Mathematical Methods Units 1 & 2

Chapter 12 — Circular functions

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Significant effort has been made to ensure this content is legal. Please send any areas of concern to admin@vwx.au with appropriate document/legal references so errors can be verified and changes be made.

Contents

Theory

Angles can be measured in **degrees** or in **radians**. Radians are the natural unit for the circular functions and calculus.

The radian. One **radian** is the angle subtended at the centre of a circle by an arc whose length equals the radius. Since the full circumference is $2\pi r$, a full revolution is 2π radians. Therefore

$$360° = 2\pi \text{ radians}, \text{so } 180° = \pi \text{ radians}.$$

Converting between units. From $180° = \pi$ radians:

- **degrees → radians:** multiply by $\frac{\pi}{180}$;
- **radians → degrees:** multiply by $\frac{180}{\pi}$.

In Mathematical Methods, angles in radians are usually left as **exact multiples of** π. Some common exact values:

Degrees	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$180°$	$270°$	$360°$
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

Arc length. For an angle of θ **radians** at the centre of a circle of radius r, the length of the arc is

$$\ell = r\theta.$$

The formula $\ell = r\theta$ requires θ in radians — convert first if the angle is given in degrees.

Worked examples

Example 1 — scaffolded

Convert $135°$ to radians, giving an exact answer.

Working	Comment
Multiply by $\frac{\pi}{180}$: $135° = 135 \times \frac{\pi}{180}$.	Degrees → radians uses the factor $\frac{\pi}{180}$.
$= \frac{135\pi}{180} = \frac{3\pi}{4}$.	Simplify the fraction (135 and 180 share a factor of 45).
$\therefore 135° = \frac{3\pi}{4}$ radians.	

Example 2 — scaffolded

Convert $\frac{5\pi}{6}$ radians to degrees.

Working	Comment
Multiply by $\frac{180}{\pi}$: $\frac{5\pi}{6} \times \frac{180}{\pi}$.	Radians → degrees uses the factor $\frac{180}{\pi}$.
$= \frac{5 \times 180}{6} = \frac{900}{6} = 150$.	The π cancels.

$\therefore \frac{5\pi}{6} = 150°$.

Example 3 — unscaffolded

A circle has radius 12 cm. Find the exact length of the arc subtended by an angle of $\frac{\pi}{3}$ radians at the centre.

The angle is already in radians, so $\ell = r\theta = 12 \times \frac{\pi}{3} = \frac{12\pi}{3} = 4\pi$.

$\therefore$ the arc length is 4π cm.

Example 4 — unscaffolded

Find the exact length of the arc subtended by an angle of $75°$ at the centre of a circle of radius 24 cm.

First convert the angle to radians: $75° = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$. Then $\ell = r\theta = 24 \times \frac{5\pi}{12} = 10\pi$.

$\therefore$ the arc length is 10π cm.

Practice questions

Scaffolded

Q1. Convert to radians, giving exact answers: (a) $60°$; (b) $90°$; (c) $45°$.

Q2. Convert to radians, giving exact answers: (a) $120°$; (b) $210°$; (c) $300°$.

Q3. Convert to degrees: (a) $\frac{\pi}{6}$; (b) $\frac{3\pi}{4}$; (c) $\frac{5\pi}{3}$.

Q4. Convert to degrees: (a) $\frac{7\pi}{6}$; (b) $\frac{5\pi}{4}$; (c) $\frac{11\pi}{6}$.

Q5. A circle has radius 9 cm. Find the exact arc length for a central angle of $\frac{2\pi}{3}$ radians.

Q6. A circle has radius 10 cm. Find the exact arc length for a central angle of $\frac{\pi}{4}$ radians.

Unscaffolded

Q7. Convert $150°$ to radians (exact).

Q8. Convert $240°$ to radians (exact).

Q9. Convert $15°$ to radians (exact).

Q10. Convert $100°$ to radians (exact).

Q11. Convert $\frac{2\pi}{9}$ to degrees.

Q12. Convert $\frac{7\pi}{4}$ to degrees.

Q13. Convert $\frac{3\pi}{10}$ to degrees.

Q14. Find the exact arc length subtended by an angle of $\frac{\pi}{5}$ at the centre of a circle of radius 15 cm.

Q15. Find the arc length subtended by an angle of 1.5 radians in a circle of radius 20 cm.

Q16. Find the exact arc length subtended by an angle of $225°$ at the centre of a circle of radius 12 cm.

Q17. A sector of a circle of radius 12 cm has an arc length of 8π cm. Find (a) the angle of the sector in radians, and (b) the angle of the sector in degrees.

Q18. The minute hand of a clock is 14 cm long. Find the exact distance travelled by the tip of the hand in 20 minutes.

Answers

Q1. (a) $\frac{\pi}{3}$; (b) $\frac{\pi}{2}$; (c) $\frac{\pi}{4}$. **Q2.** (a) $\frac{2\pi}{3}$; (b) $\frac{7\pi}{6}$; (c) $\frac{5\pi}{3}$. **Q3.** (a) $30°$; (b) $135°$; (c) $300°$. **Q4.** (a) $210°$; (b) $225°$; (c) $330°$. **Q5.** 6π cm. **Q6.** $\frac{5\pi}{2}$ cm. **Q7.** $\frac{5\pi}{6}$. **Q8.** $\frac{4\pi}{3}$. **Q9.** $\frac{\pi}{12}$. **Q10.** $\frac{5\pi}{9}$. **Q11.** $40°$. **Q12.** $315°$. **Q13.** $54°$. **Q14.** 3π cm. **Q15.** 30 cm. **Q16.** 15π cm. **Q17.** (a) $\frac{2\pi}{3}$; (b) $120°$. **Q18.** $\frac{28\pi}{3}$ cm.

Fully-worked solutions

Q1. Multiply each by $\frac{\pi}{180}$. (a) $60\times\frac{\pi}{180}=\frac{\pi}{3}$; (b) $90\times\frac{\pi}{180}=\frac{\pi}{2}$; (c) $45\times\frac{\pi}{180}=\frac{\pi}{4}$.

Q2. (a) $120\times\frac{\pi}{180}=\frac{2\pi}{3}$; (b) $210\times\frac{\pi}{180}=\frac{7\pi}{6}$; (c) $300\times\frac{\pi}{180}=\frac{5\pi}{3}$.

Q3. Multiply each by $\frac{180}{\pi}$. (a) $\frac{\pi}{6}\times\frac{180}{\pi}=30°$; (b) $\frac{3\pi}{4}\times\frac{180}{\pi}=135°$; (c) $\frac{5\pi}{3}\times\frac{180}{\pi}=300°$.

Q4. (a) $\frac{7\pi}{6}\times\frac{180}{\pi}=210°$; (b) $\frac{5\pi}{4}\times\frac{180}{\pi}=225°$; (c) $\frac{11\pi}{6}\times\frac{180}{\pi}=330°$.

Q5. $\ell=r\theta=9\times\frac{2\pi}{3}=\frac{18\pi}{3}=6\pi$ cm.

Q6. $\ell=r\theta=10\times\frac{\pi}{4}=\frac{10\pi}{4}=\frac{5\pi}{2}$ cm.

Q7. $150\times\frac{\pi}{180}=\frac{5\pi}{6}$.

Q8. $240\times\frac{\pi}{180}=\frac{4\pi}{3}$.

Q9. $15\times\frac{\pi}{180}=\frac{\pi}{12}$.

Q10. $100\times\frac{\pi}{180}=\frac{5\pi}{9}$.

Q11. $\frac{2\pi}{9}\times\frac{180}{\pi}=\frac{2\times180}{9}=40°$.

Q12. $\frac{7\pi}{4}\times\frac{180}{\pi}=\frac{7\times180}{4}=315°$.

Q13. $\frac{3\pi}{10}\times\frac{180}{\pi}=\frac{3\times180}{10}=54°$.

Q14. $\ell=r\theta=15\times\frac{\pi}{5}=\frac{15\pi}{5}=3\pi$ cm.

Q15. $\ell=r\theta=20\times1.5=30$ cm.

Q16. $225°=225\times\frac{\pi}{180}=\frac{5\pi}{4}$ radians, so $\ell=r\theta=12\times\frac{5\pi}{4}=\frac{60\pi}{4}=15\pi$ cm.

Q17. (a) From $\ell = r\theta$, $\theta = \frac{\ell}{r} = \frac{8\pi}{12} = \frac{2\pi}{3}$ radians. (b) $\frac{2\pi}{3} \times \frac{180}{\pi} = \frac{2 \times 180}{3} = 120°$.

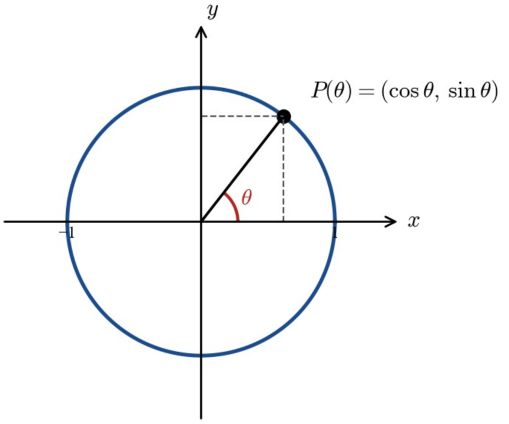
Q18. In 20 minutes the minute hand turns through $\frac{20}{60} = \frac{1}{3}$ of a full revolution, i.e. $\frac{1}{3} \times 2\pi = \frac{2\pi}{3}$ radians. The tip travels an arc of length $\ell = r\theta = 14 \times \frac{2\pi}{3} = \frac{28\pi}{3}$ cm.

Theory

The unit circle. The unit circle is the circle of radius 1 centred at the origin. For an angle θ measured **anticlockwise** from the positive x-axis, let $P(\theta)$ be the point where the rotating radius meets the circle. The **cosine** and **sine** of θ are *defined* as the coordinates of this point:

$$P(\theta)=(\cos\theta,\sin\theta).$$

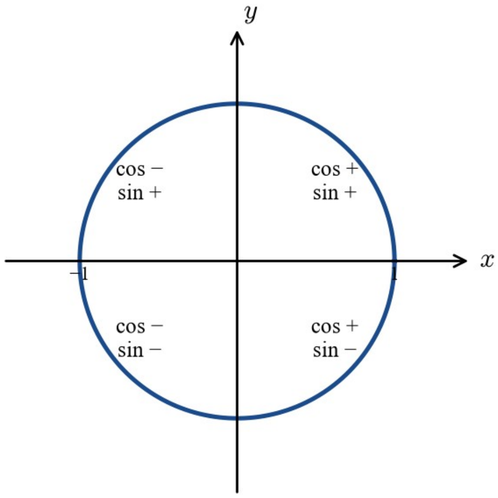
So $\cos\theta$ is the x**-coordinate** and $\sin\theta$ is the y**-coordinate** of $P(\theta)$.



Because the radius is 1, both coordinates lie between -1 and 1, so $-1\le\cos\theta\le1$ and $-1\le\sin\theta\le1$. Also, since P is on the circle $x^2+y^2=1$,

$$\cos^2\theta+\sin^2\theta=1.$$

Signs in each quadrant. The signs of $\cos\theta$ (the x-coordinate) and $\sin\theta$ (the y-coordinate) depend on which quadrant $P(\theta)$ lies in:



Values at the axes. When P lands on an axis the coordinates are easy to read:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$P(\theta)$	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$	$(1,0)$
$\cos\theta$	1	0	-1	0	1

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1	0	-1	0

Periodicity. Adding 2π (a full revolution) returns to the same point, so

$$\cos(\theta+2\pi)=\cos\theta, \sin(\theta+2\pi)=\sin\theta.$$

For an angle larger than 2π, subtract whole revolutions of 2π until $0\le\theta<2\pi$. A **negative** angle is measured **clockwise** from the positive x-axis.

Small angles (radians). When x is a *small* angle **measured in radians**, the short arc from $(1,0)$ up to $P(x)$ is almost straight and almost vertical, so its length x is very close to the height $\sin x$ of the point. Hence, for small x,

$$\sin x\approx x \text{ and } \tan x\approx x.$$

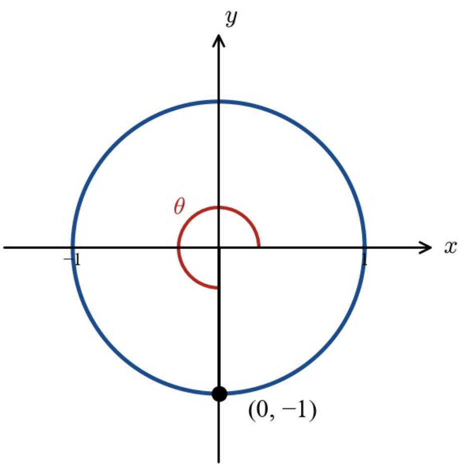
Equivalently, near $x=0$ the graph of $y=\sin x$ passes through the origin with gradient 1, so $y=\sin x$ and the line $y=x$ almost coincide there. This approximation only holds in **radians** and only for **small** x; the error grows as x moves away from 0.

Worked examples

Example 1 — scaffolded

Find $\cos\frac{3\pi}{2}$ and $\sin\frac{3\pi}{2}$ using the unit circle.

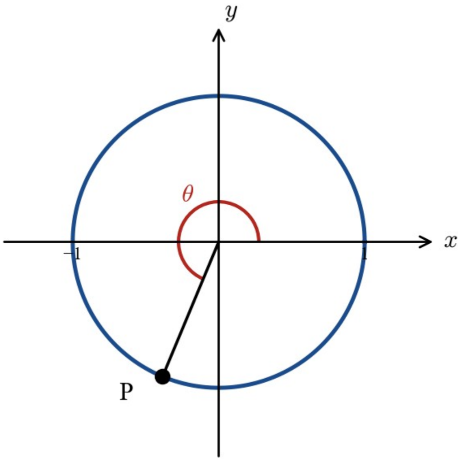
Working	Comment
$\frac{3\pi}{2}$ is three-quarters of a full revolution (2π) anticlockwise, so P is at the bottom of the circle: $(0,-1)$.	$\frac{3\pi}{2}=\frac{3}{4}\times2\pi$.
$\cos\frac{3\pi}{2}=x$-coordinate $=0$; $\sin\frac{3\pi}{2}=y$-coordinate $=-1$.	$\cos=x$, $\sin=y$ on the unit circle.



Example 2 — scaffolded

The point $P(\theta)$ lies in the **third quadrant** and $\cos\theta=-\frac{5}{13}$. Find $\sin\theta$.

Working	Comment
In the third quadrant both coordinates are negative, so $\sin\theta<0$.	P is left of and below the origin.
$\cos^2\theta+\sin^2\theta=1 \Rightarrow \left(-\frac{5}{13}\right)^2+\sin^2\theta=1$.	P is on the unit circle.
$\sin^2\theta=1-\frac{25}{169}=\frac{144}{169}$, so $\sin\theta=\pm\frac{12}{13}$.	Take the square root.
Since $\sin\theta<0$ in the third quadrant, $\sin\theta=-\frac{12}{13}$.	Choose the sign from the quadrant.



Example 3 — unscaffolded

Find the exact value of (a) $\cos 2\pi$, (b) $\sin\frac{5\pi}{2}$, (c) $\cos(-\pi)$.

(a) 2π is one full revolution, returning to $(1,0)$, so $\cos 2\pi=1$.

(b) $\frac{5\pi}{2}=2\pi+\frac{\pi}{2}$, so P is the same as for $\frac{\pi}{2}$, namely $(0,1)$; thus $\sin\frac{5\pi}{2}=1$.

(c) $-\pi$ is half a revolution **clockwise**, landing at $(-1,0)$, so $\cos(-\pi)=-1$.

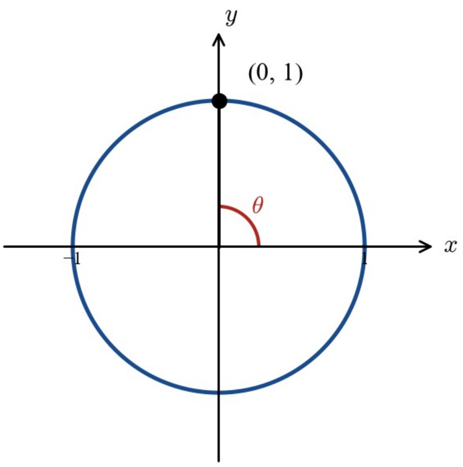
$\therefore \cos 2\pi=1, \sin\frac{5\pi}{2}=1, \cos(-\pi)=-1$.

Example 4 — unscaffolded

The point $P(\theta)$ on the unit circle has $\sin\theta=1$. For $0\le\theta\le 2\pi$, find θ and state $\cos\theta$.

$\sin\theta$ is the y-coordinate, so $\sin\theta=1$ means $P=(0,1)$, the top of the circle.

$\therefore \theta=\frac{\pi}{2}$ and $\cos\theta=0$.



Practice questions

Scaffolded

Q1. For $\theta = 0$: (a) state $P(\theta)$; (b) write $\cos 0$ and $\sin 0$.

Q2. For $\theta = \pi$: (a) state $P(\theta)$; (b) write $\cos \pi$ and $\sin \pi$.

Q3. The point $P(\theta)$ lies in the fourth quadrant and $\sin\theta = -\frac{3}{5}$. (a) State the sign of $\cos\theta$. (b) Find $\cos\theta$.

Q4. Use revolutions to find $\cos 3\pi$ and $\sin 3\pi$.

Q5. The point $P(\theta)$ lies in the first quadrant and $\cos\theta = \frac{7}{25}$. Find $\sin\theta$.

Q6. State the quadrant in which $P(\theta)$ lies for (a) $\theta = \frac{2\pi}{3}$, (b) $\theta = \frac{4\pi}{3}$, (c) $\theta = \frac{7\pi}{6}$, (d) $\theta = \frac{11\pi}{6}$.

Unscaffolded

Q7. Find $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$.

Q8. Find $\cos(-2\pi)$ and $\sin(-2\pi)$.

Q9. State the quadrant in which $P(\theta)$ lies for $\theta = \frac{5\pi}{6}$.

Q10. State the quadrant in which $P(\theta)$ lies for $\theta = \frac{7\pi}{4}$.

Q11. $P(\theta)$ lies in the second quadrant and $\sin\theta = \frac{8}{17}$. Find $\cos\theta$.

Q12. $P(\theta)$ lies in the third quadrant and $\cos\theta = -\frac{4}{5}$. Find $\sin\theta$.

Q13. Use revolutions to find $\sin\frac{7\pi}{2}$.

Q14. Find $\cos 4\pi$.

Q15. Find $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$.

Q16. Find $\sin\left(-\frac{3\pi}{2}\right)$.

Q17. For $0 \le \theta \le 2\pi$, $\sin\theta = 0$ and $\cos\theta = -1$. State θ.

Q18. $\cos\theta = \frac{3}{5}$ and $P(\theta)$ lies in the fourth quadrant. Find $\sin\theta$, and state whether $P(\theta)$ is above or below the x-axis.

Q19. The small-angle approximation says $\sin x \approx x$ for small x in radians. Test it numerically. (a) Write $\sin(0.05)$ to 6 decimal places and state the error (as a percentage) in using 0.05 in its place. (b) Write $\sin(0.1)$ to 6 decimal places and state the error in using 0.1. (c) State what happens to the accuracy of the approximation as x increases.

Answers

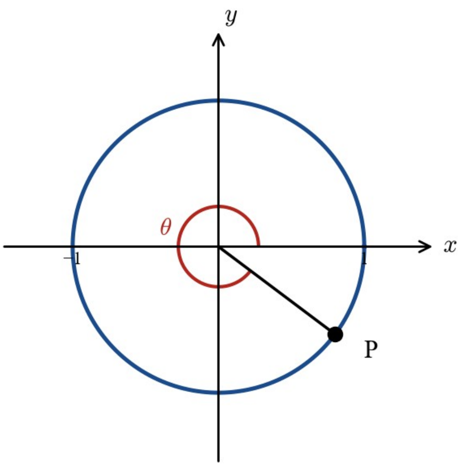
Q1. $(1,0)$; $\cos 0=1$, $\sin 0=0$. **Q2.** $(-1,0)$; $\cos\pi=-1$, $\sin\pi=0$. **Q3.** $\cos\theta>0$; $\cos\theta=\frac{4}{5}$. **Q4.** $\cos 3\pi=-1$, $\sin 3\pi=0$. **Q5.** $\sin\theta=\frac{24}{25}$. **Q6.** (a) Q2; (b) Q3; (c) Q3; (d) Q4. **Q7.** $\cos\frac{\pi}{2}=0$, $\sin\frac{\pi}{2}=1$. **Q8.** $\cos(-2\pi)=1$, $\sin(-2\pi)=0$. **Q9.** Second quadrant. **Q10.** Fourth quadrant. **Q11.** $\cos\theta=-\frac{15}{17}$. **Q12.** $\sin\theta=-\frac{3}{5}$. **Q13.** -1. **Q14.** 1. **Q15.** $\cos\left(-\frac{\pi}{2}\right)=0$, $\sin\left(-\frac{\pi}{2}\right)=-1$. **Q16.** 1. **Q17.** $\theta=\pi$. **Q18.** $\sin\theta=-\frac{4}{5}$; below the x-axis. **Q19.** (a) $\sin(0.05)=0.049979$; error $\approx 0.04\%$. (b) $\sin(0.1)=0.099833$; error $\approx 0.17\%$. (c) The approximation becomes less accurate (the error grows) as x increases.

Fully-worked solutions

Q1. $\theta=0$ gives $P=(1,0)$, so $\cos 0=1$ and $\sin 0=0$.

Q2. $\theta=\pi$ is half a revolution, giving $P=(-1,0)$, so $\cos\pi=-1$ and $\sin\pi=0$.

Q3. (a) In the fourth quadrant $\cos\theta$ (the x-coordinate) is **positive**. (b) $\cos^2\theta=1-\left(-\frac{3}{5}\right)^2=1-\frac{9}{25}=\frac{16}{25}$, so $\cos\theta=\pm\frac{4}{5}$; positive here, so $\cos\theta=\frac{4}{5}$.



Q4. $3\pi=2\pi+\pi$, so P is the same as for π, namely $(-1,0)$. Thus $\cos 3\pi=-1$ and $\sin 3\pi=0$.

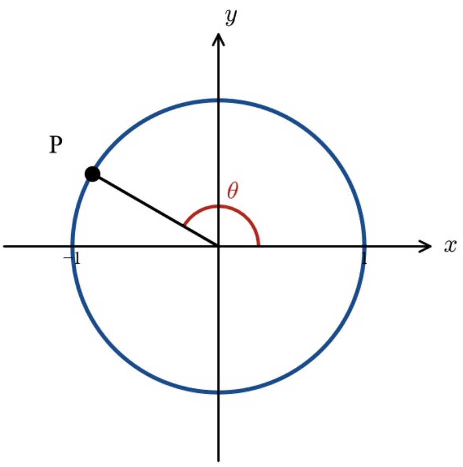
Q5. $\sin^2\theta=1-\left(\frac{7}{25}\right)^2=1-\frac{49}{625}=\frac{576}{625}$, so $\sin\theta=\pm\frac{24}{25}$; in the first quadrant $\sin\theta>0$, so $\sin\theta=\frac{24}{25}$.

Q6. Converting to degrees (or comparing with the axis angles): (a) $\frac{2\pi}{3}=120°\rightarrow$ **Q2**; (b) $\frac{4\pi}{3}=240°\rightarrow$ **Q3**; (c) $\frac{7\pi}{6}=210°\rightarrow$ **Q3**; (d) $\frac{11\pi}{6}=330°\rightarrow$ **Q4**.

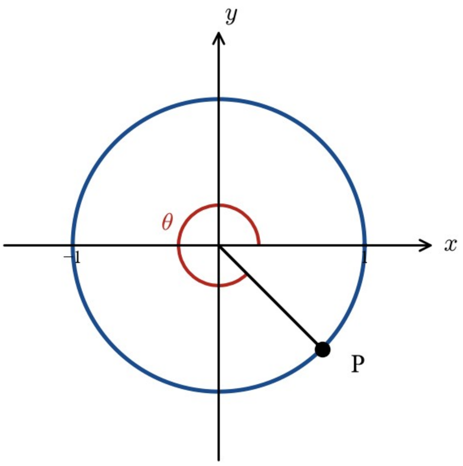
Q7. $\theta=\frac{\pi}{2}$ gives $P=(0,1)$, so $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$.

Q8. -2π is one full revolution **clockwise**, giving $P=(1,0)$, so $\cos(-2\pi)=1$ and $\sin(-2\pi)=0$.

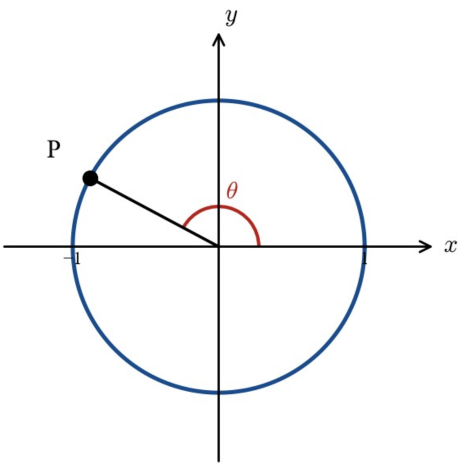
Q9. $\frac{5\pi}{6} = 150°$, which is between $90°$ and $180°$, so $P(\theta)$ is in the **second quadrant**.



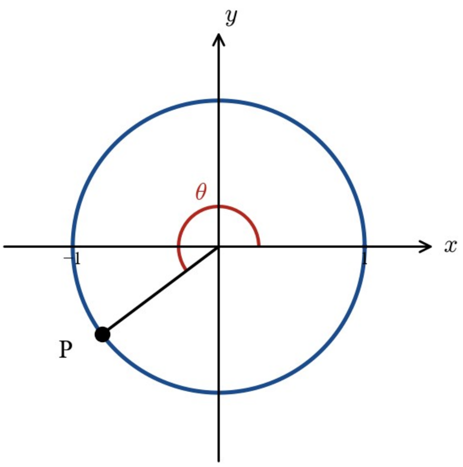
Q10. $\frac{7\pi}{4} = 315°$, which is between $270°$ and $360°$, so $P(\theta)$ is in the **fourth quadrant**.



Q11. $\cos^2\theta = 1 - \left(\frac{8}{17}\right)^2 = 1 - \frac{64}{289} = \frac{225}{289}$, so $\cos\theta = \pm\frac{15}{17}$; in the second quadrant $\cos\theta < 0$, so $\cos\theta = -\frac{15}{17}$.



Q12. $\sin^2\theta = 1 - \left(-\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}$, so $\sin\theta = \pm\frac{3}{5}$; in the third quadrant $\sin\theta < 0$, so $\sin\theta = -\frac{3}{5}$.



Q13. $\frac{7\pi}{2}=2\pi+\frac{3\pi}{2}$, so P is the same as for $\frac{3\pi}{2}$, namely $(0,-1)$; thus $\sin\frac{7\pi}{2}=-1$.

Q14. $4\pi=2\times2\pi$ is two full revolutions, returning to $(1,0)$, so $\cos4\pi=1$.

Q15. $-\frac{\pi}{2}$ is a quarter revolution **clockwise**, landing at $(0,-1)$, so $\cos\left(-\frac{\pi}{2}\right)=0$ and $\sin\left(-\frac{\pi}{2}\right)=-1$.

Q16. $-\frac{3\pi}{2}$ is three-quarters of a revolution clockwise, landing at $(0,1)$, so $\sin\left(-\frac{3\pi}{2}\right)=1$.

Q17. $\cos\theta=-1$ and $\sin\theta=0$ means $P=(-1,0)$, so $\theta=\pi$.

Q18. $\sin^2\theta=1-\left(\frac{3}{5}\right)^2=1-\frac{9}{25}=\frac{16}{25}$, so $\sin\theta=\pm\frac{4}{5}$; in the fourth quadrant $\sin\theta<0$, so $\sin\theta=-\frac{4}{5}$. Since the y-coordinate is negative, $P(\theta)$ is **below** the x-axis.

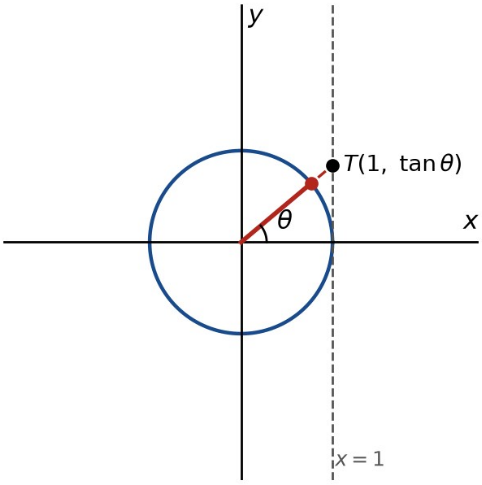
Q19. Work in radians throughout. (a) $\sin(0.05)=0.0499792$ (to 6 d.p.). The absolute error in replacing it by 0.05 is $0.05-0.0499792=0.0000208$, so the percentage error is $\frac{0.0000208}{0.05}\times100\%\approx0.04\%$. (b) $\sin(0.1)=0.0998334$ (to 6 d.p.). The absolute error in replacing it by 0.1 is $0.1-0.0998334=0.0001666$, so the percentage error is $\frac{0.0001666}{0.1}\times100\%\approx0.17\%$. (c) The estimate 0.1 is further out than 0.05 (about 0.17% vs 0.04%), so the approximation $\sin x\approx x$ becomes **less accurate as x increases** — it is only reliable for small x (the error grows roughly like $x^3/6$).

Theory

The **tangent** function is defined in terms of sine and cosine:

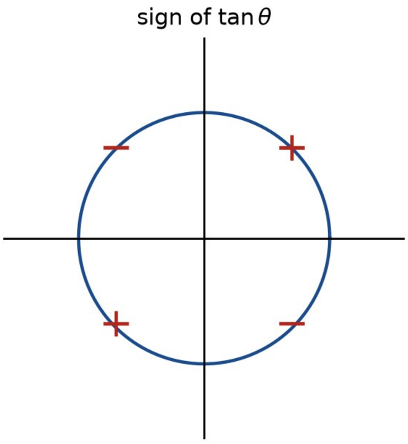
$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Unit-circle construction. For an angle θ, the point $P(\cos\theta, \sin\theta)$ lies on the unit circle. Draw the line through the origin O and P (extended if necessary); it meets the **vertical tangent line** $x=1$ at the point $T(1, \tan\theta)$. So $\tan\theta$ is the y-coordinate of that intersection.



Where tangent is undefined. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, it is **undefined** whenever $\cos\theta = 0$ — that is, at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, i.e. $\theta = \frac{\pi}{2} + n\pi$ for integer n. There the line OP is vertical, parallel to $x=1$, so it never meets it.

Sign by quadrant. $\tan\theta$ is **positive** in the 1st and 3rd quadrants (where $\sin$ and $\cos$ have the same sign) and **negative** in the 2nd and 4th. This is the **T** of the **ASTC** rule (in the 3rd quadrant only **T**an is positive).



Exact values at common angles:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Reference angle. To find $\tan\theta$ for any θ: find the reference angle (the acute angle between the terminal side and the x-axis), take the tangent of that, then attach the sign for the quadrant.

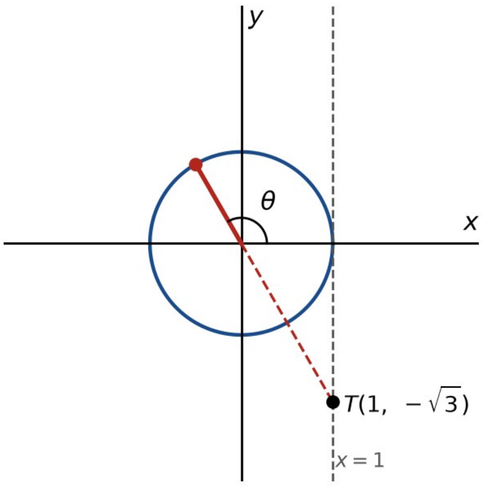
Period. Adding π to an angle gives the same line OP, so $\tan(\theta+\pi)=\tan\theta$. The tangent function has **period** π.

Worked examples

Example 1 — scaffolded

Find the exact value of $\tan\frac{2\pi}{3}$.

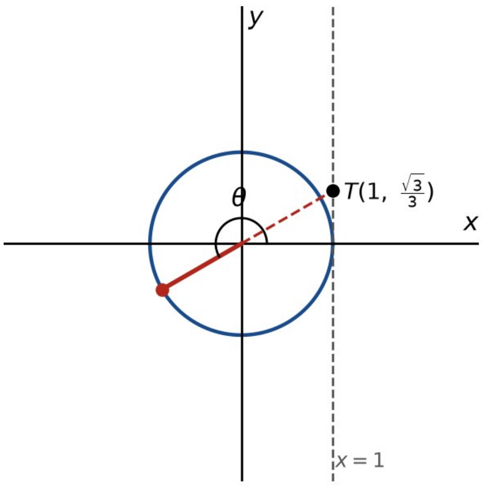
Working	Comment
$\frac{2\pi}{3}$ lies in the **2nd quadrant** (between $\frac{\pi}{2}$ and π).	Locate the angle.
In the 2nd quadrant $\tan$ is **negative**.	Sign from ASTC.
Reference angle $=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$.	The acute angle to the x-axis.
$\tan\frac{2\pi}{3}=-\tan\frac{\pi}{3}=-\sqrt{3}$.	Attach the negative sign.



