Mathematical Methods Units 1 & 2

Chapter 14 — Differentiation and anti-differentiation of polynomial functions

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Chapter 14 Differentiation and anti-differentiation of polynomial functions — 14A The derivative as a limit

Theory

The **derivative** of a function is a new function that gives the **gradient of the tangent** to the graph at any point. It is also called the **gradient function**.

The derivative by first principles. In Chapter 13 the instantaneous rate of change at a point was found as the limit of average rates over shorter and shorter intervals. Applying this at a general point x gives the derivative:

$$f'(x)=\frac{\lim_{h\to 0} f(x+h)-f(x)}{h}.$$

The method. To differentiate by first principles:

1. write out $f(x+h)$ and subtract $f(x)$;
2. simplify the numerator (the x-only terms cancel, leaving a common factor of h);
3. divide every term by h;
4. take the limit as $h\to 0$ (put $h=0$ in what remains).

Notation. If $y=f(x)$, the derivative is written $f'(x)$ or $\frac{dy}{dx}$ (and $\frac{d}{dx}(f(x))$ means "differentiate $f(x)$"). All mean the gradient function.

Gradient at a point. The number $f'(a)$ is the gradient of the graph — and of the **tangent** — at the point where $x=a$. Substitute $x=a$ into the derivative to find it.

The central difference approximation. The first-principles quotient $\frac{f(a+h)-f(a)}{h}$ — the **forward difference** — measures a gradient using the point at a and one point to its right. A more accurate estimate for the same step h samples a point on **each** side of a:

$$f'(a)\approx\frac{f(a+h)-f(a-h)}{2h}.$$

This is the **central difference** approximation. Because $a-h$ and $a+h$ sit symmetrically about a, the errors either side largely cancel, so for a small step h the central difference is closer to the true gradient $f'(a)$ than the forward difference is.

Worked examples

Example 1 — scaffolded

Find $f'(x)$ from first principles for $f(x)=x^2+3x$.

Working	Comment
$f(x+h)=(x+h)^2+3(x+h)=x^2+2xh+h^2+3x+3h$.	Substitute $x+h$ for x and expand.
$f(x+h)-f(x)=2xh+h^2+3h$.	The x^2 and $3x$ terms cancel.
$\frac{f(x+h)-f(x)}{h}=\frac{2xh+h^2+3h}{h}=2x+h+3$.	Every remaining term has a factor of h; divide.

Working	Comment
$f'(x)=\lim_{h\to 0}(2x+h+3)=2x+3$.	Let $h\to 0$.

Example 2 — scaffolded

For $f(x)=x^2-4x$, find $f'(x)$ by first principles, then find the gradient at $x=1$.

Working	Comment
$f(x+h)-f(x)=\left[(x+h)^2-4(x+h)\right]-(x^2-4x)=2x$.	Expand and subtract; the x-only terms cancel.
$\frac{2xh+h^2-4h}{h}=2x+h-4$.	Divide by h.
$f'(x)=\lim_{h\to 0}(2x+h-4)=2x-4$.	Let $h\to 0$.
$f'(1)=2(1)-4=-2$.	Substitute $x=1$: the gradient at $x=1$ is -2.

Example 3 — unscaffolded

Find $\frac{dy}{dx}$ from first principles for $y=x^3$.

$(x+h)^3-x^3=x^3+3x^2h+3xh^2+h^3-x^3=3x^2h+3xh^2+h^3$.

$\frac{3x^2h+3xh^2+h^3}{h}=3x^2+3xh+h^2$.

$\therefore \frac{dy}{dx}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$.

Example 4 — unscaffolded

For $f(x)=2x^2-5x+1$, find $f'(x)$ by first principles and hence the gradient of the tangent at $x=2$.

$f(x+h)-f(x)=\left[2(x+h)^2-5(x+h)+1\right]-(2x^2-5x+1)=4xh+2h^2-5h$.

$\frac{4xh+2h^2-5h}{h}=4x+2h-5\to 4x-5$ as $h\to 0$, so $f'(x)=4x-5$.

$\therefore f'(2)=4(2)-5=3$: the tangent at $x=2$ has gradient 3.

Example 5 — scaffolded (central difference)

For $f(x)=x^3$, use the central difference approximation with $h=0.1$ to estimate $f'(2)$, and compare with the exact value.

Working	Comment
$f'(2)\approx\frac{f(2+0.1)-f(2-0.1)}{2(0.1)}=\frac{f(2.1)-f(1.9)}{0.2}$.	Central difference with $a=2$, $h=0.1$.
$f(2.1)=2.1^3=9.261$ and $f(1.9)=1.9^3=6.859$.	Evaluate the points either side of $x=2$.
$\frac{9.261-6.859}{0.2}=\frac{2.402}{0.2}=12.01$.	Subtract, then divide by $2h=0.2$.
Exact: $f'(x)=3x^2$, so $f'(2)=12$.	The estimate 12.01 is within 0.01 of the true gradient.

Practice questions

Scaffolded

Q1. For $f(x)=x^2+5x$: (a) write $f(x+h)-f(x)$; (b) divide by h; (c) take the limit to find $f'(x)$.

Q2. For $f(x)=3x^2-x$: (a) simplify $f(x+h)-f(x)$; (b) hence find $f'(x)$.

Q3. For $f(x)=x^2-6x$: (a) find $f'(x)$ by first principles; (b) hence find $f'(2)$.

Q4. For $f(x)=x^3-x$: (a) expand $(x+h)^3$; (b) find $f'(x)$ by first principles.

Q5. For $f(x)=4-x^2$: (a) find $f'(x)$ by first principles; (b) find the gradient at $x=-1$.

Q6. For $y=x^2+2x-3$: (a) find $\frac{dy}{dx}$ by first principles; (b) find the gradient at $x=0$.

Unscaffolded

Find the derivative by first principles. (For Q14–Q18, also answer the extra question.)

Q7. $f(x)=x^2-3x$

Q8. $f(x)=2x^2+x$

Q9. $f(x)=5x^2$

Q10. $f(x)=x^2+7$

Q11. $f(x)=x^3+2x$

Q12. $f(x)=3x-x^2$

Q13. $y=2x^3$

Q14. $f(x)=x^2-x$; also find $f'(3)$.

Q15. $f(x)=6-2x^2$; also find the gradient at $x=2$.

Q16. $f(x)=x^3$; also find the gradient of the tangent at $x=1$.

Q17. $f(x)=x^2+4x$; also find the value of x where the gradient is 0.

Q18. $f(x)=x^3-3x$; also find the values of x where the gradient is 9.

Central difference

Use the central difference approximation $f'(a)\approx\frac{f(a+h)-f(a-h)}{2h}$, then compare with the exact derivative.

Q19. For $f(x)=x^2$, estimate $f'(3)$ using $h=0.1$; compare with the exact value.

Q20. For $f(x)=x^3-x$, estimate $f'(1)$ using $h=0.1$; compare with the exact value.

Q21. For $f(x)=\frac{1}{x}$, estimate $f'(2)$ using $h=0.1$; compare with the exact value (find $f'(x)$ from first principles).

Answers

Q1. $2x+5$. **Q2.** $6x-1$. **Q3.** $f'(x)=2x-6$; $f'(2)=-2$. **Q4.** $f'(x)=3x^2-1$. **Q5.** $f'(x)=-2x$; gradient at $x=-1$ is 2. **Q6.** $\frac{dy}{dx}=2x+2$; gradient at $x=0$ is 2. **Q7.** $2x-3$. **Q8.** $4x+1$. **Q9.** $10x$. **Q10.** $2x$. **Q11.** $3x^2+2$. **Q12.** $3-2x$. **Q13.** $6x^2$. **Q14.** $2x-1$; $f'(3)=5$. **Q15.** $-4x$; gradient at $x=2$ is -8. **Q16.** $3x^2$; gradient at $x=1$ is 3. **Q17.** $2x+4$; gradient 0 at $x=-2$. **Q18.** $3x^2-3$; gradient 9 at $x=-2$ and $x=2$. **Q19.** Estimate 6; exact $f'(3)=6$ (the central difference is exact for a quadratic). **Q20.** Estimate 2.01; exact $f'(1)=2$. **Q21.** Estimate $-\frac{100}{399}\approx-0.2506$; exact $f'(2)=-\frac{1}{4}=-0.25$ (the exact derivative is found by first principles in the solution).

Fully-worked solutions

Q1. $f(x+h)-f(x)=\left[(x+h)^2+5(x+h)\right]-(x^2+5x)=2xh+h^2+5h$. $\frac{2xh+h^2+5h}{h}=2x+h+5\to 2x+5$. $\therefore f'(x)=2x+5$.

Q2. $f(x+h)-f(x)=\left[3(x+h)^2-(x+h)\right]-(3x^2-x)=6xh+3h^2-h$. $\frac{6xh+3h^2-h}{h}=6x+3h-1\to 6x-1$. $\therefore f'(x)=6x-1$.

Q3. $f(x+h)-f(x)=2xh+h^2-6h$, so $\frac{\cdots}{h}=2x+h-6\to 2x-6$. $\therefore f'(x)=2x-6$ and $f'(2)=4-6=-2$.

Q4. $(x+h)^3=x^3+3x^2h+3xh^2+h^3$, so $f(x+h)-f(x)=3x^2h+3xh^2+h^3-h$. $\frac{\cdots}{h}=3x^2+3xh+h^2-1\to 3x^2-1$. $\therefore f'(x)=3x^2-1$.

Q5. $f(x+h)-f(x)=\left[4-(x+h)^2\right]-(4-x^2)=-2xh-h^2$. $\frac{-2xh-h^2}{h}=-2x-h\to -2x$. $\therefore f'(x)=-2x$ and $f'(-1)=2$.

Q6. $f(x+h)-f(x)=2xh+h^2+2h$, so $\frac{\cdots}{h}=2x+h+2\to 2x+2$. $\therefore \frac{dy}{dx}=2x+2$ and at $x=0$ the gradient is 2.

Q7. $f(x+h)-f(x)=2xh+h^2-3h$, giving $2x+h-3\to 2x-3$. $\therefore f'(x)=2x-3$.

Q8. $f(x+h)-f(x)=4xh+2h^2+h$, giving $4x+2h+1\to 4x+1$. $\therefore f'(x)=4x+1$.

Q9. $f(x+h)-f(x)=10xh+5h^2$, giving $10x+5h\to 10x$. $\therefore f'(x)=10x$.

Q10. $f(x+h)-f(x)=(x^2+2xh+h^2+7)-(x^2+7)=2xh+h^2$, giving $2x+h\to 2x$. $\therefore f'(x)=2x$ (the constant 7 has no effect on the gradient).

Q11. $f(x+h)-f(x)=3x^2h+3xh^2+h^3+2h$, giving $3x^2+3xh+h^2+2\to 3x^2+2$. $\therefore f'(x)=3x^2+2$.

Q12. $f(x+h)-f(x)=3h-(2xh+h^2)=3h-2xh-h^2$, giving $3-2x-h\to 3-2x$. $\therefore f'(x)=3-2x$.

Q13. $f(x+h)-f(x)=2\left[3x^2h+3xh^2+h^3\right]=6x^2h+6xh^2+2h^3$, giving $6x^2+6xh+2h^2\to 6x^2$. $\therefore \frac{dy}{dx}=6x^2$.

Q14. $f(x+h)-f(x)=2xh+h^2-h$, giving $2x+h-1\to 2x-1$. $\therefore f'(x)=2x-1$ and $f'(3)=6-1=5$.

Q15. $f(x+h)-f(x)=-4xh-2h^2$, giving $-4x-2h \to -4x$. $\therefore f'(x)=-4x$ and the gradient at $x=2$ is -8.

Q16. $f(x+h)-f(x)=3x^2h+3xh^2+h^3$, giving $3x^2+3xh+h^2 \to 3x^2$. $\therefore f'(x)=3x^2$ and the tangent at $x=1$ has gradient 3.

Q17. $f'(x)=2x+4$ (as in Q1's method). Setting $2x+4=0$ gives $x=-2$.

Q18. $f'(x)=3x^2-3$. Setting $3x^2-3=9$ gives $x^2=4$, so $x=-2$ or $x=2$.

Q19. $f'(3)\approx\dfrac{f(3.1)-f(2.9)}{0.2}$. With $f(3.1)=3.1^2=9.61$ and $f(2.9)=2.9^2=8.41$, this is $\dfrac{9.61-8.41}{0.2}=\dfrac{1.2}{0.2}=6$. The exact value is $f'(x)=2x$, so $f'(3)=6$: here the estimate equals the exact gradient.

Q20. $f'(1)\approx\dfrac{f(1.1)-f(0.9)}{0.2}$. With $f(1.1)=1.1^3-1.1=0.231$ and $f(0.9)=0.9^3-0.9=-0.171$, this is $\dfrac{0.231-(-0.171)}{0.2}=\dfrac{0.402}{0.2}=2.01$. The exact value is $f'(x)=3x^2-1$, so $f'(1)=2$; the estimate 2.01 is within 0.01.

