Mathematical Methods Units 1 & 2

Chapter 10 — Discrete random variables

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Chapter 10 Discrete random variables — 10A Random variables and their distributions

> **Units 3 & 4 preview (in part).** Items in this section that use the expected value $E(X)$, variance $Var(X)$ or standard deviation $sd(X)$ of a random variable go beyond the Units 1 & 2 study design, which prescribes random variables and their probability distributions only; they are included as preparation for Units 3 & 4 and are not assessed in the graded topic tests or practice examinations.

Theory

A random experiment ends in one of several outcomes, and the mathematics runs more smoothly once every outcome is recorded as a number — numbers can be added and averaged, words cannot. A **random variable** is such a number: its value is unknown until the experiment is performed, and which value appears depends on the outcome. A random variable is **discrete** when its possible values can be listed out — $0, 1, 2, 3$ and the like — rather than ranging over an interval; every random variable in this chapter is discrete. We write random variables with capitals (X) and their values with lower case (x).

Probability distribution. The distribution of X lists every value the variable can take together with the probability of that value, usually as a table of x against $Pr(X=x)$. Not every number table is a distribution — the table must pass both of these checks:

- the entries exhaust the probability: they add to one, $\sum Pr(X=x)=1$;
- each entry is a probability in its own right, so it lies between 0 and 1: $0 \le Pr(X=x) \le 1$.

The first check is the one that finds an **unknown probability**: with every other entry known, $\sum Pr(X=x)=1$ pins down the missing one. Probabilities of events are found by adding the relevant entries, e.g. $Pr(X \ge 2) = Pr(X=2) + Pr(X=3) + \cdots$

Expected value (mean). The **expected value** of X is

$$E(X) = \mu = \sum x\, Pr(X=x).$$

It is the long-run average value of X over many trials. For a linear change,

$$E(aX+b) = aE(X)+b.$$

Mode. The **mode** is the value (or values) of X with the greatest probability.

Variance and standard deviation. The spread of X is measured by the **variance**

$$Var(X) = E(X^2) - [E(X)]^2, \text{ where } E(X^2) = \sum x^2\, Pr(X=x),$$

and the **standard deviation** $sd(X) = \sqrt{Var(X)}$. For a linear change, $Var(aX+b) = a^2 Var(X)$.

Worked examples

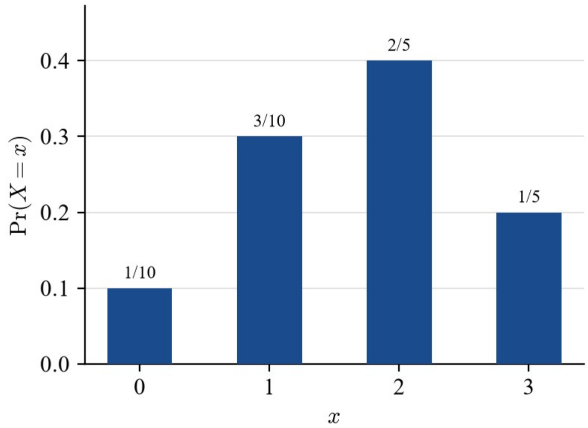
Example 1 — scaffolded

The table shows the probability distribution of X, where k is unknown.

x	0	1	2	3
$Pr(X=x)$	$\frac{1}{10}$	$\frac{3}{10}$	k	$\frac{1}{5}$

Find (a) k; (b) $Pr(X \ge 2)$; (c) $E(X)$.

Working	Comment
$\frac{1}{10}+\frac{3}{10}+k+\frac{1}{5}=1 \Rightarrow \frac{6}{10}+k=1$, so $k=\frac{2}{5}$.	The probabilities must sum to 1.
$Pr(X \geq 2)=Pr(X=2)+Pr(X=3)=\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.	Add the entries for $x=2$ and $x=3$.
$E(X)=0\times 1/10+1\times 3/10+2\times 2/5+3\times 1/5=3/10+$.	$E(X)=\sum x\,Pr(X=x)$.