Example 2 — scaffolded

Find the exact value of $\tan\frac{7\pi}{6}$.

Working	Comment
$\frac{7\pi}{6}$ lies in the **3rd quadrant** (between π and $\frac{3\pi}{2}$).	Locate the angle.
In the 3rd quadrant $\tan$ is **positive**.	Sign from ASTC.
Reference angle $=\frac{7\pi}{6}-\pi=\frac{\pi}{6}$.	Subtract π.
$\tan\frac{7\pi}{6}=+\tan\frac{\pi}{6}=\frac{\sqrt{3}}{3}$.	Positive in the 3rd quadrant.

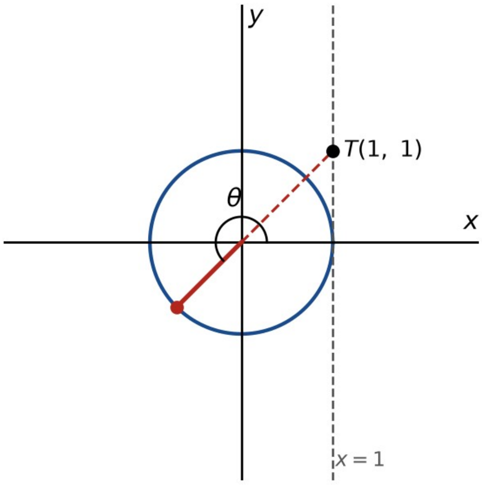


Example 3 — unscaffolded

Find the exact value of $\tan\frac{5\pi}{4}$.

$\frac{5\pi}{4}$ is in the 3rd quadrant, so $\tan$ is positive. The reference angle is $\frac{5\pi}{4}-\pi=\frac{\pi}{4}$.

$\therefore \tan\frac{5\pi}{4}=\tan\frac{\pi}{4}=1$.

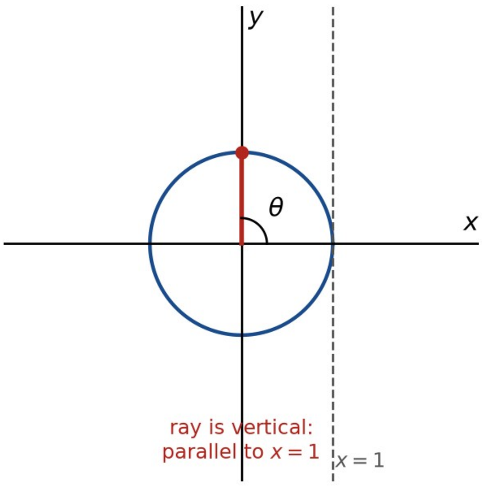


Example 4 — unscaffolded

For what values of θ in $[0,2\pi]$ is $\tan\theta$ undefined?

$\tan\theta=\frac{\sin\theta}{\cos\theta}$ is undefined where $\cos\theta=0$. In $[0,2\pi]$, $\cos\theta=0$ at $\theta=\frac{\pi}{2}$ and $\theta=\frac{3\pi}{2}$.

$\therefore \tan\theta$ is undefined at $\theta=\frac{\pi}{2}$ and $\theta=\frac{3\pi}{2}$.



Practice questions

Scaffolded

Q1. Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$: (a) state $\sin\dfrac{\pi}{6}$ and $\cos\dfrac{\pi}{6}$; (b) hence find $\tan\dfrac{\pi}{6}$.

Q2. For $\tan\dfrac{3\pi}{4}$: (a) which quadrant is $\dfrac{3\pi}{4}$ in? (b) what is the sign of $\tan$ there? (c) find the reference angle; (d) state the exact value.

Q3. Copy and complete: the sign of $\tan\theta$ is ____ in the 1st quadrant, ____ in the 2nd, ____ in the 3rd and ____ in the 4th.

Q4. Find the exact value of $\tan\dfrac{5\pi}{6}$.

Q5. Find the exact value of $\tan\dfrac{4\pi}{3}$.

Q6. State every θ in $[0, 3\pi]$ for which $\tan\theta$ is undefined, and explain why.

Unscaffolded

Find the exact value, or state that it is undefined.

Q7. $\tan\dfrac{\pi}{3}$ **Q8.** $\tan\dfrac{\pi}{4}$ **Q9.** $\tan\dfrac{11\pi}{4}$ **Q10.** $\tan\dfrac{10\pi}{3}$

Q11. $\tan\dfrac{11\pi}{6}$ **Q12.** $\tan\pi$ **Q13.** $\tan 0$ **Q14.** $\tan\dfrac{7\pi}{4}$

Q15. $\tan\dfrac{9\pi}{4}$ **Q16.** $\tan\dfrac{13\pi}{6}$ **Q17.** $\tan\dfrac{3\pi}{2}$ **Q18.** $\tan\dfrac{5\pi}{3}$

Answers

Q1. $\sin\frac{\pi}{6}=\frac{1}{2}, \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$; $\tan\frac{\pi}{6}=\frac{\sqrt{3}}{3}$. **Q2.** 2nd quadrant; negative; reference $\frac{\pi}{4}$; -1. **Q3.** $+,-,+,-$. **Q4.** $-\frac{\sqrt{3}}{3}$. **Q5.** $\sqrt{3}$. **Q6.** $\frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$ (where $\cos\theta=0$). **Q7.** $\sqrt{3}$. **Q8.** 1. **Q9.** -1. **Q10.** $\sqrt{3}$. **Q11.** $-\frac{\sqrt{3}}{3}$. **Q12.** 0. **Q13.** 0. **Q14.** -1. **Q15.** 1. **Q16.** $\frac{\sqrt{3}}{3}$. **Q17.** undefined. **Q18.** $-\sqrt{3}$.

Fully-worked solutions

Q1. (a) $\sin\frac{\pi}{6}=\frac{1}{2}$ and $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$. (b) $\tan\frac{\pi}{6}=\frac{\sin(\pi/6)}{\cos(\pi/6)}=\frac{1/2}{\sqrt{3}/2}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$.

Q2. (a) $\frac{3\pi}{4}$ is in the 2nd quadrant. (b) $\tan$ is negative there. (c) Reference angle $=\pi-\frac{3\pi}{4}=\frac{\pi}{4}$. (d) $\tan\frac{3\pi}{4}=-\tan\frac{\pi}{4}=-1$.

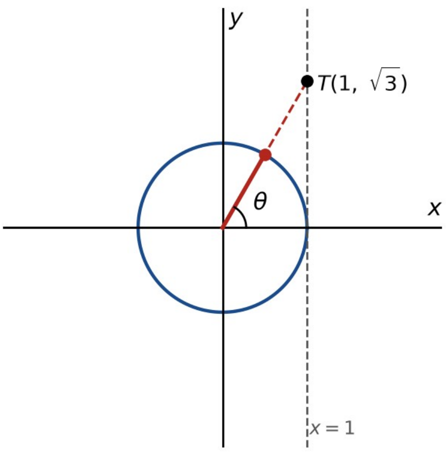
Q3. + (1st), − (2nd), + (3rd), − (4th).

Q4. $\frac{5\pi}{6}$ is in the 2nd quadrant ($\tan$ negative); reference angle $\pi-\frac{5\pi}{6}=\frac{\pi}{6}$. $\therefore \tan\frac{5\pi}{6}=-\tan\frac{\pi}{6}=-\frac{\sqrt{3}}{3}$.

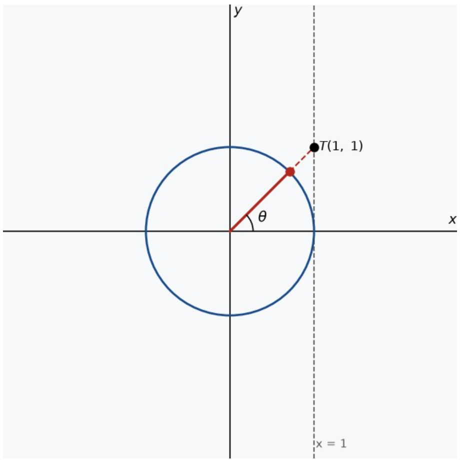
Q5. $\frac{4\pi}{3}$ is in the 3rd quadrant ($\tan$ positive); reference angle $\frac{4\pi}{3}-\pi=\frac{\pi}{3}$. $\therefore \tan\frac{4\pi}{3}=\tan\frac{\pi}{3}=\sqrt{3}$.

Q6. $\tan\theta$ is undefined where $\cos\theta=0$. Starting at $\frac{\pi}{2}$ and adding the period π each time: $\theta=\frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$ (the next value, $\frac{7\pi}{2}$, lies beyond 3π).

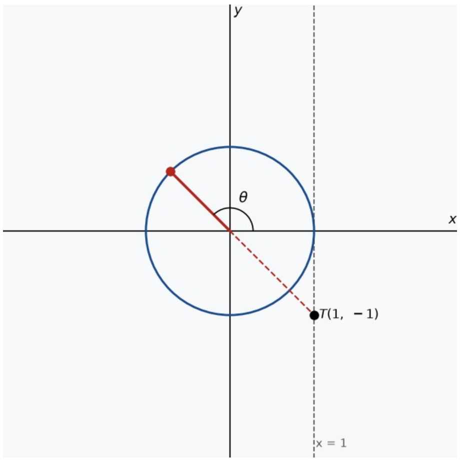
Q7. $\frac{\pi}{3}$ is in the 1st quadrant. $\tan\frac{\pi}{3}=\sqrt{3}$.



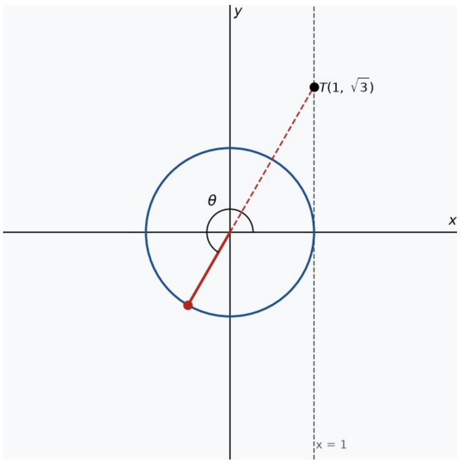
Q8. $\frac{\pi}{4}$ is in the 1st quadrant. $\tan\frac{\pi}{4}=1$.



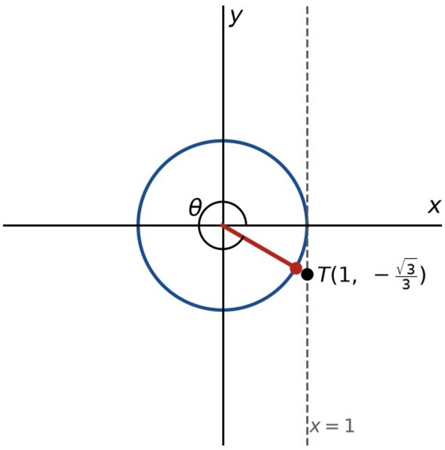
Q9. $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$: 2nd quadrant, reference $\frac{\pi}{4}$, tan negative: $\tan\frac{11\pi}{4} = -1$.



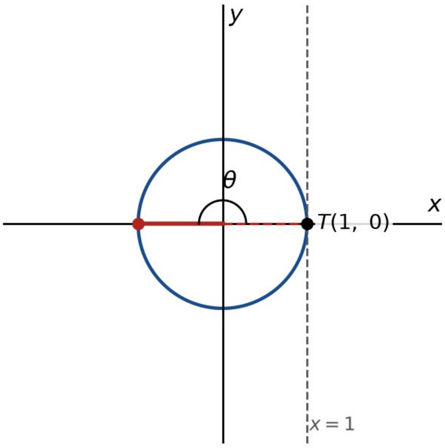
Q10. $\frac{10\pi}{3} - 2\pi = \frac{4\pi}{3}$: 3rd quadrant, reference $\frac{\pi}{3}$, tan positive: $\tan\frac{10\pi}{3} = \sqrt{3}$.



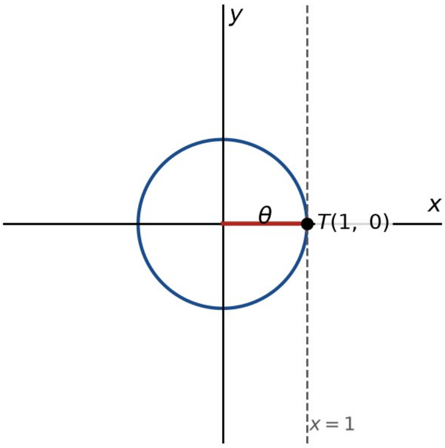
Q11. 4th quadrant, reference $\frac{\pi}{6}$, tan negative: $\tan\frac{11\pi}{6} = -\frac{\sqrt{3}}{3}$.



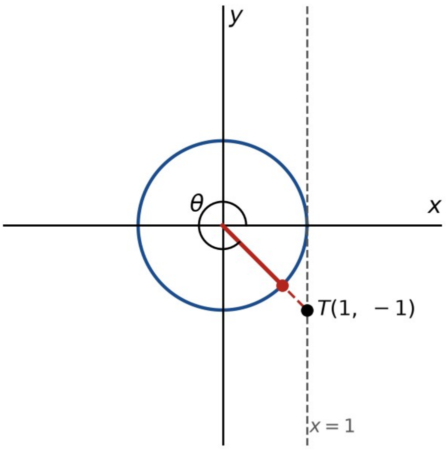
Q12. $\tan\pi = \dfrac{\sin\pi}{\cos\pi} = \dfrac{0}{-1} = 0$.



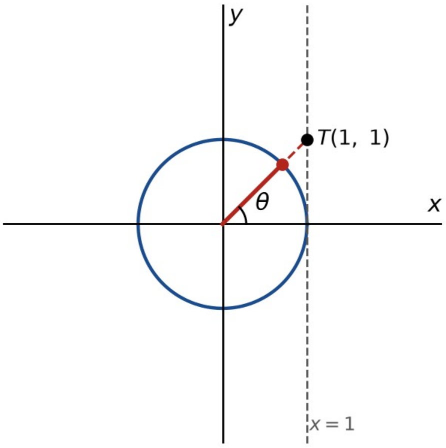
Q13. $\tan 0 = \dfrac{0}{1} = 0$.



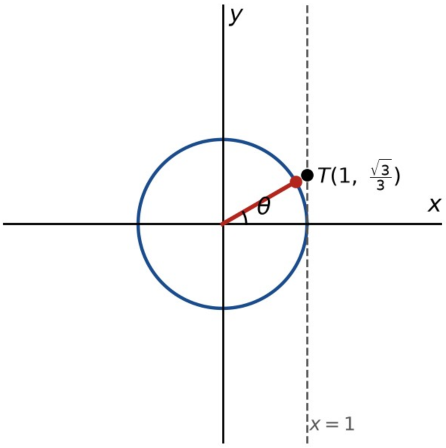
Q14. 4th quadrant, reference $\dfrac{\pi}{4}$, tan negative: $\tan\dfrac{7\pi}{4} = -1$.



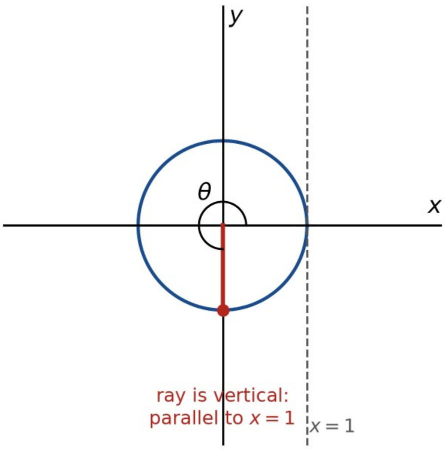
Q15. Using period π: $\dfrac{9\pi}{4}=\dfrac{\pi}{4}+2\pi$, so $\tan\dfrac{9\pi}{4}=\tan\dfrac{\pi}{4}=1$.



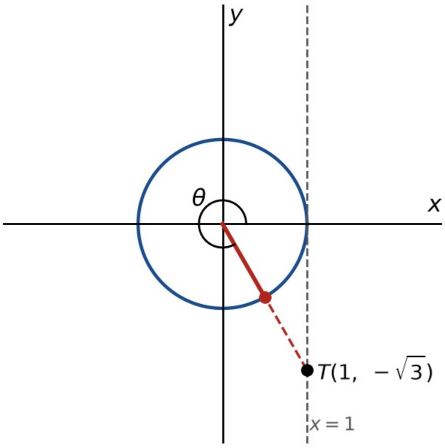
Q16. $\dfrac{13\pi}{6}=\dfrac{\pi}{6}+2\pi$, so $\tan\dfrac{13\pi}{6}=\tan\dfrac{\pi}{6}=\dfrac{\sqrt{3}}{3}$.



Q17. $\cos\dfrac{3\pi}{2}=0$, so $\tan\dfrac{3\pi}{2}$ is **undefined**.

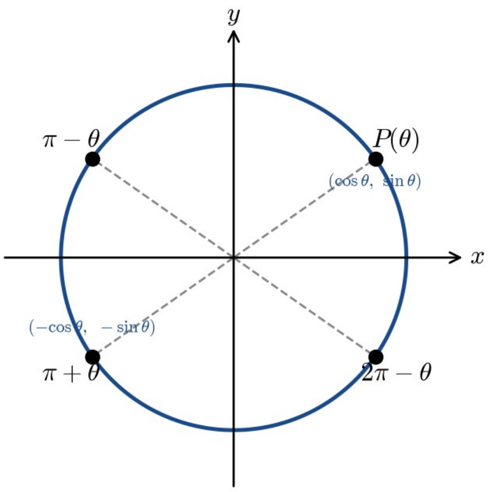


Q18. 4th quadrant, reference $\frac{\pi}{3}$, tan negative: $\tan\frac{5\pi}{3} = -\sqrt{3}$.



Theory

On the **unit circle**, the point P at an angle θ (measured anticlockwise from the positive x-axis) has coordinates $(\cos\theta, \sin\theta)$, and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Angles in the other quadrants are related to a first-quadrant angle by reflecting this point, which gives the **symmetry properties**.



Second quadrant $(\pi-\theta)$ — reflect in the y-axis:

$$\sin(\pi-\theta)=\sin\theta, \cos(\pi-\theta)=-\cos\theta, \tan(\pi-\theta)=-\tan\theta.$$

Third quadrant $(\pi+\theta)$ — reflect in the origin:

$$\sin(\pi+\theta)=-\sin\theta, \cos(\pi+\theta)=-\cos\theta, \tan(\pi+\theta)=\tan\theta.$$

Fourth quadrant $(2\pi-\theta)$ — reflect in the x-axis:

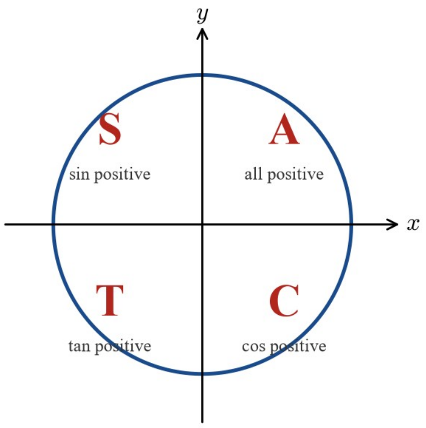
$$\sin(2\pi-\theta)=-\sin\theta, \cos(2\pi-\theta)=\cos\theta, \tan(2\pi-\theta)=-\tan\theta.$$

Negative angles — also a reflection in the x-axis, so sin and tan are **odd** and cos is **even**:

$$\sin(-\theta)=-\sin\theta, \cos(-\theta)=\cos\theta, \tan(-\theta)=-\tan\theta.$$

Complementary angles $\left(\dfrac{\pi}{2}-\theta\right)$: $\sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta$ and $\cos\left(\dfrac{\pi}{2}-\theta\right)=\sin\theta$.

The CAST diagram records which ratios are **positive** in each quadrant: **A**ll in the 1st, **S**ine in the 2nd, **T**angent in the 3rd, **C**osine in the 4th.



Reference angle. To find a ratio of any angle: (1) find the acute **reference angle** to the x-axis; (2) evaluate the ratio of the reference angle; (3) attach the sign given by the CAST diagram. The exact first-quadrant values are those of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.

Worked examples

Example 1 — scaffolded

Given $\sin\theta = \frac{3}{5}$ where θ is acute, find the exact value of (a) $\sin(\pi-\theta)$, (b) $\cos(\pi+\theta)$, (c) $\sin(2\pi-\theta)$, (d) $\tan(-\theta)$.

Working	Comment
Since θ is acute, $\cos\theta = \sqrt{1-(3/5)^2} = \frac{4}{5}$ and $\tan\theta = \frac{3/5}{4/5} = \frac{3}{4}$.	Use $\sin^2\theta + \cos^2\theta = 1$; θ acute so $\cos\theta > 0$.
(a) $\sin(\pi-\theta) = \sin\theta = \frac{3}{5}$.	Second-quadrant rule: $\sin$ unchanged.
(b) $\cos(\pi+\theta) = -\cos\theta = -\frac{4}{5}$.	Third-quadrant rule: $\cos$ negative.
(c) $\sin(2\pi-\theta) = -\sin\theta = -\frac{3}{5}$.	Fourth-quadrant rule: $\sin$ negative.
(d) $\tan(-\theta) = -\tan\theta = -\frac{3}{4}$.	$\tan$ is an odd function.

Example 2 — scaffolded

Find the exact value of (a) $\sin\frac{5\pi}{6}$, (b) $\cos\frac{3\pi}{4}$, (c) $\tan\left(-\frac{\pi}{3}\right)$.

Working	Comment
(a) $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, so $\sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$.	2nd quadrant (reference $\pi/6$); $\sin$ positive.
(b) $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$, so $\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$.	2nd quadrant; $\cos$ negative.

Working	Comment
(c) $\tan\left(-\frac{\pi}{3}\right)=-\tan\frac{\pi}{3}=-\sqrt{3}$.	tan is an odd function.

Example 3 — unscaffolded

Simplify $\dfrac{\sin(\pi-\theta)\cos(2\pi-\theta)}{\tan(\pi+\theta)}$.

Apply each symmetry rule: $\sin(\pi-\theta)=\sin\theta$, $\cos(2\pi-\theta)=\cos\theta$, and $\tan(\pi+\theta)=\tan\theta=\dfrac{\sin\theta}{\cos\theta}$. So

$$\frac{\sin\theta\cos\theta}{\tan\theta}=\sin\theta\cos\theta\times\frac{\cos\theta}{\sin\theta}=\cos^2\theta.$$

$\therefore \dfrac{\sin(\pi-\theta)\cos(2\pi-\theta)}{\tan(\pi+\theta)}=\cos^2\theta$.

Example 4 — unscaffolded

Given $\cos\alpha=-\dfrac{2}{3}$ where α is in the second quadrant, find $\sin\alpha$ and $\tan\alpha$, and hence $\cos(-\alpha)$ and $\sin(\pi+\alpha)$.

In the second quadrant $\sin\alpha>0$, so $\sin\alpha=\sqrt{1-(-2/3)^2}=\sqrt{\dfrac{5}{9}}=\dfrac{\sqrt{5}}{3}$, and $\tan\alpha=\dfrac{\sqrt{5}/3}{-2/3}=-\dfrac{\sqrt{5}}{2}$.

Then $\cos(-\alpha)=\cos\alpha=-\dfrac{2}{3}$ (cosine is even), and $\sin(\pi+\alpha)=-\sin\alpha=-\dfrac{\sqrt{5}}{3}$.

Practice questions

Scaffolded

Q1. Given $\cos\theta=\dfrac{5}{13}$ where θ is acute: (a) find $\sin\theta$; (b) find $\sin(\pi-\theta)$; (c) find $\cos(\pi+\theta)$; (d) find $\sin(2\pi-\theta)$.

Q2. Use symmetry to find the exact value of (a) $\sin\dfrac{4\pi}{3}$; (b) $\cos\dfrac{5\pi}{3}$; (c) $\tan\left(-\dfrac{\pi}{4}\right)$.

Q3. For the angle $-\dfrac{3\pi}{4}$: (a) state its quadrant and which of $\sin, \cos, \tan$ are positive there; (b) find the exact values of $\sin\left(-\dfrac{3\pi}{4}\right)$, $\cos\left(-\dfrac{3\pi}{4}\right)$ and $\tan\left(-\dfrac{3\pi}{4}\right)$.

Q4. Simplify (a) $\sin(-\theta)+\sin\theta$; (b) $\cos(\pi-\theta)+\cos\theta$; (c) $\tan(2\pi-\theta)+\tan\theta$.

Q5. Given $\sin\beta=-\dfrac{3}{5}$ where β is in the third quadrant, find $\cos\beta$ and $\tan\beta$.

Q6. Express in terms of $\sin\theta$ or $\cos\theta$: (a) $\sin(\pi+\theta)$; (b) $\cos(2\pi-\theta)$; (c) $\sin\left(\dfrac{\pi}{2}-\theta\right)$.

Unscaffolded

Find the exact value (Q7–Q12), then answer Q13–Q18.

Q7. $\sin\frac{7\pi}{4}$

Q8. $\cos\frac{7\pi}{6}$

Q9. $\tan\left(-\frac{2\pi}{3}\right)$

Q10. $\sin\frac{5\pi}{4}$

Q11. $\cos\frac{11\pi}{6}$

Q12. $\tan\left(-\frac{5\pi}{6}\right)$

Q13. Given $\sin\theta=\frac{4}{5}$ with θ acute, find $\sin(\pi+\theta)$, $\cos(\pi-\theta)$ and $\tan(2\pi-\theta)$.

Q14. Given $\cos\alpha=-\frac{1}{3}$ with α in the second quadrant, find $\sin\alpha$ and $\tan\alpha$.

Q15. Given $\tan\varphi=2$ with φ in the third quadrant, find $\sin\varphi$ and $\cos\varphi$.

Q16. Simplify $\cos(\pi-\theta)\cos(-\theta)-\sin(\pi+\theta)\sin(-\theta)$.

Q17. Simplify $\frac{\sin(\pi-\theta)}{\cos(2\pi-\theta)}\times\tan(\pi+\theta)$.

Q18. Show that $\sin(\pi+\theta)+\sin(\pi-\theta)=0$ for all θ, and hence find $\sin(\pi+\theta)$ when $\sin(\pi-\theta)=0.4$.

Answers

Q1. (a) $\frac{12}{13}$; (b) $\frac{12}{13}$; (c) $-\frac{5}{13}$; (d) $-\frac{12}{13}$. **Q2.** (a) $-\frac{\sqrt{3}}{2}$; (b) $\frac{1}{2}$; (c) -1. **Q3.** (a) Third quadrant; only $\tan$ is positive; (b) $\sin=-\frac{\sqrt{2}}{2}$, $\cos=-\frac{\sqrt{2}}{2}$, $\tan=1$. **Q4.** (a) 0; (b) 0; (c) 0. **Q5.** $\cos\beta=-\frac{4}{5}$, $\tan\beta=\frac{3}{4}$. **Q6.** (a) $-\sin\theta$; (b) $\cos\theta$; (c) $\cos\theta$. **Q7.** $-\frac{\sqrt{2}}{2}$. **Q8.** $-\frac{\sqrt{3}}{2}$. **Q9.** $\sqrt{3}$. **Q10.** $-\frac{\sqrt{2}}{2}$. **Q11.** $\frac{\sqrt{3}}{2}$. **Q12.** $\frac{\sqrt{3}}{3}$. **Q13.** $\sin(\pi+\theta)=-\frac{4}{5}$; $\cos(\pi-\theta)=-\frac{3}{5}$; $\tan(2\pi-\theta)=-\frac{4}{3}$. **Q14.** $\sin\alpha=\frac{2\sqrt{2}}{3}$, $\tan\alpha=-2\sqrt{2}$. **Q15.** $\sin\varphi=-\frac{2\sqrt{5}}{5}$, $\cos\varphi=-\frac{\sqrt{5}}{5}$. **Q16.** -1. **Q17.** $\tan^2\theta$. **Q18.** 0; $\sin(\pi+\theta)=-0.4$.

Fully-worked solutions

Q1. θ acute so $\sin\theta=\sqrt{1-(5/13)^2}=\frac{12}{13}$. (a) $\frac{12}{13}$. (b) $\sin(\pi-\theta)=\sin\theta=\frac{12}{13}$. (c) $\cos(\pi+\theta)=-\cos\theta=-\frac{5}{13}$. (d) $\sin(2\pi-\theta)=-\sin\theta=-\frac{12}{13}$.

Q2. (a) $\sin\frac{4\pi}{3}=\sin\left(\pi+\frac{\pi}{3}\right)=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$. (b) $\cos\frac{5\pi}{3}=\cos\left(2\pi-\frac{\pi}{3}\right)=\cos\frac{\pi}{3}=\frac{1}{2}$. (c) $\tan\left(-\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$.

Q3. (a) $-\frac{3\pi}{4}=-\pi+\frac{\pi}{4}$ lies in the **third quadrant**, where only $\tan$ is positive. (b) $\sin\left(-\frac{3\pi}{4}\right)=-\sin\frac{3\pi}{4}=-\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$; $\cos\left(-\frac{3\pi}{4}\right)=\cos\frac{3\pi}{4}=-\cos\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$; $\tan\left(-\frac{3\pi}{4}\right)=-\tan\frac{3\pi}{4}=\tan\frac{\pi}{4}=1$.

Q4. (a) $\sin(-\theta)+\sin\theta=-\sin\theta+\sin\theta=0$. (b) $\cos(\pi-\theta)+\cos\theta=-\cos\theta+\cos\theta=0$. (c) $\tan(2\pi-\theta)+\tan\theta=-\tan\theta+\tan\theta=0$.

Q5. β in the third quadrant, so $\cos\beta<0$: $\cos\beta=-\sqrt{1-(-3/5)^2}=-\frac{4}{5}$, and $\tan\beta=\frac{-3/5}{-4/5}=\frac{3}{4}$ (positive, as expected in the third quadrant).

Q6. (a) $\sin(\pi+\theta)=-\sin\theta$. (b) $\cos(2\pi-\theta)=\cos\theta$. (c) $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$.

Q7. $\sin\frac{7\pi}{4}=\sin\left(2\pi-\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$.

Q8. $\cos\frac{7\pi}{6}=\cos\left(\pi+\frac{\pi}{6}\right)=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$.

Q9. $\tan\left(-\frac{2\pi}{3}\right)=-\tan\frac{2\pi}{3}=-\left(-\tan\frac{\pi}{3}\right)=\sqrt{3}$.

Q10. $\sin\frac{5\pi}{4}=\sin\left(\pi+\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$.

Q11. $\cos\frac{11\pi}{6}=\cos\left(2\pi-\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$.

Q12. $\tan\left(-\frac{5\pi}{6}\right)=-\tan\frac{5\pi}{6}=-\left(-\tan\frac{\pi}{6}\right)=\frac{\sqrt{3}}{3}$.

Q13. θ acute with $\sin\theta=\frac{4}{5}$, so $\cos\theta=\frac{3}{5}$ and $\tan\theta=\frac{4}{3}$. Then $\sin(\pi+\theta)=-\sin\theta=-\frac{4}{5}$; $\cos(\pi-\theta)=-\cos\theta=-\frac{3}{5}$; $\tan(2\pi-\theta)=-\tan\theta=-\frac{4}{3}$.

Q14. α in the second quadrant, so $\sin\alpha>0$: $\sin\alpha=\sqrt{1-(-1/3)^2}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$, and $\tan\alpha=\frac{2\sqrt{2}/3}{-1/3}=-2\sqrt{2}$.

Q15. In the third quadrant $\sin$ and $\cos$ are both negative. From $\cos^2\varphi=\frac{1}{1+\tan^2\varphi}=\frac{1}{5}$, take the negative root: $\cos\varphi=-\frac{1}{\sqrt{5}}=-\frac{\sqrt{5}}{5}$, and $\sin\varphi=\tan\varphi\cos\varphi=2\times\left(-\frac{\sqrt{5}}{5}\right)=-\frac{2\sqrt{5}}{5}$.

Q16. $\cos(\pi-\theta)\cos(-\theta)-\sin(\pi+\theta)\sin(-\theta)=(-\cos\theta)(\cos\theta)-(-\sin\theta)(-\sin\theta)=-\cos^2\theta-\sin^2\theta=-1$.

Q17. $\frac{\sin(\pi-\theta)}{\cos(2\pi-\theta)}\times\tan(\pi+\theta)=\frac{\sin\theta}{\cos\theta}\times\tan\theta=\tan\theta\times\tan\theta=\tan^2\theta$.

Q18. $\sin(\pi+\theta)+\sin(\pi-\theta)=-\sin\theta+\sin\theta=0$ for all θ. Hence $\sin(\pi+\theta)=-\sin(\pi-\theta)$, so if $\sin(\pi-\theta)=0.4$ then $\sin(\pi+\theta)=-0.4$.

