**Power of a function.** The most common case is a power of a polynomial, $y=(g(x))^n$:

$$\frac{dy}{dx}=n(g(x))^{n-1}\times g'(x).$$

For example, $\frac{d}{dx}(ax+b)^n=n(ax+b)^{n-1}\times a$.

**The "let $u=\ldots$" method.** 1. Let $u$ be the inside function; write $y$ in terms of $u$. 2. Find $\frac{du}{dx}$ and $\frac{dy}{du}$. 3. Multiply: $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$, then replace $u$ by the inside function.

To find the **gradient at a point**, substitute the $x$-value into $\frac{dy}{dx}$.

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### Worked examples

#### *Example 1 — scaffolded*

Differentiate $y=(3x+2)^4$.

| Working | Comment |
|---|---|
| Let $u=3x+2$, so $y=u^4$. | Choose the inside function as $u$. |
| $\frac{du}{dx}=3$ and $\frac{dy}{du}=4u^3$. | Differentiate each part. |
| $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=4u^3\times 3=12u^3$. | Apply the chain rule. |
| $\frac{dy}{dx}=12(3x+2)^3$. | Replace $u$ by $3x+2$. |

#### *Example 2 — scaffolded*

Differentiate $y=(x^2-5x)^3$.

| Working | Comment |
| --- | --- |
| Let $u = x^2 - 5x$, so $y = u^3$. | Inside function is a quadratic. |
| $\frac{du}{dx} = 2x - 5$ and $\frac{dy}{du} = 3u^2$. | Differentiate each part. |
| $\frac{dy}{dx} = 3u^2 \times (2x - 5)$. | Chain rule. |
| $\frac{dy}{dx} = 3(2x-5)(x^2-5x)^2$. | Replace $u$; write the inner derivative in front. |

### *Example 3 — unscaffolded*

Differentiate $y = (2x^2 + 1)^5$.

Using the power-of-a-function rule with $g(x) = 2x^2 + 1$ (so $g'(x) = 4x$):

$$\frac{dy}{dx} = 5(2x^2+1)^4 \times 4x.$$

$\therefore \frac{dy}{dx} = 20x(2x^2+1)^4$.

### *Example 4 — unscaffolded*

Find the gradient of $y = (x^2 - 3)^4$ at $x = 2$.

$\frac{dy}{dx} = 4(x^2-3)^3 \times 2x = 8x(x^2-3)^3$.

At $x = 2$: $\frac{dy}{dx} = 8(2)((2)^2 - 3)^3 = 16(1)^3 = 16$. $\therefore$ the gradient is 16.

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## Practice questions

### *Scaffolded*

**Q1.** Differentiate $y = (2x+1)^5$: (a) let $u = 2x+1$ and write $y$ in terms of $u$; (b) find $\frac{du}{dx}$ and $\frac{dy}{du}$; (c) hence find $\frac{dy}{dx}$.

**Q2.** Differentiate $y = (5 - 3x)^4$ using the chain rule (note $\frac{du}{dx} = -3$).

**Q3.** Differentiate $y = (x^2 + 4)^3$.

**Q4.** Differentiate $y = (x^2 - 2x)^4$.

**Q5.** For $y = (3x^2+1)^3$, find $\frac{dy}{dx}$ and hence its value at $x = 1$.

**Q6.** Differentiate $y = (x^3 - x)^2$.

### *Unscaffolded*

Differentiate each of the following (Q7–Q12), then answer Q13–Q18 as stated.

**Q7.** $y = (4x-3)^6$

**Q8.** $y = (1-2x)^5$

**Q9.** $y=(x^2+3x)^4$

**Q10.** $y=(2x^2-x+1)^3$

**Q11.** $y=(x^3+2)^5$

**Q12.** $y=3(2x-1)^4$

**Q13.** Find the gradient of $y=(x^2-4)^3$ at $x=3$.

**Q14.** Find the gradient of $y=(x^2+1)^4$ at $x=1$.

**Q15.** Find the gradient of $y=(3x-2)^5$ at $x=1$.

**Q16.** Find the values of $x$ for which $y=(x^2-4x)^2$ has a gradient of zero.

**Q17.** Find the values of $x$ for which $y=(x^2-1)^3$ has a stationary point.

**Q18.** Find the gradient of $y=(2x-3)^4$ at the point where $x=2$.

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## Answers

**Q1.** $10(2x+1)^4$. **Q2.** $-12(5-3x)^3$. **Q3.** $6x(x^2+4)^2$. **Q4.** $4(2x-2)(x^2-2x)^3$. **Q5.** $18x(3x^2+1)^2$; at $x=1$, $288$. **Q6.** $2(3x^2-1)(x^3-x)$. **Q7.** $24(4x-3)^5$. **Q8.** $-10(1-2x)^4$. **Q9.** $4(2x+3)(x^2+3x)^3$. **Q10.** $3(4x-1)(2x^2-x+1)^2$. **Q11.** $15x^2(x^3+2)^4$. **Q12.** $24(2x-1)^3$. **Q13.** $450$. **Q14.** $64$. **Q15.** $15$. **Q16.** $x=0,2,4$. **Q17.** $x=-1,0,1$. **Q18.** $8$.

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## Fully-worked solutions

**Q1.** $u=2x+1,\ y=u^5$; $\frac{du}{dx}=2$, $\frac{dy}{du}=5u^4$; $\frac{dy}{dx}=5u^4\times 2=10(2x+1)^4$.

**Q2.** $\frac{dy}{dx}=4(5-3x)^3\times(-3)=-12(5-3x)^3$.

**Q3.** $\frac{dy}{dx}=3(x^2+4)^2\times 2x=6x(x^2+4)^2$.

**Q4.** $\frac{dy}{dx}=4(x^2-2x)^3\times(2x-2)=4(2x-2)(x^2-2x)^3$.

**Q5.** $\frac{dy}{dx}=3(3x^2+1)^2\times 6x=18x(3x^2+1)^2$. At $x=1$: $18(1)(3+1)^2=18\times 16=288$.

**Q6.** $\frac{dy}{dx}=2(x^3-x)^1\times(3x^2-1)=2(3x^2-1)(x^3-x)$.

**Q7.** $\frac{dy}{dx}=6(4x-3)^5\times 4=24(4x-3)^5$.

**Q8.** $\frac{dy}{dx}=5(1-2x)^4\times(-2)=-10(1-2x)^4$.

**Q9.** $\frac{dy}{dx}=4(x^2+3x)^3\times(2x+3)=4(2x+3)(x^2+3x)^3$.

**Q10.** $\frac{dy}{dx}=3(2x^2-x+1)^2\times(4x-1)=3(4x-1)(2x^2-x+1)^2$.

**Q11.** $\frac{dy}{dx}=5(x^3+2)^4\times 3x^2=15x^2(x^3+2)^4$.

**Q12.** $\frac{dy}{dx}=3\times 4(2x-1)^3\times 2=24(2x-1)^3$.

**Q13.** $\frac{dy}{dx}=3(x^2-4)^2\times 2x=6x(x^2-4)^2$. At $x=3$: $6(3)(9-4)^2=18\times 25=450$.

**Q14.** $\frac{dy}{dx}=4(x^2+1)^3\times 2x=8x(x^2+1)^3$. At $x=1$: $8(1)(2)^3=8\times 8=64$.

**Q15.** $\frac{dy}{dx}=5(3x-2)^4\times 3=15(3x-2)^4$. At $x=1$: $15(1)^4=15$.

**Q16.** $\frac{dy}{dx}=2(x^2-4x)(2x-4)$. Setting this to 0: $x^2-4x=0$ gives $x=0$ or $x=4$; $2x-4=0$ gives $x=2$. $\therefore x=0,2,4$.

**Q17.** $\frac{dy}{dx}=3(x^2-1)^2\times 2x=6x(x^2-1)^2$. Setting this to 0: $x=0$, or $(x^2-1)^2=0\Rightarrow x=\pm 1$. $\therefore x=-1,0,1$.

**Q18.** $\frac{dy}{dx}=4(2x-3)^3\times 2=8(2x-3)^3$. At $x=2$: $8(1)^3=8$.

# Chapter 16 Calculus beyond polynomials — 16B Derivatives of rational powers

**Units 3 & 4 preview.** This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

## Theory

The power rule extends to **rational** (fractional) indices:

$$\frac{d}{dx}\left(x^{p/q}\right)=\frac{p}{q}x^{\frac{p}{q}-1}.$$

**Rewrite surds and roots as fractional powers first**, then apply the rule:

$$\sqrt{x}=x^{1/2},\frac{1}{\sqrt{x}}=x^{-1/2},\sqrt[3]{x^2}=x^{2/3},\frac{1}{x^n}=x^{-n}.$$

Differentiate term by term (the constant-multiple and sum rules still hold). It is usual to give the answer back in **surd / positive-index form**.

**Chain rule for a linear inside.** For a bracket $(ax+b)$ raised to a rational power,

$$\frac{d}{dx}\left[(ax+b)^{p/q}\right]=\frac{p}{q}a(ax+b)^{\frac{p}{q}-1}.$$

For example $\frac{d}{dx}\sqrt{ax+b}=\frac{a}{2\sqrt{ax+b}}$.

**Domain.** $\sqrt{x}$ requires $x\geq 0$ and its derivative requires $x>0$; a term like $\frac{1}{\sqrt{x}}$ requires $x>0$.

## Worked examples

### *Example 1 — scaffolded*

Differentiate $y=\sqrt{x}$.

| Working | Comment |
|---|---|
| $y=\sqrt{x}=x^{1/2}$. | Write the surd as a fractional power. |
| $\frac{dy}{dx}=\frac{1}{2}x^{1/2-1}=\frac{1}{2}x^{-1/2}$. | Apply the power rule: multiply by the index, subtract 1. |
| $=\frac{1}{2\sqrt{x}}$. | Write the answer back in surd form. |

### *Example 2 — scaffolded*

Differentiate $f(x)=3x^{2/3}-\frac{4}{\sqrt{x}}+5$.

| Working | Comment |
|---|---|
| $f(x)=3x^{2/3}-4x^{-1/2}+5$. | Rewrite each term as a power of $x$. |
| $f'(x)=3\cdot\frac{2}{3}x^{-1/3}-4\cdot\left(-\frac{1}{2}\right)x^{-3/2}+0$. | Differentiate term by term; the constant 5 has derivative 0. |

| Working | Comment |
| --- | --- |
| $=2x^{-1/3}+2x^{-3/2}=\frac{2}{\sqrt[3]{x}}+\frac{2}{\sqrt{x^3}}$. | Simplify and write in surd form. |

***Example 3 — unscaffolded***

Differentiate $y=\sqrt{3x+1}$.

$y=(3x+1)^{1/2}$. Using the chain rule with the linear inside,

$$\frac{dy}{dx}=\frac{1}{2}(3x+1)^{-1/2}\cdot 3=\frac{3}{2}(3x+1)^{-1/2}.$$

$\therefore \frac{dy}{dx}=\frac{3}{2\sqrt{3x+1}}$.

