Mathematical Methods Units 1 & 2

Chapter 15 — Applications of differentiation

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

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Theory

The **tangent** to a curve at a point is the straight line that just touches the curve there and has the same gradient as the curve at that point.

Gradient of the tangent. For $y=f(x)$, the gradient of the tangent at $x=a$ is $f'(a)$.

Equation of the tangent. Using the point–gradient form with the point $(a, f(a))$ and gradient $m=f'(a)$:

$$y-f(a)=f'(a)(x-a).$$

The normal. The **normal** at a point is the line through that point **perpendicular** to the tangent. Since perpendicular gradients multiply to -1, the normal has gradient

$$m_{\text{normal}}=-\frac{1}{f'(a)}\left(f'(a)\neq 0\right),$$

and equation $y-f(a)=-\frac{1}{f'(a)}(x-a)$. (If $f'(a)=0$ the tangent is horizontal and the normal is the vertical line $x=a$.)

To find a tangent or normal: (1) differentiate to get $f'(x)$; (2) evaluate $f'(a)$ for the gradient; (3) find $f(a)$ for the point; (4) substitute into the point–gradient form.

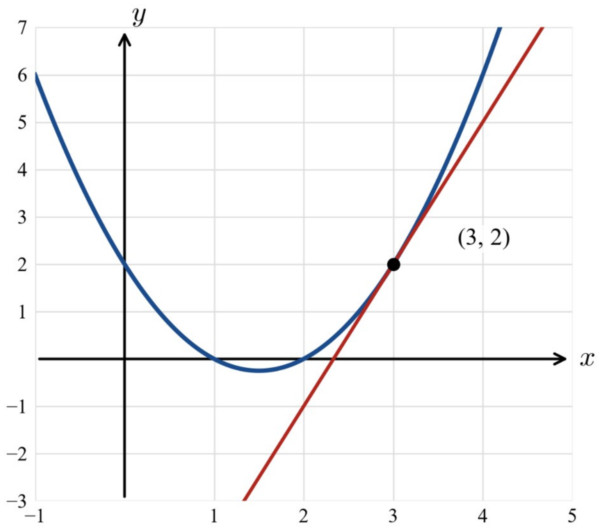
A **horizontal tangent** occurs where $f'(x)=0$; to find where a tangent has a particular gradient m, solve $f'(x)=m$.

Worked examples

Example 1 — scaffolded

Find the equation of the tangent to $y=x^2-3x+2$ at $x=3$.

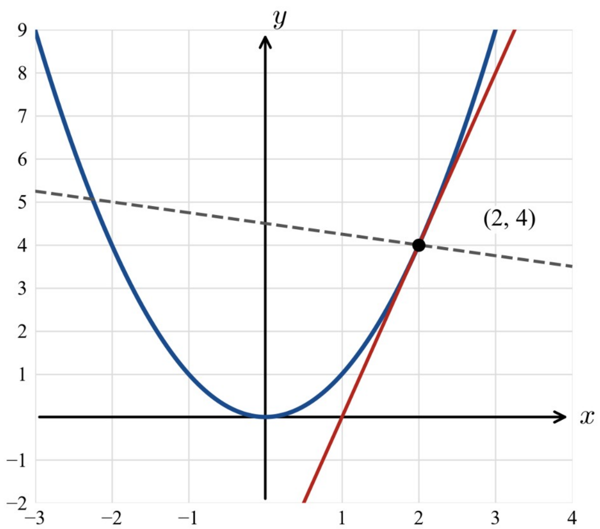
Working	Comment
$f'(x)=2x-3$.	Differentiate.
Gradient: $f'(3)=2(3)-3=3$.	Evaluate f' at $x=3$.
Point: $f(3)=9-9+2=2$, so $(3,2)$.	Find the y-coordinate.
Tangent: $y-2=3(x-3)\Rightarrow y=3x-7$.	Point–gradient form, then simplify.



Example 2 — scaffolded

Find the equations of the tangent and the normal to $y=x^2$ at the point $(2,4)$.

Working	Comment
$f'(x)=2x$, so the tangent gradient is $f'(2)=4$.	Differentiate and evaluate.
Tangent: $y-4=4(x-2)\Rightarrow y=4x-4$.	Point–gradient form.
Normal gradient $=-\dfrac{1}{4}$.	Perpendicular: $-\dfrac{1}{f'(2)}$.
Normal: $y-4=-\dfrac{1}{4}(x-2)\Rightarrow y=-\dfrac{1}{4}x+\dfrac{9}{2}$.	Point–gradient form, then simplify.



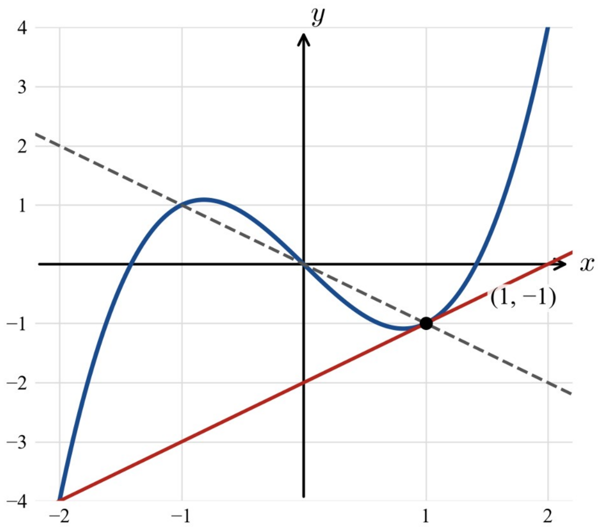
Example 3 — unscaffolded

Find the equation of the normal to $y=x^3-2x$ at $x=1$.

$f'(x)=3x^2-2$, so $f'(1)=3-2=1$; and $f(1)=1-2=-1$, giving the point $(1,-1)$.

The normal gradient is $-\dfrac{1}{1}=-1$, so $y-(-1)=-1(x-1)$.

$\therefore y=-x$.



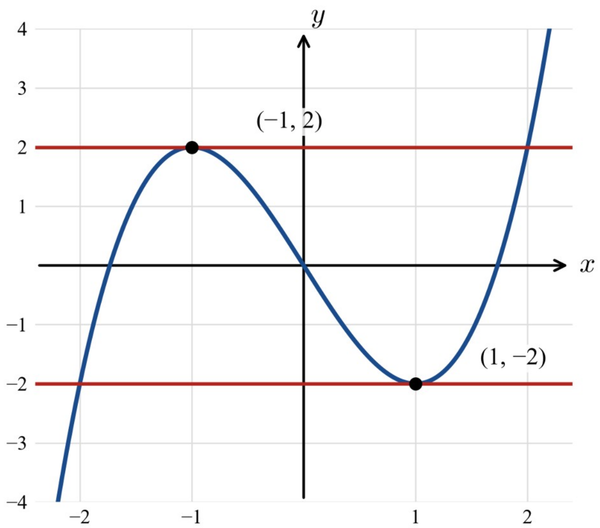
Example 4 — unscaffolded

Find the points on $y=x^3-3x$ where the tangent is horizontal.

A horizontal tangent has gradient 0: $f'(x)=3x^2-3=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$.

$f(1)=1-3=-2$ and $f(-1)=-1+3=2$.

$\therefore$ the tangent is horizontal at $(1,-2)$ and $(-1,2)$.



Practice questions

Scaffolded

Q1. For $y=x^2+1$: (a) find $f'(x)$; (b) find the gradient at $x=2$; (c) find the point; (d) write the equation of the tangent at $x=2$.

Q2. Find the equation of the tangent to $y=x^2-4x$ at $x=1$.

Q3. For $y=x^2$ at the point $(3,9)$: (a) find the tangent; (b) find the normal.

Q4. Find the equation of the tangent to $y=x^3$ at $x=1$.

Q5. Find the equation of the normal to $y=x^2-2x+3$ at $x=0$.

Q6. Find the coordinates of the point on $y=x^2-6x$ where the tangent is horizontal.

Unscaffolded

Q7. Find the tangent to $y=x^2-5x+6$ at $x=4$.

Q8. Find the tangent to $y=2x^2-3$ at $x=-1$.

Q9. Find the tangent to $y=x^3-x$ at $x=2$.

Q10. Find the normal to $y=x^2$ at $(1,1)$.

Q11. Find the normal to $y=x^2+3x$ at $x=-1$.

Q12. Find the normal to $y=x^3$ at $x=1$.

Q13. Find the tangent and the normal to $y=x^2-4x+5$ at $x=3$.

Q14. Find the point on $y=x^2-4x+1$ where the tangent has gradient 2, and find that tangent's equation.

Q15. Find the points on $y=x^3$ where the tangent is parallel to the line $y=3x$, and give those tangents.

Q16. Find the coordinates of the points on $y=x^3-12x$ where the tangent is horizontal.

Q17. Find the tangent to $y=x^2-2x$ at the point where the curve crosses the y-axis.

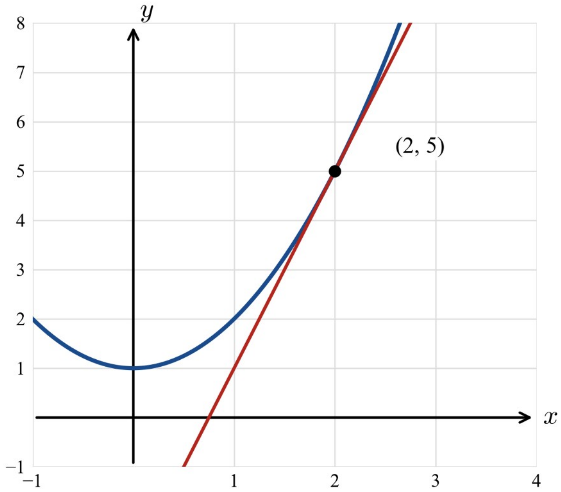
Q18. The tangent to $y=x^2$ at $(1,1)$ crosses the x-axis. Find the coordinates of that crossing point.

Answers

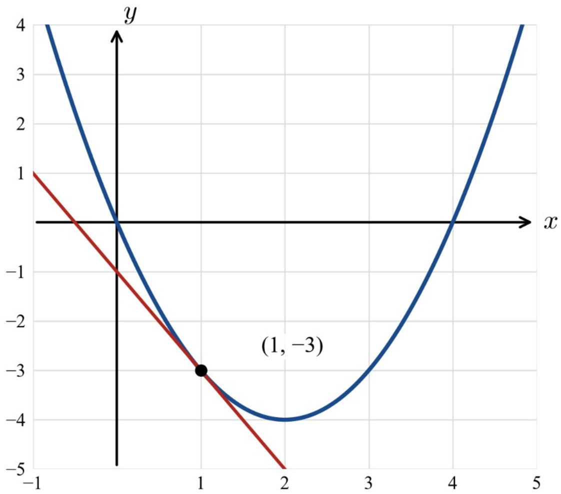
Q1. $y=4x-3$. **Q2.** $y=-2x-1$. **Q3.** Tangent $y=6x-9$; normal $y=-1/6x+19/2$. **Q4.** $y=3x-2$. **Q5.** $y=1/2x+3$. **Q6.** $(3,-9)$. **Q7.** $y=3x-10$. **Q8.** $y=-4x-5$. **Q9.** $y=11x-16$. **Q10.** $y=-1/2x+3/2$. **Q11.** $y=-x-3$. **Q12.** $y=-1/3x+4/3$. **Q13.** Tangent $y=2x-4$; normal $y=-1/2x+7/2$. **Q14.** $(3,-2)$, tangent $y=2x-8$. **Q15.** $(1,1)$ with $y=3x-2$, and $(-1,-1)$ with $y=3x+2$. **Q16.** $(2,-16)$ and $(-2,16)$. **Q17.** $(0,0)$, tangent $y=-2x$. **Q18.** $(1/2,0)$.

Fully-worked solutions

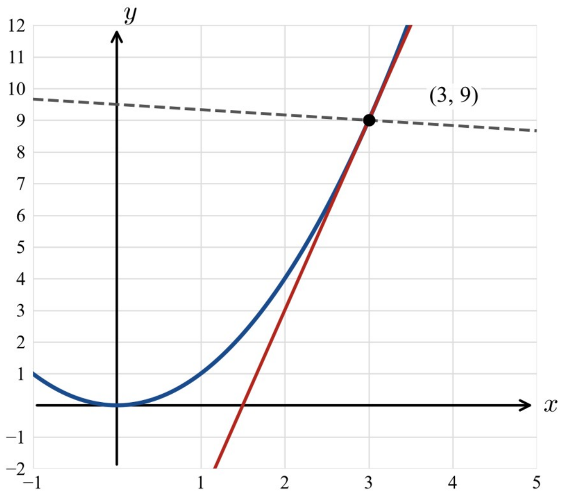
Q1. $f'(x)=2x$; (b) $f'(2)=4$; (c) $f(2)=5$, point $(2,5)$; (d) $y-5=4(x-2)\Rightarrow y=4x-3$.



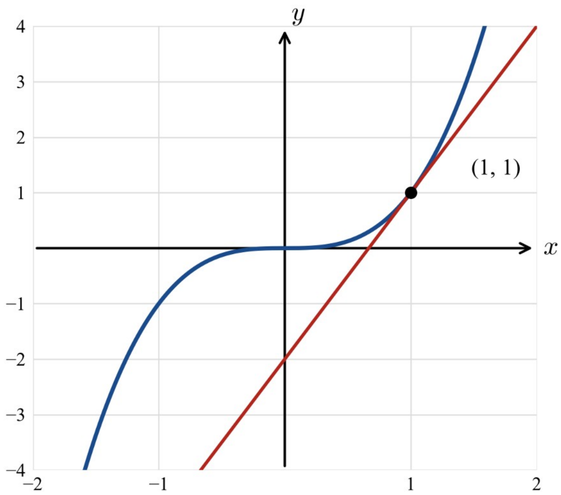
Q2. $f'(x)=2x-4$, $f'(1)=-2$; $f(1)=-3$, point $(1,-3)$; $y+3=-2(x-1)\Rightarrow y=-2x-1$.



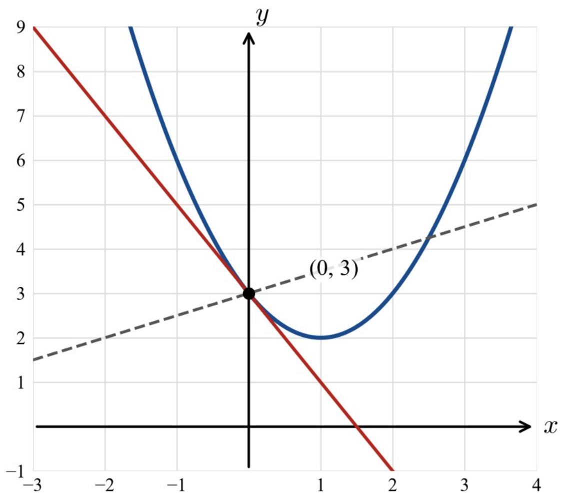
Q3. $f'(x)=2x$, $f'(3)=6$. Tangent: $y-9=6(x-3)\Rightarrow y=6x-9$. Normal gradient $-1/6$: $y-9=-1/6(x-3)\Rightarrow y=-1/6x+19/2$.



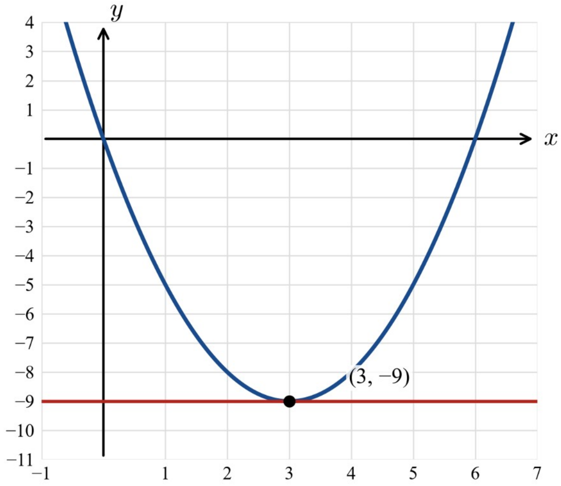
Q4. $f'(x)=3x^2$, $f'(1)=3$; $f(1)=1$; $y-1=3(x-1) \Rightarrow y=3x-2$.



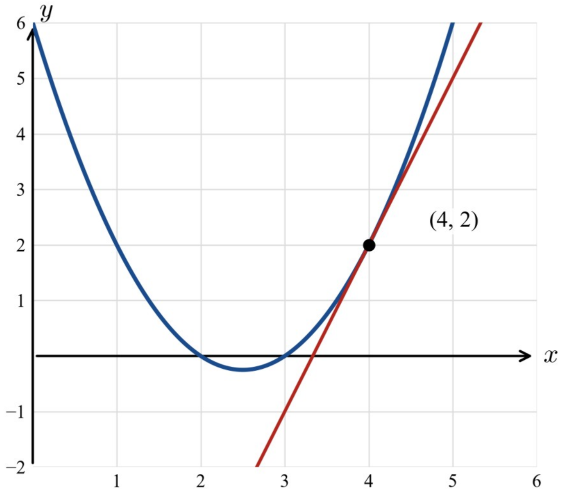
Q5. $f'(x)=2x-2$, $f'(0)=-2$; $f(0)=3$, point $(0,3)$. Normal gradient $-1/-2=1/2$:
$y-3=1/2(x-0) \Rightarrow y=1/2x+3$.



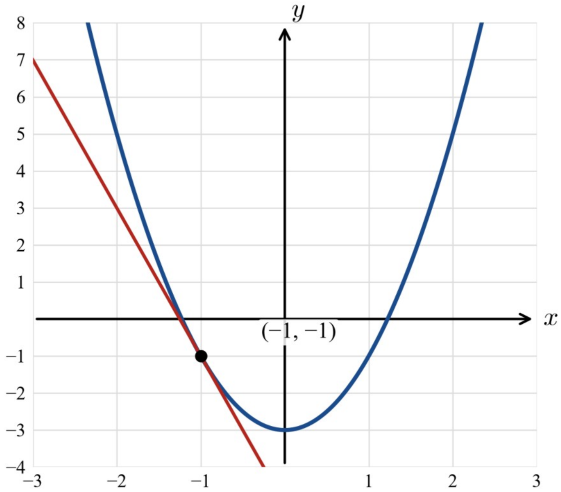
Q6. $f'(x)=2x-6=0 \Rightarrow x=3$; $f(3)=9-18=-9$. $\therefore (3,-9)$.



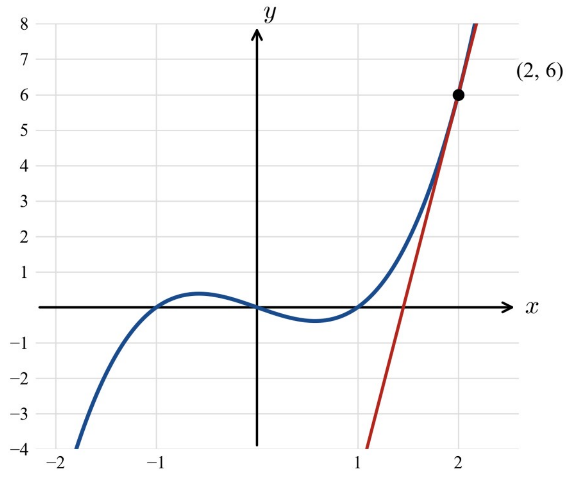
Q7. $f'(x)=2x-5$, $f'(4)=3$; $f(4)=16-20+6=2$; $y-2=3(x-4)\Rightarrow y=3x-10$.



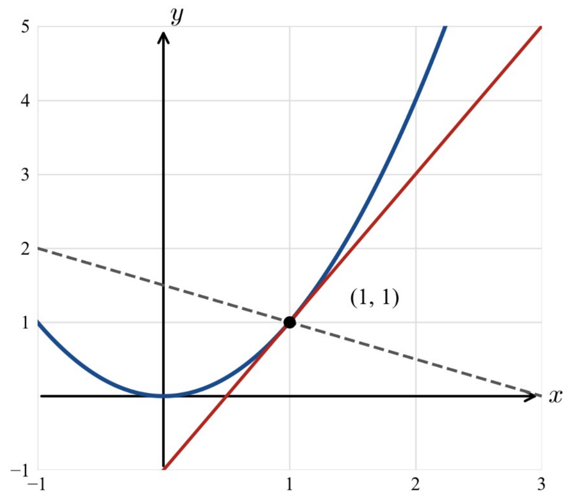
Q8. $f'(x)=4x$, $f'(-1)=-4$; $f(-1)=2-3=-1$; $y+1=-4(x+1)\Rightarrow y=-4x-5$.