Q21. $f'(2)\approx\dfrac{f(2.1)-f(1.9)}{0.2}$. With $f(2.1)=\dfrac{1}{2.1}=\dfrac{10}{21}$ and $f(1.9)=\dfrac{1}{1.9}=\dfrac{10}{19}$, the numerator is $\dfrac{10}{21}-\dfrac{10}{19}=-\dfrac{20}{399}$, so the estimate is $\dfrac{-20/399}{0.2}=-\dfrac{100}{399}\approx-0.2506$. The exact derivative, from first principles, is
$f'(x)=\dfrac{\lim\limits_{h\to 0}\dfrac{1}{x+h}-\dfrac{1}{x}}{h}=\dfrac{\lim\limits_{h\to 0}-h}{hx(x+h)}=\dfrac{\lim\limits_{h\to 0}-1}{x(x+h)}=-\dfrac{1}{x^2}$, so $f'(2)=-\dfrac{1}{4}=-0.25$; the estimate is within about 0.0006.

Chapter 14 Differentiation and anti-differentiation of polynomial functions — 14B Differentiation by rule

Theory

The **derivative** of $y=f(x)$, written $\frac{dy}{dx}$ or $f'(x)$, is the **gradient function** — it gives the gradient of the tangent to the curve at each value of x. Rather than use first principles every time, we use a set of rules.

The power rule. For any positive integer n,

$$\frac{d}{dx}(x^n)=nx^{n-1}.$$

Multiply by the index, then reduce the index by one. For example $\frac{d}{dx}(x^4)=4x^3$.

The constant rule. The derivative of a constant is zero: $\frac{d}{dx}(c)=0$ (a horizontal line has gradient 0).

The constant-multiple rule. $\frac{d}{dx}(kf(x))=kf'(x)$ — a constant factor is kept. For example $\frac{d}{dx}(5x^3)=5\times 3x^2=15x^2$.

The sum and difference rule. $\frac{d}{dx}(f(x)\pm g(x))=f'(x)\pm g'(x)$ — differentiate **term by term**.

Polynomials. Combining these rules, differentiate a polynomial one term at a time. Note that $\frac{d}{dx}(x)=1$ (since $x=x^1$).

Products must be expanded first. There is no simple "product rule" at this level — expand the brackets, then differentiate. For example, to differentiate $y=(x+1)(x-4)$, first expand to $y=x^2-3x-4$, then $\frac{dy}{dx}=2x-3$.

Using the derivative.

- To find the **gradient at a point** $x=a$, evaluate $f'(a)$.
- To find **where the gradient equals a value** k (or is **zero**), solve $f'(x)=k$.

Worked examples

Example 1 — scaffolded

Differentiate $y=3x^4-5x^2+2x-7$.

Working	Comment
Differentiate each term separately.	Use the sum/difference rule.
$\frac{d}{dx}(3x^4)=12x^3$; $\frac{d}{dx}(5x^2)=10x$; $\frac{d}{dx}(2x)=2$; $\frac{d}{dx}(7)=0$.	Power rule with the constant-multiple rule; the derivative of a constant is 0.
$\frac{dy}{dx}=12x^3-10x+2$.	Combine the terms.

Example 2 — scaffolded

For $f(x)=2x^3-4x$, find $f'(x)$ and hence $f'(2)$.

Working	Comment
$f'(x)=2\times 3x^2-4=6x^2-4.$	Differentiate term by term; $\frac{d}{dx}(4x)=4.$
$f'(2)=6(2)^2-4=24-4=20.$	Substitute $x=2$ into the gradient function.

So the gradient of the curve at $x=2$ is 20.

Example 3 — unscaffolded

Differentiate $y=(2x-1)(x+3)$.

Expand first: $y=2x^2+6x-x-3=2x^2+5x-3$.

$\therefore \frac{dy}{dx}=4x+5.$

Example 4 — unscaffolded

Find the value(s) of x where the gradient of $y=x^3-3x^2$ is zero.

$\frac{dy}{dx}=3x^2-6x$. Setting the gradient to zero:

$$3x^2-6x=0\Rightarrow 3x(x-2)=0.$$

$\therefore x=0$ or $x=2$.

Practice questions

Scaffolded

Q1. Differentiate $y=x^5$.

Q2. Differentiate $y=4x^3-2x^2+7$.

Q3. For $f(x)=x^2-6x$: (a) find $f'(x)$; (b) hence find $f'(3)$.

Q4. Differentiate $y=5x^4-x$.

Q5. Expand and then differentiate $y=x(x-4)$.

Q6. For $f(x)=x^2-8x$, find the value of x where $f'(x)=0$.

Unscaffolded

Q7. Differentiate $y=6x^2-5x+1$.

Q8. Differentiate $y=2x^4-3x^3+x$.

Q9. For $f(x)=x^3-2x^2+x-5$, find $f'(x)$.

Q10. Differentiate $y=7-3x+x^2$.

Q11. For $f(x)=1/2\,x^4-2x^2$, find $f'(x)$ and $f'(2)$.

Q12. Differentiate $y=(x+2)(x-5)$.

Q13. Differentiate $y=(2x-3)^2$.

Q14. Differentiate $y=x^2(3x-1)$.

Q15. Find the gradient of $y=x^3-4x$ at $x=1$.

Q16. Find the value of x where the gradient of $y=x^2-6x+5$ is zero.

Q17. Find the values of x where the gradient of $y=x^3-12x$ is zero.

Q18. For $y=2x^3-9x^2+12x$, find the values of x where $\frac{dy}{dx}=0$.

Answers

Q1. $5x^4$. **Q2.** $12x^2-4x$. **Q3.** $2x-6$; $f'(3)=0$. **Q4.** $20x^3-1$. **Q5.** $2x-4$. **Q6.** $x=4$. **Q7.** $12x-5$. **Q8.** $8x^3-9x^2+1$. **Q9.** $3x^2-4x+1$. **Q10.** $2x-3$. **Q11.** $2x^3-4x$; $f'(2)=8$. **Q12.** $2x-3$. **Q13.** $8x-12$. **Q14.** $9x^2-2x$. **Q15.** -1. **Q16.** $x=3$. **Q17.** $x=-2$ or $x=2$. **Q18.** $x=1$ or $x=2$.

Fully-worked solutions

Q1. $\frac{dy}{dx}=5x^4$ (power rule).

Q2. $\frac{dy}{dx}=12x^2-4x$ (the derivative of the constant 7 is 0).

Q3. (a) $f'(x)=2x-6$. (b) $f'(3)=2(3)-6=0$.

Q4. $\frac{dy}{dx}=20x^3-1$ (since $\frac{d}{dx}(x)=1$).

Q5. $y=x(x-4)=x^2-4x$, so $\frac{dy}{dx}=2x-4$.

Q6. $f'(x)=2x-8$; setting $2x-8=0$ gives $x=4$.

Q7. $\frac{dy}{dx}=12x-5$.

Q8. $\frac{dy}{dx}=8x^3-9x^2+1$.

Q9. $f'(x)=3x^2-4x+1$.

Q10. $\frac{dy}{dx}=-3+2x=2x-3$.

Q11. $f'(x)=1/2\times 4x^3-2\times 2x=2x^3-4x$; $f'(2)=2(2)^3-4(2)=16-8=8$.

Q12. $y=(x+2)(x-5)=x^2-3x-10$, so $\frac{dy}{dx}=2x-3$.

Q13. $y=(2x-3)^2=4x^2-12x+9$, so $\frac{dy}{dx}=8x-12$.

Q14. $y=x^2(3x-1)=3x^3-x^2$, so $\frac{dy}{dx}=9x^2-2x$.

Q15. $\frac{dy}{dx}=3x^2-4$; at $x=1$, $3(1)^2-4=-1$.

Q16. $\frac{dy}{dx}=2x-6$; setting $2x-6=0$ gives $x=3$.

Q17. $\frac{dy}{dx}=3x^2-12$; setting $3x^2-12=0$ gives $x^2=4$, so $x=-2$ or $x=2$.

Q18. $\frac{dy}{dx}=6x^2-18x+12=6(x^2-3x+2)=6(x-1)(x-2)$; setting this to zero gives $x=1$ or $x=2$.

Chapter 14 Differentiation and anti-differentiation of polynomial functions — 14C Derivatives of negative integer powers

Units 3 & 4 preview. This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

Theory

The **power rule** for differentiation,

$$\frac{d}{dx}\left(x^n\right)=n x^{n-1},$$

holds not only for positive whole-number powers but also for **negative integer** powers. So it can be used to differentiate expressions with x in the **denominator**, once they are rewritten using negative indices.

Rewrite first. Recall that $\frac{1}{x^n}=x^{-n}$. Before differentiating, rewrite every term with x in the denominator as a power of x with a negative index:

$$\frac{3}{x^2}=3x^{-2},\ \frac{1}{x}=x^{-1},\ \frac{5}{x^4}=5x^{-4}.$$

Split a single fraction into separate terms by dividing each term of the numerator by the denominator, then use index laws:

$$\frac{x^2+1}{x}=\frac{x^2}{x}+\frac{1}{x}=x+x^{-1}.$$

Apply the power rule. For a negative integer power,

$$\frac{d}{dx}\left(x^{-n}\right)=-n x^{-n-1}.$$

For example $\frac{d}{dx}\left(x^{-1}\right)=-x^{-2}=-\frac{1}{x^2}$ and $\frac{d}{dx}\left(x^{-2}\right)=-2x^{-3}=-\frac{2}{x^3}$. The **constant multiple** and **sum** rules still apply.

Give the answer in positive-index (fraction) form. It is usual to write the final derivative back with positive indices, e.g. $-3x^{-4}=-\frac{3}{x^4}$.

Domain. A term such as $\frac{1}{x^n}$ is undefined at $x=0$, so these functions (and their derivatives) apply only for $x\neq 0$.

Worked examples

Example 1 — scaffolded

Differentiate $y=\frac{1}{x^3}$.

Working	Comment
$y=x^{-3}$.	Rewrite with a negative index before differentiating.
$\frac{dy}{dx}=-3x^{-4}$.	Power rule: multiply by the index -3, then reduce the index by 1.

Working	Comment
$\frac{dy}{dx} = -\frac{3}{x^4}$ (for $x \neq 0$).	Write the answer back in positive-index form.

Example 2 — scaffolded

Differentiate $f(x) = \frac{3}{x^2} + \frac{2}{x}$.

Working	Comment
$f(x) = 3x^{-2} + 2x^{-1}$.	Rewrite each term with a negative index.
$f'(x) = 3(-2)x^{-3} + 2(-1)x^{-2} = -6x^{-3} - 2x^{-2}$.	Differentiate term by term.
$f'(x) = -\frac{6}{x^3} - \frac{2}{x^2}$.	Back to positive-index form.

Example 3 — unscaffolded

Differentiate $y = \frac{2x^3 - 5}{x^2}$.

Split the fraction and use index laws:

$$y = \frac{2x^3}{x^2} - \frac{5}{x^2} = 2x - 5x^{-2}.$$

Differentiate:

$$\frac{dy}{dx} = 2 - 5(-2)x^{-3} = 2 + 10x^{-3}.$$

$\therefore \frac{dy}{dx} = 2 + \frac{10}{x^3}$.

Example 4 — unscaffolded

For $f(x) = x + \frac{4}{x}$, find $f'(2)$.

Rewrite: $f(x) = x + 4x^{-1}$, so

$$f'(x) = 1 + 4(-1)x^{-2} = 1 - 4x^{-2} = 1 - \frac{4}{x^2}.$$

$\therefore f'(2) = 1 - \frac{4}{2^2} = 1 - 1 = 0$.

Practice questions

Scaffolded

Q1. Differentiate $y = \frac{1}{x^2}$ by first writing it as x^{-2}.

Q2. Differentiate $y = \frac{5}{x^4}$.

Q3. Differentiate $y=\frac{2}{x^3}+\frac{3}{x}$.

Q4. Differentiate $f(x)=\frac{x^3+2}{x}$ (split the fraction first).

Q5. Differentiate $y=x^2-\frac{1}{x}$.

Q6. For $f(x)=\frac{6}{x^2}$, find $f'(1)$.

Unscaffolded

Differentiate each of the following, giving the answer in positive-index (fraction) form.

Q7. $y=\frac{1}{x^5}$

Q8. $y=-\frac{3}{x^2}$

Q9. $y=\frac{4}{x}+\frac{1}{x^3}$

Q10. $f(x)=\frac{x^4+x^2+1}{x^2}$

Q11. $y=\frac{2x^2+3}{x}$

Q12. $y=5x-\frac{2}{x^2}$

Q13. $f(x)=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}$

Q14. For $f(x)=\frac{8}{x^2}$, find $f'(2)$.

Q15. Find the gradient of $y=\frac{3}{x}$ at $x=3$.

Q16. For $f(x)=\frac{x^3+2x}{x^2}$, find $f'(1)$.

Q17. Find the coordinates of the points where the graph of $y=x+\frac{1}{x}$ has a gradient of 0.

Q18. Find the coordinates of the points where the graph of $y=x+\frac{4}{x}$ has a gradient of 0.

