Mathematical Methods Units 1 & 2

Chapter 13 — Rates of change

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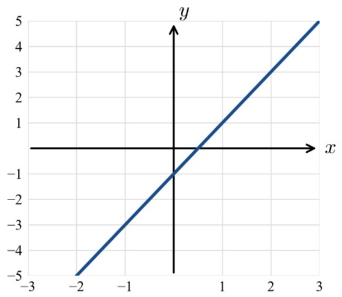
Contents

Theory

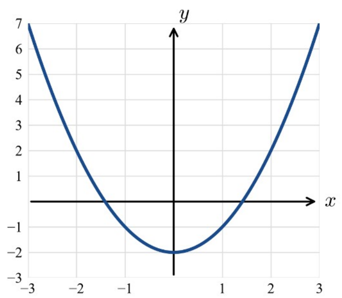
Before studying how quantities *change*, it helps to recognise the **type of relationship** between two variables from a graph, a table of values, or a worded description.

The main families (with their typical shapes):

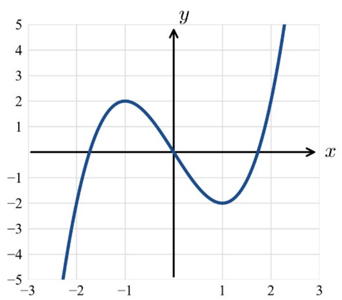
- **Linear** $y = mx + c$ — a straight line; a **constant** rate of change.



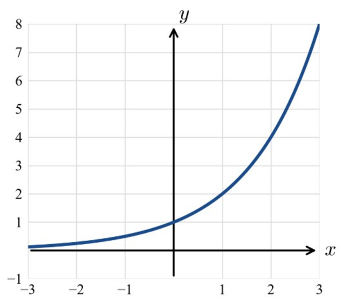
- **Quadratic** $y = ax^2 + bx + c$ — a parabola; one turning point; symmetric about a vertical axis.



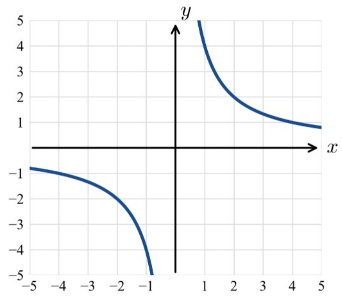
- **Cubic** $y = ax^3 + \ldots$ — up to two turning points (or a stationary point of inflection); the two ends go in opposite directions.



- **Exponential** $y = ab^x$ — rapid growth ($b > 1$) or decay ($0 < b < 1$); a horizontal asymptote.



- **Reciprocal (hyperbola)** $y = \dfrac{a}{x}$ — two branches with a vertical and a horizontal asymptote; x and y are in **inverse proportion**.



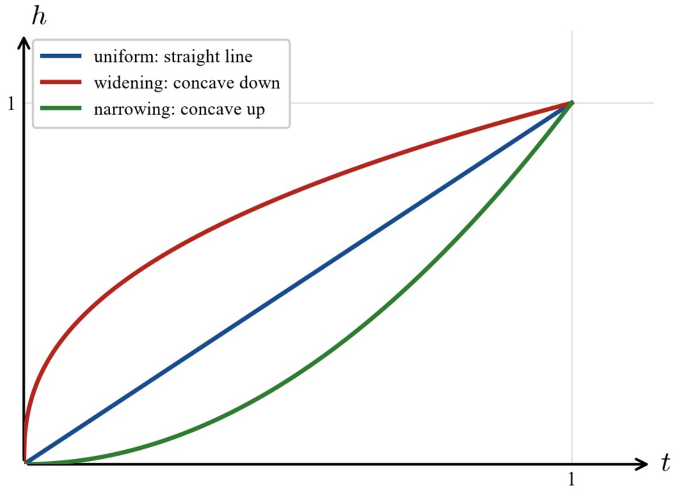
Recognising a relationship from a table. For x-values increasing in equal steps:

Pattern in the table	Relationship
First differences in y are constant	linear
Second differences in y are constant	quadratic
Successive y-values have a constant **ratio**	exponential
The **product** $x\,y$ is constant	reciprocal (inverse proportion)

Increasing and decreasing. A relationship is **increasing** on an interval if y rises as x increases, and **decreasing** if y falls. A straight line is increasing everywhere if $m > 0$; a parabola increases on one side of its turning point and decreases on the other.

Filling a container at a constant rate. Suppose liquid is poured into a container at a *constant volumetric rate* (the same number of litres every second) and we graph the **height** h of the liquid against **time** t. The *shape* of the container decides the shape of the graph:

- A **uniform cross-section** (a cylinder or rectangular tank) fills so the height rises at a **constant** rate — a **straight line**.
- A container that **widens** as it fills (a bowl) spreads each second's liquid over a larger area, so the height rises **more and more slowly** — the graph is **concave down**.
- A container that **narrows** as it fills (a funnel or vase) squeezes each second's liquid into a smaller area, so the height rises **faster and faster** — the graph is **concave up**.



The steepness of the height–time graph at any instant is the *rate* at which the height is rising; the container's changing width is what makes that rate vary.

Worked examples

Example 1 — scaffolded

The table shows a relationship between x and y. Identify its type and find a rule.

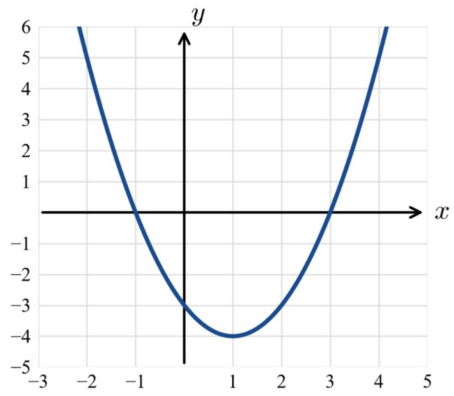
x	1	2	3	4
y	5	8	11	14

Working	Comment
First differences: $8-5=3$, $11-8=3$, $14-11=3$.	Subtract successive y-values.
The first differences are **constant** (3), so the relationship is **linear** with rate 3.	Constant first differences ⇒ linear.
The rate is 3 and at $x=1$, $y=5$, so $y=3x+c$ with $5=3(1)+c$, giving $c=2$.	Use a point to find c.

$\therefore y=3x+2$.

Example 2 — scaffolded

Name the type of relationship shown and describe its key features.



Working	Comment
The graph is a **parabola**, so the relationship is **quadratic**.	One turning point, symmetric — the parabola shape.
It opens upwards with a minimum turning point at $(1,-4)$.	The lowest point.
It has two x-intercepts, at $(-1,0)$ and $(3,0)$, and y-intercept $(0,-3)$.	Where it crosses the axes.
It is **decreasing** for $x<1$ and **increasing** for $x>1$.	Falls then rises about the turning point.

Example 3 — unscaffolded

Identify the type of relationship in the table and find a rule.

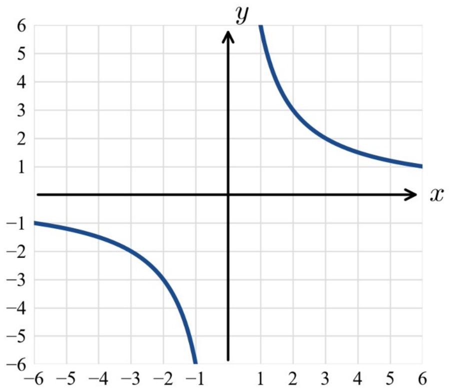
x	0	1	2	3
y	3	6	12	24

The first differences $(3,6,12)$ are **not** constant, but each y-value is 2 times the previous one: $\frac{6}{3}=\frac{12}{6}=\frac{24}{12}=2$. A **constant ratio** means the relationship is **exponential**.

The initial value (at $x=0$) is 3 and the ratio is 2. $\therefore y=3\times 2^x$.

Example 4 — unscaffolded

Name the relationship shown and describe its key features.

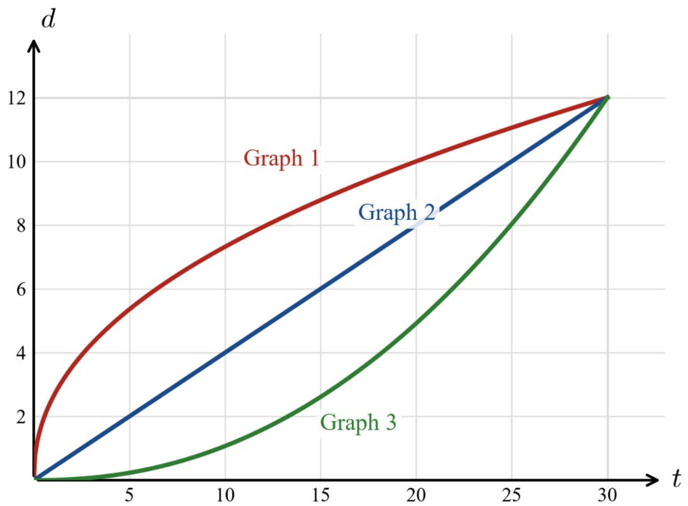


The graph has **two separate branches** approaching the axes as asymptotes — this is a **reciprocal (hyperbola)** relationship, $y=\dfrac{6}{x}$.

The x-axis ($y=0$) and the y-axis ($x=0$) are asymptotes; the graph is **decreasing** on each branch; and x and y are in inverse proportion ($x\,y=6$). ∴ there are no axis intercepts.

Example 5 — scaffolded

Dry sand runs at a constant rate from a hopper into each of three empty glass vessels of equal capacity: **(P)** a conical flask with a wide flat base and sides that slope in steadily to a narrow neck, **(Q)** a square-based jar with vertical sides, and **(R)** a goblet whose bowl is narrow above the stem and flares out towards the rim. Each vessel is full, at a depth of 12 cm, after 30 seconds. The graphs below show the depth d cm of sand after t seconds. Match each vessel to one of the graphs.



Working	Comment
(Q) has vertical sides — a uniform cross-section — so the depth rises by the same amount each second.	Constant rate ⇒ the **straight line**, Graph 2.
(P) narrows towards the neck, so each second's sand is squeezed into a smaller area and the depth rises ever faster.	Speeding-up rise ⇒ **concave up**, Graph 3.
(R) flares out towards the rim, so each second's sand spreads over a larger area and the depth rises ever more slowly.	Slowing rise ⇒ **concave down**, Graph 1.

∴ **P** → Graph 3 (concave up), **Q** → Graph 2 (straight line), **R** → Graph 1 (concave down).

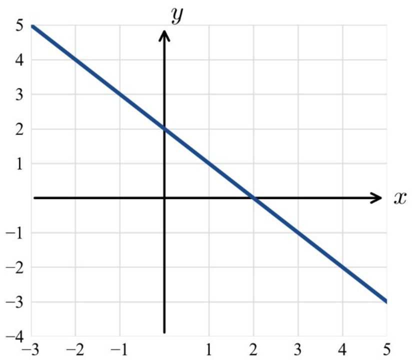
Practice questions

Scaffolded

Q1. For the table below: (a) find the first and second differences; (b) hence state the type of relationship; (c) find a rule.

x	0	1	2	3	4
y	1	2	5	10	17

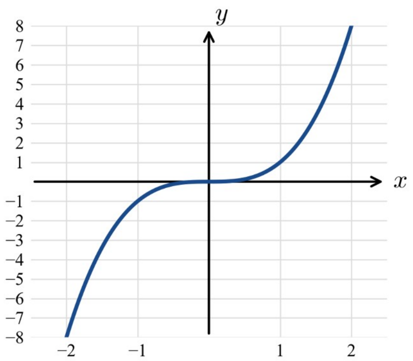
Q2. For the graph shown: (a) name the type of relationship; (b) state whether it is increasing or decreasing; (c) find its axis intercepts.



Q3. For the table below: (a) show that the product $x\,y$ is constant; (b) hence name the relationship; (c) find a rule.

x	1	2	4	8
y	24	12	6	3

Q4. For the graph shown: (a) name the type of relationship; (b) how many turning points does it have?



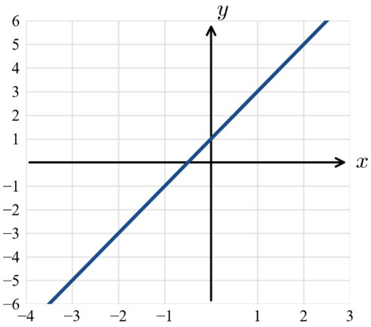
Q5. For the table below: (a) state the constant rate of change; (b) find a rule; (c) is the relationship increasing or decreasing?

x	0	1	2	3
y	7	4	1	−2

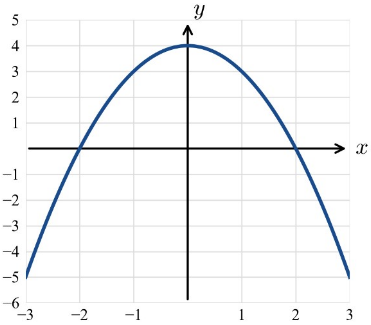
Q6. A car travels at a constant speed of 80 km/h. (a) What type of relationship connects the distance travelled and the time taken? (b) Write a rule for the distance d km after t hours.

Unscaffolded

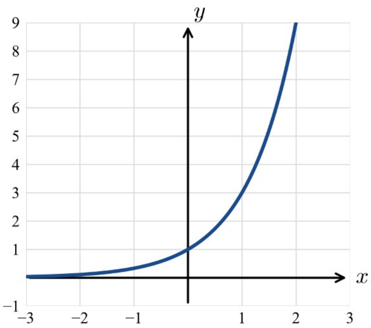
For **Q7–Q12**, name the type of relationship shown in each graph.



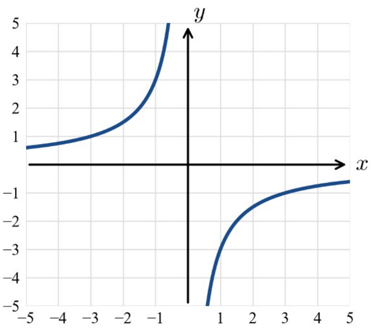
Q7.



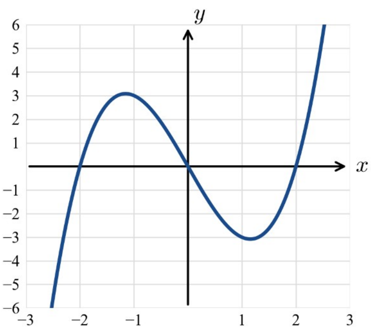
Q8.



Q9.

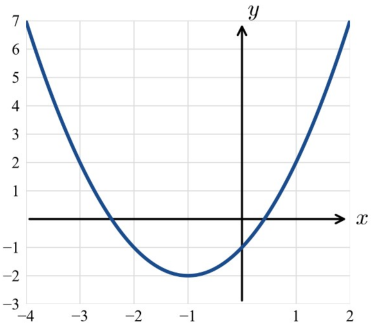


Q10.



Q11.

Q12.



For **Q13–Q16**, identify the type of relationship from the table and find a rule.

Q13.

x	1	2	3	4
y	3	8	15	24

Q14.

x	1	2	3	6
y	12	6	4	2

Q15.

x	0	1	2	3
y	10	7	4	1

Q16.

x	0	1	2	3
y	80	40	20	10

Q17. Match each rule to its relationship type: (a) $y = 5 - 2x$; (b) $y = \frac{10}{x}$; (c) $y = 3^x$; (d) $y = x^2 - 4$.

Q18. A cup of hot chocolate cools from $80\,°\mathrm{C}$ towards room temperature, halving its temperature *above the room* every few minutes. (a) Is the temperature increasing or decreasing? (b) Which relationship type — linear or exponential — best models the cooling, and why?