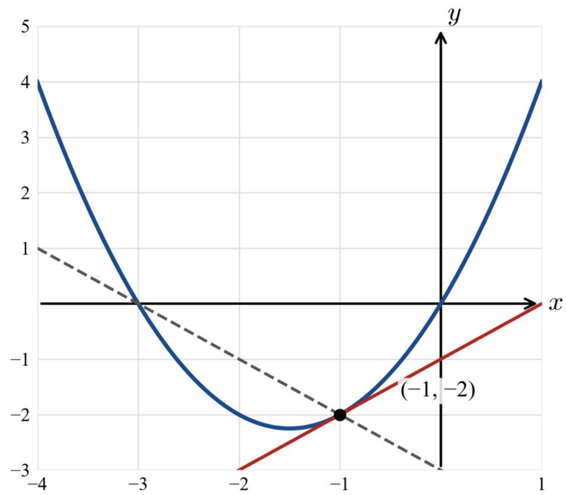
Q9. $f'(x)=3x^2-1$, $f'(2)=11$; $f(2)=8-2=6$; $y-6=11(x-2)\Rightarrow y=11x-16$.



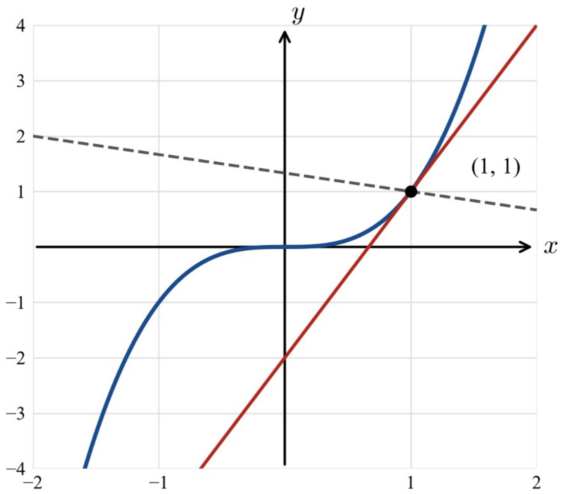
Q10. $f'(x)=2x$, $f'(1)=2$; normal gradient $-1/2$: $y-1=-1/2(x-1) \Rightarrow y=-1/2x+3/2$.



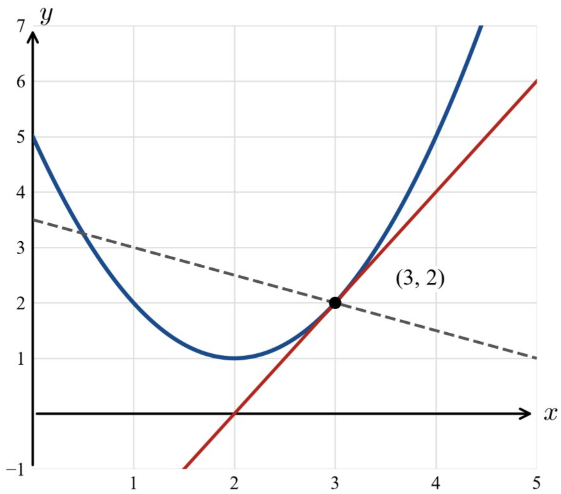
Q11. $f'(x)=2x+3$, $f'(-1)=1$; $f(-1)=1-3=-2$, point $(-1,-2)$. Normal gradient -1: $y+2=-1(x+1) \Rightarrow y=-x-3$.



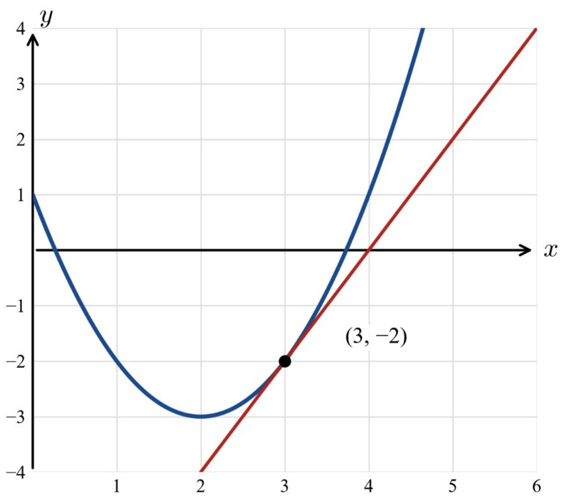
Q12. $f'(x)=3x^2$, $f'(1)=3$; normal gradient $-1/3$: $y-1=-1/3(x-1) \Rightarrow y=-1/3x+4/3$.



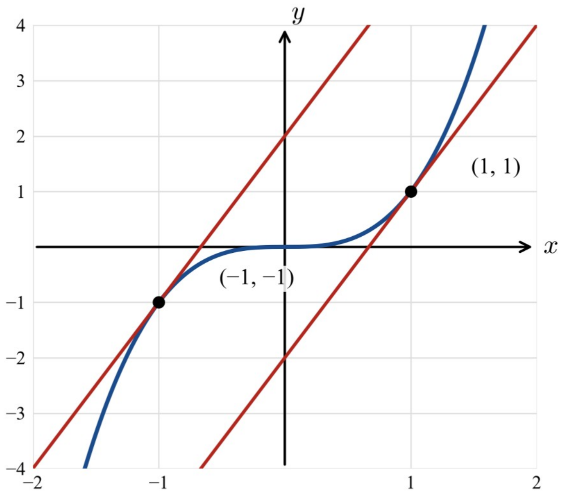
Q13. $f'(x)=2x-4$, $f'(3)=2$; $f(3)=9-12+5=2$, point $(3,2)$. Tangent $y-2=2(x-3)\Rightarrow y=2x-4$; normal gradient $-1/2$: $y-2=-1/2(x-3)\Rightarrow y=-1/2x+7/2$.



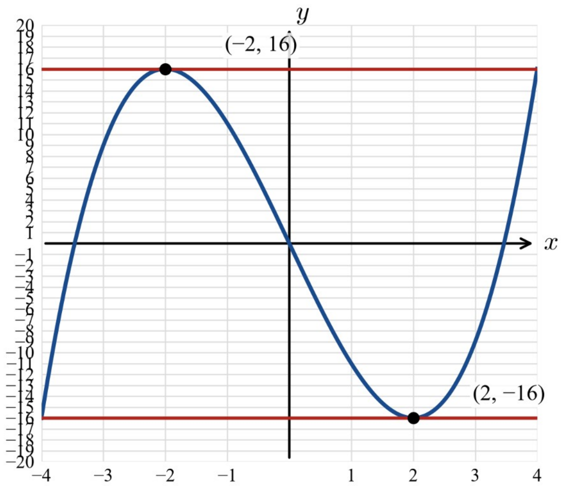
Q14. $f'(x)=2x-4=2\Rightarrow x=3$; $f(3)=9-12+1=-2$, point $(3,-2)$; tangent $y+2=2(x-3)\Rightarrow y=2x-8$.



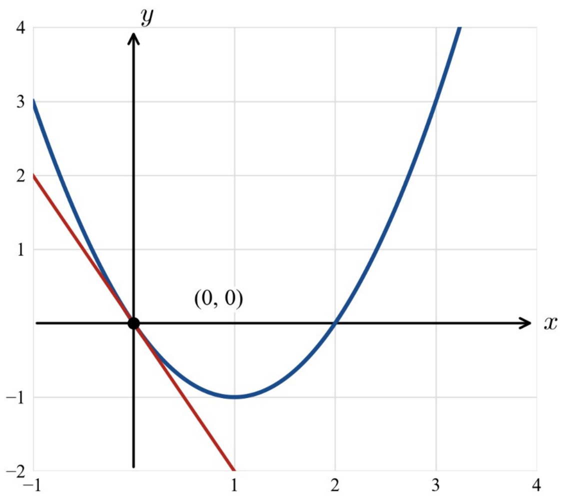
Q15. Parallel to $y=3x$ means gradient 3: $f'(x)=3x^2=3\Rightarrow x=\pm1$. At $(1,1)$: $y-1=3(x-1)\Rightarrow y=3x-2$; at $(-1,-1)$: $y+1=3(x+1)\Rightarrow y=3x+2$.



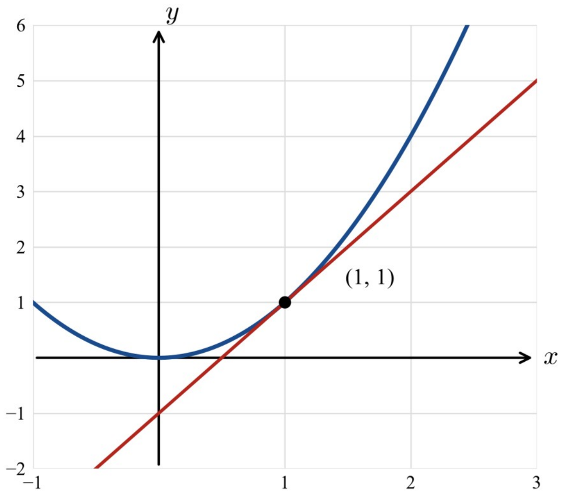
Q16. $f'(x)=3x^2-12=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$; $f(2)=8-24=-16$, $f(-2)=-8+24=16$. $\therefore (2,-16)$ and $(-2,16)$.



Q17. The curve crosses the y-axis at $x=0$: $f(0)=0$, point $(0,0)$. $f'(x)=2x-2$, $f'(0)=-2$; $y-0=-2(x-0) \Rightarrow y=-2x$.



Q18. $f'(x)=2x$, $f'(1)=2$; tangent $y-1=2(x-1) \Rightarrow y=2x-1$. It crosses the x-axis where $y=0$: $2x-1=0 \Rightarrow x=1/2$. $\therefore (1/2,0)$.


y
x
(1, 1)
6
5
4
3
2
1
−1
−2
−1
1
2
3

Theory

The derivative measures an **instantaneous rate of change**. If a quantity Q depends on a variable such as time t, then $\frac{dQ}{dt}$ (also written $Q'(t)$) is the rate at which Q is changing at that instant.

To find and use a rate:

1. **Differentiate** the quantity to obtain the rate function, e.g. $\frac{dV}{dt}$.
2. **Substitute** a value to find the rate at that instant.
3. To find *when* the rate has a particular value, **solve** $\frac{dQ}{dt}=k$ (a rate of zero means the quantity is momentarily neither increasing nor decreasing — often a maximum or minimum).

Units. A rate has the units of Q per unit of the variable — for example cm³/s, °C/min or L/min. Always state them.

Sign. A positive rate means Q is **increasing**; a negative rate means Q is **decreasing**.

Average versus instantaneous. The **average** rate of change of Q between a and b is $\frac{Q(b)-Q(a)}{b-a}$ (the gradient of a chord), whereas the **instantaneous** rate at $t=a$ is $Q'(a)$ (the gradient of the tangent).

Worked examples

Example 1 — scaffolded

A spherical balloon is inflated so that its volume is $V(t)=4t^2+3t$ cm³ after t seconds. Find the rate of change of volume, and its value when $t=2$.

Working	Comment
$\frac{dV}{dt}=8t+3$.	Differentiate V term by term.
At $t=2$: $\frac{dV}{dt}=8(2)+3=19$.	Substitute $t=2$.
∴ the volume is increasing at 19 cm³/s.	State the value with units and interpret the positive sign.

Example 2 — scaffolded

A plant's height is modelled by $h(t)=-0.1t^2+2t+5$ cm, where t is the number of weeks ($0\le t\le 10$). Find (a) the rate of growth, (b) when the plant is growing at 1 cm/week, and (c) when it stops growing.

Working	Comment
(a) $\frac{dh}{dt}=-0.2t+2$ cm/week.	Differentiate.
(b) $-0.2t+2=1 \Rightarrow 0.2t=1 \Rightarrow t=5$.	Solve rate $=1$.
(c) $-0.2t+2=0 \Rightarrow t=10$.	The plant stops growing when the rate is 0, at $t=10$ weeks.

Example 3 — unscaffolded

A draining tank holds $V(t)=1000-40t+0.4t^2$ litres, for $0\le t\le 50$ minutes. Find the rate at which water is leaving when $t=10$, and the time at which it is draining at 20 L/min.

$\frac{dV}{dt}=-40+0.8t$.

At $t=10$: $\frac{dV}{dt}=-40+0.8(10)=-32$, so water is leaving at 32 L/min.

Draining at 20 L/min means $\frac{dV}{dt}=-20$: $-40+0.8t=-20\Rightarrow 0.8t=20\Rightarrow t=25$.

∴ the tank is draining at 20 L/min when $t=25$ minutes.

Example 4 — unscaffolded

Oil spreads in a circular patch of area $A(r)=\pi r^2$ m², where r is the radius in metres. Find the rate of change of area with respect to the radius when $r=5$ m.

$\frac{dA}{dr}=2\pi r$.

At $r=5$: $\frac{dA}{dr}=2\pi(5)=10\pi$.

∴ the area is increasing at $10\pi\approx 31.4$ m² per metre of radius.

Practice questions

Scaffolded

Q1. For $V(t)=3t^2+2t$: (a) find $\frac{dV}{dt}$; (b) find the rate when $t=3$.

Q2. A ball's height is $h(t)=20t-5t^2$ m after t seconds. (a) Find the velocity $\frac{dh}{dt}$. (b) Find the velocity at $t=1$. (c) When is the velocity zero (the greatest height)?

Q3. A population is $P(t)=t^3-6t^2+20$ (hundreds) after t hours. (a) Find $\frac{dP}{dt}$. (b) Find the rate when $t=1$. (c) When is the rate zero?

Q4. A rectangle has area $A(x)=x(10-x)$. (a) Find $\frac{dA}{dx}$. (b) For what x is the rate of change of area zero?

Q5. The cost of producing x items is $C(x)=0.01x^2+5x+200$ dollars. (a) Find the marginal cost $\frac{dC}{dx}$. (b) Evaluate it at $x=100$.

Q6. A particle's position is $s(t)=2t^3-3t^2$ m. (a) Find the velocity. (b) When is the particle at rest?

Unscaffolded

Q7. For $V(t)=5t^2-t$, find the rate of change when $t=4$.

Q8. A stone's height is $h(t)=-5t^2+30t$ m after t seconds. Find its velocity at $t=2$, and the time when the velocity is zero.

Q9. A circular ripple has area $A(r)=\pi r^2$ m². Find the rate of change of area with respect to r when $r=3$ m.

Q10. A sphere has volume $V(r)=4/3\pi r^3$. Find $\frac{dV}{dr}$ when $r=2$.

Q11. A population is $P(t)=t^3-9t^2+24t$. Find the times when its rate of change is zero.

Q12. A hot object cools so its temperature is $T(t)=80-3t^2$ °C after t minutes. Find the rate of change of temperature at $t=4$.

Q13. A bacteria count is $N(t)=100+20t-t^2$ (thousands) after t hours. When is the count a maximum (rate zero)?

Q14. The cost of x units is $C(x)=x^3-12x^2+50x$ dollars. Find the marginal cost when $x=5$.

Q15. A particle's position is $s(t)=t^3-6t^2+9t$ m. Find the times when it is at rest.

Q16. A tank holds $V(t)=1200-30t+0.25t^2$ litres. Find the rate of change at $t=20$, and the time when it is draining at 10 L/min.

Q17. A ball's height is $h(t)=15t-5t^2$ m. Find its **average** velocity over the interval $[0,2]$ and its **instantaneous** velocity at $t=1$. Comment.

Q18. A circular oil slick has area $A(t)=4\pi t^2$ m² after t hours. Find the rate at which the area is increasing when $t=3$.

Answers

Q1. (a) $6t+2$; (b) 20. **Q2.** (a) $20-10t$; (b) 10 m/s; (c) $t=2$ s. **Q3.** (a) $3t^2-12t$; (b) -9 (hundred/h); (c) $t=0$ or $t=4$. **Q4.** (a) $10-2x$; (b) $x=5$. **Q5.** (a) $0.02x+5$; (b) $\$7$ per item. **Q6.** (a) $6t^2-6t$; (b) $t=0$ or $t=1$. **Q7.** 39. **Q8.** 10 m/s; $t=3$ s. **Q9.** 6π m²/m. **Q10.** 16π. **Q11.** $t=2$ and $t=4$. **Q12.** -24 °C/min. **Q13.** $t=10$ h. **Q14.** $\$5$ per unit. **Q15.** $t=1$ s and $t=3$ s. **Q16.** -20 L/min; $t=40$ min. **Q17.** average 5 m/s, instantaneous 5 m/s — equal here (for a quadratic, the average over $[0,2]$ equals the instantaneous rate at the midpoint $t=1$). **Q18.** 24π m²/h.

Fully-worked solutions

Q1. $\frac{dV}{dt}=6t+2$. At $t=3$: $6(3)+2=20$.

Q2. $\frac{dh}{dt}=20-10t$. (b) At $t=1$: $20-10=10$ m/s. (c) $20-10t=0\Rightarrow t=2$ s.

Q3. $\frac{dP}{dt}=3t^2-12t$. (b) At $t=1$: $3-12=-9$. (c) $3t^2-12t=3t(t-4)=0\Rightarrow t=0$ or $t=4$.

Q4. $A(x)=10x-x^2$, so $\frac{dA}{dx}=10-2x$. (b) $10-2x=0\Rightarrow x=5$.

Q5. $\frac{dC}{dx}=0.02x+5$. At $x=100$: $0.02(100)+5=2+5=7$, i.e. $\$7$ per extra item.

Q6. $\frac{ds}{dt}=6t^2-6t=6t(t-1)$. At rest when $6t(t-1)=0\Rightarrow t=0$ or $t=1$.

Q7. $\frac{dV}{dt}=10t-1$. At $t=4$: $10(4)-1=39$.

Q8. $\frac{dh}{dt}=-10t+30$. At $t=2$: $-20+30=10$ m/s. Velocity zero: $-10t+30=0\Rightarrow t=3$ s.

Q9. $\frac{dA}{dr}=2\pi r$. At $r=3$: 6π m²/m.

Q10. $\frac{dV}{dr}=4\pi r^2$. At $r=2$: 16π.

Q11. $\frac{dP}{dt}=3t^2-18t+24=3(t-2)(t-4)$. Zero at $t=2$ and $t=4$.

Q12. $\frac{dT}{dt}=-6t$. At $t=4$: -24 °C/min (the object is cooling).

Q13. $\frac{dN}{dt}=20-2t=0\Rightarrow t=10$ h.

Q14. $\frac{dC}{dx}=3x^2-24x+50$. At $x=5$: $75-120+50=5$, i.e. $\$5$ per unit.

Q15. $\frac{ds}{dt}=3t^2-12t+9=3(t-1)(t-3)$. At rest at $t=1$ s and $t=3$ s.

Q16. $\frac{dV}{dt} = -30 + 0.5t$. At $t = 20$: $-30 + 10 = -20$ L/min. Draining at 10 L/min:
$-30 + 0.5t = -10 \Rightarrow 0.5t = 20 \Rightarrow t = 40$ min.

Q17. Average $= \frac{h(2) - h(0)}{2 - 0} = \frac{10 - 0}{2} = 5$ m/s. Instantaneous: $\frac{dh}{dt} = 15 - 10t$, so at $t = 1$, $15 - 10 = 5$ m/s. ∴ they are equal here — for a quadratic the average rate over an interval equals the instantaneous rate at the midpoint.

Q18. $\frac{dA}{dt} = 8\pi t$. At $t = 3$: 24π m²/h.

Theory

A **stationary point** of a function f is a point on its graph where the gradient is zero — that is, where $f'(x)=0$. At such a point the tangent is **horizontal**.

To find the stationary points of $y=f(x)$:

1. Differentiate to find $f'(x)$.
2. Solve $f'(x)=0$ for x. Each solution is the x-coordinate of a stationary point.
3. Substitute each x-value back into the **original** rule $f(x)$ to find the corresponding y-coordinate.
4. State each stationary point as a coordinate pair (x, y).

A polynomial of degree n has a derivative of degree $n-1$, so it can have at most $n-1$ stationary points: a quadratic has one, a cubic has up to two, and a quartic has up to three.

This section is about **locating** stationary points. Deciding whether each one is a local **minimum**, a local **maximum**, or a **stationary point of inflection** is done in the next section (15D). (As a preview: a repeated factor of **even** multiplicity in $f'(x)$ — a squared factor, such as the $(x-1)^2$ in $f'(x)=3(x-1)^2$ — means $f'(x)$ does not change sign there, which signals a stationary point of inflection. A repeated factor of **odd** multiplicity, such as $(x-1)^3$, still changes sign and so gives a local maximum or minimum instead.)