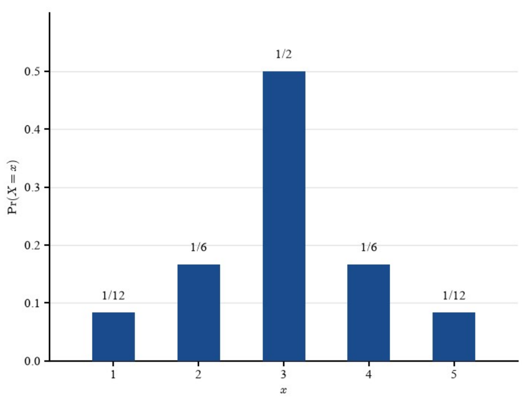


Example 2 — scaffolded

A discrete random variable X has the distribution below. Find $E(X)$, $Var(X)$ and $sd(X)$.

x	1	2	3	4	5
$Pr(X=x)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$

Working	Comment
$E(X)=1\times 1/12+2\times 1/6+3\times 1/2+4\times 1/6+5\times 1/12$.	The mean.
$E(X^2)=1\times 1/12+4\times 1/6+9\times 1/2+16\times 1/6+25\times 1$.	$E(X^2)=\sum x^2\,Pr(X=x)$.
$Var(X)=E(X^2)-[E(X)]^2=10-3^2=1$; $sd(X)=\sqrt{1}=1$.	Variance, then square root.



Example 3 — unscaffolded

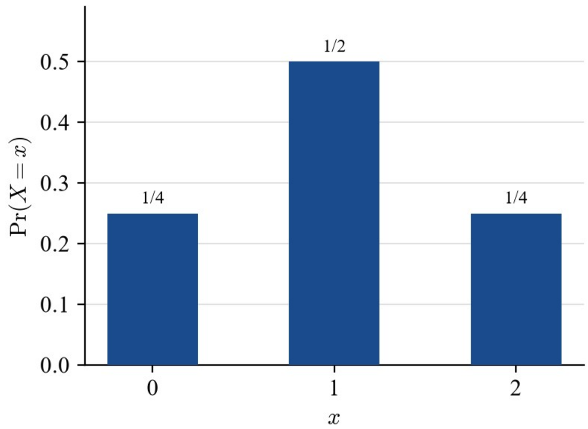
A fair coin is tossed twice. Let X be the number of heads. Find the probability distribution of X and $E(X)$.

The outcomes are TT, TH, HT, HH, each with probability $\frac{1}{4}$, so:

x	0	1	2
$Pr(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$E(X)=0\times 1/4+1\times 1/2+2\times 1/4=1/2+1/2.$

$\therefore E(X)=1.$



Example 4 — unscaffolded

For the random variable X of Example 2, $E(X)=3$ and $Var(X)=1$. Find $E(3X-1)$ and $Var(3X-1)$.

$E(3X-1)=3E(X)-1=3\times 3-1=8.$

$Var(3X-1)=3^2 Var(X)=9\times 1=9.$

$\therefore E(3X-1)=8$ and $Var(3X-1)=9.$

Practice questions

Scaffolded

Q1. For the distribution below: (a) show it is a valid probability distribution; (b) find $Pr(X\le 2)$; (c) find $E(X)$.

x	1	2	3
$Pr(X=x)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{10}$

Q2. The distribution below has an unknown k: (a) find k; (b) find $E(X)$.

x	0	1	2	3	4
$Pr(X=x)$	$\frac{1}{10}$	$\frac{1}{5}$	k	$\frac{1}{5}$	$\frac{1}{10}$

Q3. For the distribution below, find (a) $E(X)$; (b) $Var(X)$; (c) $sd(X)$.

x	2	4	6	8
$Pr(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Q4. A random variable has $Pr(X=x)=\frac{k}{x}$ for $x=1,2,3,6$. (a) Find k. (b) Find $E(X)$. (c) Find $Var(X)$.