Q19. Water is poured at a constant rate into a cylindrical drum. (a) What is the shape of the height–time graph? (b) Explain why.

Q20. A large mixing bowl is wider near the top than at the bottom. It is filled with water at a constant rate. (a) As the water rises, does the height increase at a constant, an increasing, or a decreasing rate? (b) Is the height–time graph concave up or concave down? Explain.

Q21. Olive oil is pumped at the **same constant rate** into three empty glass vessels: (i) a carafe that is widest at its base and tapers steadily to a narrow neck at the top; (ii) a straight-sided bottle with a square cross-section; (iii) a trumpet vase that is narrow at the base and flares outwards towards the rim. Match each vessel to the shape of its depth–time graph — **straight line**, **concave up**, or **concave down** — and sketch the three graphs on one set of axes.

Answers

Q1. 2nd differences constant (2) ⇒ quadratic; $y = x^2 + 1$. **Q2.** Linear; decreasing; x-int $(2, 0)$, y-int $(0, 2)$. **Q3.** $x\,y = 24$; reciprocal; $y = \dfrac{24}{x}$. **Q4.** Cubic; no turning points (a stationary point of inflection at the origin). **Q5.** Rate -3; $y = 7 - 3x$; decreasing. **Q6.** Linear; $d = 80t$. **Q7.** Linear. **Q8.** Quadratic. **Q9.** Exponential. **Q10.** Reciprocal (hyperbola). **Q11.** Cubic. **Q12.** Quadratic. **Q13.** Quadratic; $y = x^2 + 2x$. **Q14.** Reciprocal; $y = \dfrac{12}{x}$. **Q15.** Linear; $y = 10 - 3x$. **Q16.** Exponential; $y = 80 \times (1/2)^x$. **Q17.** (a) linear; (b) reciprocal; (c) exponential; (d) quadratic. **Q18.** Decreasing; exponential — the temperature falls by a constant *ratio* (halving) in equal time intervals, not by a constant amount. **Q19.** (a) A straight line; (b) a cylinder has a uniform cross-section, so a constant volume per second raises the height by the same amount each second (a constant rate). **Q20.** (a) A decreasing rate; (b) concave down — the bowl widens as it fills, so each second's water raises the height by less and less. **Q21.** (i) concave up (narrows as it fills); (ii) straight line; (iii) concave down (widens as it fills).

Fully-worked solutions

Q1. First differences: $1, 3, 5, 7$; second differences: $2, 2, 2$ (constant), so the relationship is **quadratic**. A rule of the form $y = a\,x^2 + c$ fits: at $x = 0$, $y = 1$ so $c = 1$; at $x = 1$, $y = 2 = a + 1$ so $a = 1$. $\therefore\ y = x^2 + 1$ (check $x = 3$: $9 + 1 = 10$ ✓).

Q2. The graph is a straight line, so the relationship is **linear**. It slopes down from left to right, so it is **decreasing**. It crosses the x-axis at $(2, 0)$ and the y-axis at $(0, 2)$.

Q3. $x\,y$: $1 \times 24 = 24$, $2 \times 12 = 24$, $4 \times 6 = 24$, $8 \times 3 = 24$ — the product is constant, so the relationship is a **reciprocal** (inverse proportion). Since $x\,y = 24$, $\therefore\ y = \dfrac{24}{x}$.

Q4. The graph is a **cubic**. It has **no turning points** — it flattens momentarily at a stationary point of inflection (the origin) and continues to increase.

Q5. The y-values fall by 3 each time x increases by 1, so the constant rate of change is -3. With $y = 7$ at $x = 0$, $\therefore\ y = 7 - 3x$. The rule is **decreasing** (negative rate).

Q6. At constant speed, distance = speed × time, a **linear** relationship through the origin. $\therefore\ d = 80t$.

Q7. Straight line ⇒ **linear**. **Q8.** Parabola (one turning point, symmetric) ⇒ **quadratic**. **Q9.** Rapid growth with a horizontal asymptote ⇒ **exponential**. **Q10.** Two branches with asymptotes ⇒ **reciprocal (hyperbola)**. **Q11.** Two turning points, opposite end behaviours ⇒ **cubic**. **Q12.** Parabola ⇒ **quadratic**.

Q13. First differences $5, 7, 9$; second differences $2, 2$ (constant) ⇒ **quadratic**. Trying $y = x^2 + b\,x$: at $x = 1$, $1 + b = 3$ so $b = 2$. $\therefore\ y = x^2 + 2x$ (check $x = 4$: $16 + 8 = 24$ ✓).

Q14. $x\,y = 12$ for every pair, so the relationship is **reciprocal**: $\therefore\ y = \dfrac{12}{x}$.

Q15. First differences are all -3 (constant) ⇒ **linear**; with $y = 10$ at $x = 0$, $\therefore\ y = 10 - 3x$.

Q16. Each y-value is half the previous one (constant ratio $1/2$) ⇒ **exponential**; initial value 80, so $\therefore\ y = 80 \times \left(\dfrac{1}{2}\right)^x$.

Q17. (a) $y = 5 - 2x$ is **linear**; (b) $y = \dfrac{10}{x}$ is **reciprocal**; (c) $y = 3^x$ is **exponential**; (d) $y = x^2 - 4$ is **quadratic**.

Q18. (a) The hot chocolate cools, so the temperature is **decreasing**. (b) **Exponential**: the excess temperature above the room is multiplied by a constant factor $(1/2)$ in equal time intervals — a constant *ratio*, not a constant *difference*, which is the signature of exponential decay (a linear model would fall by the same *number of degrees* each interval, which does not match "halving").

Q19. A cylinder has the **same cross-sectional area at every height**, so pouring in a constant volume each second raises the water level by the **same amount** each second — a constant rate of change. ∴ the height–time graph is a **straight line** (through the origin).

Q20. (a) Because the bowl is **wider near the top**, each extra litre must spread over a larger surface, so it raises the level by **less** than it did lower down: the rate of increase is **decreasing**. (b) A positive but decreasing rate gives a graph that rises while bending over — it is **concave down**.

Q21. (i) The carafe **narrows** as the level rises (from its wide base towards the narrow neck), so each second's oil is squeezed into a smaller area and the depth rises faster and faster ⇒ **concave up**. (ii) The square-sided bottle has a uniform cross-section ⇒ constant rate ⇒ **straight line**. (iii) The trumpet vase **widens** as the level rises (flaring towards the rim), so the depth rises more and more slowly ⇒ **concave down**. All three graphs start at the origin; for the same total volume the concave-down graph is steepest at the start, while the concave-up graph is steepest at the end.

Theory

A **rate of change** measures how one quantity changes compared with another. When a quantity changes by **equal amounts in equal intervals**, it has a **constant rate of change**, and its graph against time (or the other variable) is a **straight line** — a linear relationship.

The rate is the gradient. For a linear relationship,

$$\text{rate of change} = \text{gradient} = \frac{\text{change in } y}{\text{change in } x},$$

and it always carries **units** — for example km/h, L/min, or $ per item.

The linear rule. A constant-rate relationship can be written as

$$y = mx + c,$$

where m is the constant rate of change and c is the **initial value** (the value of y when $x = 0$). A **positive** rate means y is increasing; a **negative** rate means y is decreasing.

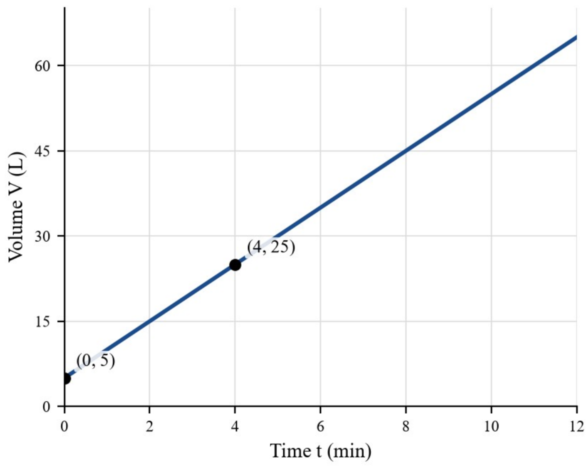
Interpreting the rule in context. The gradient m tells you *how fast* the quantity changes (with units), and the intercept c tells you the *starting* amount. Once the rule is known, you can **predict** a value by substituting, or find *when* a value occurs by solving.

Worked examples

Example 1 — scaffolded

A water tank contains 5 litres. It is then filled at a constant rate, reaching 25 litres after 4 minutes. (a) Find the rate of filling. (b) Write a rule for the volume V litres after t minutes. (c) Find the volume after 10 minutes.

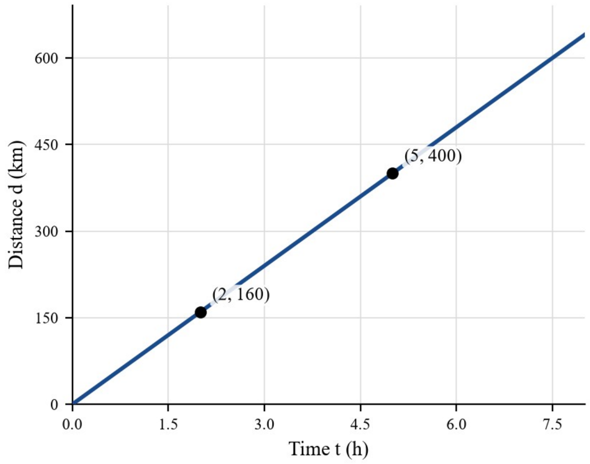
Working	Comment
Rate $= \frac{25-5}{4-0} = \frac{20}{4} = 5$ L/min.	Rate $= \frac{\text{change in volume}}{\text{change in time}}$, with units.
Initial volume is 5 L, so $V = 5 + 5t$.	c is the volume at $t = 0$; m is the rate.
$V(10) = 5 + 5(10) = 55$ L.	Substitute $t = 10$.



Example 2 — scaffolded

A car travels at a constant speed. After 2 hours it has travelled 160 km, and after 5 hours it has travelled 400 km. (a) Find the speed. (b) Write a rule for the distance d km after t hours. (c) Find the distance after 7 hours.

Working	Comment
Speed $=\dfrac{400-160}{5-2}=\dfrac{240}{3}=80$ km/h.	The gradient between the two data points.
At $t=2$, $d=160=80\times 2$, so $c=0$: $d=80t$.	The car starts from $d=0$.
$d(7)=80(7)=560$ km.	Substitute $t=7$.



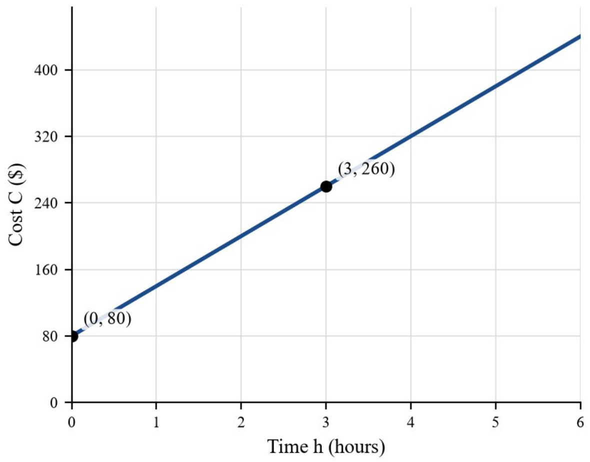
Example 3 — unscaffolded

A plumber charges an \$80 callout fee plus \$60 per hour of work. Write a rule for the total cost C dollars for h hours, and find the cost of a 3-hour job and the length of a job costing \$350.

The rate is \$60 per hour and the initial (callout) value is \$80, so

$$C=80+60h.$$

Cost of 3 hours: $C=80+60(3)=260$, i.e. \$260. For $C=350$: $350=80+60h \Rightarrow 60h=270 \Rightarrow h=4.5$. $\therefore$ a \$350 job takes 4.5 hours.



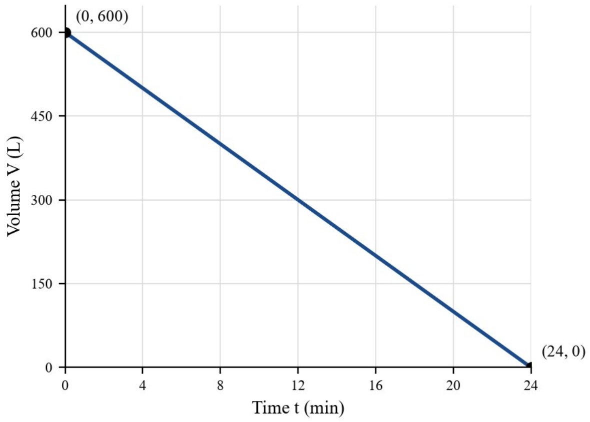
Example 4 — unscaffolded

A tank holds 600 litres of water and is drained at a constant rate of 25 litres per minute. Write a rule for the volume V litres after t minutes, find V after 10 minutes, and find when the tank is empty.

Draining gives a **negative** rate, -25 L/min, from an initial 600 L:

$$V = 600 - 25t.$$

$V(10) = 600 - 25(10) = 350$ L. Empty when $V = 0$: $600 - 25t = 0 \Rightarrow t = 24$. $\therefore$ the tank is empty after 24 minutes.



Practice questions

Scaffolded

Q1. A candle is 20 cm tall and burns down at a constant 2 cm per hour. (a) State the rate of change of height. (b) Write a rule for the height h cm after t hours. (c) Find the height after 5 hours. (d) How long until the candle burns out?

Q2. A pool contains 1000 L of water and is filled at a constant rate, reaching 4000 L after 30 minutes. (a) Find the rate. (b) Write a rule for the volume V after t minutes. (c) Find the volume after 60 minutes.

Q3. A distance–time graph is a straight line through $(0,0)$ and $(4,200)$, where distance is in km and time in hours. (a) Find the speed. (b) Write the rule. (c) Find the distance after 6 hours.

Q4. A cup of tea cools at a constant rate from 80 °C, reaching 60 °C after 10 minutes. (a) Find the rate of change. (b) Write a rule for the temperature T after t minutes. (c) When does it reach 20 °C?

Q5. A casual worker is paid $15 per hour with no other payment. (a) Write a rule for the pay W dollars for h hours. (b) State the rate. (c) Find the pay for 38 hours.

Q6. A linear cost model passes through $(2,13)$ and $(5,28)$. (a) Find the rate of change. (b) Write the rule $C = mx + c$. (c) Find C when $x = 10$.

Unscaffolded

For each, find the constant rate (with units) and a rule, then answer the question.

Q7. A car starts with a full 60 L tank and, after 300 km, has 36 L left. Find the rate of petrol use per km and a rule for the petrol L litres after d km.

Q8. A cyclist rides 45 km in 3 hours at constant speed. Find the speed, a rule for distance d after t hours, and the distance after 5 hours.

Q9. A tank drains from 500 L to 200 L in 12 minutes at a constant rate. Find the rate, a rule for V after t minutes, and when the tank is empty.

Q10. A mobile plan costs $25 per month plus $0.50 per GB of data. Write a rule for the cost C for g GB, and find the cost for 10 GB.

Q11. Snow falls at a constant 3 cm per hour onto a base depth of 12 cm. Write a rule for the depth d after t hours, find the depth after 4 hours, and find when it reaches 30 cm.

Q12. A linear relationship passes through $(1, 7)$ and $(4, 19)$. Find the rate, the rule, and the value of y when $x=10$.

Q13. A candle burns at a constant rate; its height is 18 cm after 1 hour and 6 cm after 4 hours. Find the rate, a rule for the height h after t hours, and when it burns out.