Worked examples

Example 1 — scaffolded

Find the stationary point of $f(x)=x^2-6x+5$.

Working	Comment
$f'(x)=2x-6$.	Differentiate.
$f'(x)=0$: $2x-6=0 \Rightarrow x=3$.	Solve $f'(x)=0$ for the x-coordinate.
$f(3)=3^2-6(3)+5=9-18+5=-4$.	Substitute into the original rule for the y-coordinate.
$\therefore$ the stationary point is $(3,-4)$.	State the coordinate pair.

Example 2 — scaffolded

Find the stationary points of $f(x)=x^3-3x$.

Working	Comment
$f'(x)=3x^2-3$.	Differentiate.
$f'(x)=0$: $3x^2-3=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$.	A cubic can have two stationary points.
$f(1)=1-3=-2$ and $f(-1)=-1+3=2$.	Substitute each x-value.
$\therefore$ the stationary points are $(1,-2)$ and $(-1,2)$.	State both.

Example 3 — unscaffolded

Find the stationary points of $f(x)=x^3-6x^2+9x+1$.

$f'(x)=3x^2-12x+9=3(x^2-4x+3)=3(x-1)(x-3)$.

$f'(x)=0 \Rightarrow x=1$ or $x=3$.

$f(1)=1-6+9+1=5$ and $f(3)=27-54+27+1=1$.

$\therefore$ the stationary points are $(1,5)$ and $(3,1)$.

Example 4 — unscaffolded

Find the stationary points of $f(x)=x^4-8x^2$.

$f'(x)=4x^3-16x=4x(x^2-4)=4x(x-2)(x+2)$.

$f'(x)=0 \Rightarrow x=0$, $x=2$ or $x=-2$.

$f(0)=0$, $f(2)=16-32=-16$ and $f(-2)=16-32=-16$.

$\therefore$ the stationary points are $(0,0)$, $(2,-16)$ and $(-2,-16)$.

Practice questions

Scaffolded

Q1. For $f(x)=x^2-4x+7$: (a) find $f'(x)$; (b) solve $f'(x)=0$; (c) find the y-coordinate; (d) state the stationary point.

Q2. For $f(x)=x^2+2x-3$: (a) find $f'(x)$; (b) solve $f'(x)=0$; (c) state the stationary point.

Q3. For $f(x)=x^3-12x$: (a) find $f'(x)$ and factorise it; (b) solve $f'(x)=0$; (c) find both stationary points.

Q4. For $f(x)=2x^3-3x^2-12x$: (a) find $f'(x)$; (b) solve $f'(x)=0$; (c) find both stationary points.

Q5. For $f(x)=x^3-3x^2+3x-1$: (a) find $f'(x)$; (b) show that $f'(x)=3(x-1)^2$; (c) find the stationary point (note the repeated factor).

Q6. For $f(x)=x^4-2x^2$: (a) find $f'(x)$ and factorise it fully; (b) solve $f'(x)=0$; (c) find all three stationary points.

Unscaffolded

Find the coordinates of all stationary points of each function.

Q7. $f(x)=x^2-8x+1$

Q8. $f(x)=5+4x-x^2$

Q9. $f(x)=x^3-27x$

Q10. $f(x)=x^3+3x^2-9x+2$

Q11. $f(x)=2x^3+3x^2-12x+5$

Q12. $f(x)=x^3-6x^2+12x-5$

Q13. $f(x)=x^4-4x^3$

Q14. $f(x)=3x^4-4x^3$

Q15. $f(x)=x^4-8x^2+3$

Q16. $f(x)=x^3-3x^2-9x+5$

Q17. The curve $y=x^3+ax+b$ has a stationary point at $(2,-4)$. Find the values of a and b.

Q18. The function $f(x)=x^3+ax^2+bx+c$ has stationary points where $x=-1$ and $x=3$. Find the values of a and b.

Answers

Q1. $(2,3)$. **Q2.** $(-1,-4)$. **Q3.** $(2,-16)$ and $(-2,16)$. **Q4.** $(2,-20)$ and $(-1,7)$. **Q5.** $(1,0)$. **Q6.** $(0,0)$, $(1,-1)$ and $(-1,-1)$. **Q7.** $(4,-15)$. **Q8.** $(2,9)$. **Q9.** $(3,-54)$ and $(-3,54)$. **Q10.** $(1,-3)$ and $(-3,29)$. **Q11.** $(1,-2)$ and $(-2,25)$. **Q12.** $(2,3)$. **Q13.** $(0,0)$ and $(3,-27)$. **Q14.** $(0,0)$ and $(1,-1)$. **Q15.** $(0,3)$, $(2,-13)$ and $(-2,-13)$. **Q16.** $(3,-22)$ and $(-1,10)$. **Q17.** $a=-12$, $b=12$. **Q18.** $a=-3$, $b=-9$.

Fully-worked solutions

Q1. $f'(x)=2x-4$. $f'(x)=0 \Rightarrow x=2$. $f(2)=4-8+7=3$. $\therefore$ stationary point $(2,3)$.

Q2. $f'(x)=2x+2$. $f'(x)=0 \Rightarrow x=-1$. $f(-1)=1-2-3=-4$. $\therefore$ stationary point $(-1,-4)$.

Q3. $f'(x)=3x^2-12=3(x-2)(x+2)$. $f'(x)=0 \Rightarrow x=2$ or $x=-2$. $f(2)=8-24=-16$, $f(-2)=-8+24=16$. $\therefore (2,-16)$ and $(-2,16)$.

Q4. $f'(x)=6x^2-6x-12=6(x-2)(x+1)$. $f'(x)=0 \Rightarrow x=2$ or $x=-1$. $f(2)=16-12-24=-20$, $f(-1)=-2-3+12=7$. $\therefore (2,-20)$ and $(-1,7)$.

Q5. $f'(x)=3x^2-6x+3=3(x^2-2x+1)=3(x-1)^2$. $f'(x)=0 \Rightarrow x=1$ (a repeated solution). $f(1)=1-3+3-1=0$. $\therefore$ stationary point $(1,0)$ (a stationary point of inflection).

Q6. $f'(x)=4x^3-4x=4x(x-1)(x+1)$. $f'(x)=0 \Rightarrow x=0$, $x=1$ or $x=-1$. $f(0)=0$, $f(1)=1-2=-1$, $f(-1)=1-2=-1$. $\therefore (0,0)$, $(1,-1)$ and $(-1,-1)$.

Q7. $f'(x)=2x-8=0 \Rightarrow x=4$. $f(4)=16-32+1=-15$. $\therefore (4,-15)$.

Q8. $f'(x)=4-2x=0 \Rightarrow x=2$. $f(2)=5+8-4=9$. $\therefore (2,9)$.

Q9. $f'(x)=3x^2-27=3(x-3)(x+3)=0 \Rightarrow x=\pm 3$. $f(3)=27-81=-54$, $f(-3)=-27+81=54$. $\therefore (3,-54)$ and $(-3,54)$.

Q10. $f'(x)=3x^2+6x-9=3(x-1)(x+3)=0 \Rightarrow x=1$ or $x=-3$. $f(1)=1+3-9+2=-3$, $f(-3)=-27+27+27+2=29$. $\therefore (1,-3)$ and $(-3,29)$.

Q11. $f'(x)=6x^2+6x-12=6(x-1)(x+2)=0 \Rightarrow x=1$ or $x=-2$. $f(1)=2+3-12+5=-2$, $f(-2)=-16+12+24+5=25$. $\therefore (1,-2)$ and $(-2,25)$.

Q12. $f'(x)=3x^2-12x+12=3(x-2)^2=0 \Rightarrow x=2$ (repeated). $f(2)=8-24+24-5=3$. $\therefore (2,3)$ (a stationary point of inflection).

Q13. $f'(x)=4x^3-12x^2=4x^2(x-3)=0 \Rightarrow x=0$ (repeated) or $x=3$. $f(0)=0$, $f(3)=81-108=-27$. $\therefore (0,0)$ and $(3,-27)$.

Q14. $f'(x)=12x^3-12x^2=12x^2(x-1)=0 \Rightarrow x=0$ (repeated) or $x=1$. $f(0)=0$, $f(1)=3-4=-1$. $\therefore (0,0)$ and $(1,-1)$.

Q15. $f'(x)=4x^3-16x=4x(x-2)(x+2)=0 \Rightarrow x=0$, $x=2$ or $x=-2$. $f(0)=3$, $f(2)=16-32+3=-13$, $f(-2)=16-32+3=-13$. $\therefore (0,3)$, $(2,-13)$ and $(-2,-13)$.

Q16. $f'(x)=3x^2-6x-9=3(x-3)(x+1)=0 \Rightarrow x=3$ or $x=-1$. $f(3)=27-27-27+5=-22$, $f(-1)=-1-3+9+5=10$. $\therefore (3,-22)$ and $(-1,10)$.

Q17. $f'(x)=3x^2+a$. A stationary point at $x=2$ requires $f'(2)=0$: $3(2)^2+a=12+a=0 \Rightarrow a=-12$. The point $(2,-4)$ lies on the curve: $2^3+a(2)+b=8-24+b=-4 \Rightarrow b=12$. $\therefore$ $a=-12$, $b=12$.

Q18. $f'(x)=3x^2+2ax+b$. Since $x=-1$ and $x=3$ give $f'(x)=0$: $f'(-1)=3-2a+b=0$ and $f'(3)=27+6a+b=0$. Subtracting the first from the second: $24+8a=0 \Rightarrow a=-3$. Then $3-2(-3)+b=0 \Rightarrow 9+b=0 \Rightarrow b=-9$. $\therefore$ $a=-3$, $b=-9$.

Chapter 15 Applications of differentiation — 15D Classifying stationary points

Theory

A **stationary point** of $y=f(x)$ is a point where the gradient is zero, that is, where $f'(x)=0$. The tangent there is horizontal.

There are **three types** of stationary point:

- a **local maximum** — the curve stops rising and begins to fall;
- a **local minimum** — the curve stops falling and begins to rise;
- a **stationary point of inflection** — the curve flattens momentarily but continues in the same direction.

Classifying with a sign table. To decide the type, examine the **sign of $f'(x)$ just to the left and just to the right** of the stationary point:

- f' changes from $+$ to $-$: **local maximum**;
- f' changes from $-$ to $+$: **local minimum**;
- f' has the **same sign** on both sides: **stationary point of inflection**.

A **sign table** (also called a gradient table) organises this. For example, for a stationary point at $x=a$, test a value just below and just above a.

To sketch a polynomial using calculus:

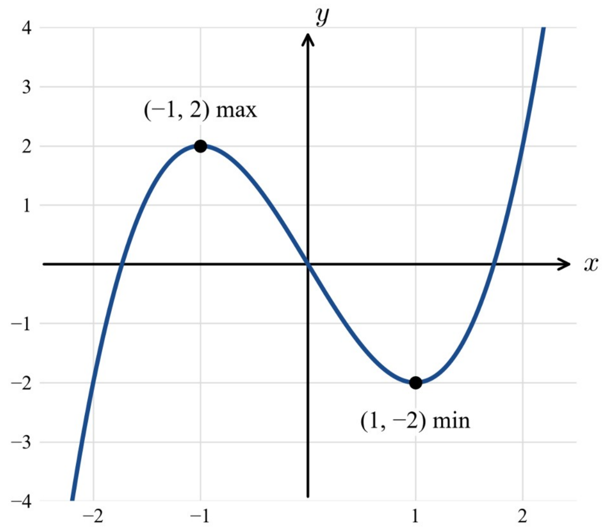
1. Find $f'(x)$ and solve $f'(x)=0$ to locate the stationary points.
2. Classify each using a sign table, and find its coordinates.
3. Find the axis intercepts.
4. Mark these features and draw a smooth curve.

Worked examples

Example 1 — scaffolded

Find and classify the stationary points of $y=x^3-3x$, then sketch the graph.

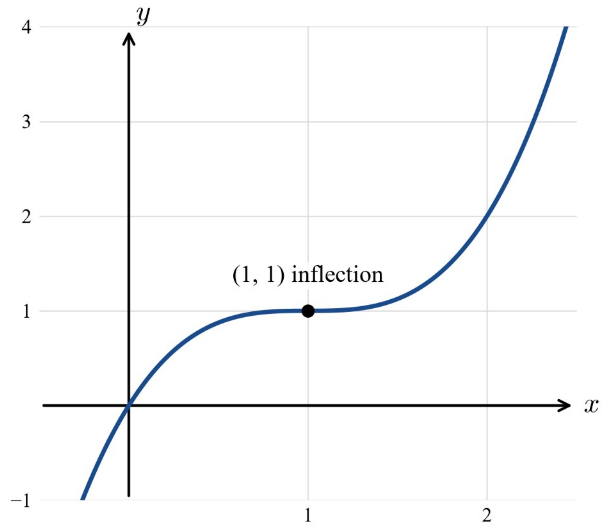
Working	Comment
$\frac{dy}{dx}=3x^2-3=3(x-1)(x+1)$.	Differentiate, then factorise.
$\frac{dy}{dx}=0$ at $x=-1$ and $x=1$.	Stationary points occur where the gradient is zero.
At $x=-1$: $f'(-2)=9>0$ and $f'(0)=-3<0$, so f' changes $+ \to -$ — a **local maximum**. $y=(-1)^3-3(-1)=2$, giving $(-1,2)$.	Sign of the gradient each side of $x=-1$.
At $x=1$: $f'(0)=-3<0$ and $f'(2)=9>0$, so f' changes $- \to +$ — a **local minimum**. $y=1-3=-2$, giving $(1,-2)$.	Sign of the gradient each side of $x=1$.



Example 2 — scaffolded

Find and classify the stationary points of $y = x^3 - 3x^2 + 3x$.

Working	Comment
$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x-1)^2$.	Differentiate and factorise.
$\frac{dy}{dx} = 0$ at $x = 1$ only.	The repeated factor gives a single stationary point.
$f'(0) = 3 > 0$ and $f'(2) = 3 > 0$: the gradient is **positive on both sides**, so this is a **stationary point of inflection**.	Same sign each side ⇒ inflection.
$y = 1 - 3 + 3 = 1$, giving $(1, 1)$.	Substitute to find the y-coordinate.



Example 3 — unscaffolded

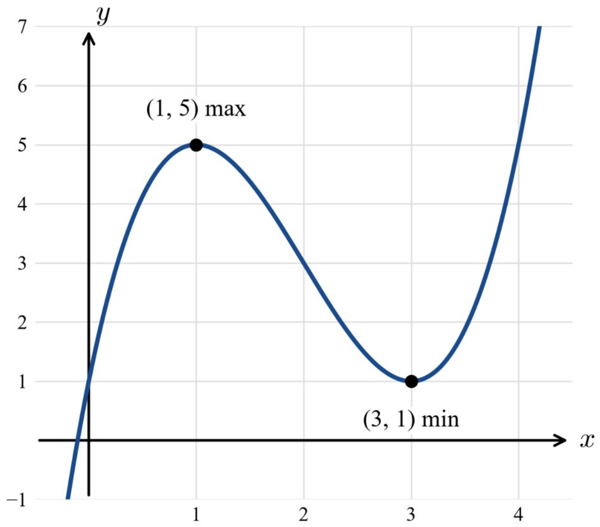
Find and classify the stationary points of $y = x^3 - 6x^2 + 9x + 1$, then sketch.

$\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, which is zero at $x = 1$ and $x = 3$.

At $x = 1$: $f'(0) = 9 > 0$, $f'(2) = -3 < 0$ $(+ \to -)$ — **local maximum** $(1, 5)$.

At $x = 3$: $f'(2) = -3 < 0$, $f'(4) = 9 > 0$ $(- \to +)$ — **local minimum** $(3, 1)$.

∴ the graph is:



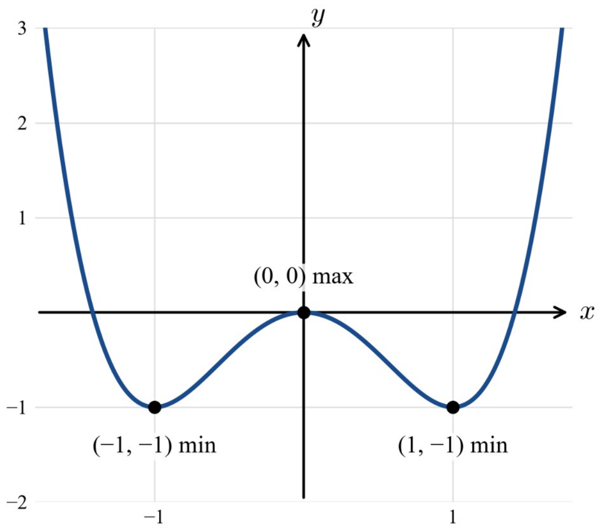
Example 4 — unscaffolded

Find and classify the stationary points of $y=x^4-2x^2$.

$\frac{dy}{dx}=4x^3-4x=4x(x-1)(x+1)$, zero at $x=-1,0,1$.

Testing the sign of f' across each: $x=-1$ gives − → + (**local minimum** $(-1,-1)$); $x=0$ gives + → − (**local maximum** $(0,0)$); $x=1$ gives − → + (**local minimum** $(1,-1)$).

∴ the graph is a "W" shape:



Practice questions

Scaffolded

Q1. For $y=x^2-4x+1$: (a) find $\frac{dy}{dx}$; (b) find the stationary point; (c) classify it with a sign table; (d) sketch.

Q2. For $y=x^3-12x$: (a) find the stationary points; (b) classify each with a sign table; (c) sketch.

Q3. For $y=-x^3+3x$: (a) find the stationary points; (b) classify each; (c) sketch.

Q4. For $y = x^3$: (a) find the stationary point; (b) show, using a sign table, that it is a stationary point of inflection; (c) sketch.

Q5. For $y = x^3 - 3x^2 + 2$: (a) find and classify the stationary points; (b) sketch.

Q6. For $y = 2x^3 - 3x^2 - 12x$: (a) find and classify the stationary points; (b) sketch.

Unscaffolded

For each function, find and classify **all** stationary points, and sketch the graph.

Q7. $y = x^2 + 6x + 5$

Q8. $y = -x^2 + 2x + 3$

Q9. $y = x^3 - 3x^2 + 4$

Q10. $y = x^3 - 6x^2 + 9x - 2$

Q11. $y = 2x^3 - 9x^2 + 12x$

Q12. $y = (x-1)^3 + 2$

Q13. $y = -x^3 + 3x^2 - 3x + 2$

Q14. $y = x^4 - 8x^2$

Q15. $y = 3x^4 - 4x^3$

Q16. $y = x^3 - x$

Q17. $y = x^3 - 3x^2 - 9x + 5$

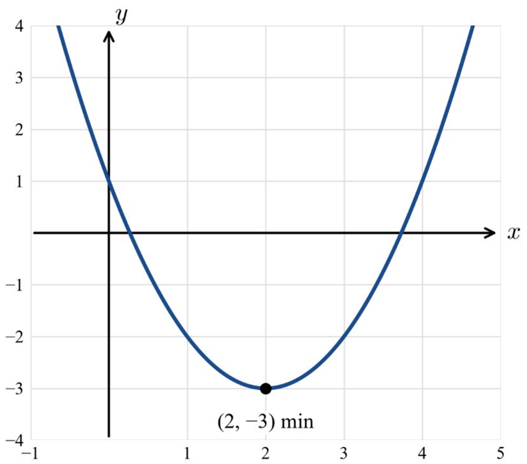
Q18. $y = (x^2 - 1)^2$

Answers

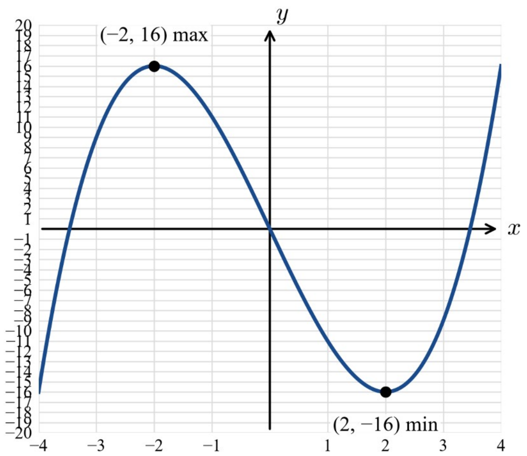
Q1. Minimum $(2,-3)$. **Q2.** Maximum $(-2,16)$; minimum $(2,-16)$. **Q3.** Minimum $(-1,-2)$; maximum $(1,2)$. **Q4.** Stationary point of inflection $(0,0)$. **Q5.** Maximum $(0,2)$; minimum $(2,-2)$. **Q6.** Maximum $(-1,7)$; minimum $(2,-20)$. **Q7.** Minimum $(-3,-4)$. **Q8.** Maximum $(1,4)$. **Q9.** Maximum $(0,4)$; minimum $(2,0)$. **Q10.** Maximum $(1,2)$; minimum $(3,-2)$. **Q11.** Maximum $(1,5)$; minimum $(2,4)$. **Q12.** Stationary point of inflection $(1,2)$. **Q13.** Stationary point of inflection $(1,1)$. **Q14.** Minima $(-2,-16)$ and $(2,-16)$; maximum $(0,0)$. **Q15.** Stationary point of inflection $(0,0)$; minimum $(1,-1)$. **Q16.** Maximum $\left(-\frac{\sqrt{3}}{3},\frac{2\sqrt{3}}{9}\right)$; minimum $\left(\frac{\sqrt{3}}{3},-\frac{2\sqrt{3}}{9}\right)$. **Q17.** Maximum $(-1,10)$; minimum $(3,-22)$. **Q18.** Minima $(-1,0)$ and $(1,0)$; maximum $(0,1)$.