Answers

Q1. $-\frac{2}{x^3}$. **Q2.** $-\frac{20}{x^5}$. **Q3.** $-\frac{6}{x^4}-\frac{3}{x^2}$. **Q4.** $2x-\frac{2}{x^2}$. **Q5.** $2x+\frac{1}{x^2}$. **Q6.** -12. **Q7.** $-\frac{5}{x^6}$. **Q8.** $\frac{6}{x^3}$. **Q9.** $-\frac{4}{x^2}-\frac{3}{x^4}$. **Q10.** $2x-\frac{2}{x^3}$. **Q11.** $2-\frac{3}{x^2}$. **Q12.** $5+\frac{4}{x^3}$. **Q13.** $-\frac{1}{x^2}-\frac{2}{x^3}-\frac{3}{x^4}$. **Q14.** -2. **Q15.** $-\frac{1}{3}$. **Q16.** -1. **Q17.** $(1,2)$ and $(-1,-2)$. **Q18.** $(2,4)$ and $(-2,-4)$.

Fully-worked solutions

Q1. $y=x^{-2}\Rightarrow\frac{dy}{dx}=-2x^{-3}=-\frac{2}{x^3}$.

Q2. $y=5x^{-4}\Rightarrow\frac{dy}{dx}=5(-4)x^{-5}=-20x^{-5}=-\frac{20}{x^5}$.

Q3. $y=2x^{-3}+3x^{-1}\Rightarrow\frac{dy}{dx}=-6x^{-4}-3x^{-2}=-\frac{6}{x^4}-\frac{3}{x^2}$.

Q4. $f(x)=\frac{x^3}{x}+\frac{2}{x}=x^2+2x^{-1}$, so $f'(x)=2x-2x^{-2}=2x-\frac{2}{x^2}$.

Q5. $y=x^2-x^{-1}\Rightarrow\frac{dy}{dx}=2x+x^{-2}=2x+\frac{1}{x^2}$.

Q6. $f(x)=6x^{-2}\Rightarrow f'(x)=-12x^{-3}=-\frac{12}{x^3}$. $\therefore f'(1)=-\frac{12}{1}=-12$.

Q7. $y=x^{-5}\Rightarrow\frac{dy}{dx}=-5x^{-6}=-\frac{5}{x^6}$.

Q8. $y=-3x^{-2}\Rightarrow\frac{dy}{dx}=-3(-2)x^{-3}=6x^{-3}=\frac{6}{x^3}$.

Q9. $y=4x^{-1}+x^{-3}\Rightarrow\frac{dy}{dx}=-4x^{-2}-3x^{-4}=-\frac{4}{x^2}-\frac{3}{x^4}$.

Q10. $f(x)=\frac{x^4}{x^2}+\frac{x^2}{x^2}+\frac{1}{x^2}=x^2+1+x^{-2}$, so $f'(x)=2x-2x^{-3}=2x-\frac{2}{x^3}$.

Q11. $y=\frac{2x^2}{x}+\frac{3}{x}=2x+3x^{-1}$, so $\frac{dy}{dx}=2-3x^{-2}=2-\frac{3}{x^2}$.

Q12. $y=5x-2x^{-2}\Rightarrow\frac{dy}{dx}=5+4x^{-3}=5+\frac{4}{x^3}$.

Q13. $f(x)=x^{-1}+x^{-2}+x^{-3}$, so $f'(x)=-x^{-2}-2x^{-3}-3x^{-4}=-\frac{1}{x^2}-\frac{2}{x^3}-\frac{3}{x^4}$.

Q14. $f(x)=8x^{-2}\Rightarrow f'(x)=-16x^{-3}=-\frac{16}{x^3}$. $\therefore f'(2)=-\frac{16}{2^3}=-\frac{16}{8}=-2$.

Q15. $y=3x^{-1}\Rightarrow\frac{dy}{dx}=-3x^{-2}=-\frac{3}{x^2}$. At $x=3$: gradient $=-\frac{3}{3^2}=-\frac{3}{9}=-\frac{1}{3}$.

Q16. $f(x)=\frac{x^3}{x^2}+\frac{2x}{x^2}=x+2x^{-1}$, so $f'(x)=1-2x^{-2}=1-\frac{2}{x^2}$. $\therefore f'(1)=1-2=-1$.

Q17. $y=x+x^{-1}\Rightarrow\frac{dy}{dx}=1-x^{-2}=1-\frac{1}{x^2}$. Set the gradient to 0: $1-\frac{1}{x^2}=0\Rightarrow x^2=1\Rightarrow x=\pm 1$. At $x=1$: $y=1+1=2$; at $x=-1$: $y=-1-1=-2$. $\therefore$ the points are $(1,2)$ and $(-1,-2)$.

Q18. $y=x+4x^{-1}\Rightarrow\frac{dy}{dx}=1-4x^{-2}=1-\frac{4}{x^2}$. Set the gradient to 0: $1-\frac{4}{x^2}=0\Rightarrow x^2=4\Rightarrow x=\pm 2$. At $x=2$: $y=2+2=4$; at $x=-2$: $y=-2-2=-4$. $\therefore$ the points are $(2,4)$ and $(-2,-4)$.

Theory

The **derivative** $f'(x)$ gives the **gradient** of $y=f(x)$ at each value of x. So the graph of $y=f'(x)$ is a picture of how the gradient of f changes. Reading one graph from the other relies on three facts.

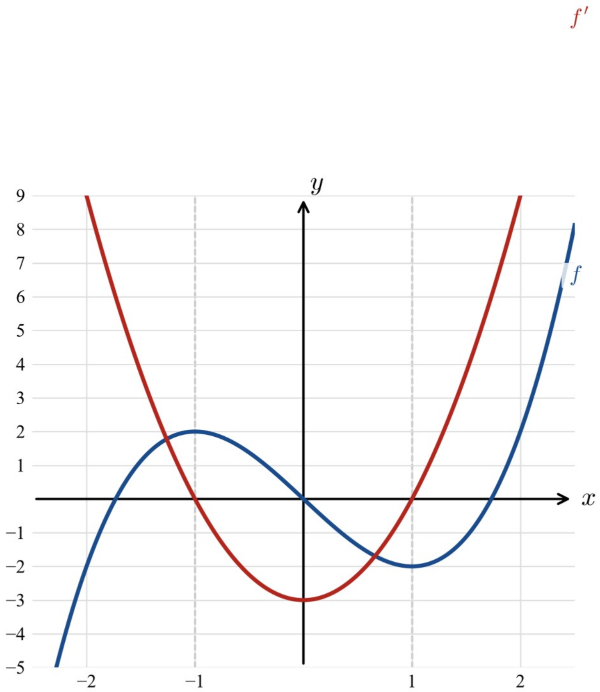
Where the gradient is zero. At a **turning point** (local maximum or minimum) of f the tangent is horizontal, so $f'(x)=0$ there. Each turning point of f gives an **x-intercept of f'**.

Sign of the gradient.

- Where f is **increasing** (sloping up), $f'(x)>0$, so the graph of f' is **above** the x-axis.
- Where f is **decreasing** (sloping down), $f'(x)<0$, so the graph of f' is **below** the x-axis.

Degree drops by one. Differentiating a polynomial of degree n gives one of degree $n-1$: a **quadratic** f gives a **linear** f'; a **cubic** f gives a **quadratic** f'; a **quartic** f gives a **cubic** f'.

For example, $f(x)=x^3-3x$ has turning points at $x=-1$ and $x=1$, and its derivative $f'(x)=3x^2-3$ is a parabola cutting the x-axis at exactly those points — negative (below the axis) between them, where f is decreasing.



To sketch $y=f'(x)$: find where f has turning points (these are the x-intercepts of f'), decide the sign of f' on each interval from whether f is increasing or decreasing, and remember the degree drops by one.

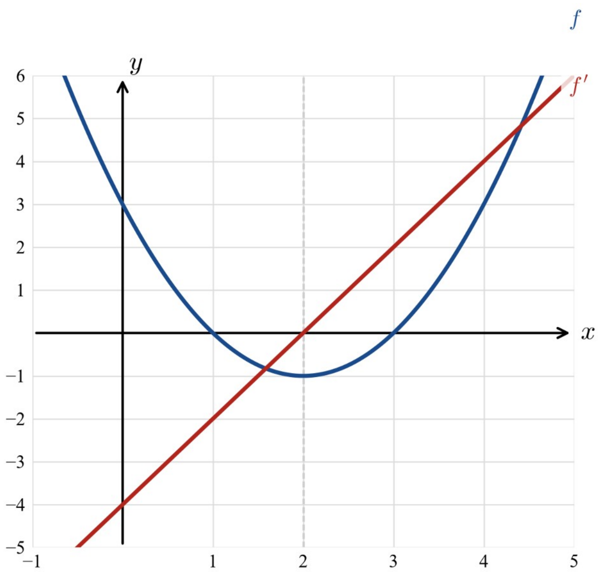
Worked examples

Example 1 — scaffolded

For $f(x)=x^2-4x+3$, find $f'(x)$ and sketch $y=f(x)$ and $y=f'(x)$ on the same axes.

Working	Comment
$f'(x)=2x-4$.	Differentiate term by term; the degree drops from 2 to 1.
$f'(x)=0 \Rightarrow 2x-4=0 \Rightarrow x=2$.	f has a minimum turning point at $x=2$, so f' cuts the x-

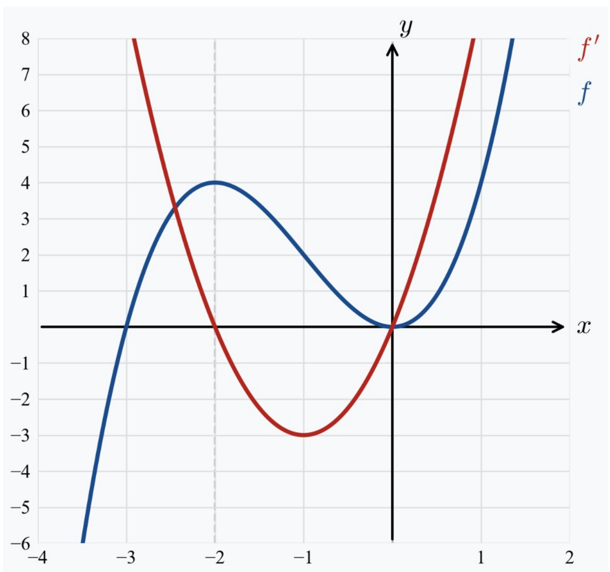
Working	Comment
	axis there.
For $x<2$, $f'(x)<0$ (f decreasing); for $x>2$, $f'(x)>0$ (f increasing).	The line f' is below the axis left of $x=2$, above it to the right.



Example 2 — scaffolded

For $f(x)=x^3+3x^2$, find $f'(x)$ and sketch both graphs.

Working	Comment
$f'(x)=3x^2+6x$.	A cubic differentiates to a quadratic.
$f'(x)=0 \Rightarrow 3x(x+2)=0 \Rightarrow x=-2$ or $x=0$.	f has turning points at $x=-2$ (max) and $x=0$ (min): these are the x-intercepts of f'.
$f'(x)<0$ for $-2<x<0$ (f decreasing), $f'(x)>0$ outside (f increasing).	The parabola f' dips below the axis between the turning points.

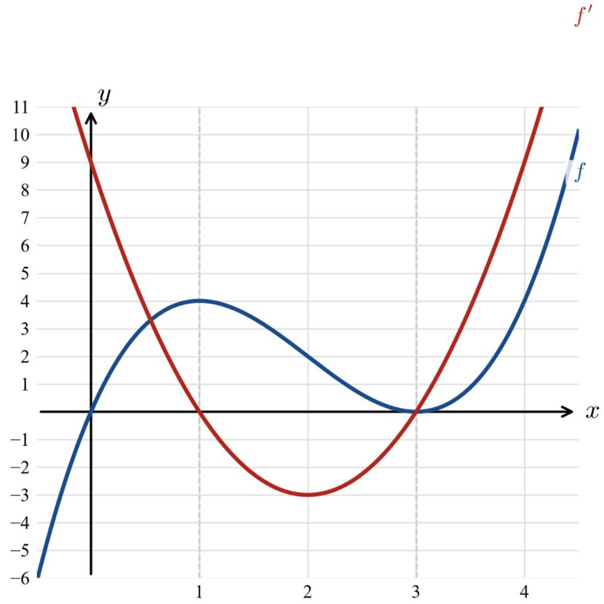


Example 3 — unscaffolded

For $f(x)=x^3-6x^2+9x$, find $f'(x)$ and sketch both graphs.

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, so $f'(x) = 0$ at $x = 1$ and $x = 3$ — the turning points of f. The parabola f' is positive (above the axis) for $x < 1$ and $x > 3$, and negative between, matching f increasing–decreasing–increasing.