***Example 4 — unscaffolded***

For $f(x)=x^{3/2}-6\sqrt{x}$, find $f'(4)$.

$f(x)=x^{3/2}-6x^{1/2}$, so

$$f'(x)=\frac{3}{2}x^{1/2}-6\cdot\frac{1}{2}x^{-1/2}=\frac{3}{2}\sqrt{x}-\frac{3}{\sqrt{x}}.$$

$f'(4)=\frac{3}{2}(2)-\frac{3}{2}=3-\frac{3}{2}$. $\therefore f'(4)=\frac{3}{2}$.

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## Practice questions

***Scaffolded***

**Q1.** Differentiate $y=x^{1/3}$.

**Q2.** Differentiate $y=x^{5/2}$.

**Q3.** Differentiate $y=\sqrt{x}+\frac{1}{\sqrt{x}}$.

**Q4.** Differentiate $y=4\sqrt[3]{x}$.

**Q5.** Differentiate $y=\sqrt{2x+3}$.

**Q6.** For $f(x)=\sqrt{x}$, find $f'(9)$.

***Unscaffolded***

Differentiate each of the following (give your answer in surd / positive-index form), unless a value is asked for.

**Q7.** $y=x^{3/4}$

**Q8.** $y=x^{-1/2}$

**Q9.** $y=2\sqrt{x}-3x$

**Q10.** $y=\frac{5}{\sqrt{x}}+\sqrt{x}$

**Q11.** $y=x^{2/3}+x^{1/3}$

**Q12.** $y = \frac{x^2+1}{\sqrt{x}}$

**Q13.** $y = \sqrt{5x-2}$

**Q14.** $y = (3x+4)^{3/2}$

**Q15.** $y = \frac{1}{\sqrt{2x+1}}$

**Q16.** For $f(x) = x^{5/2} - 4x^{1/2}$, find $f'(1)$.

**Q17.** Find the gradient of $y = \sqrt{x}$ at $x = 16$.

**Q18.** Find the value of $x$ at which the tangent to $y = x^{3/2}$ has gradient $3$.

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### Answers

**Q1.** $\frac{1}{3}x^{-2/3}=\frac{1}{3\sqrt[3]{x^2}}$. **Q2.** $\frac{5}{2}x^{3/2}$. **Q3.** $\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x^3}}$. **Q4.** $\frac{4}{3}x^{-2/3}=\frac{4}{3\sqrt[3]{x^2}}$. **Q5.** $\frac{1}{\sqrt{2x+3}}$. **Q6.** $\frac{1}{6}$. **Q7.** $\frac{3}{4}x^{-1/4}=\frac{3}{4\sqrt[4]{x}}$. **Q8.** $-\frac{1}{2}x^{-3/2}=-\frac{1}{2\sqrt{x^3}}$. **Q9.** $\frac{1}{\sqrt{x}}-3$. **Q10.** $-\frac{5}{2}x^{-3/2}+\frac{1}{2}x^{-1/2}=-\frac{5}{2\sqrt{x^3}}+\frac{1}{2\sqrt{x}}$. **Q11.** $\frac{2}{3}x^{-1/3}+\frac{1}{3}x^{-2/3}=\frac{2}{3\sqrt[3]{x}}+\frac{1}{3\sqrt[3]{x^2}}$. **Q12.** $\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-3/2}=\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x^3}}$. **Q13.** $\frac{5}{2\sqrt{5x-2}}$. **Q14.** $\frac{9}{2}\sqrt{3x+4}$. **Q15.** $-(2x+1)^{-3/2}=-\frac{1}{\sqrt{(2x+1)^3}}$. **Q16.** $\frac{1}{2}$. **Q17.** $\frac{1}{8}$. **Q18.** $x=4$.

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### Fully-worked solutions

**Q1.** $y=x^{1/3}\Rightarrow\frac{dy}{dx}=\frac{1}{3}x^{-2/3}=\frac{1}{3\sqrt[3]{x^2}}$.

**Q2.** $y=x^{5/2}\Rightarrow\frac{dy}{dx}=\frac{5}{2}x^{3/2}$.

**Q3.** $y=x^{1/2}+x^{-1/2}\Rightarrow\frac{dy}{dx}=\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}=\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x^3}}$.

**Q4.** $y=4x^{1/3}\Rightarrow\frac{dy}{dx}=4\cdot\frac{1}{3}x^{-2/3}=\frac{4}{3\sqrt[3]{x^2}}$.

**Q5.** $y=(2x+3)^{1/2}\Rightarrow\frac{dy}{dx}=\frac{1}{2}(2x+3)^{-1/2}\cdot 2=(2x+3)^{-1/2}=\frac{1}{\sqrt{2x+3}}$.

**Q6.** $f(x)=x^{1/2}$, $f'(x)=\frac{1}{2\sqrt{x}}$, so $f'(9)=\frac{1}{2\times 3}=\frac{1}{6}$.

**Q7.** $y=x^{3/4}\Rightarrow\frac{dy}{dx}=\frac{3}{4}x^{-1/4}=\frac{3}{4\sqrt[4]{x}}$.

**Q8.** $y=x^{-1/2}\Rightarrow\frac{dy}{dx}=-\frac{1}{2}x^{-3/2}=-\frac{1}{2\sqrt{x^3}}$.

**Q9.** $y=2x^{1/2}-3x\Rightarrow\frac{dy}{dx}=2\cdot\frac{1}{2}x^{-1/2}-3=x^{-1/2}-3=\frac{1}{\sqrt{x}}-3$.

**Q10.** $y=5x^{-1/2}+x^{1/2}\Rightarrow\frac{dy}{dx}=5\cdot\left(-\frac{1}{2}\right)x^{-3/2}+\frac{1}{2}x^{-1/2}=-\frac{5}{2}x^{-3/2}+\frac{1}{2}x^{-1/2}=-\frac{5}{2\sqrt{x^3}}+\frac{1}{2\sqrt{x}}$.

**Q11.** $y=x^{2/3}+x^{1/3}\Rightarrow\frac{dy}{dx}=\frac{2}{3}x^{-1/3}+\frac{1}{3}x^{-2/3}=\frac{2}{3\sqrt[3]{x}}+\frac{1}{3\sqrt[3]{x^2}}$.

**Q12.** $y=\frac{x^2+1}{\sqrt{x}}=x^{3/2}+x^{-1/2}$, so $\frac{dy}{dx}=\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-3/2}=\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x^3}}$.

**Q13.** $y=(5x-2)^{1/2}\Rightarrow\frac{dy}{dx}=\frac{1}{2}(5x-2)^{-1/2}\cdot 5=\frac{5}{2\sqrt{5x-2}}$.

**Q14.** $y=(3x+4)^{3/2}\Rightarrow\frac{dy}{dx}=\frac{3}{2}(3x+4)^{1/2}\cdot 3=\frac{9}{2}\sqrt{3x+4}$.

**Q15.** $y=(2x+1)^{-1/2} \Rightarrow \frac{dy}{dx}=-\frac{1}{2}(2x+1)^{-3/2}\cdot 2=-(2x+1)^{-3/2}=-\frac{1}{\sqrt{(2x+1)^3}}$.

**Q16.** $f(x)=x^{5/2}-4x^{1/2}$, $f'(x)=\frac{5}{2}x^{3/2}-2x^{-1/2}$, so $f'(1)=\frac{5}{2}-2=\frac{1}{2}$.

**Q17.** $y=x^{1/2}$, $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$; at $x=16$ the gradient is $\frac{1}{2\times 4}=\frac{1}{8}$.

**Q18.** $y=x^{3/2}$, $\frac{dy}{dx}=\frac{3}{2}x^{1/2}$. Set $\frac{3}{2}x^{1/2}=3$: $x^{1/2}=2$, so $\therefore x=4$.

## Chapter 16 Calculus beyond polynomials — 16C Anti-derivatives of rational powers

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### Theory

Antidifferentiation reverses differentiation. The power rule for antidifferentiation,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c,$$

holds for every **rational** index $n \neq -1$ — not just whole numbers. The constant $c$ covers every possible antiderivative.

**Rewrite first.** Surds and reciprocals must be written as powers of $x$ before the rule is applied:

$$\sqrt{x} = x^{1/2}, \frac{1}{\sqrt{x}} = x^{-1/2}, \sqrt[3]{x} = x^{1/3}, \frac{1}{x^2} = x^{-2}.$$

**Apply and simplify.** Add $1$ to the index, divide by the new index, and simplify the fraction. Dividing by a fraction means multiplying by its reciprocal — for example $\frac{x^{3/2}}{3/2} = \frac{2}{3} x^{3/2}$. It is usual to give the final answer back in **surd form** where natural, e.g. $\frac{2}{3} x^{3/2} = \frac{2}{3} \sqrt{x^3} = \frac{2}{3} x \sqrt{x}$.

**Sums and multiples.** Antidifferentiate each term separately, and a constant factor may be taken out: $\int k f(x) \, dx = k \int f(x) \, dx$.

**A specific antiderivative.** Given $f'(x)$ and the value of $f$ at one point, antidifferentiate to get the family $f(x) = \ldots + c$, then substitute the point to solve for $c$.

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### Worked examples

#### *Example 1 — scaffolded*

Find $\int \sqrt{x} \, dx$.

| Working | Comment |
|---|---|
| $\sqrt{x} = x^{1/2}$. | Rewrite the surd as a power of $x$. |
| $\int x^{1/2} \, dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c$ | Add $1$ to the index; divide by the new index. |
| $= \frac{2}{3} x^{3/2} + c$ | Dividing by $3/2$ is multiplying by $2/3$. |

$\therefore \int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + c \left(= 2/3 \, x \sqrt{x} + c\right)$.

#### *Example 2 — scaffolded*

Find $\int \frac{1}{\sqrt{x}} \, dx$.

| Working | Comment |
|---|---|
| $\frac{1}{\sqrt{x}} = x^{-1/2}$. | Rewrite the reciprocal surd as a negative power. |

| Working | Comment |
|---|---|
| $\int x^{-1/2}\,dx=\dfrac{x^{1/2}}{1/2}+c$ | $-1/2+1=1/2$. |
| $=2x^{1/2}+c=2\sqrt{x}+c$ | Dividing by $1/2$ is multiplying by 2. |

***Example 3 — unscaffolded***

Find $\int\left(6\sqrt{x}-\dfrac{4}{\sqrt{x}}\right)dx$.

Rewrite: $6\sqrt{x}-\dfrac{4}{\sqrt{x}}=6x^{1/2}-4x^{-1/2}$.

$$\int\left(6x^{1/2}-4x^{-1/2}\right)dx=6\cdot\frac{2}{3}x^{3/2}-4\cdot 2x^{1/2}+c.$$

$\therefore =4x^{3/2}-8x^{1/2}+c\left(=4x\sqrt{x}-8\sqrt{x}+c\right)$.

***Example 4 — unscaffolded***

A function has $f'(x)=\sqrt{x}$ and $f(4)=10$. Find $f(x)$.

Antidifferentiate: $f(x)=\dfrac{2}{3}x^{3/2}+c$.