Fully-worked solutions

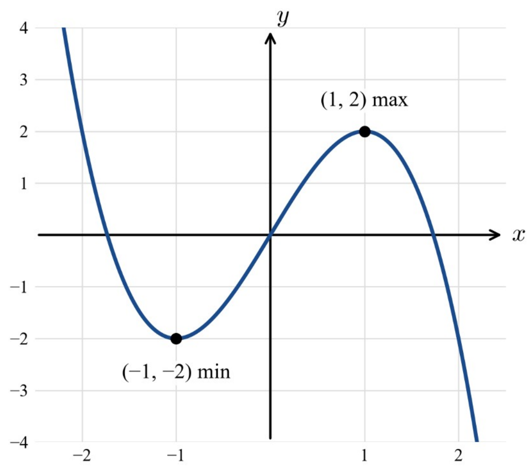
Q1. $\frac{dy}{dx}=2x-4=0\Rightarrow x=2$; $y=4-8+1=-3$. As f' changes $-\to+$, $(2,-3)$ is a **minimum**.



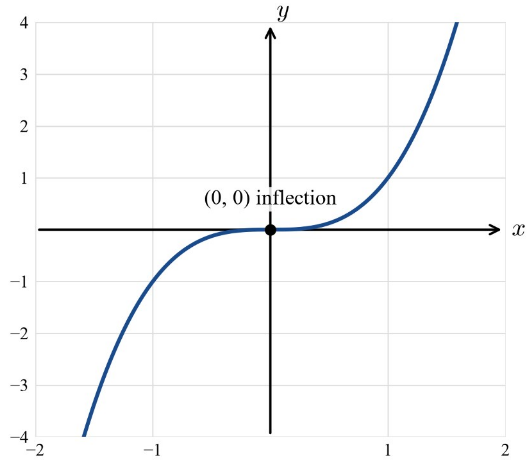
Q2. $\frac{dy}{dx}=3x^2-12=3(x-2)(x+2)=0\Rightarrow x=\pm2$. At $x=-2$ $(+\to-)$ **maximum** $(-2,16)$; at $x=2$ $(-\to+)$ **minimum** $(2,-16)$.



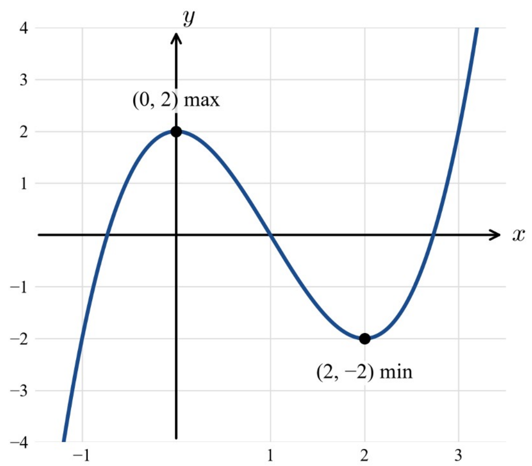
Q3. $\frac{dy}{dx}=-3x^2+3=-3(x-1)(x+1)=0\Rightarrow x=\pm1$. At $x=-1$ $(-\to+)$ **minimum** $(-1,-2)$; at $x=1$ $(+\to-)$ **maximum** $(1,2)$.



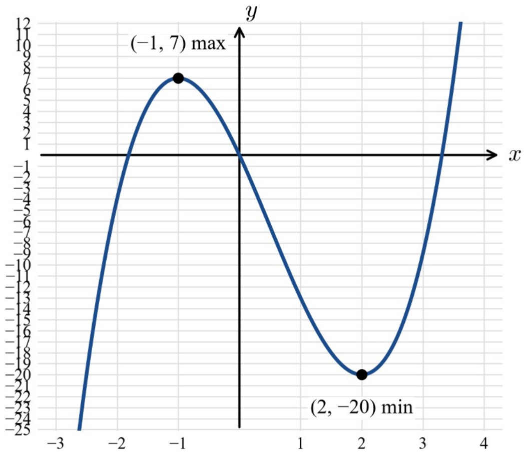
Q4. $\frac{dy}{dx}=3x^2=0 \Rightarrow x=0$. Since $f'(-1)=3>0$ and $f'(1)=3>0$ (same sign), $(0,0)$ is a **stationary point of inflection**.



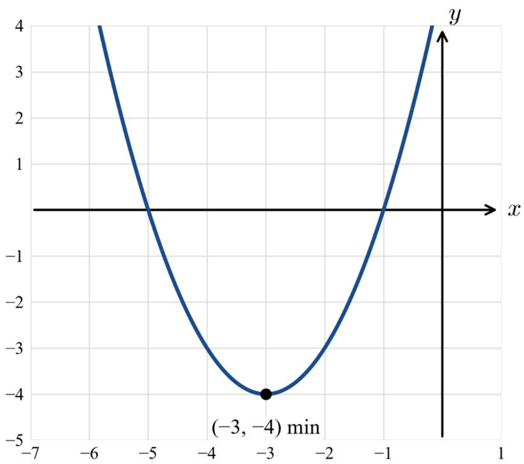
Q5. $\frac{dy}{dx}=3x^2-6x=3x(x-2)=0 \Rightarrow x=0,2$. At $x=0$ $(+\rightarrow-)$ **maximum** $(0,2)$; at $x=2$ $(-\rightarrow+)$ **minimum** $(2,-2)$.



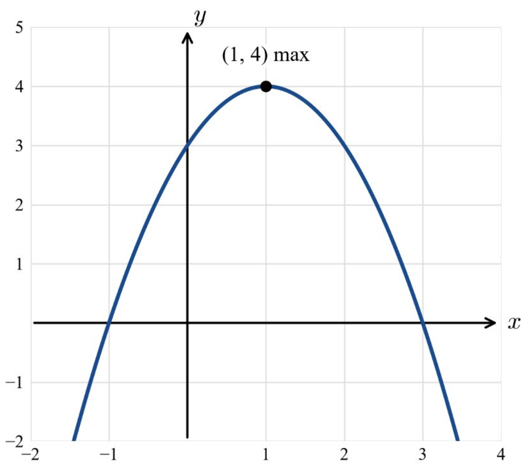
Q6. $\frac{dy}{dx}=6x^2-6x-12=6(x-2)(x+1)=0 \Rightarrow x=-1,2$. At $x=-1$ $(+\rightarrow-)$ **maximum** $(-1,7)$; at $x=2$ $(-\rightarrow+)$ **minimum** $(2,-20)$.



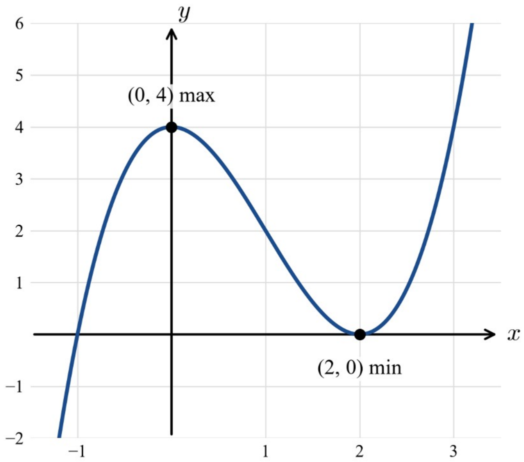
Q7. $\frac{dy}{dx}=2x+6=0 \Rightarrow x=-3$; $y=9-18+5=-4$. **Minimum** $(-3,-4)$.



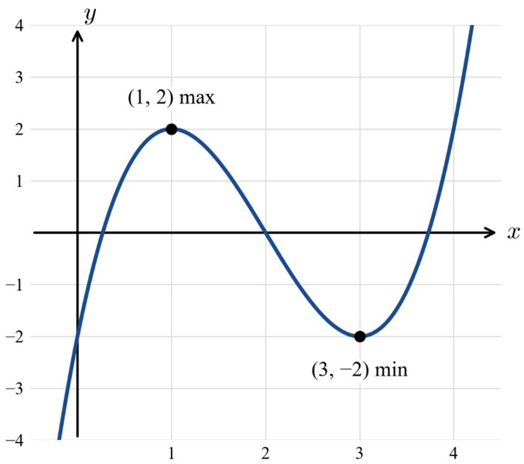
Q8. $\frac{dy}{dx}=-2x+2=0 \Rightarrow x=1$; $y=-1+2+3=4$. **Maximum** $(1,4)$.



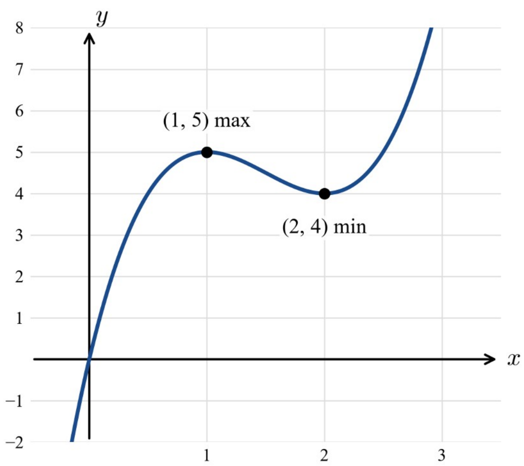
Q9. $\frac{dy}{dx}=3x^2-6x=3x(x-2)=0 \Rightarrow x=0,2$. **Maximum** $(0,4)$; **minimum** $(2,0)$.



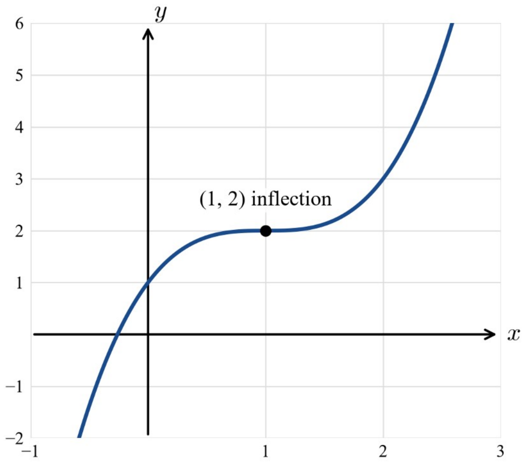
Q10. $\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0 \Rightarrow x = 1, 3$. **Maximum** $(1, 2)$; **minimum** $(3, -2)$.



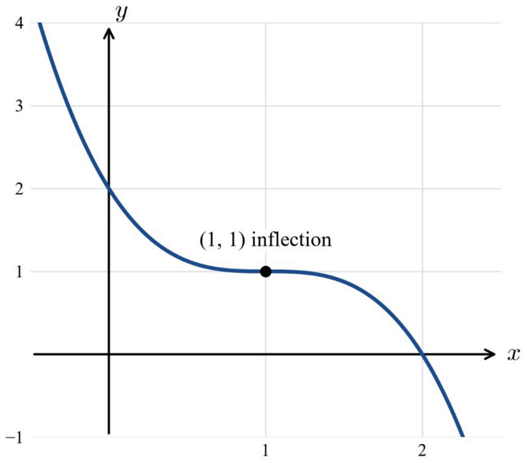
Q11. $\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x-1)(x-2) = 0 \Rightarrow x = 1, 2$. **Maximum** $(1, 5)$; **minimum** $(2, 4)$.



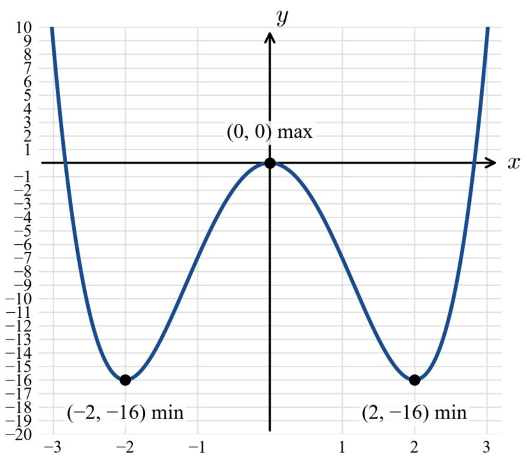
Q12. $\frac{dy}{dx} = 3(x-1)^2 = 0 \Rightarrow x = 1$. The gradient is positive on both sides, so $(1, 2)$ is a **stationary point of inflection**.



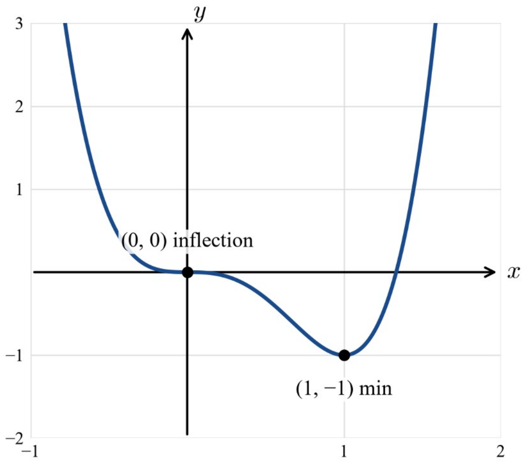
Q13. $\frac{dy}{dx}=-3x^2+6x-3=-3(x-1)^2=0\Rightarrow x=1$. The gradient is negative on both sides, so $(1,1)$ is a **stationary point of inflection**.



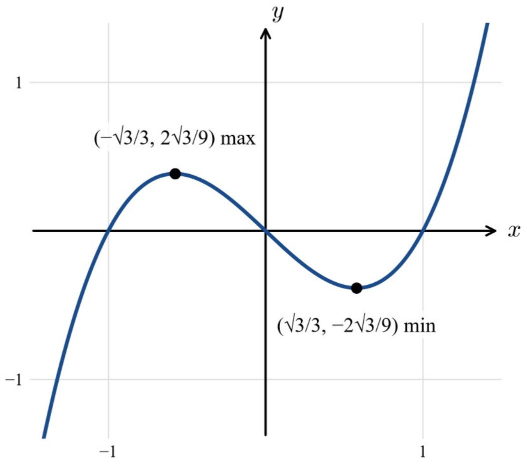
Q14. $\frac{dy}{dx}=4x^3-16x=4x(x-2)(x+2)=0\Rightarrow x=-2,0,2$. **Minima** $(-2,-16)$ and $(2,-16)$; **maximum** $(0,0)$.



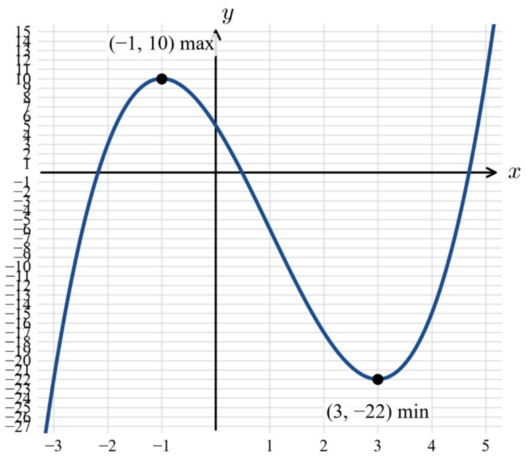
Q15. $\frac{dy}{dx}=12x^3-12x^2=12x^2(x-1)=0\Rightarrow x=0,1$. At $x=0$ the gradient is negative on both sides — a **stationary point of inflection** $(0,0)$; at $x=1$ ($-\rightarrow+$) a **minimum** $(1,-1)$.



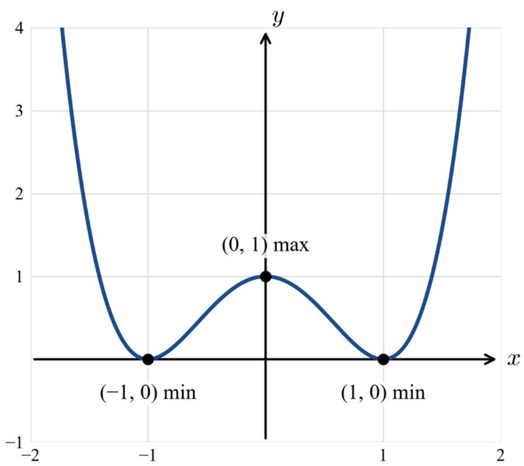
Q16. $\frac{dy}{dx}=3x^2-1=0 \Rightarrow x=\pm\frac{1}{\sqrt{3}}=\pm\frac{\sqrt{3}}{3}$. At $x=-\frac{\sqrt{3}}{3}$ $(+\to-)$ **maximum** $\left(-\frac{\sqrt{3}}{3},\frac{2\sqrt{3}}{9}\right)$; at $x=\frac{\sqrt{3}}{3}$ $(-\to+)$ **minimum** $\left(\frac{\sqrt{3}}{3},-\frac{2\sqrt{3}}{9}\right)$.



Q17. $\frac{dy}{dx}=3x^2-6x-9=3(x-3)(x+1)=0 \Rightarrow x=-1,3$. **Maximum** $(-1,10)$; **minimum** $(3,-22)$.



Q18. $\frac{dy}{dx}=4x^3-4x=4x(x-1)(x+1)=0 \Rightarrow x=-1,0,1$. **Minima** $(-1,0)$ and $(1,0)$; **maximum** $(0,1)$.



y
x
(0, 1) max
(−1, 0) min
(1, 0) min
4
3
2
1
−1
−2
−1
1
2

Units 3 & 4 preview (in part). Items in this section that require differentiating negative powers of x (the fixed-area perimeter and open-box surface-area minimisations) go beyond the Units 1 & 2 study design, which prescribes differentiation of polynomial functions only; they are included as preparation for Units 3 & 4 (see 14C) and are not assessed in the graded topic tests or practice examinations.

Theory

Many practical problems ask for the **largest** or **smallest** possible value of some quantity — the greatest area, the least cost, the maximum volume. Calculus solves these **optimisation** problems: at a maximum or minimum the gradient is zero, so we differentiate and solve $\frac{dy}{dx}=0$.

The method.

1. **Define** the variable(s) and write an expression for the quantity to be optimised.
2. Use any **constraint** to write that quantity as a function of a **single** variable.
3. State a **sensible domain** (lengths, times, etc. must be positive and physically possible).
4. **Differentiate** and solve $\frac{dQ}{dx}=0$ to find the stationary point(s) in the domain.
5. **Verify** it is the required maximum or minimum with a gradient sign check — the sign of $\frac{dQ}{dx}$ on either side of the stationary point ($+ \to -$ gives a maximum, $- \to +$ a minimum).
6. State the answer **with units**, and give the optimal value of the quantity.

Local and global (absolute) extrema. A stationary point gives a **local** (relative) maximum or minimum — the greatest or least value *in its immediate neighbourhood*. A practical problem usually wants the **global** (absolute) maximum or minimum — the greatest or least value over the **whole modelling domain**. On a **closed** domain $a \le x \le b$ the two need not coincide: a global maximum or minimum can occur at a stationary point **or at an endpoint** $x=a$ or $x=b$.

The closed-interval method. To find the global maximum and minimum of a quantity $Q(x)$ on $[a,b]$:

1. Solve $\frac{dQ}{dx}=0$ and keep only the stationary points **inside** the domain ($a<x<b$); discard any that fall outside it.
2. Evaluate Q at **every** such stationary point **and** at **both** endpoints $x=a$ and $x=b$.
3. **Compare** the values: the largest is the global maximum, the smallest is the global minimum.
4. State the answer **in context**, with units.

If a stationary point lies **outside** the modelling domain, it is ignored — the extremum is then decided by the endpoints alone. Always check the endpoints: the biggest turning point is not always the biggest value.

