Mathematical Methods Units 1 & 2

Chapter 11 — Exponential and logarithmic functions

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Contents

Theory

In the expression a^n, a is the **base** and n is the **index** (also called the *power* or *exponent*). a^n means a multiplied by itself n times.

The exponent laws (index laws). For the same base a (with $a \neq 0$):

- $a^m \times a^n = a^{m+n}$ — when multiplying, **add** the indices.
- $\dfrac{a^m}{a^n} = a^{m-n}$ — when dividing, **subtract** the indices.
- $(a^m)^n = a^{mn}$ — for a power of a power, **multiply** the indices.
- $(ab)^n = a^n b^n$ and $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.
- $a^0 = 1$ — any non-zero base to the power 0 is 1.
- $a^{-n} = \dfrac{1}{a^n}$ — a negative index means the **reciprocal**.

Simplifying. Deal with the coefficients (the numbers) separately from the powers of each pronumeral, applying the laws to each base in turn. A negative index is usually rewritten as a positive index in the denominator to give the final answer.

Equating indices. For $a > 0$, $a \neq 1$, if two powers with the **same base** are equal, their indices are equal:

$$a^x = a^y \Rightarrow x = y.$$

(The bases 1 and -1 are excluded: $1^2 = 1^5$ and $(-1)^2 = (-1)^4$, but the indices are not equal.)

To solve an exponential equation like $2^x = 32$, write both sides as powers of the same base ($32 = 2^5$), then equate the indices.

Worked examples

Example 1 — scaffolded

Simplify $3m^4 \times 4m^5$.

Working	Comment
$3m^4 \times 4m^5 = (3 \times 4)(m^4 \times m^5)$	Multiply the coefficients; group the powers of m.
$= 12m^{4+5}$	$a^m \times a^n = a^{m+n}$: add the indices.
$= 12m^9$	Simplify.

Example 2 — scaffolded

Simplify $\dfrac{10a^6b^3}{2a^2b^3}$.

Working	Comment
$\dfrac{10}{2} = 5$	Divide the coefficients.

Working	Comment
$\frac{a^6}{a^2}=a^{6-2}=a^4$; $\frac{b^3}{b^3}=b^{3-3}=b^0=1$	Subtract indices for each base; $b^0=1$.
$\therefore \frac{10a^6b^3}{2a^2b^3}=5a^4$	Combine.

Example 3 — unscaffolded

Simplify $(3x^2y^4)^2\times 2x$.

$(3x^2y^4)^2=3^2(x^2)^2(y^4)^2=9x^4y^8$.

$\therefore (3x^2y^4)^2\times 2x=9x^4y^8\times 2x=18x^{4+1}y^8=18x^5y^8$.

Example 4 — unscaffolded

Solve $3^{x+1}=81$.

Write 81 as a power of 3: $81=3^4$. So $3^{x+1}=3^4$.

Equating indices: $x+1=4$, so $\therefore x=3$.

Practice questions

Scaffolded

Q1. Simplify $a^7\times a^3$.

Q2. Simplify $\frac{x^8}{x^5}$.

Q3. Simplify $(m^3)^4$.

Q4. Simplify $4x^3\times 3x^5$.

Q5. Simplify $\frac{20a^7}{5a^2}$.

Q6. Evaluate 2^{-3}.

Unscaffolded

Simplify each of the following, expressing answers with positive indices. (Q16–Q18: solve for x.)

Q7. $5p^4\times 2p^3$

Q8. $\frac{18x^6y^4}{6x^2y}$

Q9. $(2a^3)^4$

Q10. $(3x^2y^3)^2\times 2xy$

Q11. $\frac{(a^5b^2)^3}{a^2b}$

Q12. $\dfrac{6m^5 \times 2m^{-2}}{3m}$

Q13. $3^{-2}+2^0$

Q14. $\left(\dfrac{2}{3}\right)^{-2}$

Q15. $\dfrac{4x^3y^{-2}}{2x^{-1}y}$

Q16. Solve $2^x=128$.

Q17. Solve $5^{2x-1}=125$.

Q18. Solve $9^x=27$.

Answers

Q1. a^{10}. **Q2.** x^3. **Q3.** m^{12}. **Q4.** $12x^8$. **Q5.** $4a^5$. **Q6.** $\frac{1}{8}$. **Q7.** $10p^7$. **Q8.** $3x^4y^3$. **Q9.** $16a^{12}$. **Q10.** $18x^5y^7$. **Q11.** $a^{13}b^5$. **Q12.** $4m^2$. **Q13.** $\frac{10}{9}$. **Q14.** $\frac{9}{4}$. **Q15.** $\frac{2x^4}{y^3}$. **Q16.** $x=7$. **Q17.** $x=2$. **Q18.** $x=\frac{3}{2}$.

Fully-worked solutions

Q1. $a^7\times a^3=a^{7+3}=a^{10}$.

Q2. $\frac{x^8}{x^5}=x^{8-5}=x^3$.

Q3. $(m^3)^4=m^{3\times4}=m^{12}$.

Q4. $4x^3\times3x^5=(4\times3)x^{3+5}=12x^8$.

Q5. $\frac{20a^7}{5a^2}=\frac{20}{5}a^{7-2}=4a^5$.

Q6. $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$.

Q7. $5p^4\times2p^3=(5\times2)p^{4+3}=10p^7$.

Q8. $\frac{18x^6y^4}{6x^2y}=\frac{18}{6}x^{6-2}y^{4-1}=3x^4y^3$.

Q9. $(2a^3)^4=2^4(a^3)^4=16a^{12}$.

Q10. $(3x^2y^3)^2\times2xy=9x^4y^6\times2xy=18x^{4+1}y^{6+1}=18x^5y^7$.

Q11. $\frac{(a^5b^2)^3}{a^2b}=\frac{a^{15}b^6}{a^2b}=a^{15-2}b^{6-1}=a^{13}b^5$.

Q12. $\frac{6m^5\times2m^{-2}}{3m}=\frac{12m^{5-2}}{3m}=\frac{12m^3}{3m}=4m^{3-1}=4m^2$.

Q13. $3^{-2}+2^0=\frac{1}{9}+1=\frac{1}{9}+\frac{9}{9}=\frac{10}{9}$.

Q14. $\left(\frac{2}{3}\right)^{-2}=\left(\frac{3}{2}\right)^2=\frac{9}{4}$.

Q15. $\frac{4x^3y^{-2}}{2x^{-1}y}=\frac{4}{2}x^{3-(-1)}y^{-2-1}=2x^4y^{-3}=\frac{2x^4}{y^3}$.

Q16. $2^x=128=2^7$, so $\therefore x=7$.

Q17. $5^{2x-1}=125=5^3$, so $2x-1=3\Rightarrow2x=4\Rightarrow\therefore x=2$.

Q18. $9^x=27$. Write both sides as powers of 3: $(3^2)^x=3^3$, so $3^{2x}=3^3$, giving $2x=3\Rightarrow\therefore x=\frac{3}{2}$.

Theory

The **exponent laws** (index laws) you know for whole-number powers also hold when the index is a fraction. This lets us give a meaning to powers such as $a^{1/2}$ and $a^{2/3}$.

The index laws (for $a, b > 0$ and any rational m, n):

$$a^m \times a^n = a^{m+n}, \frac{a^m}{a^n} = a^{m-n}, (a^m)^n = a^{mn},$$

$$(ab)^m = a^m b^m, a^0 = 1, a^{-m} = \frac{1}{a^m}.$$

Fractional indices as roots. A power with index $\frac{1}{n}$ means the n**th root**:

$$a^{1/n} = \sqrt[n]{a}.$$

More generally, the numerator is a power and the denominator is a root:

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}.$$

It is usually easier to take the **root first**, then apply the power — the numbers stay small. For example, $125^{2/3} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$.

Negative rational indices. A negative index means "reciprocal":

$$a^{-m/n} = \frac{1}{a^{m/n}}.$$

For example, $9^{-3/2} = \frac{1}{9^{3/2}} = \frac{1}{\left(\sqrt{9}\right)^3} = \frac{1}{3^3} = \frac{1}{27}$.

Converting between surd and index form. Because $\sqrt[n]{a^m} = a^{m/n}$, any surd can be written as a power and vice versa — e.g. $\sqrt[3]{x^4} = x^{4/3}$ and $x^{5/2} = \sqrt{x^5}$. Writing surds in index form makes them easy to multiply, divide and simplify using the index laws.

Worked examples

Example 1 — scaffolded

Evaluate $27^{2/3}$.

Working	Comment
$27^{2/3} = \left(\sqrt[3]{27}\right)^2$	The denominator 3 is the root; the numerator 2 is the power.
$= 3^2$	Take the cube root first: $\sqrt[3]{27} = 3$.
$= 9$	Then apply the power.

Example 2 — scaffolded

Evaluate $16^{-3/4}$.

Working	Comment
$16^{-3/4} = \dfrac{1}{16^{3/4}}$	A negative index means take the reciprocal.
$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$	Fourth root first ($\sqrt[4]{16} = 2$), then cube.
$\therefore 16^{-3/4} = \dfrac{1}{8}$	Substitute back.

Example 3 — unscaffolded

Simplify $\dfrac{x^{1/2}x^{2/3}}{x^{1/6}}$, giving the answer in index form.

$$\frac{x^{1/2}x^{2/3}}{x^{1/6}} = x^{1/2+2/3-1/6}.$$

The indices have a common denominator of 6: $\dfrac{1}{2} + \dfrac{2}{3} - \dfrac{1}{6} = \dfrac{3+4-1}{6} = \dfrac{6}{6} = 1$.

$\therefore \dfrac{x^{1/2}x^{2/3}}{x^{1/6}} = x$.

Example 4 — unscaffolded

(a) Write $\sqrt[4]{x^3}$ in index form and $a^{2/5}$ in surd form.

(b) Simplify $\left(16a^8\right)^{3/4}$.

(c) $\sqrt[4]{x^3} = x^{3/4}$, and $a^{2/5} = \sqrt[5]{a^2}$.

(d) $\left(16a^8\right)^{3/4} = 16^{3/4}\left(a^8\right)^{3/4} = \left(\sqrt[4]{16}\right)^3 \times a^{8\times 3/4} = 2^3 \times a^6 = 8a^6$.

$\therefore \left(16a^8\right)^{3/4} = 8a^6$.

Practice questions

Scaffolded

Q1. Evaluate $8^{2/3}$.

Q2. Evaluate $32^{2/5}$.

Q3. Evaluate $25^{-1/2}$.

Q4. Simplify $\left(x^{1/2}\right)^4$.

Q5. Simplify $\dfrac{a^{3/4}}{a^{1/4}}$, giving the answer as a surd.

Q6. Write $\sqrt[5]{x^3}$ in index form, and write $x^{3/4}$ in surd form.

Unscaffolded

Evaluate or simplify each of the following, giving exact answers.

Q7. $81^{3/4}$

Q8. $100^{3/2}$

Q9. $27^{-2/3}$

Q10. $(1/4)^{-1/2}$

Q11. $(8/27)^{2/3}$

Q12. $16^{5/4}$

Q13. $64^{-2/3}$

Q14. $(8x^6)^{1/3}$

Q15. $(27x^9)^{2/3}$

Q16. $\sqrt{x} \times \sqrt[3]{x}$

Q17. $(x^{2/3})^{3/2}$

Q18. $\dfrac{a^{5/6}}{a^{1/3}}$

Answers

Q1. 4. **Q2.** 4. **Q3.** $\frac{1}{5}$. **Q4.** x^2. **Q5.** $\sqrt{a}$. **Q6.** $x^{3/5}$; $\sqrt[4]{x^3}$. **Q7.** 27. **Q8.** 1000. **Q9.** $\frac{1}{9}$. **Q10.** 2. **Q11.** $\frac{4}{9}$. **Q12.** 32. **Q13.** $\frac{1}{16}$. **Q14.** $2x^2$. **Q15.** $9x^6$. **Q16.** $x^{5/6}$. **Q17.** x. **Q18.** $\sqrt{a}$.

Fully-worked solutions

Q1. $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$.

Q2. $32^{2/5} = \left(\sqrt[5]{32}\right)^2 = 2^2 = 4$.

Q3. $25^{-1/2} = \dfrac{1}{25^{1/2}} = \dfrac{1}{\sqrt{25}} = \dfrac{1}{5}$.

Q4. $\left(x^{1/2}\right)^4 = x^{\frac{1}{2}\times 4} = x^2$.

Q5. $\dfrac{a^{3/4}}{a^{1/4}} = a^{3/4-1/4} = a^{1/2} = \sqrt{a}$.

Q6. $\sqrt[5]{x^3} = x^{3/5}$; and $x^{3/4} = \sqrt[4]{x^3}$.

Q7. $81^{3/4} = \left(\sqrt[4]{81}\right)^3 = 3^3 = 27$.

Q8. $100^{3/2} = \left(\sqrt{100}\right)^3 = 10^3 = 1000$.

Q9. $27^{-2/3} = \dfrac{1}{27^{2/3}} = \dfrac{1}{\left(\sqrt[3]{27}\right)^2} = \dfrac{1}{3^2} = \dfrac{1}{9}$.

Q10. $(1/4)^{-1/2} = 4^{1/2} = \sqrt{4} = 2$ (a negative index inverts the fraction).

Q11. $(8/27)^{2/3} = \dfrac{8^{2/3}}{27^{2/3}} = \dfrac{\left(\sqrt[3]{8}\right)^2}{\left(\sqrt[3]{27}\right)^2} = \dfrac{2^2}{3^2} = \dfrac{4}{9}$.

Q12. $16^{5/4} = \left(\sqrt[4]{16}\right)^5 = 2^5 = 32$.

Q13. $64^{-2/3} = \dfrac{1}{64^{2/3}} = \dfrac{1}{\left(\sqrt[3]{64}\right)^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$.

Q14. $\left(8x^6\right)^{1/3} = 8^{1/3}\left(x^6\right)^{1/3} = 2 \times x^{6/3} = 2x^2$.

Q15. $\left(27x^9\right)^{2/3} = 27^{2/3}\left(x^9\right)^{2/3} = \left(\sqrt[3]{27}\right)^2 \times x^{9\times 2/3} = 3^2 \times x^6 = 9x^6$.

Q16. $\sqrt{x} \times \sqrt[3]{x} = x^{1/2} \times x^{1/3} = x^{1/2+1/3} = x^{3/6+2/6} = x^{5/6}$.

Q17. $\left(x^{2/3}\right)^{3/2} = x^{\frac{2}{3}\times\frac{3}{2}} = x^1 = x$.

Q18. $\dfrac{a^{5/6}}{a^{1/3}} = a^{5/6-1/3} = a^{5/6-2/6} = a^{3/6} = a^{1/2} = \sqrt{a}$.

Chapter 11 Exponential and logarithmic functions — 11C Exponential functions and their graphs

Theory

An **exponential function** has the form $y=a^x$, where the **base** $a>0$ and $a\neq 1$.

The basic exponential $y=a^x$.

- It passes through $(0,1)$, since $a^0=1$.
- If $a>1$ the graph is **increasing** — exponential **growth**; if $0<a<1$ it is **decreasing** — exponential **decay**.
- It has the **horizontal asymptote** $y=0$ (the x-axis): the curve approaches the axis but never reaches it.
- Domain $\mathbb{R}$; range $(0,\infty)$ — the graph is always **above** the x-axis.
- Because $a^{-x}=(1/a)^x$, the graph of $y=2^{-x}$ is the same as $y=(1/2)^x$ — a reflection in the y-axis turns growth into decay.

Transformations — the general form $y=A\,a^{x-h}+k$.

- The **horizontal asymptote** moves to $y=k$.
- Domain is still $\mathbb{R}$; the **range** is (k,∞) if $A>0$, or $(-\infty,k)$ if $A<0$.
- y**-intercept:** substitute $x=0$.
- x**-intercept:** solve $y=0$. This exists only when 0 lies strictly on the curve's side of the asymptote — that is, when A and k have opposite signs; solving it may require **logarithms** (see 11F).
- $A<0$ **reflects** the curve in the line $y=k$, so it decreases where the basic curve increases.

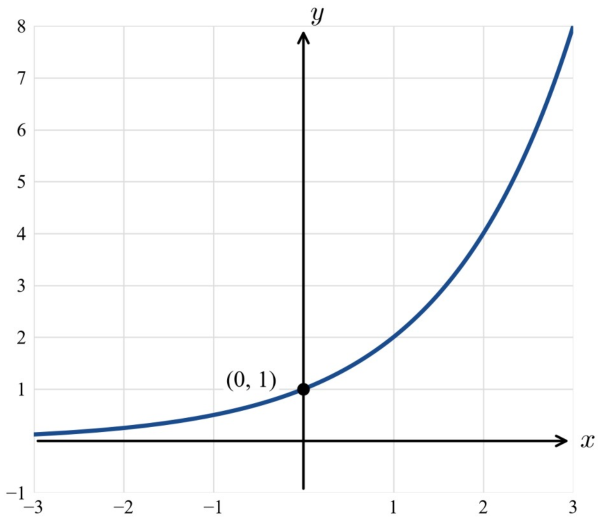
To sketch an exponential graph: (1) draw the asymptote $y=k$ (dashed); (2) find the y-intercept; (3) find the x-intercept, if there is one; (4) decide growth or decay and which side of the asymptote the curve lies; (5) draw a smooth curve approaching the asymptote.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y=2^x$, stating the asymptote, intercepts, domain and range.