Substitute $f(4)=10$, using $4^{3/2}=(\sqrt{4})^3=8$:

$$\frac{2}{3}(8)+c=10\Rightarrow\frac{16}{3}+c=10\Rightarrow c=10-\frac{16}{3}=\frac{14}{3}.$$

$\therefore f(x)=\dfrac{2}{3}x^{3/2}+\dfrac{14}{3}$.

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## Practice questions

***Scaffolded***

**Q1.** Find $\int x^{1/3}\,dx$.

**Q2.** Find $\int x^{-1/3}\,dx$.

**Q3.** Find $\int x^{3/2}\,dx$.

**Q4.** Find $\int 5\sqrt{x}\,dx$.

**Q5.** Find $\int\dfrac{3}{\sqrt{x}}\,dx$.

**Q6.** Find $\int\left(\sqrt{x}+x\right)dx$.

***Unscaffolded***

Find an antiderivative (Q7–Q14); then find the specific antiderivative satisfying the given condition (Q15–Q18).

**Q7.** $\int x^{2/3}\,dx$

**Q8.** $\int\sqrt[4]{x}\,dx$

**Q9.** $\int \frac{2}{\sqrt[3]{x}} dx$

**Q10.** $\int \left(x^{1/2}+x^{-1/2}\right) dx$

**Q11.** $\int \left(4x^{3/2}-6x^{1/2}\right) dx$

**Q12.** $\int \frac{x-2}{\sqrt{x}} dx$

**Q13.** $\int \sqrt{x}(x+3) dx$

**Q14.** $\int \left(\frac{1}{x^2}+\sqrt{x}\right) dx$

**Q15.** $f'(x)=\frac{1}{\sqrt{x}}$ and $f(9)=8$.

**Q16.** $f'(x)=3\sqrt{x}$ and $f(1)=5$.

**Q17.** $\frac{dy}{dx}=x^{1/3}$ and the curve passes through $(8,10)$.

**Q18.** $f'(x)=\sqrt{x}-\frac{1}{\sqrt{x}}$ and $f(4)=6$.

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### Answers

**Q1.** $\frac{3}{4}x^{4/3}+c$. **Q2.** $\frac{3}{2}x^{2/3}+c$. **Q3.** $\frac{2}{5}x^{5/2}+c$. **Q4.** $\frac{10}{3}x^{3/2}+c$. **Q5.** $6\sqrt{x}+c$. **Q6.** $\frac{2}{3}x^{3/2}+\frac{x^2}{2}+c$. **Q7.** $\frac{3}{5}x^{5/3}+c$. **Q8.** $\frac{4}{5}x^{5/4}+c$. **Q9.** $3x^{2/3}+c$. **Q10.** $\frac{2}{3}x^{3/2}+2\sqrt{x}+c$. **Q11.** $\frac{8}{5}x^{5/2}-4x^{3/2}+c$. **Q12.** $\frac{2}{3}x^{3/2}-4\sqrt{x}+c$. **Q13.** $\frac{2}{5}x^{5/2}+2x^{3/2}+c$. **Q14.** $-\frac{1}{x}+\frac{2}{3}x^{3/2}+c$. **Q15.** $f(x)=2\sqrt{x}+2$. **Q16.** $f(x)=2x^{3/2}+3$. **Q17.** $y=\frac{3}{4}x^{4/3}-2$. **Q18.** $f(x)=\frac{2}{3}x^{3/2}-2\sqrt{x}+\frac{14}{3}$.

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### Fully-worked solutions

**Q1.** $\int x^{1/3}\,dx=\frac{x^{4/3}}{4/3}+c=\frac{3}{4}x^{4/3}+c$.

**Q2.** $\int x^{-1/3}\,dx=\frac{x^{2/3}}{2/3}+c=\frac{3}{2}x^{2/3}+c$ (here $-1/3+1=2/3$).

**Q3.** $\int x^{3/2}\,dx=\frac{x^{5/2}}{5/2}+c=\frac{2}{5}x^{5/2}+c$.

**Q4.** $\int 5\sqrt{x}\,dx=5\int x^{1/2}\,dx=5\cdot\frac{2}{3}x^{3/2}+c=\frac{10}{3}x^{3/2}+c$.

**Q5.** $\int \frac{3}{\sqrt{x}}\,dx=3\int x^{-1/2}\,dx=3\cdot 2x^{1/2}+c=6\sqrt{x}+c$.

**Q6.** $\int \left(x^{1/2}+x\right)dx=\frac{2}{3}x^{3/2}+\frac{x^2}{2}+c$.

**Q7.** $\int x^{2/3}\,dx=\frac{x^{5/3}}{5/3}+c=\frac{3}{5}x^{5/3}+c$.

**Q8.** $\sqrt[4]{x}=x^{1/4}$, so $\int x^{1/4}\,dx=\frac{x^{5/4}}{5/4}+c=\frac{4}{5}x^{5/4}+c$.

**Q9.** $\frac{2}{\sqrt[3]{x}}=2x^{-1/3}$, so $\int 2x^{-1/3}\,dx=2\cdot\frac{3}{2}x^{2/3}+c=3x^{2/3}+c$.

**Q10.** $\int \left(x^{1/2}+x^{-1/2}\right)dx=\frac{2}{3}x^{3/2}+2x^{1/2}+c=\frac{2}{3}x^{3/2}+2\sqrt{x}+c$.

**Q11.** $\int \left(4x^{3/2}-6x^{1/2}\right)dx=4\cdot\frac{2}{5}x^{5/2}-6\cdot\frac{2}{3}x^{3/2}+c=\frac{8}{5}x^{5/2}-4x^{3/2}+c$.

**Q12.** Split the fraction: $\frac{x-2}{\sqrt{x}}=\frac{x}{\sqrt{x}}-\frac{2}{\sqrt{x}}=x^{1/2}-2x^{-1/2}$. Then $\int \left(x^{1/2}-2x^{-1/2}\right)dx=\frac{2}{3}x^{3/2}-4\sqrt{x}+c$.

**Q13.** Expand first: $\sqrt{x}(x+3)=x^{3/2}+3x^{1/2}$. Then $\int \left(x^{3/2}+3x^{1/2}\right)dx=\frac{2}{5}x^{5/2}+3\cdot\frac{2}{3}x^{3/2}+c=\frac{2}{5}x^{5/2}+2x^{3/2}+c$.

**Q14.** $\frac{1}{x^2}=x^{-2}$, so $\int \left(x^{-2}+x^{1/2}\right)dx=\frac{x^{-1}}{-1}+\frac{2}{3}x^{3/2}+c=-\frac{1}{x}+\frac{2}{3}x^{3/2}+c$ (the rule still applies since $-2\neq-1$).

**Q15.** $f(x)=\int x^{-1/2}dx=2\sqrt{x}+c$. Then $f(9)=2(3)+c=6+c=8$, so $c=2$. $\therefore f(x)=2\sqrt{x}+2$.

**Q16.** $f(x)=\int 3x^{1/2}dx=3\cdot\frac{2}{3}x^{3/2}+c=2x^{3/2}+c$. Then $f(1)=2+c=5$, so $c=3$. $\therefore f(x)=2x^{3/2}+3$.

**Q17.** $y=\int x^{1/3}dx=\frac{3}{4}x^{4/3}+c$. Using $8^{4/3}=(\sqrt[3]{8})^4=2^4=16$, the point $(8,10)$ gives $\frac{3}{4}(16)+c=12+c=10$, so $c=-2$. $\therefore y=\frac{3}{4}x^{4/3}-2$.

**Q18.** $f(x)=\int(x^{1/2}-x^{-1/2})dx=\frac{2}{3}x^{3/2}-2\sqrt{x}+c$. Using $4^{3/2}=8$ and $\sqrt{4}=2$:
$f(4)=\frac{2}{3}(8)-2(2)+c=\frac{16}{3}-4+c=\frac{4}{3}+c=6$, so $c=\frac{14}{3}$. $\therefore f(x)=\frac{2}{3}x^{3/2}-2\sqrt{x}+\frac{14}{3}$.

## Chapter 16 Calculus beyond polynomials — 16D The second derivative

**Units 3 & 4 preview.** This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

### Theory

**The second derivative.** Differentiating $f'(x)$ again gives the **second derivative**, written $f''(x)$ or $\frac{d^2y}{dx^2}$. It is the rate of change of the gradient.

**Concavity.** The sign of $f''$ describes the way a curve bends:

- where $f''(x)>0$ the gradient is increasing and the curve is **concave up** (holds water);
- where $f''(x)<0$ the gradient is decreasing and the curve is **concave down**.

**Points of inflection.** A **point of inflection** is where the concavity changes. At such a point $f''(x)=0$ **and** $f''$ changes sign. (Note $f''(x)=0$ alone is not enough — for $y=x^4$, $f''(0)=0$ but the curve stays concave up, so the origin is a minimum, not an inflection.)

**The second-derivative test.** At a stationary point ($f'(a)=0$):

- if $f''(a)>0$, the point is a **local minimum**;
- if $f''(a)<0$, the point is a **local maximum**;
- if $f''(a)=0$, the test **fails** — use a sign-of-$f'$ table instead.

**Acceleration.** For a particle with position $x(t)$, the acceleration is the second derivative $a=\frac{d^2x}{dt^2}=v'(t)$.

### Worked examples

#### *Example 1 — scaffolded*

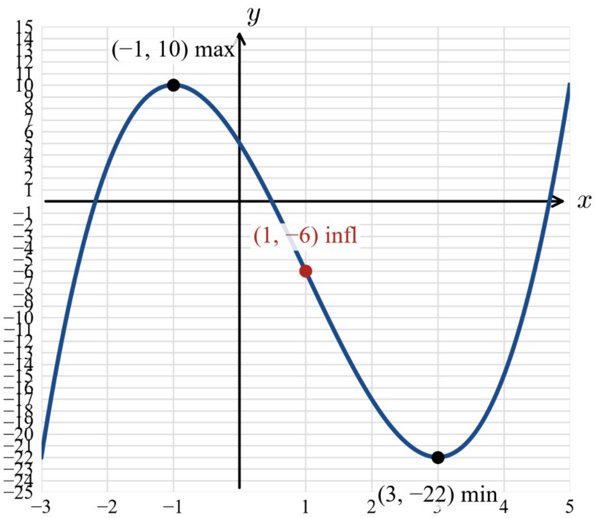
Find $f'(x)$ and $f''(x)$ for $f(x)=x^3-4x^2+5$.

| Working | Comment |
|---|---|
| $f'(x)=3x^2-8x$. | Differentiate using the power rule. |
| $f''(x)=6x-8$. | Differentiate $f'(x)$ again. |

#### *Example 2 — scaffolded*

Find the coordinates of the point of inflection of $y=x^3-3x^2-9x+5$.

| Working | Comment |
|---|---|
| $\frac{dy}{dx}=3x^2-6x-9$, so $\frac{d^2y}{dx^2}=6x-6$. | Differentiate twice. |
| $6x-6=0 \Rightarrow x=1$. | A point of inflection requires $f''=0$. |
| For $x<1$, $f''<0$ (concave down); for $x>1$, $f''>0$ (concave up) — the concavity **changes**. | Confirm the sign of $f''$ changes. |
| $y(1)=1-3-9+5=-6$, giving $(1,-6)$. | Find the $y$-coordinate. |

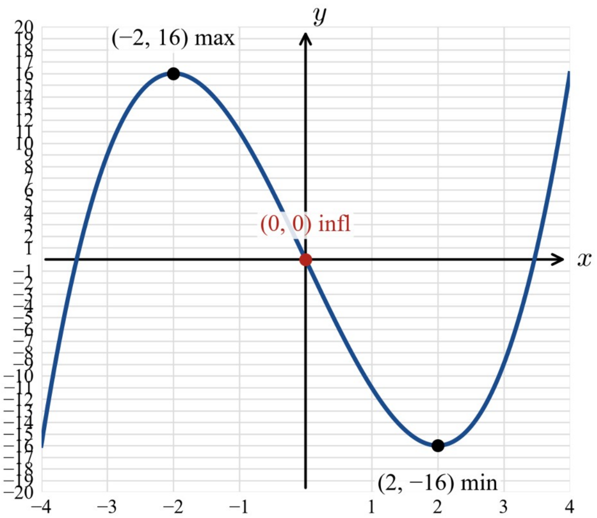



***Example 3 — unscaffolded***

Use the second-derivative test to find and classify the stationary points of $f(x)=x^3-12x$.