Worked examples

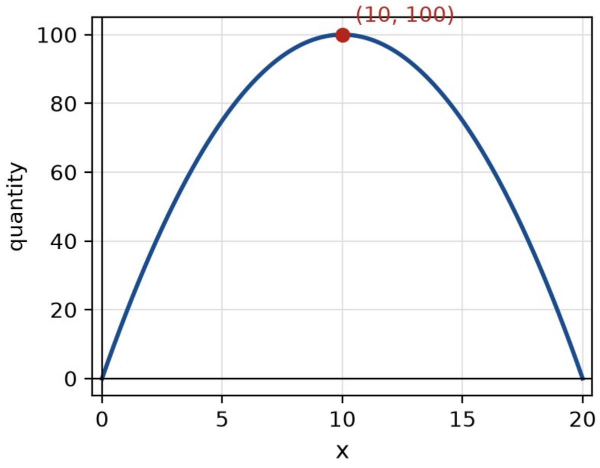
Example 1 — scaffolded

A rectangular paddock is to be enclosed with 40 m of fencing. Find the dimensions that give the **greatest area**.

Working	Comment
Let the width be x m. The perimeter is 40, so $2(x+\ell)=40$, giving length $\ell=20-x$.	Use the constraint to get everything in terms of x.
Area $A=x(20-x)=20x-x^2$, with domain $0<x<20$.	The quantity to maximise, in one variable.

Working	Comment
$\dfrac{dA}{dx}=20-2x$. Setting $=0$: $x=10$.	Solve $A'(x)=0$.
$\dfrac{dA}{dx}=20-2x$ is positive for $x<10$ and negative for $x>10$, so $x=10$ gives a **maximum**.	Gradient sign check.
Then $\ell=20-10=10$, and $A=10\times10=100$.	A $10\,\mathrm{m}\times10\,\mathrm{m}$ square.

∴ the maximum area is $100\,\mathrm{m}^2$, when the paddock is $10\,\mathrm{m}\times10\,\mathrm{m}$.

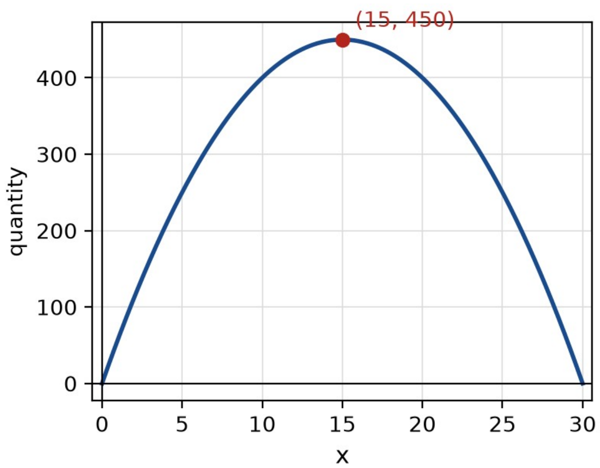


Example 2 — scaffolded

A rectangular yard is fenced on **three sides**, with the fourth side against a wall. There is 60 m of fencing. Find the maximum area.

Working	Comment
Let the two sides perpendicular to the wall each be x m; the side parallel to the wall is $60-2x$.	Only three sides are fenced.
Area $A=x(60-2x)=60x-2x^2$, domain $0<x<30$.	One-variable objective.
$A'(x)=60-4x=0\Rightarrow x=15$.	Stationary point.
$A'(x)>0$ for $x<15$ and $A'(x)<0$ for $x>15$: **maximum**.	Gradient sign check.
$A=15(60-30)=15\times30=450$.	Dimensions $15\,\mathrm{m}\times30\,\mathrm{m}$.

∴ the maximum area is $450\,\mathrm{m}^2$.



Example 3 — unscaffolded

An open box is made from a square sheet of card of side 15 cm by cutting equal squares of side x cm from the corners and folding up the sides. Find the value of x giving the **maximum volume**.

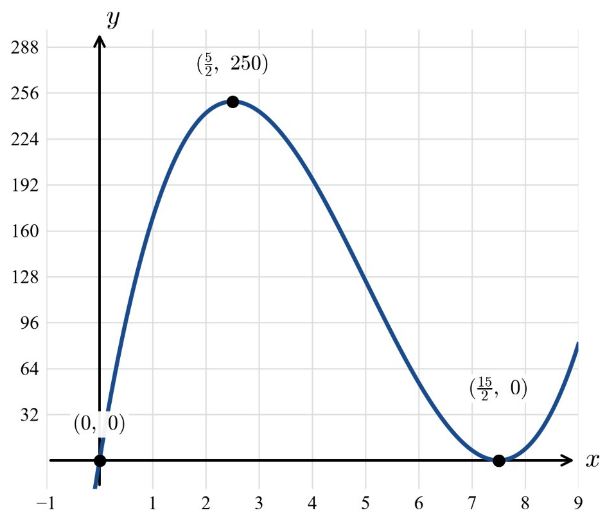
The base is $(15-2x)$ by $(15-2x)$ and the height is x, so

$$V=x(15-2x)^2=4x^3-60x^2+225x, 0<x<15/2.$$

$V'(x)=12x^2-120x+225=3(2x-5)(2x-15)$. Setting $V'(x)=0$ gives $x=\frac{15}{2}$ (rejected, outside the domain) or $x=\frac{5}{2}$.

$V'(x)$ is positive for $0<x<\frac{5}{2}$ and negative for $\frac{5}{2}<x<\frac{15}{2}$, so $x=\frac{5}{2}$ gives a maximum, and $V=\frac{5}{2}(15-5)^2=\frac{5}{2}\times 100=250$.

∴ the maximum volume is $250\,\text{cm}^3$, when $x=\frac{5}{2}$ cm.



Example 4 — unscaffolded

A rectangle has area $36\,\text{m}^2$. Find the dimensions that give the **least perimeter**.

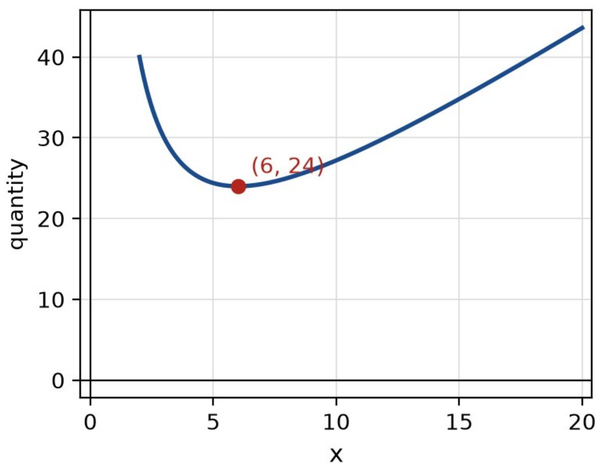
Let the width be x m; the length is $\frac{36}{x}$. The perimeter is

$$P=2x+\frac{72}{x}, x>0.$$

$P'(x)=2-\frac{72}{x^2}=0 \Rightarrow x^2=36 \Rightarrow x=6$ (taking $x>0$).

$P''(x)=\frac{144}{x^3}>0$, so $x=6$ gives a **minimum**. Then $\frac{36}{x}=6$ and $P=12+12=24$.

∴ the least perimeter is 24 m, when the rectangle is a $6\,\text{m}\times 6\,\text{m}$ square.



Example 5 — scaffolded

The volume of water V (in gigalitres, GL) stored in a reservoir over a 10-week period is modelled by

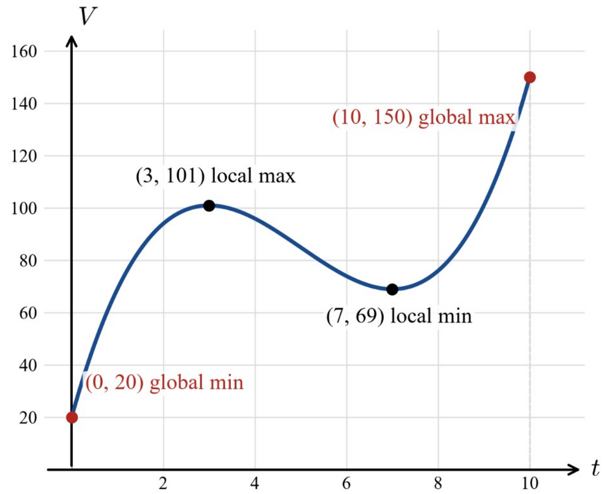
$$V(t)=t^3-15t^2+63t+20, 0\le t\le 10,$$

where t is the time in weeks. Find the **greatest** and **least** volume of water during this period.

Here the domain is a **closed** interval, so we must compare the stationary points with the endpoints.

Working	Comment
$V'(t)=3t^2-30t+63=3(t-3)(t-7)$. Setting $V'(t)=0$ gives $t=3$ or $t=7$, both inside $[0,10]$.	Stationary points from $V'(t)=0$.
$V(3)=27-135+189+20=101$ (local max), $V(7)=343-735+441+20=69$ (local min).	Evaluate V at each stationary point.
Endpoints: $V(0)=20$ and $V(10)=1000-1500+630+20=150$.	Evaluate V at **both** endpoints too.
Compare $\{20, 101, 69, 150\}$: the largest is 150 (at $t=10$) and the smallest is 20 (at $t=0$).	Both global extrema are at **endpoints**.

$\therefore$ the greatest volume is 150 GL (after 10 weeks) and the least is 20 GL (at the start) — neither occurs at a turning point.



Example 6 — unscaffolded

A pottery studio makes x ceramic pots per day. Its daily profit, in dollars, is modelled by

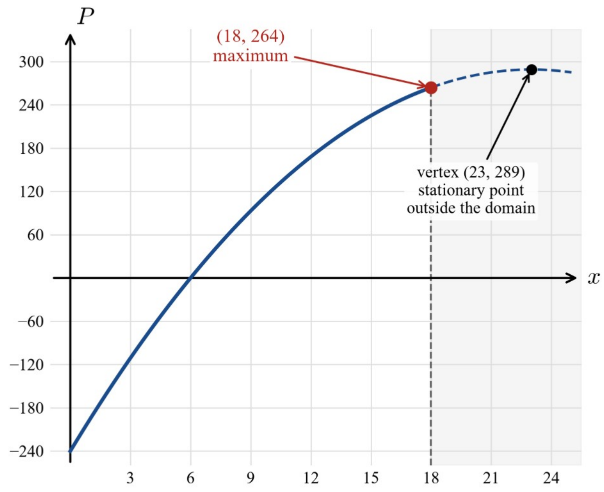
$$P(x) = -x^2 + 46x - 240, 0 \leq x \leq 18,$$

where the kiln capacity limits output to at most 18 pots. How many pots per day give the **maximum profit**?

$P'(x) = -2x + 46 = 0 \Rightarrow x = 23$. This stationary point lies **outside** the domain $[0, 18]$, so it is rejected: there is no turning point in the modelling domain, and the maximum must occur at an endpoint.

Evaluate at the endpoints: $P(0) = -240$ and $P(18) = -324 + 828 - 240 = 264$. Comparing, the greater is 264, at $x = 18$.

$\therefore$ the maximum profit is $\$264$ per day, made by producing 18 pots — the endpoint of the domain, since the unrestricted turning point at $x = 23$ is beyond the kiln's capacity.



Practice questions

Scaffolded

Q1. A rectangle has perimeter 24 cm. (a) If the width is x, write the length and the area $A(x)$. (b) Find $A'(x)$ and solve $A'(x) = 0$. (c) Verify it is a maximum. (d) State the maximum area.

Q2. A yard is fenced on three sides against a wall using 80 m of fencing. (a) Write the area $A(x)$ where x is the length of each side perpendicular to the wall. (b) Find the value of x that maximises the area. (c) State the maximum area.

Q3. Two numbers add to 16. (a) Write their product $P(x)$ where one number is x. (b) Find the value of x that maximises P. (c) State the maximum product.

Q4. An open box is made from a 18 cm square sheet by cutting squares of side x from the corners. (a) Write the volume $V(x)$. (b) Find $V'(x)$ and solve $V'(x) = 0$ in the domain $0 < x < 9$. (c) State the maximum volume.

Q5. A rectangle has area $100\,\text{m}^2$. (a) Write the perimeter $P(x)$ where x is the width. (b) Find x that minimises P. (c) State the least perimeter.

Q6. Two numbers add to 20. (a) Write the sum of their squares $S(x)$ where one number is x. (b) Find x that minimises S. (c) State the minimum.

Unscaffolded

For each problem, form a function of one variable, use calculus to find the optimum, verify it, and state the answer with units.

Q7. A rectangular paddock is enclosed with 100 m of fencing. Find the maximum area.

Q8. A rectangular pen is fenced on three sides against a wall with 40 m of fencing. Find the maximum area.

Q9. Two numbers add to 24. Find their maximum product.

Q10. Two numbers add to 12. Find the minimum value of the sum of their squares.

Q11. An open box is made from a 24 cm square sheet by cutting squares of side x from the corners. Find the maximum volume.

Q12. A rectangle has area $64\,\mathrm{m}^2$. Find its least perimeter.

Q13. A rectangle has its base on the x-axis and its two upper vertices on the graph of $y=12-x^2$ (with $y\geq0$). Find its maximum area.

Q14. The revenue from selling x items is $R=x(100-2x)$ dollars. Find the number of items that maximises revenue, and the maximum revenue.

Q15. An open-top box with a square base of side x cm is to have a volume of $4000\,\mathrm{cm}^3$. Find the value of x that minimises the amount of material (its surface area), and state that minimum area.

Q16. Two numbers differ by 10. Find the minimum value of their product.

Q17. An open box is made from a 30 cm square sheet by cutting squares of side x from the corners. Find the maximum volume.

Q18. A rectangle has area $144\,\mathrm{m}^2$. Find its least perimeter.

Closed interval — global maxima and minima

For each model, find the global maximum and minimum on the given closed domain. Find any stationary points inside the domain, evaluate the function there and at **both** endpoints, then compare. State each answer in context.

Q19. The temperature T (in °C) inside a parked car over the first 6 hours of a morning is modelled by $T(t)=t^3-9t^2+24t+10$, $0\leq t\leq6$, where t is the time in hours. Find the maximum and minimum temperature.

Q20. A market gardener's tomato yield Y (kg) from a plot depends on the number x of plants: $Y(x)=-x^2+44x$. The plot holds at most 16 plants, so $0\leq x\leq16$. Find the number of plants giving the greatest yield, and that yield.

Q21. A particle moves in a straight line with displacement $s=t^3-6t^2+9t+2$ metres, for $0\leq t\leq2$ seconds. Find its maximum and minimum displacement.

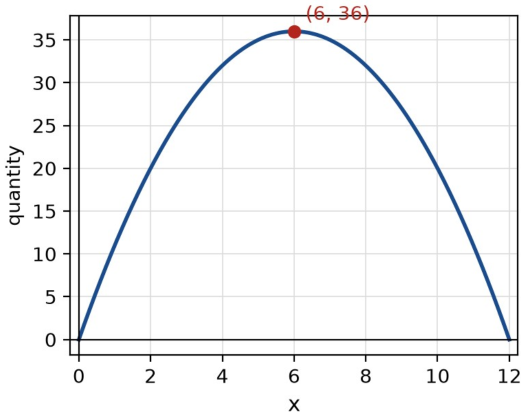
Q22. A drone's height above the ground is $h(t)=t^3-12t^2+45t$ metres, for $1\leq t\leq7$ seconds. Find its greatest and least height.

Answers

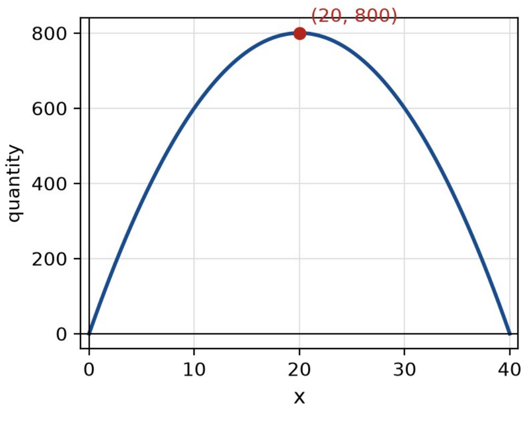
Q1. $36\,\text{cm}^2$ (6×6). **Q2.** max at $x=20$, area $800\,\text{m}^2$. **Q3.** 64 (8 and 8). **Q4.** $432\,\text{cm}^3$ ($x=3$). **Q5.** 40 m (10×10). **Q6.** 200 (10 and 10). **Q7.** $625\,\text{m}^2$ (25×25). **Q8.** $200\,\text{m}^2$ ($x=10$). **Q9.** 144 (12 and 12). **Q10.** 72 (6 and 6). **Q11.** $1024\,\text{cm}^3$ ($x=4$). **Q12.** 32 m (8×8). **Q13.** 32 (at $x=2$). **Q14.** 25 items, $\$1250$. **Q15.** $x=20$ cm, $1200\,\text{cm}^2$. **Q16.** -25 (the numbers 5 and -5). **Q17.** $2000\,\text{cm}^3$ ($x=5$). **Q18.** 48 m (12×12). **Q19.** max 46°C at $t=6$, min 10°C at $t=0$ (both endpoints). **Q20.** 16 plants, 448 kg (endpoint; stationary point $x=22$ rejected). **Q21.** max 6 m at $t=1$ (turning point), min 2 m at $t=0$ (endpoint). **Q22.** greatest 70 m at $t=7$, least 34 m at $t=1$ (both endpoints).

Fully-worked solutions

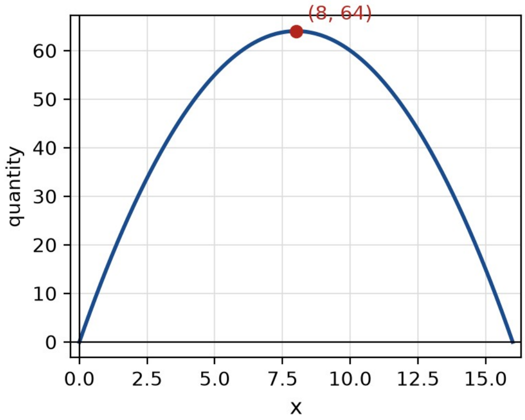
Q1. Width x, length $12-x$; $A=x(12-x)=12x-x^2$. $A'=12-2x=0\Rightarrow x=6$; $A'>0$ for $x<6$ and $A'<0$ for $x>6$ (maximum). $A=6\times6=36\,\text{cm}^2$.



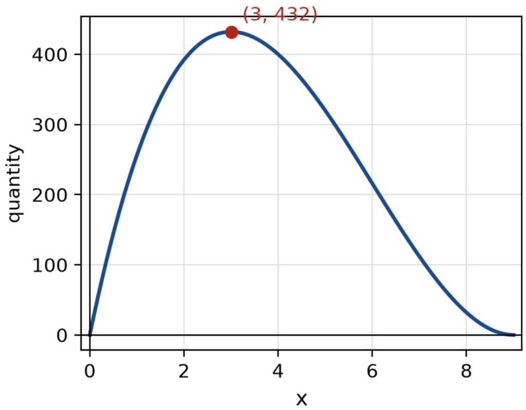
Q2. $A=x(80-2x)=80x-2x^2$, $0<x<40$. $A'=80-4x=0\Rightarrow x=20$; $A'>0$ for $x<20$ and $A'<0$ for $x>20$ (maximum). $A=20\times40=800\,\text{m}^2$.



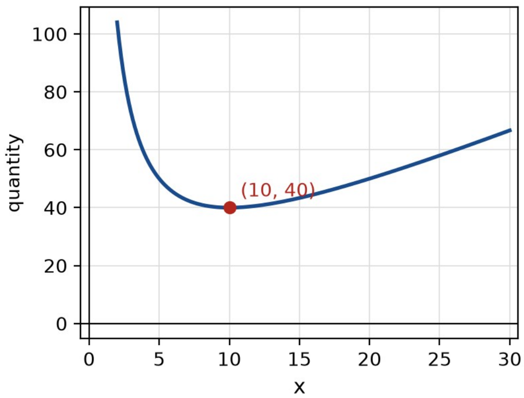
Q3. $P=x(16-x)=16x-x^2$. $P'=16-2x=0\Rightarrow x=8$; $P'>0$ for $x<8$ and $P'<0$ for $x>8$ (maximum). Maximum product $=8\times8=64$.



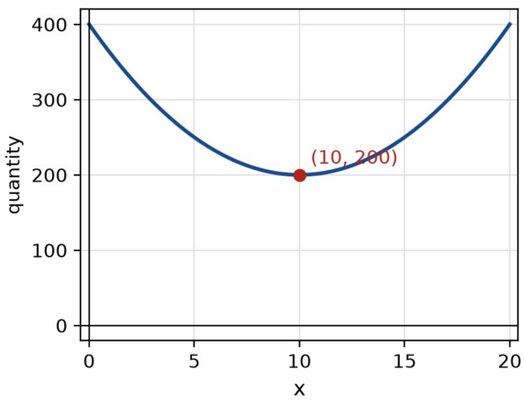
Q4. $V = x(18-2x)^2$, $0<x<9$. $V' = (18-2x)(18-6x) = 0 \Rightarrow x = 3$ (or 9, rejected). $V(3) = 3(12)^2 = 432\,\text{cm}^3$.