∴ the graphs are:

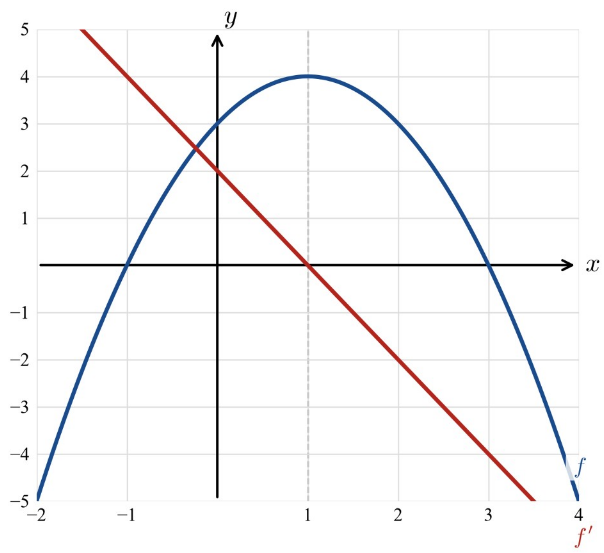


Example 4 — unscaffolded

For $f(x) = -x^2 + 2x + 3$, find $f'(x)$ and sketch both graphs.

$f'(x) = -2x + 2$, so $f'(x) = 0$ at $x = 1$ — the **maximum** turning point of f. Since f is increasing for $x < 1$ and decreasing for $x > 1$, the line f' is above the axis then below it.

∴ the graphs are:



Practice questions

Scaffolded

Q1. For $f(x) = x^2 - 2x$: (a) find $f'(x)$; (b) find where $f'(x) = 0$; (c) state the sign of f' on each side of that point; (d) sketch $y = f(x)$ and $y = f'(x)$.

Q2. For $f(x) = x^3 - 3x^2 + 1$: (a) find $f'(x)$; (b) find the x-intercepts of f'; (c) sketch both graphs.

Q3. For $f(x) = -x^2 + 4x - 1$: (a) find $f'(x)$; (b) find where $f'(x) = 0$; (c) sketch both graphs.

Q4. For $f(x) = x^2 + 2x - 3$: (a) find $f'(x)$; (b) hence state the turning point's x-value; (c) sketch both graphs.

Q5. For $f(x) = 2x^3 - 6x$: (a) find $f'(x)$; (b) find the x-intercepts of f'; (c) sketch both graphs.

Q6. For $f(x) = x^3$: (a) find $f'(x)$; (b) explain why $f'(x) \geq 0$ for all x and touches the axis at $x = 0$ (a stationary point of inflection); (c) sketch both graphs.

Unscaffolded

For each of the following, find $f'(x)$ and sketch $y = f(x)$ and $y = f'(x)$ on the same axes.

Q7. $f(x) = x^2$

Q8. $f(x) = x^2 - 6x + 5$

Q9. $f(x) = -x^2 + 8x - 12$

Q10. $f(x) = x^3 - 6x^2$

Q11. $f(x) = x^3 - 12x$

Q12. $f(x) = x^3 - 3x^2 - 9x$

Q13. $f(x) = -x^3 + 3x$

Q14. $f(x) = x^3 + 3x^2 - 9x$

Q15. $f(x) = x^4 - 2x^2$

Q16. $f(x) = 1/4\,x^4 - 2x^2$

Q17. A function $y = f(x)$ is increasing for $x < 2$, has a stationary point at $x = 2$, and is decreasing for $x > 2$. Sketch a possible graph of $y = f'(x)$, explaining its key features.

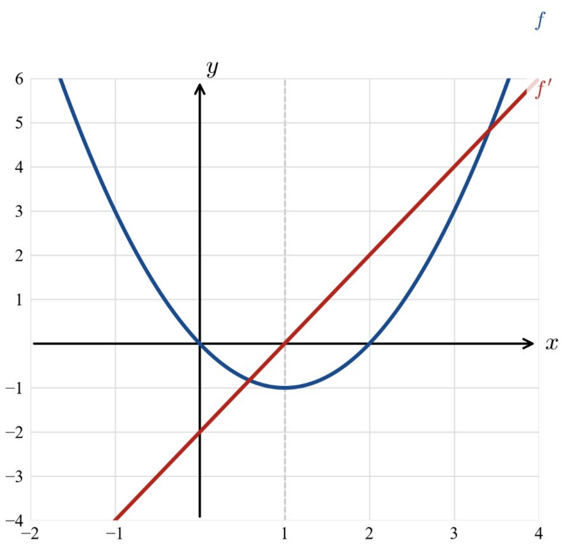
Q18. A **positive** cubic $y = f(x)$ has turning points at $x = -2$ and $x = 1$. (a) State the x-values where $f'(x) = 0$. (b) State whether $f'(x)$ is a positive or negative parabola. (c) Hence sketch $y = f'(x)$.

Answers

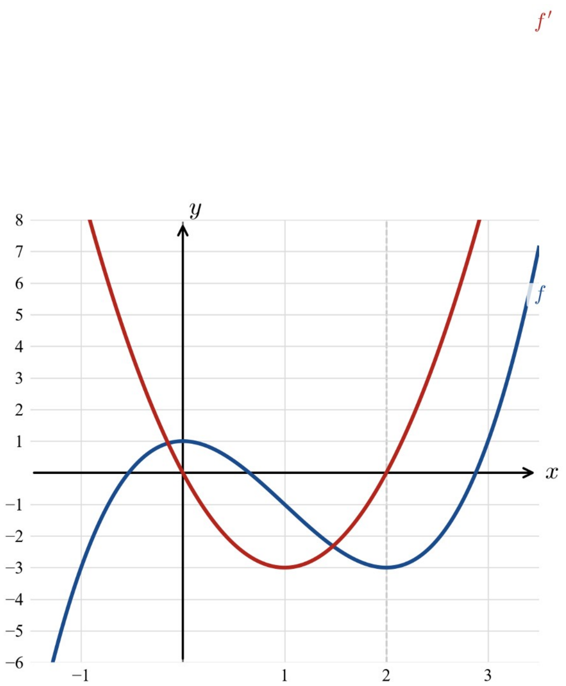
Q1. $f'(x)=2x-2$; zero at $x=1$; $f'<0$ for $x<1$, $f'>0$ for $x>1$. **Q2.** $f'(x)=3x^2-6x$; x-intercepts $x=0,2$. **Q3.** $f'(x)=-2x+4$; zero at $x=2$. **Q4.** $f'(x)=2x+2$; turning point at $x=-1$. **Q5.** $f'(x)=6x^2-6$; x-intercepts $x=\pm 1$. **Q6.** $f'(x)=3x^2$; ≥ 0 everywhere, touching the axis at $x=0$. **Q7.** $f'(x)=2x$. **Q8.** $f'(x)=2x-6$ (zero $x=3$). **Q9.** $f'(x)=-2x+8$ (zero $x=4$). **Q10.** $f'(x)=3x^2-12x$ (zeros $x=0,4$). **Q11.** $f'(x)=3x^2-12$ (zeros $x=\pm 2$). **Q12.** $f'(x)=3x^2-6x-9$ (zeros $x=-1,3$). **Q13.** $f'(x)=3-3x^2$ (zeros $x=\pm 1$). **Q14.** $f'(x)=3x^2+6x-9$ (zeros $x=-3,1$). **Q15.** $f'(x)=4x^3-4x$ (zeros $x=0,\pm 1$). **Q16.** $f'(x)=x^3-4x$ (zeros $x=0,\pm 2$). **Q17.** $f'(x)>0$ for $x<2$, $f'(2)=0$, $f'(x)<0$ for $x>2$ — a curve crossing from above to below the axis at $x=2$ (e.g. a line of negative gradient through $(2,0)$). **Q18.** (a) $x=-2$ and $x=1$; (b) a **positive** parabola (opens up); (c) a parabola through $(-2,0)$ and $(1,0)$ opening upwards.

Fully-worked solutions

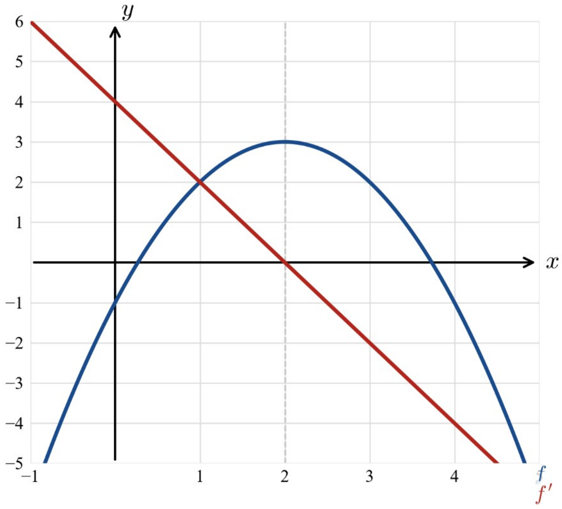
Q1. $f'(x)=2x-2$. (a)–(c) $f'(x)=0 \Rightarrow x=1$; $f'(x)<0$ for $x<1$ (f decreasing), $f'(x)>0$ for $x>1$ (f increasing). (d)



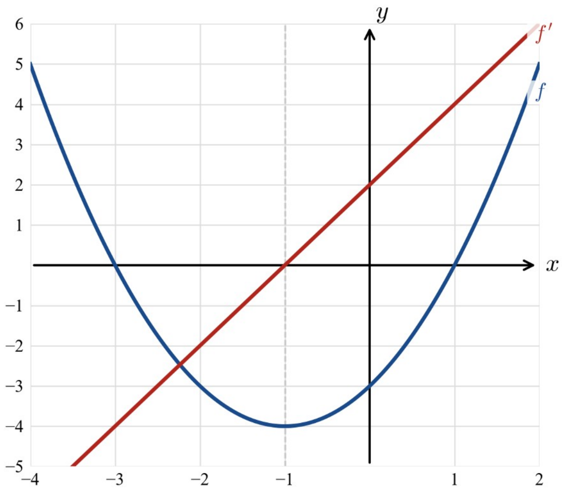
Q2. $f'(x)=3x^2-6x=3x(x-2)$, so $f'(x)=0$ at $x=0$ and $x=2$ — the turning points of f. f' is a positive parabola, below the axis for $0<x<2$.



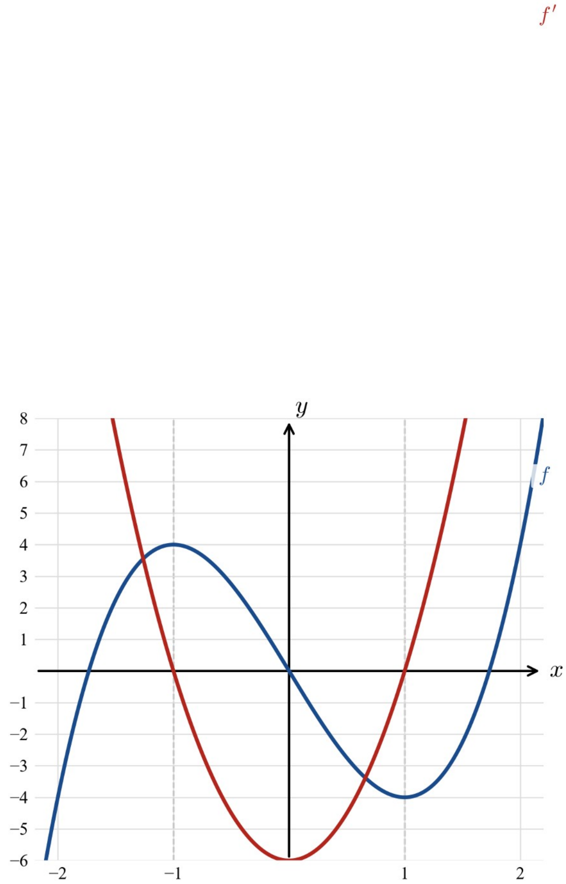
Q3. $f'(x) = -2x + 4 = 0 \Rightarrow x = 2$ (the maximum of f). f' is above the axis for $x < 2$, below for $x > 2$.



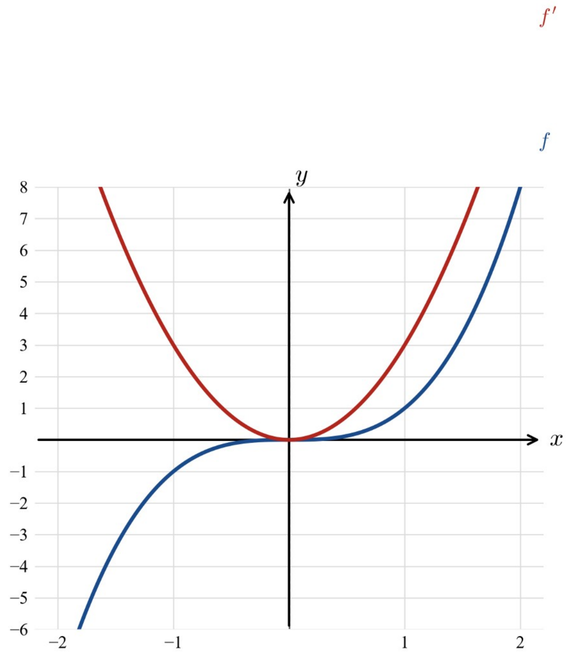
Q4. $f'(x) = 2x + 2 = 0 \Rightarrow x = -1$ (the minimum of f).