Q5. For the distribution below, find (a) $E(X)$; (b) the mode; (c) $Pr(X\geq 2)$.

x	0	1	2	3
$Pr(X=x)$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$

Q6. For the distribution below, find (a) $E(X)$; (b) $E(2X+5)$.

x	1	2	3
$Pr(X=x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Unscaffolded

Q7. Find $E(X)$ and $Var(X)$ for: $x=1,2,3,4$ with $Pr(X=x)=\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}$.

Q8. Find k and $E(X)$ for: $x=0,1,2,3$ with $Pr(X=x)=\frac{1}{6},k,\frac{1}{3},\frac{1}{4}$.

Q9. Find $E(X)$ and $Var(X)$ for: $x=-2,-1,1,2$ with $Pr(X=x)=\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{6}$.

Q10. A number X is chosen at random from $\{10,20,30,40,50\}$. Find $E(X)$ and $sd(X)$.

Q11. X is the score showing on a single roll of a fair six-sided die. Find $E(X)$ and $Var(X)$.

Q12. Find $E(X)$ and the mode for: $x=0,1,2$ with $Pr(X=x)=\frac{9}{25},\frac{12}{25},\frac{4}{25}$.

Q13. Find $E(X)$ and $Var(X)$ for: $x=2,3,5,7$, each equally likely.

Q14. In a game, the net winnings X (in dollars) has the distribution $x=-3,0,2$ with $Pr(X=x)=\frac{1}{2},\frac{3}{8},\frac{1}{8}$. Find the expected winnings and state whether the game is fair.

Q15. For the distribution $x=5,10,15,20$ with $Pr(X=x)=\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}$, find $E(X)$ and $E(5X-2)$.

Q16. Find $E(X)$ and $Var(X)$ for: $x=0,1,2,3,4$ with $Pr(X=x)=\frac{1}{16},\frac{1}{4},\frac{3}{8},\frac{1}{4},\frac{1}{16}$.

Q17. Find $E(X)$ and $Var(X)$ for: $x=-2,0,2,4$ with $Pr(X=x)=\frac{1}{4},\frac{3}{8},\frac{1}{4},\frac{1}{8}$.

Q18. A random variable has $Pr(X=x)=k,k,2k,4k$ for $x=1,2,3,4$ respectively. Find k and $E(X)$.

Answers

Q1. Valid (sum $=1$); $Pr(X \le 2) = \frac{7}{10}$; $E(X) = \frac{21}{10}$. **Q2.** $k = \frac{2}{5}$; $E(X) = 2$. **Q3.** $E(X) = 5$; $Var(X) = 3$; $sd(X) = \sqrt{3}$. **Q4.** $k = \frac{1}{2}$; $E(X) = 2$; $Var(X) = 2$. **Q5.** $E(X) = \frac{7}{5}$; mode $=1$; $Pr(X \ge 2) = \frac{2}{5}$. **Q6.** $E(X) = \frac{5}{3}$; $E(2X+5) = \frac{25}{3}$. **Q7.** $E(X) = 2$; $Var(X) = \frac{4}{3}$. **Q8.** $k = \frac{1}{4}$; $E(X) = \frac{5}{3}$. **Q9.** $E(X) = 0$; $Var(X) = 2$. **Q10.** $E(X) = 30$; $sd(X) = 10\sqrt{2}$. **Q11.** $E(X) = \frac{7}{2}$; $Var(X) = \frac{35}{12}$. **Q12.** $E(X) = \frac{4}{5}$; mode $=1$. **Q13.** $E(X) = \frac{17}{4}$; $Var(X) = \frac{59}{16}$. **Q14.** $E(X) = -\frac{5}{4}$ (dollars); **not fair** — the player expects to lose $\frac{5}{4}$ of a dollar (that is, 1.25 dollars) per game. **Q15.** $E(X) = 11$; $E(5X-2) = 53$. **Q16.** $E(X) = 2$; $Var(X) = 1$. **Q17.** $E(X) = \frac{1}{2}$; $Var(X) = \frac{15}{4}$. **Q18.** $k = \frac{1}{8}$; $E(X) = \frac{25}{8}$.