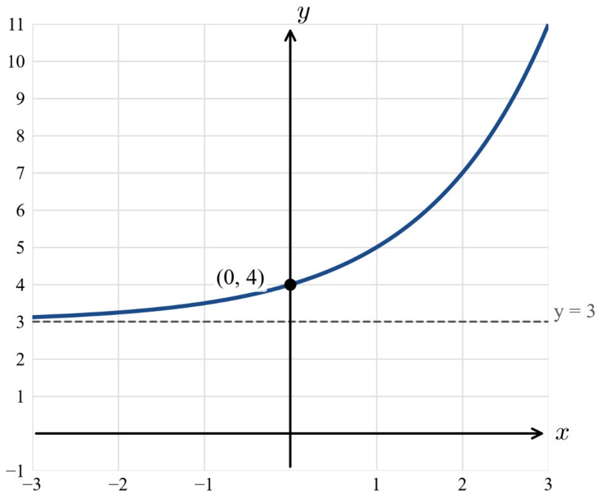
Working	Comment
The base is $2>1$, so the graph shows exponential **growth** (increasing).	The base decides growth versus decay.
Horizontal asymptote $y=0$ (the x-axis).	Every $y=a^x$ has asymptote $y=0$.
y-intercept: $x=0\Rightarrow y=2^0=1$, giving $(0,1)$.	The basic exponential passes through $(0,1)$.
No x-intercept, since $2^x>0$ for all x.	The curve stays above the asymptote.
Domain $\mathbb{R}$, range $(0,\infty)$. Extra points: $(1,2)$ and $(-1,1/2)$.	Plot a couple of points to fix the shape.



Example 2 — scaffolded

Sketch the graph of $y=2^x+3$.

Working	Comment
This is $y=2^x$ translated **up 3**, so the asymptote is $y=3$.	Adding 3 lifts the graph and its asymptote.
y-intercept: $x=0 \Rightarrow y=2^0+3=1+3=4$, giving $(0,4)$.	Substitute $x=0$.
Since $2^x>0$, $y>3$ always — there is **no x-intercept**.	The curve lies entirely above $y=3$.
Domain $\mathbb{R}$, range $(3,\infty)$.	Range starts just above the asymptote.



Example 3 — unscaffolded

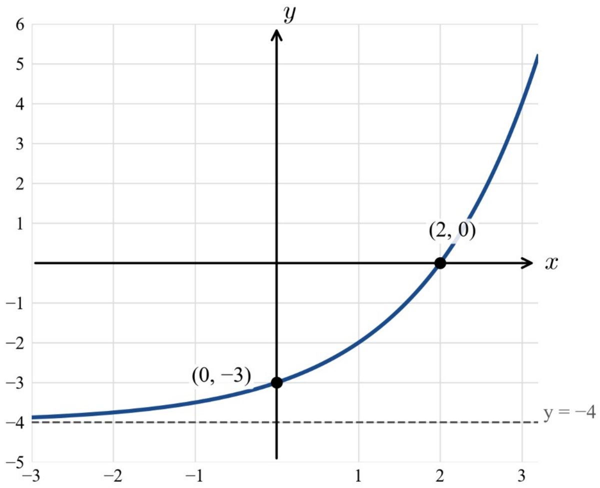
Sketch the graph of $y=2^x-4$.

This is $y=2^x$ translated **down 4**, so the asymptote is $y=-4$.

y-intercept: $x=0 \Rightarrow y=2^0-4=1-4=-3$, giving $(0,-3)$.

x-intercept: $y=0 \Rightarrow 2^x=4 \Rightarrow x=2$, giving $(2,0)$.

$\therefore$ domain $\mathbb{R}$, range $(-4,\infty)$.



Example 4 — unscaffolded

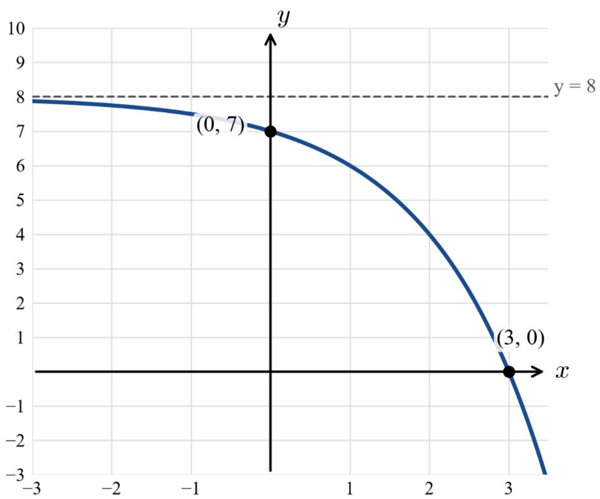
Sketch the graph of $y=8-2^x$.

The coefficient of 2^x is negative, so the curve is a **reflection** in the line $y=8$ and is **decreasing**; the asymptote is $y=8$
.

y-intercept: $x=0 \Rightarrow y=8-2^0=8-1=7$, giving $(0,7)$.

x-intercept: $y=0 \Rightarrow 2^x=8 \Rightarrow x=3$, giving $(3,0)$.

$\therefore$ domain $\mathbb{R}$, range $(-\infty,8)$.



Practice questions

Scaffolded

Q1. For $y=3^x$: (a) state whether it is growth or decay; (b) write the asymptote and the y-intercept; (c) state the domain and range; (d) sketch the graph.

Q2. For $y=(1/2)^x$: (a) state whether it is growth or decay; (b) write the asymptote and y-intercept; (c) state the range; (d) sketch the graph.

Q3. For $y=2^x-1$: (a) state the asymptote; (b) find the y-intercept; (c) find the x-intercept; (d) sketch the graph.

Q4. For $y=3^x+2$: (a) state the asymptote; (b) find the y-intercept; (c) explain why there is no x-intercept; (d) sketch the graph.

Q5. For $y = 2^{x-1}$: (a) state the asymptote; (b) find the y-intercept; (c) state the range; (d) sketch the graph.

Q6. For $y = 9 - 3^x$: (a) state the asymptote and whether the graph increases or decreases; (b) find both intercepts; (c) state the range; (d) sketch the graph.

Unscaffolded

For each of the following: sketch the graph, labelling the y-intercept, any x-intercept, and the horizontal asymptote, and state the range.

Q7. $y = 4^x$

Q8. $y = (1/3)^x$

Q9. $y = 2^x + 1$

Q10. $y = 3^x - 9$

Q11. $y = 2^{x+1}$

Q12. $y = 2^{x-2}$

Q13. $y = 2 - 2^x$

Q14. $y = 2^{-x} - 1$

Q15. $y = 3^x - 3$

Q16. $y = 3 \times 2^x$

Q17. $y = 5 - 2^x$

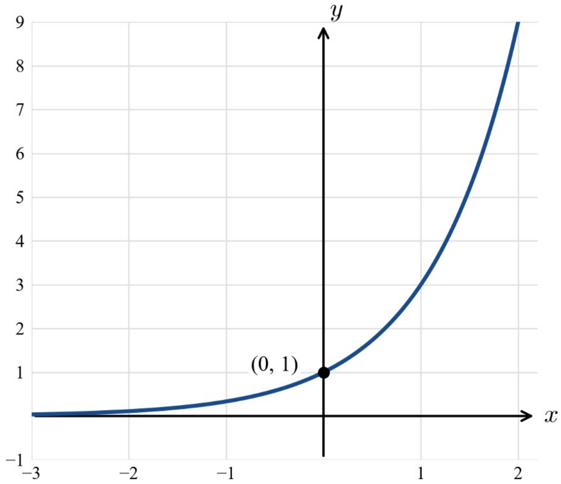
Q18. $y = 2^{-x} + 1$

Answers

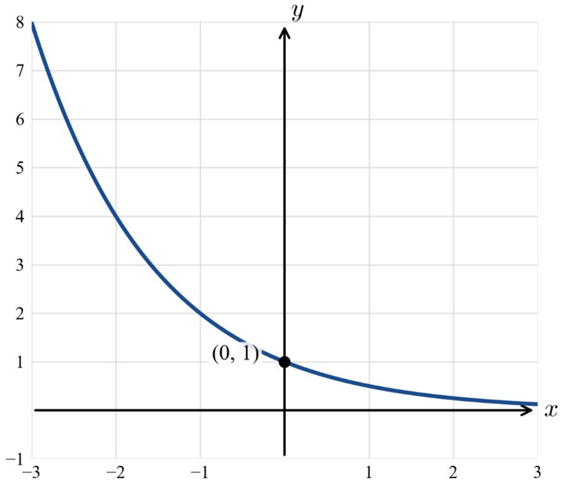
Q1. Growth; asymptote $y=0$; y-int $(0,1)$; domain $\mathbb{R}$, range $(0,\infty)$. **Q2.** Decay; asymptote $y=0$; y-int $(0,1)$; range $(0,\infty)$. **Q3.** Asymptote $y=-1$; y-int $(0,0)$; x-int $(0,0)$; range $(-1,\infty)$. **Q4.** Asymptote $y=2$; y-int $(0,3)$; no x-intercept; range $(2,\infty)$. **Q5.** Asymptote $y=0$; y-int $(0,1/2)$; range $(0,\infty)$. **Q6.** Asymptote $y=9$, decreasing; y-int $(0,8)$, x-int $(2,0)$; range $(-\infty,9)$. **Q7.** Asymptote $y=0$; y-int $(0,1)$; range $(0,\infty)$. **Q8.** Decay; asymptote $y=0$; y-int $(0,1)$; range $(0,\infty)$. **Q9.** Asymptote $y=1$; y-int $(0,2)$; range $(1,\infty)$. **Q10.** Asymptote $y=-9$; y-int $(0,-8)$; x-int $(2,0)$; range $(-9,\infty)$. **Q11.** Asymptote $y=0$; y-int $(0,2)$; range $(0,\infty)$. **Q12.** Asymptote $y=0$; y-int $(0,1/4)$; range $(0,\infty)$. **Q13.** Asymptote $y=2$, decreasing; y-int $(0,1)$; x-int $(1,0)$; range $(-\infty,2)$. **Q14.** Decay; asymptote $y=-1$; y-int $(0,0)$; x-int $(0,0)$; range $(-1,\infty)$. **Q15.** Asymptote $y=-3$; y-int $(0,-2)$; x-int $(1,0)$; range $(-3,\infty)$. **Q16.** Asymptote $y=0$; y-int $(0,3)$; range $(0,\infty)$. **Q17.** Asymptote $y=5$, decreasing; y-int $(0,4)$; x-int $(\log_2 5,0)\approx(2.32,0)$; range $(-\infty,5)$. **Q18.** Decay; asymptote $y=1$; y-int $(0,2)$; range $(1,\infty)$.

Fully-worked solutions

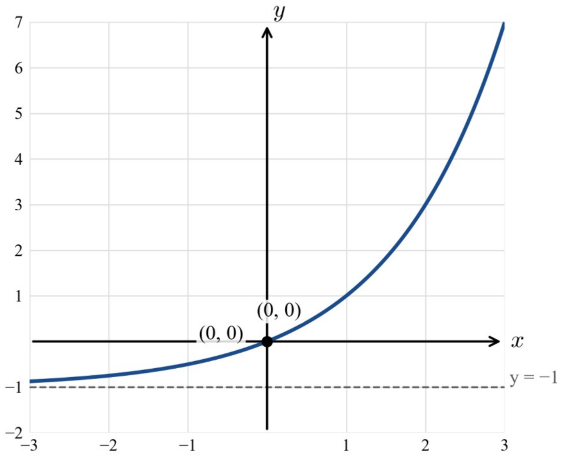
Q1. $y=3^x$. Base $3>1$: **growth**. Asymptote $y=0$; y-intercept $(0,1)$; domain $\mathbb{R}$, range $(0,\infty)$.



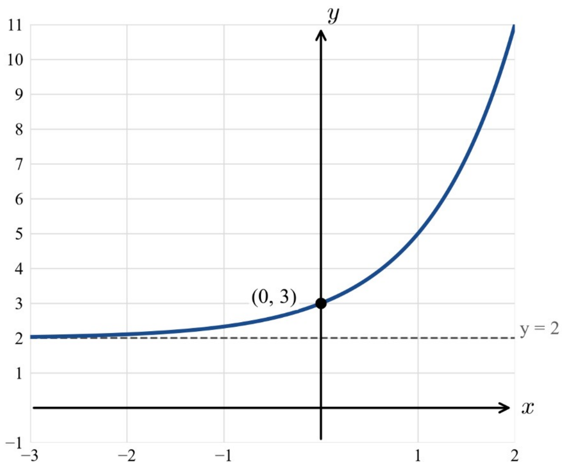
Q2. $y=(1/2)^x$. Base $1/2<1$: **decay**. Asymptote $y=0$; y-intercept $(0,1)$; range $(0,\infty)$. (Points $(1,1/2)$, $(-1,2)$.)



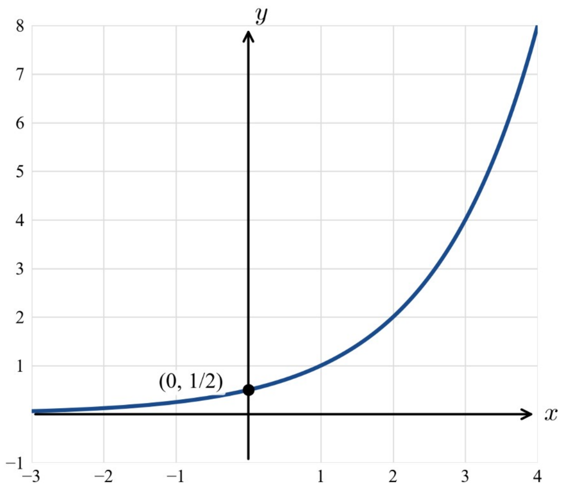
Q3. $y=2^x-1$. Asymptote $y=-1$; y-intercept $x=0\Rightarrow 1-1=0$, giving $(0,0)$; x-intercept $2^x=1\Rightarrow x=0$, giving $(0,0)$. Range $(-1,\infty)$.



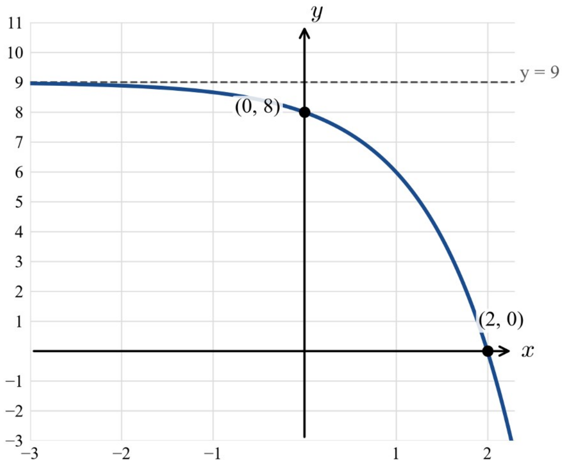
Q4. $y = 3^x + 2$. Asymptote $y = 2$; y-intercept $3^0 + 2 = 3$, giving $(0, 3)$. Since $3^x > 0$, $y > 2$ for all x, so there is **no x-intercept**. Range $(2, \infty)$.



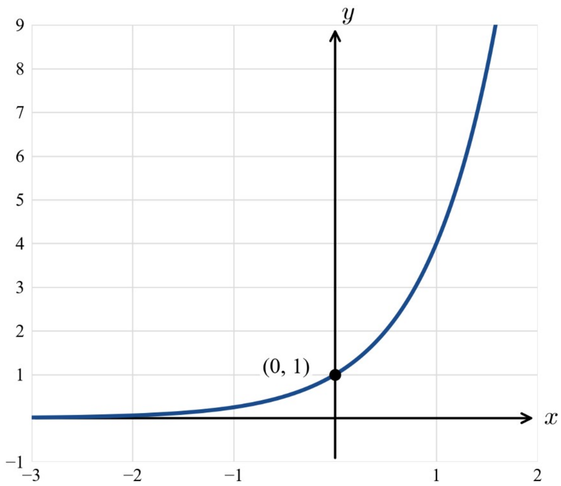
Q5. $y = 2^{x-1}$. This is $y = 2^x$ shifted right 1; asymptote $y = 0$; y-intercept $2^{-1} = 1/2$, giving $(0, 1/2)$. Range $(0, \infty)$.