Q14. A graduate's salary starts at \$50 000 and rises by \$2500 each year. Write a rule for the salary S after n years and find the salary after 8 years.

Q15. A pool is emptied at a constant rate; a graph of volume against time is a straight line from $(0, 90000)$ to $(30, 0)$ (litres, minutes). Find the rate and a rule for V after t minutes.

Q16. A car moves at constant speed; its distance–time graph passes through $(1, 40)$ and $(4, 130)$ (km, hours). Find the speed and the rule for d after t hours.

Q17. At a constant exchange rate, 100 AUD converts to 65 USD. Write a rule for the amount U USD from A AUD, and find the value of 250 AUD.

Q18. A worker's total pay is \$50 base plus a constant hourly rate; a 12-hour shift pays \$230. Find the hourly rate, a rule for the pay P for h hours, and the hours needed to earn \$350.

Answers

Q1. -2 cm/h; $h=20-2t$; 10 cm; 10 hours. **Q2.** 100 L/min; $V=1000+100t$; 7000 L. **Q3.** 50 km/h; $d=50t$; 300 km. **Q4.** -2 °C/min; $T=80-2t$; after 30 min. **Q5.** $W=15h$; $15/h; $570. **Q6.** 5; $C=5x+3$; 53. **Q7.** -0.08 L/km (8 L/100 km); $L=60-0.08d$. **Q8.** 15 km/h; $d=15t$; 75 km. **Q9.** -25 L/min; $V=500-25t$; empty after 20 min. **Q10.** $C=25+0.5g$; $30. **Q11.** $d=12+3t$; 24 cm; after 6 hours. **Q12.** 4; $y=4x+3$; 43. **Q13.** -4 cm/h; $h=22-4t$; after 5.5 hours. **Q14.** $S=50\,000+2500n$; $70 000. **Q15.** -3000 L/min; $V=90\,000-3000t$. **Q16.** 30 km/h; $d=30t+10$. **Q17.** $U=0.65A$; $162.50. **Q18.** $15/h; $P=50+15h$; 20 hours.

Fully-worked solutions

Q1. Rate $=-2$ cm/h (decreasing). $h=20-2t$. $h(5)=20-10=10$ cm. Burns out when $h=0$: $20-2t=0 \Rightarrow t=10$ hours.

Q2. Rate $=\dfrac{4000-1000}{30}=100$ L/min. $V=1000+100t$. $V(60)=1000+6000=7000$ L.

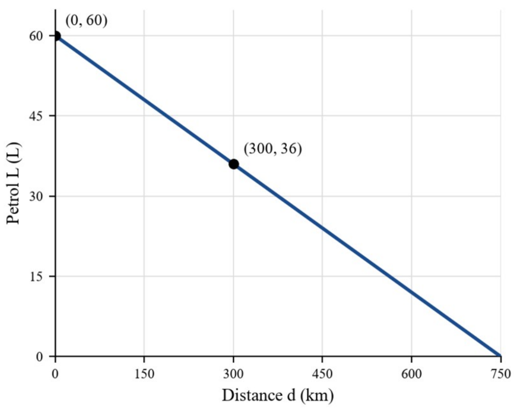
Q3. Speed $=\dfrac{200-0}{4-0}=50$ km/h. $d=50t$. $d(6)=300$ km.

Q4. Rate $=\dfrac{60-80}{10}=-2$ °C/min. $T=80-2t$. $T=20$: $80-2t=20 \Rightarrow t=30$ min.

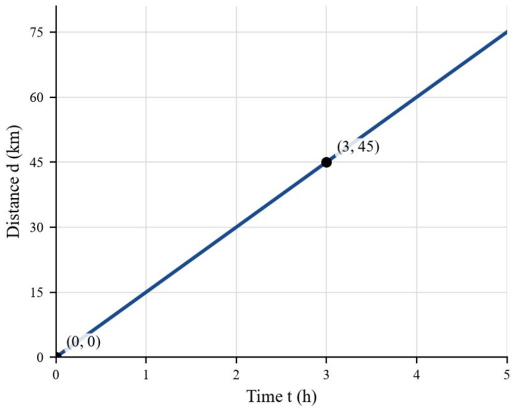
Q5. $W=15h$; the rate is $15 per hour. $W(38)=15\times 38=570$, i.e. $570.

Q6. Rate $=\dfrac{28-13}{5-2}=\dfrac{15}{3}=5$. At $(2,13)$: $13=5(2)+c \Rightarrow c=3$, so $C=5x+3$. $C(10)=53$.

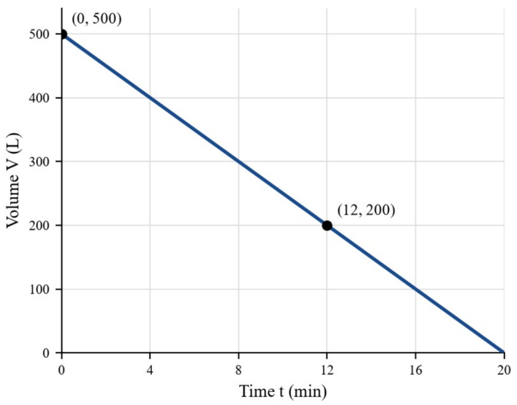
Q7. Rate $=\dfrac{36-60}{300}=\dfrac{-24}{300}=-0.08$ L/km (i.e. 8 L per 100 km). $L=60-0.08d$.



Q8. Speed $=\dfrac{45}{3}=15$ km/h. $d=15t$. $d(5)=75$ km.

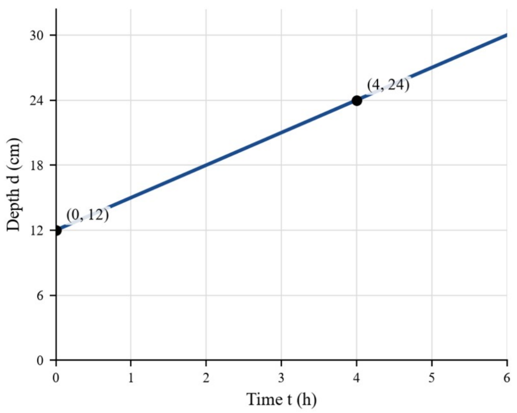


Q9. Rate $= \dfrac{200-500}{12} = -25$ L/min. $V = 500 - 25t$. Empty when $V = 0$: $t = \dfrac{500}{25} = 20$ min.



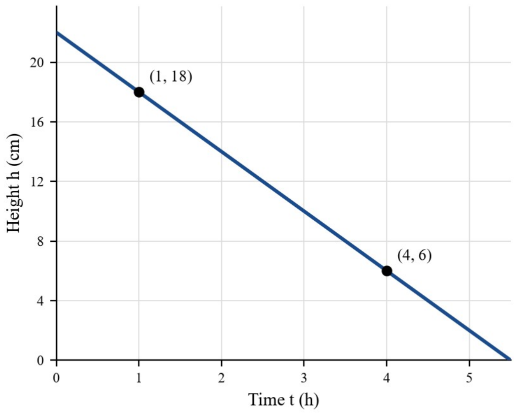
Q10. Rate \$0.50 per GB, base \$25: $C = 25 + 0.5g$. $C(10) = 25 + 5 = 30$, i.e. \$30.

Q11. $d = 12 + 3t$. $d(4) = 12 + 12 = 24$ cm. $d = 30$: $12 + 3t = 30 \Rightarrow t = 6$ hours.



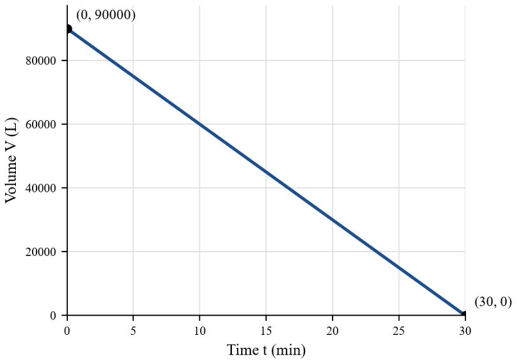
Q12. Rate $= \dfrac{19-7}{4-1} = 4$. At $(1,7)$: $7 = 4(1) + c \Rightarrow c = 3$, so $y = 4x + 3$. $y(10) = 43$.

Q13. Rate $= \dfrac{6-18}{4-1} = -4$ cm/h. At $(1,18)$: $18 = -4(1) + c \Rightarrow c = 22$, so $h = 22 - 4t$. Burns out when $h = 0$: $t = \dfrac{22}{4} = 5.5$ hours.

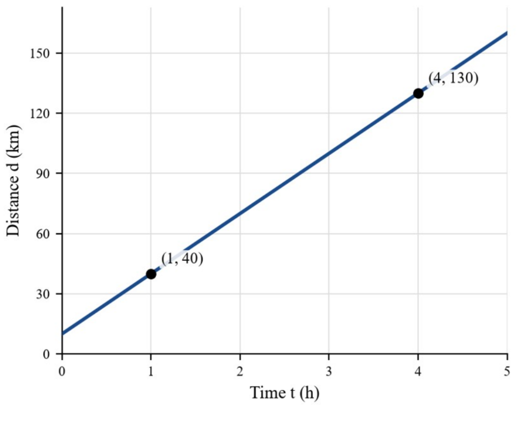


Q14. $S = 50\,000 + 2500n$. $S(8) = 50\,000 + 20\,000 = 70\,000$, i.e. \$70 000.

Q15. Rate $= \dfrac{0 - 90\,000}{30} = -3000$ L/min. $V = 90\,000 - 3000t$.

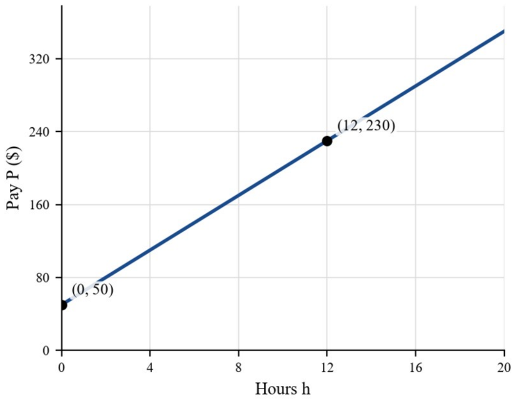


Q16. Speed $= \dfrac{130 - 40}{4 - 1} = 30$ km/h. At $(1, 40)$: $40 = 30(1) + c \Rightarrow c = 10$, so $d = 30t + 10$.



Q17. Rate $= \dfrac{65}{100} = 0.65$ USD per AUD, through the origin: $U = 0.65A$. $U(250) = 0.65 \times 250 = 162.5$, i.e. \$162.50.

Q18. $230 = 50 + 12m \Rightarrow 12m = 180 \Rightarrow m = 15$, i.e. \$15 per hour. $P = 50 + 15h$. For $P = 350$:
$350 = 50 + 15h \Rightarrow h = \dfrac{300}{15} = 20$ hours.



320
240
160
80
0
Pay P ($)
(12, 230)
(0, 50)
0
4
8
12
16
20
Hours h

Theory

The **average rate of change** of a function $y=f(x)$ between $x=a$ and $x=b$ measures how much y changes *on average* for each unit change in x over that interval:

$$\text{average rate of change}=\frac{f(b)-f(a)}{b-a}.$$

It is the gradient of a secant. The points $(a,f(a))$ and $(b,f(b))$ lie on the graph of f. The straight line joining them is called a **secant** (or **chord**), and the average rate of change is exactly the **gradient of that secant**.

Units. The average rate of change is measured in *units of* y *per unit of* x. For a distance–time graph it is the **average speed** (distance per unit time); for a position–time graph it is the **average velocity**, which uses displacement and so may be zero or negative.

Average versus instantaneous. The average rate of change describes the *whole interval* from a to b. It is generally **not** the same as the rate of change at a single instant (the gradient of the *tangent* at a point) — that instantaneous rate is developed in the next section.

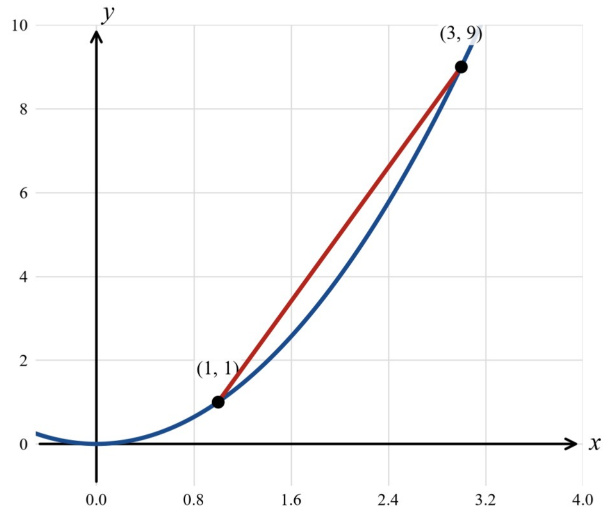
To find an average rate of change: evaluate $f(a)$ and $f(b)$, then compute $\frac{f(b)-f(a)}{b-a}$, and state the units if the problem has a context.

Worked examples

Example 1 — scaffolded

Find the average rate of change of $f(x)=x^2$ between $x=1$ and $x=3$.

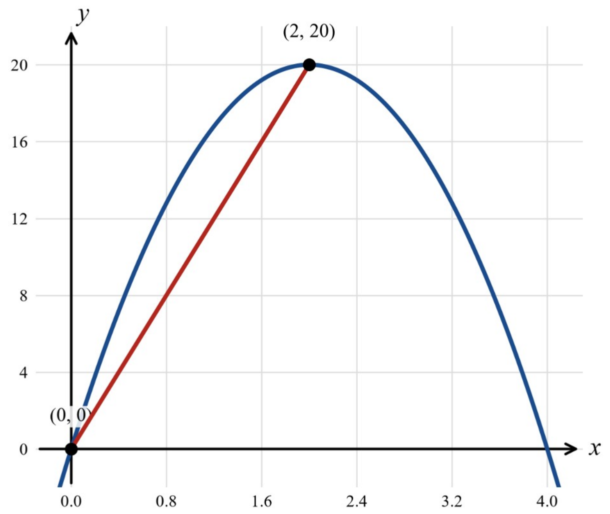
Working	Comment
$f(1)=1^2=1$ and $f(3)=3^2=9$.	Evaluate the function at each endpoint.
Average rate $=\frac{f(3)-f(1)}{3-1}=\frac{9-1}{2}=\frac{8}{2}=4$.	Use $\frac{f(b)-f(a)}{b-a}$.
This is the gradient of the secant joining $(1,1)$ and $(3,9)$.	The red chord below.



Example 2 — scaffolded

A ball is thrown so that its height after t seconds is $h(t)=20t-5t^2$ metres. Find the average velocity of the ball over the first 2 seconds.

Working	Comment
$h(0)=0$ and $h(2)=20(2)-5(2)^2=40-20=20$.	Evaluate at $t=0$ and $t=2$.
Average velocity $=\dfrac{h(2)-h(0)}{2-0}=\dfrac{20-0}{2}=10$.	Change in height over change in time.
$\therefore$ the average velocity is 10 m/s.	State the units.