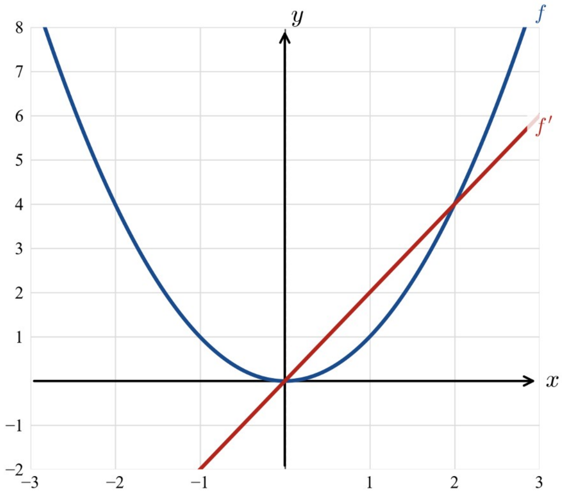
Q5. $f'(x) = 6x^2 - 6 = 6(x-1)(x+1)$, so $f'(x) = 0$ at $x = \pm 1$ — the turning points of f.



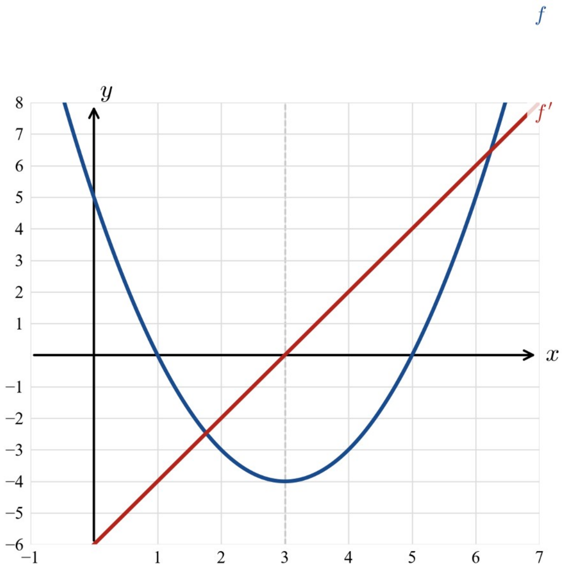
Q6. $f'(x)=3x^2\geq 0$ for all x, and $f'(0)=0$. The gradient is never negative, so f is always increasing but momentarily flat at $x=0$ — a stationary point of inflection.



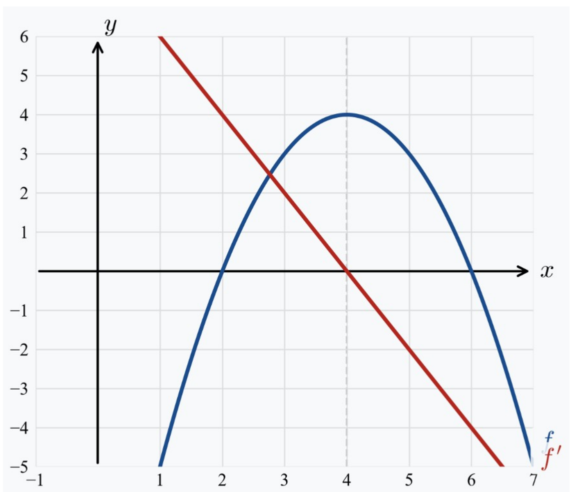
Q7. $f'(x)=2x$ (a line through the origin); f has its minimum at $x=0$, where f' cuts the axis.



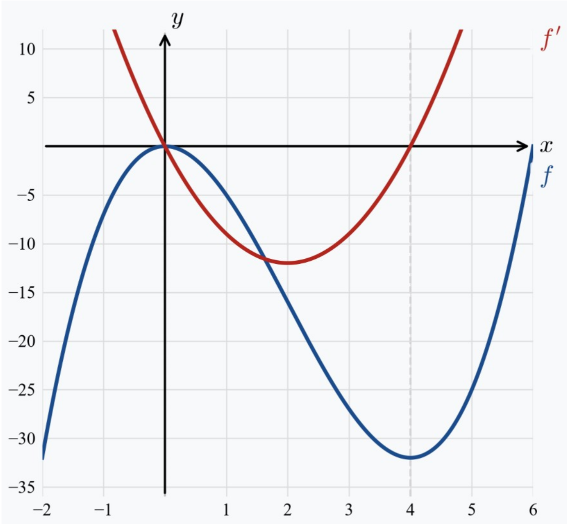
Q8. $f'(x)=2x-6$; $f'(x)=0$ at $x=3$ (the minimum of f).



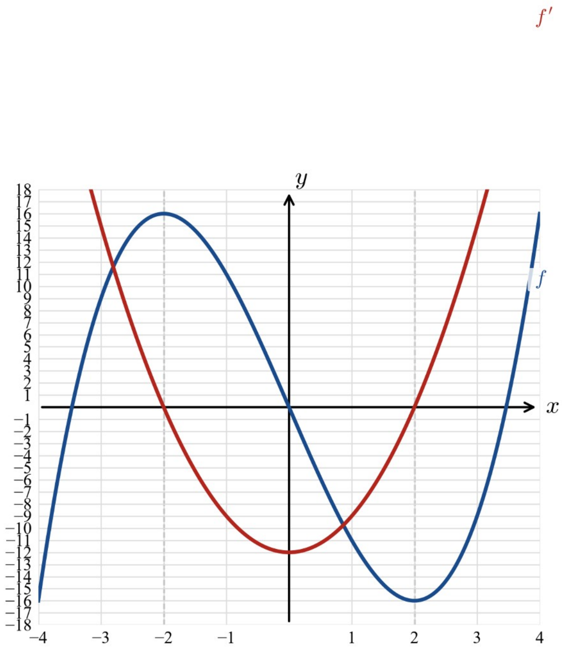
Q9. $f'(x)=-2x+8$; $f'(x)=0$ at $x=4$ (the maximum of f).



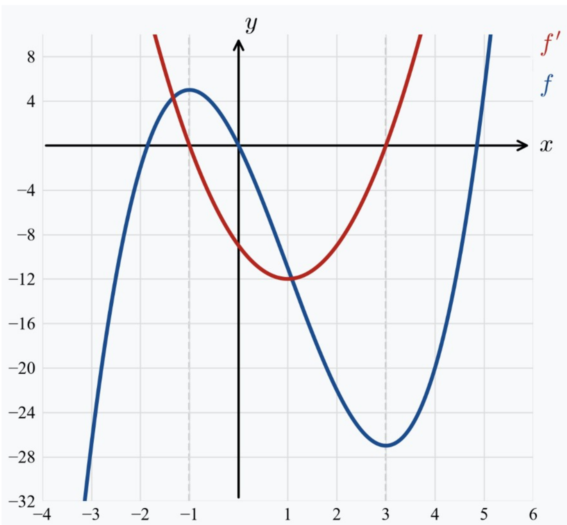
Q10. $f'(x)=3x^2-12x=3x(x-4)$; $f'(x)=0$ at $x=0$ and $x=4$ — the turning points of f. f' is a positive parabola, below the axis for $0<x<4$.



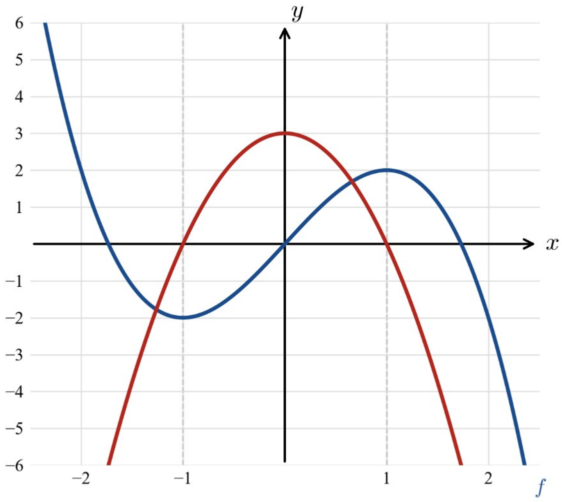
Q11. $f'(x)=3x^2-12=3(x-2)(x+2)$; $f'(x)=0$ at $x=\pm 2$ — the turning points of f.



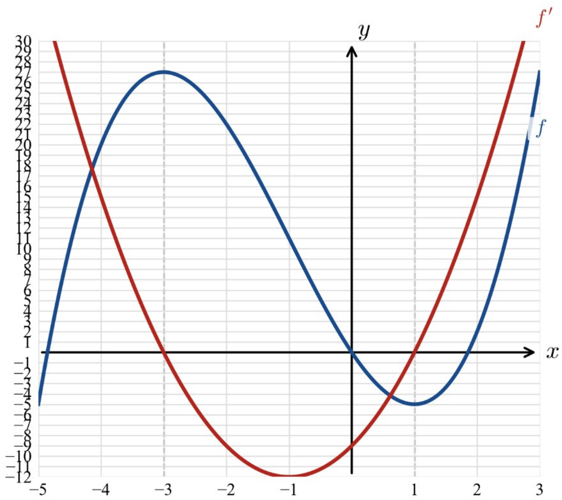
Q12. $f'(x)=3x^2-6x-9=3(x+1)(x-3)$; $f'(x)=0$ at $x=-1$ and $x=3$ — the turning points of f.



Q13. $f'(x)=3-3x^2=3(1-x)(1+x)$; $f'(x)=0$ at $x=\pm 1$. Here f' is a **negative** parabola (since f is a negative cubic).

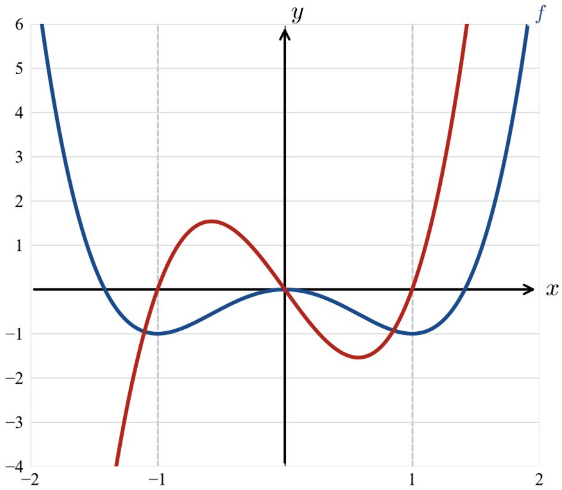


Q14. $f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$; $f'(x) = 0$ at $x = -3$ and $x = 1$.



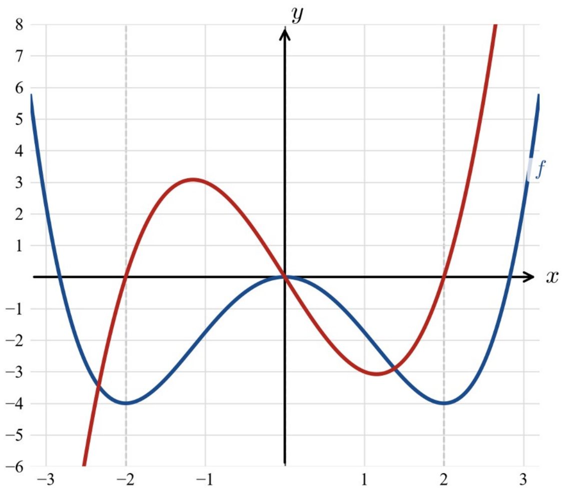
Q15. $f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$; $f'(x) = 0$ at $x = 0, \pm 1$ — the three turning points of the quartic. A quartic differentiates to a cubic.

f'



Q16. $f'(x) = x^3 - 4x = x(x-2)(x+2)$; $f'(x) = 0$ at $x = 0, \pm 2$.

f'



Q17. The gradient is positive to the left of $x=2$, zero at $x=2$, and negative to the right. So $y=f'(x)$ lies above the x-axis for $x<2$, passes through $(2,0)$, and lies below for $x>2$ — for instance a straight line of negative gradient crossing the axis at $x=2$.

Q18. (a) The turning points give $f'(x)=0$ at $x=-2$ and $x=1$. (b) Because f is a positive cubic, f' is a **positive** (upward) parabola. (c) It is a parabola through $(-2,0)$ and $(1,0)$ opening upwards, negative between the roots and positive outside them.

Chapter 14 Differentiation and anti-differentiation of polynomial functions — 14E Anti-differentiation of polynomial functions

Theory

Antidifferentiation reverses differentiation. If $F'(x)=f(x)$, then F is an **antiderivative** of f. Because the derivative of a constant is 0, antiderivatives come as a **family** differing only by a constant.

The basic rule. For any $n\neq -1$,

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c,$$

where c is an **arbitrary constant** (the *constant of integration*). In words: increase the index by 1, then divide by the new index.

Linearity. Antidifferentiate a sum term by term, and a constant multiple keeps its factor:

$$\int (af(x)+bg(x))\,dx=a\int f(x)\,dx+b\int g(x)\,dx.$$

A constant k antidifferentiates to kx (since $k=kx^0$).

Checking. Always **differentiate your answer** — you should get back the original function.

The specific antiderivative. A single extra piece of information (a point the curve passes through, or an initial value) lets you find c: substitute the values and solve. This picks out one member of the family.

From a rate to a quantity. If a quantity Q changes at a known rate $\frac{dQ}{dt}$, antidifferentiating gives Q as a function of t (with c found from the initial amount).