Fully-worked solutions

Q1. (a) All entries are between 0 and 1 and $\frac{1}{5} + \frac{1}{2} + \frac{3}{10} = \frac{2}{10} + \frac{5}{10} + \frac{3}{10} = 1$, so it is valid. (b) $Pr(X \le 2) = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$. (c) $E(X) = 1 \times 1/5 + 2 \times 1/2 + 3 \times 3/10 = 1/5 + 1 + 9/10 = \frac{21}{10}$.

Q2. (a) $\frac{1}{10} + \frac{1}{5} + k + \frac{1}{5} + \frac{1}{10} = 1 \Rightarrow \frac{3}{5} + k = 1$, so $k = \frac{2}{5}$. (b) By symmetry about $x = 2$, $E(X) = 2$ (or compute $\sum x\, Pr(X = x) = 2$).

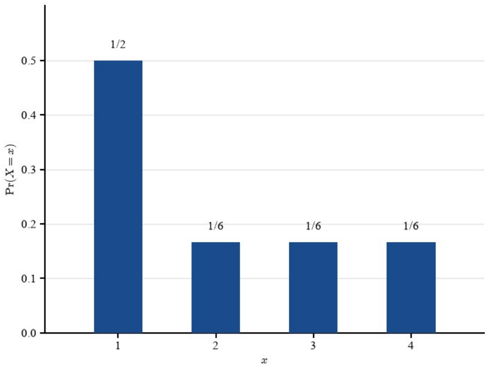
Q3. (a) $E(X) = 2 \times 1/8 + 4 \times 3/8 + 6 \times 3/8 + 8 \times 1/8 = 2 + 12 + 18 + 8/8 = 5$. (b) $E(X^2) = 4 + 48 + 108 + 64/8 = 28$, so $Var(X) = 28 - 25 = 3$. (c) $sd(X) = \sqrt{3}$.

Q4. (a) $k + k/2 + k/3 + k/6 = 2k = 1$, so $k = \frac{1}{2}$; the probabilities are $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}$. (b) $E(X) = 1 \times 1/2 + 2 \times 1/4 + 3 \times 1/6 + 6 \times 1/12 = 1/2 + 1/2 + 1/2 + 1/2 = 2$. (c) $E(X^2) = 1/2 + 1 + 3/2 + 3 = 6$, so $Var(X) = 6 - 4 = 2$.

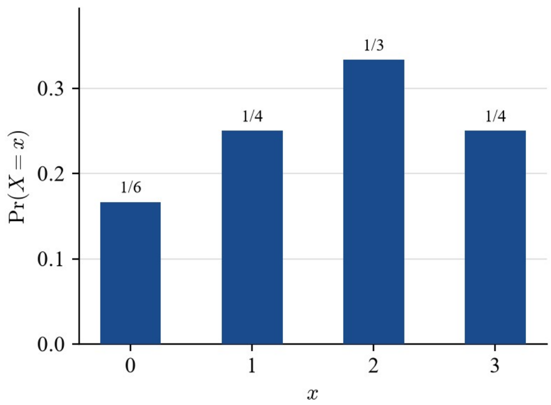
Q5. (a) $E(X) = 0 \times 1/10 + 1 \times 1/2 + 2 \times 3/10 + 3 \times 1/10 = 5/10 + 6/10 + 3/10 = \frac{7}{5}$. (b) The greatest probability $\frac{1}{2}$ occurs at $x = 1$, so the mode is 1. (c) $Pr(X \ge 2) = \frac{3}{10} + \frac{1}{10} = \frac{2}{5}$.

Q6. (a) $E(X) = 1 \times 1/2 + 2 \times 1/3 + 3 \times 1/6 = 1/2 + 2/3 + 1/2 = \frac{5}{3}$. (b) $E(2X+5) = 2E(X) + 5 = 2 \times 5/3 + 5 = \frac{25}{3}$.