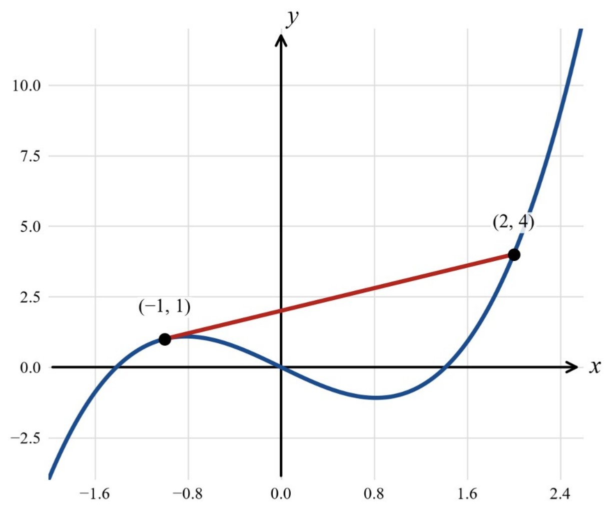
Example 3 — unscaffolded

Find the average rate of change of $f(x)=x^3-2x$ between $x=-1$ and $x=2$.

$f(-1)=(-1)^3-2(-1)=-1+2=1$ and $f(2)=2^3-2(2)=8-4=4$.

Average rate $=\dfrac{f(2)-f(-1)}{2-(-1)}=\dfrac{4-1}{3}=1$.

$\therefore$ the average rate of change is 1 (the gradient of the secant joining $(-1,1)$ and $(2,4)$).



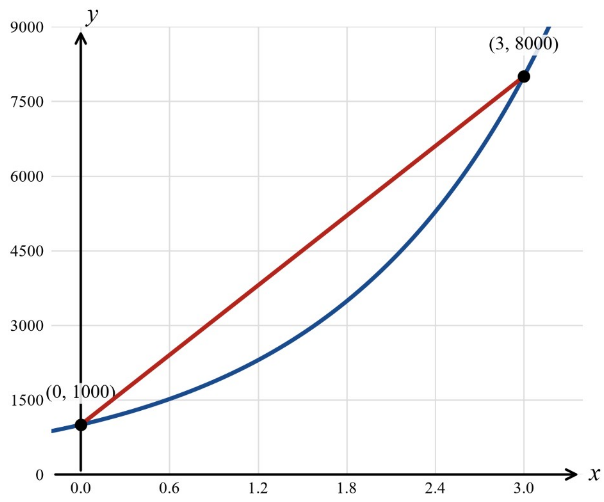
Example 4 — unscaffolded

A colony of bacteria numbers $P(t)=1000\times 2^t$ after t hours. Find the average rate of growth over the first 3 hours.

$P(0)=1000\times 2^0=1000$ and $P(3)=1000\times 2^3=8000$.

Average rate $=\dfrac{P(3)-P(0)}{3-0}=\dfrac{8000-1000}{3}=\dfrac{7000}{3}\approx 2333$.

$\therefore$ the colony grows on average by about 2333 bacteria per hour.



Practice questions

Scaffolded

Q1. For $f(x)=x^2$ between $x=2$ and $x=5$: (a) find $f(2)$ and $f(5)$; (b) find the average rate of change; (c) sketch the graph and the secant.

Q2. For $f(x)=x^2+1$ between $x=0$ and $x=4$: (a) evaluate the endpoints; (b) find the average rate of change; (c) sketch the secant.

Q3. For $g(x)=x^2-3x$ between $x=1$ and $x=4$: (a) find $g(1)$ and $g(4)$; (b) find the average rate of change.

Q4. A car travels so that its distance from a point is $d(t)=t^2$ metres after t seconds. Find its average speed between $t=2$ and $t=5$ (state the units).

Q5. For $f(x)=-x^2+6x$ between $x=1$ and $x=3$: (a) evaluate the endpoints; (b) find the average rate of change.

Q6. For $f(x)=x^3$ between $x=0$ and $x=2$: (a) evaluate the endpoints; (b) find the average rate of change.

Unscaffolded

Find the average rate of change of each function over the interval given (state units where a context is given).

Q7. $f(x)=x^2$, from $x=1$ to $x=4$.

Q8. $f(x)=x^2-2x$, from $x=0$ to $x=3$.

Q9. $f(x)=x^3$, from $x=-1$ to $x=2$.

Q10. $f(x)=-x^2+4x$, from $x=1$ to $x=4$.

Q11. $f(x)=2^x$, from $x=0$ to $x=3$.

Q12. A stone's height is $h(t)=25t-5t^2$ metres after t seconds. Find its average velocity from $t=1$ to $t=4$.

Q13. A particle's distance is $d(t)=t^2+2t$ metres after t seconds. Find its average speed from $t=0$ to $t=5$.

Q14. A population is $P(t)=500\times 3^t$ after t years. Find the average rate of growth from $t=0$ to $t=2$.

Q15. $f(x)=\frac{1}{x}$, from $x=1$ to $x=4$.

Q16. $f(x)=x^2-5x+6$, from $x=2$ to $x=5$.

Q17. A cup of tea cools so that its temperature is $T(t)=20+60\times(1/2)^t$ degrees Celsius after t minutes. Find the average rate of change of temperature from $t=0$ to $t=2$.

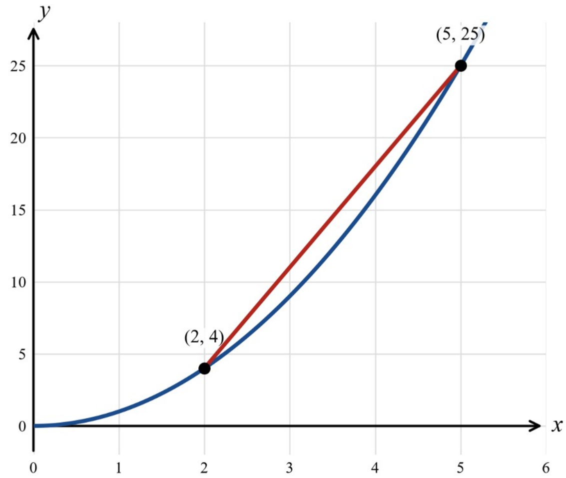
Q18. $f(x)=\sqrt{x}$, from $x=1$ to $x=9$.

Answers

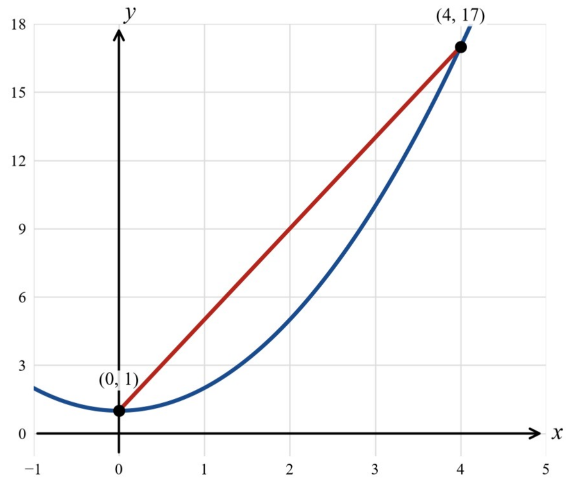
Q1. $f(2)=4$, $f(5)=25$; rate $=7$. **Q2.** 1 and 17; rate $=4$. **Q3.** $g(1)=-2$, $g(4)=4$; rate $=2$. **Q4.** 7 m/s. **Q5.** 5 and 9; rate $=2$. **Q6.** 0 and 8; rate $=4$. **Q7.** 5. **Q8.** 1. **Q9.** 3. **Q10.** -1. **Q11.** $\frac{7}{3}$. **Q12.** 0 m/s. **Q13.** 7 m/s. **Q14.** 2000 per year. **Q15.** $-\frac{1}{4}$. **Q16.** 2. **Q17.** -22.5 °C/min. **Q18.** $\frac{1}{4}$.

Fully-worked solutions

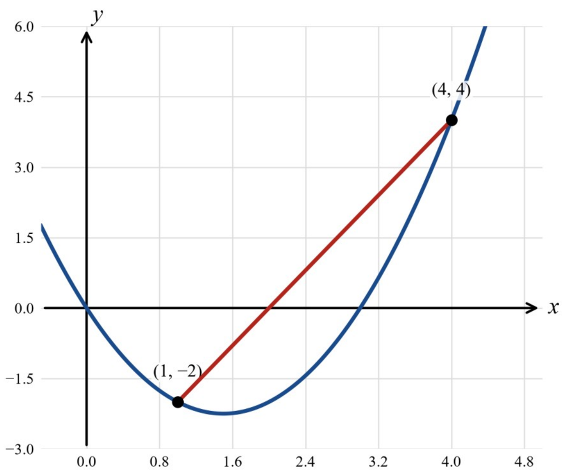
Q1. $f(2)=2^2=4$, $f(5)=5^2=25$. Rate $=\frac{25-4}{5-2}=\frac{21}{3}=7$.



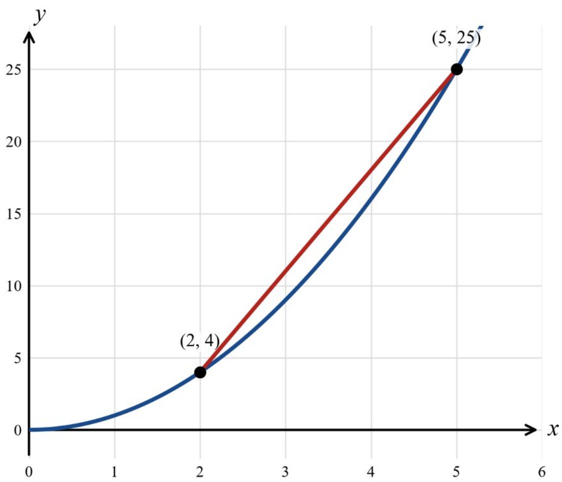
Q2. $f(0)=1$, $f(4)=4^2+1=17$. Rate $=\frac{17-1}{4-0}=\frac{16}{4}=4$.



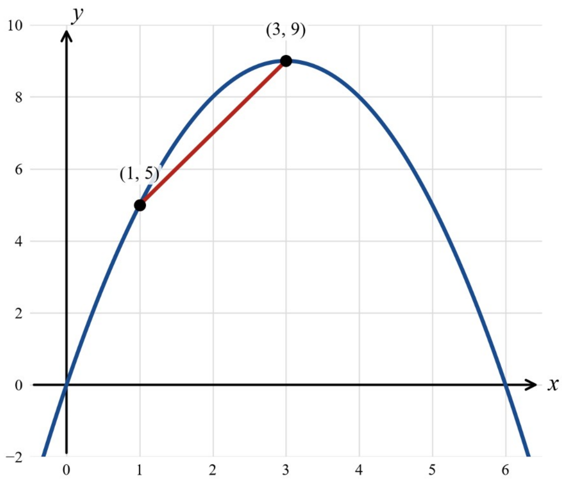
Q3. $g(1)=1-3=-2$, $g(4)=16-12=4$. Rate $=\frac{4-(-2)}{4-1}=\frac{6}{3}=2$.



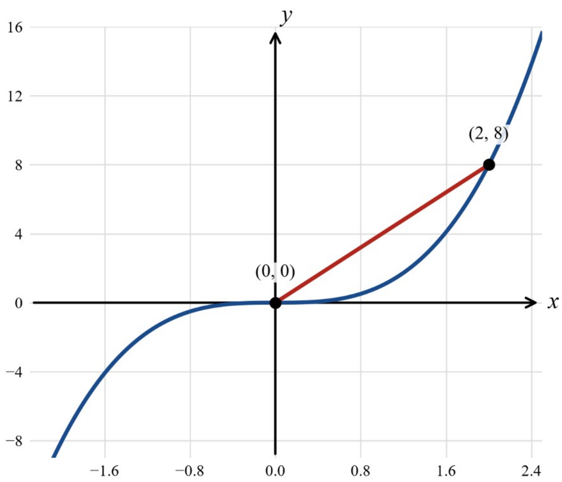
Q4. $d(2)=4$, $d(5)=25$. Average speed $=\dfrac{25-4}{5-2}=\dfrac{21}{3}=7$. $\therefore$ the average speed is 7 m/s.



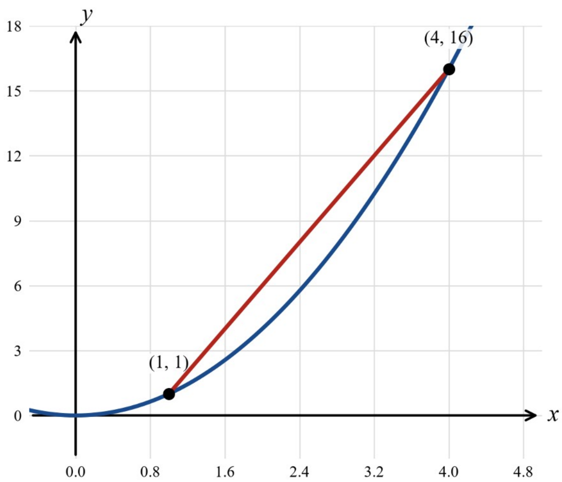
Q5. $f(1)=-1+6=5$, $f(3)=-9+18=9$. Rate $=\dfrac{9-5}{3-1}=\dfrac{4}{2}=2$.



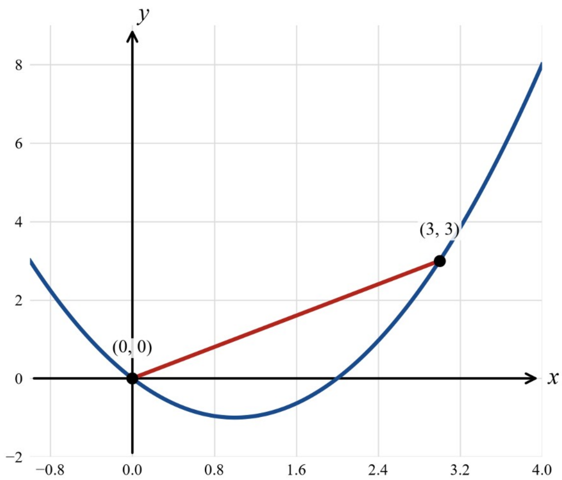
Q6. $f(0)=0$, $f(2)=8$. Rate $=\dfrac{8-0}{2-0}=4$.



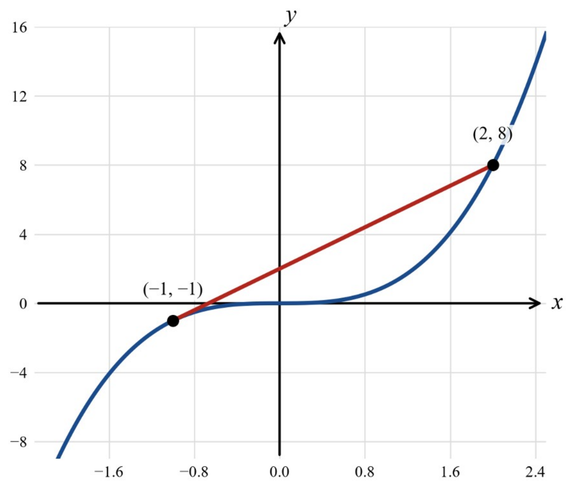
Q7. $f(1) = 1$, $f(4) = 16$. Rate $= \dfrac{16 - 1}{4 - 1} = \dfrac{15}{3} = 5$.



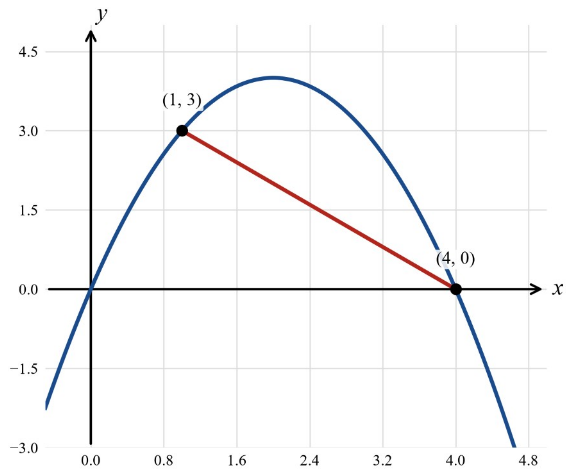
Q8. $f(0) = 0$, $f(3) = 9 - 6 = 3$. Rate $= \dfrac{3 - 0}{3 - 0} = 1$.