Worked examples

Example 1 — scaffolded

Find the general antiderivative of $f'(x)=6x^2-4x+5$.

Working	Comment
Antidifferentiate each term: $\int 6x^2\,dx=\frac{6x^3}{3}=2x^3$.	Index $2\to 3$, divide by 3.
$\int(-4x)\,dx=\frac{-4x^2}{2}=-2x^2$, and $\int 5\,dx=5x$.	Do each term; the constant 5 gives $5x$.
$f(x)=2x^3-2x^2+5x+c$.	Add ONE constant c for the whole expression.
Check: $\frac{d}{dx}(2x^3-2x^2+5x+c)=6x^2-4x+5$. ✓	Differentiate back to confirm.

Example 2 — scaffolded

Find $\int(x^3-2x+1)\,dx$.

Working	Comment
$\int x^3\,dx=\frac{x^4}{4}$.	Index $3\to 4$, divide by 4.
$\int(-2x)\,dx=-x^2$, and $\int 1\,dx=x$.	Term by term.

Working	Comment
$\int(x^3-2x+1)\,dx=\frac{x^4}{4}-x^2+x+c$.	Include $+c$.

Example 3 — unscaffolded

Given $f'(x)=3x^2-2$ and $f(1)=4$, find $f(x)$.

Antidifferentiating: $f(x)=x^3-2x+c$.

Using $f(1)=4$: $(1)^3-2(1)+c=4\Rightarrow -1+c=4\Rightarrow c=5$.

$\therefore f(x)=x^3-2x+5$.

Example 4 — unscaffolded

Water flows into a tank at a rate of $\frac{dV}{dt}=4t+3$ litres per minute. Initially the tank holds 10 litres. Find the volume V after t minutes.

Antidifferentiating: $V=2t^2+3t+c$.

At $t=0$, $V=10$: $2(0)^2+3(0)+c=10\Rightarrow c=10$.

$\therefore V=2t^2+3t+10$ litres.

Practice questions

Scaffolded

Q1. Find $\int(4x+3)\,dx$.

Q2. Find $\int 6x^2\,dx$.

Q3. Find $\int(x^2-4x+2)\,dx$.

Q4. Given $f'(x)=2x+1$ and $f(0)=3$: (a) find the general antiderivative; (b) use the condition to find c; (c) state $f(x)$.

Q5. Find $\int(5x^4-3x^2)\,dx$.

Q6. The gradient of a curve is $\frac{dy}{dx}=6x^2-2$, and $y=5$ when $x=1$. Find y in terms of x.

Unscaffolded

Find each antiderivative (Q7–Q12), and find the specific function (Q13–Q18).

Q7. $\int(8x^3-6x)\,dx$

Q8. $\int(3x^2+2x-5)\,dx$

Q9. $\int x^5\,dx$

Q10. $\int(10x-7)\,dx$

Q11. $\int(4-x^2)\,dx$

Q12. $\int (x^3 + x^2 + x + 1)\,dx$

Q13. $f'(x) = 4x - 3$ and $f(2) = 5$. Find $f(x)$.

Q14. $f'(x) = 3x^2 + 2x$ and $f(-1) = 0$. Find $f(x)$.

Q15. $\frac{dy}{dx} = x^2 - 4$ and the curve passes through $(3, 2)$. Find y.

Q16. A curve has gradient $6x - 4$ and passes through $(1, -2)$. Find its equation.

Q17. A ball's velocity is $v(t) = 10 - 2t$ metres per second, and its position is $x = 0$ when $t = 0$. Find $x(t)$.

Q18. The volume V of water in a tank changes at a rate of $\frac{dV}{dt} = -6t + 12$ litres per minute, and $V = 40$ when $t = 0$. Find $V(t)$.

Answers

Q1. $2x^2+3x+c$. **Q2.** $2x^3+c$. **Q3.** $\frac{x^3}{3}-2x^2+2x+c$. **Q4.** $f(x)=x^2+x+3$. **Q5.** x^5-x^3+c. **Q6.** $y=2x^3-2x+5$. **Q7.** $2x^4-3x^2+c$. **Q8.** x^3+x^2-5x+c. **Q9.** $\frac{x^6}{6}+c$. **Q10.** $5x^2-7x+c$. **Q11.** $4x-\frac{x^3}{3}+c$. **Q12.** $\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+c$. **Q13.** $f(x)=2x^2-3x+3$. **Q14.** $f(x)=x^3+x^2$. **Q15.** $y=\frac{x^3}{3}-4x+5$. **Q16.** $y=3x^2-4x-1$. **Q17.** $x(t)=10t-t^2$. **Q18.** $V(t)=-3t^2+12t+40$.

Fully-worked solutions

Q1. $\int(4x+3)\,dx=\frac{4x^2}{2}+3x+c=2x^2+3x+c$.

Q2. $\int 6x^2\,dx=\frac{6x^3}{3}+c=2x^3+c$.

Q3. $\int(x^2-4x+2)\,dx=\frac{x^3}{3}-\frac{4x^2}{2}+2x+c=\frac{x^3}{3}-2x^2+2x+c$.

Q4. (a) $f(x)=x^2+x+c$. (b) $f(0)=0+0+c=3$, so $c=3$. (c) $\therefore f(x)=x^2+x+3$.

Q5. $\int(5x^4-3x^2)\,dx=\frac{5x^5}{5}-\frac{3x^3}{3}+c=x^5-x^3+c$.

Q6. $y=2x^3-2x+c$. At $(1,5)$: $2-2+c=5\Rightarrow c=5$. $\therefore y=2x^3-2x+5$.

Q7. $\int(8x^3-6x)\,dx=\frac{8x^4}{4}-\frac{6x^2}{2}+c=2x^4-3x^2+c$.

Q8. $\int(3x^2+2x-5)\,dx=x^3+x^2-5x+c$.

Q9. $\int x^5\,dx=\frac{x^6}{6}+c$.

Q10. $\int(10x-7)\,dx=5x^2-7x+c$.

Q11. $\int(4-x^2)\,dx=4x-\frac{x^3}{3}+c$.

Q12. $\int(x^3+x^2+x+1)\,dx=\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+c$.

Q13. $f(x)=2x^2-3x+c$. $f(2)=8-6+c=2+c=5\Rightarrow c=3$. $\therefore f(x)=2x^2-3x+3$.

Q14. $f(x)=x^3+x^2+c$. $f(-1)=-1+1+c=c=0$. $\therefore f(x)=x^3+x^2$.

Q15. $y=\frac{x^3}{3}-4x+c$. At $(3,2)$: $9-12+c=-3+c=2\Rightarrow c=5$. $\therefore y=\frac{x^3}{3}-4x+5$.

Q16. $y=3x^2-4x+c$. At $(1,-2)$: $3-4+c=-1+c=-2\Rightarrow c=-1$. $\therefore y=3x^2-4x-1$.

Q17. $x(t)=10t-t^2+c$. At $t=0$, $x=0\Rightarrow c=0$. $\therefore x(t)=10t-t^2$.

Q18. $V(t) = -3t^2 + 12t + c$. At $t = 0$, $V = 40 \Rightarrow c = 40$. $\therefore V(t) = -3t^2 + 12t + 40$.

Chapter 14 Differentiation and anti-differentiation of polynomial functions — 14F Limits and continuity

Theory

Limits. The statement

$$\lim_{x \to a} f(x) = L$$

means that $f(x)$ can be made as close as we like to L by taking x close enough to a (but not equal to a). It describes the value $f(x)$ **approaches** near $x = a$, whether or not $f(a)$ itself is defined.

Evaluating limits.

- **Polynomials** are *continuous* everywhere, so $\lim_{x \to a} f(x) = f(a)$ — just substitute.
- For a **rational** function, substitute provided the denominator is not zero.
- If substitution gives the indeterminate form $\frac{0}{0}$, **factorise and cancel** the common factor first, then substitute.

  For example,

$$\frac{\lim_{x \to 2} x^2 - 4}{x-2} = \frac{\lim_{x \to 2} (x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = 4.$$

Left and right limits. $\lim_{x \to a^-} f(x)$ is the value approached from the **left** ($x < a$); $\lim_{x \to a^+} f(x)$ is the value from the **right** ($x > a$). The limit $\lim_{x \to a} f(x)$ **exists only if the two one-sided limits are equal**. If they differ, the graph has a **jump** and the limit does not exist.

Continuity. A function f is **continuous at** $x = a$ if all three conditions hold:

1. $f(a)$ is defined;
2. $\lim_{x \to a} f(x)$ exists (left limit = right limit);
3. $\lim_{x \to a} f(x) = f(a)$.

A function that is continuous at every point of an interval can be drawn without lifting the pen. Common discontinuities are a **hole** (removable), a **jump**, and a vertical **asymptote**.

Making a piecewise function continuous. For a hybrid function joined at $x = a$, set the **left-hand rule's value at a equal to the right-hand rule's value at a** and solve for the unknown constant.

Worked examples

Example 1 — scaffolded

Evaluate $\lim_{x \to 3} (x^2 - 2x + 4)$.

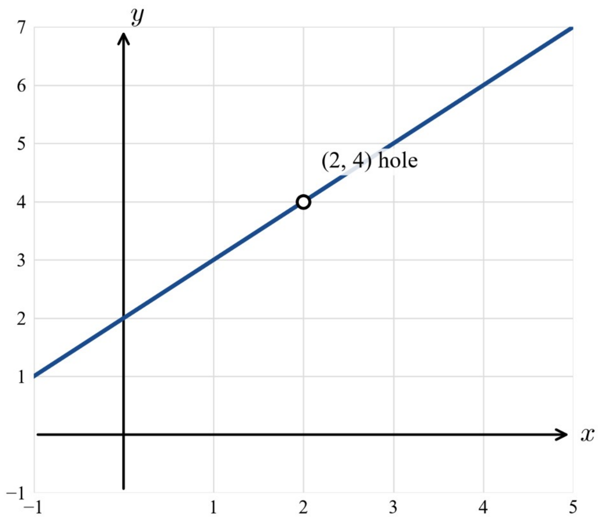
Working	Comment
$x^2 - 2x + 4$ is a polynomial, so it is continuous.	Polynomials are continuous everywhere.
$\lim_{x \to 3} (x^2 - 2x + 4) = 3^2 - 2(3) + 4$.	Substitute $x = 3$.
$= 9 - 6 + 4 = 7$.	$\therefore$ the limit is 7.

Example 2 — scaffolded

Evaluate $\dfrac{\lim\limits_{x \to 2} x^2 - 4}{x-2}$.

Working	Comment
At $x=2$: $\dfrac{2^2-4}{2-2}=\dfrac{0}{0}$ — indeterminate.	Substitution fails, so factorise first.
$\dfrac{x^2-4}{x-2}=\dfrac{(x-2)(x+2)}{x-2}=x+2$ (for $x \neq 2$).	Difference of two squares, then cancel.
$\lim\limits_{x \to 2}(x+2)=4$.	$\therefore$ the limit is 4.

The graph is the line $y=x+2$ with a **hole** at $(2,4)$ — the function approaches 4 even though it is undefined there.



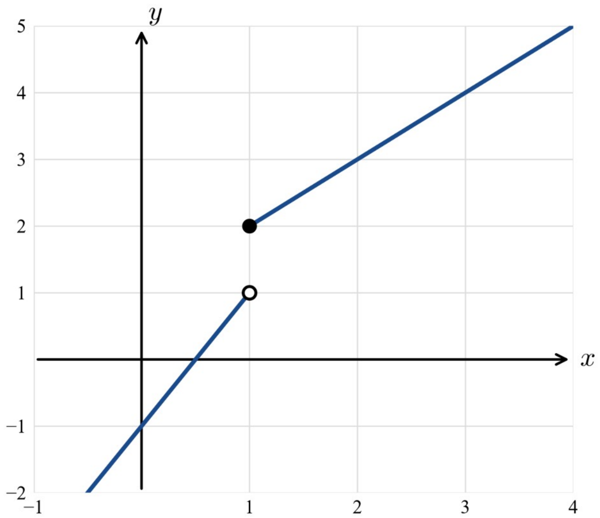
Example 3 — unscaffolded

For $f(x)=\begin{cases} 2x-1 & x<1 \\ x+1 & x \geq 1 \end{cases}$, find $\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$, and state whether $\lim\limits_{x \to 1} f(x)$ exists.

Left: $\lim\limits_{x \to 1^-} f(x)=\lim\limits_{x \to 1}(2x-1)=2(1)-1=1$.

Right: $\lim\limits_{x \to 1^+} f(x)=\lim\limits_{x \to 1}(x+1)=1+1=2$.

The one-sided limits differ ($1 \neq 2$), so $\therefore \lim\limits_{x \to 1} f(x)$ **does not exist** — the graph has a jump at $x=1$.