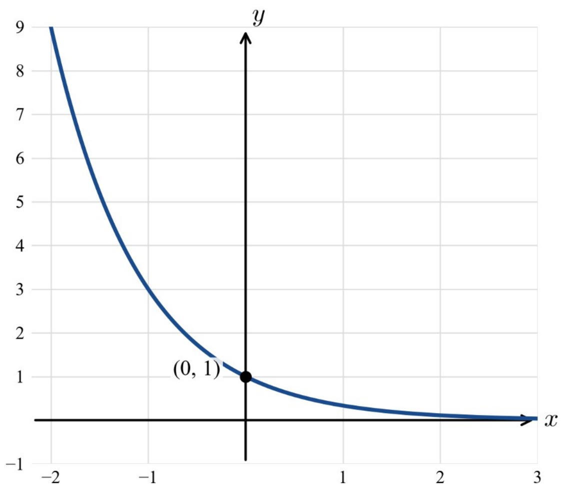
Q6. $y = 9 - 3^x$. Negative coefficient of 3^x: **decreasing**, asymptote $y = 9$. y-intercept $9 - 3^0 = 8$, giving $(0, 8)$; x-intercept $3^x = 9 \Rightarrow x = 2$, giving $(2, 0)$. Range $(-\infty, 9)$.



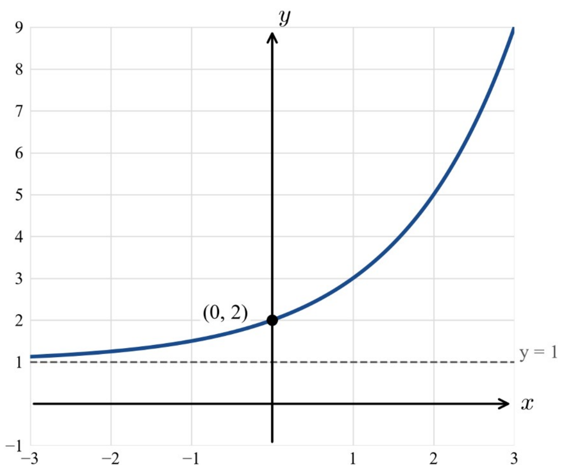
Q7. $y = 4^x$. Growth; asymptote $y = 0$; y-intercept $(0, 1)$; range $(0, \infty)$.



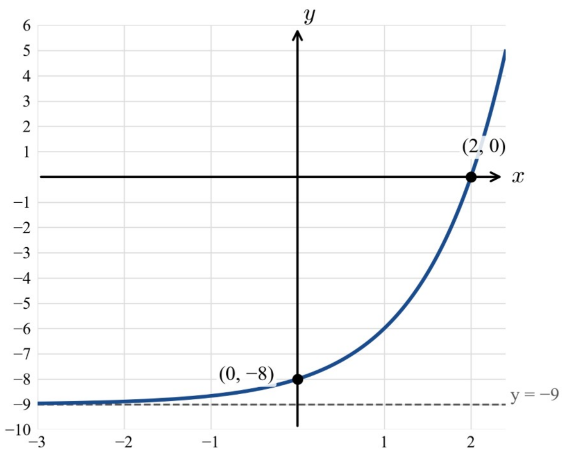
Q8. $y = (1/3)^x$. Decay; asymptote $y = 0$; y-intercept $(0, 1)$; range $(0, \infty)$.



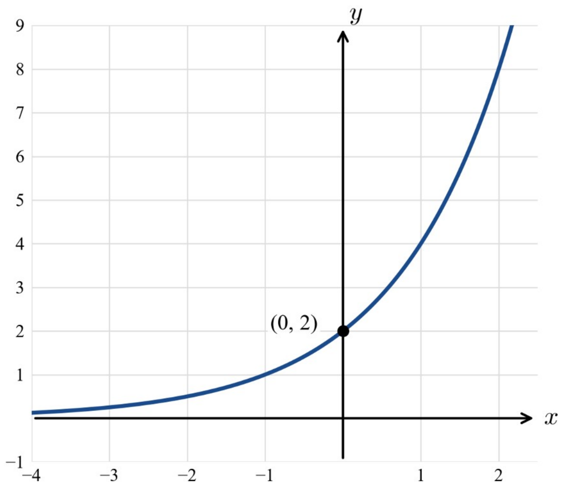
Q9. $y = 2^x + 1$. Asymptote $y = 1$; y-intercept $2^0 + 1 = 2$, giving $(0, 2)$; no x-intercept ($y > 1$). Range $(1, \infty)$.



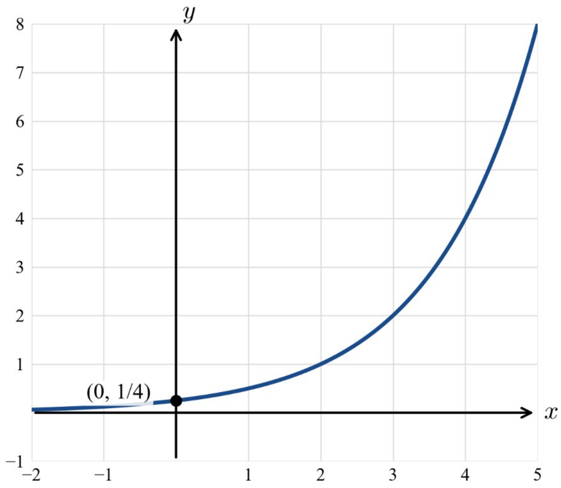
Q10. $y=3^x-9$. Asymptote $y=-9$; y-intercept $3^0-9=-8$, giving $(0,-8)$; x-intercept $3^x=9 \Rightarrow x=2$, giving $(2,0)$. Range $(-9,\infty)$.



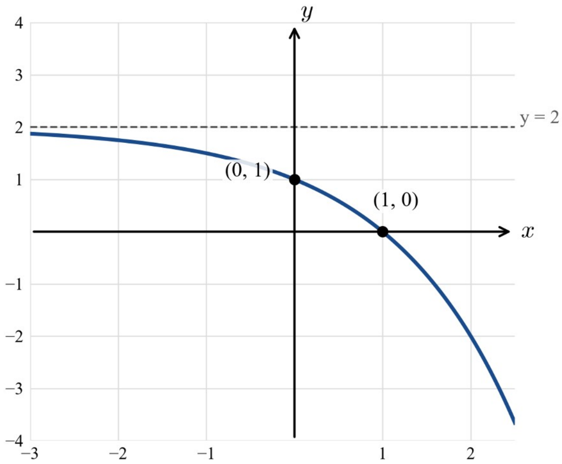
Q11. $y=2^{x+1}=2\times 2^x$. Asymptote $y=0$; y-intercept $2^1=2$, giving $(0,2)$; range $(0,\infty)$. (Shifted left 1 from $y=2^x$.)



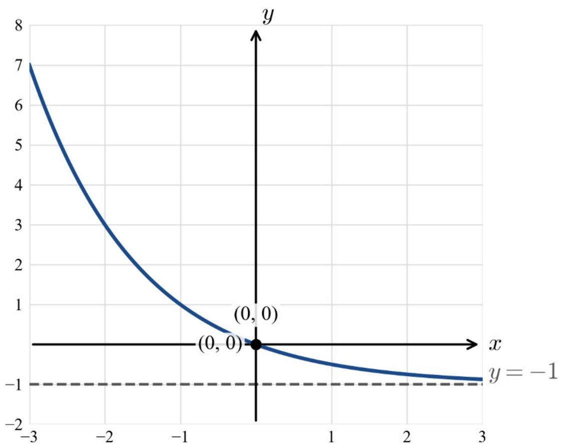
Q12. $y=2^{x-2}$. Asymptote $y=0$; y-intercept $2^{-2}=1/4$, giving $(0,1/4)$; range $(0,\infty)$. (Shifted right 2.)



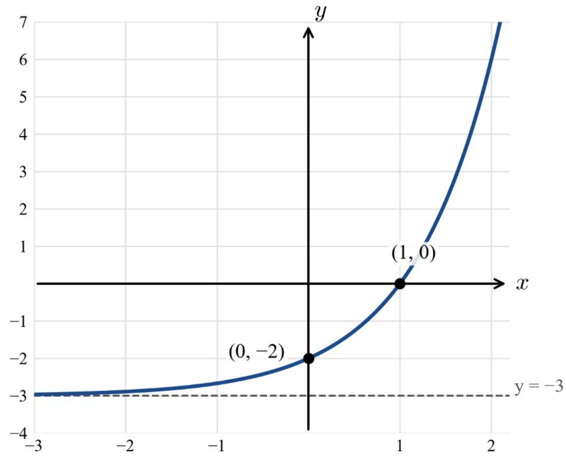
Q13. $y=2-2^{x}$. Negative coefficient: **decreasing**, asymptote $y=2$. y-intercept $2-2^{0}=1$, giving $(0,1)$; x-intercept $2^{x}=2 \Rightarrow x=1$, giving $(1,0)$. Range $(-\infty,2)$.



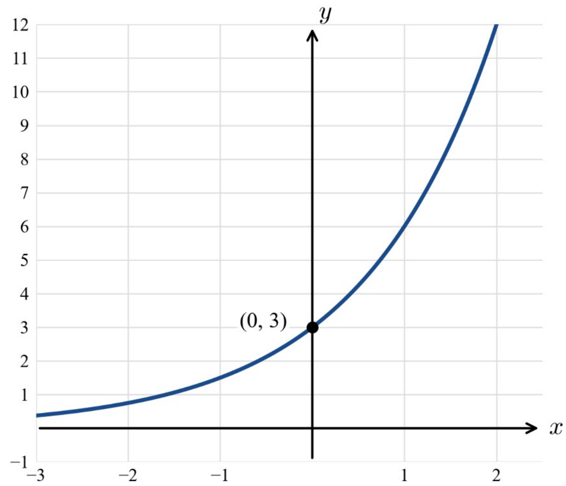
Q14. $y=2^{-x}-1$. This is $y=2^{-x}$ translated **down 1**, so the asymptote is $y=-1$ and the curve **decays** towards it. y-intercept $2^{0}-1=0$, giving $(0,0)$; x-intercept $2^{-x}=1 \Rightarrow x=0$, giving $(0,0)$. Range $(-1,\infty)$.



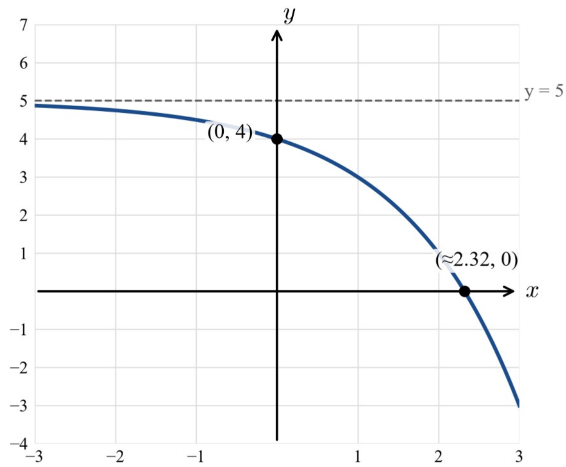
Q15. $y=3^{x}-3$. Asymptote $y=-3$; y-intercept $3^{0}-3=-2$, giving $(0,-2)$; x-intercept $3^{x}=3 \Rightarrow x=1$, giving $(1,0)$. Range $(-3,\infty)$.



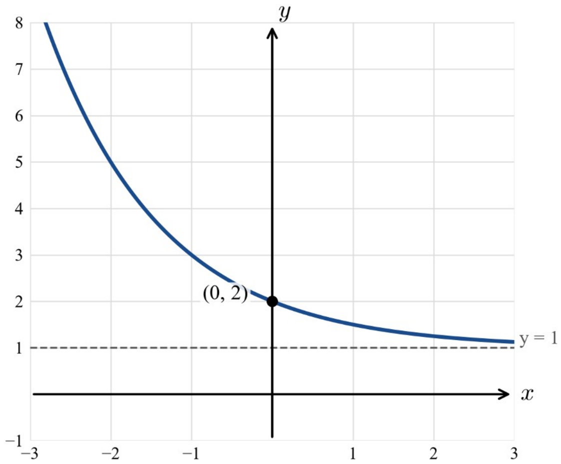
Q16. $y=3\times 2^x$. A dilation of $y=2^x$ by factor 3 from the x-axis; asymptote $y=0$; y-intercept $3\times 2^0=3$, giving $(0,3)$; range $(0,\infty)$.



Q17. $y=5-2^x$. Negative coefficient: **decreasing**, asymptote $y=5$. y-intercept $5-2^0=4$, giving $(0,4)$; x-intercept $2^x=5\Rightarrow x=\log_2 5\approx 2.32$, giving $(\log_2 5,0)$. Range $(-\infty,5)$.



Q18. $y=2^{-x}+1$. Decay with asymptote $y=1$; y-intercept $2^0+1=2$, giving $(0,2)$; no x-intercept ($y>1$). Range $(1,\infty)$.


y
8
7
6
5
4
3
2
1
−1
(0, 2)
y = 1
x
−3
−2
−1
1
2
3

Chapter 11 Exponential and logarithmic functions — 11D Exponential equations and inequalities

Theory

An **exponential equation** has the unknown in an index (exponent). This section solves such equations *without logarithms*, using the **same-base method**.

The same-base principle. For $a > 0$, $a \neq 1$,

$$a^m = a^n \Leftrightarrow m = n.$$

Because $y = a^x$ is a one-to-one function, two powers of the same base are equal only when their indices are equal.

Method. (1) Write both sides as powers of the **same base**; (2) equate the indices; (3) solve the resulting equation. The **index laws** are the main tool:

$$a^m \times a^n = a^{m+n}, (a^m)^n = a^{mn}, a^{-n} = \frac{1}{a^n}, a^0 = 1.$$

Equations that are quadratic in a^x. If an equation contains a^{2x} and a^x, substitute $y = a^x$ to get a **quadratic** in y. Solve it, then return to $a^x = y$ for each solution. **Reject any** $y \leq 0$, because $a^x > 0$ for all x.

Exponential inequalities. Since $y = a^x$ is **increasing** when $a > 1$,

$$a^x > a^k \Leftrightarrow x > k\,(a > 1).$$

When $0 < a < 1$ the function is **decreasing**, so the inequality **reverses**: $a^x > a^k \Leftrightarrow x < k$.

Worked examples

Example 1 — scaffolded

Solve $3^{2x-1} = 81$.

Working	Comment
$81 = 3^4$, so $3^{2x-1} = 3^4$.	Write the right-hand side as a power of 3.
$2x - 1 = 4$.	Equate the indices (same base).
$2x = 5$, so $x = \frac{5}{2}$.	Solve the linear equation.

Example 2 — scaffolded

Solve $4^x = 8$.

Working	Comment
$4^x = (2^2)^x = 2^{2x}$ and $8 = 2^3$.	Write both sides as powers of 2.
$2^{2x} = 2^3 \Rightarrow 2x = 3$.	Equate indices.
$x = \frac{3}{2}$.	Solve.

Example 3 — unscaffolded

Solve $25^x = 125^{x-1}$.

Write both sides as powers of 5: $25=5^2$ and $125=5^3$, so

$$5^{2x}=5^{3(x-1)} \Rightarrow 2x=3(x-1)=3x-3.$$

Then $-x=-3$. $\therefore x=3$.

Example 4 — unscaffolded

Solve $2^{2x}-5\times 2^x+4=0$.

Let $y=2^x$ (so $y>0$). Since $2^{2x}=(2^x)^2=y^2$,

$$y^2-5y+4=0 \Rightarrow (y-1)(y-4)=0 \Rightarrow y=1 \text{ or } y=4.$$

Returning to $2^x=y$: $2^x=1 \Rightarrow x=0$; $2^x=4=2^2 \Rightarrow x=2$. $\therefore x=0$ or $x=2$.

Practice questions

Scaffolded

Q1. Solve $2^x=64$.

Q2. Solve $3^x=\frac{1}{9}$.

Q3. Solve $5^{2x}=125$.

Q4. Solve $8^x=32$ (write both sides as powers of 2).

Q5. Solve $2^{x+1}=16$.

Q6. Solve $9^x=27^{x-1}$ (write both sides as powers of 3).

Unscaffolded

Solve each of the following. Give exact solutions.