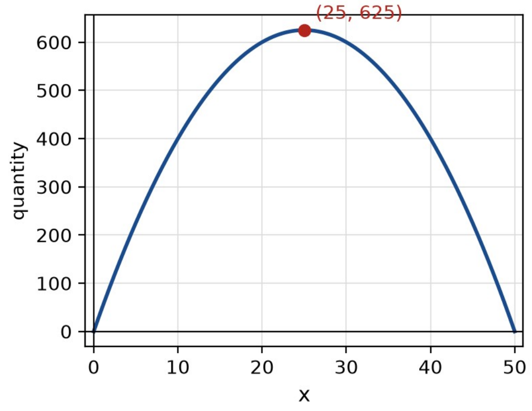
Q5. $P = 2x + \dfrac{200}{x}$, $x>0$. $P' = 2 - \dfrac{200}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = 10$; $P'' > 0$ (minimum). $P = 20 + 20 = 40$ m.



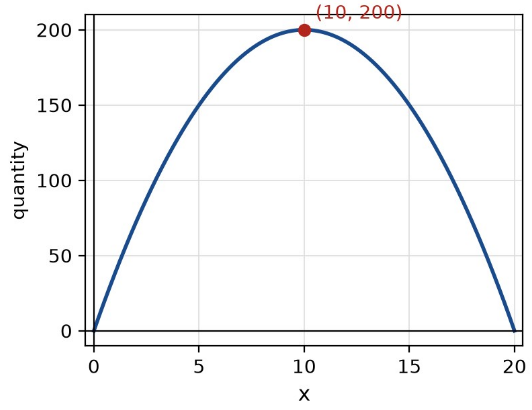
Q6. $S = x^2 + (20-x)^2 = 2x^2 - 40x + 400$. $S' = 4x - 40 = 0 \Rightarrow x = 10$; $S' < 0$ for $x<10$ and $S' > 0$ for $x>10$ (minimum). $S = 100 + 100 = 200$.



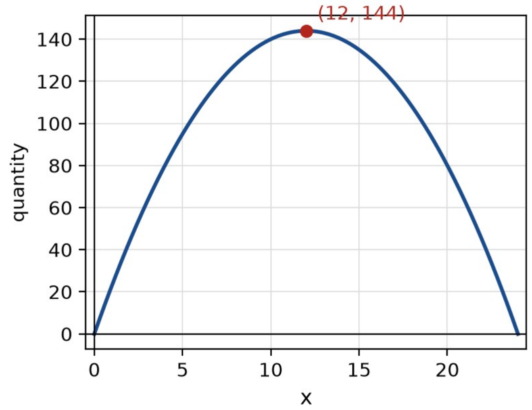
Q7. Width x, length $50-x$; $A=x(50-x)$. $A'=50-2x=0 \Rightarrow x=25$ (maximum). $A=25\times 25=625\,\mathrm{m}^2$.



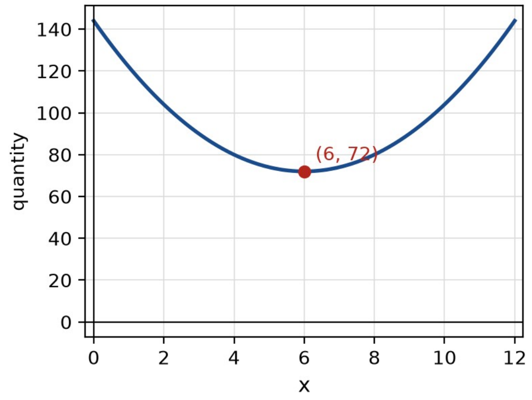
Q8. $A=x(40-2x)$, $0<x<20$. $A'=40-4x=0 \Rightarrow x=10$ (maximum). $A=10\times 20=200\,\mathrm{m}^2$.



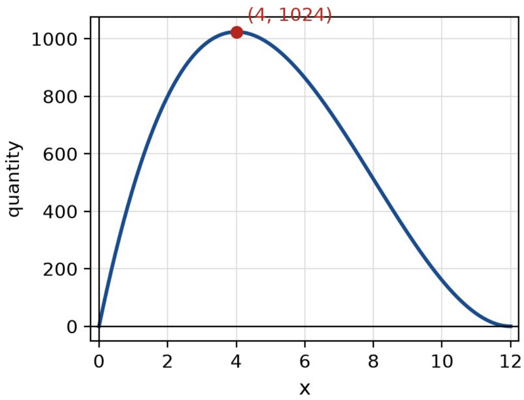
Q9. $P=x(24-x)$. $P'=24-2x=0 \Rightarrow x=12$ (maximum). Product $=12\times 12=144$.



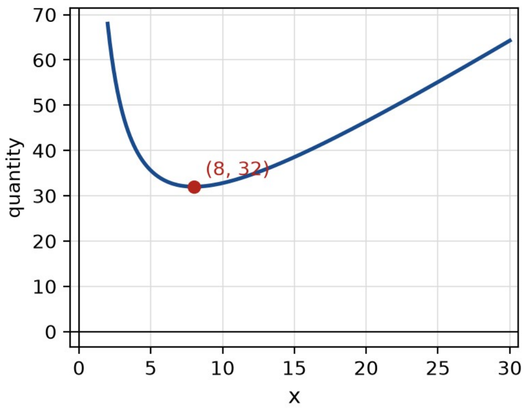
Q10. $S=x^2+(12-x)^2=2x^2-24x+144$. $S'=4x-24=0 \Rightarrow x=6$ (minimum). $S=36+36=72$.



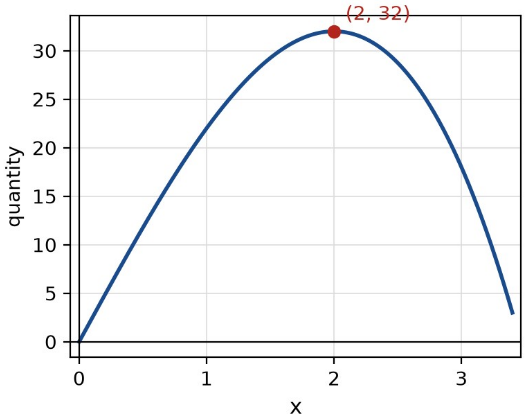
Q11. $V = x(24-2x)^2$, $0<x<12$. $V' = (24-2x)(24-6x) = 0 \Rightarrow x=4$ (or 12, rejected). $V(4) = 4(16)^2 = 1024\,\text{cm}^3$.



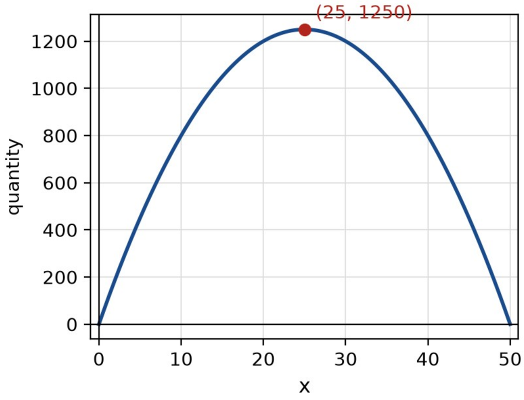
Q12. $P = 2x + \dfrac{128}{x}$. $P' = 2 - \dfrac{128}{x^2} = 0 \Rightarrow x^2 = 64 \Rightarrow x = 8$ (minimum). $P = 16+16 = 32$ m.



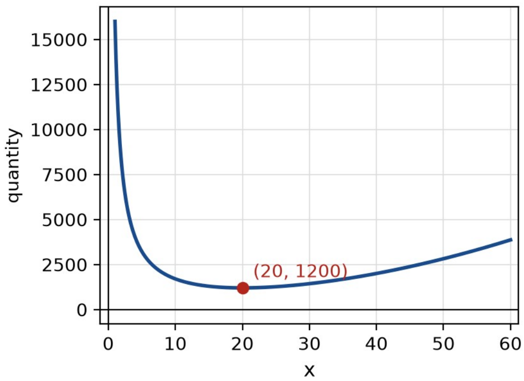
Q13. Upper vertices at $(\pm x, 12-x^2)$, so width $2x$ and height $12-x^2$: $A = 2x(12-x^2) = 24x - 2x^3$, $0<x<\sqrt{12}$. $A' = 24 - 6x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$; $A'>0$ for $0<x<2$ and $A'<0$ for $2<x<\sqrt{12}$ (maximum). $A = 2(2)(12-4) = 32$
.



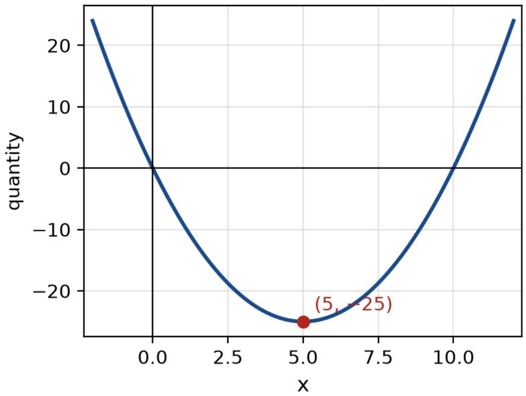
Q14. $R = x(100-2x) = 100x - 2x^2$. $R' = 100 - 4x = 0 \Rightarrow x = 25$; $R'>0$ for $x<25$ and $R'<0$ for $x>25$ (maximum). $R = 25 \times 50 = \$1250$ at 25 items.



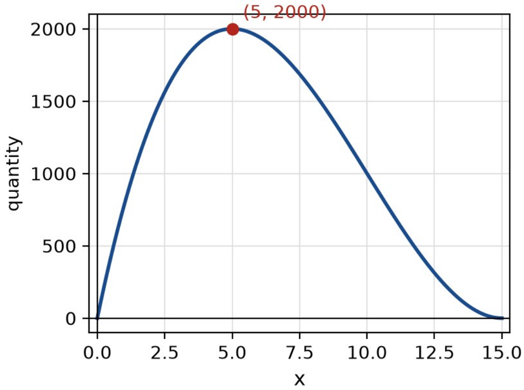
Q15. Base x, height h with $x^2 h=4000$, so $h=\dfrac{4000}{x^2}$. Surface area (open top) $S=x^2+4xh=x^2+\dfrac{16000}{x}$.
$S'=2x-\dfrac{16000}{x^2}=0 \Rightarrow x^3=8000 \Rightarrow x=20$; $S''>0$ (minimum). $S=400+800=1200\,\text{cm}^2$.



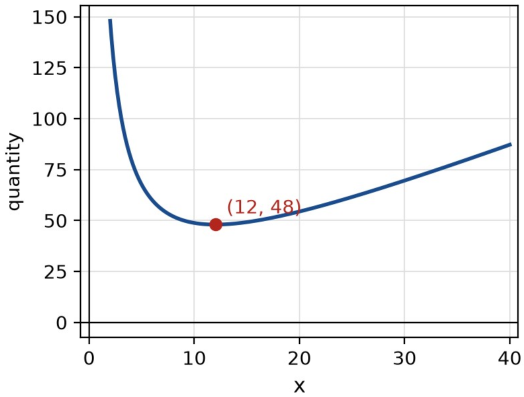
Q16. Let the larger number be x, so the other is $x-10$; product $P=x(x-10)=x^2-10x$. $P'=2x-10=0 \Rightarrow x=5$; $P'<0$ for $x<5$ and $P'>0$ for $x>5$ (minimum). The numbers are 5 and -5, product $=-25$.



Q17. $V=x(30-2x)^2$, $0<x<15$. $V'=(30-2x)(30-6x)=0 \Rightarrow x=5$ (or 15, rejected).
$V(5)=5(20)^2=2000\,\text{cm}^3$.



Q18. $P=2x+\dfrac{288}{x}$. $P'=2-\dfrac{288}{x^2}=0\Rightarrow x^2=144\Rightarrow x=12$ (minimum). $P=24+24=48$ m.



Q19. $T'(t)=3t^2-18t+24=3(t-2)(t-4)=0\Rightarrow t=2$ or $t=4$ (both in $[0,6]$). Stationary values $T(2)=30$, $T(4)=26$; endpoints $T(0)=10$, $T(6)=46$. Comparing $\{10,30,26,46\}$: maximum 46°C at $t=6$ (an endpoint), minimum 10°C at $t=0$ (an endpoint). ∴ the temperature ranges from 10°C to 46°C.

Q20. $Y'(x)=-2x+44=0\Rightarrow x=22$, which lies **outside** $[0,16]$, so it is rejected. With no stationary point in the domain, compare the endpoints: $Y(0)=0$ and $Y(16)=-256+704=448$. ∴ the greatest yield is 448 kg, from 16 plants (an endpoint).

Q21. $s'(t)=3t^2-12t+9=3(t-1)(t-3)=0\Rightarrow t=1$ (in $[0,2]$) or $t=3$ (rejected, outside). Stationary value $s(1)=6$; endpoints $s(0)=2$, $s(2)=4$. Comparing $\{2,6,4\}$: maximum 6 m at $t=1$ (a turning point), minimum 2 m at $t=0$ (an endpoint). ∴ the displacement ranges from 2 m to 6 m.

Q22. $h'(t)=3t^2-24t+45=3(t-3)(t-5)=0\Rightarrow t=3$ or $t=5$ (both in $[1,7]$). Stationary values $h(3)=54$, $h(5)=50$; endpoints $h(1)=34$, $h(7)=70$. Comparing $\{34,54,50,70\}$: greatest 70 m at $t=7$ (an endpoint), least 34 m at $t=1$ (an endpoint). ∴ the height ranges from 34 m to 70 m.

Theory

For an object moving in a straight line, its **position** $x(t)$ (the displacement from a fixed origin at time t) is a function of time. Calculus links position, velocity and acceleration.

Velocity is the rate of change of position:

$$v=\frac{dx}{dt}.$$

If $v>0$ the object moves in the positive direction; if $v<0$ it moves in the negative direction; if $v=0$ the object is **instantaneously at rest**.

Acceleration is the rate of change of velocity:

$$a=\frac{dv}{dt}.$$

Key events.

- **At rest:** solve $v=0$.
- **Change of direction:** occurs where $v=0$ *and* v changes sign.
- **Maximum or minimum velocity:** occurs where $\frac{dv}{dt}=a=0$.

Working backwards (antidifferentiation). Given the acceleration, antidifferentiate to recover velocity, then position, using the initial conditions to find each constant:

$$a \xrightarrow{\int} v \xrightarrow{\int} x.$$

Displacement vs distance travelled. The **displacement** over $[t_1,t_2]$ is $x(t_2)-x(t_1)$. The **distance travelled** is found by splitting the motion at each change of direction (where $v=0$) and adding the distances of the separate legs — because distance ignores direction.

Units. If position is in metres and time in seconds, velocity is in m/s and acceleration in m/s².

Worked examples

Example 1 — scaffolded

A particle moves in a straight line with position $x=t^3-6t^2+9t$ metres at time t seconds ($t\ge 0$). Find (a) the velocity, (b) the acceleration, (c) when the particle is at rest, and (d) the acceleration at each of those times.

Working	Comment
(a) $v=\frac{dx}{dt}=3t^2-12t+9$.	Differentiate the position.
(b) $a=\frac{dv}{dt}=6t-12$.	Differentiate the velocity.
(c) At rest: $3t^2-12t+9=0 \Rightarrow 3(t-1)(t-3)=0$, so $t=1$ or $t=3$ s.	Solve $v=0$; both are ≥ 0.
(d) $a(1)=6(1)-12=-6$ m/s²; $a(3)=6(3)-12=6$ m/s².	Substitute into a.

Example 2 — scaffolded

A particle has position $x=2t^3-9t^2+12t$ metres ($t\ge 0$). Find the velocity and acceleration, state when the particle is at rest, and find the minimum velocity.

Working	Comment
$v=6t^2-18t+12=6(t-1)(t-2)$; $a=12t-18$.	Differentiate twice.
At rest: $v=0\Rightarrow t=1$ or $t=2$ s.	Null factor law.
Minimum velocity where $a=0$: $12t-18=0\Rightarrow t=3/2$.	The velocity is least when its rate of change is zero.
$v(3/2)=6(3/2-1)(3/2-2)=6\times 1/2\times(-1/2)=-3/2$ m/s.	The minimum velocity is $-3/2$ m/s.

Example 3 — unscaffolded

A particle moves with acceleration $a=6t-4$ m/s². When $t=0$ its velocity is 5 m/s and its position is 0. Find $v(t)$ and $x(t)$.

$v=\int(6t-4)\,dt=3t^2-4t+c$. Since $v(0)=5$, $c=5$, so $v=3t^2-4t+5$.

$x=\int(3t^2-4t+5)\,dt=t^3-2t^2+5t+c$. Since $x(0)=0$, $c=0$.

$\therefore v=3t^2-4t+5$ and $x=t^3-2t^2+5t$.

Example 4 — unscaffolded

A particle has position $x=t^3-3t^2$ metres ($t\ge 0$). Find the distance travelled in the first 3 seconds.

$v=3t^2-6t=3t(t-2)$, so $v=0$ at $t=0$ and $t=2$: the particle changes direction at $t=2$.

Positions: $x(0)=0$, $x(2)=8-12=-4$, $x(3)=27-27=0$.

Leg 0 → 2: distance $=|-4-0|=4$ m. Leg 2 → 3: distance $=|0-(-4)|=4$ m.

$\therefore$ distance travelled $=4+4=8$ m. (The displacement is only $x(3)-x(0)=0$.)

Practice questions

Scaffolded

Q1. A particle has position $x=t^2-4t+3$ m. (a) Find v. (b) Find a. (c) When is the particle at rest? (d) State the initial velocity.

Q2. For $x=t^3-3t^2-9t$ m ($t\ge 0$): (a) find v; (b) find when the particle is at rest; (c) find the acceleration at that time.

Q3. A particle has velocity $v=3t^2-12$ m/s ($t\ge 0$). (a) Find a. (b) When is the velocity zero? (c) When is the acceleration zero?

Q4. A particle has acceleration $a=12t-6$ m/s², with $v(0)=0$ and $x(0)=2$. (a) Find $v(t)$. (b) Find $x(t)$.

Q5. For $x=2t^3-15t^2+24t$ m ($t\ge 0$): (a) find v; (b) find when the particle is at rest; (c) find the minimum velocity.

Q6. A particle has position $x=t^3-6t^2+9t+2$ m ($t\ge 0$). Find the distance travelled in the first 4 seconds.

Unscaffolded

Q7. For $x=t^2-6t$ m, find v, a, and when the particle is at rest.

Q8. For $x=t^3-12t$ m, find the velocity when $t=2$ and the acceleration when $t=1$.

Q9. For $x=t^3-9t^2+24t$ m ($t\geq 0$), find when the particle is at rest.

Q10. For $x=-t^3+6t^2$ m ($t\geq 0$), find the maximum velocity.

Q11. A particle has acceleration $a=6t+2$ m/s² with $v(0)=-3$ m/s. Find $v(t)$.

Q12. A ball is thrown so that its acceleration is $a=-10$ m/s², with $v(0)=20$ m/s and $x(0)=0$. Find $v(t)$ and $x(t)$, and state when $v=0$.

Q13. A particle has velocity $v=t^2-5t+6$ m/s ($t\geq 0$). When is it at rest? When is the acceleration zero?

Q14. For $x=t^3-9t^2+15t$ m ($t\geq 0$), find the distance travelled in the first 3 seconds.

Q15. For $x=t^3-6t^2+5t$ m, find when the velocity is a minimum and the value of that minimum velocity.

Q16. A ball thrown vertically has height $x=15t-5t^2$ m ($t\geq 0$). Find (a) its velocity at $t=1$; (b) its maximum height; (c) when it returns to the ground.

Q17. A particle has acceleration $a=6t-2$ m/s², with $v(1)=4$ m/s and $x(1)=0$. Find $v(t)$ and $x(t)$.

Q18. For $x=t^4-4t^3$ m ($t\geq 0$), find when the particle is at rest and when the acceleration is zero.