Theory

Certain angles have **exact** values of sine, cosine and tangent. These come from two special right-angled triangles, together with the reference-angle and quadrant-sign rules.

The special triangles.

- The **half-square** (isosceles right-angled) triangle has angles $45°$, $45°$, $90°$ and sides 1, 1, $\sqrt{2}$. It gives $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ and $\tan\frac{\pi}{4}=1$.
- The **half-equilateral** triangle has angles $30°$, $60°$, $90°$ and sides 1, $\sqrt{3}$, 2. It gives $\sin\frac{\pi}{6}=\frac{1}{2}$, $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, $\tan\frac{\pi}{6}=\frac{\sqrt{3}}{3}$, and $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3}=\frac{1}{2}$, $\tan\frac{\pi}{3}=\sqrt{3}$.

Exact-value table (first quadrant):

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Quadrant signs (CAST). Reading anticlockwise from the first quadrant, the functions that are **positive** are: quadrant 1 — **A**ll; quadrant 2 — **S**ine only; quadrant 3 — **T**angent only; quadrant 4 — **C**osine only.

Reference angle. The reference angle is the acute angle between the terminal side and the x-axis. For θ in $[0,2\pi]$:

- quadrant 1: reference $=\theta$; quadrant 2: reference $=\pi-\theta$;
- quadrant 3: reference $=\theta-\pi$; quadrant 4: reference $=2\pi-\theta$.

To find an exact value: (1) decide the quadrant and hence the **sign** (CAST); (2) find the **reference angle**; (3) the exact value is that sign times the first-quadrant value of the reference angle.

Periodicity. Sine and cosine repeat every 2π, and tangent every π. Add or subtract full turns until the angle lies in $[0,2\pi]$. Also $\sin(-\theta)=-\sin\theta$ (odd), $\cos(-\theta)=\cos\theta$ (even), $\tan(-\theta)=-\tan\theta$ (odd).

Worked examples

Example 1 — scaffolded

Find the exact value of $\sin\frac{2\pi}{3}$.

Working	Comment
$\frac{2\pi}{3}$ is in quadrant 2, where **sine is positive**.	$\frac{\pi}{2}<\frac{2\pi}{3}<\pi$; apply CAST.

Working	Comment
Reference angle $= \pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$.	Quadrant 2: reference $= \pi - \theta$.
$\sin\dfrac{2\pi}{3} = +\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.	Sign (positive) × first-quadrant value.

Example 2 — scaffolded

Find the exact value of $\cos\dfrac{5\pi}{4}$.

Working	Comment
$\dfrac{5\pi}{4}$ is in quadrant 3, where **cosine is negative**.	$\pi < \dfrac{5\pi}{4} < \dfrac{3\pi}{2}$; apply CAST.
Reference angle $= \dfrac{5\pi}{4} - \pi = \dfrac{\pi}{4}$.	Quadrant 3: reference $= \theta - \pi$.
$\cos\dfrac{5\pi}{4} = -\cos\dfrac{\pi}{4} = -\dfrac{\sqrt{2}}{2}$.	Sign (negative) × first-quadrant value.

Example 3 — unscaffolded

Find the exact value of $\tan\left(-\dfrac{4\pi}{3}\right)$.

Tangent is odd, so $\tan\left(-\dfrac{4\pi}{3}\right) = -\tan\dfrac{4\pi}{3}$. Now $\dfrac{4\pi}{3}$ is in quadrant 3, where tangent is **positive**, and the reference angle is $\dfrac{4\pi}{3} - \pi = \dfrac{\pi}{3}$.

$\therefore \tan\left(-\dfrac{4\pi}{3}\right) = -\tan\dfrac{\pi}{3} = -\sqrt{3}$.

Example 4 — unscaffolded

Find the exact value of $\sin\dfrac{9\pi}{4}$.

Using periodicity, $\dfrac{9\pi}{4} - 2\pi = \dfrac{\pi}{4}$, so $\sin\dfrac{9\pi}{4} = \sin\dfrac{\pi}{4}$.

$\therefore \sin\dfrac{9\pi}{4} = \dfrac{\sqrt{2}}{2}$.

Practice questions

Scaffolded

Q1. For $\sin\dfrac{3\pi}{4}$: (a) state the quadrant and the sign of sine there; (b) find the reference angle; (c) write the exact value.

Q2. For $\cos\dfrac{2\pi}{3}$: (a) state the quadrant and sign; (b) find the reference angle; (c) write the exact value.

Q3. For $\tan\dfrac{17\pi}{6}$: (a) state the quadrant and sign; (b) find the reference angle; (c) write the exact value.

Q4. For $\sin\frac{7\pi}{6}$: (a) state the quadrant and sign; (b) find the reference angle; (c) write the exact value.

Q5. For $\cos\frac{7\pi}{4}$: (a) state the quadrant and sign; (b) find the reference angle; (c) write the exact value.

Q6. For $\tan\left(-\frac{5\pi}{3}\right)$: (a) state the quadrant and sign; (b) find the reference angle; (c) write the exact value.

Unscaffolded

Find the exact value of each of the following.

Q7. $\cos\frac{5\pi}{6}$

Q8. $\sin\frac{5\pi}{3}$

Q9. $\tan\frac{8\pi}{3}$

Q10. $\cos\frac{4\pi}{3}$

Q11. $\sin\frac{11\pi}{6}$

Q12. $\tan\left(-\frac{5\pi}{4}\right)$

Q13. $\sin\left(-\frac{\pi}{4}\right)$

Q14. $\cos\left(-\frac{2\pi}{3}\right)$

Q15. $\tan\left(-\frac{\pi}{6}\right)$

Q16. $\sin\frac{13\pi}{6}$

Q17. $\cos\frac{9\pi}{4}$

Q18. $\tan\frac{7\pi}{3}$

Answers

Q1. $\frac{\sqrt{2}}{2}$. **Q2.** $-\frac{1}{2}$. **Q3.** $-\frac{\sqrt{3}}{3}$. **Q4.** $-\frac{1}{2}$. **Q5.** $\frac{\sqrt{2}}{2}$. **Q6.** $\sqrt{3}$. **Q7.** $-\frac{\sqrt{3}}{2}$. **Q8.** $-\frac{\sqrt{3}}{2}$. **Q9.** $-\sqrt{3}$. **Q10.** $-\frac{1}{2}$. **Q11.** $-\frac{1}{2}$. **Q12.** -1. **Q13.** $-\frac{\sqrt{2}}{2}$. **Q14.** $-\frac{1}{2}$. **Q15.** $-\frac{\sqrt{3}}{3}$. **Q16.** $\frac{1}{2}$. **Q17.** $\frac{\sqrt{2}}{2}$. **Q18.** $\sqrt{3}$.

Fully-worked solutions

Q1. $\frac{3\pi}{4}$ is in quadrant 2, sine **positive**; reference $=\pi-\frac{3\pi}{4}=\frac{\pi}{4}$. $\therefore \sin\frac{3\pi}{4}=\frac{\sqrt{2}}{2}$.

Q2. $\frac{2\pi}{3}$ is in quadrant 2, cosine **negative**; reference $=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$. $\therefore \cos\frac{2\pi}{3}=-\cos\frac{\pi}{3}=-\frac{1}{2}$.

Q3. Periodicity: $\frac{17\pi}{6}-2\pi=\frac{5\pi}{6}$. Then $\frac{5\pi}{6}$ is in quadrant 2, tangent **negative**; reference $=\pi-\frac{5\pi}{6}=\frac{\pi}{6}$.
$\therefore \tan\frac{17\pi}{6}=\tan\frac{5\pi}{6}=-\tan\frac{\pi}{6}=-\frac{\sqrt{3}}{3}$.

Q4. $\frac{7\pi}{6}$ is in quadrant 3, sine **negative**; reference $=\frac{7\pi}{6}-\pi=\frac{\pi}{6}$. $\therefore \sin\frac{7\pi}{6}=-\sin\frac{\pi}{6}=-\frac{1}{2}$.

Q5. $\frac{7\pi}{4}$ is in quadrant 4, cosine **positive**; reference $=2\pi-\frac{7\pi}{4}=\frac{\pi}{4}$. $\therefore \cos\frac{7\pi}{4}=\frac{\sqrt{2}}{2}$.

Q6. Tangent is odd, so $\tan\left(-\frac{5\pi}{3}\right)=-\tan\frac{5\pi}{3}$. Now $\frac{5\pi}{3}$ is in quadrant 4, tangent **negative**; reference $=2\pi-\frac{5\pi}{3}=\frac{\pi}{3}$. $\therefore \tan\left(-\frac{5\pi}{3}\right)=-\left(-\tan\frac{\pi}{3}\right)=\sqrt{3}$.

Q7. Quadrant 2, cosine negative; reference $\pi-\frac{5\pi}{6}=\frac{\pi}{6}$. $\therefore \cos\frac{5\pi}{6}=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$.

Q8. Quadrant 4, sine negative; reference $2\pi-\frac{5\pi}{3}=\frac{\pi}{3}$. $\therefore \sin\frac{5\pi}{3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$.

Q9. Periodicity: $\frac{8\pi}{3}-2\pi=\frac{2\pi}{3}$. Quadrant 2, tangent negative; reference $\pi-\frac{2\pi}{3}=\frac{\pi}{3}$. $\therefore \tan\frac{8\pi}{3}=-\tan\frac{\pi}{3}=-\sqrt{3}$.

Q10. Quadrant 3, cosine negative; reference $\frac{4\pi}{3}-\pi=\frac{\pi}{3}$. $\therefore \cos\frac{4\pi}{3}=-\cos\frac{\pi}{3}=-\frac{1}{2}$.

Q11. Quadrant 4, sine negative; reference $2\pi-\frac{11\pi}{6}=\frac{\pi}{6}$. $\therefore \sin\frac{11\pi}{6}=-\sin\frac{\pi}{6}=-\frac{1}{2}$.

Q12. Tangent is odd: $\tan\left(-\frac{5\pi}{4}\right)=-\tan\frac{5\pi}{4}$. Quadrant 3, tangent positive; reference $\frac{5\pi}{4}-\pi=\frac{\pi}{4}$.
$\therefore \tan\left(-\frac{5\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$.

Q13. Sine is odd: $\sin\left(-\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$.

Q14. Cosine is even: $\cos\left(-\frac{2\pi}{3}\right)=\cos\frac{2\pi}{3}=-\frac{1}{2}$ (quadrant 2, cosine negative).

Q15. Tangent is odd: $\tan\left(-\frac{\pi}{6}\right)=-\tan\frac{\pi}{6}=-\frac{\sqrt{3}}{3}$.

Q16. Periodicity: $\frac{13\pi}{6}-2\pi=\frac{\pi}{6}$, so $\sin\frac{13\pi}{6}=\sin\frac{\pi}{6}=\frac{1}{2}$.

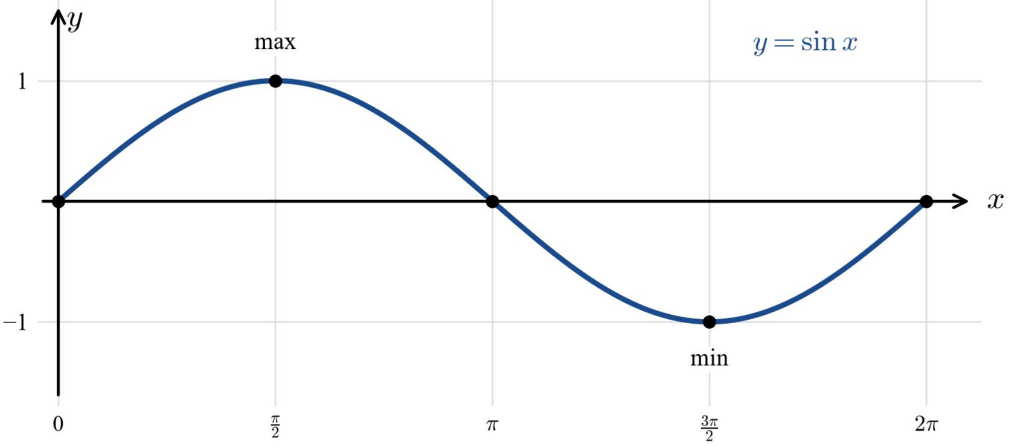
Q17. Periodicity: $\frac{9\pi}{4}-2\pi=\frac{\pi}{4}$, so $\cos\frac{9\pi}{4}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$.

Q18. Tangent has period π: $\frac{7\pi}{3}-2\pi=\frac{\pi}{3}$, so $\tan\frac{7\pi}{3}=\tan\frac{\pi}{3}=\sqrt{3}$.

Theory

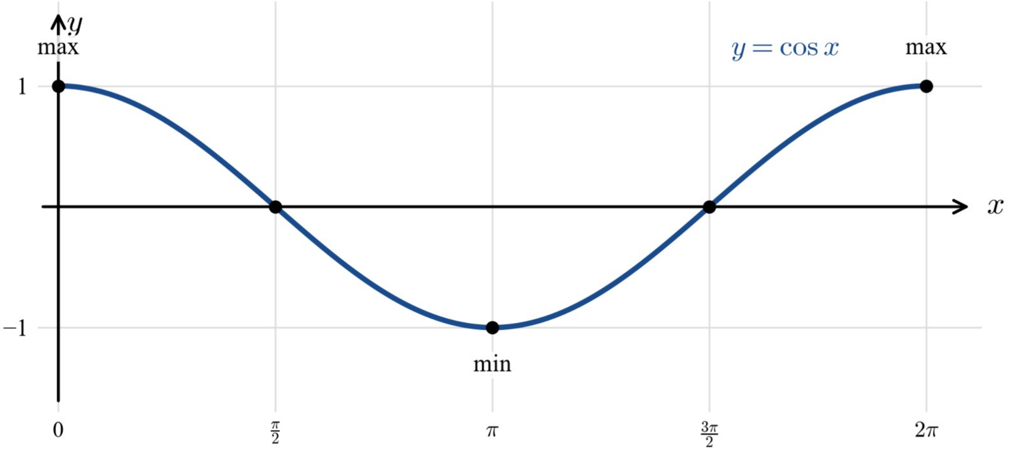
The circular functions $\sin x$ and $\cos x$ are **periodic** — their graphs repeat the same shape over and over.

The graph of $y = \sin x$. Over one cycle $0 \le x \le 2\pi$ the graph starts at the origin, rises to a maximum, returns through the axis, falls to a minimum, and returns:



The key points of one cycle are $(0,0)$, $(\pi/2,1)$, $(\pi,0)$, $(3\pi/2,-1)$ and $(2\pi,0)$.

The graph of $y = \cos x$ has the same wave shape but begins at its maximum:

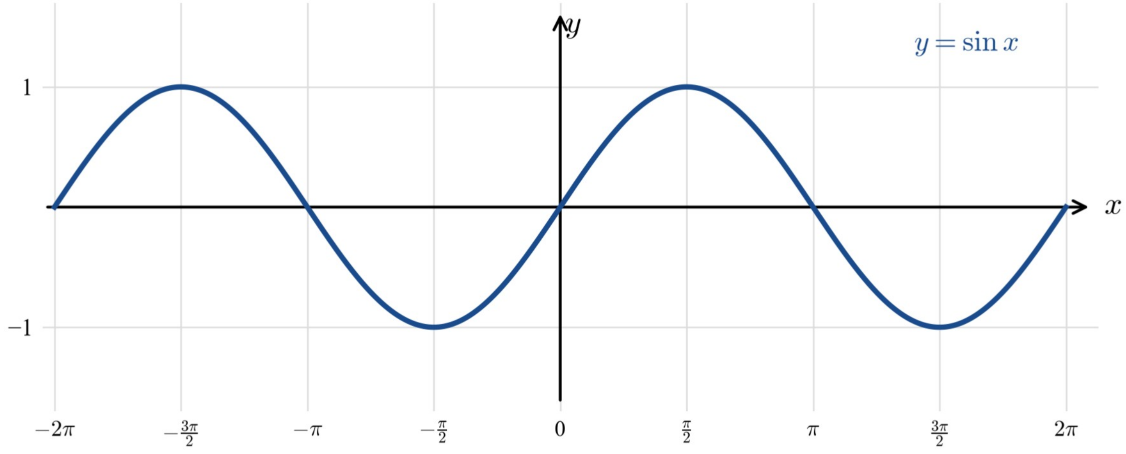


Its key points are $(0,1)$, $(\pi/2,0)$, $(\pi,-1)$, $(3\pi/2,0)$ and $(2\pi,1)$.

Amplitude, period, domain and range. For both $y = \sin x$ and $y = \cos x$:

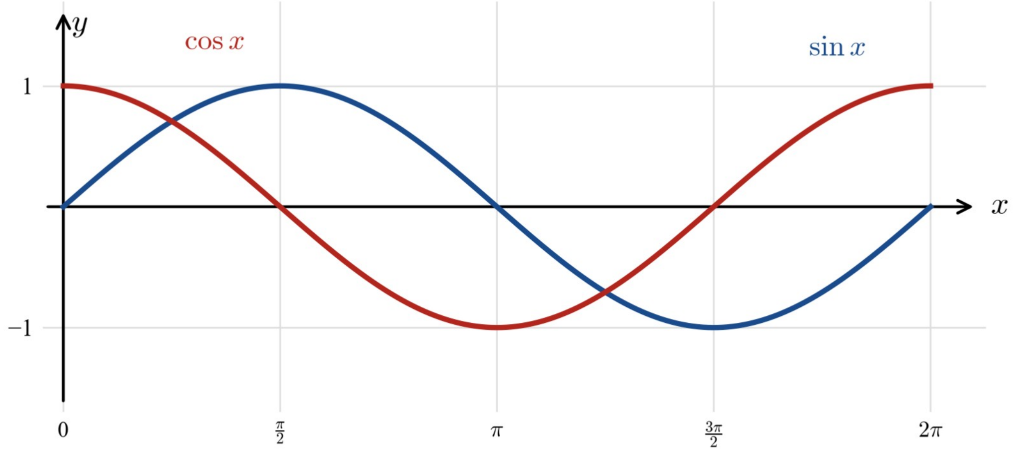
- the **amplitude** is 1 (half the distance between the maximum 1 and minimum -1);
- the **period** is 2π (the length of one complete cycle);
- the **domain** is $\mathbb{R}$ and the **range** is $[-1,1]$.

Periodicity. Because the period is 2π, $\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$ — the graph repeats every 2π in both directions:



Symmetry. sin is an **odd** function: $\sin(-x) = -\sin x$, so its graph has point symmetry about the origin. cos is an **even** function: $\cos(-x) = \cos x$, so its graph is symmetric about the y-axis.

Relationship between the graphs. The cosine graph is the sine graph translated $\pi/2$ to the **left**: $\cos x = \sin(x + \pi/2)$.



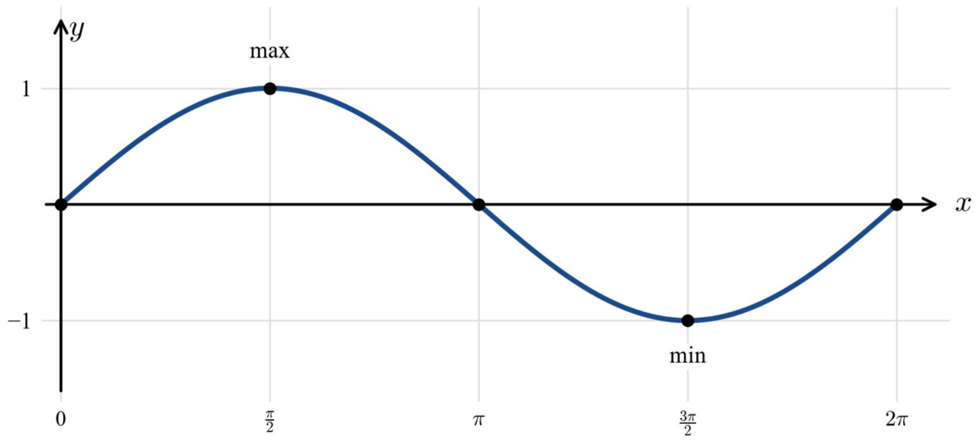
Reading values from the graph. A horizontal line $y = c$ (with $-1 \le c \le 1$) usually meets one cycle **twice**, so equations such as $\sin x = 1/2$ have two solutions in $[0, 2\pi]$.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = \sin x$ for $0 \le x \le 2\pi$, showing the key points, and state its amplitude and period.

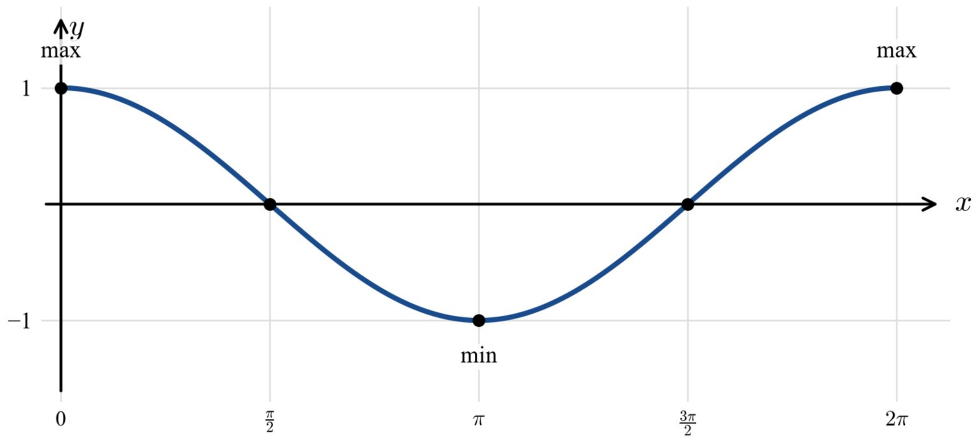
Working	Comment
Maximum $(\pi/2, 1)$, minimum $(3\pi/2, -1)$.	Read from the shape of one cycle.
x-intercepts at $x = 0, \pi, 2\pi$.	$\sin x = 0$ at multiples of π.
Amplitude $= 1$; period $= 2\pi$.	Amplitude $= 1/2(\max - \min)$; period = length of one cycle.



Example 2 — scaffolded

Sketch $y = \cos x$ for $0 \le x \le 2\pi$, and state the coordinates of the maximum and minimum points and the x-intercepts.

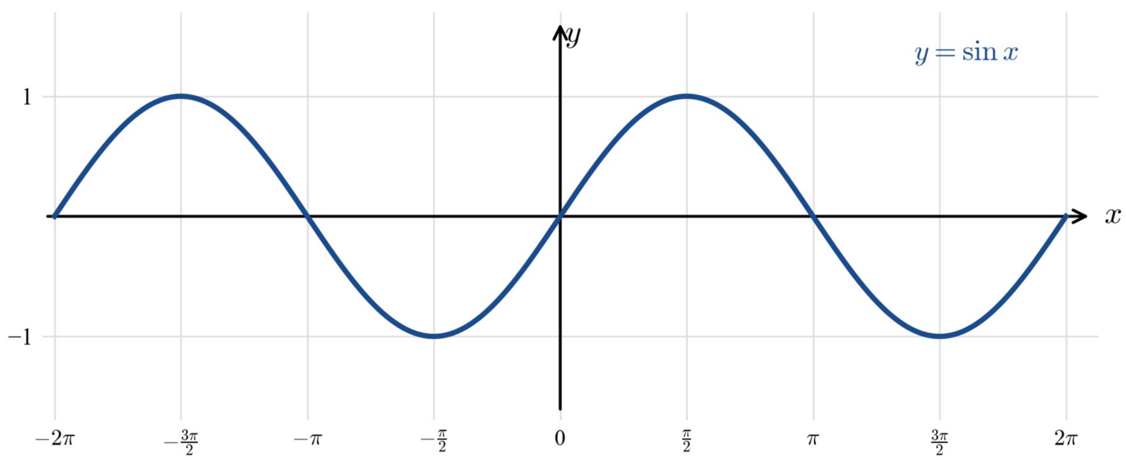
Working	Comment
Maximum points $(0, 1)$ and $(2\pi, 1)$; minimum point $(\pi, -1)$.	Cosine starts at its maximum.
x-intercepts where $\cos x = 0$: $x = \pi/2$ and $x = 3\pi/2$.	Read the axis crossings.



Example 3 — unscaffolded

Sketch $y = \sin x$ for $-2\pi \le x \le 2\pi$.

Using periodicity ($\sin(x + 2\pi) = \sin x$), the cycle on $[0, 2\pi]$ is repeated on $[-2\pi, 0]$. The graph is symmetric about the origin (sine is odd).



∴ there are maxima at $x = \pi/2$ and $x = -3\pi/2$, and minima at $x = -\pi/2$ and $x = 3\pi/2$.

Example 4 — unscaffolded

Use the graph of $y = \cos x$ to find all x in $[-\pi, \pi]$ for which $\cos x = 0$.

The cosine graph crosses the x-axis where it equals 0. From the graph this occurs at $x = -\pi/2$ and $x = \pi/2$.

$\therefore x = -\pi/2$ or $x = \pi/2$.

Practice questions

Scaffolded

Q1. For $y = \sin x$: (a) state the amplitude; (b) state the period; (c) give the coordinates of the maximum in $[0, 2\pi]$; (d) give the coordinates of the minimum in $[0, 2\pi]$.

Q2. For $y = \cos x$: (a) state the amplitude; (b) state the period; (c) state the y-intercept; (d) find the value of x in $[0, 2\pi]$ for which $\cos x = -1$.

Q3. Sketch $y = \sin x$ for $-\pi \le x \le \pi$, marking the three points where it crosses the x-axis.

Q4. Sketch $y = \cos x$ for $-\pi \le x \le \pi$, marking the maximum and the endpoints.

Q5. Use the graph of $y = \sin x$ to solve, for $0 \le x \le 2\pi$: (a) $\sin x = 1$; (b) $\sin x = 0$.

Q6. State whether each function is odd or even, and what symmetry its graph has: (a) $\sin x$; (b) $\cos x$.

Unscaffolded

Q7. State the domain and range of $y = \sin x$.

Q8. State the domain and range of $y = \cos x$.

Q9. Find all x in $[-2\pi, 0]$ for which $\sin x = -1$.

Q10. Find all x in $[0, 2\pi]$ for which $\cos x = 1$.

Q11. Use the graph of $y = \sin x$ to solve $\sin x = 1/2$ for $-2\pi \le x \le 2\pi$.

Q12. Use the graph of $y = \cos x$ to solve $\cos x = 1/2$ for $0 \le x \le 4\pi$.

Q13. On which interval(s) of $[0, 2\pi]$ is $y = \sin x$ **increasing**?

Q14. On which interval of $[0, 2\pi]$ is $y = \cos x$ **decreasing**?

Q15. Sketch $y = \cos x$ for $-2\pi \le x \le 2\pi$.

Q16. Using the symmetry of the graphs, evaluate: (a) $\sin(-\pi/2)$; (b) $\cos(-\pi)$.

Q17. Explain, by referring to the two graphs, why $\cos x = \sin(x + \pi/2)$.

Q18. Use the graphs of $y = \sin x$ and $y = \cos x$ to find all x in $[0, 2\pi]$ for which $\sin x = \cos x$.

Answers

Q1. (a) 1; (b) 2π; (c) $(\pi/2, 1)$; (d) $(3\pi/2, -1)$. **Q2.** (a) 1; (b) 2π; (c) $(0, 1)$; (d) π. **Q3.** Crossings at $x = -\pi, 0, \pi$ (see sketch). **Q4.** Maximum $(0, 1)$, endpoints $(\pm\pi, -1)$ (see sketch). **Q5.** (a) $x = \pi/2$; (b) $x = 0, \pi, 2\pi$. **Q6.** (a) odd — point symmetry about the origin; (b) even — symmetric about the y-axis. **Q7.** Domain $\mathbb{R}$, range $[-1, 1]$. **Q8.** Domain $\mathbb{R}$, range $[-1, 1]$. **Q9.** $x = -\pi/2$. **Q10.** $x = 0$ or $x = 2\pi$. **Q11.** $x = -11\pi/6, -7\pi/6, \pi/6, 5\pi/6$. **Q12.** $x = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$. **Q13.** $[0, \pi/2]$ and $[3\pi/2, 2\pi]$. **Q14.** $[0, \pi]$. **Q15.** See sketch (two cycles). **Q16.** (a) -1; (b) -1. **Q17.** Cosine is sine shifted $\pi/2$ left. **Q18.** $x = \pi/4$ or $x = 5\pi/4$.

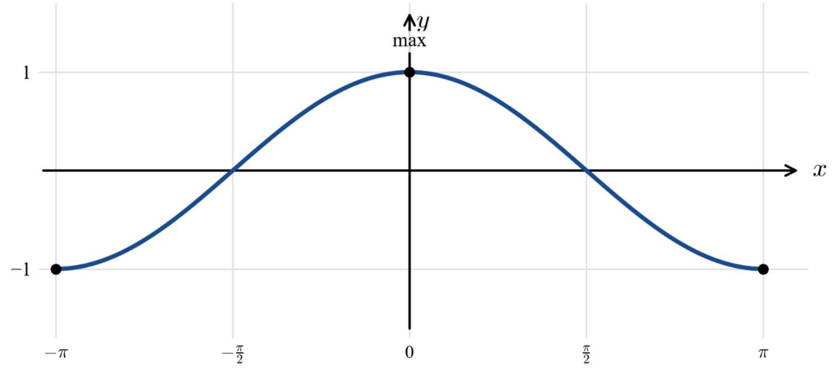
Fully-worked solutions

Q1. For $y = \sin x$: (a) amplitude $= 1$; (b) period $= 2\pi$; (c) the maximum is at $(\pi/2, 1)$; (d) the minimum is at $(3\pi/2, -1)$.

Q2. For $y = \cos x$: (a) amplitude $= 1$; (b) period $= 2\pi$; (c) y-intercept: put $x = 0$, $\cos 0 = 1$, so $(0, 1)$; (d) $\cos x = -1$ at the minimum of the cycle, so $x = \pi$.

Q3. The sine curve crosses the x-axis at $x = -\pi$, $x = 0$ and $x = \pi$.

Q4. Cosine is even, so the graph on $[-\pi, \pi]$ is symmetric about the y-axis, with a maximum at $(0, 1)$ and reaching -1 at both endpoints $x = \pm\pi$.



Q5. From the sine graph: (a) $\sin x = 1$ at the maximum, so $x = \pi/2$; (b) $\sin x = 0$ at the axis crossings, so $x = 0, \pi, 2\pi$.

Q6. (a) $\sin(-x) = -\sin x$, so sine is **odd** — its graph has point (rotational) symmetry about the origin. (b) $\cos(-x) = \cos x$, so cosine is **even** — its graph is symmetric about the y-axis.

Q7. Domain $\mathbb{R}$; range $[-1, 1]$.

Q8. Domain $\mathbb{R}$; range $[-1, 1]$.

Q9. $\sin x = -1$ at the minima of the sine curve; in $[-2\pi, 0]$ the only such point is $x = -\pi/2$.

Q10. $\cos x = 1$ at the maxima, so $x = 0$ or $x = 2\pi$.

Q11. The line $y = 1/2$ meets the sine graph at $x = \pi/6$ and $x = 5\pi/6$ in $[0, 2\pi]$ (these are symmetric about the peak at $x = \pi/2$); subtracting one period 2π gives $x = -11\pi/6$ and $x = -7\pi/6$ in $[-2\pi, 0]$.

Q12. The line $y = 1/2$ meets the cosine graph at $x = \pi/3$ and, by symmetry about the minimum at $x = \pi$, at $x = 5\pi/3$ in $[0, 2\pi]$; adding one period 2π gives $x = 7\pi/3$ and $x = 11\pi/3$ in $[2\pi, 4\pi]$.

Q13. $y = \sin x$ is increasing where the graph rises: on $[0, \pi/2]$ and $[3\pi/2, 2\pi]$.

Q14. $y = \cos x$ falls from its maximum at $x = 0$ to its minimum at $x = \pi$, so it is decreasing on $[0, \pi]$.

Q15. Using $\cos(x+2\pi)=\cos x$, repeat the cycle to the left: maxima at $x=-2\pi, 0, 2\pi$, minima at $x=-\pi, \pi$, and x-intercepts at $x=\pm\pi/2, \pm 3\pi/2$.

Q16. (a) $\sin(-\pi/2)=-\sin\pi/2=-1$ (sine is odd); (b) $\cos(-\pi)=\cos\pi=-1$ (cosine is even).

Q17. Sliding the sine graph $\pi/2$ units to the left moves its maximum from $x=\pi/2$ to $x=0$ — exactly where the cosine graph has its maximum. The two graphs then coincide, so $\cos x=\sin(x+\pi/2)$.

Q18. $\sin x=\cos x$ where the two graphs intersect. From the graphs, the intersections in $[0, 2\pi]$ are at $x=\pi/4$ (where both values are $\sqrt{2}/2$) and $x=5\pi/4$ (where both values are $-\sqrt{2}/2$).

Theory

A trigonometric equation such as $\sin x = 1/2$ has **infinitely many** solutions, so a domain (for example $x \in [0, 2\pi]$) is always given. Because the circular functions are periodic, we find every solution in that domain using a **base angle** and the **quadrant signs**.