$f'(x)=3x^2-12=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$, and $f''(x)=6x$.

At $x=2$: $f''(2)=12>0$, a **local minimum** at $(2,-16)$. At $x=-2$: $f''(-2)=-12<0$, a **local maximum** at $(-2,16)$.

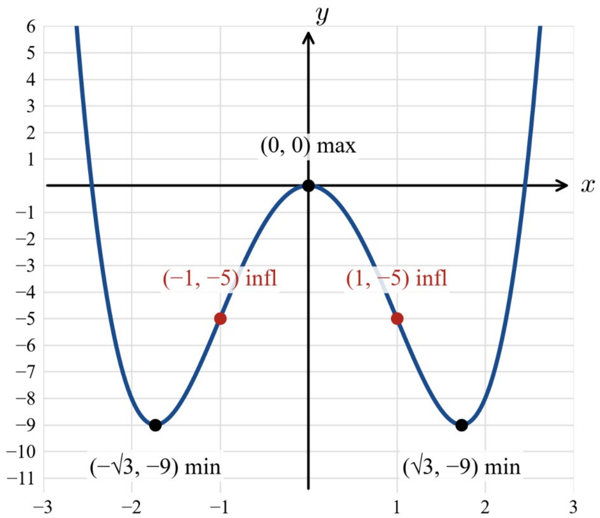



***Example 4 — unscaffolded***

For $y=x^4-6x^2$, find the intervals where the graph is concave up.

$\frac{d^2y}{dx^2}=12x^2-12=12(x-1)(x+1)$.

This is positive when $x<-1$ or $x>1$, so the graph is **concave up** on $(-\infty,-1)\cup(1,\infty)$ and concave down on $(-1,1)$. The concavity changes at $x=\pm 1$, giving points of inflection $(-1,-5)$ and $(1,-5)$.




## Practice questions

### *Scaffolded*

**Q1.** For $f(x)=x^3+2x^2-x$, find $f'(x)$ and $f''(x)$.

**Q2.** For $y=2x^4-3x^2+1$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

**Q3.** Find the coordinates of the point of inflection of $y=x^3-6x^2+5$.

**Q4.** Find and classify the stationary points of $f(x)=x^3-3x^2+4$ using the second-derivative test.

**Q5.** Find the values of $x$ for which $y=x^3-3x^2+2x$ is concave down.

**Q6.** A particle has position $x(t)=t^3-6t^2+9t$ metres at time $t$ seconds. Find its acceleration $a(t)$ and the time at which $a=0$.

### *Unscaffolded*

**Q7.** Find $f''(x)$ for $f(x)=x^5-2x^3+x$.

**Q8.** Find $\frac{d^2y}{dx^2}$ for $y=(2x-1)^3$.

**Q9.** Find the point of inflection of $y=x^3+3x^2-9x-2$.

**Q10.** Find the points of inflection of $y=x^4-4x^3$.

**Q11.** Find and classify the stationary points of $f(x)=x^3-6x^2+9x+1$ using the second-derivative test.

**Q12.** Find and classify the stationary points of $y=x^4-2x^2$ using the second-derivative test.

**Q13.** Find the intervals of concavity of $y=x^3-3x$ and state its point of inflection.

**Q14.** Find the values of $x$ for which $y=x^4-6x^2+8x$ is concave up.

**Q15.** Show that $f(x)=x^3$ has a stationary point of inflection at the origin.

**Q16.** A particle has position $x(t)=t^3-3t^2-9t$ metres ($t\geq 0$). Find the time at which its velocity is a minimum, and the minimum velocity.

**Q17.** For $y = x^4$, find $\dfrac{d^2y}{dx^2}$ and explain why the origin is a **minimum**, not a point of inflection.

**Q18.** The curve $y = x^3 + ax^2 + bx + 2$ has a stationary point of inflection at $x = 1$. Find the values of $a$ and $b$.

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### Answers

**Q1.** $f'(x)=3x^2+4x-1$; $f''(x)=6x+4$. **Q2.** $\frac{dy}{dx}=8x^3-6x$; $\frac{d^2y}{dx^2}=24x^2-6$. **Q3.** $(2,-11)$. **Q4.** Local max $(0,4)$; local min $(2,0)$. **Q5.** $x<1$, i.e. $(-\infty,1)$. **Q6.** $a(t)=6t-12$; $a=0$ at $t=2$. **Q7.** $f''(x)=20x^3-12x$. **Q8.** $\frac{d^2y}{dx^2}=48x-24$. **Q9.** $(-1,9)$. **Q10.** $(0,0)$ and $(2,-16)$. **Q11.** Local max $(1,5)$; local min $(3,1)$. **Q12.** Local max $(0,0)$; local minima $(-1,-1)$ and $(1,-1)$. **Q13.** Concave up for $x>0$, concave down for $x<0$; inflection $(0,0)$. **Q14.** $x<-1$ or $x>1$, i.e. $(-\infty,-1)\cup(1,\infty)$. **Q15.** $f'(0)=0$ and $f''$ changes sign at 0 — a stationary point of inflection at $(0,0)$. **Q16.** Minimum velocity $-12$ m/s at $t=1$ s. **Q17.** $\frac{d^2y}{dx^2}=12x^2\geq 0$ everywhere; $f''$ does not change sign, so $(0,0)$ is a minimum. **Q18.** $a=-3$, $b=3$.

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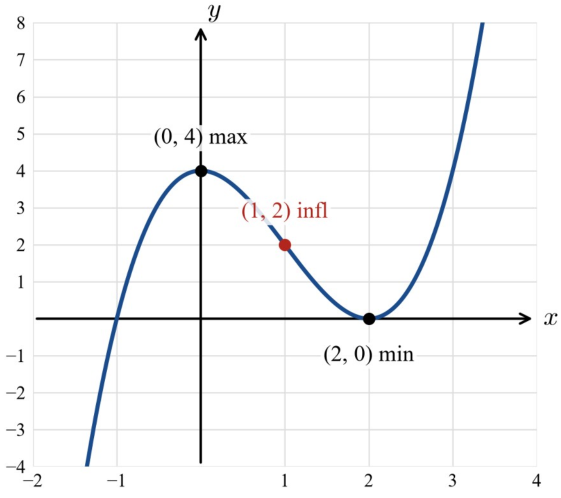
### Fully-worked solutions

**Q1.** $f'(x)=3x^2+4x-1$; differentiating again, $f''(x)=6x+4$.

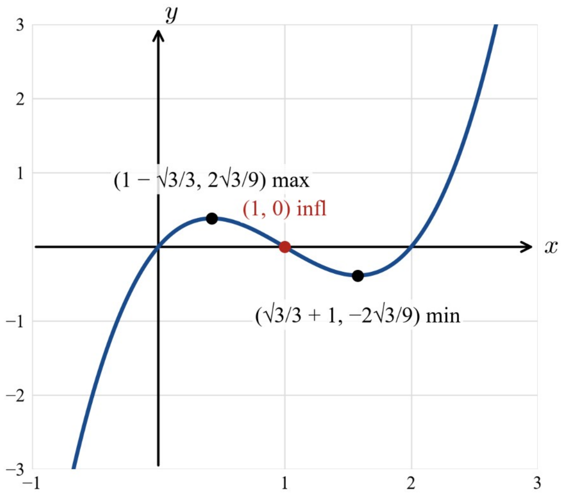
**Q2.** $\frac{dy}{dx}=8x^3-6x$; $\frac{d^2y}{dx^2}=24x^2-6$.

**Q3.** $y'=3x^2-12x$, $y''=6x-12=0\Rightarrow x=2$; $y''$ changes from $-$ to $+$, so it is an inflection. $y(2)=8-24+5=-11$, giving $(2,-11)$.

**Q4.** $f'(x)=3x^2-6x=3x(x-2)=0\Rightarrow x=0$ or 2; $f''(x)=6x-6$. $f''(0)=-6<0\rightarrow$ local max $(0,4)$; $f''(2)=6>0\rightarrow$ local min $(2,0)$.




**Q5.** $y''=6x-6<0\Rightarrow x<1$. The graph is concave down on $(-\infty,1)$.

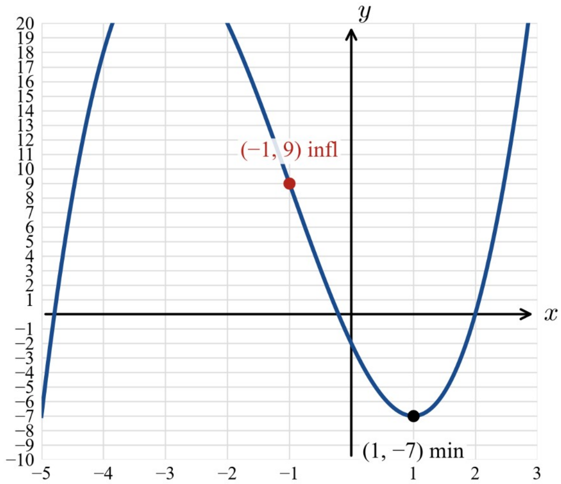



**Q6.** $v(t)=3t^2-12t+9$, so $a(t)=6t-12$. $a=0 \Rightarrow t=2$ seconds.

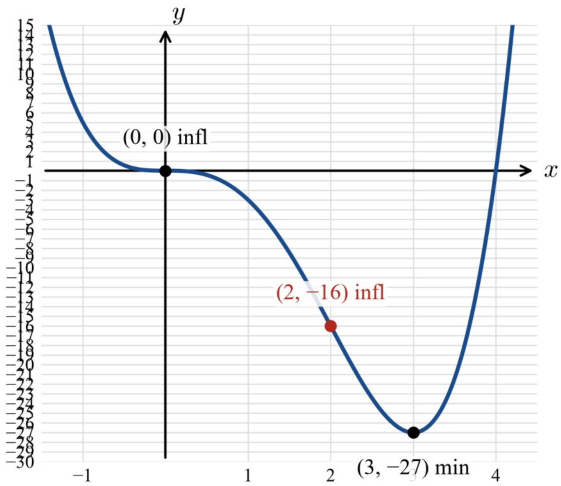
**Q7.** $f'(x)=5x^4-6x^2+1$, so $f''(x)=20x^3-12x$.