Q7. $3^x=243$

Q8. $2^x=\frac{1}{32}$

Q9. $4^{x-1}=64$

Q10. $16^x=8$

Q11. $25^x=\frac{1}{5}$

Q12. $27^{x+1}=9^x$

Q13. $2^{2x}-10\times 2^x+16=0$

Q14. $3^{2x}-4\times 3^x+3=0$

Q15. $2^{2x}-2^x-2=0$

Q16. $2^x>16$

Q17. $3^x \le \frac{1}{27}$

Q18. $\left(\frac{1}{2}\right)^x > 8$

Answers

Q1. $x=6$. **Q2.** $x=-2$. **Q3.** $x=\frac{3}{2}$. **Q4.** $x=\frac{5}{3}$. **Q5.** $x=3$. **Q6.** $x=3$. **Q7.** $x=5$. **Q8.** $x=-5$. **Q9.** $x=4$. **Q10.** $x=\frac{3}{4}$.
Q11. $x=-\frac{1}{2}$. **Q12.** $x=-3$. **Q13.** $x=1$ or $x=3$. **Q14.** $x=0$ or $x=1$. **Q15.** $x=1$. **Q16.** $x>4$. **Q17.** $x\le-3$. **Q18.** $x<-3$.

Fully-worked solutions

Q1. $2^x=64=2^6\Rightarrow x=6$.

Q2. $3^x=\frac{1}{9}=3^{-2}\Rightarrow x=-2$.

Q3. $5^{2x}=125=5^3\Rightarrow 2x=3\Rightarrow x=\frac{3}{2}$.

Q4. $8^x=(2^3)^x=2^{3x}$ and $32=2^5$, so $2^{3x}=2^5\Rightarrow 3x=5\Rightarrow x=\frac{5}{3}$.

Q5. $2^{x+1}=16=2^4\Rightarrow x+1=4\Rightarrow x=3$.

Q6. $9^x=3^{2x}$ and $27^{x-1}=3^{3(x-1)}$, so $2x=3(x-1)=3x-3\Rightarrow x=3$.

Q7. $3^x=243=3^5\Rightarrow x=5$.

Q8. $2^x=\frac{1}{32}=2^{-5}\Rightarrow x=-5$.

Q9. $4^{x-1}=64$; $4^{x-1}=2^{2(x-1)}$ and $64=2^6$, so $2(x-1)=6\Rightarrow x-1=3\Rightarrow x=4$.

Q10. $16^x=2^{4x}$ and $8=2^3$, so $4x=3\Rightarrow x=\frac{3}{4}$.

Q11. $25^x=5^{2x}$ and $\frac{1}{5}=5^{-1}$, so $2x=-1\Rightarrow x=-\frac{1}{2}$.

Q12. $27^{x+1}=3^{3(x+1)}$ and $9^x=3^{2x}$, so $3(x+1)=2x\Rightarrow 3x+3=2x\Rightarrow x=-3$.

Q13. Let $y=2^x$: $y^2-10y+16=0\Rightarrow(y-2)(y-8)=0\Rightarrow y=2$ or $y=8$. Then $2^x=2\Rightarrow x=1$; $2^x=8=2^3\Rightarrow x=3$. $\therefore x=1$ or $x=3$.

Q14. Let $y=3^x$: $y^2-4y+3=0\Rightarrow(y-1)(y-3)=0\Rightarrow y=1$ or $y=3$. Then $3^x=1\Rightarrow x=0$; $3^x=3\Rightarrow x=1$. $\therefore x=0$ or $x=1$.

Q15. Let $y=2^x$: $y^2-y-2=0\Rightarrow(y-2)(y+1)=0\Rightarrow y=2$ or $y=-1$. Since $2^x>0$, reject $y=-1$. So $2^x=2\Rightarrow x=1$.

Q16. $2^x>16=2^4$. The base $2>1$, so $x>4$.

Q17. $3^x\le\frac{1}{27}=3^{-3}$. The base $3>1$, so $x\le-3$.

Q18. $\left(\frac{1}{2}\right)^x>8$. Write $8=2^3=\left(\frac{1}{2}\right)^{-3}$, so $\left(\frac{1}{2}\right)^x>\left(\frac{1}{2}\right)^{-3}$. The base $\frac{1}{2}<1$, so the inequality reverses: $x<-3$.

Theory

A **logarithm** is an index (power). The statement

$$\log_a b = c \text{ means } a^c = b,$$

where the **base** $a > 0$, $a \neq 1$, and $b > 0$. In words, $\log_a b$ is *the power to which the base a must be raised to give b.*

Index form ↔ log form. Every index statement can be rewritten as a logarithm and vice versa:

$$a^c = b \Leftrightarrow \log_a b = c.$$

For example, $2^5 = 32$ becomes $\log_2 32 = 5$, and $\log_3 81 = 4$ becomes $3^4 = 81$.

Evaluating a logarithm. To evaluate $\log_a b$, ask "a to what power gives b?" — often by writing b as a power of a. For example, $\log_2 \frac{1}{16} = \log_2 2^{-4} = -4$.

Special values. For any valid base a:

$$\log_a 1 = 0, \log_a a = 1, \log_a a^x = x.$$

The logarithm laws. For $m, n > 0$:

- **Product:** $\log_a (mn) = \log_a m + \log_a n$
- **Quotient:** $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
- **Power:** $\log_a (m^p) = p \log_a m$

These follow from the index laws. The common logarithm has base 10 (written $\log$ or $\log_{10}$) and the natural logarithm has base e (written ln).

Solving simple logarithmic equations. Rewrite in index form: $\log_a x = c \Leftrightarrow x = a^c$. Always check that any solution keeps every logarithm's argument **positive**.

Change of base. $\log_a b = \frac{\log_c b}{\log_c a}$ for any valid base c — useful for evaluating a logarithm on a calculator that only has $\log_{10}$ or ln.

Worked examples

Example 1 — scaffolded

Evaluate (a) $\log_2 32$ and (b) $\log_3 \frac{1}{9}$.

Working	Comment
(a) Let $\log_2 32 = c$, so $2^c = 32$.	Write the logarithm in index form.
$32 = 2^5$, so $2^c = 2^5$ and $c = 5$. $\therefore \log_2 32 = 5$.	Write 32 as a power of the base.
(b) Let $\log_3 \frac{1}{9} = c$, so $3^c = \frac{1}{9}$.	Index form again.

Working	Comment
$\frac{1}{9}=3^{-2}$, so $c=-2$. $\therefore \log_3 \frac{1}{9}=-2$.	A negative index gives a fraction.

Example 2 — scaffolded

Simplify $\log_{10} 4+\log_{10} 25$.

Working	Comment
$\log_{10} 4+\log_{10} 25=\log_{10}(4\times 25)$	Product law: a sum of logs is the log of the product.
$=\log_{10} 100$	Multiply inside the logarithm.
$=\log_{10} 10^2=2$	$100=10^2$, and $\log_a a^x=x$.

Example 3 — unscaffolded

Simplify $2\log_a 3+\log_a 2-\log_a 6$.

$$2\log_a 3+\log_a 2-\log_a 6=\log_a 3^2+\log_a 2-\log_a 6=\log_a 9+\log_a 2-\log_a 6.$$

$$=\log_a\left(\frac{9\times 2}{6}\right)=\log_a 3.$$

$\therefore 2\log_a 3+\log_a 2-\log_a 6=\log_a 3$.

Example 4 — unscaffolded

Solve for x: (a) $\log_2 x=5$ and (b) $\log_5(x-1)=2$.

(a) In index form, $x=2^5=32$. $\therefore x=32$.

(b) In index form, $x-1=5^2=25$, so $x=26$. (Check: $x-1=25>0$.) $\therefore x=26$.

Practice questions

Scaffolded

Q1. Write in logarithm form: (a) $6^2=36$; (b) $10^3=1000$; (c) $5^{-2}=\frac{1}{25}$.

Q2. Write in index form: (a) $\log_2 16=4$; (b) $\log_5 25=2$; (c) $\log_4 2=\frac{1}{2}$.

Q3. Evaluate: (a) $\log_7 49$; (b) $\log_7 \frac{1}{49}$; (c) $\log_7 7$.

Q4. Simplify using the log laws: (a) $\log_6 2+\log_6 3$; (b) $\log_2 40-\log_2 5$.

Q5. Simplify: (a) $\log_{10} 5+\log_{10} 20$; (b) $3\log_2 2$.

Q6. Solve for x: (a) $\log_3 x=2$; (b) $\log_2 x=-3$.

Unscaffolded

Q7. Evaluate $\log_5 125$.

Q8. Evaluate $\log_4 \frac{1}{64}$.

Q9. Evaluate $\log_9 27$.

Q10. Simplify $\log_2 6 + \log_2 \frac{2}{3}$.

Q11. Simplify $\log_3 54 - \log_3 2$.

Q12. Simplify $2\log_{10} 5 + \log_{10} 4$.

Q13. Simplify $\log_2 32 - 3\log_2 2$.

Q14. Express as a single logarithm: $2\log_a x + \log_a y - 3\log_a z$.

Q15. Solve $\log_2 x = 6$.

Q16. Solve $\log_5(2x-1) = 2$.

Q17. Solve $\log_3 x + \log_3(x-2) = 1$.

Q18. Given that $\log_{10} 2 \approx 0.301$ and $\log_{10} 3 \approx 0.477$, find $\log_{10} 12$.

Answers

Q1. (a) $\log_6 36=2$; (b) $\log_{10} 1000=3$; (c) $\log_5 1/25=-2$. **Q2.** (a) $2^4=16$; (b) $5^2=25$; (c) $4^{1/2}=2$. **Q3.** (a) 2; (b) -2; (c) 1. **Q4.** (a) 1; (b) 3. **Q5.** (a) 2; (b) 3. **Q6.** (a) $x=9$; (b) $x=1/8$. **Q7.** 3. **Q8.** -3. **Q9.** $3/2$. **Q10.** 2. **Q11.** 3. **Q12.** 2. **Q13.** 2. **Q14.** $\log_a \frac{x^2 y}{z^3}$. **Q15.** $x=64$. **Q16.** $x=13$. **Q17.** $x=3$. **Q18.** $\log_{10} 12 \approx 1.079$.

Fully-worked solutions

Q1. Using $a^c=b \Leftrightarrow \log_a b=c$: (a) $\log_6 36=2$; (b) $\log_{10} 1000=3$; (c) $\log_5 \frac{1}{25}=-2$.

Q2. Using $\log_a b=c \Leftrightarrow a^c=b$: (a) $2^4=16$; (b) $5^2=25$; (c) $4^{1/2}=2$.

Q3. (a) $49=7^2$, so $\log_7 49=2$. (b) $\frac{1}{49}=7^{-2}$, so $\log_7 \frac{1}{49}=-2$. (c) $\log_7 7=1$.

Q4. (a) $\log_6 2+\log_6 3=\log_6(2\times 3)=\log_6 6=1$. (b) $\log_2 40-\log_2 5=\log_2 \frac{40}{5}=\log_2 8=3$.

Q5. (a) $\log_{10} 5+\log_{10} 20=\log_{10} 100=2$. (b) $3\log_2 2=3\times 1=3$.

Q6. (a) $x=3^2=9$. (b) $x=2^{-3}=\frac{1}{8}$.

Q7. $125=5^3$, so $\log_5 125=3$.

Q8. $\frac{1}{64}=4^{-3}$, so $\log_4 \frac{1}{64}=-3$.

Q9. $27=9^{3/2}$ (since $9^{1/2}=3$ and $3^3=27$), so $\log_9 27=\frac{3}{2}$.

Q10. $\log_2 6+\log_2 \frac{2}{3}=\log_2\left(6\times\frac{2}{3}\right)=\log_2 4=2$.

Q11. $\log_3 54-\log_3 2=\log_3 \frac{54}{2}=\log_3 27=3$.

Q12. $2\log_{10} 5+\log_{10} 4=\log_{10} 25+\log_{10} 4=\log_{10} 100=2$.

Q13. $\log_2 32-3\log_2 2=5-3(1)=2$.

Q14. $2\log_a x+\log_a y-3\log_a z=\log_a x^2+\log_a y-\log_a z^3=\log_a \frac{x^2 y}{z^3}$.

Q15. $\log_2 x=6 \Rightarrow x=2^6=64$.

Q16. $\log_5(2x-1)=2 \Rightarrow 2x-1=5^2=25 \Rightarrow 2x=26 \Rightarrow x=13$. (Check: $2x-1=25>0$.)

Q17. $\log_3 x+\log_3(x-2)=\log_3(x(x-2))=1$, so $x(x-2)=3^1=3$. Then $x^2-2x-3=0 \Rightarrow (x-3)(x+1)=0$, so $x=3$ or $x=-1$. Since the arguments must be positive, reject $x=-1$. $\therefore x=3$.

Q18. $\log_{10} 12=\log_{10}(4\times 3)=\log_{10} 4+\log_{10} 3=2\log_{10} 2+\log_{10} 3\approx 2(0.301)+0.477=1.079$.

Theory

Some exponential equations can be solved by writing both sides as powers of the **same base** (for example, $2^x = 8 \Rightarrow 2^x = 2^3 \Rightarrow x = 3$). When the bases cannot easily be matched, take **logarithms of both sides**.

The key idea. For $a > 0$, $a \neq 1$ and $b > 0$,

$$a^x = b \Rightarrow x = \log_a b = \frac{\log_{10} b}{\log_{10} a}.$$

The second form (the **change-of-base rule**) lets you evaluate the answer on a calculator using $\log_{10}$ (or $\ln$).

Method.

1. Isolate the power (divide off any coefficient first).
2. Take $\log_{10}$ (or $\ln$) of both sides.
3. Use the log law $\log\left(a^x\right) = x \log a$ to bring the power down.
4. Solve the resulting linear equation for x.

Give an **exact** answer in logarithm form, and a **decimal approximation** to the number of places asked for.

Equations of the form $A\,a^{kx} = B$. Divide by A, then take logarithms: $a^{kx} = \dfrac{B}{A} \Rightarrow k\,x \log a = \log \dfrac{B}{A}$.

Equations quadratic in a^x. An equation such as $a^{2x} + p\,a^x + q = 0$ can be solved by letting $y = a^x$, solving the quadratic in y, then solving $a^x = y$ for each **positive** value of y (reject $y \leq 0$, since $a^x > 0$ always).

Inequalities. Solve as for an equation. Because $y = \log x$ is increasing, taking logarithms of both sides of an inequality (base > 1) **keeps** the inequality direction.

Worked examples

Example 1 — scaffolded

Solve $2^x = 20$, giving the answer (a) exactly and (b) correct to 2 decimal places.

Working	Comment
Take $\log_{10}$ of both sides: $\log 2^x = \log 20$.	The bases 2 and 20 cannot be matched, so use logarithms.
$x \log 2 = \log 20$.	Bring the power down: $\log\left(2^x\right) = x \log 2$.
$x = \dfrac{\log 20}{\log 2} = \log_2 20$.	Divide by $\log 2$. This is the **exact** answer.
$x \approx 4.32$.	Evaluate on a calculator, to 2 decimal places.

Example 2 — scaffolded

Solve $5 \times 3^x = 100$.

Working	Comment
$3^x = \dfrac{100}{5} = 20$.	Isolate the power first: divide both sides by 5.

Working	Comment
$\log 3^x = \log 20 \Rightarrow x \log 3 = \log 20$.	Take logarithms and bring the power down.
$x = \dfrac{\log 20}{\log 3} = \log_3 20$.	Exact answer.
$x \approx 2.73$.	To 2 decimal places.

Example 3 — unscaffolded

Solve $2^{x+1} = 7$, giving the answer exactly and correct to 3 decimal places.

$$\log 2^{x+1} = \log 7 \Rightarrow (x+1)\log 2 = \log 7 \Rightarrow x+1 = \log_2 7.$$

$\therefore x = \log_2 7 - 1 \approx 1.807$.

Example 4 — unscaffolded

Solve $3^{2x} - 7 \times 3^x + 10 = 0$.

Let $y = 3^x$. Since $3^{2x} = (3^x)^2 = y^2$, the equation becomes

$$y^2 - 7y + 10 = 0 \Rightarrow (y-2)(y-5) = 0 \Rightarrow y = 2 \text{ or } y = 5.$$

Both values are positive, so $3^x = 2$ or $3^x = 5$.

$\therefore x = \log_3 2 \approx 0.631$ or $x = \log_3 5 \approx 1.465$.

Practice questions

Scaffolded

Q1. Solve $2^x = 15$: (a) give the exact answer as a logarithm; (b) evaluate it to 2 decimal places.

Q2. Solve $10^x = 4.5$: (a) write the exact answer; (b) give it to 3 decimal places.

Q3. Solve $3^x = 50$, exactly and to 2 decimal places.

Q4. Solve $4 \times 2^x = 52$. (a) First divide both sides by 4. (b) Hence solve for x, exactly and to 2 decimal places.

Q5. Solve $5^{2x} = 30$, exactly and to 3 decimal places.

Q6. Solve $3^{x-2} = 11$, exactly and to 3 decimal places.

Unscaffolded

Solve each equation. Give the exact answer in logarithm form and, unless stated otherwise, a decimal approximation to 2 decimal places.