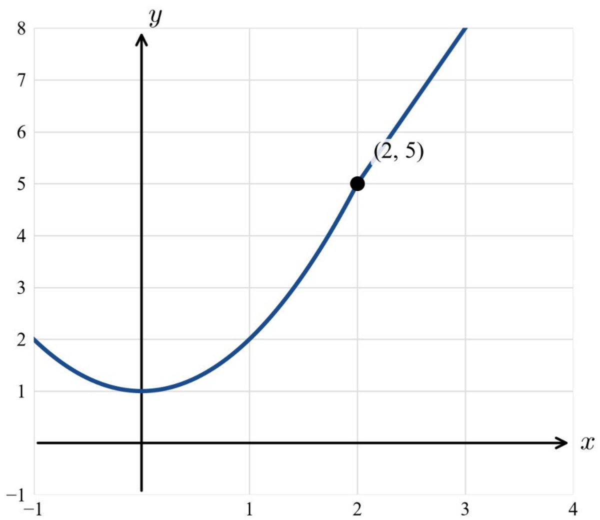


Example 4 — unscaffolded

Find the value of k for which $f(x)=\begin{cases} x^2+k & x\le 2 \\ 3x-1 & x>2 \end{cases}$ is continuous at $x=2$.

Left-hand value at 2: $2^2+k=4+k$. Right-hand value at 2: $3(2)-1=5$.

For continuity these must be equal: $4+k=5$, so $\therefore k=1$.



Practice questions

Scaffolded

Q1. Evaluate $\lim_{x\to 1}(3x^2+2x-1)$.

Q2. Evaluate $\lim_{x\to -2}(x^3+x)$.

Q3. Evaluate $\dfrac{\lim_{x\to 5} x^2-25}{x-5}$ (factorise first).

Q4. Evaluate $\dfrac{\lim_{x\to -1} x^2+3x+2}{x+1}$.

Q5. For $f(x)=\begin{cases} 3x+1 & x<1 \\ 2x+2 & x\geq 1 \end{cases}$, find the left and right limits at $x=1$ and state whether $\lim_{x\to 1} f(x)$ exists.

Q6. Find k so that $f(x)=\begin{cases} 2x+k & x\leq 3 \\ x^2-2 & x>3 \end{cases}$ is continuous at $x=3$.

Unscaffolded

Q7. $\lim_{x\to 2}\left(x^2+3x-1\right)$

Q8. $\lim_{x\to 0}\left(x^3-2x^2+5\right)$

Q9. $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$

Q10. $\lim_{x\to 3}\dfrac{x^2-x-6}{x-3}$

Q11. $\lim_{x\to -2}\dfrac{x^2-4}{x^2+2x}$

Q12. $\lim_{x\to 1}\dfrac{x^2-1}{x^2+x-2}$

Q13. For $f(x)=\begin{cases} x^2+1 & x<2 \\ 2x & x\geq 2 \end{cases}$, find the left and right limits at $x=2$. Does $\lim_{x\to 2} f(x)$ exist?

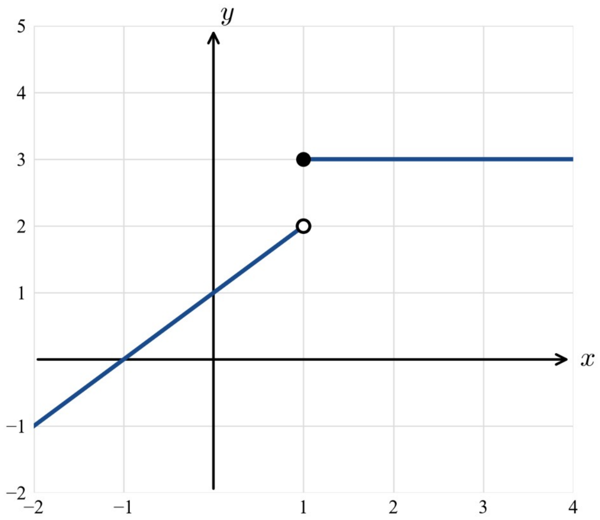
Q14. For $f(x)=\begin{cases} x+2 & x<1 \\ 3x & x\geq 1 \end{cases}$, does $\lim_{x\to 1} f(x)$ exist? If so, state its value.

Q15. Is $f(x)=\begin{cases} x^2 & x\leq 1 \\ 3x-2 & x>1 \end{cases}$ continuous at $x=1$? Justify your answer using the three conditions.

Q16. Find k so that $f(x)=\begin{cases} kx+1 & x\leq 2 \\ x^2-1 & x>2 \end{cases}$ is continuous at $x=2$.

Q17. Find a so that $f(x)=\begin{cases} ax^2 & x\leq 1 \\ x+2 & x>1 \end{cases}$ is continuous at $x=1$.

Q18. The graph below shows a function f. State the type of discontinuity at $x=1$ and give $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.


y
x
5
4
3
2
1
−1
−2
−2
−1
1
2
3
4

Answers

Q1. 4. **Q2.** -10. **Q3.** 10. **Q4.** 1. **Q5.** left 4, right 4; limit exists, $=4$. **Q6.** $k=1$. **Q7.** 9. **Q8.** 5. **Q9.** 8. **Q10.** 5. **Q11.** 2. **Q12.** $\frac{2}{3}$. **Q13.** left 5, right 4; limit does not exist (jump). **Q14.** yes, 3. **Q15.** yes, continuous. **Q16.** $k=1$. **Q17.** $a=3$. **Q18.** jump discontinuity; left 2, right 3.

Fully-worked solutions

Q1. Polynomial, so substitute: $3(1)^2+2(1)-1=3+2-1=4$.

Q2. $(-2)^3+(-2)=-8-2=-10$.

Q3. $\frac{x^2-25}{x-5}=\frac{(x-5)(x+5)}{x-5}=x+5\to 5+5=10$.

Q4. $\frac{x^2+3x+2}{x+1}=\frac{(x+1)(x+2)}{x+1}=x+2\to -1+2=1$.

Q5. Left: $3(1)+1=4$. Right: $2(1)+2=4$. The one-sided limits are equal, so the limit **exists** and $\lim_{x\to 1} f(x)=4$.

Q6. Left value at 3: $2(3)+k=6+k$. Right value: $3^2-2=7$. Set $6+k=7$, so $\therefore k=1$.

Q7. $2^2+3(2)-1=4+6-1=9$.

Q8. $0-0+5=5$.

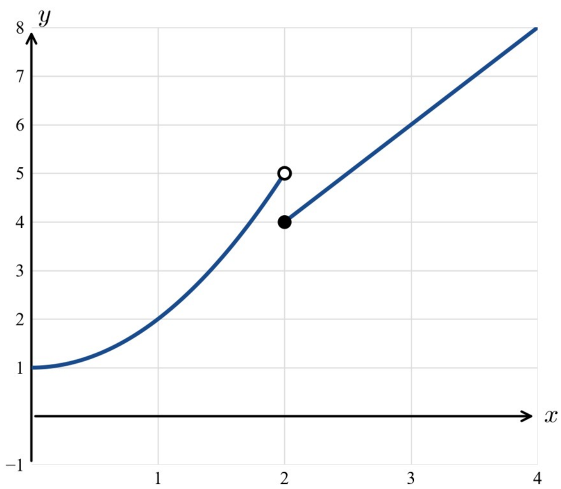
Q9. $\frac{(x-4)(x+4)}{x-4}=x+4\to 8$.

Q10. $\frac{(x-3)(x+2)}{x-3}=x+2\to 5$.

Q11. $\frac{x^2-4}{x^2+2x}=\frac{(x-2)(x+2)}{x(x+2)}=\frac{x-2}{x}\to\frac{-2-2}{-2}=\frac{-4}{-2}=2$.

Q12. $\frac{x^2-1}{x^2+x-2}=\frac{(x-1)(x+1)}{(x-1)(x+2)}=\frac{x+1}{x+2}\to\frac{2}{3}$.

Q13. Left: $2^2+1=5$. Right: $2(2)=4$. Since $5\neq 4$, $\lim_{x\to 2} f(x)$ **does not exist** (jump).



Q14. Left: $1+2=3$. Right: $3(1)=3$. Equal, so the limit **exists** and equals 3.

Q15. (1) $f(1)=1^2=1$ is defined. (2) Left $=1^2=1$, right $=3(1)-2=1$, so the limit exists and equals 1. (3) $\lim_{x \to 1} f(x)=1=f(1)$. All three conditions hold, so f **is continuous** at $x=1$.

Q16. Left value at 2: $2k+1$. Right value: $2^2-1=3$. Set $2k+1=3$, so $\therefore k=1$.

Q17. Left value at 1: $a(1)^2=a$. Right value: $1+2=3$. Set $a=3$, so $\therefore a=3$.

Q18. The left branch approaches 2 and the right branch (a closed dot) is at 3, so $\lim_{x \to 1^-} f(x)=2$ and $\lim_{x \to 1^+} f(x)=3$. As the one-sided limits differ, this is a **jump discontinuity**.

Theory

A function f is **differentiable** at $x=a$ if it has a well-defined tangent (gradient) there — that is, the derivative

$$f'(a)=\frac{\lim_{h\to 0} f(a+h)-f(a)}{h}$$

exists (the limits from the left and from the right must agree).

Where differentiability fails. A function is *not* differentiable at a point if its graph has:

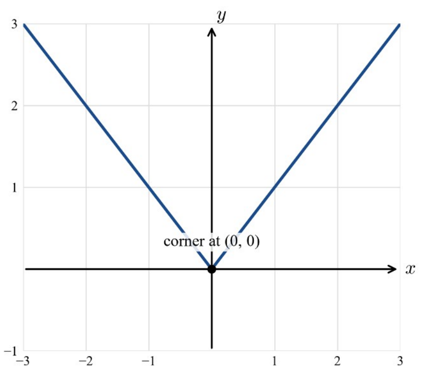
- a **corner** — the gradient from the left and the gradient from the right are different (finite but unequal), e.g. $y=|x|$ at $x=0$ (left gradient -1, right gradient $+1$);
- a **cusp** — the gradient tends to $+\infty$ on one side and $-\infty$ on the other, e.g. $y=x^{2/3}$ at $x=0$;
- a **vertical tangent** — the gradient tends to ∞, e.g. $y=x^{1/3}$ at $x=0$;
- a **discontinuity** — a jump, hole or asymptote (the graph breaks).

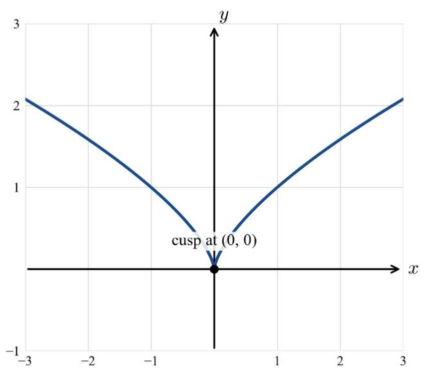
Differentiable ⇒ continuous. If f is differentiable at a then it is continuous at a. The converse is **not** true: $y=|x|$ is continuous at 0 but not differentiable there (a corner).

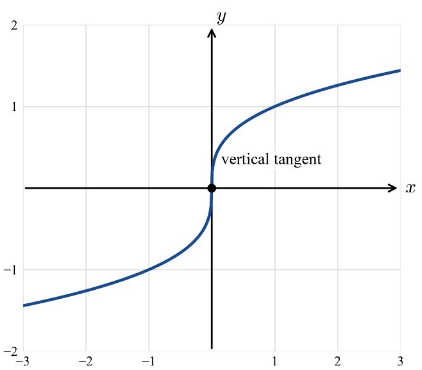
Piecewise functions. A hybrid function with a join at $x=a$ is differentiable at the join when **both** of these hold:

1. it is **continuous** there — the two pieces meet: $\text{left}(a)=\text{right}(a)$; and
2. the join is **smooth** — the left and right derivatives are equal: $\frac{d}{dx}[\text{left}]|_a=\frac{d}{dx}[\text{right}]|_a$.

Reading a graph. A function is differentiable wherever its graph is a smooth, unbroken curve; it is not differentiable at corners, cusps, vertical tangents or breaks.





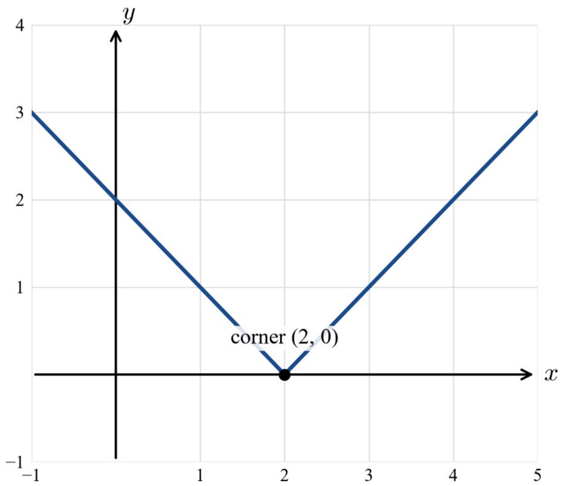


Worked examples

Example 1 — scaffolded

State where $f(x)=|x-2|$ is not differentiable, and find the left and right gradients there.