**Q8.** $y'=3(2x-1)^2\times 2=6(2x-1)^2$, so $y''=12(2x-1)\times 2=24(2x-1)=48x-24$.

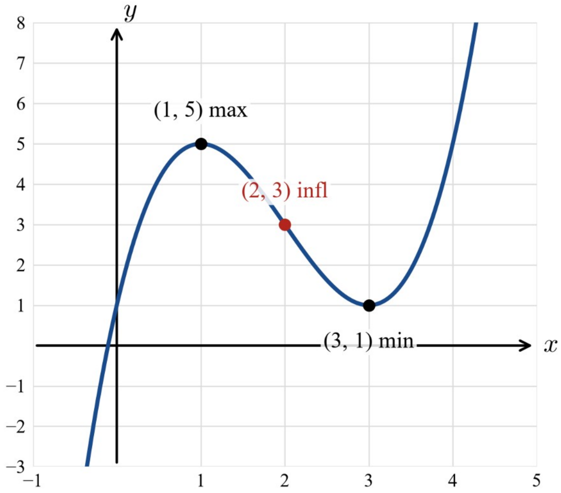
**Q9.** $y'=3x^2+6x-9$, $y''=6x+6=0 \Rightarrow x=-1$ (sign of $y''$ changes). $y(-1)=-1+3+9-2=9$, giving $(-1,9)$.



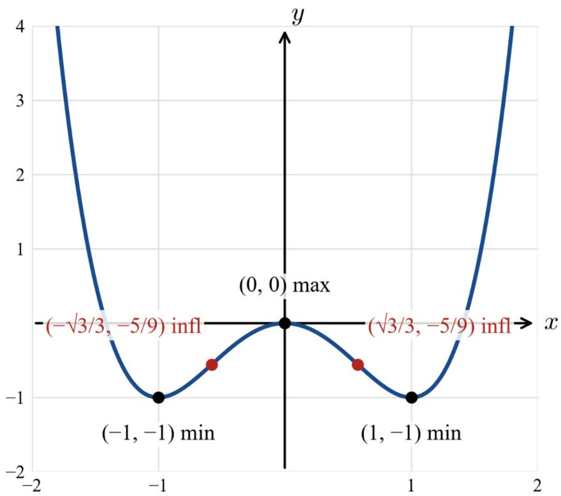

**Q10.** $y'=4x^3-12x^2$, $y''=12x^2-24x=12x(x-2)=0 \Rightarrow x=0$ or 2; the sign of $y''$ changes at each. $y(0)=0$ and $y(2)=16-32=-16$, giving $(0,0)$ and $(2,-16)$.



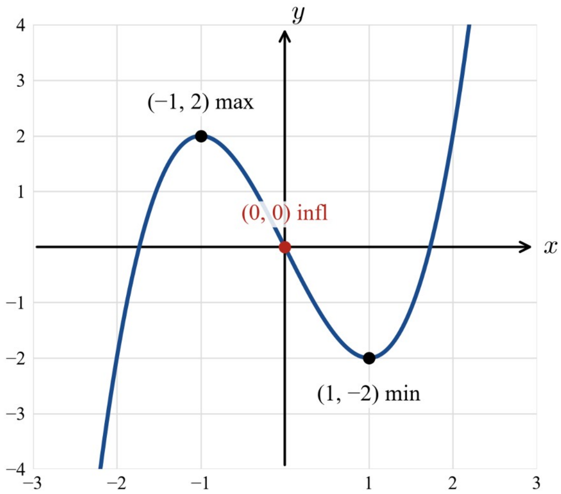
**Q11.** $f'(x)=3x^2-12x+9=3(x-1)(x-3)=0 \Rightarrow x=1$ or $3$; $f''(x)=6x-12$. $f''(1)=-6<0 \rightarrow$ local max $(1,5)$; $f''(3)=6>0 \rightarrow$ local min $(3,1)$.




**Q12.** $y'=4x^3-4x=4x(x-1)(x+1)=0 \Rightarrow x=0,\pm 1$; $y''=12x^2-4$. $y''(0)=-4<0 \rightarrow$ local max $(0,0)$; $y''(\pm 1)=8>0 \rightarrow$ local minima $(-1,-1)$ and $(1,-1)$.

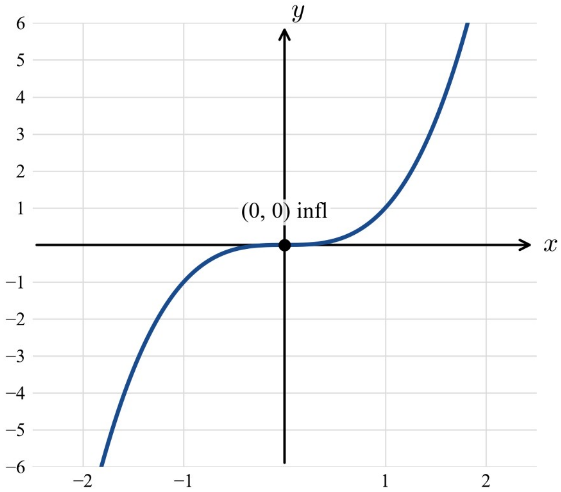



**Q13.** $y''=6x$: concave down for $x<0$, concave up for $x>0$; the concavity changes at $x=0$, giving the point of inflection $(0,0)$.



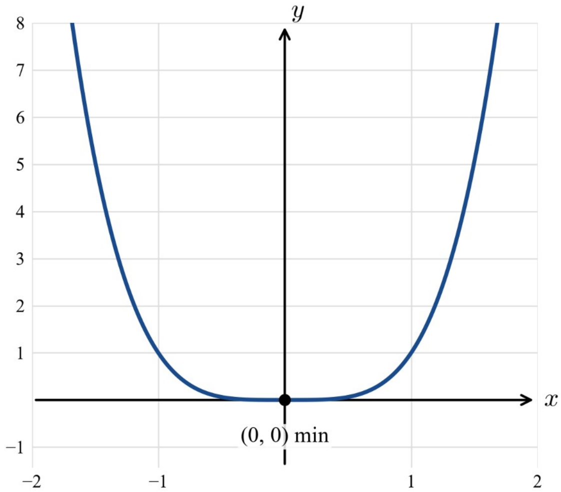

**Q14.** $y'=4x^3-12x+8$, $y''=12x^2-12=12(x-1)(x+1)>0 \Rightarrow x<-1$ or $x>1$.

**Q15.** $f'(x)=3x^2$, so $f'(0)=0$ (stationary). $f''(x)=6x$ changes from negative to positive at $x=0$, so the concavity changes — a **stationary point of inflection** at $(0,0)$.




**Q16.** $v(t)=3t^2-6t-9$. The velocity is stationary where $\frac{dv}{dt}=a=6t-6=0$, i.e. $t=1$; since $a$ changes from $-$ to $+$, this is a minimum. $v(1)=3-6-9=-12$. $\therefore$ minimum velocity $-12$ m/s at $t=1$ s.

**Q17.** $y''=12x^2$, which is $\geq 0$ for all $x$. Although $y''(0)=0$, the second derivative does **not** change sign, so the concavity does not change: the curve is concave up throughout and the stationary point $(0,0)$ is a **local minimum**, not a point of inflection.




**Q18.** $y'=3x^2+2ax+b$ and $y''=6x+2a$. A stationary point of inflection at $x=1$ needs both $y''(1)=0$ and $y'(1)=0$:

$$y''(1)=6+2a=0 \Rightarrow a=-3;\, y'(1)=3+2a+b=3-6+b=0 \Rightarrow b=3.$$

$\therefore a=-3, b=3$ (the curve is $y=(x-1)^3+3$).

### Theory

Calculus lets us sketch a polynomial graph accurately by locating its **stationary points** and **points of inflection**, not just its intercepts.

**The sketching procedure.**

1. **Axis intercepts.** The $y$-intercept is $f(0)$. The $x$-intercepts solve $f(x)=0$ (factorise where possible).
2. **Stationary points.** Solve $f'(x)=0$ for $x$, then find each $y$-coordinate. A stationary point is where the tangent is horizontal.
3. **Classify** each stationary point as a **local maximum**, **local minimum**, or **stationary point of inflection**, using the **second derivative**: $f''>0 \Rightarrow$ minimum, $f''<0 \Rightarrow$ maximum, and $f''=0 \Rightarrow$ investigate further (a sign test on $f'$).
4. **Points of inflection.** Solve $f''(x)=0$; a point of inflection occurs where $f''$ **changes sign** (the concavity changes).
5. **Long-run behaviour.** Read the end behaviour from the leading term (e.g. a positive cubic rises to the right, a positive quartic rises at both ends).
6. **Draw** a smooth curve through all the marked features.

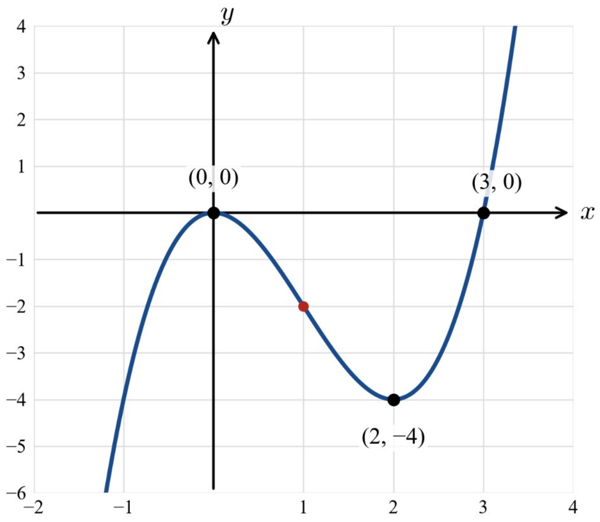
A degree-$n$ polynomial has at most $n-1$ stationary points.

### Worked examples

#### *Example 1 — scaffolded*

Sketch the graph of $y=x^3-3x^2$, showing the intercepts, stationary points (and their nature), and the point of inflection.