Q7. $2^x = 100$

Q8. $5^x = 200$

Q9. $10^x = 0.02$

Q10. $3 \times 2^x = 100$

Q11. $7^x = 343$

Q12. $2^{x+3} = 50$

Q13. $5^{2x-1}=40$ (to 3 decimal places)

Q14. $6\times 3^{x}=48$

Q15. $2^{2x}-12\times 2^{x}+32=0$

Q16. $3^{2x}-10\times 3^{x}+9=0$

Q17. $2^{2x}-9\times 2^{x}+14=0$

Q18. Find the smallest integer x for which $2^{x}>1000$.

Answers

Q1. $\log_2 15 \approx 3.91$. **Q2.** $\log_{10} 4.5 \approx 0.653$. **Q3.** $\log_3 50 \approx 3.56$. **Q4.** $\log_2 13 \approx 3.70$. **Q5.** $1/2 \log_5 30 \approx 1.057$. **Q6.** $\log_3 11 + 2 \approx 4.183$. **Q7.** $\log_2 100 \approx 6.64$. **Q8.** $\log_5 200 \approx 3.29$. **Q9.** $\log_{10} 0.02 = -\log_{10} 50 \approx -1.70$. **Q10.** $\log_2 100/3 \approx 5.06$. **Q11.** $x = 3$. **Q12.** $\log_2 50 - 3 \approx 2.64$. **Q13.** $1/2(\log_5 40 + 1) \approx 1.646$. **Q14.** $\log_3 8 \approx 1.89$. **Q15.** $x = 2$ or $x = 3$. **Q16.** $x = 0$ or $x = 2$. **Q17.** $x = 1$ or $x = \log_2 7 \approx 2.81$. **Q18.** $x = 10$.

Fully-worked solutions

Q1. $2^x = 15 \Rightarrow x \log 2 = \log 15 \Rightarrow x = \dfrac{\log 15}{\log 2} = \log_2 15 \approx 3.91$.

Q2. $10^x = 4.5 \Rightarrow x = \log_{10} 4.5 \approx 0.653$ (base 10, so no change of base is needed).

Q3. $3^x = 50 \Rightarrow x = \dfrac{\log 50}{\log 3} = \log_3 50 \approx 3.56$.

Q4. (a) $2^x = \dfrac{52}{4} = 13$. (b) $x = \dfrac{\log 13}{\log 2} = \log_2 13 \approx 3.70$.

Q5. $5^{2x} = 30 \Rightarrow 2x \log 5 = \log 30 \Rightarrow x = \dfrac{\log 30}{2 \log 5} = 1/2 \log_5 30 \approx 1.057$.

Q6. $3^{x-2} = 11 \Rightarrow (x-2) \log 3 = \log 11 \Rightarrow x - 2 = \log_3 11$, so $x = \log_3 11 + 2 \approx 4.183$.

Q7. $2^x = 100 \Rightarrow x = \log_2 100 \approx 6.64$.

Q8. $5^x = 200 \Rightarrow x = \log_5 200 \approx 3.29$.

Q9. $10^x = 0.02 \Rightarrow x = \log_{10} 0.02 = -\log_{10} 50 \approx -1.70$.

Q10. $2^x = \dfrac{100}{3} \Rightarrow x = \log_2 \dfrac{100}{3} \approx 5.06$.

Q11. $7^x = 343 = 7^3$, so the bases match: $x = 3$ exactly (no logarithm needed).

Q12. $2^{x+3} = 50 \Rightarrow (x+3) \log 2 = \log 50 \Rightarrow x + 3 = \log_2 50$, so $x = \log_2 50 - 3 \approx 2.64$.

Q13. $5^{2x-1} = 40 \Rightarrow (2x-1) \log 5 = \log 40 \Rightarrow 2x - 1 = \log_5 40$, so $x = \dfrac{\log_5 40 + 1}{2} \approx 1.646$.

Q14. $3^x = \dfrac{48}{6} = 8 \Rightarrow x = \log_3 8 \approx 1.89$.

Q15. Let $y = 2^x$: $y^2 - 12y + 32 = 0 \Rightarrow (y-4)(y-8) = 0 \Rightarrow y = 4$ or $y = 8$. Then $2^x = 4 \Rightarrow x = 2$, and $2^x = 8 \Rightarrow x = 3$.

Q16. Let $y = 3^x$: $y^2 - 10y + 9 = 0 \Rightarrow (y-1)(y-9) = 0 \Rightarrow y = 1$ or $y = 9$. Then $3^x = 1 \Rightarrow x = 0$, and $3^x = 9 \Rightarrow x = 2$.

Q17. Let $y = 2^x$: $y^2 - 9y + 14 = 0 \Rightarrow (y-2)(y-7) = 0 \Rightarrow y = 2$ or $y = 7$. Then $2^x = 2 \Rightarrow x = 1$, and $2^x = 7 \Rightarrow x = \log_2 7 \approx 2.81$.

Q18. $2^x > 1000 \Rightarrow x > \log_2 1000 \approx 9.97$. The smallest integer satisfying this is $x = 10$ (check: $2^{10} = 1024 > 1000$, while $2^9 = 512 < 1000$).

Chapter 11 Exponential and logarithmic functions — 11G Logarithmic functions and their graphs

Theory

The **logarithm function** $y=\log_a x$ (with base $a>1$) is the inverse of the exponential function $y=a^x$. This section shows how to sketch it and its transformations.

The basic logarithm graph. For $y=\log_a x$ with $a>1$:

- **Domain** $(0,\infty)$ — you can only take the logarithm of a positive number.
- **Range** $\mathbb{R}$.
- **Vertical asymptote** the y-axis, $x=0$ (as $x\to 0^+$, $y\to -\infty$).
- **x-intercept** $(1,0)$, since $\log_a 1=0$; the graph also passes through $(a,1)$.
- The curve is **increasing**.

Inverse of $y=a^x$. Because $y=\log_a x$ is the inverse of $y=a^x$, its graph is the **reflection of** $y=a^x$ **in the line** $y=x$. The domain and range swap: $y=a^x$ has domain $\mathbb{R}$ and range $(0,\infty)$; $y=\log_a x$ has domain $(0,\infty)$ and range $\mathbb{R}$.

General form. The graph of

$$y=\log_a(x-h)+k$$

is the basic curve translated h units horizontally and k units vertically:

- **Vertical asymptote** $x=h$; **domain** (h,∞); **range** $\mathbb{R}$.
- **x-intercept:** put $y=0$: $\log_a(x-h)=-k \Rightarrow x-h=a^{-k}$, so $x=h+a^{-k}$.
- **y-intercept** (only when $h<0$, so that $x=0$ is in the domain): $y=\log_a(-h)+k$.

Reflections: $y=-\log_a x$ is a reflection in the x-axis (decreasing); $y=\log_a(-x)$ is a reflection in the y-axis (domain $(-\infty,0)$).

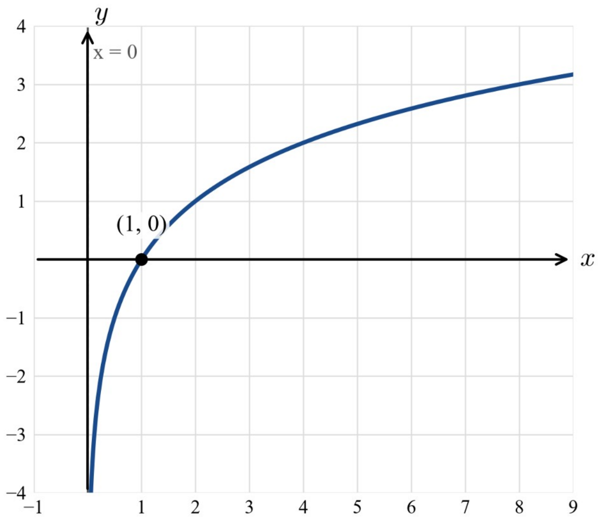
To sketch a logarithm graph: (1) draw the asymptote $x=h$ as a dashed line; (2) find the x-intercept; (3) find the y-intercept if $x=0$ is in the domain; (4) state the domain (h,∞) and range $\mathbb{R}$; (5) draw a smooth increasing curve rising away from the asymptote.

Worked examples

Example 1 — scaffolded

Sketch $y=\log_2 x$, stating the domain, range, asymptote and axis intercept.

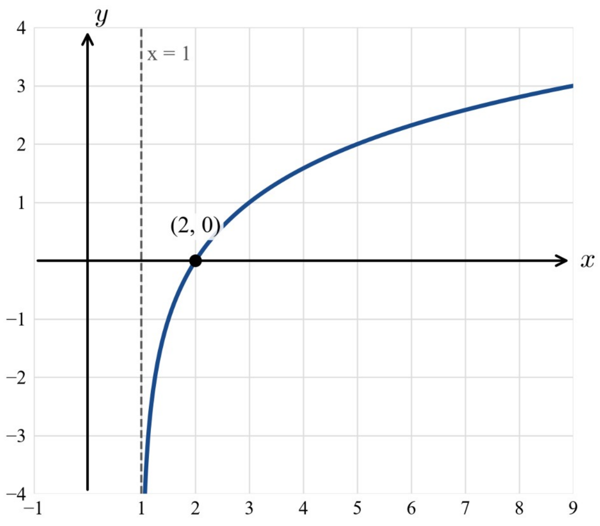
Working	Comment
Base $a=2>1$, so the curve is increasing.	It is the inverse of $y=2^x$.
Domain $(0,\infty)$, range $\mathbb{R}$; vertical asymptote $x=0$.	You can only take $\log$ of a positive number.
x-intercept: $\log_2 x=0 \Rightarrow x=2^0=1$, giving $(1,0)$.	$\log_a 1=0$ for every base.
The graph also passes through $(2,1)$ and $(4,2)$.	$\log_2 2=1$, $\log_2 4=2$.



Example 2 — scaffolded

Sketch $y = \log_2(x-1)$.

Working	Comment
The argument is $x-1$, so the vertical asymptote is $x=1$; domain $(1,\infty)$, range $\mathbb{R}$.	The graph of $y=\log_2 x$ is translated 1 unit right.
x-intercept: $\log_2(x-1)=0 \Rightarrow x-1=1 \Rightarrow x=2$, giving $(2,0)$.	Put $y=0$ and solve.
No y-intercept, since $x=0$ is not in the domain. Passes through $(3,1)$.	$x=0\le 1$ lies outside $(1,\infty)$.



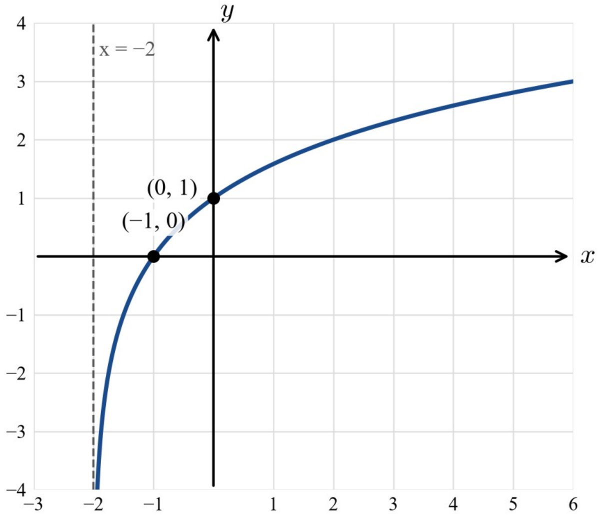
Example 3 — unscaffolded

Sketch $y=\log_2(x+2)$.

Vertical asymptote $x=-2$; domain $(-2,\infty)$, range $\mathbb{R}$.

x-intercept: $\log_2(x+2)=0 \Rightarrow x+2=1 \Rightarrow x=-1$, giving $(-1,0)$.

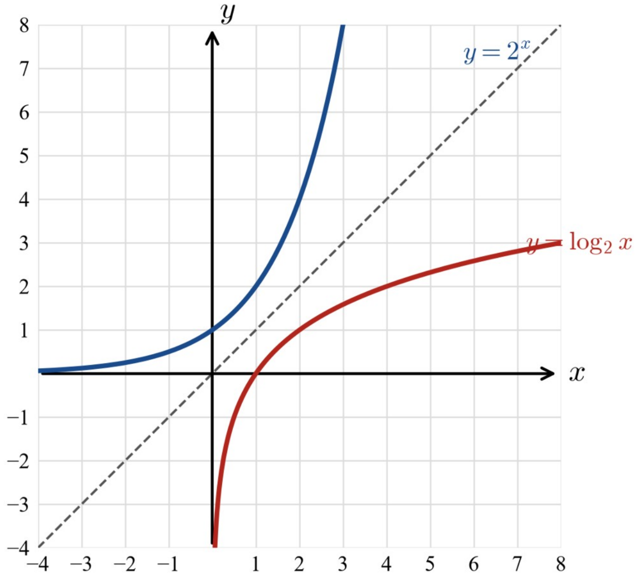
y-intercept: $x=0 \Rightarrow y=\log_2 2=1$, giving $(0,1)$.



Example 4 — unscaffolded

On the one set of axes, sketch $y = 2^x$ and its inverse $y = \log_2 x$, and show the line of symmetry.

$y = 2^x$ has horizontal asymptote $y = 0$ and passes through $(0, 1)$ and $(1, 2)$. Its inverse $y = \log_2 x$ has vertical asymptote $x = 0$ and passes through $(1, 0)$ and $(2, 1)$. Each is the reflection of the other in the line $y = x$.



Practice questions

Scaffolded

Q1. For $y = \log_3 x$: (a) state the domain and range; (b) state the asymptote; (c) find the x-intercept; (d) sketch the graph.

Q2. For $y = \log_2(x - 3)$: (a) state the asymptote and domain; (b) find the x-intercept; (c) sketch the graph.

Q3. For $y = \log_2 x + 2$: (a) state the asymptote; (b) find the x-intercept; (c) sketch the graph.

Q4. For $y = \log_3(x + 1)$: (a) state the asymptote and domain; (b) find the axis intercept(s); (c) sketch the graph.

Q5. For $y = \log_2(x - 1) + 1$: (a) state the asymptote and domain; (b) find the x-intercept; (c) sketch the graph.

Q6. The graph of $y = 3^x$ is reflected in the line $y = x$. (a) State the rule of the image. (b) State its domain, range and asymptote. (c) Sketch both graphs and $y = x$.

Unscaffolded

For each of the following: sketch the graph, showing the asymptote and axis intercept(s), and state the domain and range.

Q7. $y=\log_5 x$

Q8. $y=\log_2(x+4)$

Q9. $y=\log_3(x-2)$

Q10. $y=\log_2 x-3$

Q11. $y=\log_3 x+2$

Q12. $y=\log_2(x+1)-2$

Q13. $y=\log_2(-x)$

Q14. $y=-\log_2 x$

Q15. $y=2\log_2 x$

Q16. $y=\log_2(x-1)+3$

Q17. State the inverse of $y=4^x$, then sketch both graphs and the line $y=x$.

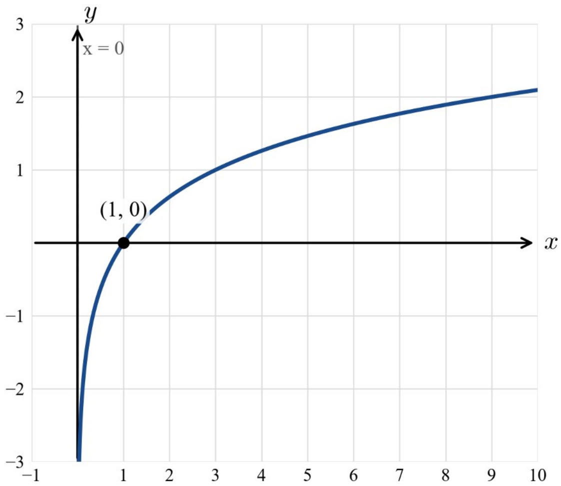
Q18. $y=\log_3(x+3)-1$

Answers

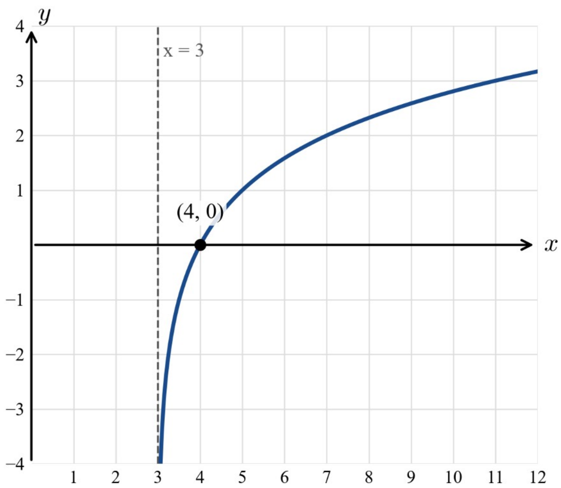
Q1. Domain $(0,\infty)$, range $\mathbb{R}$; asymptote $x=0$; x-int $(1,0)$. **Q2.** Asymptote $x=3$, domain $(3,\infty)$; x-int $(4,0)$. **Q3.** Asymptote $x=0$; x-int $(1/4,0)$. **Q4.** Asymptote $x=-1$, domain $(-1,\infty)$; intercept $(0,0)$. **Q5.** Asymptote $x=1$, domain $(1,\infty)$; x-int $(3/2,0)$. **Q6.** $y=\log_3 x$; domain $(0,\infty)$, range $\mathbb{R}$, asymptote $x=0$. **Q7.** Asymptote $x=0$; x-int $(1,0)$; domain $(0,\infty)$, range $\mathbb{R}$. **Q8.** Asymptote $x=-4$; x-int $(-3,0)$, y-int $(0,2)$; domain $(-4,\infty)$, range $\mathbb{R}$. **Q9.** Asymptote $x=2$; x-int $(3,0)$; domain $(2,\infty)$, range $\mathbb{R}$. **Q10.** Asymptote $x=0$; x-int $(8,0)$; domain $(0,\infty)$, range $\mathbb{R}$. **Q11.** Asymptote $x=0$; x-int $(1/9,0)$; domain $(0,\infty)$, range $\mathbb{R}$. **Q12.** Asymptote $x=-1$; x-int $(3,0)$, y-int $(0,-2)$; domain $(-1,\infty)$, range $\mathbb{R}$. **Q13.** Asymptote $x=0$; x-int $(-1,0)$; domain $(-\infty,0)$, range $\mathbb{R}$. **Q14.** Asymptote $x=0$; x-int $(1,0)$; domain $(0,\infty)$, range $\mathbb{R}$ (decreasing). **Q15.** Asymptote $x=0$; x-int $(1,0)$; domain $(0,\infty)$, range $\mathbb{R}$. **Q16.** Asymptote $x=1$; x-int $(9/8,0)$; domain $(1,\infty)$, range $\mathbb{R}$. **Q17.** $y=\log_4 x$; domain $(0,\infty)$, range $\mathbb{R}$, asymptote $x=0$. **Q18.** Asymptote $x=-3$; intercept $(0,0)$; domain $(-3,\infty)$, range $\mathbb{R}$.