Answers

Q1. $v=2t-4$; $a=2$; at rest $t=2$ s; initial velocity -4 m/s. **Q2.** $v=3t^2-6t-9$; at rest $t=3$ s; $a(3)=12$ m/s². **Q3.** $a=6t$; $v=0$ at $t=2$ s; $a=0$ at $t=0$. **Q4.** $v=6t^2-6t$; $x=2t^3-3t^2+2$. **Q5.** $v=6t^2-30t+24$; at rest $t=1,4$ s; minimum velocity $-27/2$ m/s (at $t=5/2$). **Q6.** 12 m. **Q7.** $v=2t-6$; $a=2$; at rest $t=3$ s. **Q8.** $v(2)=0$ m/s; $a(1)=6$ m/s². **Q9.** $t=2$ s and $t=4$ s. **Q10.** 12 m/s (at $t=2$ s). **Q11.** $v=3t^2+2t-3$. **Q12.** $v=20-10t$; $x=20t-5t^2$; $v=0$ at $t=2$ s. **Q13.** At rest $t=2,3$ s; acceleration zero at $t=5/2$ s. **Q14.** 23 m. **Q15.** Minimum at $t=2$ s; minimum velocity -7 m/s. **Q16.** (a) 5 m/s; (b) $45/4=11.25$ m; (c) $t=3$ s. **Q17.** $v=3t^2-2t+3$; $x=t^3-t^2+3t-3$. **Q18.** At rest $t=0$ and $t=3$ s; acceleration zero at $t=0$ and $t=2$ s.

Fully-worked solutions

Q1. $x=t^2-4t+3$. (a) $v=2t-4$. (b) $a=2$. (c) $v=0 \Rightarrow t=2$ s. (d) $v(0)=-4$ m/s.

Q2. $v=3t^2-6t-9=3(t-3)(t+1)$. At rest $v=0 \Rightarrow t=3$ (rejecting $t=-1$ since $t\geq 0$). $a=6t-6$, so $a(3)=12$ m/s².

Q3. $v=3t^2-12$. (a) $a=\dfrac{dv}{dt}=6t$. (b) $v=0 \Rightarrow t^2=4 \Rightarrow t=2$ s (rejecting $t=-2$). (c) $a=0 \Rightarrow t=0$.

Q4. $v=\int(12t-6)\,dt=6t^2-6t+c$; $v(0)=0 \Rightarrow c=0$, so $v=6t^2-6t$. $x=\int(6t^2-6t)\,dt=2t^3-3t^2+c$; $x(0)=2 \Rightarrow c=2$, so $x=2t^3-3t^2+2$.

Q5. $v=6t^2-30t+24=6(t-1)(t-4)$. At rest $t=1$ or $t=4$ s. $a=12t-30$; $a=0 \Rightarrow t=5/2$. $v(5/2)=6(25/4)-30(5/2)+24=75/2-75+24=-27/2$ m/s.

Q6. $v=3t^2-12t+9=3(t-1)(t-3)$, so $v=0$ at $t=1,3$. Positions: $x(0)=2$, $x(1)=6$, $x(3)=2$, $x(4)=6$. Distances: $|6-2|+|2-6|+|6-2|=4+4+4=12$ m.

Q7. $v=2t-6$; $a=2$; $v=0 \Rightarrow t=3$ s.

Q8. $v=3t^2-12$, so $v(2)=12-12=0$ m/s. $a=6t$, so $a(1)=6$ m/s².

Q9. $v=3t^2-18t+24=3(t-2)(t-4)$, so at rest at $t=2$ s and $t=4$ s.

Q10. $v=-3t^2+12t$; $a=-6t+12$. Maximum velocity where $a=0 \Rightarrow t=2$, giving $v(2)=-12+24=12$ m/s.

Q11. $v=\int(6t+2)\,dt=3t^2+2t+c$; $v(0)=-3 \Rightarrow c=-3$, so $v=3t^2+2t-3$.

Q12. $v=\int(-10)\,dt=-10t+c$; $v(0)=20 \Rightarrow c=20$, so $v=20-10t$. $x=\int(20-10t)\,dt=20t-5t^2+c$; $x(0)=0 \Rightarrow c=0$. $v=0 \Rightarrow t=2$ s (the top of the flight).

Q13. $v=t^2-5t+6=(t-2)(t-3)$, so at rest at $t=2,3$ s. $a=2t-5$; $a=0 \Rightarrow t=5/2$ s.

Q14. $v=3t^2-18t+15=3(t-1)(t-5)$, so $v=0$ at $t=1,5$; only $t=1$ lies in the first 3 s. Positions: $x(0)=0$, $x(1)=1-9+15=7$, $x(3)=27-81+45=-9$. Distances: $|7-0|+|-9-7|=7+16=23$ m.

Q15. $v=3t^2-12t+5$; $a=6t-12=0 \Rightarrow t=2$. $v(2)=12-24+5=-7$ m/s.

Q16. $v=15-10t$. (a) $v(1)=5$ m/s. (b) Maximum height where $v=0 \Rightarrow t=3/2$: $x(3/2)=15(3/2)-5(9/4)=45/2-45/4=45/4=11.25$ m. (c) Ground: $x=0 \Rightarrow t(15-5t)=0 \Rightarrow t=3$ s (taking $t>0$).

Q17. $v=\int(6t-2)\,dt=3t^2-2t+c$; $v(1)=3-2+c=4 \Rightarrow c=3$, so $v=3t^2-2t+3$. $x=\int(3t^2-2t+3)\,dt=t^3-t^2+3t+c$; $x(1)=1-1+3+c=0 \Rightarrow c=-3$, so $x=t^3-t^2+3t-3$.

Q18. $v = 4t^3 - 12t^2 = 4t^2(t-3)$, so at rest at $t = 0$ and $t = 3$ s. $a = 12t^2 - 24t = 12t(t-2)$, so acceleration is zero at $t = 0$ and $t = 2$ s.

Theory

Many function rules contain unknown constants called **parameters**, for example $f(x)=x^2+ax+b$ or $y=ax^2+bx+c$. Each choice of the parameters gives a different member of a **family** of curves that all share the same form. When we differentiate, a parameter is treated as a **constant**: for $f(x)=x^3+ax+b$,

$$f'(x)=3x^2+a.$$

Finding parameters from conditions. Translate each piece of given information into an equation, then solve. The common conditions are:

- **passes through the point** $(p,q)\Rightarrow f(p)=q$;
- **has a stationary point at** $x=p\Rightarrow f'(p)=0$;
- **the tangent at** $x=p$ **has gradient** m (e.g. parallel to a line of gradient m) $\Rightarrow f'(p)=m$;
- **has a stationary point at** $(p,q)\Rightarrow$ **both** $f(p)=q$ **and** $f'(p)=0$.

Set up as many equations as there are unknown parameters and solve them simultaneously. To find n parameters you need n independent conditions.

Worked examples

Example 1 — scaffolded

The curve $f(x)=x^2+ax+3$ has a stationary point at $x=2$. Find the value of a.

Working	Comment
$f'(x)=2x+a$.	Differentiate, treating a as a constant.
Stationary point at $x=2$, so $f'(2)=0$.	A stationary point means the gradient is zero.
$2(2)+a=0\Rightarrow 4+a=0$.	Substitute $x=2$ into f'.
$\therefore a=-4$.	Solve for a.

Example 2 — scaffolded

The curve $y=ax^2+bx$ passes through the point $(1,5)$ and has gradient 8 at $x=1$. Find a and b.

Working	Comment
Passes through $(1,5)$: $a(1)^2+b(1)=5$, so $a+b=5$.	Use $f(1)=5$.
$\frac{dy}{dx}=2ax+b$; gradient 8 at $x=1$: $2a+b=8$.	Use $f'(1)=8$.
Subtracting the first equation from the second: $a=3$.	Two equations, two unknowns.
Then $3+b=5$, so $b=2$.	Back-substitute.

$\therefore a=3$, $b=2$, giving $y=3x^2+2x$.

Example 3 — unscaffolded

The curve $f(x)=x^3+ax+b$ has a stationary point at $(1,-1)$. Find a and b.

$f'(x)=3x^2+a$. A stationary point at $x=1$ gives $f'(1)=3+a=0$, so $a=-3$.

The curve passes through $(1,-1)$: $f(1)=1+a+b=1-3+b=-1$, so $b=1$.

$\therefore a=-3$, $b=1$, and $f(x)=x^3-3x+1$.

Example 4 — unscaffolded

The tangent to $y=x^2+px+q$ at the point $(2,3)$ has gradient 5. Find p and q.

$\frac{dy}{dx}=2x+p$. The gradient at $x=2$ is 5: $2(2)+p=5$, so $p=1$.

The curve passes through $(2,3)$: $(2)^2+p(2)+q=3$, so $4+2+q=3$ and $q=-3$.

$\therefore p=1$, $q=-3$, and $y=x^2+x-3$.

Practice questions

Scaffolded

Q1. $f(x)=x^2+kx+1$ has a stationary point at $x=3$. (a) Find $f'(x)$. (b) Hence find k.

Q2. $y=x^2+bx+c$ passes through $(0,4)$ and has a stationary point at $x=1$. (a) Use the point to find c. (b) Find b.

Q3. $f(x)=ax^2+2x$ has gradient 10 at $x=2$. (a) Find $f'(x)$. (b) Find a.

Q4. The curve $y=x^3+ax$ has a stationary point at $x=2$. Find a.

Q5. $y=ax^2+bx+1$ has a turning point at $(2,-3)$. (a) Use the point to write one equation in a and b. (b) Use $\frac{dy}{dx}=0$ at $x=2$ to write a second equation. (c) Hence find a and b.

Q6. The tangent to $y=x^2+kx$ at $x=1$ is parallel to the line $y=3x+2$. Find k.

Unscaffolded

Q7. $f(x)=x^2+ax-5$ has a stationary point at $x=3$. Find a.

Q8. $y=x^3+bx+2$ has a stationary point at $x=1$. Find b.

Q9. $f(x)=ax^2+bx$ passes through $(2,2)$ and has gradient -1 at $x=2$. Find a and b.

Q10. $y=x^3+ax^2+bx$ has stationary points at $x=1$ and $x=3$. Find a and b.

Q11. $f(x)=x^2+px+q$ has a stationary point at $(-1,4)$. Find p and q.

Q12. The tangent to $y=x^3+ax$ at $x=1$ has gradient 5. Find a.

Q13. $f(x)=ax^3+bx+1$ has a stationary point at $(1,-1)$. Find a and b.

Q14. The tangent to $y=x^2+bx+c$ at the point $(1,2)$ is horizontal. Find b and c.

Q15. $y=ax^2+bx+c$ passes through $(0,1)$ and $(1,-2)$ and has a stationary point at $x=2$. Find a, b and c.

Q16. $y=x^3+ax^2+bx+c$ passes through $(0,2)$ and has a stationary point at $(1,0)$. Find a, b and c.

Q17. Consider the family $y=x^2-2kx+k^2$. Show that the turning point lies on the x-axis for **every** value of k, and state its coordinates.

Q18. A cubic $f(x)=x^3+ax^2+bx+c$ has stationary points at $x=-1$ and $x=2$ and passes through the origin. Find a, b and c.

Answers

Q1. $f'(x)=2x+k$; $k=-6$. **Q2.** $c=4$; $b=-2$. **Q3.** $f'(x)=2ax+2$; $a=2$. **Q4.** $a=-12$. **Q5.** $a=1$, $b=-4$. **Q6.** $k=1$. **Q7.** $a=-6$. **Q8.** $b=-3$. **Q9.** $a=-1$, $b=3$. **Q10.** $a=-6$, $b=9$. **Q11.** $p=2$, $q=5$. **Q12.** $a=2$. **Q13.** $a=1$, $b=-3$. **Q14.** $b=-2$, $c=3$. **Q15.** $a=1$, $b=-4$, $c=1$. **Q16.** $a=0$, $b=-3$, $c=2$. **Q17.** Turning point $(k,0)$, on the x-axis for all k. **Q18.** $a=-\frac{3}{2}$, $b=-6$, $c=0$.

Fully-worked solutions

Q1. (a) $f'(x)=2x+k$. (b) Stationary at $x=3$: $f'(3)=6+k=0$, so $k=-6$.

Q2. (a) $y(0)=c=4$. (b) $\frac{dy}{dx}=2x+b$; stationary at $x=1$ gives $2+b=0$, so $b=-2$. (The curve is $y=x^2-2x+4$.)

Q3. (a) $f'(x)=2ax+2$. (b) $f'(2)=4a+2=10$, so $4a=8$ and $a=2$.

Q4. $\frac{dy}{dx}=3x^2+a$; stationary at $x=2$ gives $12+a=0$, so $a=-12$.

Q5. (a) $y(2)=4a+2b+1=-3$, so $2a+b=-2$. (b) $\frac{dy}{dx}=2ax+b$; $y'(2)=4a+b=0$. (c) Subtracting, $2a=2$, so $a=1$ and then $b=-4$.

Q6. $\frac{dy}{dx}=2x+k$. Parallel to $y=3x+2$ means gradient 3: $y'(1)=2+k=3$, so $k=1$.

Q7. $f'(x)=2x+a$; $f'(3)=6+a=0$, so $a=-6$.

Q8. $\frac{dy}{dx}=3x^2+b$; at $x=1$, $3+b=0$, so $b=-3$.

Q9. Through $(2,2)$: $4a+2b=2$, so $2a+b=1$. Gradient -1 at $x=2$: $f'(x)=2ax+b$, so $4a+b=-1$. Subtracting, $2a=-2$, giving $a=-1$ and $b=3$.

Q10. $\frac{dy}{dx}=3x^2+2ax+b$. Since the stationary points are at $x=1$ and $x=3$, $3x^2+2ax+b=3(x-1)(x-3)=3x^2-12x+9$. Matching coefficients, $2a=-12$ so $a=-6$, and $b=9$.

Q11. $f'(x)=2x+p$; $f'(-1)=-2+p=0$, so $p=2$. Then $f(-1)=1-2+q=4$, so $q=5$.

Q12. $\frac{dy}{dx}=3x^2+a$; $y'(1)=3+a=5$, so $a=2$.

Q13. $f'(x)=3ax^2+b$; $f'(1)=3a+b=0$. Also $f(1)=a+b+1=-1$, so $a+b=-2$. Subtracting, $2a=2$, giving $a=1$ and $b=-3$.

Q14. A horizontal tangent means gradient 0. $\frac{dy}{dx}=2x+b$; $y'(1)=2+b=0$, so $b=-2$. Then $y(1)=1-2+c=2$, so $c=3$.

Q15. $y(0)=c=1$. $\frac{dy}{dx}=2ax+b$; stationary at $x=2$ gives $4a+b=0$. $y(1)=a+b+c=-2$, so $a+b=-3$. From $4a+b=0$, $b=-4a$; substituting, $a-4a=-3$, so $-3a=-3$, $a=1$, $b=-4$, $c=1$.

Q16. $y(0)=c=2$. $\frac{dy}{dx}=3x^2+2ax+b$; a stationary point at $(1,0)$ gives both $y'(1)=3+2a+b=0$ and $y(1)=1+a+b+2=0 \Rightarrow a+b=-3$. Subtracting $a+b=-3$ from $2a+b=-3$ gives $a=0$, then $b=-3$. So $a=0$, $b=-3$, $c=2$.

Q17. $\frac{dy}{dx}=2x-2k=0$ gives $x=k$. Then $y(k)=k^2-2k(k)+k^2=k^2-2k^2+k^2=0$. So the turning point is $(k,0)$, which lies on the x-axis for every value of k. (Indeed $y=(x-k)^2$, a parabola with its minimum on the x-axis.)

Q18. Through the origin: $f(0)=c=0$. $f'(x)=3x^2+2ax+b$; the stationary points are at $x=-1$ and $x=2$, so $3x^2+2ax+b=3(x+1)(x-2)=3x^2-3x-6$. Matching coefficients, $2a=-3$ so $a=-\frac{3}{2}$, and $b=-6$. Thus $a=-\frac{3}{2}$, $b=-6$, $c=0$, giving $f(x)=x^3-\frac{3}{2}x^2-6x$.

Chapter 15 Applications of differentiation — 15H The Newton's method algorithm

Units 3 & 4 preview (in part). Items in this section that apply Newton's method to quadratic or quartic equations — the square-root approximations in the theory, Example 1, Example 4 and Questions 1, 2, 5, 7, 8, 12 and 16, and the quartic equations in Questions 14 and 18 — go beyond the Units 1 & 2 study design, which prescribes the Newton's method algorithm for roots of cubic polynomial functions only; they are included as preparation for Units 3 & 4 and are not assessed in the graded topic tests or practice examinations.

Theory

Newton's method (or the Newton–Raphson method) uses tangent lines to find an **approximate** solution of an equation $f(x)=0$.

The idea. Start with an estimate x_0 near the root. The tangent to $y=f(x)$ at $(x_0, f(x_0))$ crosses the x-axis at a point that is usually **closer** to the root — call it x_1. Repeating from x_1 gives a still better estimate x_2, and so on.

The formula. The tangent at x_n is $y-f(x_n)=f'(x_n)(x-x_n)$. Putting $y=0$ and solving for x gives the next estimate:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

Iterate until successive estimates agree to the **required number of decimal places**.

Approximating surds. An irrational number is often a root of a polynomial equation — for example $\sqrt{7}$ is the positive solution of $x^2-7=0$ — so Newton's method can be used to approximate it.

When it may fail. If $f'(x_n)=0$ the tangent is horizontal and never meets the x-axis, so the method breaks down. A poor starting estimate can also make the estimates diverge, or converge to a *different* root than the one intended.

Worked examples

Example 1 — scaffolded

Use Newton's method with $x_0=2$ to approximate $\sqrt{7}$, correct to 4 decimal places.

Working	Comment
$\sqrt{7}$ is the positive root of $f(x)=x^2-7$, and $f'(x)=2x$.	Turn the surd into an equation $f(x)=0$.
$x_{n+1}=x_n-\dfrac{x_n^2-7}{2x_n}$.	Substitute f and f' into Newton's formula.
Iterate from $x_0=2$ (table below).	Keep going until two estimates agree to 4 dp.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	2	-3	4	2.75
1	2.75	0.5625	5.5	2.647727
2	2.647727	0.010460	5.295455	2.645752
3	2.645752	0.000004	5.291504	2.645751

x_3 and x_4 agree to 4 dp, so $\sqrt{7}\approx 2.6458$.

Example 2 — scaffolded

Use Newton's method with $x_0 = 1.5$ to approximate the root of $x^3 - x - 3 = 0$, correct to 4 decimal places.

Working	Comment
$f(x) = x^3 - x - 3$, so $f'(x) = 3x^2 - 1$.	Differentiate to get f'.
$x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 3}{3x_n^2 - 1}$.	Newton's formula for this f.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1.5	-1.125	5.75	1.695652
1	1.695652	0.179749	7.625709	1.672081
2	1.672081	0.002813	7.387563	1.671700
3	1.671700	0.000001	7.383742	1.671700

$\therefore$ the root is $x \approx 1.6717$.

Example 3 — unscaffolded

Use Newton's method with $x_0 = 1$ to approximate the root of $x^3 + 2x - 5 = 0$, correct to 4 decimal places.

$f(x) = x^3 + 2x - 5$, $f'(x) = 3x^2 + 2$, so $x_{n+1} = x_n - \dfrac{x_n^3 + 2x_n - 5}{3x_n^2 + 2}$.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1	-2	5	1.4
1	1.4	0.544	7.88	1.330964
2	1.330964	0.019688	7.314399	1.328273
3	1.328273	0.000029	7.292926	1.328269

$\therefore x \approx 1.3283$.

Example 4 — unscaffolded

Use Newton's method with $x_0 = 3$ to approximate $\sqrt{10}$, correct to 4 decimal places.

$\sqrt{10}$ is the positive root of $f(x) = x^2 - 10$, $f'(x) = 2x$, so $x_{n+1} = x_n - \dfrac{x_n^2 - 10}{2x_n}$.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	3	-1	6	3.166667
1	3.166667	0.027778	6.333333	3.162281
2	3.162281	0.000019	6.324561	3.162278

$\therefore \sqrt{10} \approx 3.1623$.

Practice questions

Scaffolded

Q1. For $f(x)=x^2-5$ (whose positive root is $\sqrt{5}$), with $x_0=2$: (a) write $f'(x)$ and Newton's formula; (b) find x_1; (c) find x_2; (d) hence state $\sqrt{5}$ to 4 decimal places.

Q2. For $f(x)=x^2-3$, with $x_0=2$: (a) find x_1; (b) find x_2; (c) state $\sqrt{3}$ to 4 decimal places.

Q3. For $f(x)=x^3-20$, with $x_0=3$: (a) write Newton's formula; (b) find x_1; (c) find x_2 and x_3, and hence $\sqrt[3]{20}$ to 4 decimal places.

Q4. For $f(x)=x^3+x-1$, with $x_0=1$: (a) find x_1; (b) find x_2; (c) find x_3 and hence the root to 4 decimal places.