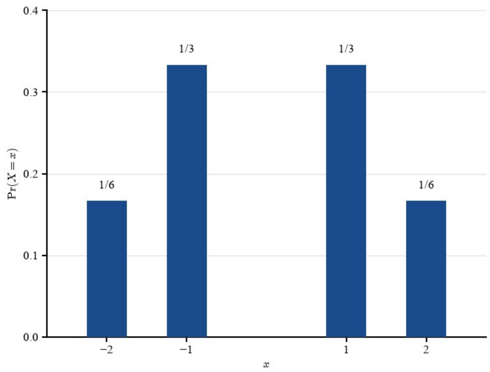
Q7. $E(X) = 1 \times 1/2 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 = 3/6 + 2/6 + 3/6 + 4/6 = 2$.
$E(X^2) = 1 \times 1/2 + 4 \times 1/6 + 9 \times 1/6 + 16 \times 1/6 = \frac{3+4+9+16}{6} = \frac{16}{3}$, so $Var(X) = \frac{16}{3} - 4 = \frac{4}{3}$.



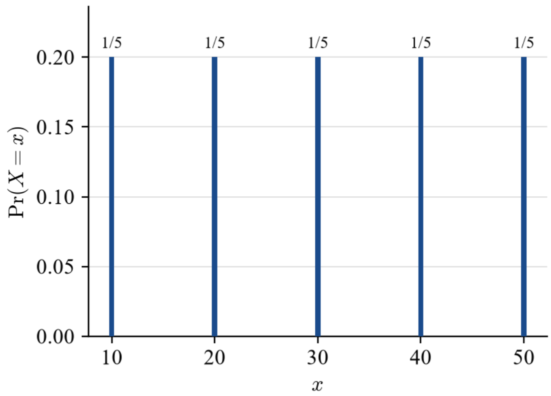
Q8. $\frac{1}{6}+k+\frac{1}{3}+\frac{1}{4}=1 \Rightarrow k=1-\frac{3}{4}=\frac{1}{4}$. Then $E(X)=0\times 1/6+1\times 1/4+2\times 1/3+3\times 1/4=1/4+2/3+3/4=\frac{5}{3}$.



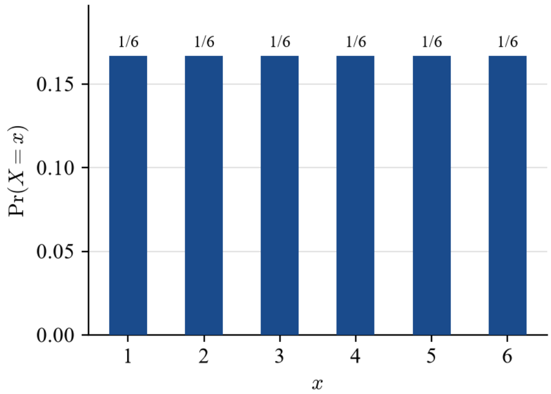
Q9. By symmetry about 0, $E(X)=0$. $E(X^2)=4\times 1/6+1\times 1/3+1\times 1/3+4\times 1/6=\frac{4+2+2+4}{6}=2$, so $Var(X)=2-0=2$.



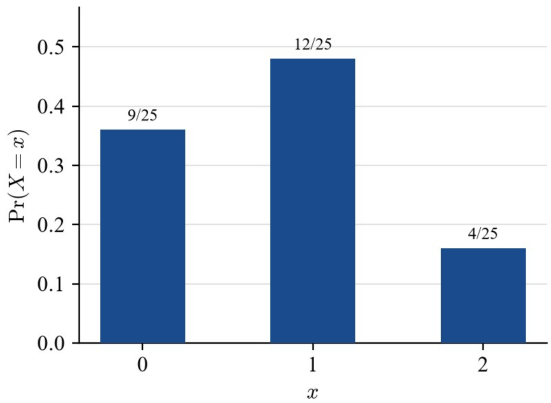
Q10. Each value has probability $\frac{1}{5}$. $E(X)=1/5(10+20+30+40+50)=30$.
$E(X^2)=1/5(100+400+900+1600+2500)=1100$, so $Var(X)=1100-900=200$ and $sd(X)=\sqrt{200}=10\sqrt{2}$.