Fully-worked solutions

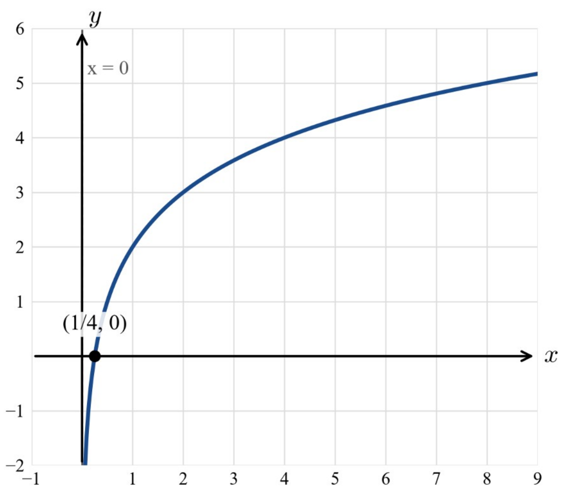
Q1. $y=\log_3 x$. (a) Domain $(0,\infty)$, range $\mathbb{R}$. (b) Asymptote $x=0$. (c) $\log_3 x=0 \Rightarrow x=1$, giving $(1,0)$. Passes through $(3,1)$.



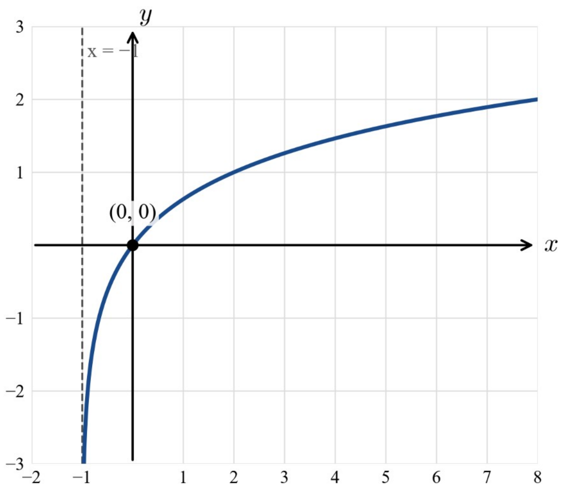
Q2. $y=\log_2(x-3)$. (a) Asymptote $x=3$; domain $(3,\infty)$. (b) $\log_2(x-3)=0 \Rightarrow x-3=1 \Rightarrow x=4$, giving $(4,0)$.



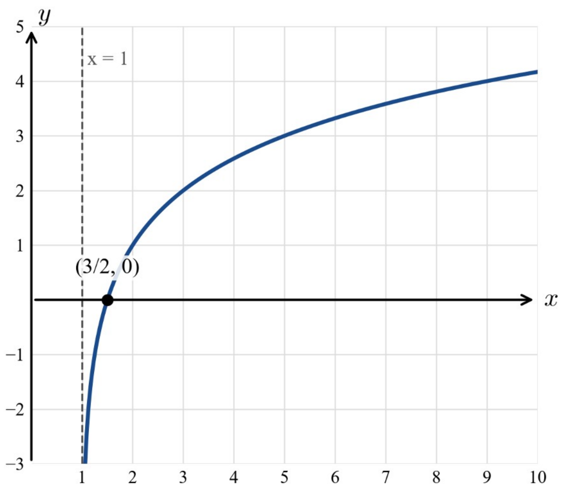
Q3. $y=\log_2 x+2$. (a) Asymptote $x=0$. (b) $\log_2 x+2=0 \Rightarrow \log_2 x=-2 \Rightarrow x=2^{-2}=1/4$, giving $(1/4,0)$.



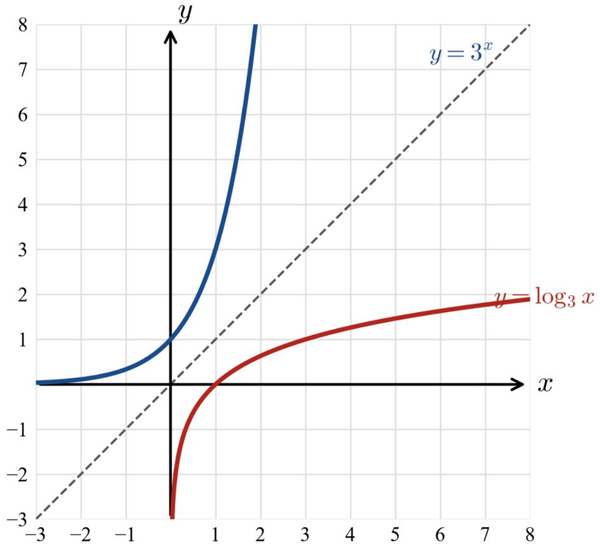
Q4. $y=\log_3(x+1)$. (a) Asymptote $x=-1$; domain $(-1,\infty)$. (b) $\log_3(x+1)=0 \Rightarrow x+1=1 \Rightarrow x=0$: the graph passes through the origin $(0,0)$ (both the x- and y-intercept).



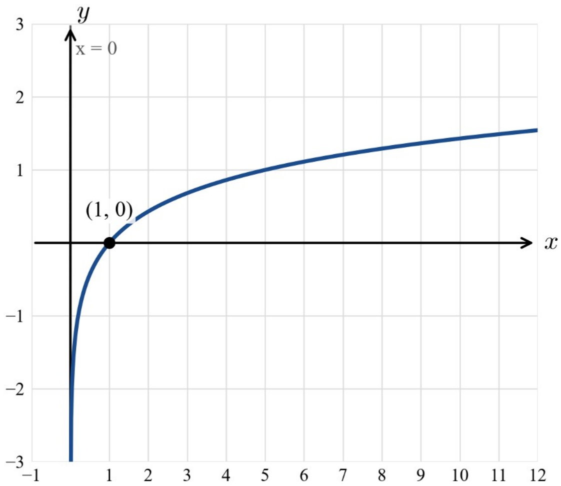
Q5. $y=\log_2(x-1)+1$. (a) Asymptote $x=1$; domain $(1,\infty)$. (b) $\log_2(x-1)+1=0 \Rightarrow \log_2(x-1)=-1 \Rightarrow x-1=2^{-1}=1/2$, so $x=3/2$, giving $(3/2,0)$.



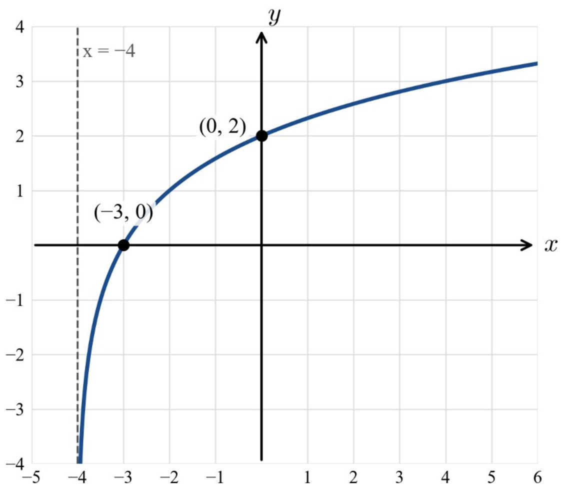
Q6. Reflecting $y=3^x$ in $y=x$ gives its inverse $y=\log_3 x$. (b) Domain $(0,\infty)$, range $\mathbb{R}$, asymptote $x=0$; it passes through $(1,0)$ and $(3,1)$.



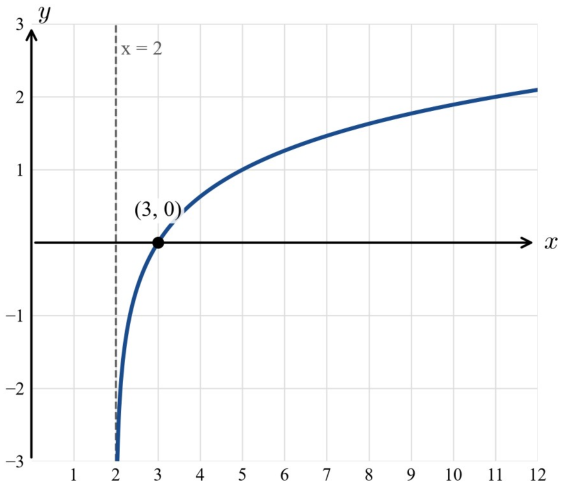
Q7. $y = \log_5 x$. Asymptote $x = 0$; $\log_5 x = 0 \Rightarrow x = 1$, giving $(1, 0)$; domain $(0, \infty)$, range $\mathbb{R}$. Passes through $(5, 1)$.



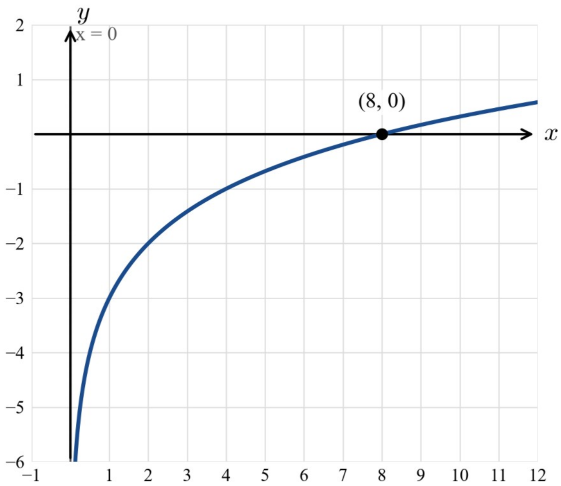
Q8. $y = \log_2(x + 4)$. Asymptote $x = -4$; $x + 4 = 1 \Rightarrow x = -3$, giving $(-3, 0)$; y-intercept $x = 0 \Rightarrow y = \log_2 4 = 2$, giving $(0, 2)$; domain $(-4, \infty)$.



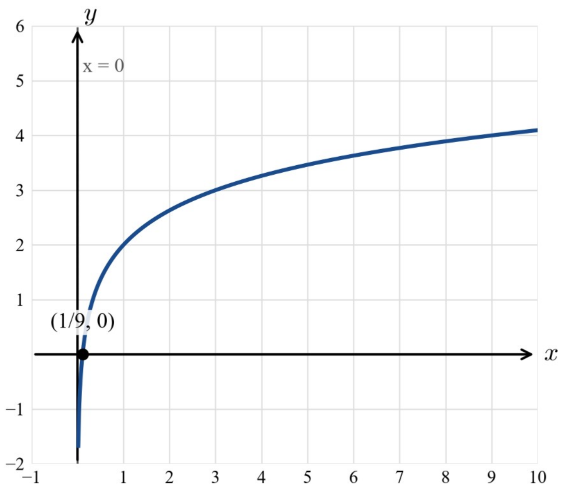
Q9. $y = \log_3(x - 2)$. Asymptote $x = 2$; $x - 2 = 1 \Rightarrow x = 3$, giving $(3, 0)$; domain $(2, \infty)$.



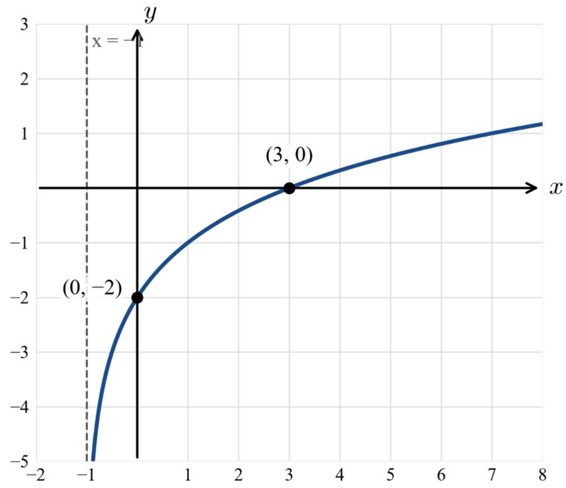
Q10. $y=\log_2 x-3$. Asymptote $x=0$; $\log_2 x=3 \Rightarrow x=2^3=8$, giving $(8,0)$; domain $(0,\infty)$.



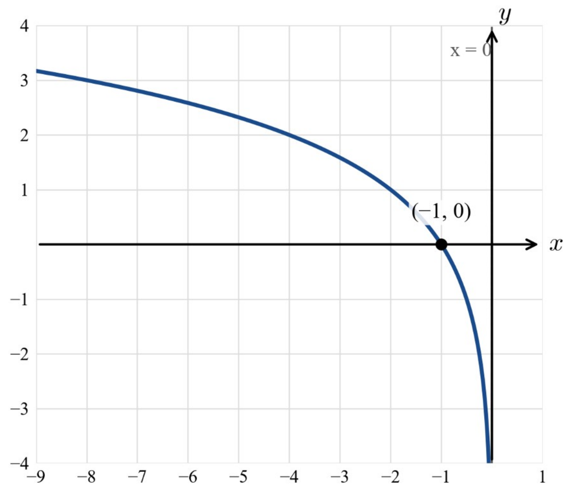
Q11. $y=\log_3 x+2$. Asymptote $x=0$; $\log_3 x=-2 \Rightarrow x=3^{-2}=1/9$, giving $(1/9,0)$; domain $(0,\infty)$.



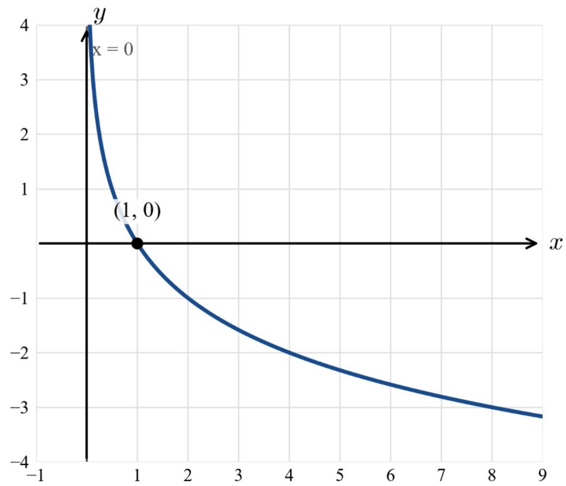
Q12. $y=\log_2(x+1)-2$. Asymptote $x=-1$; $\log_2(x+1)=2 \Rightarrow x+1=4 \Rightarrow x=3$, giving $(3,0)$; y-intercept $x=0 \Rightarrow y=\log_2 1-2=-2$, giving $(0,-2)$; domain $(-1,\infty)$.