The base (reference) angle. The base angle α is the *acute* angle whose sine, cosine or tangent equals the **positive** value of the ratio. The exact base angles come from the standard triangles:

Ratio	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	$1/\sqrt{3}$	1	$\sqrt{3}$
$\sin^{-1}$	$\pi/6$	$\pi/4$	$\pi/3$	—	—	—
$\cos^{-1}$	$\pi/3$	$\pi/4$	$\pi/6$	—	—	—
$\tan^{-1}$	—	—	—	$\pi/6$	$\pi/4$	$\pi/3$

Quadrant signs (CAST). In $[0, 2\pi]$: **all** ratios are positive in the 1st quadrant; **sine** only in the 2nd; **tangent** only in the 3rd; **cosine** only in the 4th. So, with base angle α:

- $\sin x = k$ ($k > 0$): $x = \alpha$ (Q1) and $x = \pi - \alpha$ (Q2).
- $\cos x = k$ ($k > 0$): $x = \alpha$ (Q1) and $x = 2\pi - \alpha$ (Q4).
- $\tan x = k$ ($k > 0$): $x = \alpha$ (Q1) and $x = \pi + \alpha$ (Q3).

If the ratio is **negative**, use the quadrants where that function is negative instead (for $\sin$: Q3, Q4; for $\cos$: Q2, Q3; for $\tan$: Q2, Q4).

Method. (1) Isolate the ratio; (2) find the base angle from the *positive* value; (3) decide the quadrants from the *sign*; (4) write every solution in the domain; (5) for a larger domain, add or subtract 2π (or π for $\tan$) to reach the other solutions.

Transformed argument $A f(nx) + c = k$. When the angle is a multiple nx (with n a positive integer), work in stages. First **isolate** the circular function, so that $f(nx) = k - c/A$. Then **substitute** $u = nx$ and **scale the domain**: if $x \in [a, b]$ then $u = nx \in [na, nb]$. Solve $f(u) = k - c/A$ over this *widened* interval exactly as before — base angle, quadrant signs, and adding 2π (or π for $\tan$) as many times as the interval now allows. Finally **divide every u-solution by n** to recover x. Because the u-interval is n times as wide, the **number of solutions grows with n** (about n times as many as for the single-argument equation).

Worked examples

Example 1 — scaffolded

Solve $2\sin x = 1$ for $x \in [0, 2\pi]$.

Working	Comment
$2\sin x = 1 \Rightarrow \sin x = 1/2$.	Isolate the trigonometric ratio.
Base angle: $\sin^{-1} 1/2 = \pi/6$.	The base angle uses the positive value.
$\sin x > 0$, so x is in the 1st and 2nd quadrants.	Sine is positive in Q1 and Q2.
$x = \pi/6$ (Q1) or $x = \pi - \pi/6 = 5\pi/6$ (Q2).	Q1: α; Q2: $\pi - \alpha$.

$\therefore x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

Example 2 — scaffolded

Solve $\sqrt{2}\cos x+1=0$ for $x\in[0,2\pi]$.

Working	Comment
$\sqrt{2}\cos x+1=0\Rightarrow\cos x=-1/\sqrt{2}$.	Isolate the ratio.
Base angle: $\cos^{-1}1/\sqrt{2}=\pi/4$.	Use the *positive* value $1/\sqrt{2}$.
$\cos x<0$, so x is in the 2nd and 3rd quadrants.	Cosine is negative in Q2 and Q3.
$x=\pi-\pi/4=3\pi/4$ (Q2) or $x=\pi+\pi/4=5\pi/4$ (Q3).	Q2: $\pi-\alpha$; Q3: $\pi+\alpha$.

$\therefore x=\frac{3\pi}{4},\frac{5\pi}{4}$.

Example 3 — unscaffolded

Solve $\tan x=\sqrt{3}$ for $x\in[0,2\pi]$.

Base angle: $\tan^{-1}\sqrt{3}=\frac{\pi}{3}$. Since $\tan x>0$, x is in the 1st and 3rd quadrants: $x=\frac{\pi}{3}$ or $x=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$.

$\therefore x=\frac{\pi}{3},\frac{4\pi}{3}$.

Example 4 — unscaffolded

Solve $2\cos x=\sqrt{3}$ for $x\in[-\pi,\pi]$.

$\cos x=\frac{\sqrt{3}}{2}$; base angle $\frac{\pi}{6}$. Cosine is positive in the 1st and 4th quadrants. Over $[-\pi,\pi]$ the 4th-quadrant solution is the *negative* angle: $x=\frac{\pi}{6}$ or $x=-\frac{\pi}{6}$.

$\therefore x=-\frac{\pi}{6},\frac{\pi}{6}$.

Example 5 — scaffolded

Solve $2\sin(2x)=\sqrt{3}$ for $x\in[0,2\pi]$.

Working	Comment
$2\sin(2x)=\sqrt{3}\Rightarrow\sin(2x)=\sqrt{3}/2$.	Isolate the trigonometric ratio.
Let $u=2x$. As $x\in[0,2\pi]$, $u\in[0,4\pi]$.	Substitute and **scale the domain** ($\times 2$).
Base angle: $\sin^{-1}\sqrt{3}/2=\pi/3$; $\sin u>0$ in Q1, Q2.	Solve for u over the widened domain.
$u=\pi/3,2\pi/3$, and adding 2π: $u=7\pi/3,8\pi/3$.	$[0,4\pi]$ is two periods, so two extra solutions.
$x=u/2=\pi/6,\pi/3,7\pi/6,4\pi/3$.	Divide every u-solution by $n=2$.

$\therefore x=\frac{\pi}{6},\frac{\pi}{3},\frac{7\pi}{6},\frac{4\pi}{3}$.

Example 6 — unscaffolded

Solve $2\cos(2x)+1=0$ for $x\in[0,\pi]$.

Isolate the ratio: $\cos(2x)=-1/2$. Let $u=2x$; as $x\in[0,\pi]$, the domain scales to $u\in[0,2\pi]$. Base angle $\cos^{-1}1/2=\pi/3$; cosine is negative in the 2nd and 3rd quadrants, so $u=\pi-\pi/3=2\pi/3$ or $u=\pi+\pi/3=4\pi/3$. Dividing by $n=2$: $x=\pi/3,2\pi/3$.

$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}$.

Practice questions

Scaffolded

Q1. Solve $2\sin x = \sqrt{3}$ for $x \in [0, 2\pi]$. (a) Write the equation as $\sin x = \ldots$ (b) State the base angle. (c) In which quadrants is $\sin x$ positive? (d) Write all solutions.

Q2. Solve $\sqrt{2}\sin x - 1 = 0$ for $x \in [0, 2\pi]$. (a) Make $\sin x$ the subject. (b) State the base angle. (c) State the quadrants. (d) Write all solutions.

Q3. Solve $2\cos x + 1 = 0$ for $x \in [0, 2\pi]$. (a) Make $\cos x$ the subject. (b) State the base angle (from the positive value). (c) In which quadrants is $\cos x$ negative? (d) Solve.

Q4. Solve $\tan x = 1$ for $x \in [0, 2\pi]$. (a) State the base angle. (b) In which quadrants is $\tan x$ positive? (c) Write all solutions.

Q5. Solve $\sqrt{3}\tan x + 1 = 0$ for $x \in [0, 2\pi]$. (a) Make $\tan x$ the subject. (b) State the base angle. (c) In which quadrants is $\tan x$ negative? (d) Solve.

Q6. Solve $2\sin x + 1 = 0$ for $x \in [0, 2\pi]$. (a) Make $\sin x$ the subject. (b) State the base angle. (c) In which quadrants is $\sin x$ negative? (d) Solve.

Unscaffolded

Solve each equation over the given domain, giving all solutions as exact values.

Q7. $\cos x = \frac{1}{2}$, $x \in [0, 2\pi]$

Q8. $\sin x = \frac{\sqrt{3}}{2}$, $x \in [0, 2\pi]$

Q9. $\tan x = -\sqrt{3}$, $x \in [0, 2\pi]$

Q10. $2\cos x = -\sqrt{3}$, $x \in [0, 2\pi]$

Q11. $\sqrt{2}\sin x + 1 = 0$, $x \in [0, 2\pi]$

Q12. $\tan x = \frac{1}{\sqrt{3}}$, $x \in [0, 2\pi]$

Q13. $\sin x = -1$, $x \in [0, 2\pi]$

Q14. $\cos x = 0$, $x \in [0, 2\pi]$

Q15. $2\sin x - \sqrt{3} = 0$, $x \in [-\pi, \pi]$

Q16. $\sqrt{2}\cos x = -1$, $x \in [-\pi, \pi]$

Q17. $\tan x = 1$, $x \in [-\pi, \pi]$

Q18. $2\cos x + 1 = 0$, $x \in [0, 4\pi]$

Q19. $\sin(2x) = \frac{1}{2}$, $x \in [0, 2\pi]$

Q20. $2\cos(2x) = \sqrt{2}$, $x \in [0, 2\pi]$

Q21. $\tan(2x)=1,\ x\in[0,\pi]$

Q22. $\sin(3x)=\frac{\sqrt{3}}{2},\ x\in[0,\pi]$

Q23. $2\sin(2x)+\sqrt{2}=0,\ x\in[0,2\pi]$

Q24. $2\sin(3x)+1=0,\ x\in[0,2\pi]$

Answers

Q1. $\frac{\pi}{3}, \frac{2\pi}{3}$. **Q2.** $\frac{\pi}{4}, \frac{3\pi}{4}$. **Q3.** $\frac{2\pi}{3}, \frac{4\pi}{3}$. **Q4.** $\frac{\pi}{4}, \frac{5\pi}{4}$. **Q5.** $\frac{5\pi}{6}, \frac{11\pi}{6}$. **Q6.** $\frac{7\pi}{6}, \frac{11\pi}{6}$. **Q7.** $\frac{\pi}{3}, \frac{5\pi}{3}$. **Q8.** $\frac{\pi}{3}, \frac{2\pi}{3}$. **Q9.** $\frac{2\pi}{3}, \frac{5\pi}{3}$. **Q10.** $\frac{5\pi}{6}, \frac{7\pi}{6}$. **Q11.** $\frac{5\pi}{4}, \frac{7\pi}{4}$. **Q12.** $\frac{\pi}{6}, \frac{7\pi}{6}$. **Q13.** $\frac{3\pi}{2}$. **Q14.** $\frac{\pi}{2}, \frac{3\pi}{2}$. **Q15.** $\frac{\pi}{3}, \frac{2\pi}{3}$. **Q16.** $-\frac{3\pi}{4}, \frac{3\pi}{4}$. **Q17.** $-\frac{3\pi}{4}, \frac{\pi}{4}$. **Q18.** $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$. **Q19.** $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$. **Q20.** $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$. **Q21.** $\frac{\pi}{8}, \frac{5\pi}{8}$. **Q22.** $\frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$. **Q23.** $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$. **Q24.** $\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$.

Fully-worked solutions

Q1. $\sin x = \frac{\sqrt{3}}{2}$; base angle $\frac{\pi}{3}$; sine positive in Q1, Q2: $x = \frac{\pi}{3}$ or $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Q2. $\sin x = \frac{1}{\sqrt{2}}$; base angle $\frac{\pi}{4}$; Q1, Q2: $x = \frac{\pi}{4}$ or $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

Q3. $\cos x = -\frac{1}{2}$; base angle $\frac{\pi}{3}$; cosine negative in Q2, Q3: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Q4. Base angle $\frac{\pi}{4}$; tangent positive in Q1, Q3: $x = \frac{\pi}{4}$ or $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Q5. $\tan x = -\frac{1}{\sqrt{3}}$; base angle $\frac{\pi}{6}$; tangent negative in Q2, Q4: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ or $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Q6. $\sin x = -\frac{1}{2}$; base angle $\frac{\pi}{6}$; sine negative in Q3, Q4: $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ or $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Q7. Base angle $\frac{\pi}{3}$; cosine positive in Q1, Q4: $x = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Q8. Base angle $\frac{\pi}{3}$; sine positive in Q1, Q2: $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

Q9. Base angle $\frac{\pi}{3}$; tangent negative in Q2, Q4: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Q10. $\cos x = -\frac{\sqrt{3}}{2}$; base angle $\frac{\pi}{6}$; cosine negative in Q2, Q3: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ or $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

Q11. $\sin x = -\frac{1}{\sqrt{2}}$; base angle $\frac{\pi}{4}$; sine negative in Q3, Q4: $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ or $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Q12. Base angle $\frac{\pi}{6}$; tangent positive in Q1, Q3: $x = \frac{\pi}{6}$ or $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

Q13. $\sin x = -1$ at the single point $x = \frac{3\pi}{2}$ in $[0, 2\pi]$ (the minimum of the sine curve).

Q14. $\cos x = 0$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ in $[0, 2\pi]$.

Q15. $\sin x = \frac{\sqrt{3}}{2}$; base angle $\frac{\pi}{3}$; Q1, Q2. Both solutions $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ lie in $[-\pi, \pi]$.

Q16. $\cos x = -\frac{1}{\sqrt{2}}$; base angle $\frac{\pi}{4}$; cosine negative in Q2, Q3. Over $[-\pi, \pi]$ the Q3 solution is the negative angle: $x = \frac{3\pi}{4}$ or $x = -\frac{3\pi}{4}$.

Q17. Base angle $\frac{\pi}{4}$; tangent positive in Q1, Q3. Over $[-\pi, \pi]$ the Q3 solution is $-\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$: $x = -\frac{3\pi}{4}, \frac{\pi}{4}$.

Q18. $\cos x = -\frac{1}{2}$; base angle $\frac{\pi}{3}$; Q2, Q3 gives $\frac{2\pi}{3}, \frac{4\pi}{3}$ in $[0, 2\pi]$. Adding 2π (period) gives the rest of $[0, 4\pi]$: $\frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$ and $\frac{4\pi}{3} + 2\pi = \frac{10\pi}{3}$. $\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$.

Q19. Let $u = 2x$; scaling, $u \in [0, 4\pi]$. Base angle $\frac{\pi}{6}$; sine positive in Q1, Q2: $u = \frac{\pi}{6}, \frac{5\pi}{6}$, and adding 2π, $\frac{13\pi}{6}, \frac{17\pi}{6}$. Divide by 2: $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

Q20. $\cos(2x) = \frac{1}{\sqrt{2}}$. Let $u = 2x \in [0, 4\pi]$; base angle $\frac{\pi}{4}$; cosine positive in Q1, Q4: $u = \frac{\pi}{4}$, $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$, and adding 2π, $\frac{9\pi}{4}, \frac{15\pi}{4}$. Divide by 2: $x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$.

Q21. Let $u = 2x \in [0, 2\pi]$; base angle $\frac{\pi}{4}$; tangent positive in Q1, Q3: $u = \frac{\pi}{4}$ or $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$. Divide by 2: $x = \frac{\pi}{8}, \frac{5\pi}{8}$.

Q22. Let $u = 3x$; scaling, $u \in [0, 3\pi]$. Base angle $\frac{\pi}{3}$; sine positive in Q1, Q2: $u = \frac{\pi}{3}, \frac{2\pi}{3}$, and adding 2π, $\frac{7\pi}{3}, \frac{8\pi}{3}$ (both $\leq 3\pi$). Divide by 3: $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$.

Q23. $\sin(2x) = -\frac{1}{\sqrt{2}}$. Let $u = 2x \in [0, 4\pi]$; base angle $\frac{\pi}{4}$; sine negative in Q3, Q4: $u = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$, $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$, and adding 2π, $\frac{13\pi}{4}, \frac{15\pi}{4}$. Divide by 2: $x = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$.

Q24. $\sin(3x) = -\frac{1}{2}$. Let $u = 3x$; scaling, $u \in [0, 6\pi]$. Base angle $\frac{\pi}{6}$; sine negative in Q3, Q4: $u = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$, $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$, and adding 2π then 4π, $\frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}, \frac{35\pi}{6}$ (all $\leq 6\pi$). Divide by 3: $x = \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$. $\therefore$ six solutions — three times as many as a single-argument equation, reflecting $n = 3$.

Theory

The graphs of $y=\sin t$ and $y=\cos t$ each have **period** 2π, **amplitude** 1 and range $[-1,1]$. Applying a dilation, a possible reflection and a horizontal translation gives the general forms

$$y=a\sin n(t-\varepsilon),\ y=a\cos n(t-\varepsilon).$$

Amplitude. The amplitude is $|a|$ — a dilation of factor $|a|$ from the t-axis. The graph oscillates between $y=|a|$ and $y=-|a|$, so the **range is** $[-|a|,|a|]$. If $a<0$ the graph is also **reflected in the t-axis**.

Period. The period is

$$\text{period}=\frac{2\pi}{n}.$$

A value $n>1$ compresses the graph horizontally (more cycles); $0<n<1$ stretches it.

Phase (horizontal translation). In the form $y=a\sin n(t-\varepsilon)$ the graph of $y=a\sin nt$ is translated ε units to the **right**; $y=a\sin n(t+\varepsilon)$ translates it ε units to the **left**. The same applies to cosine.

If the rule is given as $y=a\sin(nt+c)$, first **factor out** n: $y=a\sin n\left(t+\frac{c}{n}\right)$, then read the translation as $\frac{c}{n}$.

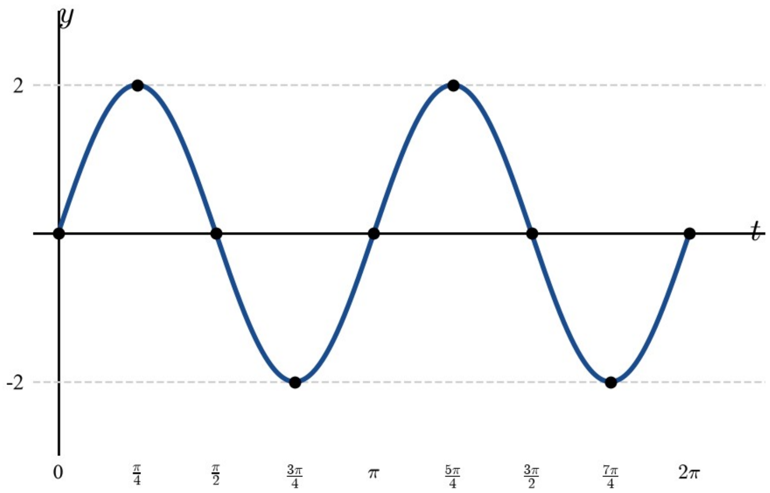
To sketch: (1) state the amplitude and draw the guide lines $y=\pm|a|$; (2) find the period and mark the **key points** at quarter-period intervals ($1/4\times\text{period}$), starting from the translated start; (3) for $\sin$ the cycle starts on the mean line $y=0$, for $\cos$ it starts at a maximum (or minimum if $a<0$); (4) draw a smooth curve over one or two periods.

Worked examples

Example 1 — scaffolded

Sketch $y=2\sin 2t$ for $t\in[0,2\pi]$, showing the amplitude and period.

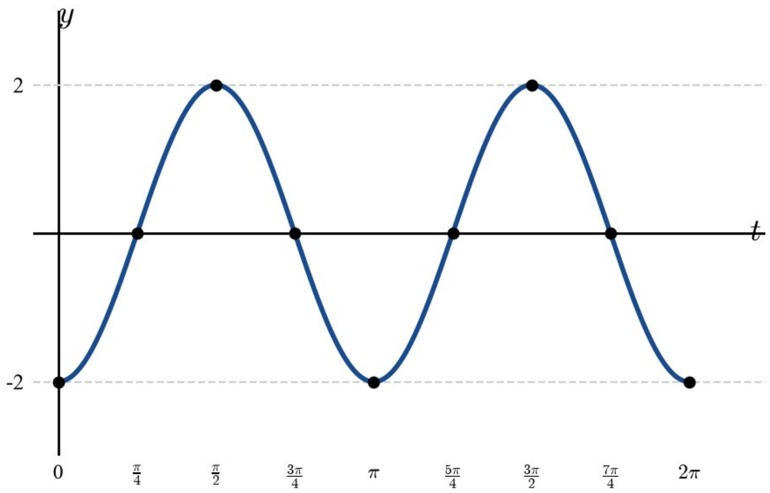
Working	Comment
Amplitude $=|2|=2$, so the graph oscillates between $y=2$ and $y=-2$.	Read $a=2$.
Period $=\frac{2\pi}{2}=\pi$.	Here $n=2$.
Quarter-period $=\frac{\pi}{4}$. Key points from $t=0$: $(0,0)$, $(\pi/4,2)$, $(\pi/2,0)$, $(3\pi/4,-2)$, $(\pi,0)$.	A sine cycle: mean → max → mean → min → mean.
Repeat the cycle on $[\pi,2\pi]$.	Two full periods fit in $[0,2\pi]$.



Example 2 — scaffolded

Sketch $y = 2\sin 2\left(t - \frac{\pi}{4}\right)$ for $t \in [0, 2\pi]$.

Working	Comment
Amplitude $= 2$; period $= \frac{2\pi}{2} = \pi$.	Same a and n as Example 1.
Translation: $\frac{\pi}{4}$ units to the **right** (the form is $2\sin 2(t - \pi/4)$).	$\varepsilon = \pi/4$ with a minus sign $\Rightarrow$ right.
The cycle now starts at $t = \frac{\pi}{4}$: $(\pi/4, 0)$, $(\pi/2, 2)$, $(3\pi/4, 0)$, $(\pi, -2)$, $(5\pi/4, 0)$.	Shift every key point of Example 1 right by $\pi/4$.
At $t = 0$: $y = 2\sin(-\pi/2) = -2$ (the graph starts at its minimum).	Check the endpoint.

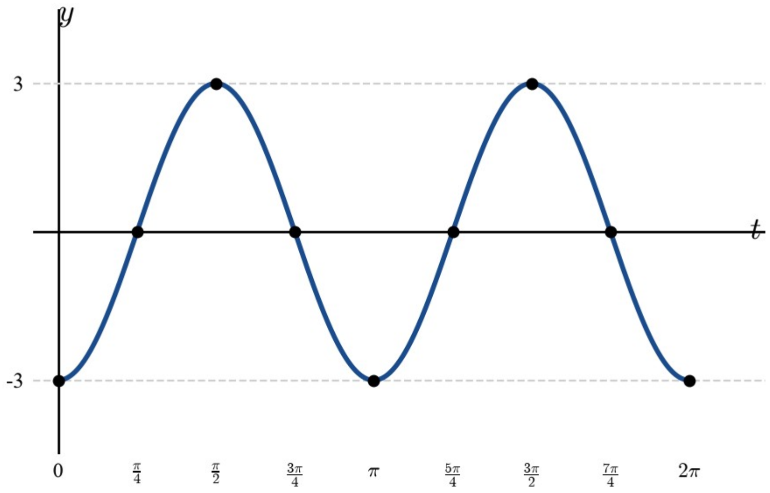


Example 3 — unscaffolded

Sketch $y = -3\cos 2t$ for $t \in [0, 2\pi]$.

Amplitude $= |-3| = 3$; period $= \frac{2\pi}{2} = \pi$. Since $a = -3 < 0$, the graph is **reflected in the t-axis**, so it starts at a **minimum**: at $t = 0$, $y = -3$.

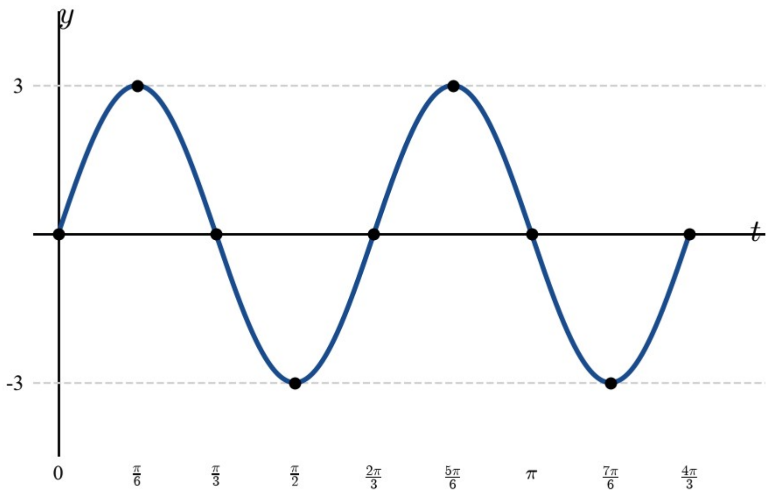
Key points: $(0,-3)$, $(\pi/4,0)$, $(\pi/2,3)$, $(3\pi/4,0)$, $(\pi,-3)$, repeating on $[\pi,2\pi]$. Range $[-3,3]$.



Example 4 — unscaffolded

Sketch $y=3\cos 3\left(t-\frac{\pi}{6}\right)$ for $t\in[0,4\pi/3]$.

Amplitude $=3$; period $=\frac{2\pi}{3}$; translation $\frac{\pi}{6}$ to the **right**. A cosine cycle starts at a maximum, so the maximum is at $t=\frac{\pi}{6}$: the point $(\pi/6,3)$. Quarter-period $=\frac{\pi}{6}$, giving key points $(\pi/6,3)$, $(\pi/3,0)$, $(\pi/2,-3)$, $(2\pi/3,0)$, $(5\pi/6,3)$. Range $[-3,3]$.



Practice questions

Scaffolded

For each: (a) state the amplitude; (b) state the period; (c) state any reflection or translation; (d) sketch the graph over the given interval.

Q1. $y=4\sin 3t$, $t\in[0,4\pi/3]$.

Q2. $y=2\cos 4t$, $t\in[0,\pi]$.

Q3. $y=-3\sin t$, $t\in[0,2\pi]$.

Q4. $y = 3\sin 2(t - \pi/4)$, $t \in [0, 2\pi]$.

Q5. $y = 2\cos 3(t - \pi/6)$, $t \in [0, 4\pi/3]$.

Q6. $y = -2\cos 2t$, $t \in [0, 2\pi]$.

Unscaffolded

State the amplitude, the period and any reflection/translation, then sketch each graph over one or two complete periods.

Q7. $y = 3\sin 4t$

Q8. $y = 5\cos 2t$

Q9. $y = 2\sin \frac{t}{2}$

Q10. $y = -4\sin 2t$

Q11. $y = -\cos 3t$

Q12. $y = 2\cos 2(t - \pi/4)$

Q13. $y = 3\sin 2(t + \pi/4)$

Q14. $y = 4\sin 3(t - \pi/6)$

Q15. $y = 2\cos 3(t + \pi/6)$

Q16. $y = 3\sin(2t - \pi/2)$ *(factor out the* 2 *first)*

Q17. $y = 2\cos(2t + \pi/2)$ *(factor out the* 2 *first)*

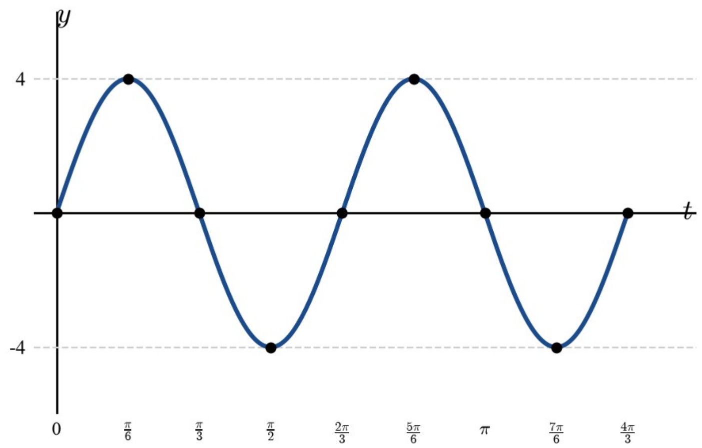
Q18. $y = -3\sin 2(t - \pi/4)$

Answers

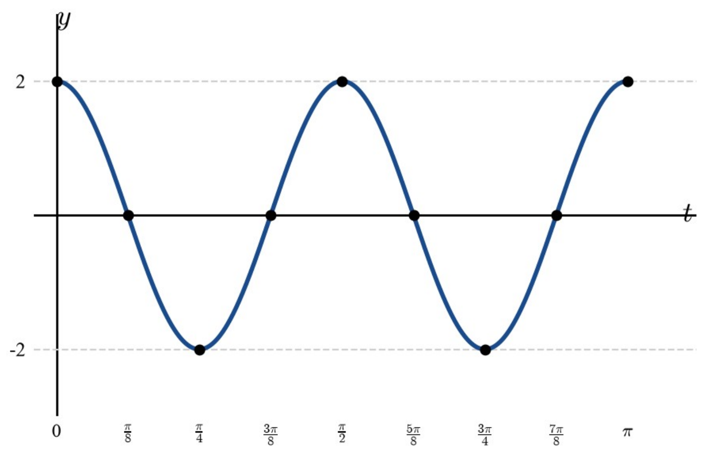
Q1. Amplitude 4; period $2\pi/3$. **Q2.** Amplitude 2; period $\pi/2$. **Q3.** Amplitude 3; period 2π; reflected in the t-axis. **Q4.** Amplitude 3; period π; translated $\pi/4$ right. **Q5.** Amplitude 2; period $2\pi/3$; translated $\pi/6$ right. **Q6.** Amplitude 2; period π; reflected in the t-axis. **Q7.** Amplitude 3; period $\pi/2$. **Q8.** Amplitude 5; period π. **Q9.** Amplitude 2; period 4π. **Q10.** Amplitude 4; period π; reflected in the t-axis. **Q11.** Amplitude 1; period $2\pi/3$; reflected in the t-axis. **Q12.** Amplitude 2; period π; translated $\pi/4$ right. **Q13.** Amplitude 3; period π; translated $\pi/4$ left. **Q14.** Amplitude 4; period $2\pi/3$; translated $\pi/6$ right. **Q15.** Amplitude 2; period $2\pi/3$; translated $\pi/6$ left. **Q16.** $=3\sin 2(t-\pi/4)$: amplitude 3; period π; translated $\pi/4$ right. **Q17.** $=2\cos 2(t+\pi/4)$: amplitude 2; period π; translated $\pi/4$ left. **Q18.** Amplitude 3; period π; reflected in the t-axis and translated $\pi/4$ right.

Fully-worked solutions

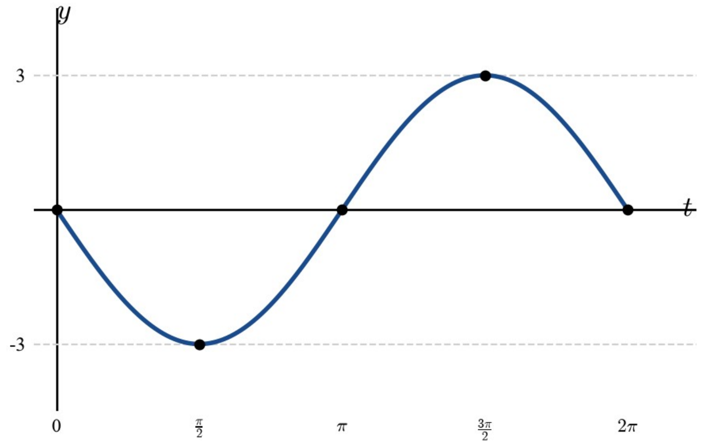
Q1. $y=4\sin 3t$. Amplitude $|4|=4$; period $\dfrac{2\pi}{3}$; no translation. Key points at quarter-period $\pi/6$: $(0,0),(\pi/6,4),(\pi/3,0),(\pi/2,-4),(2\pi/3,0)$, then repeat.



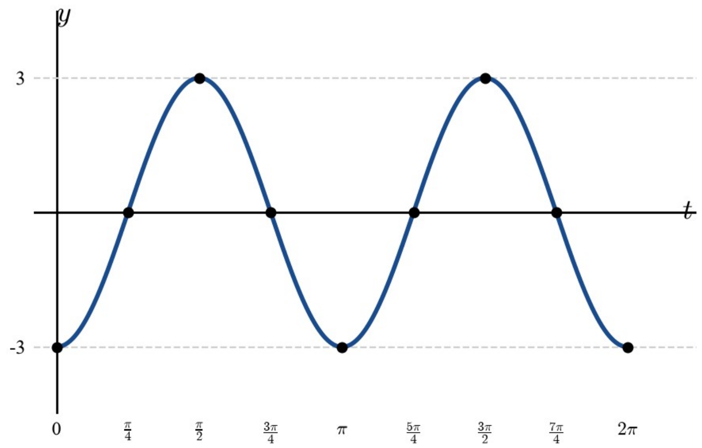
Q2. $y=2\cos 4t$. Amplitude 2; period $\dfrac{2\pi}{4}=\dfrac{\pi}{2}$; cosine starts at a maximum $(0,2)$. Quarter-period $\pi/8$.