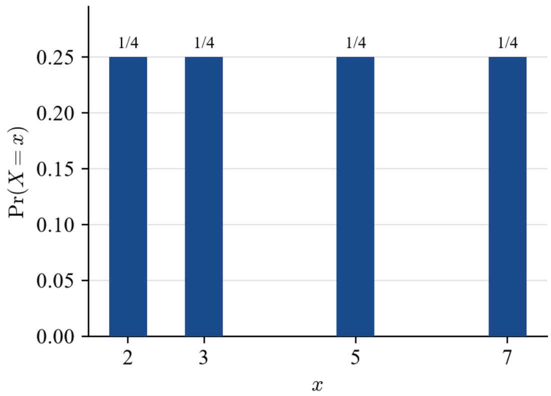
Q11. $E(X)=1/6(1+2+3+4+5+6)=\frac{7}{2}$. $E(X^2)=1/6(1+4+9+16+25+36)=\frac{91}{6}$, so $Var(X)=\frac{91}{6}-\left(\frac{7}{2}\right)^2=\frac{91}{6}-\frac{49}{4}=\frac{182-147}{12}=\frac{35}{12}$.



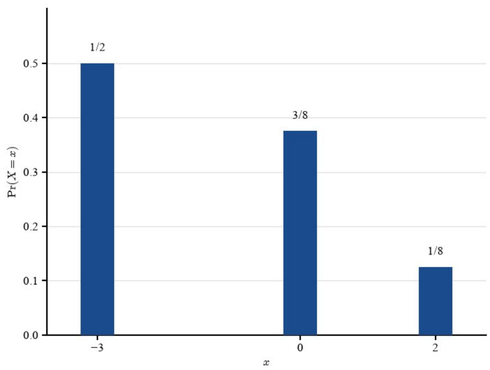
Q12. $E(X)=0\times 9/25+1\times 12/25+2\times 4/25=\frac{12+8}{25}=\frac{4}{5}$. The greatest probability $\frac{12}{25}$ is at $x=1$, so the mode is 1.



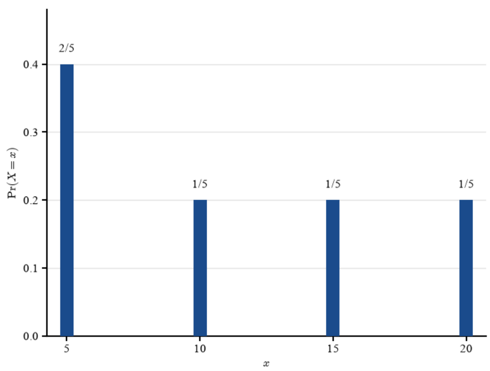
Q13. Each value has probability $\frac{1}{4}$. $E(X)=1/4(2+3+5+7)=\frac{17}{4}$. $E(X^2)=1/4(4+9+25+49)=\frac{87}{4}$, so $Var(X)=\frac{87}{4}-\left(\frac{17}{4}\right)^2=\frac{348-289}{16}=\frac{59}{16}$.



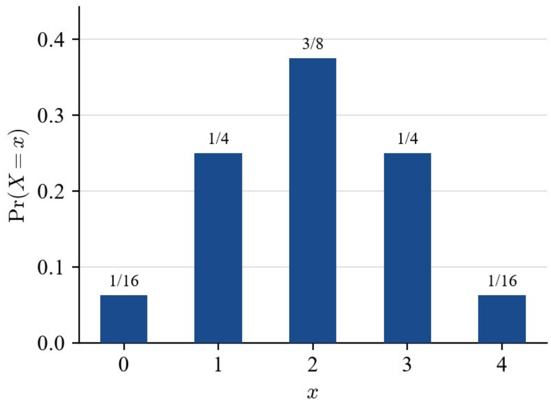
Q14. $E(X) = -3 \times 1/2 + 0 \times 3/8 + 2 \times 1/8 = -3/2 + 1/4 = -\frac{5}{4}$. Since the expected winnings are not 0 (they are negative), the game is **not fair** — on average the player loses $\frac{5}{4}$ of a dollar (that is, 1.25 dollars) per game.