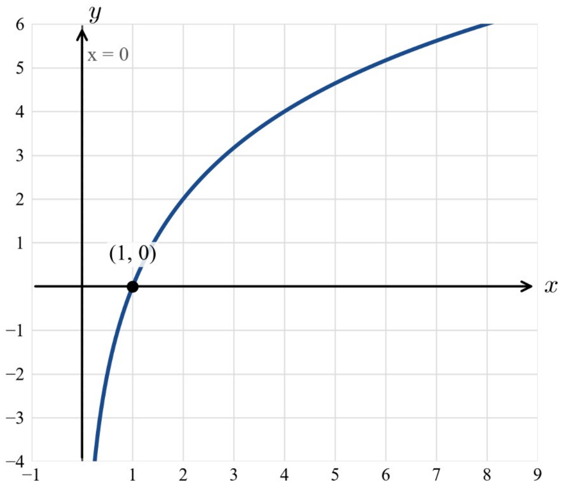
Q13. $y = \log_2(-x)$. The argument $-x > 0$ requires $x < 0$: domain $(-\infty, 0)$, asymptote $x = 0$. $\log_2(-x) = 0 \Rightarrow -x = 1 \Rightarrow x = -1$, giving $(-1, 0)$. This is $y = \log_2 x$ reflected in the y-axis.



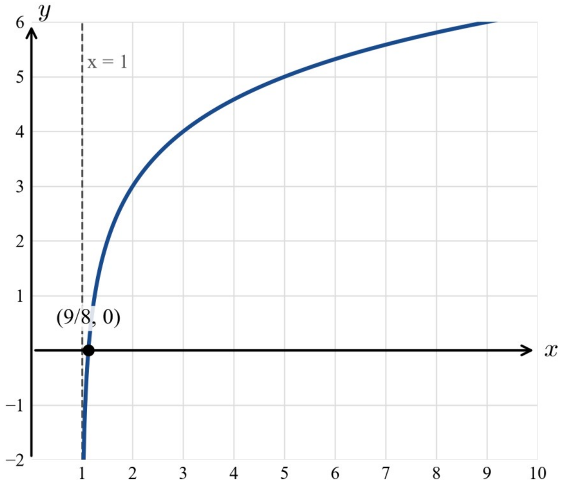
Q14. $y = -\log_2 x$. Reflection of $y = \log_2 x$ in the x-axis, so it is **decreasing**; asymptote $x = 0$; x-intercept still $(1, 0)$; domain $(0, \infty)$.



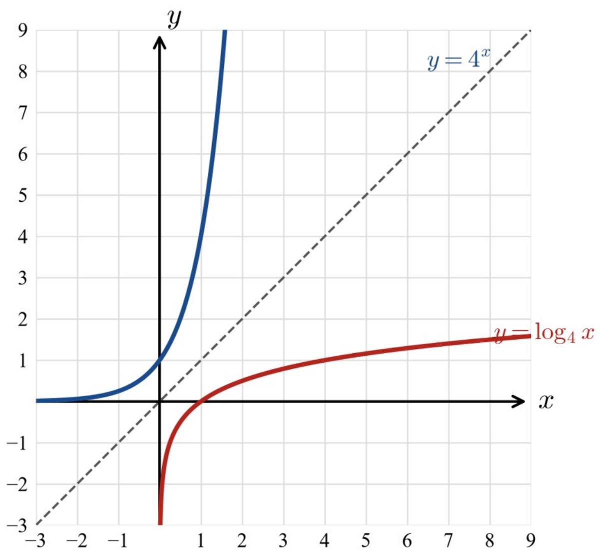
Q15. $y = 2\log_2 x$. A dilation of factor 2 from the x-axis; asymptote $x = 0$; x-intercept $(1, 0)$; domain $(0, \infty)$. Passes through $(2, 2)$ and $(4, 4)$.



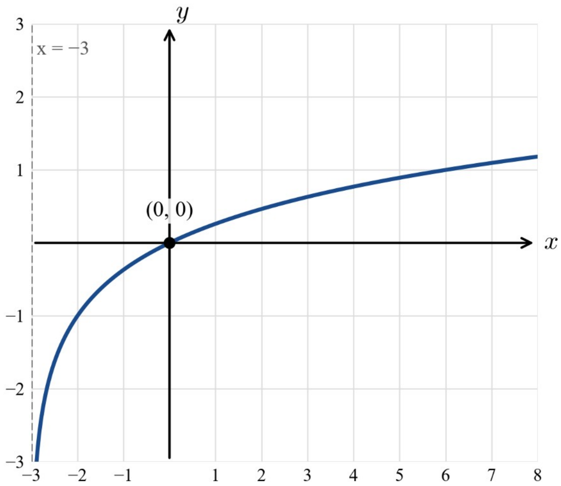
Q16. $y = \log_2(x-1)+3$. Asymptote $x=1$; $\log_2(x-1) = -3 \Rightarrow x-1 = 2^{-3} = 1/8 \Rightarrow x = 9/8$, giving $(9/8, 0)$; domain $(1, \infty)$.



Q17. The inverse of $y = 4^x$ is $y = \log_4 x$ (swap x and y and make y the subject). Its graph is the reflection of $y = 4^x$ in $y = x$: asymptote $x = 0$, domain $(0, \infty)$, range $\mathbb{R}$, through $(1, 0)$ and $(4, 1)$.



Q18. $y = \log_3(x+3)-1$. Asymptote $x = -3$; $\log_3(x+3) = 1 \Rightarrow x+3 = 3 \Rightarrow x = 0$: the graph passes through the origin $(0, 0)$; domain $(-3, \infty)$.



y
x
x = −3
(0, 0)
3
2
1
−1
−2
−3
−3
−2
−1
1
2
3
4
5
6
7
8

Theory

Many real quantities change by a constant **factor** in equal time intervals — these are modelled by **exponential functions**.

Exponential growth and decay. A quantity with initial value A that is multiplied by a factor b each unit of time is modelled by

$$y = A\,b^t,$$

where $A = y(0)$ is the **initial value** and b is the **growth (or decay) factor**.

- If $b > 1$ the quantity **grows**; writing $b = 1 + r$ gives $y = A(1+r)^t$, where r is the growth rate (as a decimal).
- If $0 < b < 1$ the quantity **decays**; writing $b = 1 - r$ gives $y = A(1-r)^t$.

Compound interest. An amount P invested at rate r per period, compounded each period, grows to $A = P(1+r)^n$ after n periods.

Half-life. A decaying quantity with **half-life** h (the time to halve) is modelled by

$$y = A(1/2)^{t/h}.$$

Doubling time. In the same way, a growing quantity with **doubling time** d (the time taken to double) is modelled by

$$y = A \cdot 2^{t/d}.$$

Solving for time using logarithms. To find when the model reaches a given value, isolate the power and take logarithms: $b^t = k \Rightarrow t = \log_b k = \dfrac{\log k}{\log b}$.

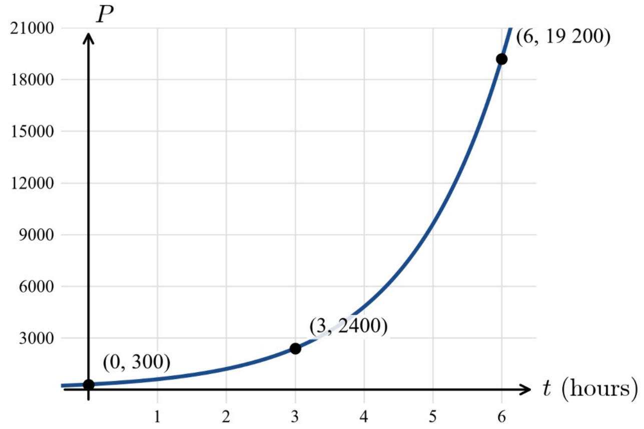
Modelling. State the initial value and the factor from the information given, keep a **sensible domain** (usually $t \ge 0$), and interpret answers in context with units. Values found using logarithms are usually given as a rounded decimal.

Worked examples

Example 1 — scaffolded

A colony of bacteria initially numbers 300 and doubles every hour. Let P be the number after t hours.

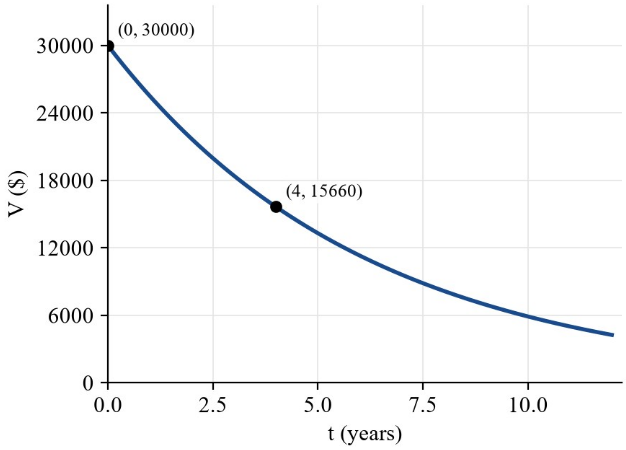
Working	Comment
$P = 300 \times 2^t$.	Initial value 300; factor 2 (doubles) each hour.
After 3 hours: $P = 300 \times 2^3 = 300 \times 8 = 2400$.	Substitute $t = 3$.
Reaches $19\,200$: $300 \times 2^t = 19\,200 \Rightarrow 2^t = 64 = 2^6$, so $t = 6$ hours.	Divide, then equate the powers.



Example 2 — scaffolded

A car bought for $\$30\,000$ depreciates by $15\,\%$ each year. Let V be its value after t years.

Working	Comment
$V = 30\,000 \times (0.85)^t$.	Losing $15\,\%$ leaves $85\,\%$, so the factor is 0.85.
After 4 years: $V = 30\,000 \times 0.85^4 \approx \$15\,660.19$.	Substitute $t = 4$.
Falls to $\$10\,000$: $0.85^t = 1/3$, so $t = \dfrac{\log 1/3}{\log 0.85} \approx 6.76$ years — during the 7th year.	Take logarithms to solve for t.



Example 3 — unscaffolded

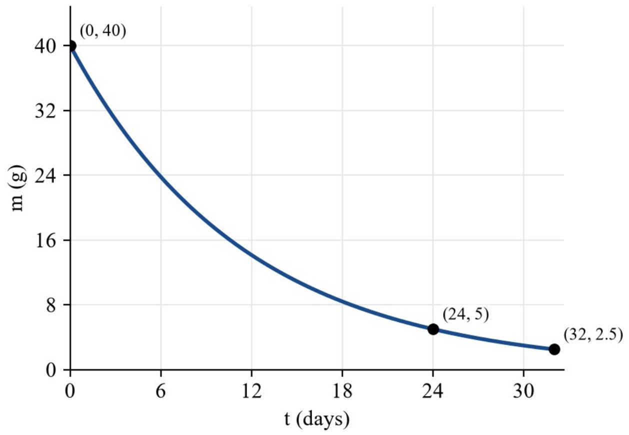
A radioactive isotope has a half-life of 8 days. A sample has initial mass 40 g.

The model is $m = 40(1/2)^{t/8}$, where m is the mass (g) after t days.

After 24 days: $m = 40(1/2)^3 = 40 \times 1/8 = 5$ g.

Reaches 2.5 g: $(1/2)^{t/8} = \dfrac{2.5}{40} = \dfrac{1}{16} = (1/2)^4$, so $\dfrac{t}{8} = 4$.

$\therefore t = 32$ days.



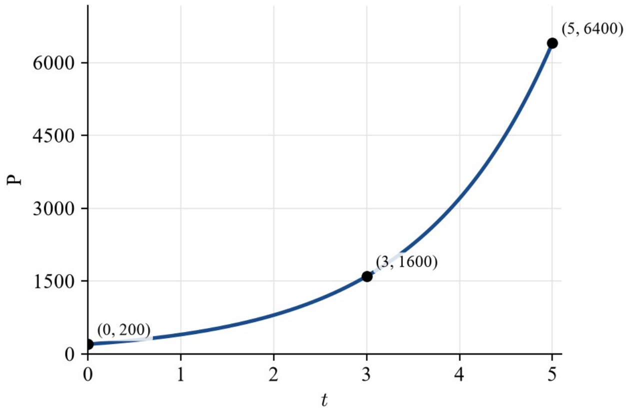
Example 4 — unscaffolded

A population is modelled by $P = A\,b^t$. When $t=0$ the population is 200, and when $t=3$ it is 1600.

At $t=0$: $A=200$. At $t=3$: $200\,b^3=1600 \Rightarrow b^3=8 \Rightarrow b=2$.

So $P=200\times 2^t$. When $t=5$: $P=200\times 2^5=200\times 32=6400$.

$\therefore$ the model is $P=200(2)^t$ and $P(5)=6400$.



Practice questions

Scaffolded

Q1. A population of insects is modelled by $P=200\times 3^t$ (t in weeks). (a) State the initial population. (b) Find the population after 2 weeks. (c) Find when the population reaches $16\,200$.

Q2. A drug in the bloodstream is modelled by $M=80\times(1/2)^t$ mg (t in hours). (a) State the initial amount. (b) Find the amount after 3 hours. (c) Find when 5 mg remains.

Q3. $\$2000$ is invested at 6% per annum, compounded annually, so $A=2000(1.06)^n$. Find the value after 5 years, to the nearest cent.

Q4. An isotope of half-life 5 years has initial mass 100 g, so $m=100(1/2)^{t/5}$. Find the mass after (a) 10 years and (b) 15 years.

Q5. A quantity is modelled by $y=A\,b^t$ and passes through $(0,3)$ and $(2,27)$. (a) State A. (b) Find b. (c) Write the rule.

Q6. Solve $5^t=100$ for t, giving your answer to two decimal places.

Unscaffolded

For each model, answer the question(s); give times found with logarithms to two decimal places.

Q7. Bacteria triple every hour from an initial 50: $N = 50 \times 3^t$. Find N after 4 hours, and find when $N = 1350$.

Q8. $\$4000$ is invested at 7% p.a. compounded annually. Find its value after 8 years, to the nearest cent.

Q9. A machine worth $\$50\,000$ depreciates 20% per year: $V = 50\,000(0.8)^t$. Find its value after 3 years, and find when it falls below $\$20\,000$.

Q10. A radioactive substance has half-life 12 hours and initial mass 64 mg. Find the mass after 36 hours, and find when 1 mg remains.

Q11. A town of 8000 people grows 3% per year: $P = 8000(1.03)^t$. Find the population after 10 years, and find when it doubles to $16\,000$.

Q12. A cup of tea cools according to $T = 20 + 60(0.9)^t$ (°C, t in minutes). Find the temperature after 5 minutes, and state the temperature it approaches in the long run.

Q13. A quantity $y = A\,b^t$ passes through $(0, 5)$ and $(4, 80)$. Find the rule, then find y when $t = 6$.

Q14. A sample decays from 200 g to 50 g in 10 years. Find the half-life, and write the model $m = A(1/2)^{t/h}$.

Q15. $\$1000$ is invested at 5% p.a. compounded annually. Find when the investment doubles.

Q16. Bacteria double every 4 hours from 250: $N = 250 \times 2^{t/4}$. Find N after 8 hours, and find when $N = 16\,000$.

Q17. A drug leaves the body at 25% per hour from an initial 200 mg: $M = 200(0.75)^t$. Find the amount after 5 hours, and find when it falls below 10 mg.

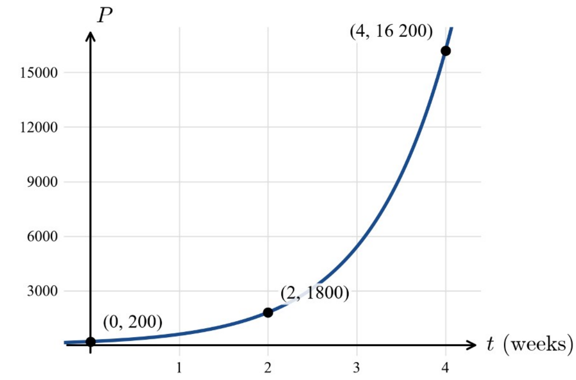
Q18. An account's followers are modelled by $F = A\,b^t$ (t in days), passing through $(0, 100)$ and $(2, 900)$. Find the rule, and find F after 5 days.

Answers

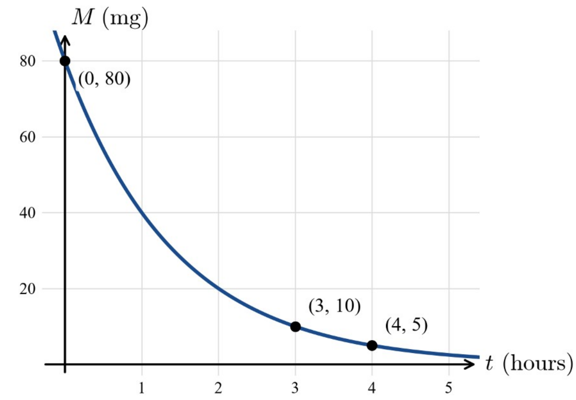
Q1. (a) 200; (b) 1800; (c) $t=4$ weeks. **Q2.** (a) 80 mg; (b) 10 mg; (c) $t=4$ hours. **Q3.** $\$2676.45$. **Q4.** (a) 25 g; (b) 12.5 g. **Q5.** (a) $A=3$; (b) $b=3$; (c) $y=3\times3^t$. **Q6.** $t\approx2.86$. **Q7.** $N(4)=4050$; $N=1350$ at $t=3$ hours. **Q8.** $\$6872.74$. **Q9.** $\$25\,600$; below $\$20\,000$ after $t\approx4.11$ years (during the 5th year). **Q10.** 8 mg; 1 mg at $t=72$ hours. **Q11.** 10 751 (nearest person); doubles at $t\approx23.45$ years. **Q12.** ≈55.43°C; approaches 20°C. **Q13.** $y=5\times2^t$; $y(6)=320$. **Q14.** Half-life 5 years; $m=200(1/2)^{t/5}$. **Q15.** $t\approx14.21$ years (during the 15th year). **Q16.** $N(8)=1000$; $N=16\,000$ at $t=24$ hours. **Q17.** ≈47.46 mg; below 10 mg after $t\approx10.41$ hours (during the 11th hour). **Q18.** $F=100\times3^t$; $F(5)=24\,300$.