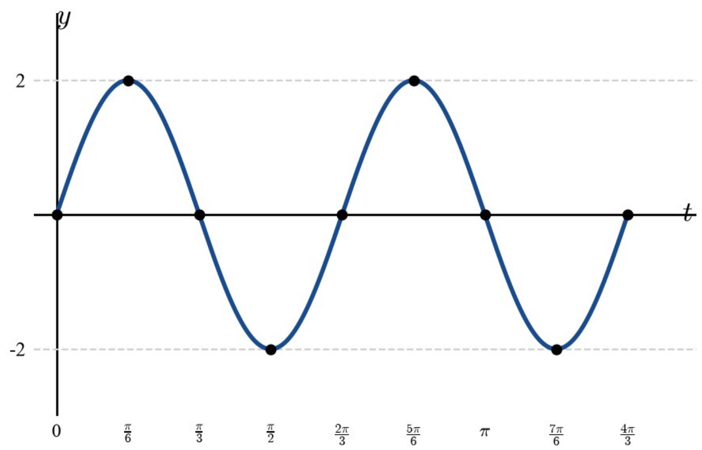
Q3. $y=-3\sin t$. Amplitude 3; period 2π; $a<0$ so reflected in the t-axis (starts by going **down** from $(0,0)$): $(0,0),(\pi/2,-3),(\pi,0),(3\pi/2,3),(2\pi,0)$.



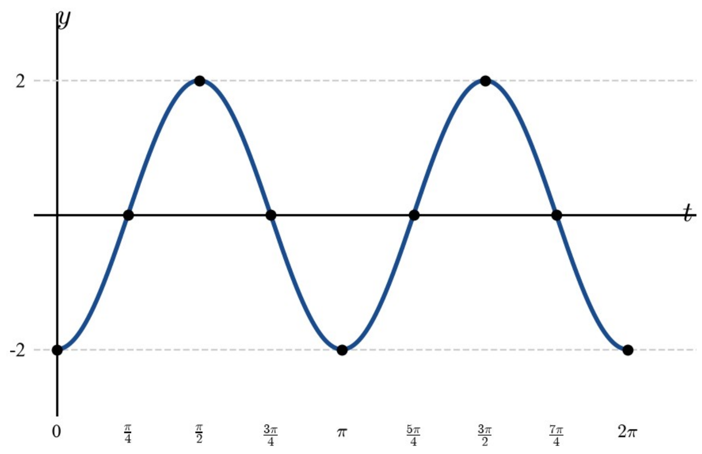
Q4. $y = 3\sin 2(t - \pi/4)$. Amplitude 3; period π; translated $\pi/4$ right. The cycle starts on the mean line at $t = \pi/4$; at $t = 0$, $y = 3\sin(-\pi/2) = -3$.



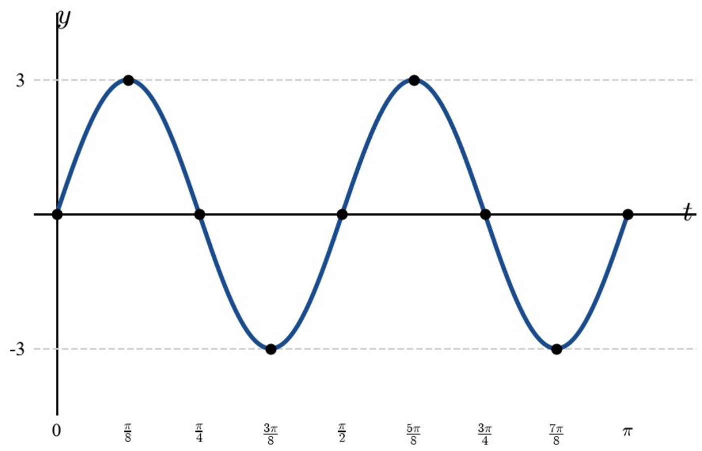
Q5. $y = 2\cos 3(t - \pi/6)$. Amplitude 2; period $\dfrac{2\pi}{3}$; translated $\pi/6$ right, so the maximum is at $t = \pi/6$: $(\pi/6, 2)$.



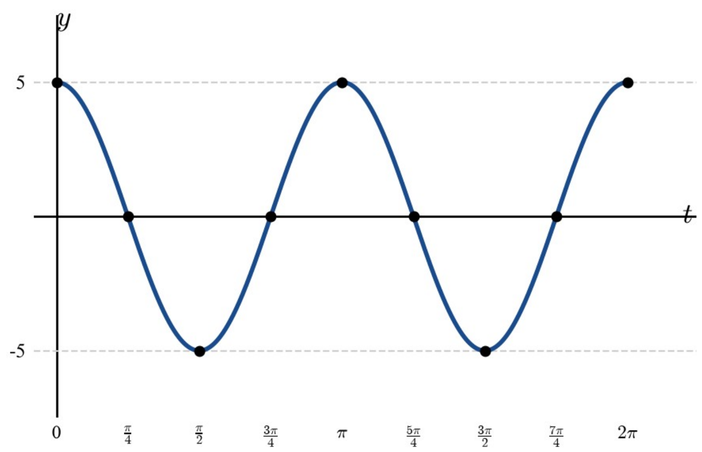
Q6. $y = -2\cos 2t$. Amplitude 2; period π; reflected, so it starts at a **minimum** $(0, -2)$: $(0, -2), (\pi/4, 0), (\pi/2, 2), (3\pi/4, 0), (\pi, -2)$.



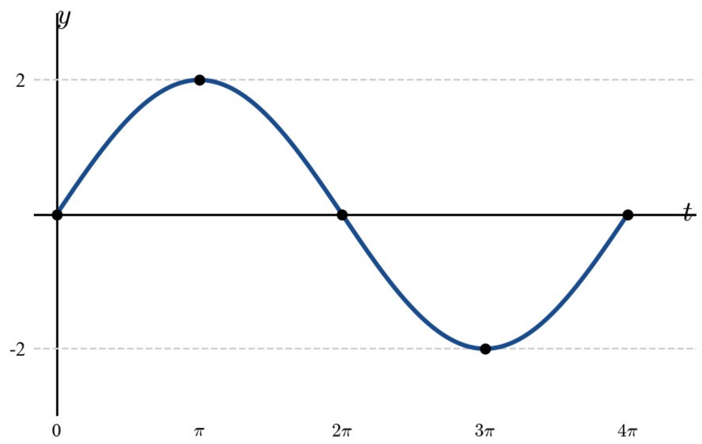
Q7. $y=3\sin 4t$. Amplitude 3; period $\frac{2\pi}{4}=\frac{\pi}{2}$; quarter-period $\pi/8$; starts on the mean line rising.



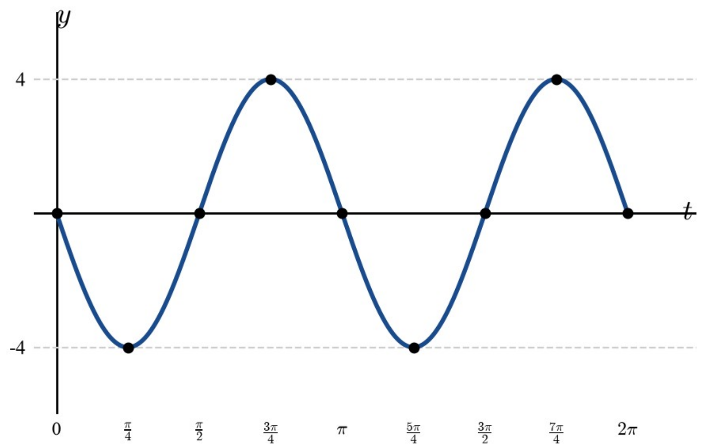
Q8. $y=5\cos 2t$. Amplitude 5; period π; maximum at $(0,5)$.



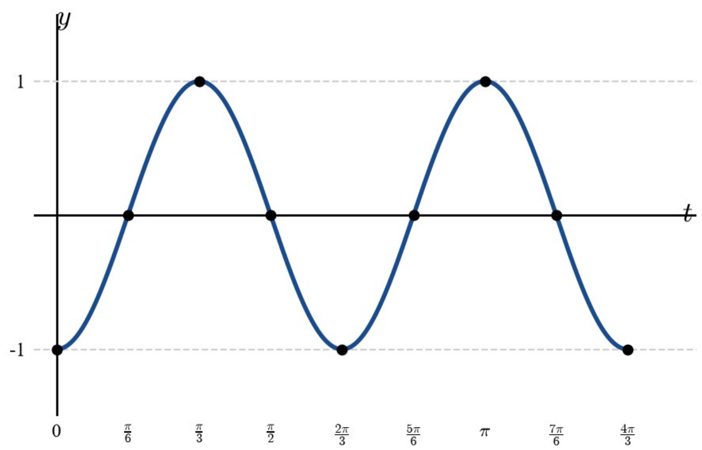
Q9. $y=2\sin\frac{t}{2}$. Here $n=1/2$, so period $=\frac{2\pi}{1/2}=4\pi$; amplitude 2. Key points at quarter-period π: $(0,0),(\pi,2),(2\pi,0),(3\pi,-2),(4\pi,0)$.



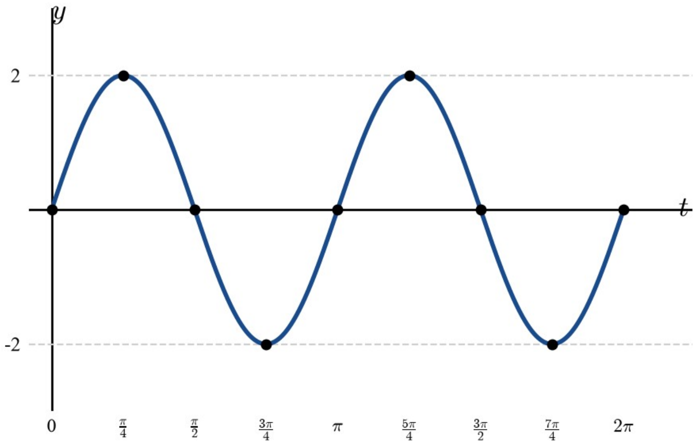
Q10. $y = -4\sin 2t$. Amplitude 4; period π; reflected, so it starts by going **down** from $(0, 0)$.



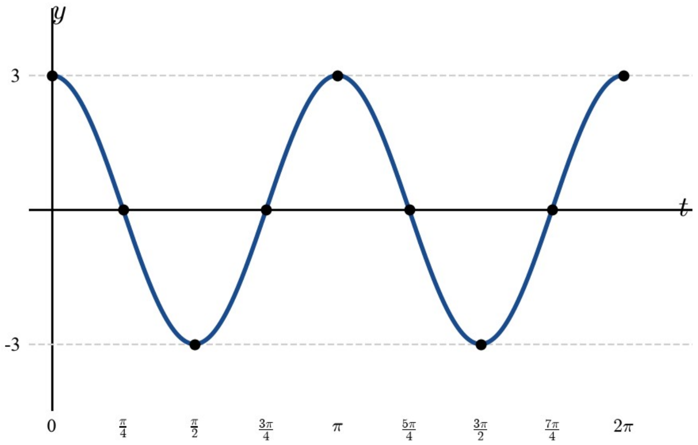
Q11. $y = -\cos 3t$. Amplitude 1; period $\dfrac{2\pi}{3}$; reflected, so it starts at a **minimum** $(0, -1)$.



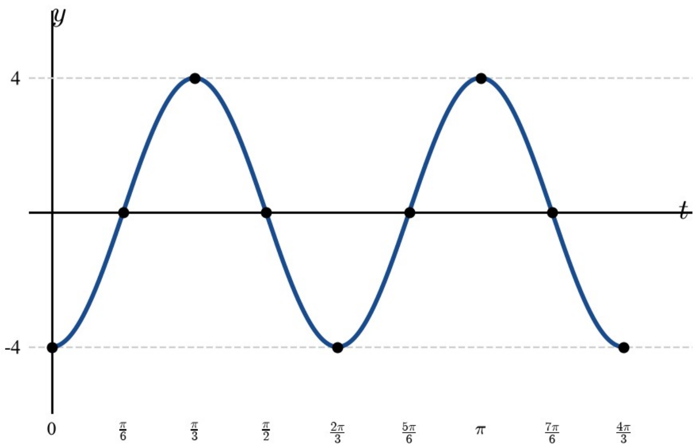
Q12. $y = 2\cos 2(t - \pi/4)$. Amplitude 2; period π; translated $\pi/4$ right, so the maximum is at $t = \pi/4$; at $t = 0$, $y = 2\cos(-\pi/2) = 0$.



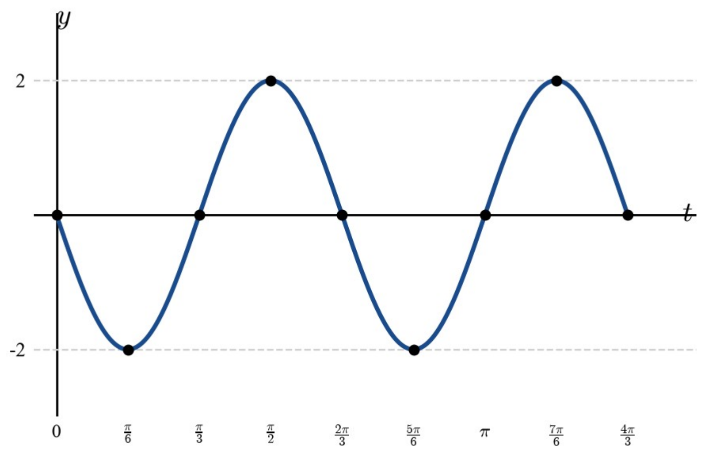
Q13. $y = 3\sin 2(t + \pi/4)$. Amplitude 3; period π; translated $\pi/4$ **left**. At $t = 0$, $y = 3\sin \pi/2 = 3$ (a maximum).



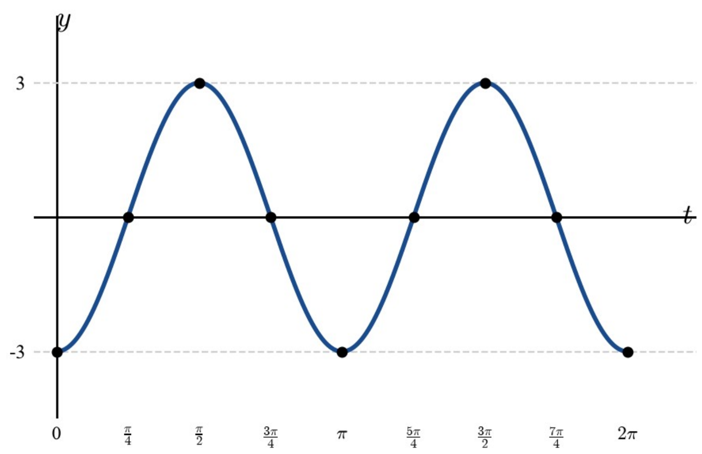
Q14. $y = 4\sin 3(t - \pi/6)$. Amplitude 4; period $\dfrac{2\pi}{3}$; translated $\pi/6$ right; the cycle starts on the mean line at $t = \pi/6$.



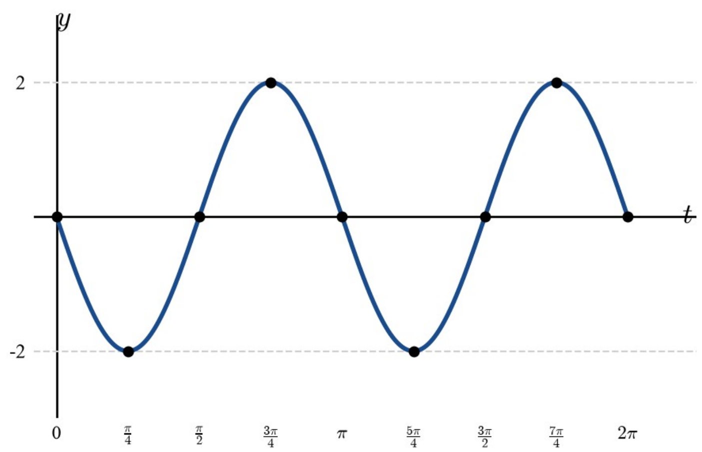
Q15. $y = 2\cos 3(t + \pi/6)$. Amplitude 2; period $\dfrac{2\pi}{3}$; translated $\pi/6$ **left**. At $t = 0$, $y = 2\cos \pi/2 = 0$; the maximum is at $t = -\pi/6 + 2\pi/3 = \pi/2$ within the first period shown.



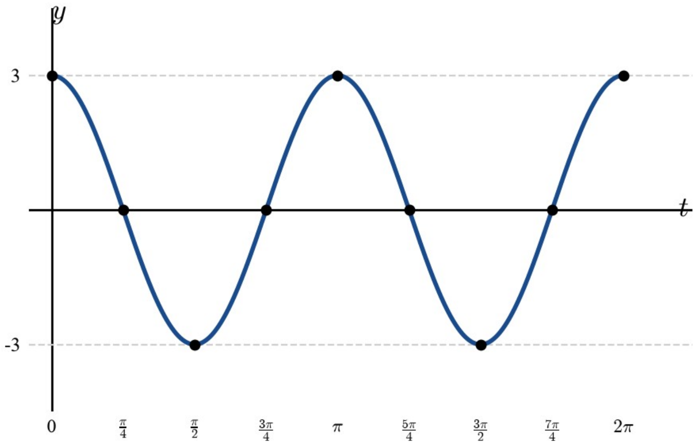
Q16. $y = 3\sin(2t - \pi/2)$. Factor the 2: $y = 3\sin 2(t - \pi/4)$. Amplitude 3; period π; translated $\pi/4$ right (identical to Q4's graph).



Q17. $y = 2\cos(2t + \pi/2)$. Factor the 2: $y = 2\cos 2(t + \pi/4)$. Amplitude 2; period π; translated $\pi/4$ **left**. At $t = 0$, $y = 2\cos \pi/2 = 0$.



Q18. $y = -3\sin 2(t - \pi/4)$. Amplitude 3; period π; translated $\pi/4$ right **and** reflected in the t-axis. At $t = 0$, $y = -3\sin(-\pi/2) = 3$ (a maximum).



y
3
-3
t
0
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{4}$
$\frac{3\pi}{2}$
$\frac{7\pi}{4}$
2π

Theory

This section combines all of the sinusoid transformations. For

$$y = a\sin\left(n\left(t-\varepsilon\right)\right)+b \text{ and } y = a\cos\left(n\left(t-\varepsilon\right)\right)+b\,,$$

the constants have these meanings.

- **Amplitude** $|a|$ — half the distance between the maximum and minimum. If $a<0$ the basic graph is reflected in its midline.
- **Period** $\frac{2\pi}{n}$ — the length of one full cycle ($n>0$).
- **Phase (horizontal translation)** ε — the graph of $y = a\sin nt$ is moved ε units in the t-direction (right if $\varepsilon>0$, left if $\varepsilon<0$).
- **Vertical translation** b — the whole graph moves up by b, so the **midline** (the mean position the graph oscillates about) becomes $y=b$ instead of $y=0$.

Key consequences of the $+b$**.** The graph oscillates $|a|$ above and below the line $y=b$, so

$$\text{maximum} = b+|a|,\ \text{minimum} = b-|a|,\ \text{range} = \left[b-|a|,\, b+|a|\right].$$

To sketch: draw the midline $y=b$ (dashed); mark the levels $y=b+|a|$ and $y=b-|a|$; mark one period $\frac{2\pi}{n}$ starting from the phase shift; then draw the sine or cosine shape oscillating between the two levels. The y-intercept is found by substituting $t=0$, and any t-intercepts by solving $y=0$.

Recognising a circular model from data. A set of measurements suggests a **circular (sinusoidal) model** when the values **repeat over a fixed cycle** and **oscillate steadily between a maximum and a minimum about a central level** (the midline). Tides, daily temperature, hours of daylight, and the height of a Ferris-wheel seat all behave this way. The three key numbers can be read straight from the data:

$$\text{amplitude} = \frac{1}{2}\left(\max-\min\right),\ \text{period} = \text{length of one cycle}\,,\ \text{midline (mean)} = \frac{1}{2}\left(\max+\min\right).$$

A model $y = a\cos\left(n\left(t-\varepsilon\right)\right)+b$ (or the sine form) then has $|a|=$ amplitude, $b=$ midline and $n=\frac{2\pi}{\text{period}}$. Use cos when the cycle **starts at a maximum or minimum**, and sin when it **starts on the midline**; the phase ε shifts the graph so it begins where the data does.

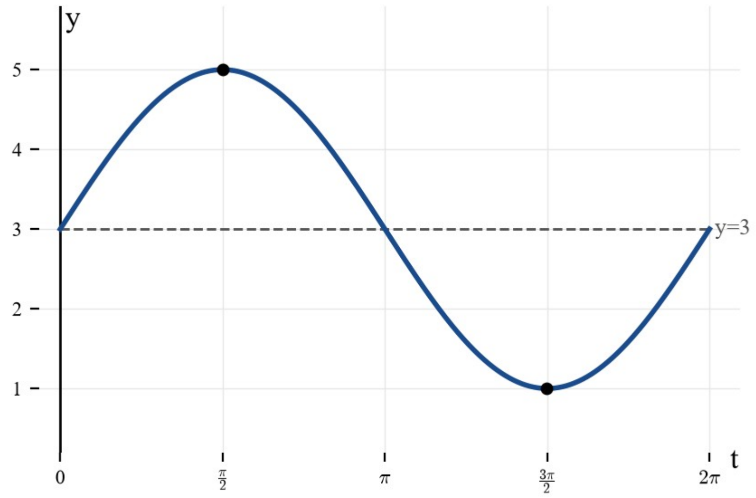
Worked examples

Example 1 — scaffolded

Sketch $y = 2\sin t+3$ for $t\in\left[0,2\pi\right]$, and state its amplitude, period, range, and its maximum and minimum values.

Working	Comment
$a=2$, $n=1$, $\varepsilon=0$, $b=3$. Amplitude $=2$, period $=\frac{2\pi}{1}=2\pi$.	Read the constants from the rule.
Midline $y=3$; the graph oscillates 2 above and below it.	The $+3$ raises the mean position to $y=3$.
Maximum $=3+2=5$ (at $t=\pi/2$); minimum $=3-2=1$ (at $t=3\pi/2$).	$max=b+\lvert a\rvert$, $min=b-\lvert a\rvert$.
Range $=\left[1,5\right]$. y-intercept: $t=0 \Rightarrow y=2\sin 0+3=3$.	Since the minimum $1>0$, there are no t

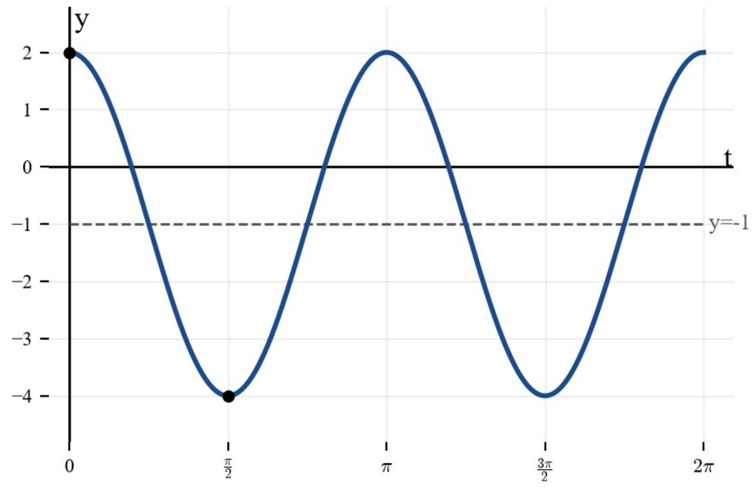
Working	Comment
	-intercepts.



Example 2 — scaffolded

Sketch $y=3\cos 2t-1$ for $t\in[0,2\pi]$, and state its amplitude, period and range.

Working	Comment
$a=3$, $n=2$, $b=-1$. Amplitude $=3$, period $=\dfrac{2\pi}{2}=\pi$.	Two full cycles fit in $[0,2\pi]$.
Midline $y=-1$. Maximum $=-1+3=2$; minimum $=-1-3=-4$.	$max=b+\|a\|$, $min=b-\|a\|$.
Cosine starts at its maximum: $y=2$ at $t=0,\pi,2\pi$; minimum $y=-4$ at $t=\pi/2,3\pi/2$.	$\cos 0=1$, so $t=0$ gives the maximum.
Range $=[-4,2]$; y-intercept $(0,2)$.	



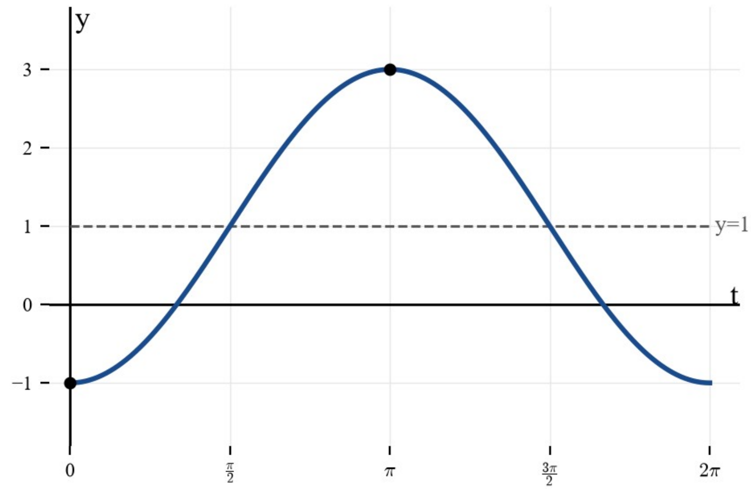
Example 3 — unscaffolded

Sketch $y=1-2\cos t$ for $t\in[0,2\pi]$, stating the range and all axis intercepts.

Rewrite as $y=-2\cos t+1$: $a=-2$ (a reflection), $b=1$, period 2π. Amplitude 2, midline $y=1$, so the range is $[1-2,1+2]=[-1,3]$.

Because $a<0$, the maximum $y=3$ occurs where $\cos t=-1$, i.e. $t=\pi$; the minimum $y=-1$ where $\cos t=1$, i.e. $t=0$ and $t=2\pi$.

y-intercept: $t=0\Rightarrow y=1-2=-1$. t-intercepts: $1-2\cos t=0\Rightarrow \cos t=1/2\Rightarrow t=\pi/3$ or $t=5\pi/3$.

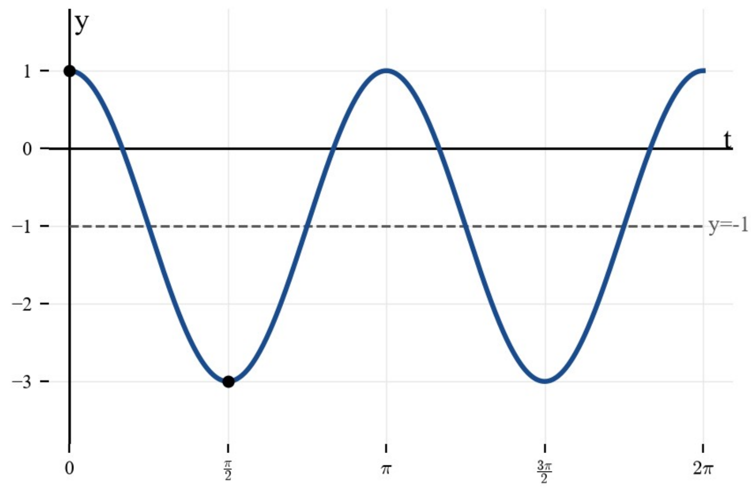


Example 4 — unscaffolded

Sketch $y=2\sin\left(2\left(t+\pi/4\right)\right)-1$ for $t\in[0,2\pi]$.

Here $a=2$, $n=2$, $\varepsilon=-\pi/4$ (a translation $\pi/4$ to the **left**), $b=-1$. Amplitude 2, period $\frac{2\pi}{2}=\pi$, midline $y=-1$, range $[-3,1]$.

$\therefore$ maximum 1, minimum -3. (As a check, $\sin\left(2\left(t+\pi/4\right)\right)=\sin\left(2t+\pi/2\right)=\cos 2t$, so the rule is also $y=2\cos 2t-1$, which starts at its maximum at $t=0$.) t-intercepts occur at $t=\pi/6,5\pi/6,7\pi/6,11\pi/6$.



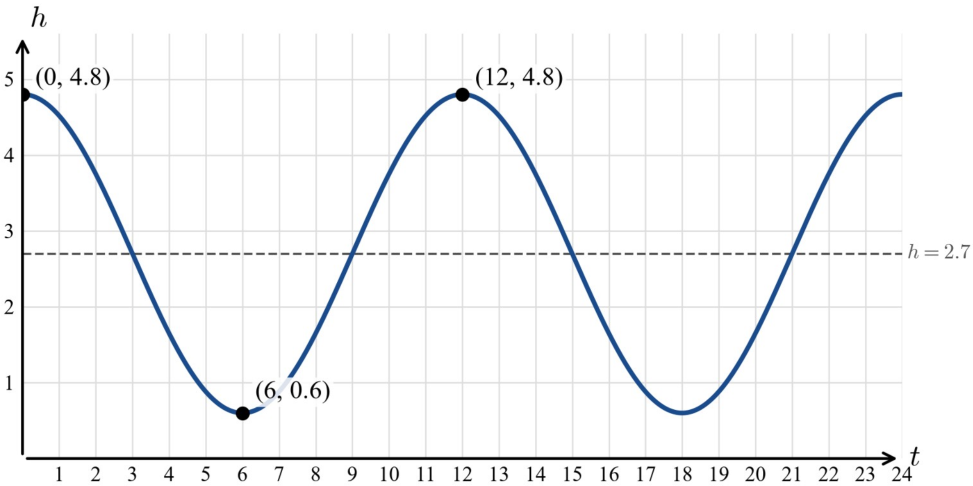
Example 5 — scaffolded

On a certain day the tide height at a jetty peaks at 4.8 m, drops to a low of 0.6 m, and one full rise-and-fall cycle takes 12 hours; the pattern repeats through the day. Model the height h metres as a function of the time t hours after a high tide, and state the amplitude, period and midline.

Working	Comment
The height repeats every 12 h between a fixed maximum 4.8 and minimum 0.6, oscillating about a mean, so a circular model fits.	Repeats over a fixed cycle about a midline.
Amplitude $=\frac{1}{2}(4.8-0.6)=\frac{1}{2}(4.2)=2.1$.	Half the max–min gap.
Midline $b=\frac{1}{2}(4.8+0.6)=\frac{1}{2}(5.4)=2.7$.	Mean of the max and min.

Working	Comment
Period $=12$, so $n=\frac{2\pi}{12}=\frac{\pi}{6}$.	$n=\frac{2\pi}{\text{period}}$.
At $t=0$ the tide is at its **maximum**, so use cosine with no phase shift: $h=2.1\cos\left(\frac{\pi}{6}t\right)+2.7$.	$\cos$ starts at a maximum.

Check: $t=0 \Rightarrow h=2.1(1)+2.7=4.8$ (high tide); $t=6 \Rightarrow h=2.1\cos\pi+2.7=-2.1+2.7=0.6$ (low tide). The graph confirms the model repeats every 12 h between 4.8 m and 0.6 m about the midline $h=2.7$. (Equivalently, since a high tide is at $t=0$, one may write $h=2.1\sin\left(\frac{\pi}{6}(t+3)\right)+2.7$.)



Practice questions

Scaffolded

Q1. For $y=4\sin t-2$: (a) state the amplitude and period; (b) state the midline and hence the maximum and minimum values; (c) state the range; (d) sketch the graph for $t\in[0,2\pi]$.

Q2. For $y=2\cos t+2$: (a) state the amplitude, period and midline; (b) find the maximum and minimum values; (c) find the axis intercepts; (d) sketch the graph for $t\in[0,2\pi]$.

Q3. For $y=3\sin 2t+1$: (a) state the amplitude and period; (b) state the range; (c) sketch the graph for $t\in[0,2\pi]$.

Q4. For $y=\cos t-3$: (a) state the amplitude, period and midline; (b) state the range; (c) explain why the graph has no t-intercepts; (d) sketch the graph for $t\in[0,2\pi]$.

Q5. For $y=2-3\sin t$: (a) state the amplitude and midline; (b) find the maximum and minimum values and where they occur; (c) sketch the graph for $t\in[0,2\pi]$.

Q6. For $y=2\cos(t-\pi/2)+1$: (a) state the amplitude, period and midline; (b) show that the rule can be written $y=2\sin t+1$; (c) state the range; (d) sketch the graph for $t\in[0,2\pi]$.

Unscaffolded

For each of the following, state the amplitude, period and range, and sketch the graph over one stated interval, labelling the maximum and minimum.