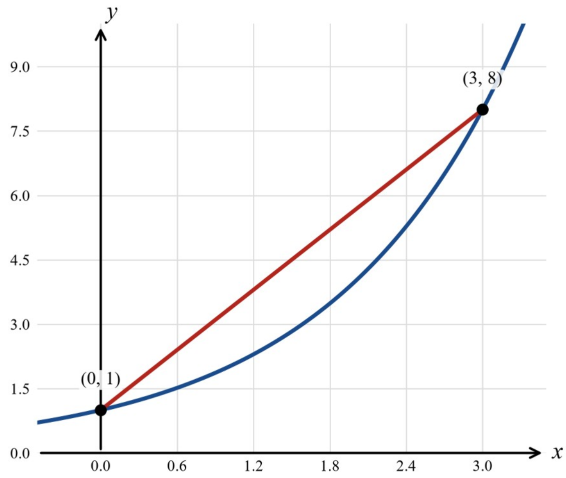
Q9. $f(-1) = -1$, $f(2) = 8$. Rate $= \dfrac{8 - (-1)}{2 - (-1)} = \dfrac{9}{3} = 3$.



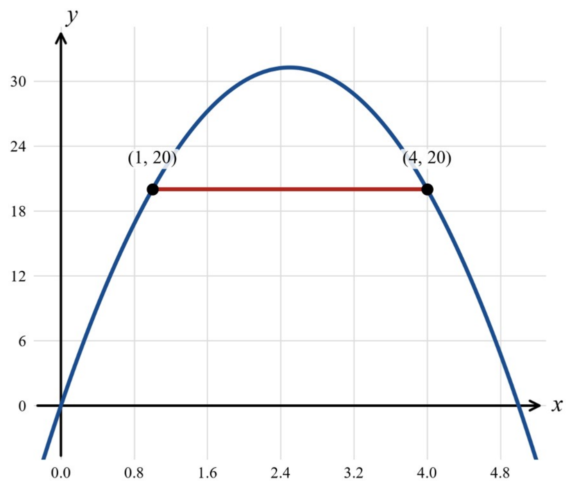
Q10. $f(1)=-1+4=3$, $f(4)=-16+16=0$. Rate $=\dfrac{0-3}{4-1}=\dfrac{-3}{3}=-1$.



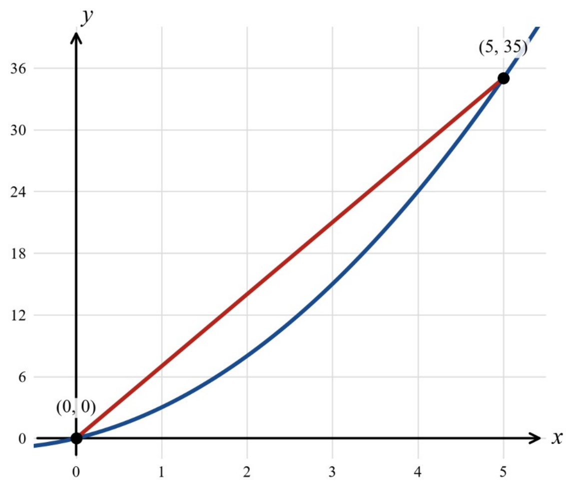
Q11. $f(0)=1$, $f(3)=8$. Rate $=\dfrac{8-1}{3-0}=\dfrac{7}{3}$.



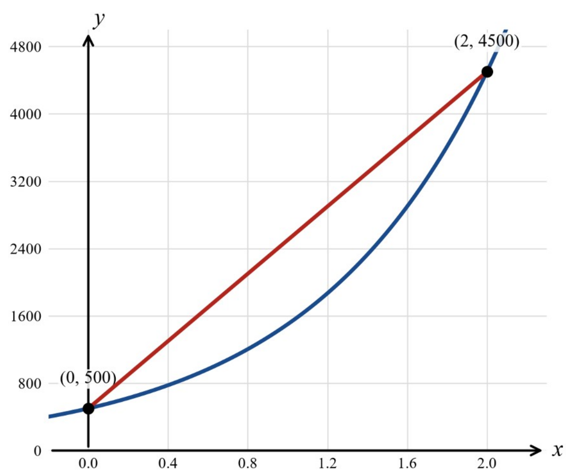
Q12. $h(1)=25-5=20$, $h(4)=100-80=20$. Average velocity $=\dfrac{20-20}{4-1}=0$. $\therefore$ the average velocity is 0 m/s (the stone returns to the same height).



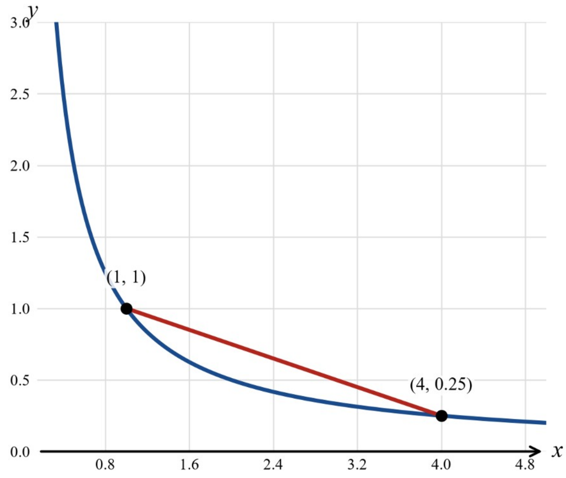
Q13. $d(0)=0$, $d(5)=25+10=35$. Average speed $=\dfrac{35-0}{5-0}=7$. $\therefore$ the average speed is 7 m/s.



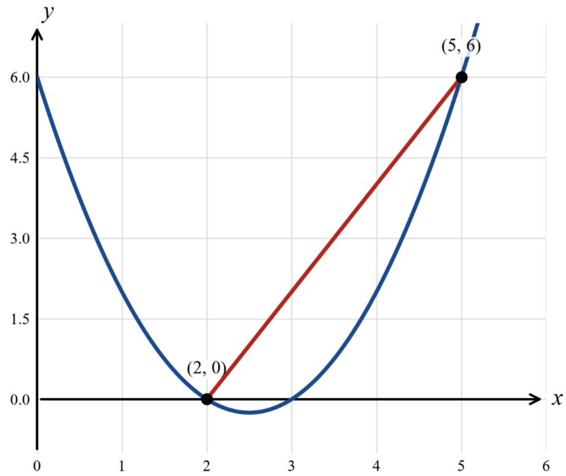
Q14. $P(0)=500$, $P(2)=500\times 9=4500$. Rate $=\dfrac{4500-500}{2-0}=\dfrac{4000}{2}=2000$. $\therefore$ the population grows on average by 2000 per year.



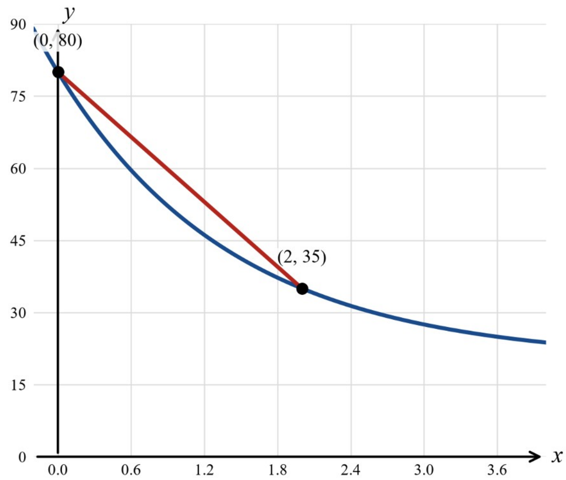
Q15. $f(1)=1$, $f(4)=1/4$. Rate $=\dfrac{1/4-1}{4-1}=\dfrac{-3/4}{3}=-\dfrac{1}{4}$.



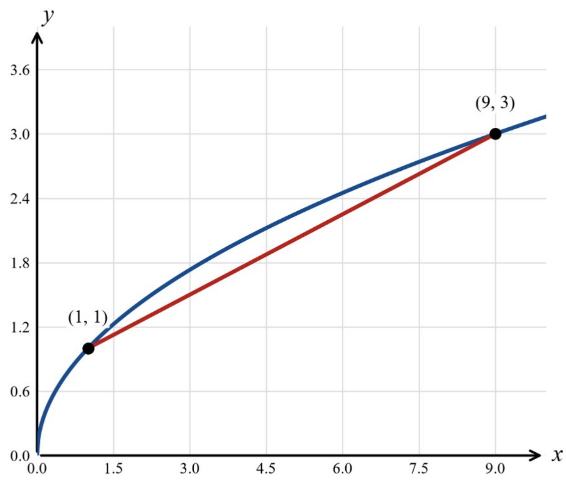
Q16. $f(2) = 4 - 10 + 6 = 0$, $f(5) = 25 - 25 + 6 = 6$. Rate $= \dfrac{6-0}{5-2} = \dfrac{6}{3} = 2$.



Q17. $T(0) = 20 + 60 = 80$, $T(2) = 20 + 60 \times 1/4 = 20 + 15 = 35$. Rate $= \dfrac{35-80}{2-0} = \dfrac{-45}{2} = -22.5$. $\therefore$ the tea cools on average by 22.5 °C/min.



Q18. $f(1) = 1$, $f(9) = 3$. Rate $= \dfrac{3-1}{9-1} = \dfrac{2}{8} = \dfrac{1}{4}$.

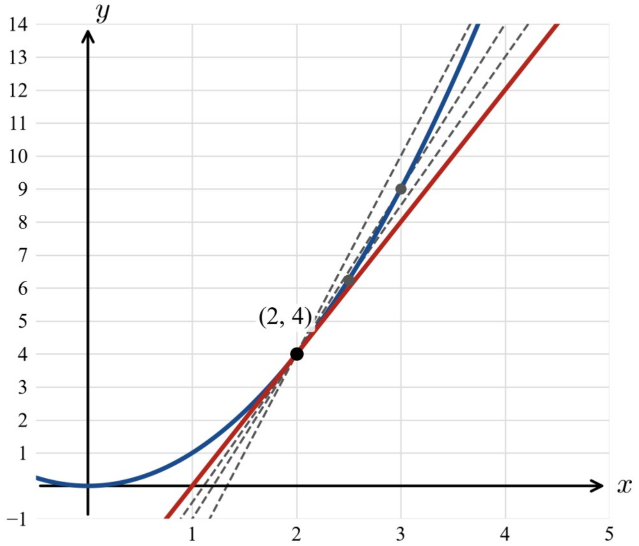

y
x
3.6
3.0
2.4
1.8
1.2
0.6
0.0
0.0
1.5
3.0
4.5
6.0
7.5
9.0
(9, 3)
(1, 1)

Theory

The **average rate of change** of $y=f(x)$ over the interval $[a, a+h]$ is the gradient of the **secant** joining the points $(a, f(a))$ and $(a+h, f(a+h))$:

$$\text{average rate} = \frac{f(a+h)-f(a)}{h}.$$

The **instantaneous rate of change** at $x=a$ is the gradient of the **tangent** to the curve at that point. As h gets smaller, the point $(a+h, f(a+h))$ slides towards $(a, f(a))$ and the secant rotates towards the tangent:



So the instantaneous rate is the **limiting value** of the average rate as $h \to 0$.

Estimating numerically. Build a table of average rates over shorter and shorter intervals and see what value they approach.

By first principles. The instantaneous rate of change of f at $x=a$ is

$$\frac{\lim\limits_{h\to 0} f(a+h)-f(a)}{h}.$$

For a polynomial, expand and simplify the difference quotient (the h in the denominator cancels), then let $h \to 0$.

Estimating graphically. Given an accurate graph, you can estimate the instantaneous rate at a point by hand: **draw the tangent** there (a straight line that just touches the curve, matching its direction), then pick **two points that lie on the tangent** — far apart and on grid lines for accuracy — and compute $\frac{\text{rise}}{\text{run}}$. This gives an *estimate*; the exact value comes from first principles.

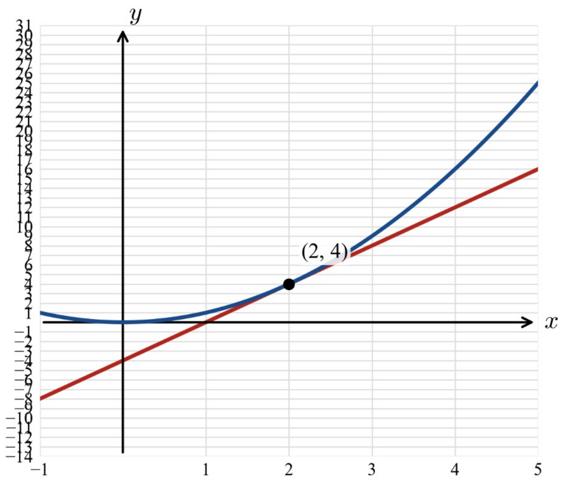
Interpreting. The instantaneous rate carries the units of $\frac{y\text{-units}}{x\text{-units}}$. For a position–time function $s(t)$, the instantaneous rate of change is the **velocity**.

Worked examples

Example 1 — scaffolded

For $f(x)=x^2$, estimate the instantaneous rate of change at $x=2$ using a table, then find it exactly by first principles.

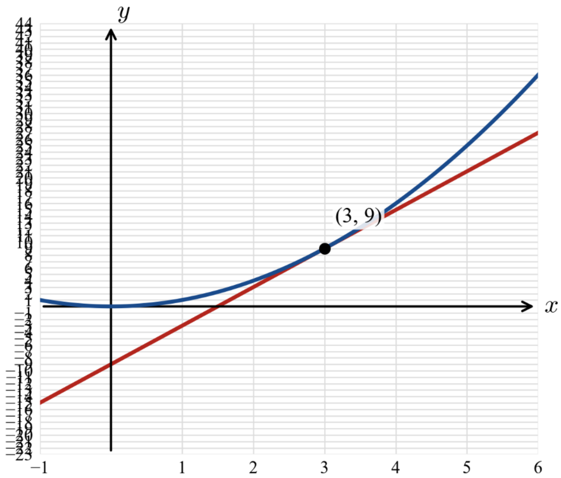
Working	Comment
Average rate over $[2,2+h]$ is $\dfrac{f(2+h)-f(2)}{h}=\dfrac{(2+h)^2-4}{h}$.	Set up the difference quotient.
$h=1\to 5$, $h=0.5\to 4.5$, $h=0.1\to 4.1$, $h=0.01\to 4.01$, $h=0.001\to 4.001$.	Evaluate for shrinking h; the values approach 4.
First principles: $\dfrac{(2+h)^2-4}{h}=\dfrac{4+4h+h^2-4}{h}=\dfrac{4h+h^2}{h}=4+h$.	Expand and cancel the h.
As $h\to 0$, $4+h\to 4$.	∴ the instantaneous rate at $x=2$ is 4.



Example 2 — scaffolded

A particle has position $s(t)=t^2$ metres after t seconds. Find its velocity at $t=3$.

Working	Comment
Velocity is the instantaneous rate of change of position.	Use first principles at $t=3$.
$\dfrac{s(3+h)-s(3)}{h}=\dfrac{(3+h)^2-9}{h}=\dfrac{6h+h^2}{h}=6+h$.	Expand and cancel.
As $h\to 0$, $6+h\to 6$.	∴ the velocity at $t=3$ is 6 m/s.