Q5. For $f(x)=x^2-x-1$, with $x_0=1.5$: (a) find x_1; (b) find x_2; (c) find x_3 and hence state the root to 4 decimal places. *(This root is the golden ratio.)*

Q6. For $f(x)=x^3+x-3$, with $x_0=1$: (a) find x_1; (b) find x_2; (c) find x_3 and hence the root to 4 decimal places.

Unscaffolded

Use Newton's method with the given starting value to approximate the root, correct to 4 decimal places. Show your iteration table.

Q7. $f(x)=x^2-11$, $x_0=3$.

Q8. $f(x)=x^2-2$, $x_0=1$.

Q9. $f(x)=x^3-7$, $x_0=2$.

Q10. $f(x)=x^3-x-1$, $x_0=1$.

Q11. $f(x)=x^3+2x^2-5$, $x_0=1$.

Q12. $f(x)=x^2-x-3$, $x_0=2$.

Q13. $f(x)=x^3-3x-1$, $x_0=2$.

Q14. $f(x)=x^4-5$, $x_0=1.5$.

Q15. $f(x)=x^3+x^2-1$, $x_0=0.8$.

Q16. $f(x)=x^2-6$, $x_0=2$.

Q17. $f(x)=x^3-2x-2$, $x_0=1.8$.

Q18. $f(x)=x^4-x-1$, $x_0=1.2$.

Answers

Q1. $x_1 = 2.25$, $x_2 = 2.236111$; $\sqrt{5} \approx 2.2361$. **Q2.** $x_1 = 1.75$, $x_2 = 1.732143$; $\sqrt{3} \approx 1.7321$. **Q3.** $x_1 = 2.740741$, $x_2 = 2.714670$, $x_3 = 2.714418$; $\sqrt[3]{20} \approx 2.7144$. **Q4.** $x_1 = 0.75$, $x_2 = 0.686047$, $x_3 = 0.682340$; root ≈ 0.6823. **Q5.** $x_1 = 1.625$, $x_2 = 1.618056$, $x_3 = 1.618034$; root ≈ 1.6180. **Q6.** $x_1 = 1.25$, $x_2 = 1.214286$, $x_3 = 1.213412$; root ≈ 1.2134. **Q7.** 3.3166. **Q8.** 1.4142. **Q9.** 1.9129. **Q10.** 1.3247. **Q11.** 1.2419. **Q12.** 2.3028. **Q13.** 1.8794. **Q14.** 1.4953. **Q15.** 0.7549. **Q16.** 2.4495. **Q17.** 1.7693. **Q18.** 1.2207.

Fully-worked solutions

Q1. $f'(x) = 2x$, so $x_{n+1} = x_n - \dfrac{x_n^2 - 5}{2x_n}$. $x_1 = 2 - \dfrac{-1}{4} = 2.25$; $x_2 = 2.25 - \dfrac{0.0625}{4.5} = 2.236111$; $x_3 = 2.236068$. $\therefore \sqrt{5} \approx 2.2361$.

Q2. $x_{n+1} = x_n - \dfrac{x_n^2 - 3}{2x_n}$. $x_1 = 2 - \dfrac{1}{4} = 1.75$; $x_2 = 1.75 - \dfrac{0.0625}{3.5} = 1.732143$; $x_3 = 1.732051$. $\therefore \sqrt{3} \approx 1.7321$.

Q3. $f'(x) = 3x^2$, so $x_{n+1} = x_n - \dfrac{x_n^3 - 20}{3x_n^2}$. $x_1 = 3 - \dfrac{7}{27} = 2.740741$; $x_2 = 2.740741 - \dfrac{0.587512}{22.534979} = 2.714670$; $x_3 = 2.714418$. $\therefore \sqrt[3]{20} \approx 2.7144$.

Q4. $f'(x) = 3x^2 + 1$. $x_1 = 1 - \dfrac{1}{4} = 0.75$; $x_2 = 0.75 - \dfrac{0.171875}{2.6875} = 0.686047$; $x_3 = 0.682340$, $x_4 = 0.682328$. $\therefore$ root ≈ 0.6823.

Q5. $f'(x) = 2x - 1$. $x_1 = 1.5 - \dfrac{-0.25}{2} = 1.625$; $x_2 = 1.625 - \dfrac{0.015625}{2.25} = 1.618056$; $x_3 = 1.618034$. $\therefore$ root ≈ 1.6180 (the golden ratio $1 + \sqrt{5}/2$).

Q6. $f'(x) = 3x^2 + 1$. $x_1 = 1 - \dfrac{-1}{4} = 1.25$; $x_2 = 1.25 - \dfrac{0.203125}{5.6875} = 1.214286$; $x_3 = 1.213412$. $\therefore$ root ≈ 1.2134.

Q7. $x_{n+1} = x_n - \dfrac{x_n^2 - 11}{2x_n}$: $x_1 = 3.333333$, $x_2 = 3.316667$, $x_3 = 3.316625$. $\therefore \sqrt{11} \approx 3.3166$.

Q8. $x_1 = 1.5$, $x_2 = 1.416667$, $x_3 = 1.414216$, $x_4 = 1.414214$. $\therefore \sqrt{2} \approx 1.4142$.

Q9. $f'(x) = 3x^2$: $x_1 = 1.916667$, $x_2 = 1.912938$, $x_3 = 1.912931$. $\therefore \sqrt[3]{7} \approx 1.9129$.

Q10. $f'(x) = 3x^2 - 1$: $x_1 = 1.5$, $x_2 = 1.347826$, $x_3 = 1.325200$, $x_4 = 1.324718$. $\therefore$ root ≈ 1.3247.

Q11. $f'(x) = 3x^2 + 4x$: $x_1 = 1.285714$, $x_2 = 1.243001$, $x_3 = 1.241897$. $\therefore$ root ≈ 1.2419.

Q12. $f'(x) = 2x - 1$: $x_1 = 2.333333$, $x_2 = 2.303030$, $x_3 = 2.302776$. $\therefore$ root ≈ 2.3028.

Q13. $f'(x) = 3x^2 - 3$: $x_1 = 1.888889$, $x_2 = 1.879452$, $x_3 = 1.879385$. $\therefore$ root ≈ 1.8794.

Q14. $f'(x) = 4x^3$: $x_1 = 1.495370$, $x_2 = 1.495349$. $\therefore \sqrt[4]{5} \approx 1.4953$.

Q15. $f'(x) = 3x^2 + 2x$: $x_1 = 0.756818$, $x_2 = 0.754881$, $x_3 = 0.754878$. $\therefore$ root ≈ 0.7549.

Q16. $x_1 = 2.5$, $x_2 = 2.45$, $x_3 = 2.449490$. $\therefore \sqrt{6} \approx 2.4495$.

Q17. $f'(x) = 3x^2 - 2$: $x_1 = 1.769948$, $x_2 = 1.769293$, $x_3 = 1.769292$. $\therefore$ root ≈ 1.7693.

Q18. $f'(x)=4x^3-1$: $x_1=1.221380$, $x_2=1.220745$, $x_3=1.220744$. $\therefore$ root ≈ 1.2207.

Topic 4 Test A — Technology-active

Topic 4 covers Chapters 13–15 (rates of change, differentiation and anti-differentiation, applications of differentiation). Reading time: 5 minutes. Writing time: 50 minutes. Total: 50 marks. One bound reference, one approved CAS calculator or CAS software, and one scientific calculator are permitted. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (6 marks)

For $f(x) = 2x^3 - 5x^2 + 4x - 1$:

a. Find $f'(x)$. 2 marks

b. Find the gradient of the curve at $x = 2$. 2 marks

c. Find the values of x at which the gradient is 4. 2 marks

Question 2 (7 marks)

Consider $y = x^3 + 3x^2 - 24x$.

a. Find the coordinates of the stationary points and classify each. 4 marks

b. Find the equation of the tangent to the curve at $x = 1$. 3 marks

Question 3 (6 marks)

A tank holds $V(t) = 2t^3 - 24t^2 + 72t$ litres of water, $0 \le t \le 6$ (with t in minutes).

a. Find $\frac{dV}{dt}$. 1 mark

b. Find the rate of change of volume at $t = 1$ and interpret it. 2 marks

c. Find the times at which the rate of change of volume is zero. 3 marks

Question 4 (8 marks)

A rectangular enclosure is built against a straight wall using 56 m of fencing for the three remaining sides. Let the width (perpendicular to the wall) be x metres.

a. Show that the area is $A = 56x - 2x^2$. 2 marks

b. Find the value of x that maximises the area. 3 marks

c. Find the maximum area. 2 marks

d. State the domain of A. 1 mark

Question 5 (7 marks)

A particle moves in a straight line with position $x(t) = t^3 - 15t^2 + 48t$ metres, $t \ge 0$ seconds.

a. Find the velocity $v(t)$ and the acceleration $a(t)$. 2 marks

b. Find the times at which the particle is at rest. 2 marks

c. Find the acceleration at each of those times. 2 marks

d. Find the position of the particle when $t = 3$. 1 mark

Question 6 (5 marks)

Use Newton's method with the starting value $x_0 = 2$ to find a root of $f(x) = x^3 - 2x - 5$, correct to 4 decimal places. Set out your iterations in a table.

Question 7 (5 marks)

a. Find $f(x)$ given that $f'(x) = 6x^2 + 2x - 3$ and $f(1) = 5$. 3 marks

b. Find $\int (4x^3 - 2x)\,dx$. 2 marks

Question 8 (6 marks)

For $f(x) = 5x^2 - 2x + 3$:

a. Find the average rate of change between $x = 1$ and $x = 4$. 2 marks

b. Find the instantaneous rate of change at $x = 1$. 2 marks

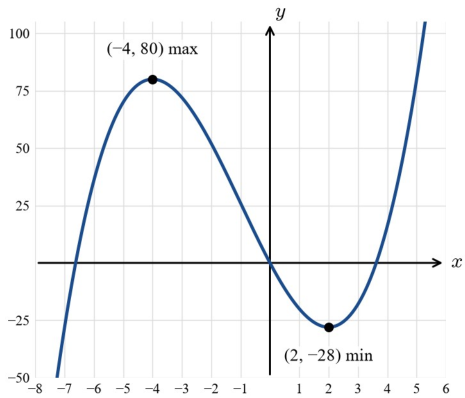
c. Explain why the two values differ. 2 marks

End of Topic 4 Test A

Topic 4 Test A — solutions

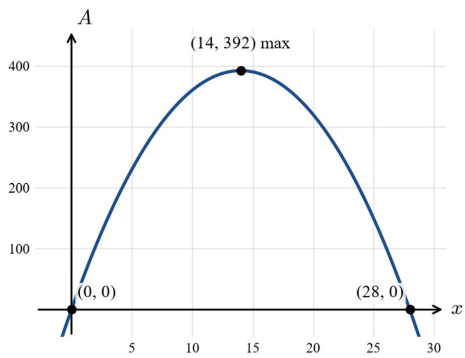
Q1. (a) $f'(x)=6x^2-10x+4$. (b) $f'(2)=6(4)-10(2)+4=24-20+4=8$. (c) $6x^2-10x+4=4 \Rightarrow 6x^2-10x=0 \Rightarrow 2x(3x-5)=0$, so $x=0$ or $x=5/3$.

Q2. (a) $y'=3x^2+6x-24=3(x+4)(x-2)=0$ at $x=-4, 2$. The gradient is positive for $x<-4$, negative for $-4<x<2$ and positive for $x>2$, so $(-4, 80)$ is a **local maximum** and, since $y(2)=8+12-48=-28$, $(2,-28)$ is a **local minimum**. (b) $y(1)=1+3-24=-20$; $y'(1)=3+6-24=-15$. Tangent: $y-(-20)=-15(x-1)$, i.e. $\therefore y=-15x-5$.

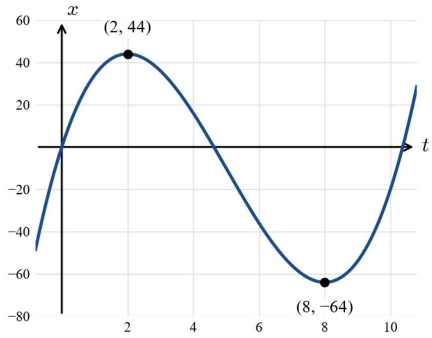


Q3. (a) $\dfrac{dV}{dt}=6t^2-48t+72$. (b) At $t=1$: $6-48+72=30$, so the volume is increasing at 30 L/min. (c) $6t^2-48t+72=0 \Rightarrow t^2-8t+12=0 \Rightarrow (t-2)(t-6)=0$, so $t=2$ or $t=6$ minutes.

Q4. (a) The wall side needs no fence, so the two widths and one length use the fencing: $2x+\ell=56 \Rightarrow \ell=56-2x$. Area $A=x\ell=x(56-2x)=56x-2x^2$. (b) $A'=56-4x=0 \Rightarrow x=14$; A' changes from positive to negative at $x=14$ (e.g. $A'(13)=4>0$ and $A'(15)=-4<0$), so this is a maximum. (c) $A(14)=56(14)-2(196)=392\,\text{m}^2$. (d) $0<x<28$.



Q5. (a) $v(t)=3t^2-30t+48$; $a(t)=6t-30$. (b) $v=3(t^2-10t+16)=3(t-2)(t-8)=0$ at $t=2$ and $t=8$ s. (c) $a(2)=-18\,\text{m/s}^2$, $a(8)=18\,\text{m/s}^2$. (d) $x(3)=27-135+144=36$ m.



Q6. Newton's method: $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	2	-1	10	2.1
1	2.1	0.061	11.23	2.094568
2	2.094568	0.000186	11.1616	2.094551

$\therefore$ the root is $x \approx 2.0946$ (to 4 decimal places).

Q7. (a) $f(x) = 2x^3 + x^2 - 3x + c$. $f(1) = 2 + 1 - 3 + c = c = 5$; $\therefore f(x) = 2x^3 + x^2 - 3x + 5$. (b) $\int (4x^3 - 2x)\,dx = x^4 - x^2 + c$.

Q8. (a) $f(1) = 6$ and $f(4) = 75$; average rate $= \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{75 - 6}{3} = 23$. (b) $f'(x) = 10x - 2$, so $f'(1) = 8$. (c) The average rate is the gradient of the **secant** (chord) from $x = 1$ to $x = 4$, while the instantaneous rate is the gradient of the **tangent** at $x = 1$; because the curve bends, these gradients differ.

Topic 4 Test B — Technology-free

Topic 4 covers Chapters 13–15 (rates of change, differentiation and anti-differentiation, applications of differentiation). Reading time: 5 minutes. Writing time: 45 minutes. Total: 40 marks. Students are not permitted to bring any technology (calculators or software), or notes of any kind. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (5 marks)

For $f(x)=4x^2-3x$:

a. Use **first principles** to find $f'(x)$. 4 marks

b. Hence find $f'(2)$. 1 mark

Question 2 (6 marks)

Differentiate each of the following.

a. $y=3x^4-2x^2+7$ 2 marks

b. $y=(3x-2)(x+4)$ 2 marks

c. $y=x^2(3-x)$ 2 marks

Question 3 (6 marks)

Consider $y=x^3-48x$.

a. Find the coordinates of the stationary points. 3 marks

b. Classify each stationary point using the sign of the gradient on either side. 3 marks

Question 4 (6 marks)

For the curve $y=x^2-6x+4$ at the point where $x=5$:

a. Find the equation of the tangent. 3 marks

b. Find the equation of the normal. 3 marks

Question 5 (5 marks)

A particle moves in a straight line with position $x(t)=t^2-2t-8$ metres, $t\geq 0$ seconds.

a. Find the velocity $v(t)$. 1 mark

b. Find the time at which the particle is at rest. 1 mark

c. Find the displacement in the first 3 seconds. 2 marks

d. Find the distance travelled in the first 3 seconds. 1 mark

Question 6 (5 marks)

a. Find $\int(3x^2+4x-2)\,dx$. 2 marks

b. Given $f'(x)=6x-7$ and $f(2)=3$, find $f(x)$. 3 marks

Question 7 (4 marks)

a. Evaluate $\dfrac{\lim\limits_{x \to 3} x^2 - 9}{x - 3}$. 2 marks

b. For what value of k is the function $f(x) = \begin{cases} x^2 & x \le 2 \\ kx & x > 2 \end{cases}$ continuous at $x = 2$? 2 marks

Question 8 (3 marks)

The sum of two numbers is 26. Find the two numbers that give the maximum product, and state the maximum product.

End of Topic 4 Test B

Topic 4 Test B — solutions

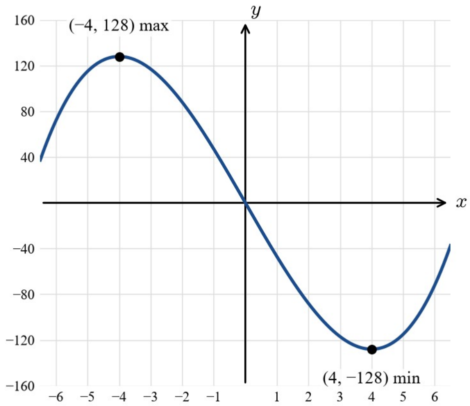
Q1. (a)

$$f'(x)=\frac{\lim_{h\to 0} f(x+h)-f(x)}{h}=\frac{\lim_{h\to 0}\left[4(x+h)^2-3(x+h)\right]-(4x^2-3x)}{h}=\frac{\lim_{h\to 0} 8xh+4h^2-3h}{h}=\lim_{h\to 0}(8x+4h-3)=8x-3$$

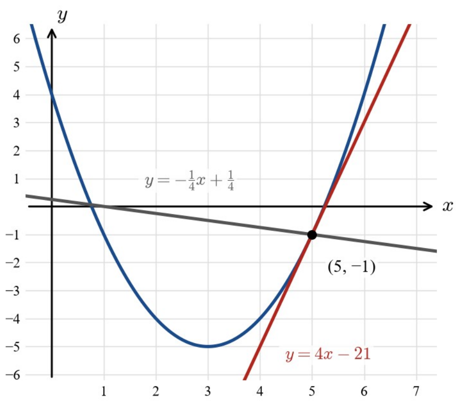
. (b) $f'(2)=8(2)-3=13$.

Q2. (a) $\frac{dy}{dx}=12x^3-4x$. (b) $y=3x^2+10x-8$, so $\frac{dy}{dx}=6x+10$. (c) $y=3x^2-x^3$, so $\frac{dy}{dx}=6x-3x^2$.

Q3. (a) $y'=3x^2-48=3(x-4)(x+4)=0$ at $x=\pm 4$. $y(4)=64-192=-128$ and $y(-4)=-64+192=128$, giving stationary points $(4,-128)$ and $(-4,128)$. (b) $y'=3(x-4)(x+4)$ is positive for $x<-4$, negative for $-4<x<4$ and positive for $x>4$: the gradient changes $+\to-$ at $x=-4$ and $-\to+$ at $x=4$, so $(-4,128)$ is a **local maximum** and $(4,-128)$ is a **local minimum**.



Q4. $y(5)=25-30+4=-1$; $y'=2x-6$, so $y'(5)=4$. (a) Tangent: $y-(-1)=4(x-5)$, i.e. $\therefore y=4x-21$. (b) Normal gradient $=-1/4$: $y+1=-1/4(x-5)$, i.e. $\therefore y=-1/4x+1/4$.



Q5. (a) $v(t)=2t-2$. (b) $v=0\Rightarrow t=1$ s. (c) Displacement $=x(3)-x(0)=-5-(-8)=3$ m. (d) The particle changes direction at $t=1$: $x(0)=-8$, $x(1)=-9$, $x(3)=-5$. Distance $=|-9-(-8)|+|-5-(-9)|=1+4=5$ m.

Q6. (a) $\int(3x^2+4x-2)\,dx=x^3+2x^2-2x+c$. (b) $f(x)=3x^2-7x+c$; $f(2)=12-14+c=-2+c=3$, so $c=5$; $\therefore f(x)=3x^2-7x+5$.

Q7. (a) $\frac{x^2-9}{x-3}=\frac{(x-3)(x+3)}{x-3}=x+3$ for $x\neq 3$, so the limit is $3+3=6$. (b) For continuity at $x=2$ the two rules must agree there: $2^2=k(2)\Rightarrow 4=2k\Rightarrow k=2$.

Q8. Let the numbers be x and $26-x$. Product $P=x(26-x)=26x-x^2$. $P'=26-2x=0 \Rightarrow x=13$; P' is positive for $x<13$ and negative for $x>13$, so the gradient changes $+ \to -$: a maximum. The numbers are 13 and 13, and the maximum product is 169.