Fully-worked solutions

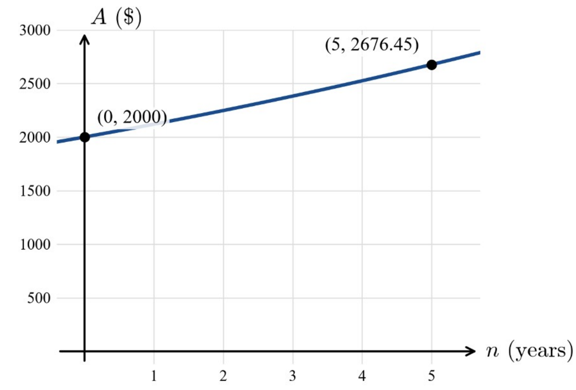
Q1. $P=200\times3^t$. (a) Initial population 200. (b) $P(2)=200\times3^2=200\times9=1800$. (c) $200\times3^t=16\,200\Rightarrow3^t=81=3^4$, so $t=4$ weeks.



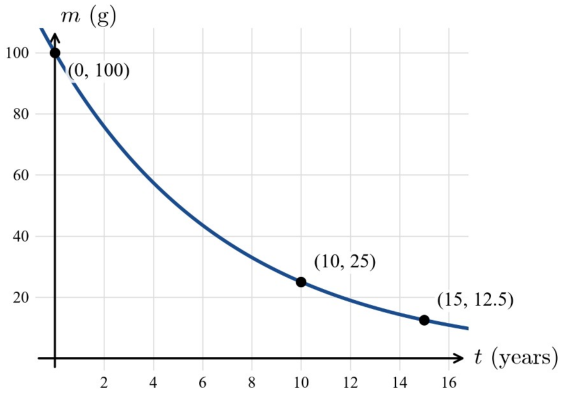
Q2. $M=80(1/2)^t$. (a) Initial amount 80 mg. (b) $M(3)=80\times1/8=10$ mg. (c) $80(1/2)^t=5\Rightarrow(1/2)^t=1/16=(1/2)^4$, so $t=4$ hours.



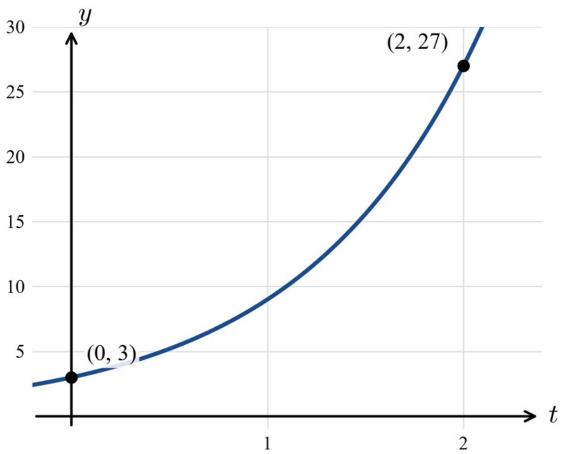
Q3. $A=2000(1.06)^5\approx\$2676.45$ (to the nearest cent).



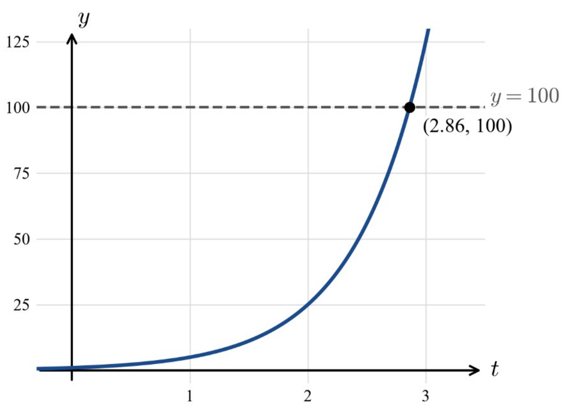
Q4. $m = 100(1/2)^{t/5}$. (a) $m(10) = 100(1/2)^2 = 100 \times 1/4 = 25$ g. (b) $m(15) = 100(1/2)^3 = 100 \times 1/8 = 12.5$ g.



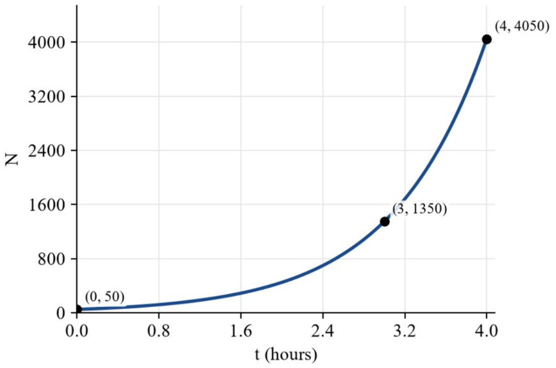
Q5. (a) $A = 3$ (from $t = 0$). (b) $3b^2 = 27 \Rightarrow b^2 = 9 \Rightarrow b = 3$. (c) $y = 3 \times 3^t$.



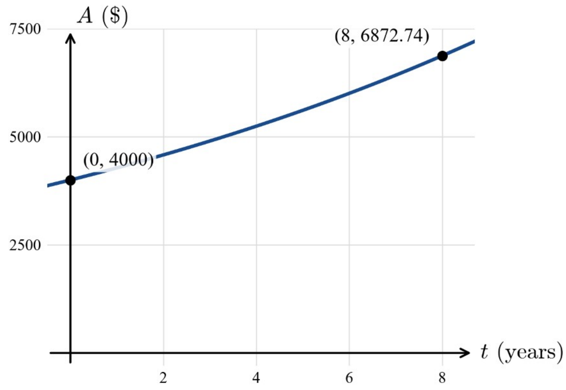
Q6. $5^t = 100 \Rightarrow t = \dfrac{\log 100}{\log 5} \approx 2.86$.



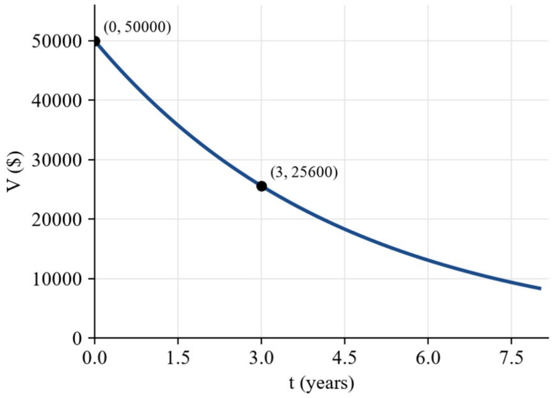
Q7. $N = 50 \times 3^t$. $N(4) = 50 \times 3^4 = 50 \times 81 = 4050$. $50 \times 3^t = 1350 \Rightarrow 3^t = 27 = 3^3$, so $t = 3$ hours.



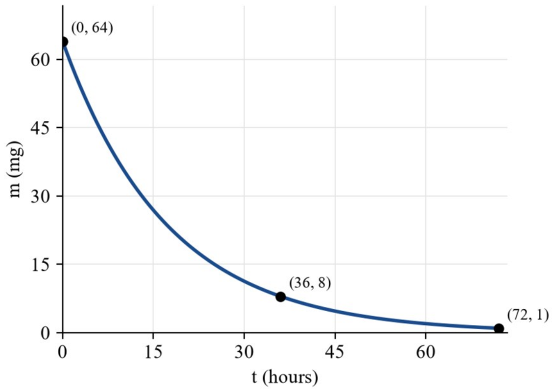
Q8. $A = 4000(1.07)^8 \approx \$6872.74$.



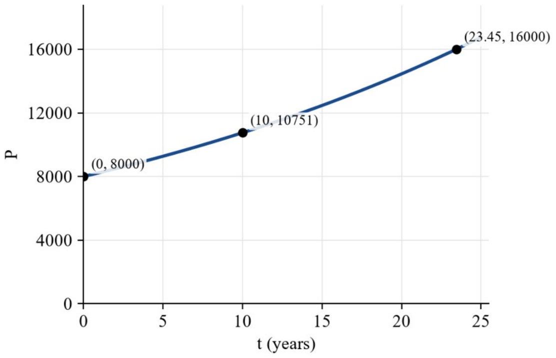
Q9. $V = 50\,000(0.8)^t$. $V(3) = 50\,000 \times 0.8^3 = 50\,000 \times 0.512 = \$25\,600$. Below $\$20\,000$: $0.8^t = 0.4 \Rightarrow t = \dfrac{\log 0.4}{\log 0.8} \approx 4.11$ years, so during the 5th year.



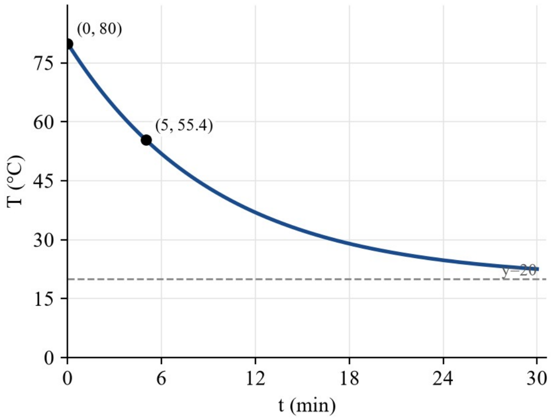
Q10. $m = 64(1/2)^{t/12}$. $m(36) = 64(1/2)^3 = 64 \times 1/8 = 8$ mg. 1 mg: $(1/2)^{t/12} = 1/64 = (1/2)^6$, so $t/12 = 6$, $t = 72$ hours.



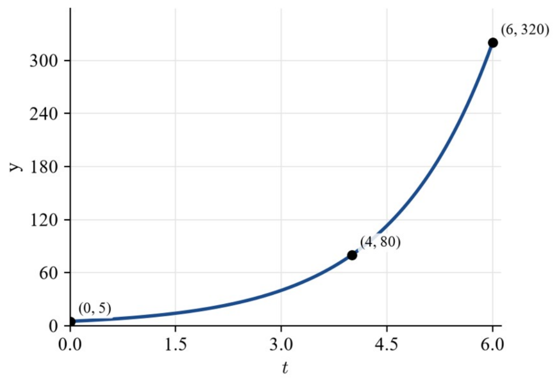
Q11. $P = 8000(1.03)^t$. $P(10) = 8000 \times 1.03^{10} \approx 10\,751$ people. Doubles: $1.03^t = 2 \Rightarrow t = \dfrac{\log 2}{\log 1.03} \approx 23.45$ years.



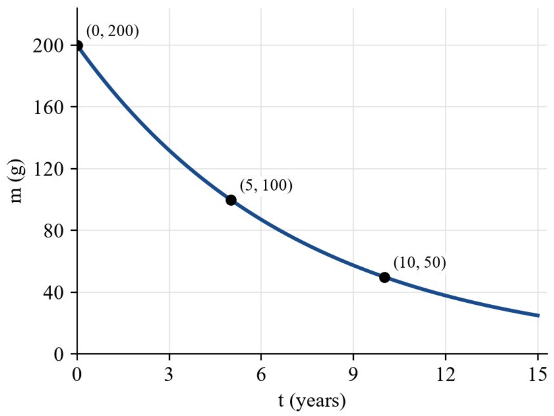
Q12. $T = 20 + 60(0.9)^t$. $T(5) = 20 + 60 \times 0.9^5 \approx 20 + 35.43 = 55.43$°C. As $t \to \infty$, $(0.9)^t \to 0$, so $T \to 20$°C (room temperature).



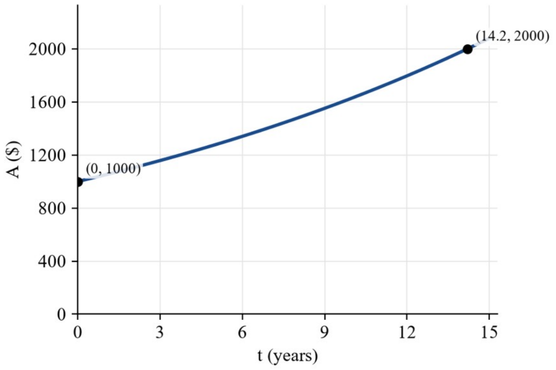
Q13. $A = 5$ (from $t = 0$). $5b^4 = 80 \Rightarrow b^4 = 16 \Rightarrow b = 2$. So $y = 5 \times 2^t$, and $y(6) = 5 \times 2^6 = 5 \times 64 = 320$.



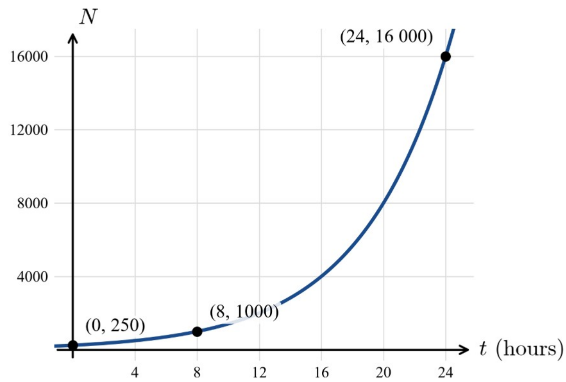
Q14. After 10 years the mass is $\frac{50}{200} = \frac{1}{4} = (1/2)^2$ of the original — that is 2 half-lives in 10 years, so the half-life is 5 years. Hence $m = 200(1/2)^{t/5}$.



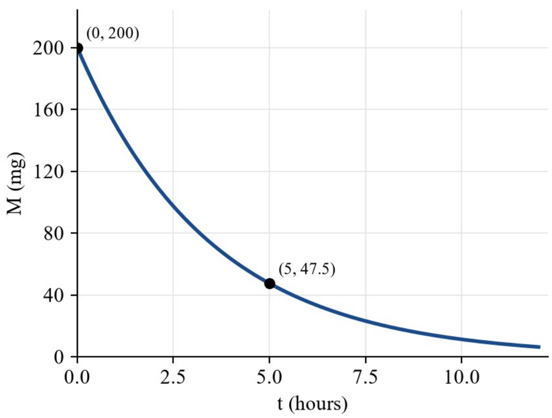
Q15. $1000(1.05)^n = 2000 \Rightarrow 1.05^n = 2 \Rightarrow n = \dfrac{\log 2}{\log 1.05} \approx 14.21$ years, so the investment first exceeds $\$2000$ during the 15th year.



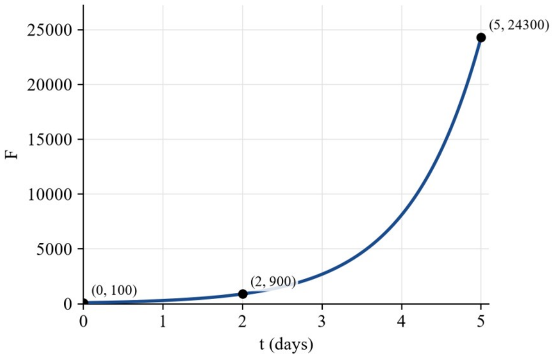
Q16. $N = 250 \times 2^{t/4}$. $N(8) = 250 \times 2^2 = 250 \times 4 = 1000$. $16\,000$: $2^{t/4} = 64 = 2^6$, so $\dfrac{t}{4} = 6$, $t = 24$ hours.



Q17. $M = 200(0.75)^t$. $M(5) = 200 \times 0.75^5 \approx 47.46$ mg. Below 10 mg: $0.75^t = 0.05 \Rightarrow t = \dfrac{\log 0.05}{\log 0.75} \approx 10.41$ hours, so during the 11th hour.



Q18. $A = 100$ (from $t = 0$). $100b^2 = 900 \Rightarrow b^2 = 9 \Rightarrow b = 3$. So $F = 100 \times 3^t$, and $F(5) = 100 \times 3^5 = 100 \times 243 = 24\,300$.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(0, 100)
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10000
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F
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t (days)