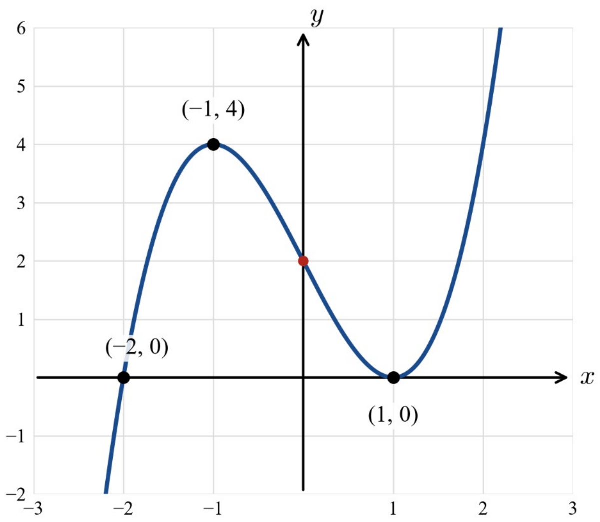
| Working | Comment |
|---|---|
| $y$-intercept: $x=0 \Rightarrow y=0$, giving $(0,0)$. | Put $x=0$. |
| $x$-intercepts: $x^3-3x^2=x^2(x-3)=0$, so $x=0$ or $x=3$, giving $(0,0)$ and $(3,0)$. | Factorise. |
| $f'(x)=3x^2-6x=3x(x-2)=0 \Rightarrow x=0$ or $x=2$. | Stationary points where $f'(x)=0$. |
| $f''(x)=6x-6$. $f''(0)=-6<0$: $(0,0)$ is a local **maximum**. $f''(2)=6>0$: $(2,-4)$ is a local **minimum**. | Second-derivative test. |
| $f''(x)=0 \Rightarrow x=1$; $f''$ changes sign, so $(1,-2)$ is a **point of inflection**. | Concavity changes here. |




### *Example 2 — scaffolded*

Sketch the graph of $y=x^3-3x+2$.

| Working | Comment |
| --- | --- |
| $y$-intercept $(0,2)$; $x^3-3x+2=(x-1)^2(x+2)=0\Rightarrow x=1$ (repeated) or $x=-2$, giving $(1,0)$ and $(-2,0)$. | The repeated factor means the curve touches at $x=1$. |
| $f'(x)=3x^2-3=3(x-1)(x+1)=0\Rightarrow x=\pm1$. | Stationary points. |
| $f''(x)=6x$. $f''(-1)=-6<0$: $(-1,4)$ local **maximum**. $f''(1)=6>0$: $(1,0)$ local **minimum**. | Second-derivative test. |
| $f''(x)=0\Rightarrow x=0$: $(0,2)$ is a **point of inflection**. | Concavity changes. |

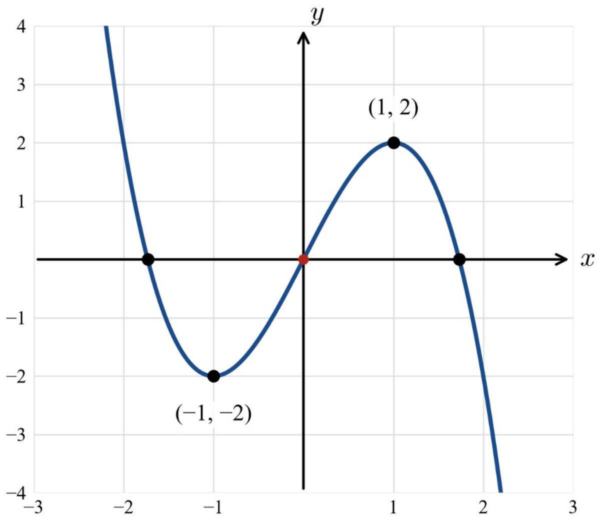



### *Example 3 — unscaffolded*

Sketch the graph of $y=-x^3+3x$.

Intercepts: $y$-intercept $(0,0)$; $-x^3+3x=-x(x^2-3)=0\Rightarrow x=0,\pm\sqrt{3}$, giving $(0,0)$, $(-\sqrt{3},0)$ and $(\sqrt{3},0)$.

$f'(x)=-3x^2+3=-3(x-1)(x+1)=0\Rightarrow x=\pm1$. $f''(x)=-6x$: $f''(-1)=6>0$ so $(-1,-2)$ is a local **minimum**; $f''(1)=-6<0$ so $(1,2)$ is a local **maximum**. $f''(x)=0\Rightarrow x=0$: $(0,0)$ is a **point of inflection**.

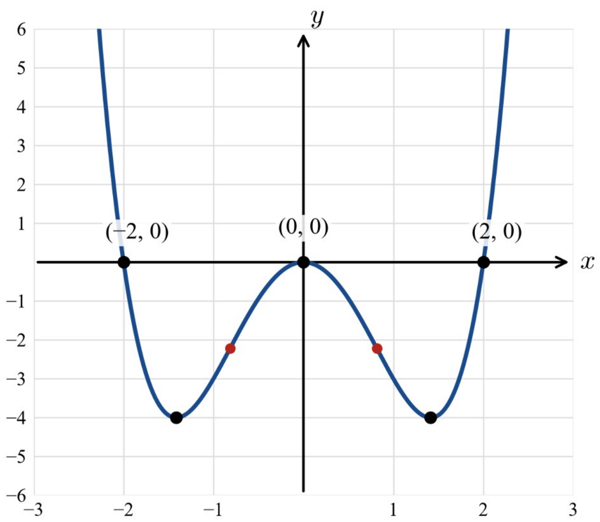



### *Example 4 — unscaffolded*

Sketch the graph of $y=x^4-4x^2$.

Intercepts: $y$-intercept $(0,0)$; $x^4-4x^2=x^2(x^2-4)=0 \Rightarrow x=0,\pm 2$, giving $(0,0)$, $(-2,0)$ and $(2,0)$.

$f'(x)=4x^3-8x=4x(x^2-2)=0 \Rightarrow x=0,\pm\sqrt{2}$. $f''(x)=12x^2-8$: $f''(0)=-8<0$ so $(0,0)$ is a local **maximum**; $f''(\pm\sqrt{2})=16>0$ so $(-\sqrt{2},-4)$ and $(\sqrt{2},-4)$ are local **minima**. Points of inflection where $f''(x)=0 \Rightarrow x=\pm\dfrac{\sqrt{6}}{3}$, at $y=-\dfrac{20}{9}$.

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## Practice questions

### *Scaffolded*

**Q1.** For $y=x^3-3x$: (a) find the axis intercepts; (b) find and classify the stationary points; (c) find the point of inflection; (d) sketch the graph.

**Q2.** For $y=x^3-6x^2+9x$: (a) find the intercepts; (b) find and classify the stationary points; (c) find the point of inflection; (d) sketch the graph.

**Q3.** For $y=-x^3+3x^2$: (a) find the intercepts; (b) find and classify the stationary points; (c) find the point of inflection; (d) sketch the graph.

**Q4.** For $y = x^4 - 2x^2$: (a) find the intercepts; (b) find and classify the stationary points; (c) sketch the graph.

**Q5.** For $y = x^3 + 1$: (a) find the intercepts; (b) show that the only stationary point is a stationary point of inflection at $(0, 1)$; (c) sketch the graph.

**Q6.** For $y = x^4 - 8x^2$: (a) find the intercepts; (b) find and classify the stationary points; (c) sketch the graph.

### *Unscaffolded*

Sketch each graph, labelling the axis intercepts, the stationary points (with their nature), and any point of inflection.

**Q7.** $y = x^3 - 12x$

**Q8.** $y = x^3 - 3x^2 - 9x$

**Q9.** $y = -x^3 + 12x$

**Q10.** $y = 2x^3 - 3x^2$

**Q11.** $y = x^4 - 4x^3$

**Q12.** $y = x^4 - 2x^2 + 1$

**Q13.** $y = x^3 - 3x^2 + 4$

**Q14.** $y = -x^4 + 4x^2$

**Q15.** $y = x^3 + 3x^2$

**Q16.** $y = x^4 - 4x$

**Q17.** $y = x^3$

**Q18.** $y = x^4 - 6x^2 + 8$

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### Answers

**Q1.** Int $(0,0),(\pm\sqrt{3},0)$; max $(-1,2)$, min $(1,-2)$; inflection $(0,0)$. **Q2.** Int $(0,0),(3,0)$; max $(1,4)$, min $(3,0)$; inflection $(2,2)$. **Q3.** Int $(0,0),(3,0)$; min $(0,0)$, max $(2,4)$; inflection $(1,2)$. **Q4.** Int $(0,0),(\pm\sqrt{2},0)$; max $(0,0)$, min $(\pm 1,-1)$; inflections $(\pm\sqrt{3}/3,-5/9)$. **Q5.** Int $(0,1),(-1,0)$; stationary point of inflection $(0,1)$. **Q6.** Int $(0,0),(\pm 2\sqrt{2},0)$; max $(0,0)$, min $(\pm 2,-16)$. **Q7.** Int $(0,0),(\pm 2\sqrt{3},0)$; max $(-2,16)$, min $(2,-16)$; inflection $(0,0)$. **Q8.** Int $(0,0),(3\pm 3\sqrt{5}/2,0)$; max $(-1,5)$, min $(3,-27)$; inflection $(1,-11)$. **Q9.** Int $(0,0),(\pm 2\sqrt{3},0)$; min $(-2,-16)$, max $(2,16)$; inflection $(0,0)$. **Q10.** Int $(0,0),(3/2,0)$; max $(0,0)$, min $(1,-1)$; inflection $(1/2,-1/2)$. **Q11.** Int $(0,0),(4,0)$; stationary point of inflection $(0,0)$, min $(3,-27)$; inflection also at $(2,-16)$. **Q12.** Int $(0,1),(\pm 1,0)$; max $(0,1)$, min $(\pm 1,0)$; inflections $(\pm\sqrt{3}/3,4/9)$. **Q13.** Int $(0,4),(-1,0),(2,0)$; max $(0,4)$, min $(2,0)$; inflection $(1,2)$. **Q14.** Int $(0,0),(\pm 2,0)$; min $(0,0)$, max $(\pm\sqrt{2},4)$; inflections $(\pm\sqrt{6}/3,20/9)$. **Q15.** Int $(0,0),(-3,0)$; max $(-2,4)$, min $(0,0)$; inflection $(-1,2)$. **Q16.** Int $(0,0),(\sqrt[3]{4},0)$; min $(1,-3)$; no point of inflection. **Q17.** Int $(0,0)$; stationary point of inflection $(0,0)$. **Q18.** Int $(\pm\sqrt{2},0),(\pm 2,0)$, $y$-int $(0,8)$; max $(0,8)$, min $(\pm\sqrt{3},-1)$; inflections $(\pm 1,3)$.