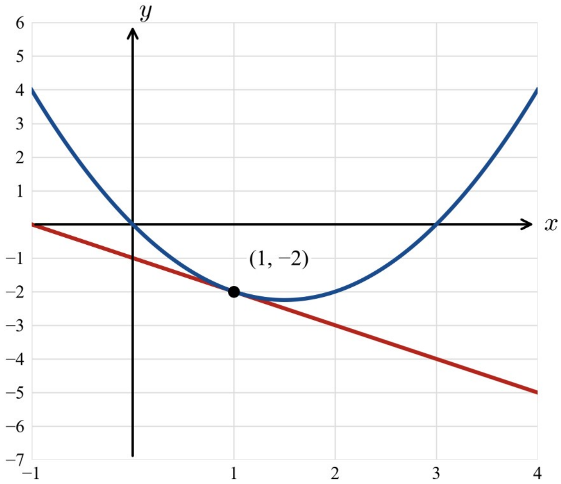


Example 3 — unscaffolded

Use first principles to find the instantaneous rate of change of $f(x) = x^2 - 3x$ at $x = 1$.

$$\frac{f(1+h) - f(1)}{h} = \frac{\left[(1+h)^2 - 3(1+h)\right] - (-2)}{h} = \frac{1 + 2h + h^2 - 3 - 3h + 2}{h} = \frac{h^2 - h}{h} = h - 1.$$

As $h \to 0$, $h - 1 \to -1$. $\therefore$ the instantaneous rate at $x = 1$ is -1.

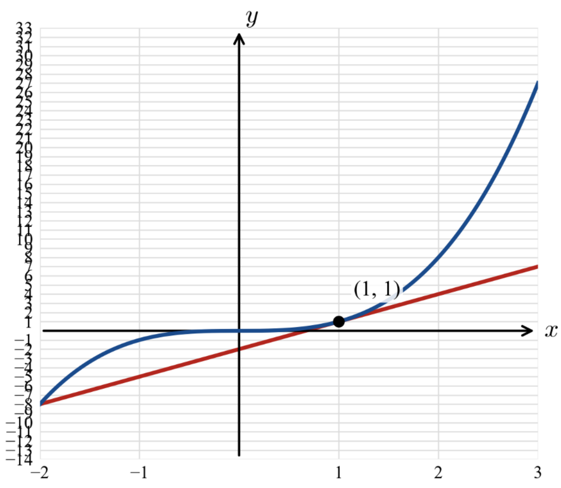


Example 4 — unscaffolded

Use first principles to find the instantaneous rate of change of $f(x) = x^3$ at $x = 1$.

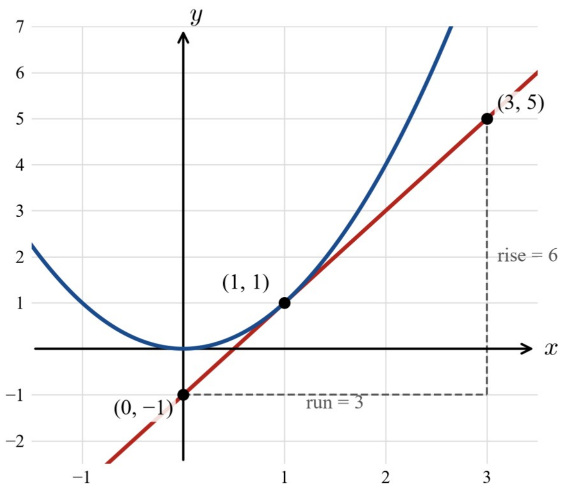
$$\frac{(1+h)^3 - 1}{h} = \frac{1 + 3h + 3h^2 + h^3 - 1}{h} = \frac{3h + 3h^2 + h^3}{h} = 3 + 3h + h^2.$$

As $h \to 0$, this $\to 3$. $\therefore$ the instantaneous rate at $x = 1$ is 3.



Example 5 — scaffolded

The graph shows $y = x^2$ with the tangent drawn at the point $(1, 1)$. Estimate the gradient of the curve at $x = 1$ by reading two points off the tangent, then compare with the exact value.



Working	Comment
Read two points that lie on the **tangent** (not the curve): $(0, -1)$ and $(3, 5)$.	Choose points far apart and on grid lines for accuracy.
$\text{rise} = 5 - (-1) = 6$ and $\text{run} = 3 - 0 = 3$.	Rise is the change in y; run is the change in x.
Gradient $\approx \dfrac{\text{rise}}{\text{run}} = \dfrac{6}{3} = 2$.	The estimated gradient of the tangent.
First principles: $\dfrac{(1+h)^2 - 1}{h} = 2 + h \to 2$.	The exact instantaneous rate at $x = 1$ is 2 — the graphical estimate matches.

Practice questions

Scaffolded

Q1. For $f(x) = x^2$ at $x = 1$: (a) simplify $\dfrac{f(1+h) - f(1)}{h}$; (b) hence find the instantaneous rate of change at $x = 1$.

Q2. For $f(x) = x^2 + 2$ at $x = 2$: (a) show that $\dfrac{f(2+h) - f(2)}{h} = 4 + h$; (b) hence find the instantaneous rate at $x = 2$.

Q3. For $f(x) = x^2 - x$ at $x = 2$: (a) simplify the difference quotient; (b) find the instantaneous rate at $x = 2$.

Q4. For $f(x)=x^3$ at $x=2$: (a) expand $(2+h)^3$ and simplify $\frac{f(2+h)-f(2)}{h}$; (b) find the instantaneous rate at $x=2$.

Q5. For $f(x)=4-x^2$ at $x=1$: (a) simplify the difference quotient; (b) find the instantaneous rate at $x=1$. (What does the negative sign mean?)

Q6. For $f(x)=2x^2$ at $x=3$: (a) simplify $\frac{f(3+h)-f(3)}{h}$; (b) find the instantaneous rate at $x=3$.

Unscaffolded

Use first principles to find the instantaneous rate of change at the given point.

Q7. $f(x)=x^2$ at $x=3$

Q8. $f(x)=x^2$ at $x=-1$

Q9. $f(x)=x^2+1$ at $x=2$

Q10. $f(x)=3x^2$ at $x=1$

Q11. $f(x)=x^2-4x$ at $x=3$

Q12. $f(x)=x^2+2x$ at $x=0$

Q13. $f(x)=x^3$ at $x=3$

Q14. $f(x)=x^3-x$ at $x=2$

Q15. $f(x)=5-x^2$ at $x=2$

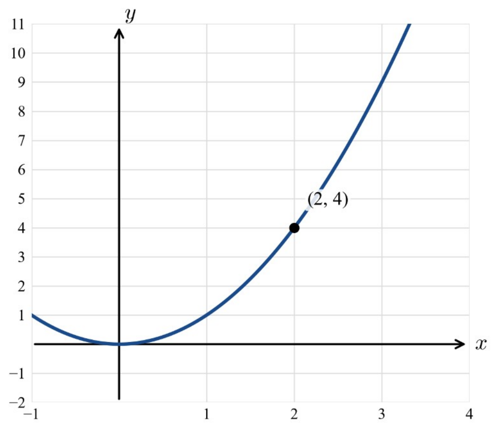
Q16. A ball has height $s(t)=2t^2$ metres after t seconds. Find its velocity at $t=2$.

Q17. A particle has position $s(t)=t^2-2t$ metres after t seconds. Find its velocity at $t=1$, and describe what is happening at that instant.

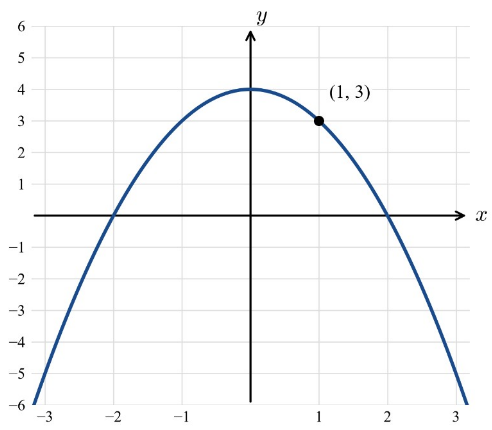
Q18. $f(x)=2x^2-x$ at $x=2$

For **Q19–Q20**, estimate the gradient **graphically**: draw the tangent at the marked point, read two points off your tangent, and estimate the gradient by $\frac{\text{rise}}{\text{run}}$. Then find the exact instantaneous rate by first principles and compare.

Q19. The graph shows $y=x^2$. Estimate the gradient of the curve at the point $(2,4)$.



Q20. The graph shows $y = 4 - x^2$. Estimate the gradient of the curve at the point $(1, 3)$. (What does the sign of your answer tell you?)



Answers

Q1. $2+h \to 2$. **Q2.** 4. **Q3.** 3. **Q4.** 12. **Q5.** -2 (the quantity is decreasing). **Q6.** 12. **Q7.** 6. **Q8.** -2. **Q9.** 4. **Q10.** 6. **Q11.** 2. **Q12.** 2. **Q13.** 27. **Q14.** 11. **Q15.** -4. **Q16.** 8 m/s. **Q17.** 0 m/s (the particle is momentarily at rest). **Q18.** 7. **Q19.** Estimate $\approx \frac{8-0}{3-1}=4$; exact 4 (they agree). **Q20.** Estimate $\approx \frac{1-5}{2-0}=-2$; exact -2 (the negative sign shows the curve is **decreasing** at $x=1$).

Fully-worked solutions

Q1. $\frac{(1+h)^2-1}{h}=\frac{2h+h^2}{h}=2+h$. As $h \to 0$, $\to 2$.

Q2. $\frac{(2+h)^2+2-6}{h}=\frac{4h+h^2}{h}=4+h \to 4$.

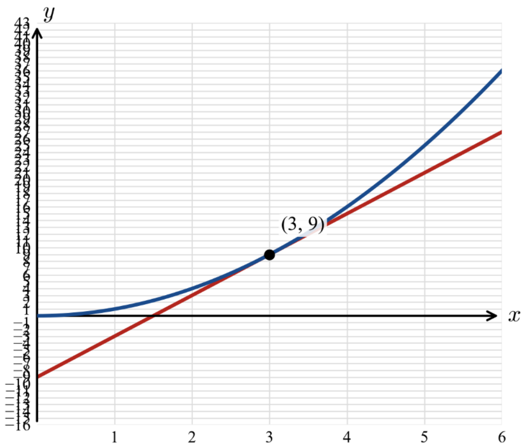
Q3. $\frac{[(2+h)^2-(2+h)]-2}{h}=\frac{3h+h^2}{h}=3+h \to 3$.

Q4. $\frac{(2+h)^3-8}{h}=\frac{12h+6h^2+h^3}{h}=12+6h+h^2 \to 12$.

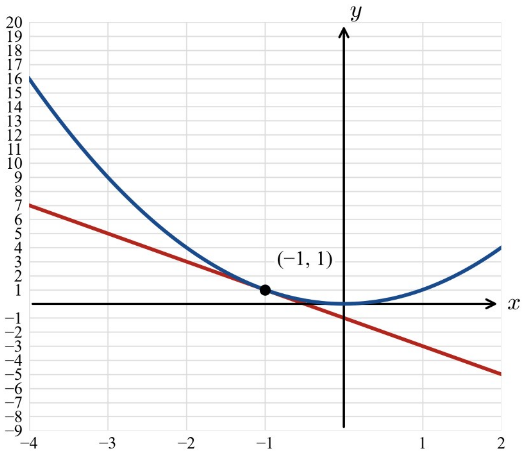
Q5. $\frac{[4-(1+h)^2]-3}{h}=\frac{-2h-h^2}{h}=-2-h \to -2$. The negative value means f is **decreasing** at $x=1$.

Q6. $\frac{2(3+h)^2-18}{h}=\frac{12h+2h^2}{h}=12+2h \to 12$.

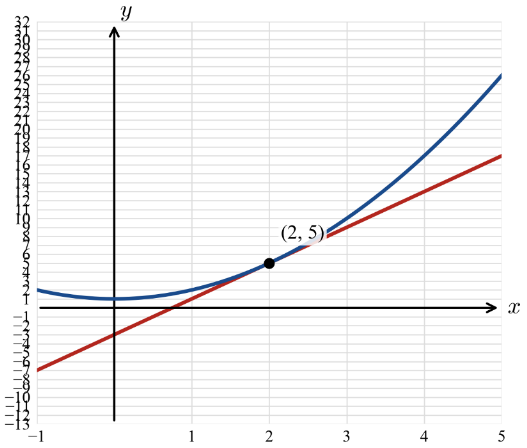
Q7. $\frac{(3+h)^2-9}{h}=6+h \to 6$.



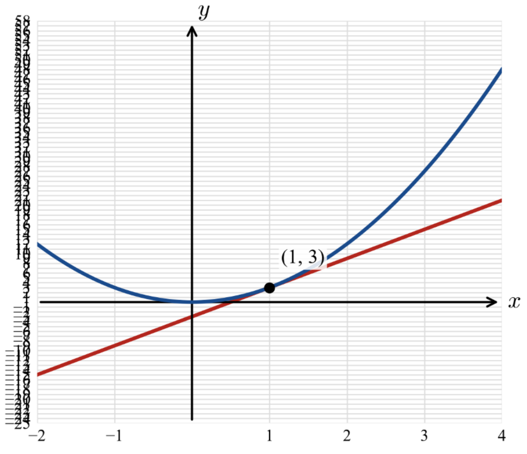
Q8. $\frac{(-1+h)^2-1}{h}=\frac{-2h+h^2}{h}=-2+h \to -2$.