Q15. $E(X) = 5 \times 2/5 + 10 \times 1/5 + 15 \times 1/5 + 20 \times 1/5 = 2 + 2 + 3 + 4 = 11$. Then
$E(5X - 2) = 5E(X) - 2 = 5 \times 11 - 2 = 53$.

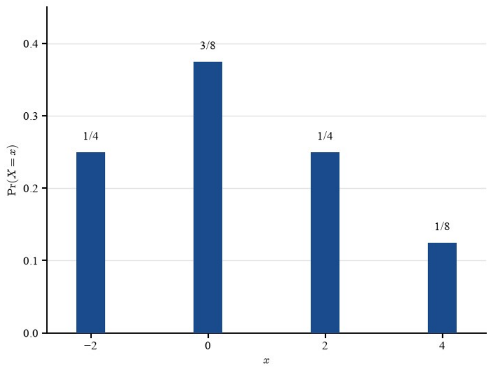


Q16. By symmetry about $x = 2$, $E(X) = 2$.
$E(X^2) = 0 + 1 \times 1/4 + 4 \times 3/8 + 9 \times 1/4 + 16 \times 1/16 = 1/4 + 3/2 + 9/4 + 1 = 5$, so $Var(X) = 5 - 4 = 1$.



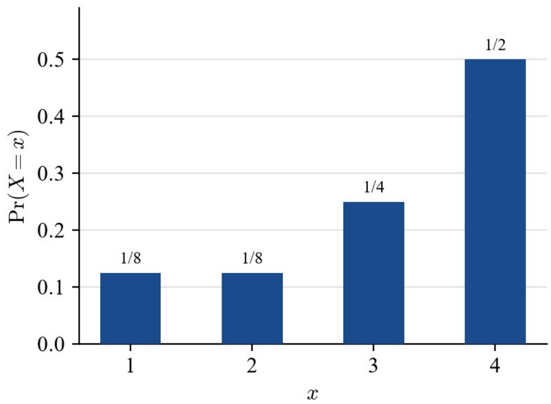
Q17. $E(X) = -2 \times 1/4 + 0 \times 3/8 + 2 \times 1/4 + 4 \times 1/8 = -1/2 + 1/2 + 1/2 = \dfrac{1}{2}$.

$E(X^2) = 4 \times 1/4 + 0 + 4 \times 1/4 + 16 \times 1/8 = 1 + 1 + 2 = 4$, so $Var(X) = 4 - 1/4 = \dfrac{15}{4}$.



Q18. $k + k + 2k + 4k = 8k = 1$, so $k = \dfrac{1}{8}$; the probabilities are $\dfrac{1}{8}, \dfrac{1}{8}, \dfrac{1}{4}, \dfrac{1}{2}$.

$E(X) = 1 \times 1/8 + 2 \times 1/8 + 3 \times 1/4 + 4 \times 1/2 = 1/8 + 2/8 + 3/4 + 2 = \dfrac{25}{8}$.



Chapter 10 Discrete random variables — 10B Selection without replacement

Units 3 & 4 preview. This section goes beyond the Units 1 & 2 study design. The hypergeometric distribution is not part of the study design at Units 3 & 4 level either; the section is included as extension practice in specifying probability distributions and calculating their means, and is not assessed in the graded topic tests or practice examinations in this series.

Theory

When items are sampled **without replacement**, the draws are **not independent** — each selection changes what remains. Suppose a group of N items contains D **successes** (and $N-D$ failures), and a sample of n items is chosen without replacement. Let X be the number of successes in the sample. Then

$$Pr(X=x)=\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}},$$

for the possible values $max(0, n-(N-D)) \le x \le min(n, D)$. This is a **hypergeometric** distribution.

Why the formula works. There are $\binom{N}{n}$ equally likely samples in total. A sample with exactly x successes is built by choosing x of the D successes **and** $n-x$ of the $N-D$ failures — that is, $\binom{D}{x}\binom{N-D}{n-x}$ favourable samples.