Q7. $y=3\sin t+4$, $t\in[0,2\pi]$

Q8. $y=2\cos t-3$, $t\in[0,2\pi]$

Q9. $y = 4\cos 2t + 1,\ t \in [0, 2\pi]$

Q10. $y = 2\sin 3t - 1,\ t \in [0, 2\pi]$

Q11. $y = 3 - 2\sin t,\ t \in [0, 2\pi]$

Q12. $y = 3\cos t - 1,\ t \in [0, 2\pi]$

Q13. $y = 2\sin\left(2\left(t - \pi/6\right)\right) + 1,\ t \in [0, 2\pi]$

Q14. $y = 4\cos\left(t + \pi/3\right) - 2,\ t \in [0, 2\pi]$

Q15. $y = \sin 2t + 3,\ t \in [0, 2\pi]$

Q16. $y = 2 - \cos 2t,\ t \in [0, 2\pi]$

Q17. $y = 3\sin t/2 + 1,\ t \in [0, 4\pi]$

Q18. $y = 2\cos\left(3\left(t + \pi/6\right)\right) - 1,\ t \in [0, 2\pi]$

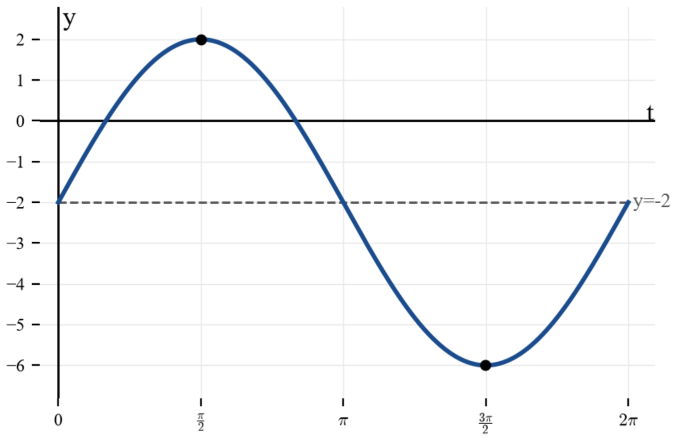
Q19. A Ferris wheel carries a passenger whose height above the ground is recorded each minute. The height rises to a maximum of 43 m, falls to a minimum of 3 m, and one complete revolution takes 8 minutes; this pattern repeats for the whole ride. (a) Explain why a circular (sinusoidal) model is appropriate for these data. (b) State the amplitude, period and midline. (c) Taking $t = 0$ at the lowest point, write a rule of the form $h = a\cos\left(n\left(t - \varepsilon\right)\right) + b$ (or a sine form) for the height h metres after t minutes.

Answers

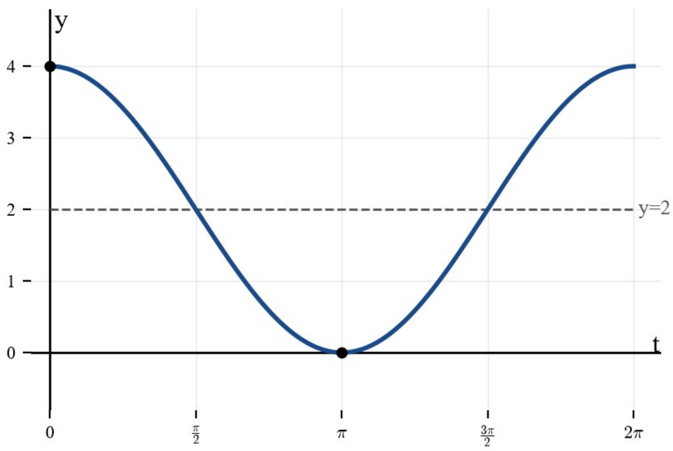
Q1. Amp 4, period 2π, midline -2; max 2, min -6; range $[-6,2]$. **Q2.** Amp 2, period 2π, midline 2; max 4, min 0; intercepts $(0,4)$ and $t=\pi$; range $[0,4]$. **Q3.** Amp 3, period π; range $[-2,4]$. **Q4.** Amp 1, period 2π, midline -3; range $[-4,-2]$; no t-intercepts (whole graph below the t-axis). **Q5.** Amp 3, midline 2; max 5 at $t=3\pi/2$, min -1 at $t=\pi/2$. **Q6.** Amp 2, period 2π, midline 1; range $[-1,3]$. **Q7.** Amp 3, period 2π, range $[1,7]$. **Q8.** Amp 2, period 2π, range $[-5,-1]$. **Q9.** Amp 4, period π, range $[-3,5]$. **Q10.** Amp 2, period $2\pi/3$, range $[-3,1]$. **Q11.** Amp 2, period 2π, range $[1,5]$. **Q12.** Amp 3, period 2π, range $[-4,2]$. **Q13.** Amp 2, period π, range $[-1,3]$. **Q14.** Amp 4, period 2π, range $[-6,2]$. **Q15.** Amp 1, period π, range $[2,4]$. **Q16.** Amp 1, period π, range $[1,3]$. **Q17.** Amp 3, period 4π, range $[-2,4]$. **Q18.** Amp 2, period $2\pi/3$, range $[-3,1]$. **Q19.** (a) The height repeats every 8 min between a fixed max and min about a mean → circular model. (b) Amp 20, period 8 min, midline 23. (c) $h=-20\cos\left(\frac{\pi}{4}t\right)+23$ (equivalently $h=20\sin\left(\frac{\pi}{4}(t-2)\right)+23$).

Fully-worked solutions

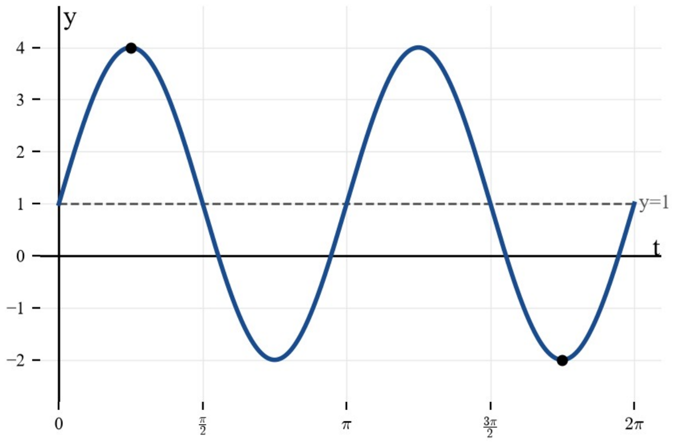
Q1. $y=4\sin t-2$. Amplitude 4, period $\frac{2\pi}{1}=2\pi$, midline $y=-2$. Maximum $=-2+4=2$ (at $t=\pi/2$); minimum $=-2-4=-6$ (at $t=3\pi/2$). Range $[-6,2]$; y-intercept $(0,-2)$.



Q2. $y=2\cos t+2$. Amplitude 2, period 2π, midline $y=2$. Maximum $=4$ (at $t=0,2\pi$); minimum $=0$ (at $t=\pi$). The graph just touches the t-axis at $t=\pi$; y-intercept $(0,4)$. Range $[0,4]$.

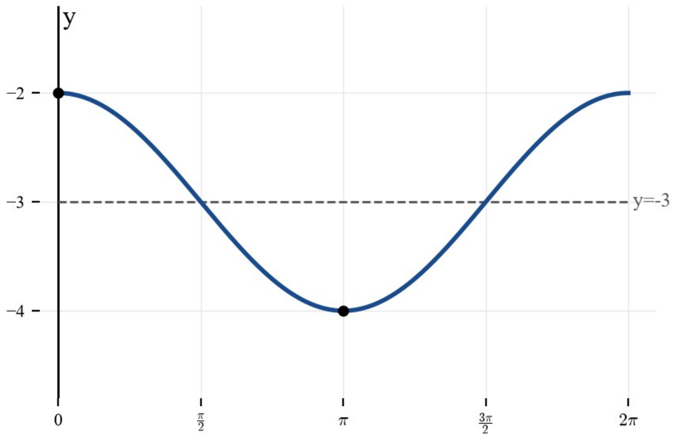


Q3. $y=3\sin 2t+1$. Amplitude 3, period $\frac{2\pi}{2}=\pi$ (two cycles in $[0,2\pi]$), midline $y=1$. Maximum 4, minimum -2, range $[-2,4]$; y-intercept $(0,1)$.

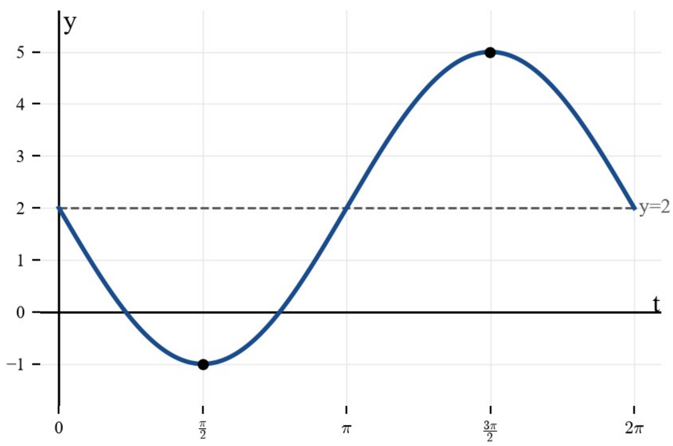


Q4. $y = \cos t - 3$. Amplitude 1, period 2π, midline $y = -3$. Maximum $= -2$ (at $t = 0, 2\pi$); minimum $= -4$ (at $t = \pi$). Range $[-4, -2]$. Since the maximum value $-2 < 0$, the entire graph lies below the t-axis, so there are **no** t**-intercepts**.

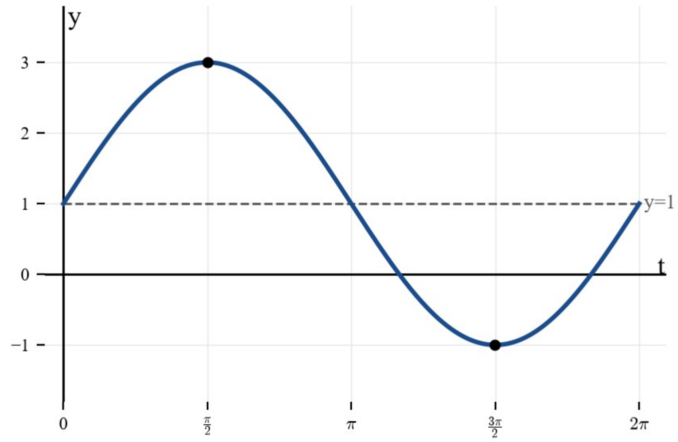
t



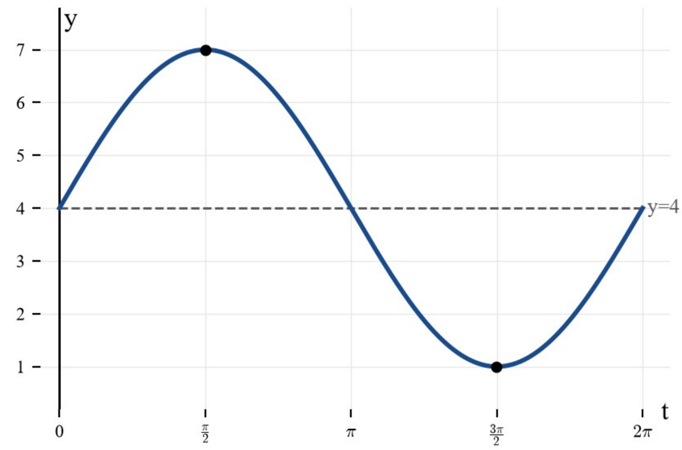
Q5. $y = 2 - 3\sin t = -3\sin t + 2$. Amplitude 3, midline $y = 2$, range $[-1, 5]$. Because $a = -3 < 0$, the maximum $y = 5$ occurs where $\sin t = -1$, i.e. $t = 3\pi/2$, and the minimum $y = -1$ where $\sin t = 1$, i.e. $t = \pi/2$. y-intercept $(0, 2)$.



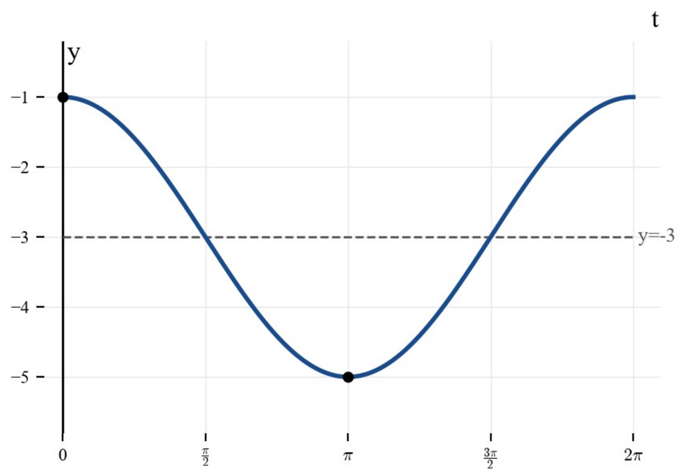
Q6. $y = 2\cos(t - \pi/2) + 1$. Since $\cos(t - \pi/2) = \sin t$, the rule is $y = 2\sin t + 1$. Amplitude 2, period 2π, midline $y = 1$, range $[-1, 3]$; y-intercept $(0, 1)$.



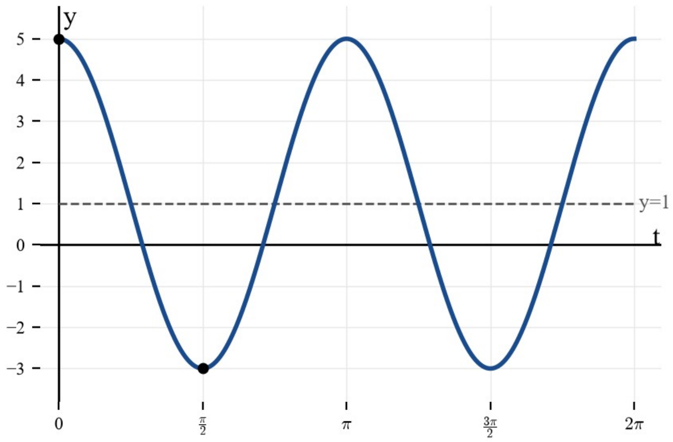
Q7. $y = 3\sin t + 4$. Amplitude 3, period 2π, midline $y = 4$; max 7 (at $t = \pi/2$), min 1 (at $t = 3\pi/2$); range $[1, 7]$.



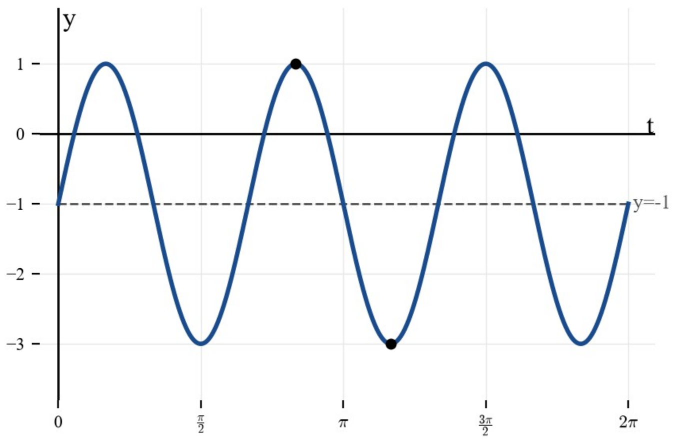
Q8. $y = 2\cos t - 3$. Amplitude 2, period 2π, midline $y = -3$; max -1, min -5; range $[-5, -1]$ (entirely below the t-axis).



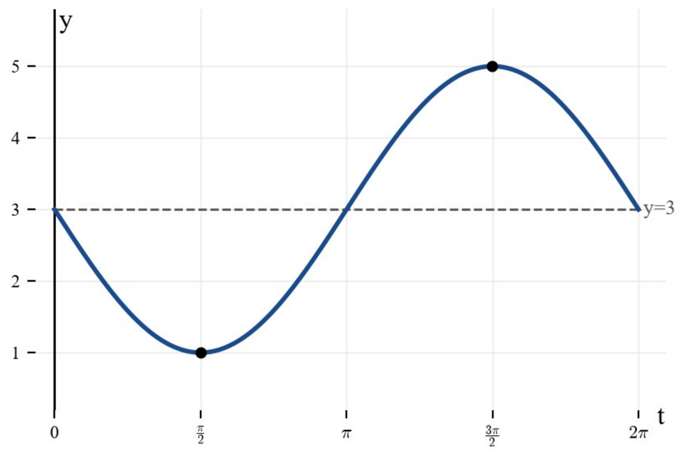
Q9. $y = 4\cos 2t + 1$. Amplitude 4, period π, midline $y = 1$; max 5, min -3; range $[-3, 5]$; y-intercept $(0, 5)$.



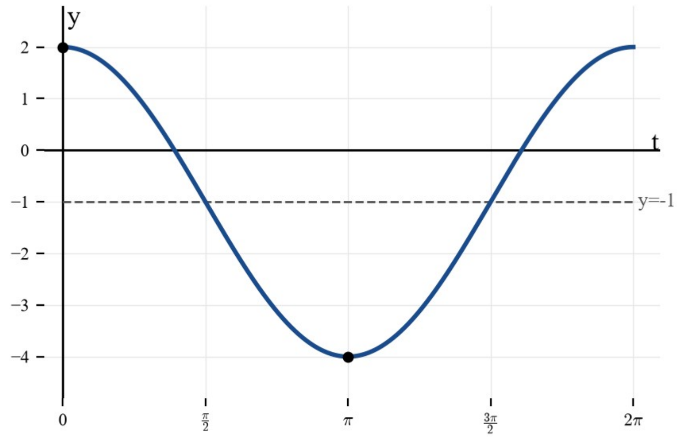
Q10. $y=2\sin 3t-1$. Amplitude 2, period $\frac{2\pi}{3}$ (three cycles in $[0,2\pi]$), midline $y=-1$; max 1, min -3; range $[-3,1]$.



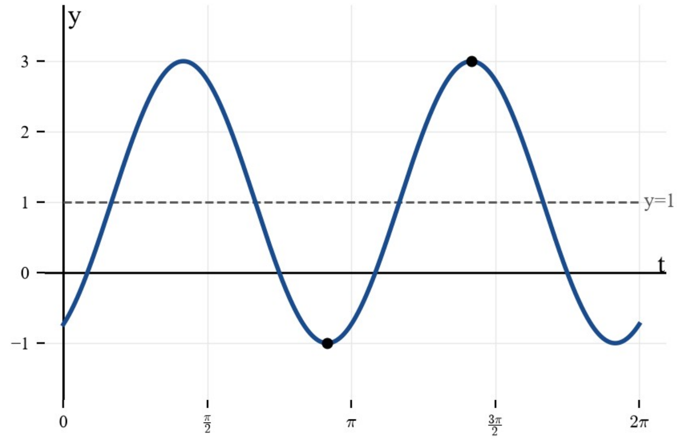
Q11. $y=3-2\sin t$. Amplitude 2, period 2π, midline $y=3$; max 5 (at $t=3\pi/2$, where $\sin t=-1$), min 1 (at $t=\pi/2$); range $[1,5]$.



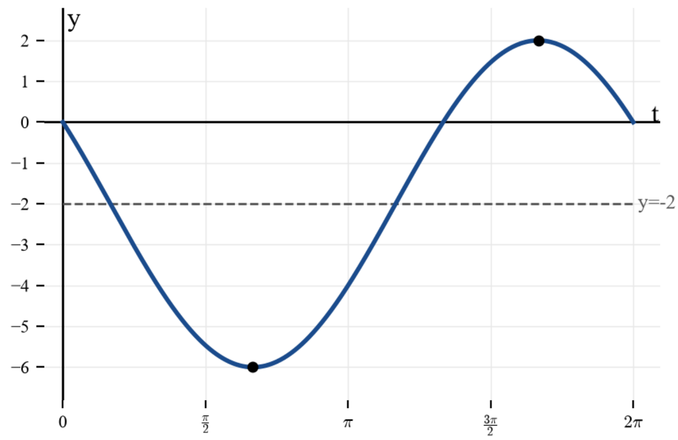
Q12. $y=3\cos t-1$. Amplitude 3, period 2π, midline $y=-1$; max 2, min -4; range $[-4,2]$; y-intercept $(0,2)$.



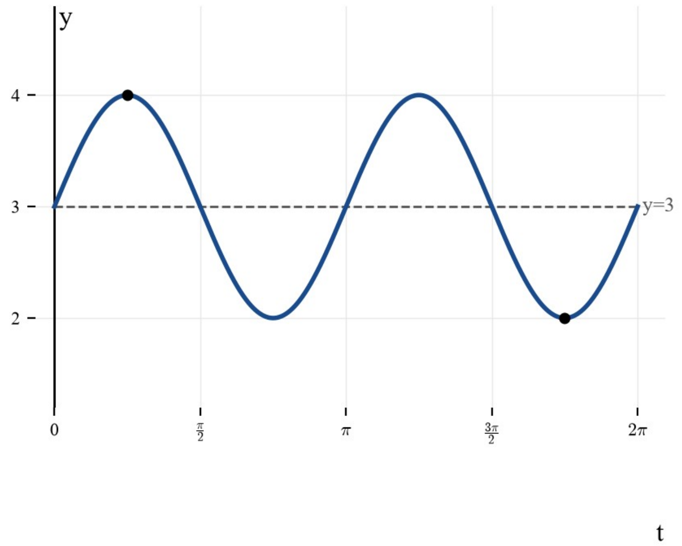
Q13. $y=2\sin\left(2\left(t-\pi/6\right)\right)+1$. Amplitude 2, period π, phase $\pi/6$ right, midline $y=1$; max 3, min -1; range $[-1,3]$.



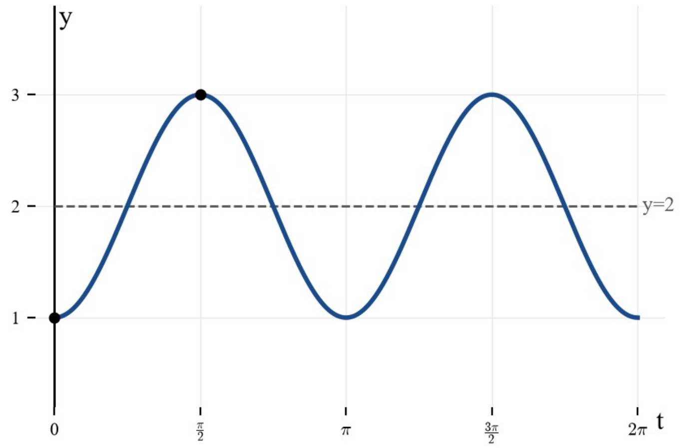
Q14. $y=4\cos\left(t+\pi/3\right)-2$. Amplitude 4, period 2π, phase $\pi/3$ left, midline $y=-2$; max 2, min -6; range $[-6,2]$; y-intercept $(0,0)$.



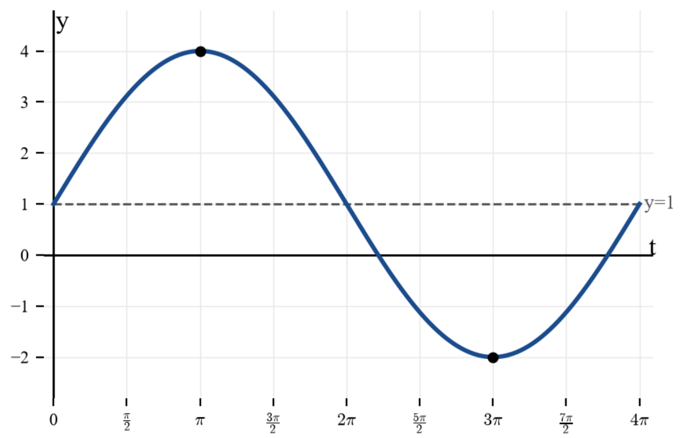
Q15. $y=\sin 2t+3$. Amplitude 1, period π, midline $y=3$; max 4, min 2; range $[2,4]$; y-intercept $(0,3)$.



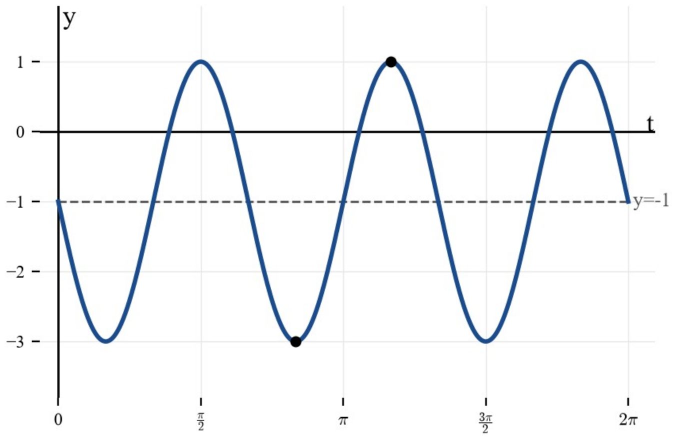
Q16. $y=2-\cos 2t=-\cos 2t+2$. Amplitude 1, period π, midline $y=2$; max 3 (where $\cos 2t=-1$), min 1 (where $\cos 2t=1$); range $[1,3]$; y-intercept $(0,1)$.



Q17. $y=3\sin t/2+1$. Here $n=1/2$, so the period is $\dfrac{2\pi}{1/2}=4\pi$ (one cycle over $[0,4\pi]$). Amplitude 3, midline $y=1$; max 4 (at $t=\pi$), min -2 (at $t=3\pi$); range $[-2,4]$; y-intercept $(0,1)$.



Q18. $y=2\cos\left(3\left(t+\pi/6\right)\right)-1$. Amplitude 2, period $\dfrac{2\pi}{3}$, phase $\pi/6$ left, midline $y=-1$; max 1, min -3; range $[-3,1]$.



Q19. (a) The recorded heights **repeat over a fixed cycle** (every 8 minutes) and **oscillate between a constant maximum** 43 **m and minimum** 3 **m about a central level**, which is exactly the behaviour a sinusoid describes, so a circular model is appropriate. (b) Amplitude $=\frac{1}{2}(43-3)=\frac{1}{2}(40)=20$ m; period = one revolution $=8$ min; midline $=\frac{1}{2}(43+3)=\frac{1}{2}(46)=23$ m. (c) From the period, $n=\frac{2\pi}{8}=\frac{\pi}{4}$, with $b=23$ and $|a|=20$. The passenger starts at the **minimum** at $t=0$, so a reflected cosine (which starts at its minimum) fits with no phase shift: $h=-20\cos\left(\frac{\pi}{4}t\right)+23$. Check: $t=0\Rightarrow h=-20+23=3$ (lowest) and $t=4\Rightarrow h=-20\cos\pi+23=20+23=43$ (highest), as required. Equivalently $h=20\sin\left(\frac{\pi}{4}(t-2)\right)+23$.

Chapter 12 Circular functions — 12J Complementary relations and the Pythagorean identity

Theory

The Pythagorean identity. For any angle θ, the point $(\cos\theta, \sin\theta)$ lies on the unit circle $x^2+y^2=1$. Therefore

$$\sin^2\theta+\cos^2\theta=1,$$

where $\sin^2\theta$ means $(\sin\theta)^2$. Rearranging gives the two useful forms

$$\sin^2\theta=1-\cos^2\theta, \cos^2\theta=1-\sin^2\theta.$$

Together with $\tan\theta=\frac{\sin\theta}{\cos\theta}$, this identity lets you find all three ratios from any one of them.

Finding the other ratios. Given one ratio and the quadrant of θ:

1. use the Pythagorean identity to find the **magnitude** of a second ratio;
2. use the quadrant to choose the correct **sign** (recall **ASTC**: in quadrant 1 **A**ll ratios are positive, in quadrant 2 only **S**ine, in quadrant 3 only **T**angent, in quadrant 4 only **C**osine);
3. find the third ratio using $\tan\theta=\frac{\sin\theta}{\cos\theta}$.

Complementary relations. Because the angles θ and $\pi/2-\theta$ are complementary, the “co”-functions are interchanged:

$$\sin(\pi/2-\theta)=\cos\theta, \cos(\pi/2-\theta)=\sin\theta.$$

Simplifying and proving. Expressions can be simplified, and simple identities proved, by substituting these relations and factorising — for example $1-\cos^2\theta=\sin^2\theta$, and the difference of squares $1-\sin^2\theta=(1-\sin\theta)(1+\sin\theta)$.

Worked examples

Example 1 — scaffolded

Given $\cos\theta=\frac{3}{5}$ and θ is in the fourth quadrant, find $\sin\theta$ and $\tan\theta$.

Working	Comment
$\sin^2\theta=1-\cos^2\theta=1-\frac{9}{25}=\frac{16}{25}$.	Rearrange the Pythagorean identity.
$\sin\theta=\pm\frac{4}{5}$; in quadrant 4 sine is negative, so $\sin\theta=-\frac{4}{5}$.	Take the square root, then fix the sign by the quadrant (ASTC).
$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{-4/5}{3/5}=-\frac{4}{3}$.	Divide sine by cosine.

Example 2 — scaffolded

Given $\sin\theta=\frac{5}{13}$ and θ is in the second quadrant, find $\cos\theta$ and $\tan\theta$.

Working	Comment
$\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{25}{169} = \frac{144}{169}$.	Pythagorean identity.
$\cos\theta = \pm\frac{12}{13}$; in quadrant 2 cosine is negative, so $\cos\theta = -\frac{12}{13}$.	Quadrant 2: only sine is positive.
$\tan\theta = \frac{5/13}{-12/13} = -\frac{5}{12}$.	$\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Example 3 — unscaffolded

Prove that $\frac{1-\cos^2\theta}{\sin\theta} = \sin\theta$ (for $\sin\theta \neq 0$).

$$\text{LHS} = \frac{1-\cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta = \text{RHS}.$$

$\therefore \frac{1-\cos^2\theta}{\sin\theta} = \sin\theta$.

Example 4 — unscaffolded

Given $\tan\theta = -2$ and θ is in the fourth quadrant, find the exact values of $\sin\theta$ and $\cos\theta$.

Since $\tan\theta = \frac{\sin\theta}{\cos\theta} = -2$, we have $\sin\theta = -2\cos\theta$. Substituting into $\sin^2\theta + \cos^2\theta = 1$:

$$4\cos^2\theta + \cos^2\theta = 1 \Rightarrow 5\cos^2\theta = 1 \Rightarrow \cos^2\theta = 1/5.$$

In quadrant 4 cosine is positive, so $\cos\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$, and $\sin\theta = -2\cos\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$.

$\therefore \sin\theta = -\frac{2\sqrt{5}}{5}, \cos\theta = \frac{\sqrt{5}}{5}$.

Practice questions

Scaffolded

Q1. Given $\cos\theta = \frac{4}{5}$ and θ is in the first quadrant: (a) find $\sin^2\theta$; (b) hence find $\sin\theta$; (c) find $\tan\theta$.

Q2. Given $\sin\theta = -\frac{8}{17}$ and θ is in the third quadrant, find $\cos\theta$ and $\tan\theta$.

Q3. (a) Write $\sin(\pi/2 - \theta)$ in terms of $\cos\theta$. (b) Hence simplify $\sin(\pi/2 - \theta)\tan\theta$.

Q4. Given $\tan\theta = \frac{3}{4}$ and θ is in the first quadrant, find $\sin\theta$ and $\cos\theta$.

Q5. (a) Explain why $1 - \sin^2\theta = \cos^2\theta$. (b) Hence simplify $\frac{1-\sin^2\theta}{\cos\theta}$.

Q6. Given $\cos\theta = -\frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$, find the exact values of $\sin\theta$ and $\tan\theta$.

Unscaffolded

Q7. Given $\sin\theta = \frac{3}{5}$ and θ is in the second quadrant, find $\cos\theta$ and $\tan\theta$.

Q8. Given $\cos\theta = -\frac{5}{13}$ and θ is in the third quadrant, find $\sin\theta$ and $\tan\theta$.

Q9. Given $\tan\theta = -\frac{3}{4}$ and θ is in the fourth quadrant, find $\sin\theta$ and $\cos\theta$.

Q10. Given $\sin\theta = -\frac{1}{2}$ and θ is in the fourth quadrant, find the exact values of $\cos\theta$ and $\tan\theta$.

Q11. Simplify $\frac{\cos(\pi/2-\theta)}{\sin\theta}$.

Q12. Simplify $\frac{\sin^2\theta}{1-\cos\theta}$.

Q13. Prove that $\frac{\cos\theta}{\sin(\pi/2-\theta)} = 1$.

Q14. Prove that $(\sin\theta+\cos\theta)^2 = 1+2\sin\theta\cos\theta$.

Q15. Given $\cos\theta = \frac{1}{4}$ and θ is in the fourth quadrant, find the exact values of $\sin\theta$ and $\tan\theta$.

Q16. Simplify $\frac{1-\cos^2\theta}{1-\sin^2\theta}$.

Q17. Given $\sin\theta = \frac{2}{3}$ and θ is obtuse, find the exact values of $\cos\theta$ and $\tan\theta$.

Q18. Prove that $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = \frac{2}{\cos^2\theta}$.