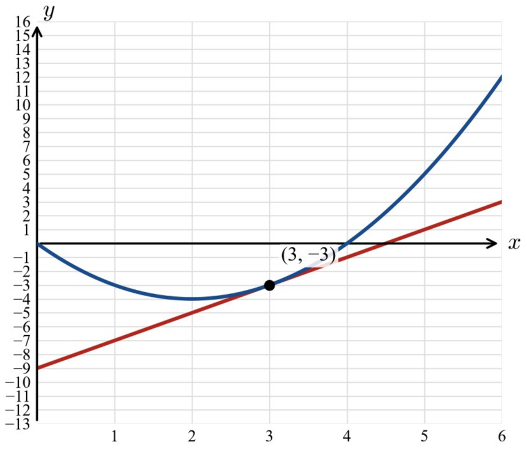
Q9. $\dfrac{(2+h)^2+1-5}{h}=4+h \to 4.$



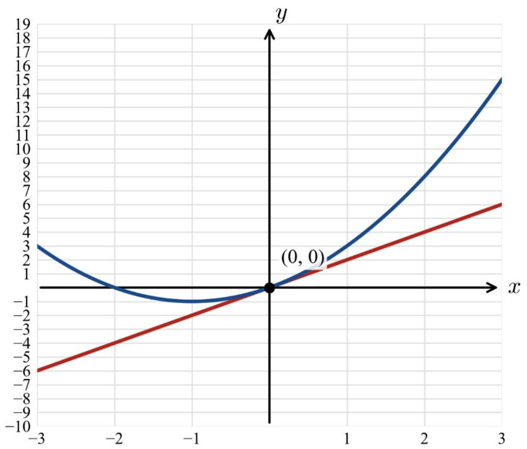
Q10. $\dfrac{3(1+h)^2-3}{h}=\dfrac{6h+3h^2}{h}=6+3h \to 6.$



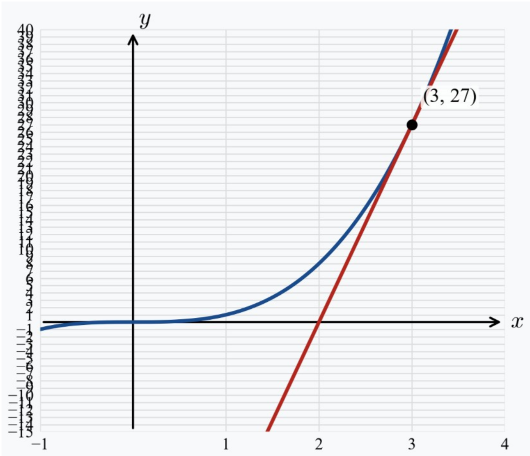
Q11. $\dfrac{\left[(3+h)^2-4(3+h)\right]-(-3)}{h}=\dfrac{2h+h^2}{h}=2+h \to 2.$



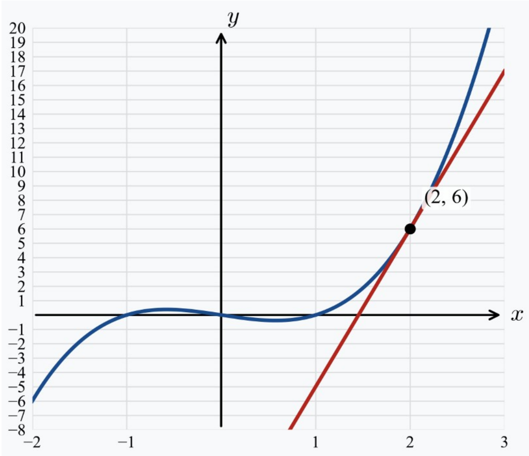
Q12. $\dfrac{\left[(0+h)^2+2(0+h)\right]-0}{h}=\dfrac{h^2+2h}{h}=h+2\to 2.$



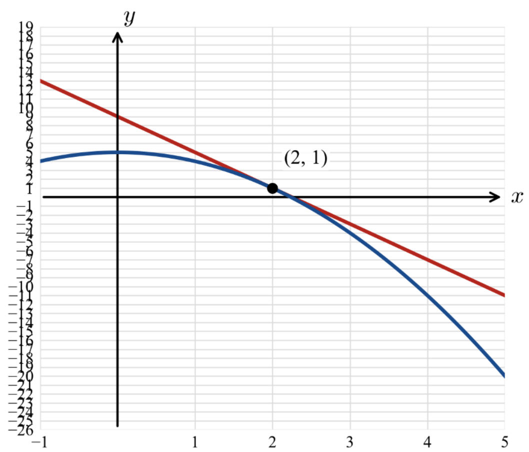
Q13. $\dfrac{(3+h)^3-27}{h}=\dfrac{27h+9h^2+h^3}{h}=27+9h+h^2\to 27.$



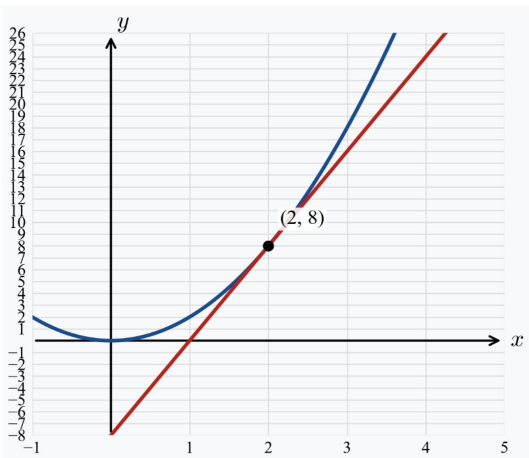
Q14. $\dfrac{\left[(2+h)^3-(2+h)\right]-6}{h}=\dfrac{11h+6h^2+h^3}{h}=11+6h+h^2\to 11.$



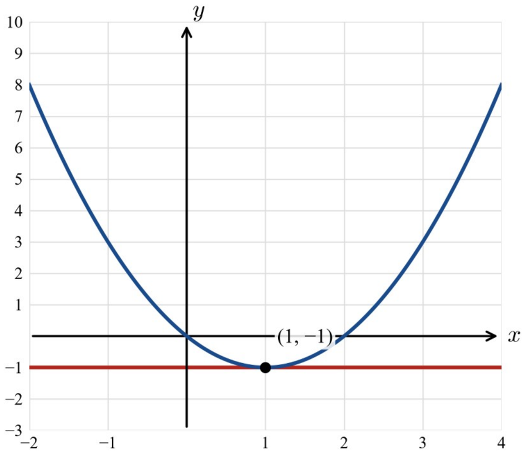
Q15. $\dfrac{\left[5-(2+h)^2\right]-1}{h}=\dfrac{-4h-h^2}{h}=-4-h\rightarrow -4$.



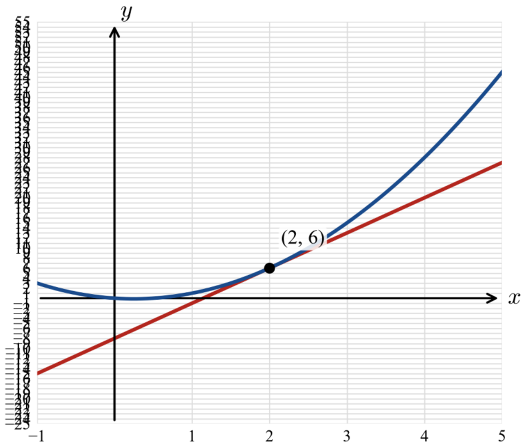
Q16. Velocity $=\dfrac{\lim\limits_{h\to 0} 2(2+h)^2-8}{h}=\lim\limits_{h\to 0}(8+2h)=8$ m/s.



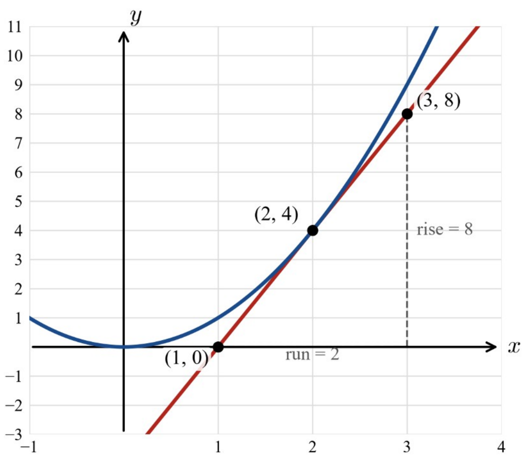
Q17. Velocity $=\dfrac{\lim\limits_{h\to 0}\left[(1+h)^2-2(1+h)\right]-(-1)}{h}=\dfrac{\lim\limits_{h\to 0} h^2}{h}=\lim\limits_{h\to 0} h=0$ m/s. The particle is **momentarily at rest** (it is turning around).



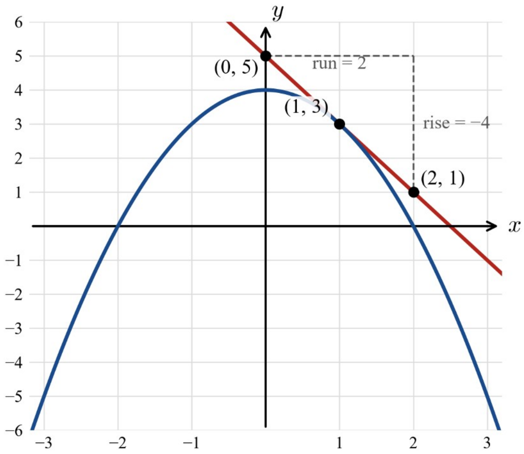
Q18. $\dfrac{\left[2(2+h)^2-(2+h)\right]-6}{h}=\dfrac{7h+2h^2}{h}=7+2h\to 7.$



Q19. Draw the tangent at $(2,4)$ and read two points on it, e.g. $(1,0)$ and $(3,8)$: gradient $\approx\dfrac{8-0}{3-1}=\dfrac{8}{2}=4$. Exact (first principles): $\dfrac{(2+h)^2-4}{h}=4+h\to 4$. The estimate matches the exact rate of 4.



Q20. Draw the tangent at $(1,3)$ and read two points on it, e.g. $(0,5)$ and $(2,1)$: gradient $\approx\dfrac{1-5}{2-0}=\dfrac{-4}{2}=-2$. Exact (first principles): $\dfrac{\left[4-(1+h)^2\right]-3}{h}=\dfrac{-2h-h^2}{h}=-2-h\to -2$. The estimate matches; the negative value shows the curve is **decreasing** at $x=1$.



y
x
(0, 5)
run = 2
(1, 3)
rise = −4
(2, 1)
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−3
−2
−1
1
2
3

Theory

The **position** of an object moving in a straight line is given as a function of time, $x(t)$, measured from a chosen origin with a chosen positive direction (position in metres, time in seconds). Its graph is the **position–time graph**, with t on the horizontal axis and x on the vertical axis.

Displacement. The **displacement** from time t_1 to time t_2 is the *change in position*

$$\Delta x = x(t_2) - x(t_1).$$

It may be positive, negative or zero; its sign gives the direction of the net movement.

Distance travelled. If the object **changes direction**, the total **distance travelled** is longer than $|\Delta x|$. To find it, split the journey at each turning point and add the distance covered on each leg.

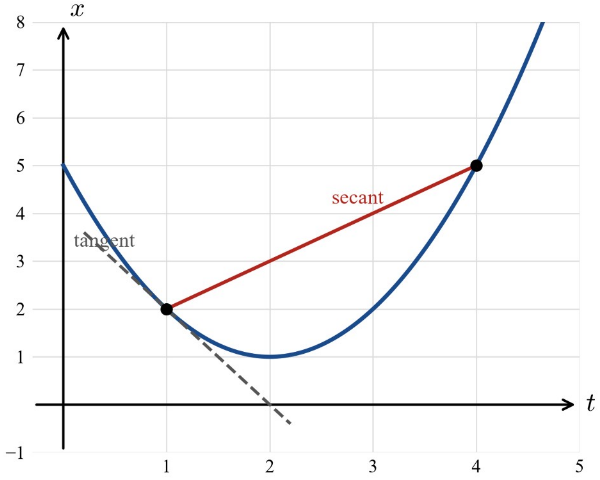
Average velocity. The **average velocity** over $[t_1, t_2]$ is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time taken}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

On the position–time graph this is the **gradient of the secant** (the chord) joining the two points. Its units are m/s and its sign gives the direction.

Average speed. $\text{average speed} = \dfrac{\text{distance travelled}}{\text{time taken}}$, which is positive or zero — it is 0 exactly when the object does not move.

Instantaneous velocity. As t_2 moves closer to t_1, the secant approaches the **tangent** at t_1. The gradient of the tangent is the **instantaneous velocity** — the velocity at that instant. This limiting process is the idea behind differentiation (the next chapters).

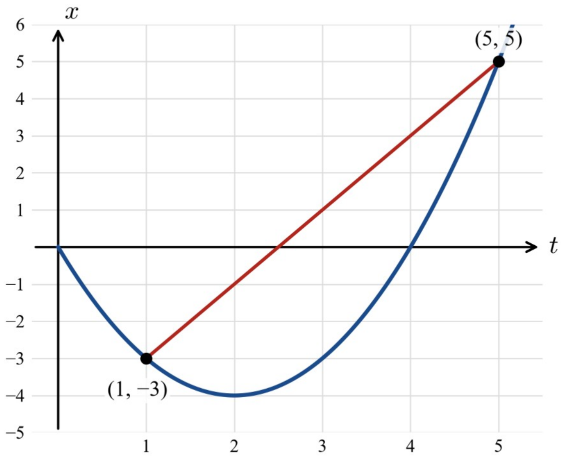


Worked examples

Example 1 — scaffolded

A particle moves in a straight line with position $x(t) = t^2 - 4t$ metres at time t seconds ($t \geq 0$). Find (a) its position at $t = 1$ and $t = 5$; (b) the displacement from $t = 1$ to $t = 5$; (c) the average velocity over this interval.

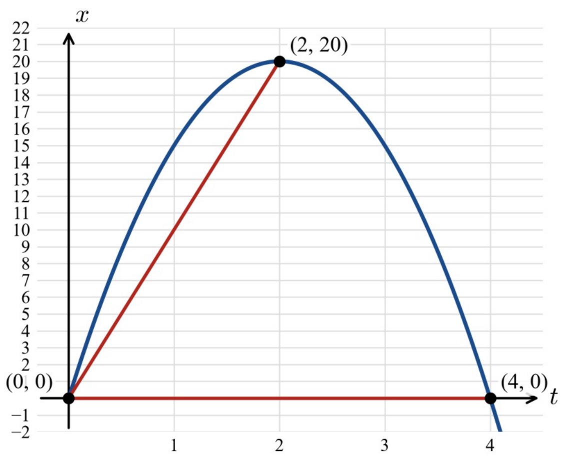
Working	Comment
$x(1)=1^2-4(1)=-3$ m; $x(5)=5^2-4(5)=5$ m.	Substitute each time into $x(t)$.
Displacement $=x(5)-x(1)=5-(-3)=8$ m.	Displacement is the change in position.
Average velocity $=\dfrac{x(5)-x(1)}{5-1}=\dfrac{8}{4}=2$ m/s.	The gradient of the secant from $(1,-3)$ to $(5,5)$.



Example 2 — scaffolded

A ball rolls so that its position is $x(t)=20t-5t^2$ metres. Find the average velocity over (a) $[0,2]$ and (b) $[0,4]$.

Working	Comment
$x(0)=0$, $x(2)=40-20=20$, $x(4)=80-80=0$.	Evaluate the positions needed.
(a) Average velocity $=\dfrac{20-0}{2-0}=10$ m/s.	Secant gradient over $[0,2]$.
(b) Average velocity $=\dfrac{0-0}{4-0}=0$ m/s.	The ball returns to its start, so the displacement is 0 — but the distance travelled is **not** 0.



Example 3 — unscaffolded

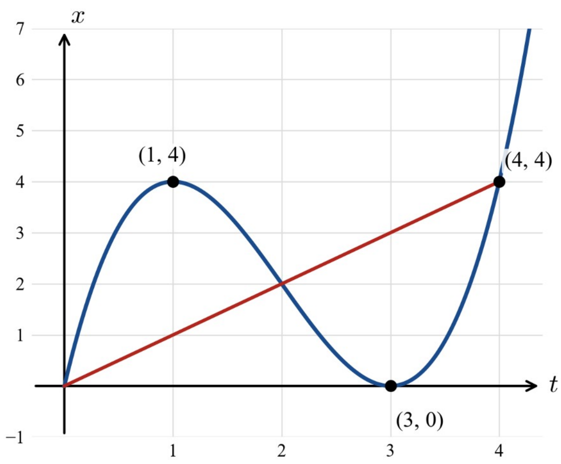
A particle has position $x(t)=t^3-6t^2+9t$ metres ($t\ge 0$). It is at rest (turns around) at $t=1$ and $t=3$. Find, over $[0,4]$: (a) the displacement; (b) the average velocity; (c) the distance travelled.

$x(0)=0$, $x(1)=1-6+9=4$, $x(3)=27-54+27=0$, $x(4)=64-96+36=4$.

(a) Displacement $=x(4)-x(0)=4$ m.

(b) Average velocity $=\dfrac{4}{4}=1$ m/s.

(c) The particle goes $0 \to 4$ (distance 4), then $4 \to 0$ (distance 4), then $0 \to 4$ (distance 4): total distance $= 4+4+4 = 12$ m. $\therefore$ average speed $= \dfrac{12}{4} = 3$ m/s.