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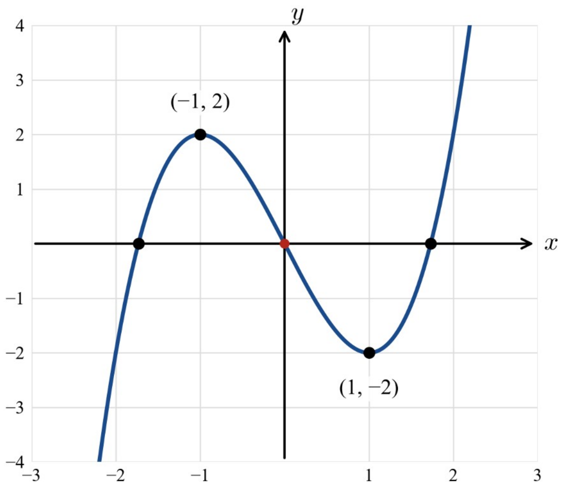
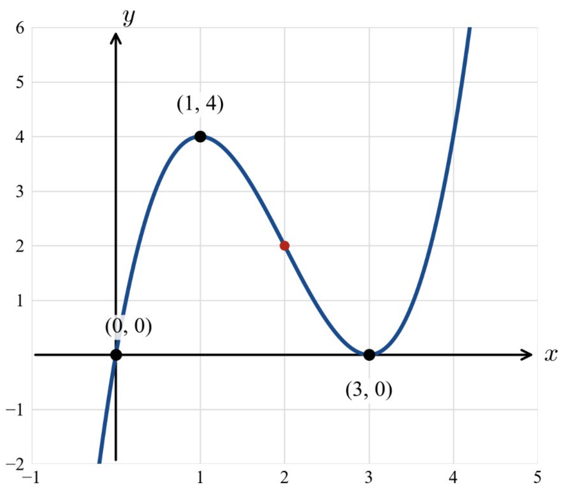
### Fully-worked solutions

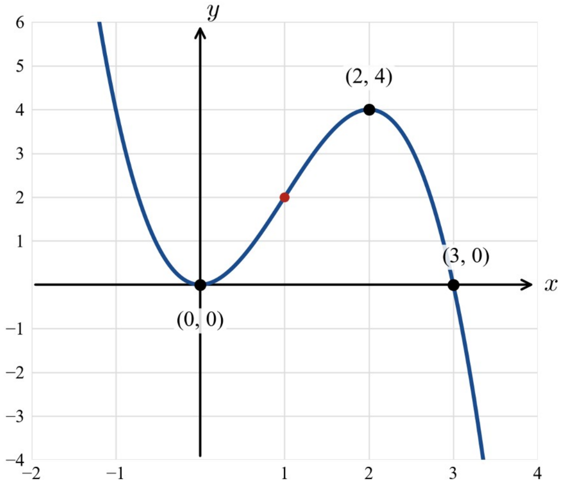
**Q1.** $y=x^3-3x$. Int $(0,0)$ and $x(x^2-3)=0 \Rightarrow (\pm\sqrt{3},0)$. $f'(x)=3x^2-3=0 \Rightarrow x=\pm 1$; $f''(x)=6x$, so $(-1,2)$ max, $(1,-2)$ min. $f''(x)=0 \Rightarrow x=0$: inflection $(0,0)$.



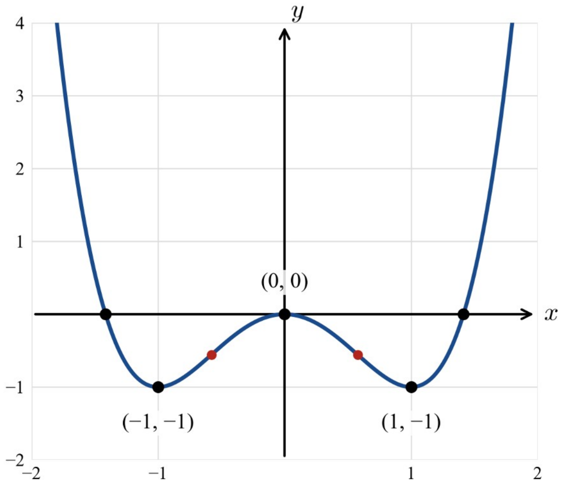

**Q2.** $y=x^3-6x^2+9x=x(x-3)^2$. Int $(0,0)$, $(3,0)$ (touch). $f'(x)=3x^2-12x+9=3(x-1)(x-3)=0 \Rightarrow x=1,3$; $f''(x)=6x-12$, so $(1,4)$ max, $(3,0)$ min. $f''(x)=0 \Rightarrow x=2$: inflection $(2,2)$.



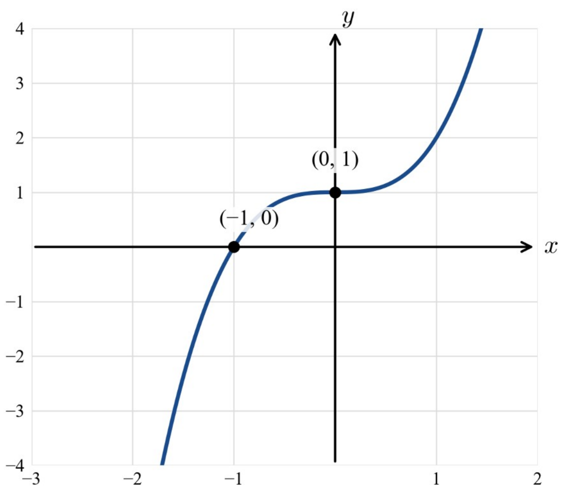
**Q3.** $y=-x^3+3x^2=-x^2(x-3)$. Int $(0,0)$, $(3,0)$. $f'(x)=-3x^2+6x=-3x(x-2)=0 \Rightarrow x=0,2$; $f''(x)=-6x+6$, so $(0,0)$ min, $(2,4)$ max. $f''(x)=0 \Rightarrow x=1$: inflection $(1,2)$.



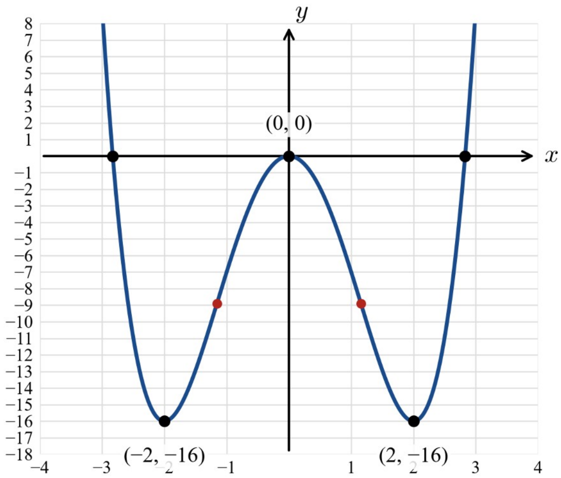

**Q4.** $y=x^4-2x^2=x^2(x^2-2)$. Int $(0,0)$, $(\pm\sqrt{2},0)$. $f'(x)=4x^3-4x=4x(x^2-1)=0 \Rightarrow x=0,\pm1$; $f''(x)=12x^2-4$, so $(0,0)$ max, $(\pm1,-1)$ minima. Inflections where $f''(x)=0 \Rightarrow x=\pm\frac{\sqrt{3}}{3}$, $y=-\frac{5}{9}$.



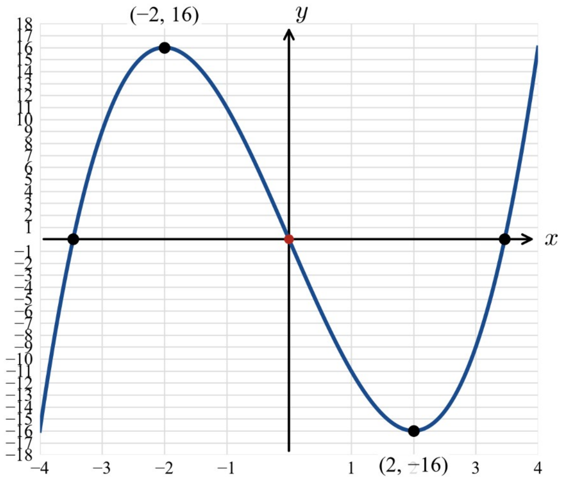

**Q5.** $y=x^3+1$. Int $(0,1)$, $(-1,0)$. $f'(x)=3x^2=0 \Rightarrow x=0$ only; since $f'(x)\geq 0$ everywhere, the gradient does not change sign, so $(0,1)$ is a **stationary point of inflection** ($f''(x)=6x$ changes sign there).




**Q6.** $y=x^4-8x^2=x^2(x^2-8)$. Int $(0,0)$, $(\pm2\sqrt{2},0)$. $f'(x)=4x^3-16x=4x(x^2-4)=0 \Rightarrow x=0,\pm2$; $f''(x)=12x^2-16$, so $(0,0)$ max, $(\pm2,-16)$ minima.

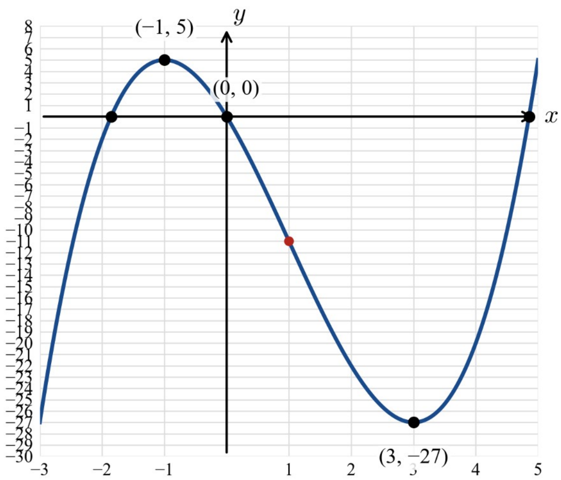



**Q7.** $y=x^3-12x$. Int $(0,0)$, $(\pm 2\sqrt{3},0)$. $f'(x)=3x^2-12=0 \Rightarrow x=\pm 2$; $f''(x)=6x$, so $(-2,16)$ max, $(2,-16)$ min; inflection $(0,0)$.

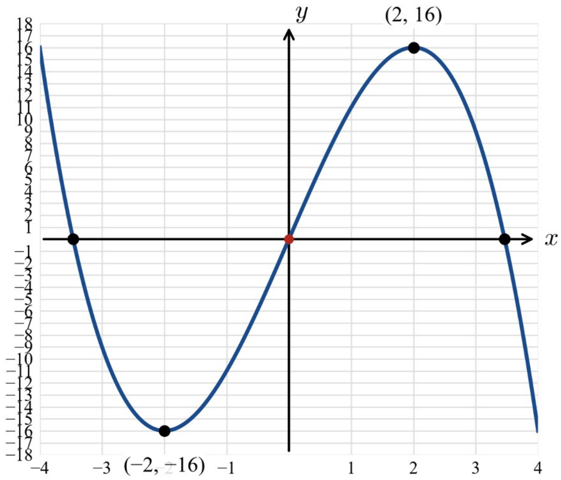



**Q8.** $y=x^3-3x^2-9x$. Int $(0,0)$ and $x^2-3x-9=0 \Rightarrow x=\dfrac{3\pm 3\sqrt{5}}{2}$.

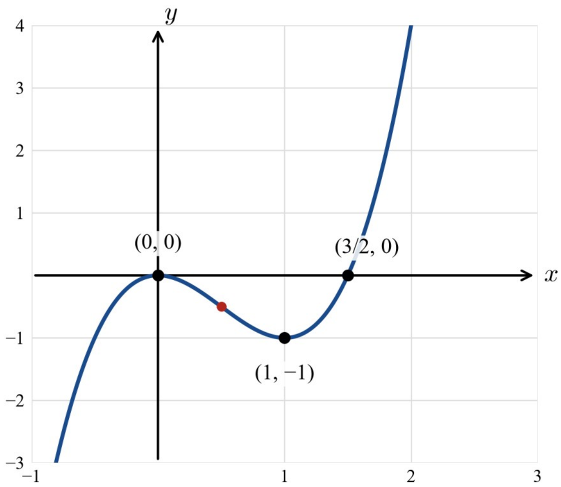
$f'(x)=3x^2-6x-9=3(x-3)(x+1)=0 \Rightarrow x=-1,3$; $f''(x)=6x-6$, so $(-1,5)$ max, $(3,-27)$ min; inflection $(1,-11)$.