Answers

Q1. (a) $\frac{9}{25}$; (b) $\frac{3}{5}$; (c) $\frac{3}{4}$. **Q2.** $\cos\theta=-\frac{15}{17}$, $\tan\theta=\frac{8}{15}$. **Q3.** (a) $\cos\theta$; (b) $\sin\theta$. **Q4.** $\sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$. **Q5.** (a) from $\sin^2\theta+\cos^2\theta=1$; (b) $\cos\theta$. **Q6.** $\sin\theta=\frac{\sqrt{5}}{3}$, $\tan\theta=-\frac{\sqrt{5}}{2}$. **Q7.** $\cos\theta=-\frac{4}{5}$, $\tan\theta=-\frac{3}{4}$. **Q8.** $\sin\theta=-\frac{12}{13}$, $\tan\theta=\frac{12}{5}$. **Q9.** $\sin\theta=-\frac{3}{5}$, $\cos\theta=\frac{4}{5}$. **Q10.** $\cos\theta=\frac{\sqrt{3}}{2}$, $\tan\theta=-\frac{\sqrt{3}}{3}$. **Q11.** 1. **Q12.** $1+\cos\theta$. **Q13.** proof. **Q14.** proof. **Q15.** $\sin\theta=-\frac{\sqrt{15}}{4}$, $\tan\theta=-\sqrt{15}$. **Q16.** $\tan^2\theta$. **Q17.** $\cos\theta=-\frac{\sqrt{5}}{3}$, $\tan\theta=-\frac{2\sqrt{5}}{5}$. **Q18.** proof.

Fully-worked solutions

Q1. $\cos\theta=\frac{4}{5}$, quadrant 1. (a) $\sin^2\theta=1-\frac{16}{25}=\frac{9}{25}$. (b) Quadrant 1: sine positive, so $\sin\theta=\frac{3}{5}$. (c) $\tan\theta=\frac{3/5}{4/5}=\frac{3}{4}$.

Q2. $\cos^2\theta=1-\frac{64}{289}=\frac{225}{289}$, so $\cos\theta=\pm\frac{15}{17}$. Quadrant 3: cosine negative, $\cos\theta=-\frac{15}{17}$. $\tan\theta=\frac{-8/17}{-15/17}=\frac{8}{15}$.

Q3. (a) $\sin(\pi/2-\theta)=\cos\theta$. (b) $\sin(\pi/2-\theta)\tan\theta=\cos\theta\cdot\frac{\sin\theta}{\cos\theta}=\sin\theta$.

Q4. Let $\sin\theta=3k$, $\cos\theta=4k$ (same $k>0$ in quadrant 1, since $\tan\theta=3/4$). Then $9k^2+16k^2=1\Rightarrow 25k^2=1\Rightarrow k=1/5$. $\therefore \sin\theta=\frac{3}{5}, \cos\theta=\frac{4}{5}$.

Q5. (a) Rearranging $\sin^2\theta+\cos^2\theta=1$ gives $1-\sin^2\theta=\cos^2\theta$. (b) $\frac{1-\sin^2\theta}{\cos\theta}=\frac{\cos^2\theta}{\cos\theta}=\cos\theta$.

Q6. $\sin^2\theta=1-\frac{4}{9}=\frac{5}{9}$. In quadrant 2 sine is positive, so $\sin\theta=\frac{\sqrt{5}}{3}$. $\tan\theta=\frac{\sqrt{5}/3}{-2/3}=-\frac{\sqrt{5}}{2}$.

Q7. $\cos^2\theta=1-\frac{9}{25}=\frac{16}{25}$; quadrant 2 cosine negative, so $\cos\theta=-\frac{4}{5}$. $\tan\theta=\frac{3/5}{-4/5}=-\frac{3}{4}$.

Q8. $\sin^2\theta=1-\frac{25}{169}=\frac{144}{169}$; quadrant 3 sine negative, so $\sin\theta=-\frac{12}{13}$. $\tan\theta=\frac{-12/13}{-5/13}=\frac{12}{5}$.

Q9. $\tan\theta=-\frac{3}{4}$, so take $\sin\theta=3k$, $\cos\theta=-4k$ ($\tan\theta=3k/-4k=-3/4$). Then $9k^2+16k^2=1\Rightarrow k=1/5$. In quadrant 4 cosine is positive and sine negative, so $k=1/5$ gives $\cos\theta=\frac{4}{5}$, $\sin\theta=-\frac{3}{5}$.

Q10. $\cos^2\theta=1-\frac{1}{4}=\frac{3}{4}$; quadrant 4 cosine positive, so $\cos\theta=\frac{\sqrt{3}}{2}$. $\tan\theta=\frac{-1/2}{\sqrt{3}/2}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$.

Q11. $\frac{\cos(\pi/2-\theta)}{\sin\theta}=\frac{\sin\theta}{\sin\theta}=1$.

Q12. $\frac{\sin^2\theta}{1-\cos\theta}=\frac{1-\cos^2\theta}{1-\cos\theta}=\frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}=1+\cos\theta$.

Q13. $\sin(\pi/2-\theta)=\cos\theta$, so $\frac{\cos\theta}{\sin(\pi/2-\theta)}=\frac{\cos\theta}{\cos\theta}=1$.

Q14. $\left(\sin\theta+\cos\theta\right)^2=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=\left(\sin^2\theta+\cos^2\theta\right)+2\sin\theta\cos\theta=1+2\sin\theta\cos\theta$.

Q15. $\sin^2\theta=1-\dfrac{1}{16}=\dfrac{15}{16}$; quadrant 4 sine negative, so $\sin\theta=-\dfrac{\sqrt{15}}{4}$. $\tan\theta=\dfrac{-\sqrt{15}/4}{1/4}=-\sqrt{15}$.

Q16. $\dfrac{1-\cos^2\theta}{1-\sin^2\theta}=\dfrac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta$.

Q17. θ obtuse means quadrant 2. $\cos^2\theta=1-\dfrac{4}{9}=\dfrac{5}{9}$; cosine negative, so $\cos\theta=-\dfrac{\sqrt{5}}{3}$.

$\tan\theta=\dfrac{2/3}{-\sqrt{5}/3}=-\dfrac{2}{\sqrt{5}}=-\dfrac{2\sqrt{5}}{5}$.

Q18. LHS $=\dfrac{\left(1-\sin\theta\right)+\left(1+\sin\theta\right)}{\left(1+\sin\theta\right)\left(1-\sin\theta\right)}=\dfrac{2}{1-\sin^2\theta}=\dfrac{2}{\cos^2\theta}=$ RHS.

Theory

The **tangent function** is defined by

$$\tan x = \frac{\sin x}{\cos x}.$$

Asymptotes. $\tan x$ is undefined wherever $\cos x = 0$, that is at $x = \frac{\pi}{2} + n\pi$ for integer n. The graph has a **vertical asymptote** at each of these values.

x-intercepts. $\tan x = 0$ wherever $\sin x = 0$, that is at $x = n\pi$.

Period. The tangent function repeats every π, so its **period is** π (half the period of $\sin$ and $\cos$).

Range. Between consecutive asymptotes the curve increases from $-\infty$ to ∞, so the **range is** $\mathbb{R}$. There is no amplitude.

Exact values (first quadrant): $\tan 0 = 0$, $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $\tan \frac{\pi}{4} = 1$, $\tan \frac{\pi}{3} = \sqrt{3}$. Using symmetry, $\tan \frac{2\pi}{3} = -\sqrt{3}$ and $\tan \frac{3\pi}{4} = -1$.

Transformations.

- $y = a\tan(nx)$ has **period** $\frac{\pi}{n}$; its asymptotes are where $nx = \frac{\pi}{2} + k\pi$. The factor a is a dilation from the x-axis (the graph still rises through every value).
- $y = \tan(x - h)$ translates the graph h units horizontally, moving every asymptote by h.
- $y = -\tan x$ reflects the graph in the x-axis, so each branch **decreases**.

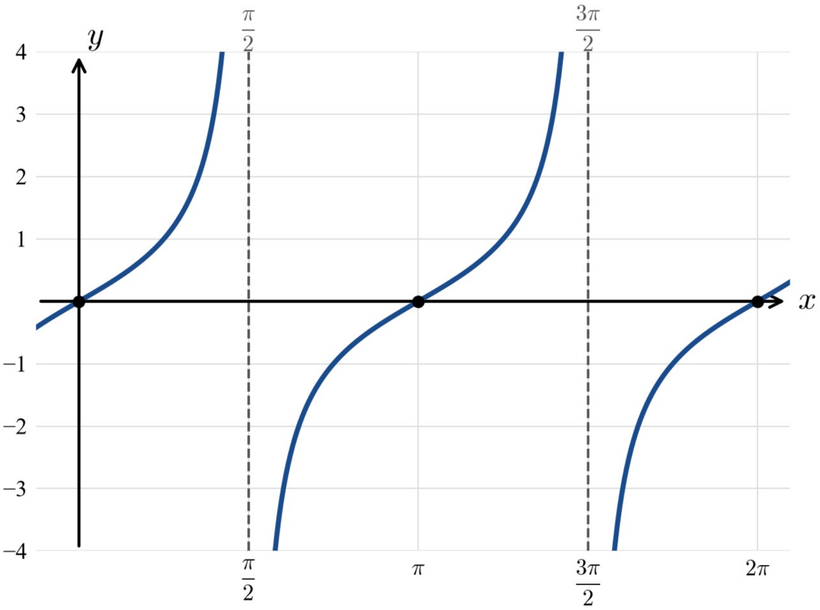
Solving $\tan x = k$**.** Because the period is π, once one solution is found every other solution is obtained by adding multiples of π: $x = \tan^{-1} k + n\pi$.

Worked examples

Example 1 — scaffolded

Sketch $y = \tan x$ for $x \in [0, 2\pi]$, showing the asymptotes and x-intercepts.

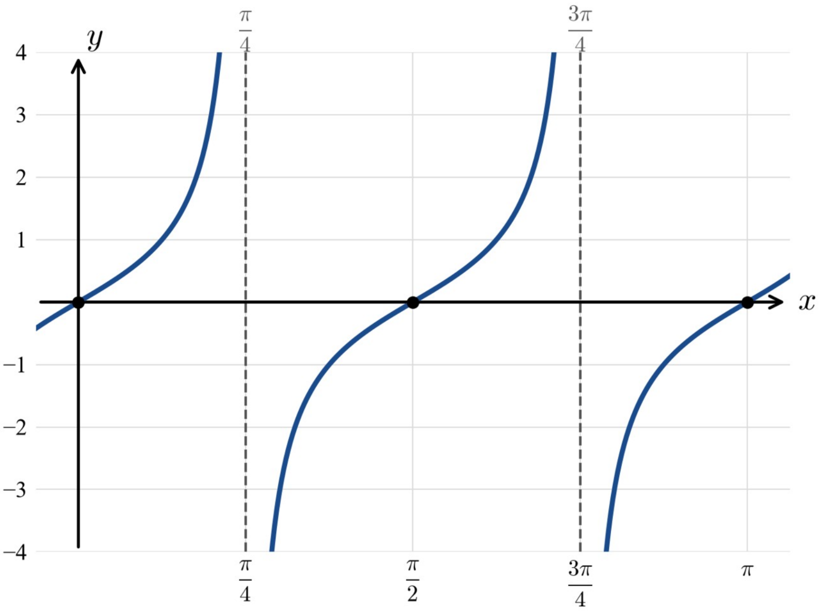
Working	Comment
Asymptotes where $\cos x = 0$: $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.	$\frac{\pi}{2} + n\pi$ inside $[0, 2\pi]$.
x-intercepts where $\sin x = 0$: $x = 0, \pi, 2\pi$.	$\tan x = 0$ at multiples of π.
Between each pair of asymptotes the curve rises from $-\infty$ to ∞, passing through the intercept.	Period π; range $\mathbb{R}$.



Example 2 — scaffolded

For $y = \tan 2x$, state the period and the asymptotes on $[0, \pi]$, and sketch the graph.

Working	Comment
Period $= \dfrac{\pi}{2}$.	For $y = \tan nx$ the period is $\dfrac{\pi}{n}$, here $n = 2$.
Asymptotes where $2x = \dfrac{\pi}{2} + k\pi$, so $x = \dfrac{\pi}{4}$ and $x = \dfrac{3\pi}{4}$.	Divide by 2.
x-intercepts where $2x = m\pi$: $x = 0, \dfrac{\pi}{2}, \pi$.	$\tan 2x = 0$.



Example 3 — unscaffolded

Solve $\tan x = \sqrt{3}$ for $x \in [0, 2\pi]$.

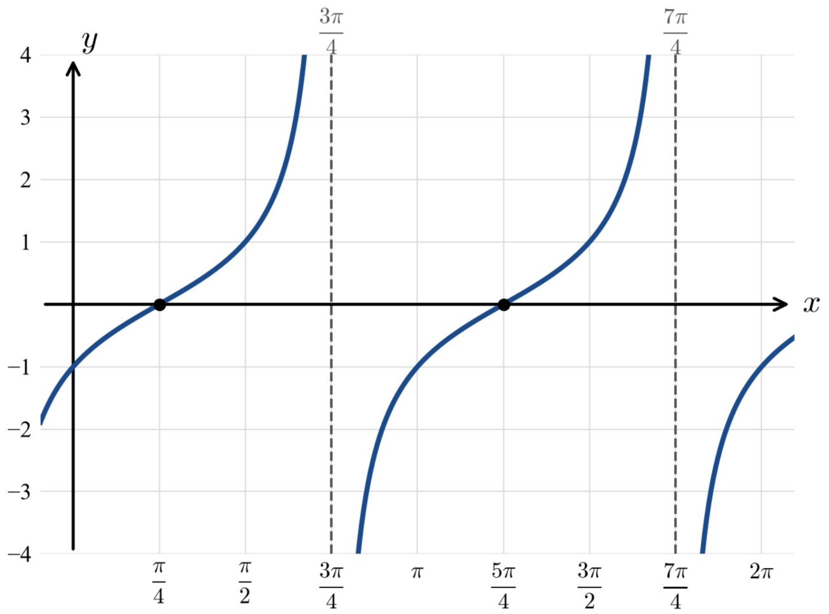
$\tan \dfrac{\pi}{3} = \sqrt{3}$, so one solution is $x = \dfrac{\pi}{3}$. Since the period is π, the next solution is $x = \dfrac{\pi}{3} + \pi = \dfrac{4\pi}{3}$.

$\therefore x = \frac{\pi}{3}$ or $x = \frac{4\pi}{3}$.

Example 4 — unscaffolded

Sketch $y = \tan\left(x - \frac{\pi}{4}\right)$ for $x \in [0, 2\pi]$.

This is $y = \tan x$ translated $\frac{\pi}{4}$ to the right. Asymptotes: $x - \frac{\pi}{4} = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$. x-intercepts: $x - \frac{\pi}{4} = m\pi \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$.



Practice questions

Scaffolded

Q1. For $y = \tan x$: (a) state the period; (b) give the asymptotes on $[0, 2\pi]$; (c) give the x-intercepts on $[0, 2\pi]$; (d) state $\tan\frac{\pi}{4}$.

Q2. For $y = \tan 3x$: (a) state the period; (b) find the asymptotes on $[0, \pi]$; (c) sketch the graph on $[0, \pi]$.

Q3. Solve $\tan x = 1$ for $x \in [0, 2\pi]$.

Q4. Solve $\tan x = -1$ for $x \in [0, 2\pi]$.

Q5. For $y = \tan\frac{x}{2}$: (a) state the period; (b) find the asymptote on $[0, 2\pi]$; (c) sketch the graph on $[0, 2\pi]$.

Q6. For $y = 3\tan x$: (a) state the period; (b) does the factor of 3 change the positions of the asymptotes? (c) find y when $x = \frac{\pi}{4}$.

Unscaffolded

Q7. Sketch $y = \tan x$ for $x \in [-\pi, \pi]$, showing asymptotes and intercepts.

Q8. State the period and the equations of the asymptotes of $y = \tan 4x$.

Q9. Solve $\tan x = \sqrt{3}$ for $x \in [0, 2\pi]$.

Q10. Solve $\tan x = \dfrac{1}{\sqrt{3}}$ for $x \in [0, 2\pi]$.

Q11. Solve $\tan x = 0$ for $x \in [0, 2\pi]$.

Q12. Solve $\tan 2x = 1$ for $x \in [0, \pi]$.

Q13. Sketch $y = 2\tan x$ for $x \in [0, 2\pi]$.

Q14. State the period and the equations of the asymptotes of $y = \tan\left(x + \dfrac{\pi}{3}\right)$.

Q15. Solve $\tan x = -\sqrt{3}$ for $x \in [0, 2\pi]$.

Q16. Find the exact value of (a) $\tan\dfrac{2\pi}{3}$; (b) $\tan\dfrac{3\pi}{4}$.

Q17. Sketch $y = -\tan x$ for $x \in [0, 2\pi]$.

Q18. Solve $\tan 3x = \sqrt{3}$ for $x \in [0, \pi]$.

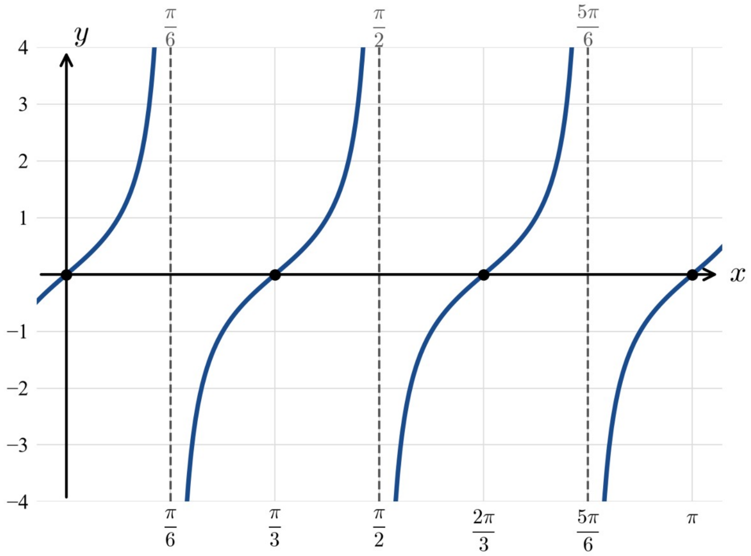
Answers

Q1. (a) π; (b) $x=\frac{\pi}{2},\frac{3\pi}{2}$; (c) $x=0,\pi,2\pi$; (d) 1. **Q2.** (a) $\frac{\pi}{3}$; (b) $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}$. **Q3.** $x=\frac{\pi}{4},\frac{5\pi}{4}$. **Q4.** $x=\frac{3\pi}{4},\frac{7\pi}{4}$. **Q5.** (a) 2π; (b) $x=\pi$. **Q6.** (a) π; (b) no; (c) 3. **Q7.** Asymptotes $x=\pm\frac{\pi}{2}$; intercepts $x=-\pi,0,\pi$. **Q8.** Period $\frac{\pi}{4}$; asymptotes $x=\frac{\pi}{8}+\frac{k\pi}{4}$. **Q9.** $x=\frac{\pi}{3},\frac{4\pi}{3}$. **Q10.** $x=\frac{\pi}{6},\frac{7\pi}{6}$. **Q11.** $x=0,\pi,2\pi$. **Q12.** $x=\frac{\pi}{8},\frac{5\pi}{8}$. **Q13.** Same asymptotes ($x=\frac{\pi}{2},\frac{3\pi}{2}$); passes through $\left(\frac{\pi}{4},2\right)$. **Q14.** Period π; asymptotes $x=\frac{\pi}{6}+k\pi$. **Q15.** $x=\frac{2\pi}{3},\frac{5\pi}{3}$. **Q16.** (a) $-\sqrt{3}$; (b) -1. **Q17.** Asymptotes $x=\frac{\pi}{2},\frac{3\pi}{2}$; each branch decreases. **Q18.** $x=\frac{\pi}{9},\frac{4\pi}{9},\frac{7\pi}{9}$.

Fully-worked solutions

Q1. (a) The period of $\tan x$ is π. (b) Asymptotes where $\cos x=0$: $x=\frac{\pi}{2},\frac{3\pi}{2}$. (c) Intercepts where $\sin x=0$: $x=0,\pi,2\pi$. (d) $\tan\frac{\pi}{4}=1$.

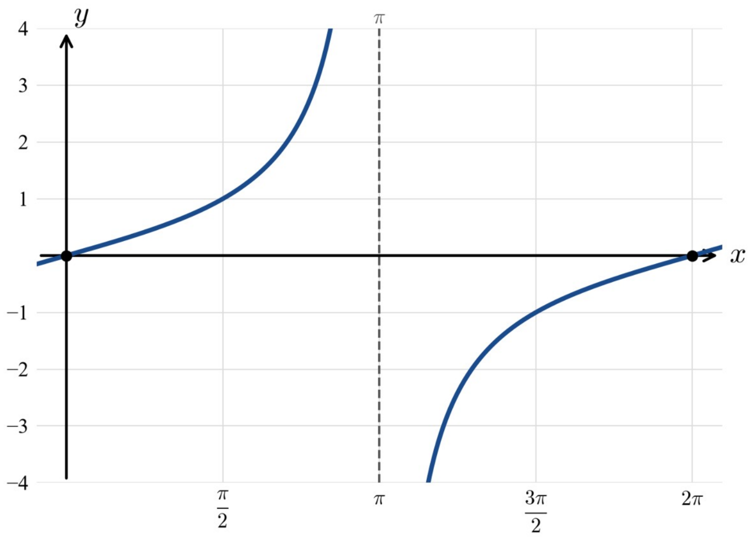
Q2. (a) Period $=\frac{\pi}{3}$. (b) Asymptotes where $3x=\frac{\pi}{2}+k\pi$, so $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}$. (c)



Q3. $\tan\frac{\pi}{4}=1$, so $x=\frac{\pi}{4}$; adding the period, $x=\frac{\pi}{4}+\pi=\frac{5\pi}{4}$. $\therefore x=\frac{\pi}{4},\frac{5\pi}{4}$.

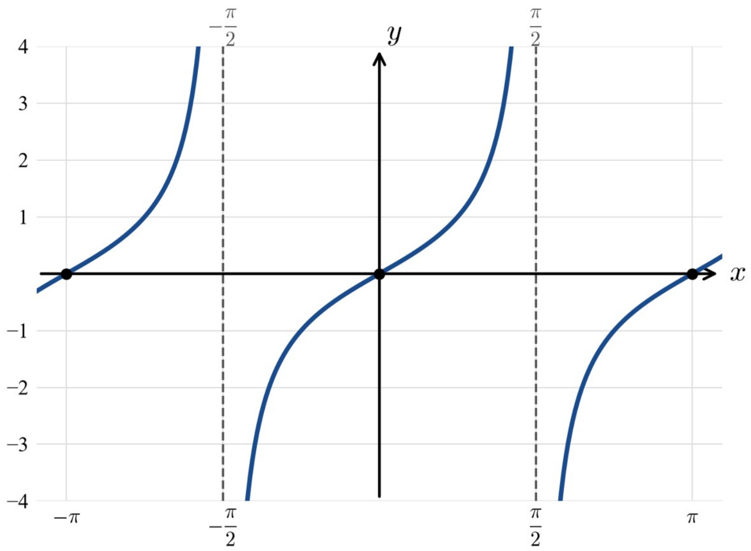
Q4. $\tan x=-1$ first occurs at $x=\frac{3\pi}{4}$ (second quadrant); adding π, $x=\frac{7\pi}{4}$. $\therefore x=\frac{3\pi}{4},\frac{7\pi}{4}$.

Q5. (a) Period $=\frac{\pi}{1/2}=2\pi$. (b) Asymptote where $\frac{x}{2}=\frac{\pi}{2}$, so $x=\pi$. (c)



Q6. (a) The period is unchanged: π. (b) No — the factor 3 dilates the graph from the x-axis but the asymptotes stay at $x=\frac{\pi}{2}+n\pi$. (c) $y=3\tan\frac{\pi}{4}=3(1)=3$.

Q7. Asymptotes $x=-\frac{\pi}{2},\frac{\pi}{2}$; x-intercepts $x=-\pi,0,\pi$.



Q8. Period $=\frac{\pi}{4}$. Asymptotes where $4x=\frac{\pi}{2}+k\pi$, so $x=\frac{\pi}{8}+\frac{k\pi}{4}$.

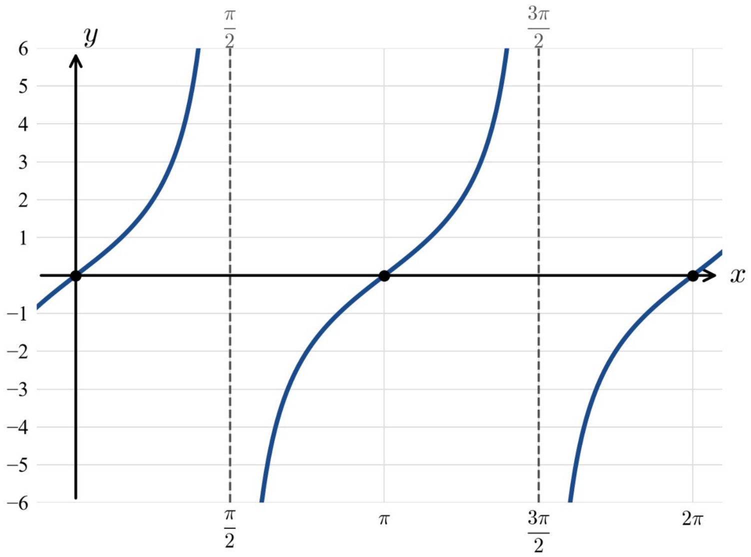
Q9. $\tan\frac{\pi}{3}=\sqrt{3}$, so $x=\frac{\pi}{3}$; adding π, $x=\frac{4\pi}{3}$. $\therefore x=\frac{\pi}{3},\frac{4\pi}{3}$.

Q10. $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$, so $x=\frac{\pi}{6}$; adding π, $x=\frac{7\pi}{6}$. $\therefore x=\frac{\pi}{6},\frac{7\pi}{6}$.

Q11. $\tan x=0$ where $\sin x=0$: $x=0,\pi,2\pi$.

Q12. $\tan 2x=1\Rightarrow 2x=\frac{\pi}{4}+n\pi$, so $x=\frac{\pi}{8}+\frac{n\pi}{2}$. On $[0,\pi]$: $x=\frac{\pi}{8},\frac{5\pi}{8}$.

Q13. The factor 2 dilates the graph from the x-axis; the asymptotes ($x=\frac{\pi}{2},\frac{3\pi}{2}$) and intercepts are unchanged. The point $\left(\frac{\pi}{4},1\right)$ on $y=\tan x$ maps to $\left(\frac{\pi}{4},2\right)$.

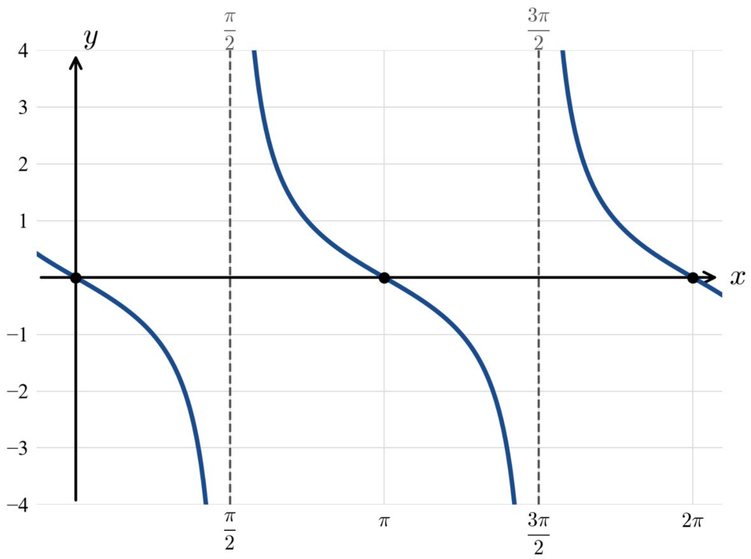


Q14. Period $=\pi$. Asymptotes where $x+\frac{\pi}{3}=\frac{\pi}{2}+k\pi$, so $x=\frac{\pi}{6}+k\pi$.

Q15. $\tan x=-\sqrt{3}$ first occurs at $x=\frac{2\pi}{3}$ (second quadrant); adding π, $x=\frac{5\pi}{3}$. $\therefore x=\frac{2\pi}{3},\frac{5\pi}{3}$.

Q16. (a) $\tan\frac{2\pi}{3}=-\tan\frac{\pi}{3}=-\sqrt{3}$. (b) $\tan\frac{3\pi}{4}=-\tan\frac{\pi}{4}=-1$.

Q17. Reflecting $y=\tan x$ in the x-axis leaves the asymptotes ($x=\frac{\pi}{2},\frac{3\pi}{2}$) and intercepts ($x=0,\pi,2\pi$) unchanged, but each branch now **decreases**.



Q18. $\tan 3x=\sqrt{3}\Rightarrow 3x=\frac{\pi}{3}+n\pi$, so $x=\frac{\pi}{9}+\frac{n\pi}{3}$. On $[0,\pi]$: $x=\frac{\pi}{9},\frac{4\pi}{9},\frac{7\pi}{9}$.

Theory

Many trigonometric equations can be solved **exactly** when the right-hand side is a special value — for example $\cos x = 1/2$ gives $x = \pi/3$. But when the right-hand side is **not** a special value (such as $\cos x = 0.3$), or when the equation mixes a circular function with a polynomial (such as $2\sin x = x$), there is no exact method and we use a **CAS calculator** to find **approximate (decimal)** solutions.

Two CAS approaches.

- **Solve command.** Enter the equation with the domain specified, e.g. solve $\cos x = 0.3$ for x with $0 \le x \le 2\pi$. The calculator returns the decimal solutions.
- **Graphical (intersection).** Graph each side as a separate function ($y = \cos x$ and $y = 0.3$) and find the coordinates of every **intersection point** inside the domain.

Key points.

- Always work in **radians** for circular functions, and give answers to the **required number of decimal places**.
- A quick graph shows **how many** solutions to expect in the domain — make sure you find them **all**.
- For a simple equation the symmetry of the graph gives the extra solutions once you have one: in $[0, 2\pi]$, $\sin x = k$ has solutions x and $\pi - x$; $\cos x = k$ has solutions x and $2\pi - x$; $\tan x = k$ repeats every π.
- For a mixed equation (trig = polynomial) there is no such symmetry — read **every** intersection from the graph.

Worked examples

Example 1 — scaffolded

Solve $\cos x = 0.3$ for $x \in [0, 2\pi]$, correct to 3 decimal places.

Working	Comment
0.3 is not a special value, so a numerical (CAS) solution is needed.	No exact method applies.
Graphing $y = \cos x$ and $y = 0.3$ over $[0, 2\pi]$ shows **two** intersections.	Decide how many solutions to expect.
CAS gives the first solution $x \approx 1.266$.	This is $\cos^{-1}(0.3)$.
By symmetry the second is $x \approx 2\pi - 1.266 \approx 5.017$ (CAS confirms).	$\cos x = \cos(2\pi - x)$.
$\therefore x \approx 1.266$ or $x \approx 5.017$.	Both lie in $[0, 2\pi]$.

Example 2 — scaffolded

Solve $3\sin x = 2$ for $x \in [0, 2\pi]$, correct to 2 decimal places.

Working	Comment
Rearrange to $\sin x = 2/3$; $2/3$ is not a special value.	Isolate the circular function first.
A graph shows **two** solutions in $[0, 2\pi]$.	Both are above the x-axis ($\sin x > 0$).
CAS gives $x \approx 0.73$.	The first solution, $\sin^{-1} 2/3$.
The second is $x \approx \pi - 0.73 \approx 2.41$.	$\sin x = \sin(\pi - x)$.
$\therefore x \approx 0.73$ or $x \approx 2.41$.	

Example 3 — unscaffolded

Solve $2\sin x = x$ for $x \in [0, \pi]$, correct to 3 decimal places.

This mixes a circular function with a linear function, so there is no exact method. Graphing $y = 2\sin x$ and $y = x$ over $[0, \pi]$ shows they intersect at the origin and at one other point. The CAS gives $x = 0$ or $x \approx 1.895$.

$\therefore x = 0$ or $x \approx 1.895$.

Example 4 — unscaffolded

Solve $\tan x = x + 1$ for $x \in [0, \pi/2)$, correct to 3 decimal places.

Graphing $y = \tan x$ and $y = x + 1$ over $[0, \pi/2)$ shows a single intersection. The CAS gives $x \approx 1.132$.

$\therefore x \approx 1.132$.

Practice questions

Scaffolded

Q1. Solve $\sin x = 0.8$ for $x \in [0, 2\pi]$, correct to 3 decimal places. (a) How many solutions are there in the domain? (b) Find the first solution. (c) Use the symmetry $\sin x = \sin(\pi - x)$ to find the second.

Q2. Solve $\cos x = -0.4$ for $x \in [0, 2\pi]$, correct to 3 decimal places. (a) State the number of solutions. (b) Find the first. (c) Use $\cos x = \cos(2\pi - x)$ for the second.

Q3. Solve $\tan x = 3$ for $x \in [0, 2\pi]$, correct to 3 decimal places. (a) State the number of solutions. (b) Find the first. (c) Add π to find the second.

Q4. Solve $5\cos x = 2$ for $x \in [0, 2\pi]$, correct to 2 decimal places.

Q5. Solve $2\sin x + 1 = 2.5$ for $x \in [0, 2\pi]$, correct to 2 decimal places.

Q6. Solve $\cos x = x$ for $x \in [0, \pi/2]$, correct to 3 decimal places. (This is a mixed equation — expect a single solution.)