Example 4 — unscaffolded

A particle has position $x(t) = t^2 + 2t$ metres. Find its average velocity over the interval $[t, t+h]$, simplify, and hence state its instantaneous velocity at $t = 3$.

$$\frac{x(t+h) - x(t)}{h} = \frac{\left[(t+h)^2 + 2(t+h)\right] - \left[t^2 + 2t\right]}{h} = \frac{2th + h^2 + 2h}{h} = 2t + h + 2.$$

As $h \to 0$, this approaches $2t + 2$. $\therefore$ the instantaneous velocity at $t = 3$ is $2(3) + 2 = 8$ m/s.

Practice questions

Scaffolded

Q1. For $x(t) = 3t + 2$: (a) find $x(0)$ and $x(4)$; (b) find the displacement over $[0, 4]$; (c) find the average velocity.

Q2. For $x(t) = t^2 - 2t$: (a) find $x(1)$ and $x(4)$; (b) find the displacement over $[1, 4]$; (c) find the average velocity.

Q3. For $x(t) = 10t - t^2$: (a) find $x(2)$ and $x(5)$; (b) find the average velocity over $[2, 5]$.

Q4. For $x(t) = t^2 - 6t + 5$: (a) find the times when $x = 0$; (b) find the average velocity over $[0, 6]$, and explain your answer.

Q5. A particle has $x(t) = t^2 - 4t$ and turns around at $t = 2$. Over $[0, 4]$ find (a) the displacement and (b) the distance travelled.

Q6. For $x(t) = t^2 + t$: (a) find the average velocity over $[2, 2+h]$ and simplify; (b) hence state the instantaneous velocity at $t = 2$.

Unscaffolded

Positions are in metres and times in seconds.

Q7. For $x(t) = 5t - 3$, find the average velocity over $[0, 4]$.

Q8. For $x(t) = t^2 + 3t$, find the average velocity over $[1, 4]$.

Q9. For $x(t) = t^2 - 5t + 6$, find the displacement and the average velocity over $[0, 3]$.

Q10. For $x(t) = 8t - t^2$, find the average velocity over (a) $[0, 4]$ and (b) $[0, 8]$.

Q11. For $x(t)=t^3-3t^2$, find the average velocity over $[0,3]$.

Q12. A particle has $x(t)=t^2-6t$ and turns around at $t=3$. Over $[0,5]$ find the displacement, the distance travelled, and the average speed.

Q13. A particle has $x(t)=t^2-2t$ and turns around at $t=1$. Over $[0,3]$ find the displacement, the average velocity, and the distance travelled.

Q14. For $x(t)=3t^2$, find the average velocity over $[t,t+h]$, simplify, and hence find the instantaneous velocity at $t=2$.

Q15. For $x(t)=t^2+4t$, find the average velocity over $[1,1+h]$ and simplify.

Q16. Particle A has $x_A(t)=2t+1$ and particle B has $x_B(t)=t^2-t+3$. (a) Find the times when they are at the same position. (b) Find and compare their average velocities over $[0,3]$.

Q17. A ball is thrown up so that its height is $h(t)=15t-5t^2$ metres. Find the average velocity over (a) $[0,1]$ and (b) $[0,3]$, and (c) the greatest height reached.

Q18. For $x(t)=t^2-t$, find the average velocity over $[a,a+h]$ and simplify; hence find the instantaneous velocity at $t=a$ and the time when it is zero.

Answers

Q1. $x(0)=2$, $x(4)=14$; displacement 12 m; average velocity 3 m/s. **Q2.** $x(1)=-1$, $x(4)=8$; displacement 9 m; average velocity 3 m/s. **Q3.** $x(2)=16$, $x(5)=25$; average velocity 3 m/s. **Q4.** $t=1$ and $t=5$; average velocity 0 (the particle returns to its starting position). **Q5.** Displacement 0 m; distance travelled 8 m. **Q6.** $2t+1+h$; instantaneous velocity 5 m/s. **Q7.** 5 m/s. **Q8.** 8 m/s. **Q9.** Displacement -6 m; average velocity -2 m/s. **Q10.** (a) 4 m/s; (b) 0 m/s. **Q11.** 0 m/s. **Q12.** Displacement -5 m; distance 13 m; average speed 2.6 m/s. **Q13.** Displacement 3 m; average velocity 1 m/s; distance 5 m. **Q14.** $6t+3h$; instantaneous velocity 12 m/s. **Q15.** $6+h$. **Q16.** (a) $t=1$ and $t=2$; (b) both average velocities are 2 m/s. **Q17.** (a) 10 m/s; (b) 0 m/s; (c) 11.25 m (at $t=1.5$). **Q18.** $2a-1+h$; instantaneous velocity $2a-1$; zero at $t=1/2$.

Fully-worked solutions

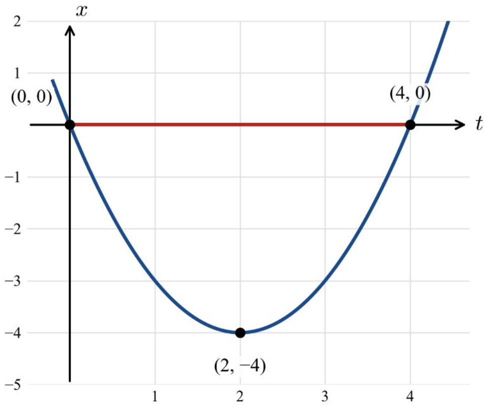
Q1. $x(0)=2$, $x(4)=3(4)+2=14$. Displacement $=14-2=12$ m. Average velocity $=\frac{12}{4}=3$ m/s (constant, since the graph is a straight line).

Q2. $x(1)=1-2=-1$, $x(4)=16-8=8$. Displacement $=8-(-1)=9$ m. Average velocity $=\frac{9}{3}=3$ m/s.

Q3. $x(2)=20-4=16$, $x(5)=50-25=25$. Average velocity $=\frac{25-16}{5-2}=\frac{9}{3}=3$ m/s.

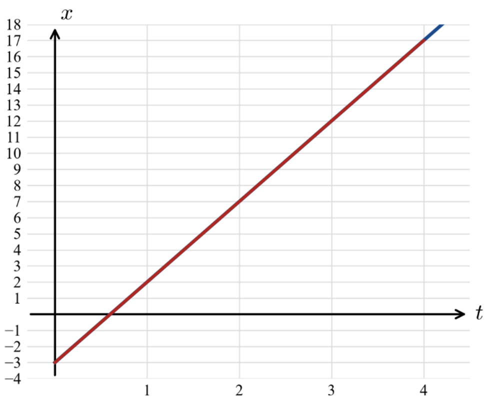
Q4. (a) $t^2-6t+5=(t-1)(t-5)=0 \Rightarrow t=1$ or $t=5$. (b) $x(0)=5$ and $x(6)=36-36+5=5$, so displacement $=0$ and average velocity $=0$ — the particle finishes where it started.

Q5. $x(0)=0$, $x(2)=4-8=-4$, $x(4)=16-16=0$. (a) Displacement $=0$ m. (b) $0 \to -4$ (distance 4) then $-4 \to 0$ (distance 4): total distance $=8$ m.

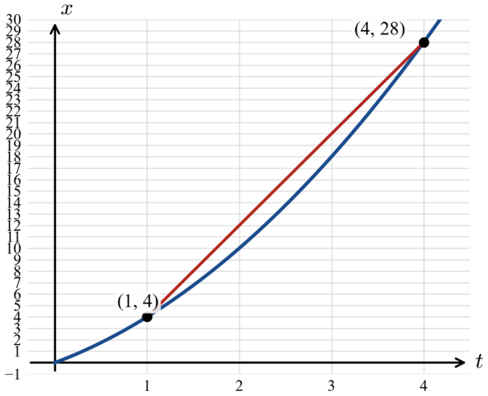


Q6. $\frac{x(2+h)-x(2)}{h}=\frac{\left[(2+h)^2+(2+h)\right]-6}{h}=\frac{5h+h^2}{h}=2t+1+h$ (with $t=2$, i.e. $5+h$). As $h \to 0$, the instantaneous velocity is 5 m/s.

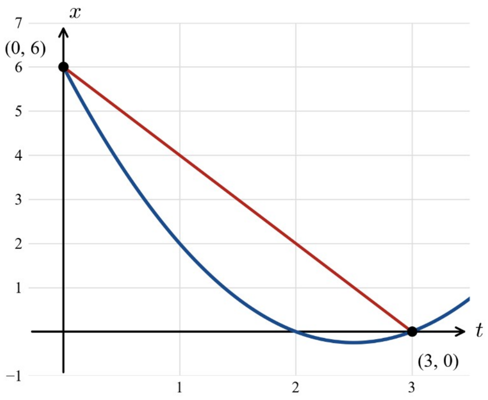
Q7. Average velocity $=\frac{x(4)-x(0)}{4}=\frac{17-(-3)}{4}=\frac{20}{4}=5$ m/s.



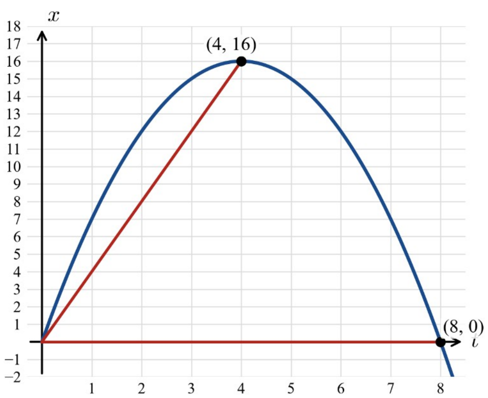
Q8. $x(1)=4$, $x(4)=16+12=28$. Average velocity $=\dfrac{28-4}{3}=8$ m/s.



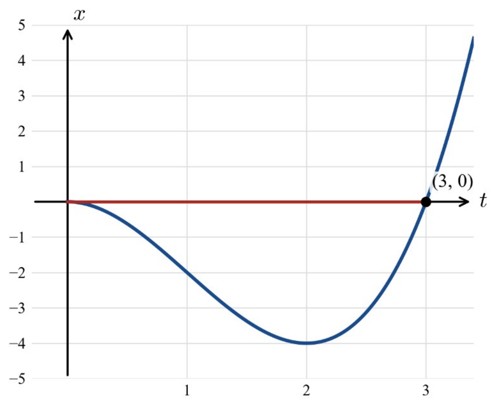
Q9. $x(0)=6$, $x(3)=9-15+6=0$. Displacement $=0-6=-6$ m; average velocity $=\dfrac{-6}{3}=-2$ m/s.



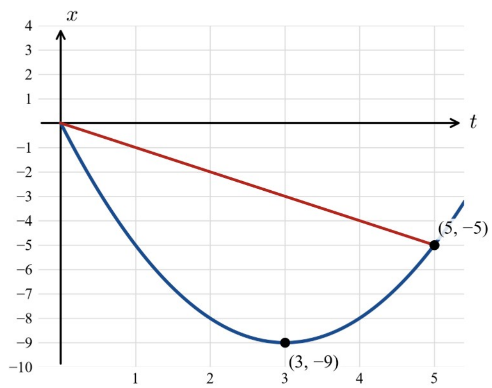
Q10. $x(0)=0$, $x(4)=32-16=16$, $x(8)=64-64=0$. (a) $\dfrac{16}{4}=4$ m/s; (b) $\dfrac{0}{8}=0$ m/s.



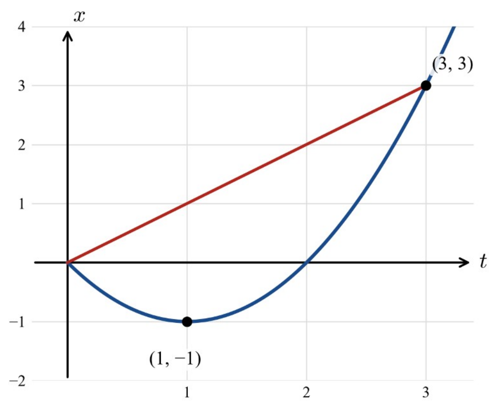
Q11. $x(0)=0$, $x(3)=27-27=0$. Average velocity $=\frac{0}{3}=0$ m/s.



Q12. $x(0)=0$, $x(3)=9-18=-9$, $x(5)=25-30=-5$. Displacement $=-5-0=-5$ m. Distance: $0 \to -9$ (distance 9) then $-9 \to -5$ (distance 4), total 13 m. Average speed $=\frac{13}{5}=2.6$ m/s.



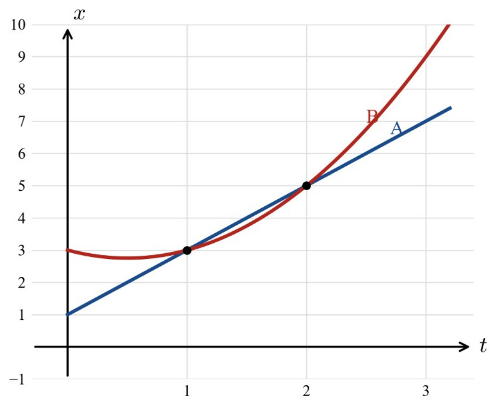
Q13. $x(0)=0$, $x(1)=-1$, $x(3)=3$. Displacement $=3$ m; average velocity $=1$ m/s. Distance: $0 \to -1$ (distance 1) then $-1 \to 3$ (distance 4), total 5 m.



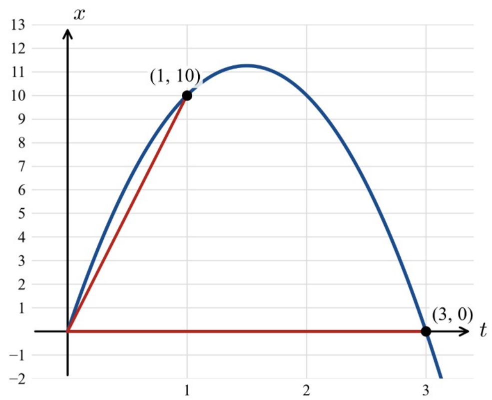
Q14. $\frac{3(t+h)^2-3t^2}{h}=\frac{6th+3h^2}{h}=6t+3h$. As $h \to 0$, the instantaneous velocity is $6t$; at $t=2$ it is 12 m/s.

Q15. $\frac{x(1+h)-x(1)}{h}=\frac{\left[(1+h)^2+4(1+h)\right]-5}{h}=\frac{6h+h^2}{h}=6+h$.

Q16. (a) Same position when $2t+1=t^2-t+3 \Rightarrow t^2-3t+2=0 \Rightarrow (t-1)(t-2)=0$, so $t=1$ or $t=2$. (b) A: $\frac{x_A(3)-x_A(0)}{3}=\frac{7-1}{3}=2$ m/s; B: $\frac{x_B(3)-x_B(0)}{3}=\frac{9-3}{3}=2$ m/s. $\therefore$ they have the same average velocity over $[0,3]$.



Q17. $h(0)=0$, $h(1)=10$, $h(3)=45-45=0$. (a) $\frac{10}{1}=10$ m/s; (b) $\frac{0}{3}=0$ m/s (it lands back at ground level). (c) The maximum is at the turning point $t=\frac{15}{2\times 5}=1.5$; $h(1.5)=22.5-11.25=11.25$ m.



Q18. $\frac{x(a+h)-x(a)}{h}=\frac{\left[(a+h)^2-(a+h)\right]-\left[a^2-a\right]}{h}=\frac{2ah+h^2-h}{h}=2a-1+h$. As $h\to 0$, the instantaneous velocity is $2a-1$. This is zero when $2a-1=0$, i.e. $t=\frac{1}{2}$ s.