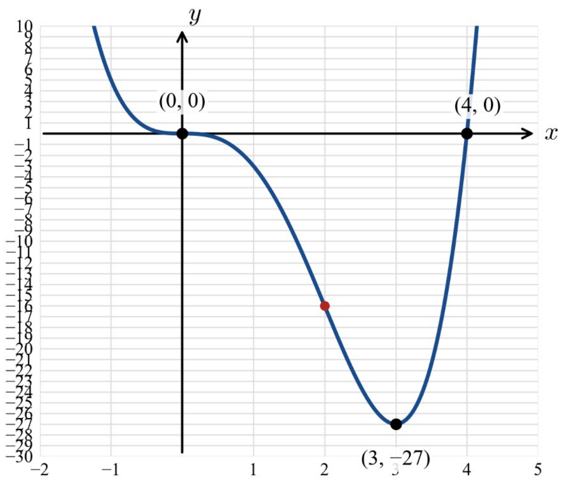

**Q9.** $y=-x^3+12x$. Int $(0,0)$, $(\pm 2\sqrt{3},0)$. $f'(x)=-3x^2+12=0 \Rightarrow x=\pm 2$; $f''(x)=-6x$, so $(-2,-16)$ min, $(2,16)$ max; inflection $(0,0)$.



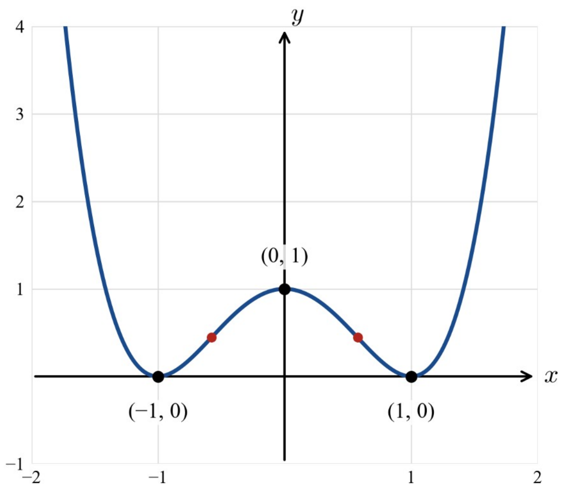

**Q10.** $y=2x^3-3x^2=x^2(2x-3)$. Int $(0,0)$, $\left(\frac{3}{2},0\right)$. $f'(x)=6x^2-6x=6x(x-1)=0\Rightarrow x=0,1$; $f''(x)=12x-6$, so $(0,0)$ max, $(1,-1)$ min; inflection $\left(\frac{1}{2},-\frac{1}{2}\right)$.



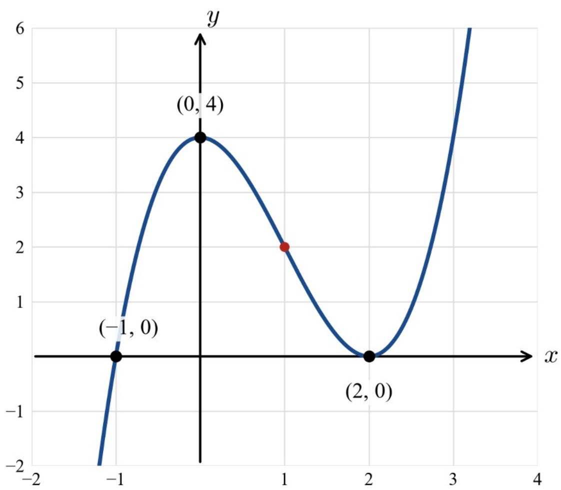

**Q11.** $y=x^4-4x^3=x^3(x-4)$. Int $(0,0)$, $(4,0)$. $f'(x)=4x^3-12x^2=4x^2(x-3)=0\Rightarrow x=0,3$;
$f''(x)=12x^2-24x$. At $x=0$, $f''=0$ and $f'$ does not change sign, so $(0,0)$ is a **stationary point of inflection**; $(3,-27)$ is a **minimum**. $f''(x)=0\Rightarrow x=0,2$, giving a second point of inflection at $(2,-16)$.



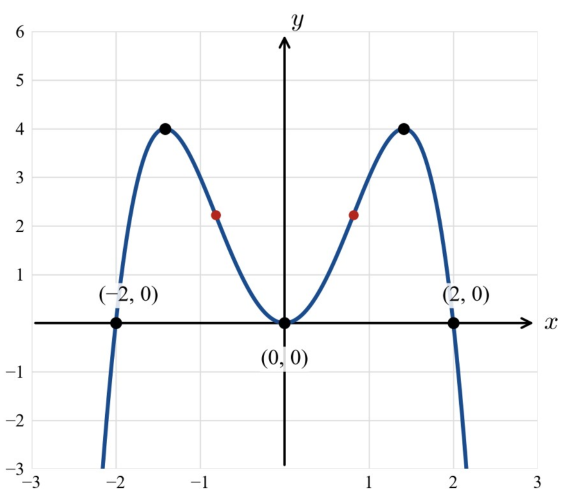
**Q12.** $y = x^4 - 2x^2 + 1 = (x^2 - 1)^2$. Int $(0,1)$, $(\pm 1, 0)$ (touch). $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1$; $f''(x) = 12x^2 - 4$, so $(0,1)$ max, $(\pm 1, 0)$ minima; inflections $\left(\pm\dfrac{\sqrt{3}}{3}, \dfrac{4}{9}\right)$.



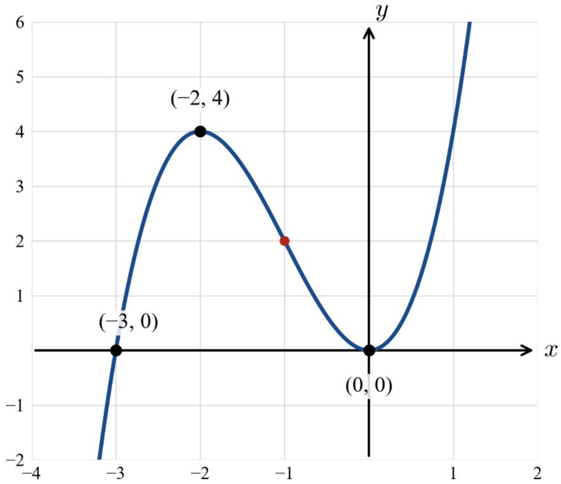

**Q13.** $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$. Int $(0,4)$, $(-1,0)$, $(2,0)$ (touch).
$f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \Rightarrow x = 0, 2$; $f''(x) = 6x - 6$, so $(0,4)$ max, $(2,0)$ min; inflection $(1,2)$.



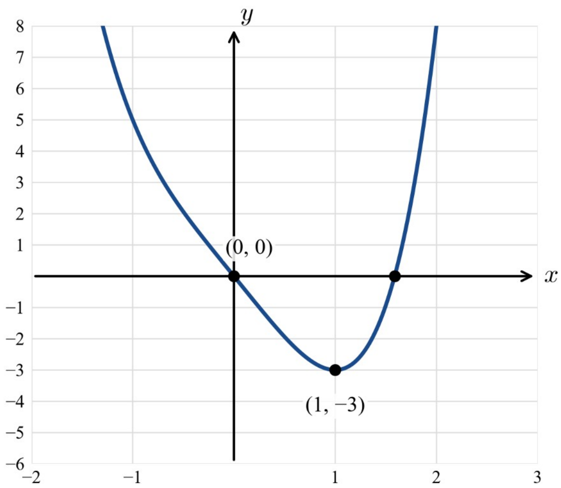

**Q14.** $y = -x^4 + 4x^2 = -x^2(x^2 - 4)$. Int $(0,0)$, $(\pm 2, 0)$. $f'(x) = -4x^3 + 8x = -4x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$; $f''(x) = -12x^2 + 8$, so $(0,0)$ min, $(\pm\sqrt{2}, 4)$ maxima; inflections $\left(\pm\dfrac{\sqrt{6}}{3}, \dfrac{20}{9}\right)$.



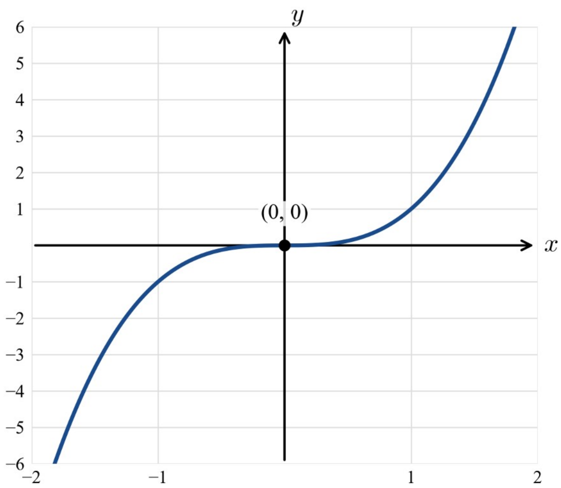
**Q15.** $y = x^3 + 3x^2 = x^2(x+3)$. Int $(0,0)$, $(-3,0)$. $f'(x) = 3x^2 + 6x = 3x(x+2) = 0 \Rightarrow x = 0, -2$; $f''(x) = 6x + 6$, so $(-2,4)$ max, $(0,0)$ min; inflection $(-1,2)$.




**Q16.** $y = x^4 - 4x$. Int $(0,0)$ and $x^4 = 4x \Rightarrow x = \sqrt[3]{4}$, giving $(\sqrt[3]{4}, 0)$. $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 0 \Rightarrow x = 1$ only; $f''(1) = 12 > 0$, so $(1,-3)$ is a **minimum**. $f''(x) = 12x^2 = 0$ at $x = 0$ but $f''$ does not change sign, so there is **no point of inflection**.




**Q17.** $y = x^3$. Int $(0,0)$. $f'(x) = 3x^2 = 0 \Rightarrow x = 0$; the gradient does not change sign, so $(0,0)$ is a **stationary point of inflection**.



**Q18.** $y = x^4 - 6x^2 + 8 = (x^2 - 2)(x^2 - 4)$. Int $y$-int $(0, 8)$; $x$-ints $(\pm\sqrt{2}, 0)$ and $(\pm 2, 0)$.
$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{3}$; $f''(x) = 12x^2 - 12$, so $(0, 8)$ max, $(\pm\sqrt{3}, -1)$ minima; inflections $(\pm 1, 3)$.