Unscaffolded

Use a CAS calculator to solve each equation over the given domain, correct to the number of decimal places indicated.

Q7. $\sin x = 0.35$, $x \in [0, 2\pi]$ (3 dp)

Q8. $\cos x = 0.85$, $x \in [0, 2\pi]$ (3 dp)

Q9. $\tan x = -2$, $x \in [0, 2\pi]$ (3 dp)

Q10. $7\cos x = 4$, $x \in [0, 2\pi]$ (2 dp)

Q11. $3\cos x + 1 = 0$, $x \in [0, 2\pi]$ (3 dp)

Q12. $\sin x = x - 1$, $x \in [0, 2\pi]$ (3 dp)

Q13. $\cos x = x/2$, $x \in [0, \pi]$ (3 dp)

Q14. $\tan x = x$ — find the smallest positive solution greater than $\pi/2$ (3 dp).

Q15. $x^2 = 4\cos x$, $x \in [0, \pi/2]$ (3 dp)

Q16. $3\sin x = x$ — find the non-zero solution in $[0, \pi]$ (3 dp).

Q17. $\sin 2x = 0.4$, $x \in [0, \pi]$ (3 dp)

Q18. $2\cos x = x^2 - 1$, $x \in [0, 2\pi]$ (3 dp)

Answers

Q1. Two; $x \approx 0.927$ or $x \approx 2.214$. **Q2.** Two; $x \approx 1.982$ or $x \approx 4.301$. **Q3.** Two; $x \approx 1.249$ or $x \approx 4.391$. **Q4.** $x \approx 1.16$ or $x \approx 5.12$. **Q5.** $x \approx 0.85$ or $x \approx 2.29$. **Q6.** $x \approx 0.739$. **Q7.** $x \approx 0.358$ or $x \approx 2.784$. **Q8.** $x \approx 0.555$ or $x \approx 5.728$. **Q9.** $x \approx 2.034$ or $x \approx 5.176$. **Q10.** $x \approx 0.96$ or $x \approx 5.32$. **Q11.** $x \approx 1.911$ or $x \approx 4.373$. **Q12.** $x \approx 1.935$. **Q13.** $x \approx 1.030$. **Q14.** $x \approx 4.493$. **Q15.** $x \approx 1.202$. **Q16.** $x \approx 2.279$. **Q17.** $x \approx 0.206$ or $x \approx 1.365$. **Q18.** $x \approx 1.265$.

Fully-worked solutions

Q1. $\sin x = 0.8$. The graph of $y = \sin x$ meets $y = 0.8$ **twice** in $[0, 2\pi]$ (both where $\sin x > 0$). CAS: $x \approx 0.927$; by symmetry $x \approx \pi - 0.927 \approx 2.214$. $\therefore x \approx 0.927$ or $x \approx 2.214$.

Q2. $\cos x = -0.4$. Two solutions in $[0, 2\pi]$. CAS: $x \approx 1.982$; by symmetry $x \approx 2\pi - 1.982 \approx 4.301$. $\therefore x \approx 1.982$ or $x \approx 4.301$.

Q3. $\tan x = 3$. The tangent graph meets $y = 3$ once per period π, so **twice** in $[0, 2\pi]$. CAS: $x \approx 1.249$; the next is $x \approx 1.249 + \pi \approx 4.391$. $\therefore x \approx 1.249$ or $x \approx 4.391$.

Q4. $5\cos x = 2 \Rightarrow \cos x = 0.4$. Two solutions. CAS: $x \approx 1.16$; by symmetry $x \approx 2\pi - 1.16 \approx 5.12$. $\therefore x \approx 1.16$ or $x \approx 5.12$.

Q5. $2\sin x + 1 = 2.5 \Rightarrow \sin x = 0.75$. Two solutions. CAS: $x \approx 0.85$; $x \approx \pi - 0.85 \approx 2.29$. $\therefore x \approx 0.85$ or $x \approx 2.29$.

Q6. $\cos x = x$. Graph $y = \cos x$ and $y = x$: one intersection in $[0, \pi/2]$. CAS: $x \approx 0.739$. $\therefore x \approx 0.739$.

Q7. $\sin x = 0.35$: two solutions. CAS $x \approx 0.358$; $\pi - 0.358 \approx 2.784$. $\therefore x \approx 0.358$ or $x \approx 2.784$.

Q8. $\cos x = 0.85$: two solutions. CAS $x \approx 0.555$; $2\pi - 0.555 \approx 5.728$. $\therefore x \approx 0.555$ or $x \approx 5.728$.

Q9. $\tan x = -2$: the tangent is negative in the 2nd and 4th quadrants, giving two solutions in $[0, 2\pi]$. CAS: $x \approx 2.034$ and $x \approx 2.034 + \pi \approx 5.176$. $\therefore x \approx 2.034$ or $x \approx 5.176$.

Q10. $7\cos x = 4 \Rightarrow \cos x = 4/7$: two solutions. CAS $x \approx 0.96$; $2\pi - 0.96 \approx 5.32$. $\therefore x \approx 0.96$ or $x \approx 5.32$.

Q11. $3\cos x + 1 = 0 \Rightarrow \cos x = -1/3$: two solutions. CAS $x \approx 1.911$; $2\pi - 1.911 \approx 4.373$. $\therefore x \approx 1.911$ or $x \approx 4.373$.

Q12. $\sin x = x - 1$ (mixed). Graphing $y = \sin x$ and $y = x - 1$ over $[0, 2\pi]$ gives a single intersection. CAS: $x \approx 1.935$. $\therefore x \approx 1.935$.

Q13. $\cos x = x/2$ (mixed). One intersection in $[0, \pi]$. CAS: $x \approx 1.030$. $\therefore x \approx 1.030$.

Q14. $\tan x = x$. Beyond the asymptote at $\pi/2$, the branch of $y = \tan x$ on $(\pi/2, 3\pi/2)$ meets $y = x$ once. CAS: $x \approx 4.493$. $\therefore x \approx 4.493$.

Q15. $x^2 = 4\cos x$ (mixed). Graph $y = x^2$ and $y = 4\cos x$ over $[0, \pi/2]$: one intersection. CAS: $x \approx 1.202$. $\therefore x \approx 1.202$.

Q16. $3\sin x = x$. Besides $x = 0$, the graphs meet once more in $[0, \pi]$. CAS: $x \approx 2.279$. $\therefore x \approx 2.279$.

Q17. $\sin 2x = 0.4$. As x runs over $[0, \pi]$, $2x$ runs over $[0, 2\pi]$, so there are **two** solutions. CAS: $x \approx 0.206$ and $x \approx 1.365$. $\therefore x \approx 0.206$ or $x \approx 1.365$.

Q18. $2\cos x = x^2 - 1$ (mixed). Graph $y = 2\cos x$ and $y = x^2 - 1$ over $[0, 2\pi]$: a single intersection. CAS: $x \approx 1.265$. $\therefore x \approx 1.265$.

Chapter 12 Circular functions — 12M General solutions of trigonometric equations

Units 3 & 4 preview. This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

Theory

Because the circular functions are **periodic**, an equation such as $\cos x = 1/2$ has infinitely many solutions. The **general solution** describes *all* of them using an integer parameter $n \in \mathbb{Z}$.

Let α be a **base solution** — the value returned by the inverse function ($\alpha = \cos^{-1} a$ in $[0, \pi]$; $\alpha = \sin^{-1} a$ in $[-\pi/2, \pi/2]$; $\alpha = \tan^{-1} a$ in $(-\pi/2, \pi/2)$). Then:

- **Cosine:** $\cos x = a$ (with $-1 \le a \le 1$) has general solution

$$x = 2n\pi \pm \alpha, n \in \mathbb{Z}.$$

- **Sine:** $\sin x = a$ (with $-1 \le a \le 1$) has general solution

$$x = 2n\pi + \alpha \text{ or } x = (2n+1)\pi - \alpha, n \in \mathbb{Z}$$

- which can be written compactly as $x = n\pi + (-1)^n \alpha$.
- **Tangent:** $\tan x = a$ has general solution

$$x = n\pi + \alpha, n \in \mathbb{Z}.$$

Listing solutions in an interval. Substitute successive integer values of n into the general solution and keep those values of x that lie in the required interval.

A coefficient of x. For an equation such as $\cos(kx) = a$, first write the general solution for kx, then divide every term by k. For example $\cos 2x = a \Rightarrow 2x = 2n\pi \pm \alpha \Rightarrow x = n\pi \pm \alpha/2$.

Worked examples

Example 1 — scaffolded

Find the general solution of $\cos x = 1/2$.

Working	Comment
The base solution is $\alpha = \cos^{-1} 1/2 = \pi/3$.	Read α from the exact-value table; $\pi/3 \in [0, \pi]$.
$\cos x = a$ has general solution $x = 2n\pi \pm \alpha$.	Use the cosine rule with the $\pm$.
$\therefore x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$.	Both the + and − families are needed.

Example 2 — scaffolded

Find the general solution of $\sin x = \sqrt{3}/2$, and hence list all solutions in $[0, 2\pi]$.

Working	Comment
$\alpha = \sin^{-1} \sqrt{3}/2 = \pi/3$.	Base solution in $[-\pi/2, \pi/2]$.
General solution: $x = n\pi + (-1)^n \frac{\pi}{3}$.	The sine rule (compact form).
$n=0: x = \pi/3$; $n=1: x = \pi - \pi/3 = 2\pi/3$; $n=2: x = 2\pi + \pi/3$ (too big).	Substitute integers and keep those in $[0, 2\pi]$.

$\therefore$ in $[0, 2\pi]$ the solutions are $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$.

Example 3 — unscaffolded

Find the general solution of $\tan x = 1$.

$\alpha = \tan^{-1} 1 = \frac{\pi}{4}$. For tangent, $x = n\pi + \alpha$.

$\therefore x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$.

Example 4 — unscaffolded

Find the general solution of $\cos 2x = 1/2$.

Let the argument be $2x$. Since $\cos\theta = 1/2$ has $\theta = 2n\pi \pm \pi/3$,

$$2x = 2n\pi \pm \frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{6}.$$

$\therefore x = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$.

Practice questions

Scaffolded

Q1. Find the general solution of $\cos x = \frac{\sqrt{3}}{2}$.

Q2. Find the general solution of $\sin x = \frac{1}{2}$.

Q3. Find the general solution of $\tan x = \sqrt{3}$.

Q4. Find the general solution of $\cos x = -\frac{1}{2}$.

Q5. Find the general solution of $\cos 2x = \frac{\sqrt{3}}{2}$, and hence list all solutions in $[0, 2\pi]$.

Q6. Find the general solution of $\sin x = -\frac{1}{2}$, and hence list all solutions in $[0, 2\pi]$.

Unscaffolded

Find the general solution of each equation (with $n \in \mathbb{Z}$); where an interval is given, also list all solutions in it.

Q7. $\cos x = 0$

Q8. $\sin x = 1$

Q9. $\tan x = -1$

Q10. $\cos x = \frac{\sqrt{2}}{2}$

Q11. $\sin x = \frac{\sqrt{2}}{2}$

Q12. $\cos x = -\frac{\sqrt{3}}{2}$

Q13. $\tan x = \frac{1}{\sqrt{3}}$

Q14. $\sin x = -\frac{\sqrt{3}}{2}$, in $[0, 2\pi]$

Q15. $\cos 3x = \frac{1}{2}$

Q16. $\sin 2x = \frac{\sqrt{3}}{2}$

Q17. $\tan 2x = 1$

Q18. $2\cos x + 1 = 0$, in $[0, 2\pi]$

Answers

Q1. $x=2n\pi\pm\frac{\pi}{6}$. **Q2.** $x=n\pi+(-1)^n\frac{\pi}{6}$. **Q3.** $x=n\pi+\frac{\pi}{3}$. **Q4.** $x=2n\pi\pm\frac{2\pi}{3}$. **Q5.** $x=n\pi\pm\frac{\pi}{12}$; in $[0,2\pi]$: $\frac{\pi}{12},\frac{11\pi}{12},\frac{13\pi}{12},\frac{23\pi}{12}$. **Q6.** $x=n\pi+(-1)^n\left(-\frac{\pi}{6}\right)$; in $[0,2\pi]$: $\frac{7\pi}{6},\frac{11\pi}{6}$. **Q7.** $x=\frac{\pi}{2}+n\pi$. **Q8.** $x=\frac{\pi}{2}+2n\pi$. **Q9.** $x=n\pi-\frac{\pi}{4}$. **Q10.** $x=2n\pi\pm\frac{\pi}{4}$. **Q11.** $x=n\pi+(-1)^n\frac{\pi}{4}$. **Q12.** $x=2n\pi\pm\frac{5\pi}{6}$. **Q13.** $x=n\pi+\frac{\pi}{6}$. **Q14.** $x=n\pi+(-1)^n\left(-\frac{\pi}{3}\right)$; in $[0,2\pi]$: $\frac{4\pi}{3},\frac{5\pi}{3}$. **Q15.** $x=\frac{2n\pi}{3}\pm\frac{\pi}{9}$. **Q16.** $x=\frac{n\pi}{2}+(-1)^n\frac{\pi}{6}$. **Q17.** $x=\frac{n\pi}{2}+\frac{\pi}{8}$. **Q18.** $x=2n\pi\pm\frac{2\pi}{3}$; in $[0,2\pi]$: $\frac{2\pi}{3},\frac{4\pi}{3}$.

Fully-worked solutions

Q1. $\alpha=\cos^{-1}\sqrt{3}/2=\pi/6$. Cosine: $x=2n\pi\pm\alpha$, so $x=2n\pi\pm\frac{\pi}{6}$.

Q2. $\alpha=\sin^{-1}1/2=\pi/6$. Sine: $x=n\pi+(-1)^n\frac{\pi}{6}$.

Q3. $\alpha=\tan^{-1}\sqrt{3}=\pi/3$. Tangent: $x=n\pi+\frac{\pi}{3}$.

Q4. $\cos x=-1/2$ gives $\alpha=\cos^{-1}(-1/2)=2\pi/3$. So $x=2n\pi\pm\frac{2\pi}{3}$.

Q5. $\cos 2x=\sqrt{3}/2\Rightarrow 2x=2n\pi\pm\pi/6\Rightarrow x=n\pi\pm\pi/12$. Testing integers: $n=0\Rightarrow\pi/12$ (and $-\pi/12$, out); $n=1\Rightarrow 11\pi/12, 13\pi/12$; $n=2\Rightarrow 23\pi/12$ (and $25\pi/12$, out). In $[0,2\pi]$: $\frac{\pi}{12},\frac{11\pi}{12},\frac{13\pi}{12},\frac{23\pi}{12}$.

Q6. $\alpha=\sin^{-1}(-1/2)=-\pi/6$, so $x=n\pi+(-1)^n(-\pi/6)$. $n=1\Rightarrow\pi+\pi/6=7\pi/6$; $n=2\Rightarrow 2\pi-\pi/6=11\pi/6$. In $[0,2\pi]$: $\frac{7\pi}{6},\frac{11\pi}{6}$.

Q7. $\cos x=0$ at $x=\pi/2$, and again every π thereafter, so $x=\frac{\pi}{2}+n\pi$.

Q8. $\sin x=1$ has the single base solution $\alpha=\pi/2$ (a maximum), so $x=\frac{\pi}{2}+2n\pi$.

Q9. $\alpha=\tan^{-1}(-1)=-\pi/4$. Tangent: $x=n\pi-\frac{\pi}{4}$.

Q10. $\alpha=\cos^{-1}\sqrt{2}/2=\pi/4$, so $x=2n\pi\pm\frac{\pi}{4}$.

Q11. $\alpha=\sin^{-1}\sqrt{2}/2=\pi/4$, so $x=n\pi+(-1)^n\frac{\pi}{4}$.

Q12. $\alpha=\cos^{-1}(-\sqrt{3}/2)=5\pi/6$, so $x=2n\pi\pm\frac{5\pi}{6}$.

Q13. $\alpha=\tan^{-1}1/\sqrt{3}=\pi/6$, so $x=n\pi+\frac{\pi}{6}$.

Q14. $\alpha=\sin^{-1}(-\sqrt{3}/2)=-\pi/3$, so $x=n\pi+(-1)^n(-\pi/3)$. $n=1\Rightarrow\pi+\pi/3=4\pi/3$; $n=2\Rightarrow2\pi-\pi/3=5\pi/3$. In $[0,2\pi]$: $\frac{4\pi}{3},\frac{5\pi}{3}$.

Q15. $\cos3x=1/2\Rightarrow3x=2n\pi\pm\pi/3\Rightarrow x=\frac{2n\pi}{3}\pm\frac{\pi}{9}$.

Q16. $\sin2x=\sqrt{3}/2\Rightarrow2x=n\pi+(-1)^n\pi/3\Rightarrow x=\frac{n\pi}{2}+(-1)^n\frac{\pi}{6}$.

Q17. $\tan2x=1\Rightarrow2x=n\pi+\pi/4\Rightarrow x=\frac{n\pi}{2}+\frac{\pi}{8}$.

Q18. $2\cos x+1=0\Rightarrow\cos x=-1/2$, so (as in Q4) $x=2n\pi\pm2\pi/3$. In $[0,2\pi]$: $n=0\Rightarrow2\pi/3$; $2\pi-2\pi/3=4\pi/3$. Solutions $\frac{2\pi}{3},\frac{4\pi}{3}$.

Topic 3 Test A — Technology-active

Topic 3 covers Chapters 11–12 (exponential and logarithmic functions, circular functions). Reading time: 5 minutes. Writing time: 50 minutes. Total: 50 marks. One bound reference, one approved CAS calculator or CAS software, and one scientific calculator are permitted. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (4 marks)

a. Simplify $\frac{6a^3b^{-2}}{2a^{-1}b}$, giving your answer with positive indices. 2 marks

b. Evaluate $32^{3/5}+27^{-1/3}$. 2 marks

Question 2 (5 marks)

Consider the exponential function $y=2^{x-1}-4$.

a. State the equation of the horizontal asymptote. 1 mark

b. Find the coordinates of the axis intercepts. 2 marks

c. State the range. 1 mark

d. Sketch the graph, labelling the asymptote and intercepts. 1 mark

Question 3 (5 marks)

a. Solve $\log_2(x-1)+\log_2(x+2)=2$ for x. 3 marks

b. Evaluate $\log_3 45-\log_3 5$. 2 marks

Question 4 (5 marks)

a. Solve $4\times 3^x=1000$, giving your answer exact and correct to 3 decimal places. 2 marks

b. Solve $2^{2x}-6(2^x)+8=0$. 3 marks

Question 5 (6 marks)

A bacterial colony grows according to $N=500\times 2^{t/3}$, where t is the time in hours.

a. State the initial number of bacteria. 1 mark

b. Find the number of bacteria after 9 hours. 2 marks

c. Find, to the nearest 0.1 hour, the time for the colony to reach 5000. 3 marks

Question 6 (5 marks)

a. Convert $105°$ to radians, as an exact multiple of π. 1 mark

b. Find the exact values of $\cos\frac{11\pi}{4}$ and $\tan\left(-\frac{11\pi}{6}\right)$. 2 marks

c. A sector of a circle has radius 6 cm and angle $\frac{2\pi}{3}$. Find the exact arc length. 2 marks

Question 7 (7 marks)

Consider $y=5\sin(2x)-2$ for $x\in[0,2\pi]$.

a. State the amplitude. 1 mark

b. State the period. 1 mark

c. State the range. 1 mark

d. Find the maximum value and the values of x at which it occurs. 2 marks

e. Sketch the graph over $[0, 2\pi]$. 2 marks

Question 8 (6 marks)

Solve each equation over $[0, 2\pi]$, giving exact values.

a. $\cos(2x) = -\frac{\sqrt{3}}{2}$ 2 marks

b. $\sin(2x) = \frac{\sqrt{2}}{2}$ 2 marks

c. $\tan(2x) = -\sqrt{3}$ 2 marks

Question 9 (7 marks)

The temperature in a greenhouse is modelled by $T = 18 - 6\cos\frac{\pi t}{12}$ °C, where t is the number of hours after midnight ($0 \leq t \leq 24$).

a. Find the temperature at midnight. 1 mark

b. State the amplitude and the period of the model. 2 marks

c. Find the maximum temperature and the time at which it occurs. 2 marks

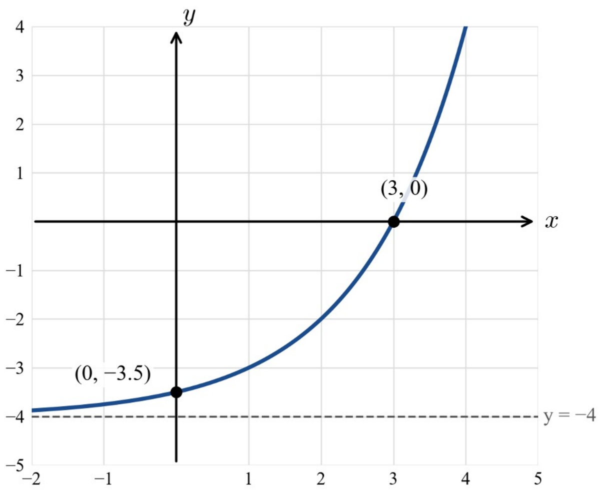
d. Find the times at which the temperature is 21 °C. 2 marks

End of Test A

Topic 3 Test A — solutions

Q1. (a) $\dfrac{6a^3b^{-2}}{2a^{-1}b} = 3a^{3-(-1)}b^{-2-1} = 3a^4b^{-3} = \dfrac{3a^4}{b^3}$. (b) $32^{3/5} = (\sqrt[5]{32})^3 = 2^3 = 8$ and $27^{-1/3} = \dfrac{1}{\sqrt[3]{27}} = \dfrac{1}{3}$, so the sum is $8\,1/3 = \dfrac{25}{3}$.

Q2. (a) Asymptote $y = -4$. (b) y-intercept $x = 0$: $y = 2^{-1} - 4 = -\dfrac{7}{2}$, giving $\left(0, -\dfrac{7}{2}\right)$; x-intercept $y = 0$: $2^{x-1} = 4 = 2^2 \Rightarrow x - 1 = 2 \Rightarrow x = 3$, giving $(3, 0)$. (c) Range $(-4, \infty)$. (d)



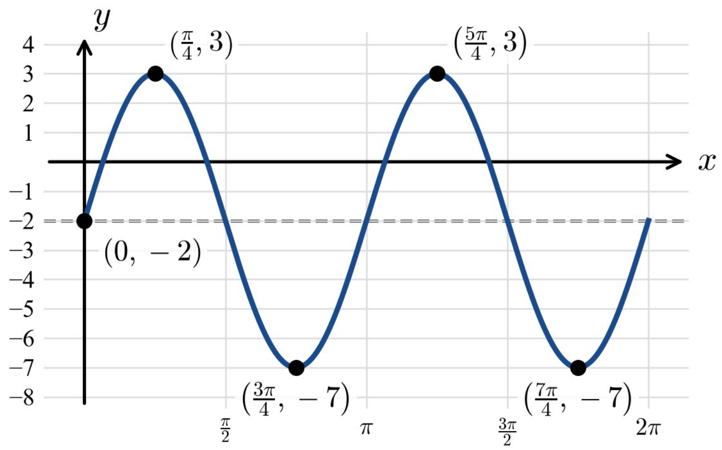
Q3. (a) $\log_2((x-1)(x+2)) = 2 \Rightarrow (x-1)(x+2) = 2^2 = 4 \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$. Since the domain requires $x > 1$, reject $x = -3$; $\therefore x = 2$. (b) $\log_3 45 - \log_3 5 = \log_3 \dfrac{45}{5} = \log_3 9 = 2$.

Q4. (a) $3^x = 250 \Rightarrow x = \log_3 250 = \dfrac{\log_{10} 250}{\log_{10} 3} \approx 5.026$. (b) Let $y = 2^x$: $y^2 - 6y + 8 = 0 \Rightarrow (y-2)(y-4) = 0 \Rightarrow y = 2$ or $y = 4$, so $2^x = 2^1$ or $2^x = 2^2$; $\therefore x = 1$ or $x = 2$.

Q5. (a) $N(0) = 500 \times 2^0 = 500$. (b) $N(9) = 500 \times 2^3 = 500 \times 8 = 4000$. (c) $500 \times 2^{t/3} = 5000 \Rightarrow 2^{t/3} = 10 \Rightarrow \dfrac{t}{3} = \log_2 10 \Rightarrow t = 3\log_2 10 \approx 10.0$ hours.

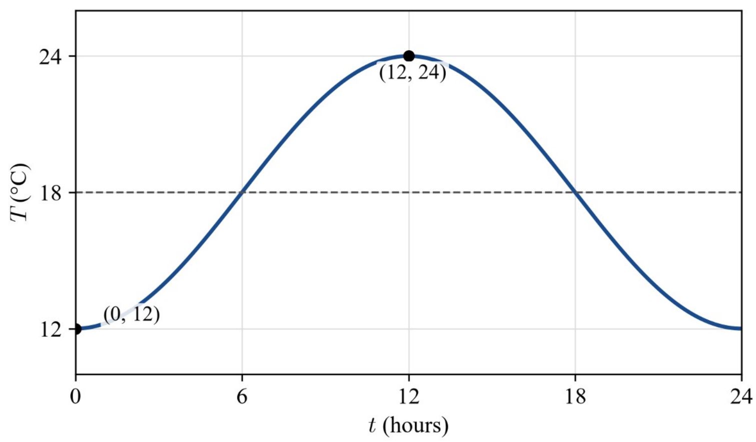
Q6. (a) $105° = 105 \times \dfrac{\pi}{180} = \dfrac{7\pi}{12}$. (b) $\dfrac{11\pi}{4} - 2\pi = \dfrac{3\pi}{4}$, second quadrant (reference $\dfrac{\pi}{4}$): $\cos\dfrac{11\pi}{4} = -\dfrac{\sqrt{2}}{2}$; tangent is odd, so $\tan\left(-\dfrac{11\pi}{6}\right) = -\tan\dfrac{11\pi}{6}$, and $\dfrac{11\pi}{6}$ is in the fourth quadrant (reference $\dfrac{\pi}{6}$): $\tan\dfrac{11\pi}{6} = -\dfrac{\sqrt{3}}{3}$, so $\tan\left(-\dfrac{11\pi}{6}\right) = \dfrac{\sqrt{3}}{3}$. (c) $\ell = r\theta = 6 \times \dfrac{2\pi}{3} = 4\pi$ cm.

Q7. (a) Amplitude 5. (b) Period $\dfrac{2\pi}{2} = \pi$. (c) Range $[-7, 3]$. (d) Maximum value 3, when $\sin(2x) = 1 \Rightarrow 2x = \dfrac{\pi}{2}, \dfrac{5\pi}{2} \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$. (e)



Q8. (a) $\cos(2x) = -\frac{\sqrt{3}}{2} \Rightarrow 2x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6} \Rightarrow x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$. (b) $\sin(2x) = \frac{\sqrt{2}}{2} \Rightarrow 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4} \Rightarrow x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$. (c) $\tan(2x) = -\sqrt{3} \Rightarrow 2x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \frac{11\pi}{3} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$.

Q9. (a) $T(0) = 18 - 6\cos 0 = 18 - 6 = 12$ °C. (b) Amplitude 6; period $\frac{2\pi}{\pi/12} = 24$ hours. (c) T is maximum when $\cos\frac{\pi t}{12} = -1$, i.e. $\frac{\pi t}{12} = \pi \Rightarrow t = 12$; maximum $T = 18 + 6 = 24$ °C at $t = 12$ (noon). (d) $18 - 6\cos\frac{\pi t}{12} = 21 \Rightarrow \cos\frac{\pi t}{12} = -\frac{1}{2} \Rightarrow \frac{\pi t}{12} = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 8$ or $t = 16$.



Topic 3 Test B — Technology-free

Topic 3 covers Chapters 11–12 (exponential and logarithmic functions, circular functions). Reading time: 5 minutes. Writing time: 45 minutes. Total: 40 marks. Students are not permitted to bring any technology (calculators or software), or notes of any kind. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (4 marks)

a. Simplify $\frac{a^5 \times a^{-2}}{a^{-1}}$. 2 marks

b. Evaluate $25^{3/2} \times 49^{1/2}$. 2 marks

Question 2 (4 marks)

a. Evaluate $\log_4 32$ and $\log_9 \frac{1}{27}$. 2 marks

b. Solve $\log_4 x = -2$. 1 mark

c. Write $5^4 = 625$ in logarithmic form. 1 mark

Question 3 (5 marks)

For $y = 4^x - 16$:

a. state the asymptote 1 mark

b. find the y-intercept 1 mark

c. find the x-intercept 2 marks

d. state the range 1 mark

Question 4 (4 marks)

Solve for x:

a. $5^{3x} = 625$ 2 marks

b. $16^x = 32^{x-1}$ 2 marks

Question 5 (5 marks)

a. Convert $\frac{2\pi}{5}$ to degrees. 1 mark

b. Find the exact values of $\sin\left(-\frac{2\pi}{3}\right)$, $\cos\left(-\frac{\pi}{4}\right)$ and $\tan\left(-\frac{7\pi}{6}\right)$. 3 marks

c. State the quadrant in which $\frac{19\pi}{4}$ lies. 1 mark

Question 6 (5 marks)

Given $\sin\theta = \frac{20}{29}$ and θ lies in the second quadrant, find the exact values of $\cos\theta$ and $\tan\theta$.

Question 7 (5 marks)

Solve each equation over $[0, 2\pi]$, giving exact values.

a. $\sin(2x) = -1$ 2 marks

b. $\cos(2x) = -1$ 2 marks

c. $\tan\frac{x}{2} = \sqrt{3}$ 1 mark

Question 8 (4 marks)

For $y = 2\cos\frac{x}{2}$: state

a. the amplitude 1 mark

b. the period 1 mark

c. the range 1 mark

d. the value of y when $x = 0$ 1 mark

Question 9 (4 marks)

Consider $y = \sin x$ for $x \in [0, 4\pi]$.

a. State the exact solutions of $\sin x = 0$ on this interval. 2 marks

b. State the coordinates of the maximum and minimum points. 2 marks

End of Test B

Topic 3 Test B — solutions

Q1. (a) $\frac{a^5 \times a^{-2}}{a^{-1}} = a^{5-2-(-1)} = a^4$. (b) $25^{3/2} \times 49^{1/2} = (\sqrt{25})^3 \times \sqrt{49} = 5^3 \times 7 = 125 \times 7 = 875$.

Q2. (a) $\log_4 32 = \frac{5}{2}$ (since $4^{5/2} = (\sqrt{4})^5 = 2^5 = 32$); $\log_9 \frac{1}{27} = -\frac{3}{2}$ (since $9^{-3/2} = \frac{1}{(\sqrt{9})^3} = \frac{1}{27}$). (b) $\log_4 x = -2 \Rightarrow x = 4^{-2} = \frac{1}{16}$. (c) $\log_5 625 = 4$.

Q3. (a) Asymptote $y = -16$. (b) $x = 0$: $y = 4^0 - 16 = -15$, giving $(0, -15)$. (c) $4^x = 16 = 4^2 \Rightarrow x = 2$, giving $(2, 0)$. (d) Range $(-16, \infty)$.

Q4. (a) $5^{3x} = 625 = 5^4 \Rightarrow 3x = 4 \Rightarrow x = \frac{4}{3}$. (b) $2^{4x} = 2^{5(x-1)} \Rightarrow 4x = 5x - 5 \Rightarrow x = 5$.

Q5. (a) $\frac{2\pi}{5} = \frac{2 \times 180°}{5} = 72°$. (b) $\sin\left(-\frac{2\pi}{3}\right) = -\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\tan\left(-\frac{7\pi}{6}\right) = -\tan\frac{7\pi}{6} = -\frac{\sqrt{3}}{3}$. (c) $\frac{19\pi}{4} - 4\pi = \frac{3\pi}{4}$, which lies in the second quadrant.

Q6. $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \frac{400}{841} = \frac{441}{841}$, so $\cos\theta = \pm\frac{21}{29}$. In the second quadrant cosine is negative, so $\cos\theta = -\frac{21}{29}$. Then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{20/29}{-21/29} = -\frac{20}{21}$.

Q7. (a) $\sin(2x) = -1 \Rightarrow 2x = \frac{3\pi}{2}, \frac{7\pi}{2} \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$. (b) $\cos(2x) = -1 \Rightarrow 2x = \pi, 3\pi \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$. (c) Since $x \in [0, 2\pi]$, $\frac{x}{2} \in [0, \pi]$; $\tan\frac{x}{2} = \sqrt{3} \Rightarrow \frac{x}{2} = \frac{\pi}{3} \Rightarrow x = \frac{2\pi}{3}$.

Q8. (a) Amplitude 2. (b) Period $\frac{2\pi}{1/2} = 4\pi$. (c) Range $[-2, 2]$. (d) $y = 2\cos 0 = 2$.

Q9. (a) $\sin x = 0 \Rightarrow x = 0, \pi, 2\pi, 3\pi, 4\pi$. (b) Maxima $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{5\pi}{2}, 1\right)$; minima $\left(\frac{3\pi}{2}, -1\right)$ and $\left(\frac{7\pi}{2}, -1\right)$.