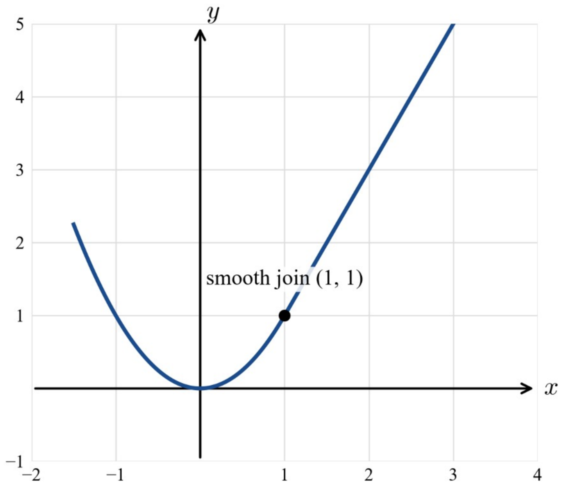
Working	Comment
$\|x-2\|=0$ when $x=2$, so the graph has a **corner** at $x=2$.	The modulus graph bends where the inside is zero.
For $x<2$: $f(x)=-(x-2)=2-x$, gradient -1.	To the left the branch has gradient -1.
For $x>2$: $f(x)=x-2$, gradient $+1$.	To the right the gradient is $+1$.
The gradients (-1 and $+1$) differ, so f is **not differentiable at** $x=2$.	A corner: no single tangent.



Example 2 — scaffolded

Show that $f(x)=\begin{cases} x^2 & x\le 1 \\ 2x-1 & x>1 \end{cases}$ is differentiable at $x=1$.

Working	Comment
Continuity: left value $1^2=1$; right value $2(1)-1=1$. Equal, so continuous at $x=1$.	First check the pieces meet.
Left derivative: $\frac{d}{dx}(x^2)=2x$, which is 2 at $x=1$.	Differentiate the left rule.
Right derivative: $\frac{d}{dx}(2x-1)=2$.	Differentiate the right rule.
Both derivatives equal 2, so the join is **smooth**: f is differentiable at $x=1$, with $f'(1)=2$.	Continuous **and** equal gradients.



Example 3 — unscaffolded

Determine whether $f(x)=\begin{cases} x^2 & x\le 1 \\ 3x-1 & x>1 \end{cases}$ is differentiable at $x=1$.

Continuity: left value $1^2=1$; right value $3(1)-1=2$. Since $1\ne 2$ the function is **not continuous** at $x=1$ (there is a jump).

$\therefore f$ is **not differentiable** at $x=1$ (a function must be continuous there to be differentiable).

Example 4 — unscaffolded

Find the values of a and b so that $f(x)=\begin{cases} x^2 & x\le 1 \\ ax+b & x>1 \end{cases}$ is differentiable at $x=1$.

Smooth (equal derivatives): $\frac{d}{dx}(x^2)|_1=2$ and $\frac{d}{dx}(ax+b)=a$, so $a=2$.

Continuous (pieces meet): $1^2=a(1)+b$, so $1=a+b$. With $a=2$, $b=1-2=-1$.

$\therefore a=2, b=-1$.

Practice questions

Scaffolded

Q1. For $f(x)=|x+3|$: (a) state where it is not differentiable; (b) find the left and right gradients there; (c) name the type of point.

Q2. For $f(x)=\begin{cases} x^2 & x\le 2 \\ 4x-4 & x>2 \end{cases}$: (a) check continuity at $x=2$; (b) find the left and right derivatives; (c) is f differentiable at $x=2$?

Q3. For $f(x)=\begin{cases} x^2 & x\le 0 \\ x & x>0 \end{cases}$: (a) check continuity at $x=0$; (b) find the left and right derivatives; (c) is f differentiable at $x=0$?

Q4. For $f(x)=\begin{cases} 2x & x\le 1 \\ 3x-1 & x>1 \end{cases}$: (a) check continuity at $x=1$; (b) compare the left and right gradients; (c) state whether f is differentiable at $x=1$ and why.

Q5. Find a and b so that $f(x)=\begin{cases} x^2 & x\le 2 \\ ax+b & x>2 \end{cases}$ is differentiable at $x=2$.

Q6. Explain, using its gradient, why $f(x)=x^{1/3}$ is not differentiable at $x=0$.

Unscaffolded

Q7. State where $f(x)=|x-5|$ is not differentiable.

Q8. State where $f(x)=|2x+4|$ is not differentiable.

Q9. Is $f(x)=\begin{cases} x^2 & x\le 3 \\ 6x-9 & x>3 \end{cases}$ differentiable at $x=3$? Justify.

Q10. Is $f(x)=\begin{cases} x^2 & x\le 2 \\ 5x-6 & x>2 \end{cases}$ differentiable at $x=2$? Justify.

Q11. Is $f(x)=\begin{cases} 2x+1 & x\le 1 \\ x^2+2 & x>1 \end{cases}$ differentiable at $x=1$? Justify.

Q12. Is $f(x)=\begin{cases} x^3 & x\le 0 \\ x & x>0 \end{cases}$ differentiable at $x=0$? Justify.

Q13. Find a so that $f(x)=\begin{cases} x^2 & x\le 1 \\ ax-1 & x>1 \end{cases}$ is differentiable at $x=1$.

Q14. Find a and b so that $f(x)=\begin{cases} x^2 & x\le -1 \\ ax+b & x>-1 \end{cases}$ is differentiable at $x=-1$.

Q15. Explain why $f(x)=x^{2/3}$ is not differentiable at $x=0$, and name the type of point.

Q16. Is $f(x)=\begin{cases} x^2 & x<2 \\ 3 & x\ge 2 \end{cases}$ differentiable at $x=2$? Justify.

Q17. State where $f(x)=|x^2-4|$ is not differentiable.

Q18. Find a and b so that $f(x)=\begin{cases} ax^2+b & x\le 1 \\ 4x-1 & x>1 \end{cases}$ is differentiable at $x=1$.

Answers

Q1. Corner at $x=-3$; left gradient -1, right $+1$; a corner. **Q2.** Continuous $(4=4)$; $L'=4$, $R'=4$; **differentiable**, $f'(2)=4$. **Q3.** Continuous $(0=0)$; $L'=0$, $R'=1$; **not** differentiable (corner). **Q4.** Continuous $(2=2)$; $L'=2$, $R'=3$; **not** differentiable (corner — gradients differ). **Q5.** $a=4, b=-4$. **Q6.** chord gradients $\frac{1}{\sqrt[3]{h^2}} \to \infty$ as $h \to 0$: a vertical tangent, so not differentiable. **Q7.** $x=5$ (corner). **Q8.** $x=-2$ (corner). **Q9.** Yes — continuous $(9=9)$ and $L'=R'=6$; $f'(3)=6$. **Q10.** No — continuous $(4=4)$ but $L'=4\neq5=R'$ (corner). **Q11.** Yes — continuous $(3=3)$ and $L'=R'=2$; $f'(1)=2$. **Q12.** No — continuous $(0=0)$ but $L'=0\neq1=R'$ (corner). **Q13.** $a=2$. **Q14.** $a=-2, b=-1$. **Q15.** chord gradients $\frac{1}{\sqrt[3]{h}} \to +\infty$ (from the right) and $-\infty$ (from the left): a **cusp**, so not differentiable. **Q16.** No — not continuous $(4\neq3)$, so not differentiable (a jump). **Q17.** $x=-2$ and $x=2$ (corners, where $x^2-4=0$). **Q18.** $a=2, b=1$.

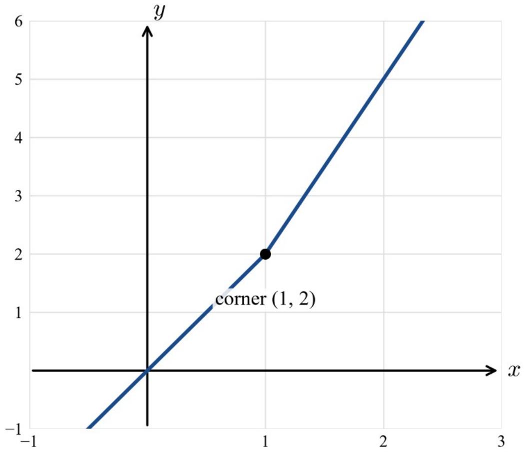
Fully-worked solutions

Q1. $|x+3|=0$ at $x=-3$. For $x<-3$, $f=-(x+3)$, gradient -1; for $x>-3$, $f=x+3$, gradient $+1$. The gradients differ, so there is a **corner** at $x=-3$ and f is not differentiable there.

Q2. Continuity: $2^2=4$ and $4(2)-4=4$ — equal, so continuous. $L'=d/dx(x^2)=2x=4$ at $x=2$; $R'=d/dx(4x-4)=4$. Since $4=4$, f is **differentiable** at $x=2$ with $f'(2)=4$.

Q3. Continuity: $0^2=0$ and $0=0$ — continuous. $L'=2x=0$ at $x=0$; $R'=d/dx(x)=1$. Since $0\neq1$, there is a **corner**: **not differentiable** at $x=0$.

Q4. Continuity: $2(1)=2$ and $3(1)-1=2$ — continuous. $L'=2$, $R'=3$. The gradients differ $(2\neq3)$, so the graph has a **corner** at $(1,2)$ and f is **not differentiable** at $x=1$.



Q5. Smooth: $d/dx(x^2)|_2=4=a$, so $a=4$. Continuous: $2^2=a(2)+b \Rightarrow 4=8+b \Rightarrow b=-4$. $\therefore a=4, b=-4$.

Q6. The chord from the origin to the point $(h, h^{1/3})$ on the graph has gradient $\frac{h^{1/3}}{h}=\frac{1}{h^{2/3}}=\frac{1}{\sqrt[3]{h^2}}$ (index laws). Since $\sqrt[3]{h^2}>0$ and $\to 0$ as $h\to0$ from either side, the chord gradients grow without bound: the graph has a **vertical tangent** at the origin and no finite gradient exists — f is not differentiable at $x=0$.

Q7. $|x-5|=0$ at $x=5$: a corner, so not differentiable at $x=5$.

Q8. $|2x+4|=0$ at $x=-2$: a corner, so not differentiable at $x=-2$.

Q9. Continuity: $3^2=9$ and $6(3)-9=9$ — continuous. $L'=2x=6$ at $x=3$; $R'=6$. Equal, so **differentiable** at $x=3$ with $f'(3)=6$.

Q10. Continuity: $2^2=4$ and $5(2)-6=4$ — continuous. $L'=2x=4$; $R'=5$. Since $4\neq 5$, there is a **corner**: **not differentiable** at $x=2$.

Q11. Continuity: $2(1)+1=3$ and $1^2+2=3$ — continuous. $L'=d/dx(2x+1)=2$; $R'=d/dx(x^2+2)=2x=2$ at $x=1$. Equal, so **differentiable**, $f'(1)=2$.

Q12. Continuity: $0^3=0$ and $0=0$ — continuous. $L'=3x^2=0$ at $x=0$; $R'=1$. Since $0\neq 1$, a **corner**: **not differentiable** at $x=0$.

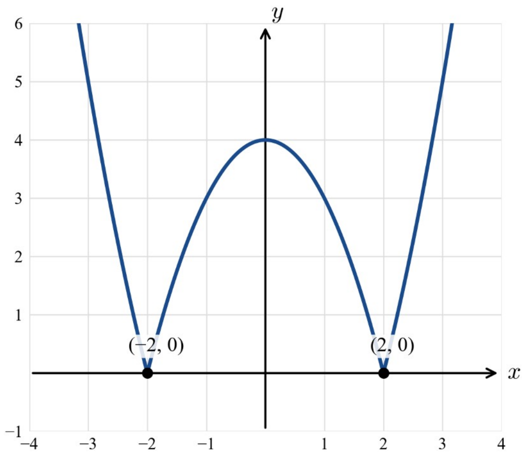
Q13. Smooth: $d/dx(x^2)|_1=2=a$. (Check continuity: $1^2=1$ and $a(1)-1=2-1=1$ — consistent.) $\therefore a=2$.

Q14. Smooth: $d/dx(x^2)|_{-1}=2(-1)=-2=a$, so $a=-2$. Continuous: $(-1)^2=a(-1)+b \Rightarrow 1=-a+b=2+b \Rightarrow b=-1$. $\therefore a=-2, b=-1$.

Q15. The chord from the origin to $(h, h^{2/3})$ has gradient $\dfrac{h^{2/3}}{h}=\dfrac{1}{h^{1/3}}=\dfrac{1}{\sqrt[3]{h}}$ (index laws). As $h\to 0^+$ this $\to +\infty$; as $h\to 0^-$ it $\to -\infty$. The one-sided gradients disagree in sign — a **cusp** — so f is not differentiable at $x=0$.

Q16. Continuity: as $x\to 2^-$, $x^2\to 4$; at $x=2$, $f=3$. Since $4\neq 3$ the function is **not continuous** at $x=2$, so it is **not differentiable** there.

Q17. $|x^2-4|=0$ when $x^2-4=0$, i.e. $x=-2$ or $x=2$. At each of these the graph turns sharply (a corner), so f is **not differentiable at $x=-2$ or $x=2$.**



Q18. Smooth: $d/dx(ax^2+b)|_1=2a$ and $d/dx(4x-1)=4$, so $2a=4\Rightarrow a=2$. Continuous: $a(1)^2+b=4(1)-1\Rightarrow a+b=3\Rightarrow b=3-2=1$. $\therefore a=2, b=1$.