- The probabilities of any distribution **sum to** 1.
- The **mean** (expected value) is $\mathrm{E}(X)=\sum x\, Pr(X=x)$, which for this distribution simplifies to $\mathrm{E}(X)=\frac{nD}{N}$.
- For small cases a probability **tree** gives the same answers (multiply along branches, add the paths).

To find a distribution: identify N, D and n; list the possible values of x; compute each $Pr(X=x)$; tabulate; and check the probabilities sum to 1.

Worked examples

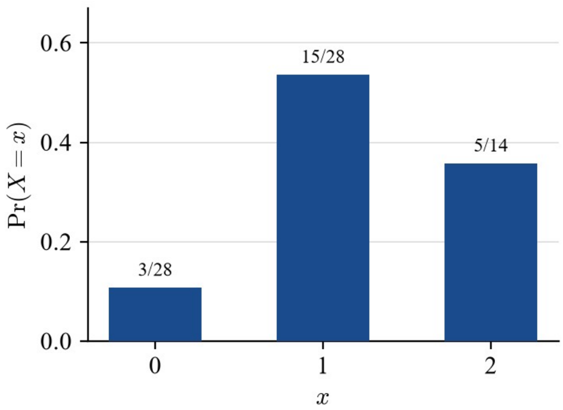
Example 1 — scaffolded

A box contains 5 red and 3 blue marbles. Two marbles are drawn without replacement. Let X be the number of red marbles drawn. Find the probability distribution of X and its mean.

Working	Comment
Here $N=8$, $D=5$ (red), $n=2$; X may be $0, 1, 2$.	Identify the totals; X counts red.
Total number of samples $=\binom{8}{2}=28$.	Equally likely selections of 2 from 8.
$Pr(X=0)=\frac{\binom{5}{0}\binom{3}{2}}{28}=\frac{3}{28}$.	0 red means both blue.
$Pr(X=1)=\frac{\binom{5}{1}\binom{3}{1}}{28}=\frac{15}{28}$.	1 red and 1 blue.

Working	Comment
$Pr(X=2)=\dfrac{\binom{5}{2}\binom{3}{0}}{28}=\dfrac{10}{28}=\dfrac{5}{14}$.	Both red.
Sum $=\dfrac{3+15+10}{28}=1$; mean $=\dfrac{nD}{N}=\dfrac{2\times 5}{8}=\dfrac{5}{4}$.	The probabilities sum to 1.

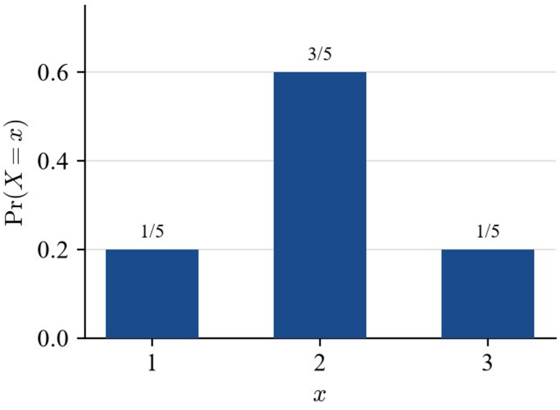
x	$Pr(X=x)$
0	$\dfrac{3}{28}$
1	$\dfrac{15}{28}$
2	$\dfrac{5}{14}$



Example 2 — scaffolded

A committee of 3 is chosen from 4 women and 2 men. Let X be the number of women. Find the distribution of X and its mean.

Working	Comment
$N=6$, $D=4$ (women), $n=3$. The smallest possible value is 1 (only 2 men).	$max(0,3-2)=1$, so $X\in\{1,2,3\}$.
Total $=\binom{6}{3}=20$.	Equally likely committees.
$Pr(X=1)=\dfrac{\binom{4}{1}\binom{2}{2}}{20}=\dfrac{4}{20}=\dfrac{1}{5}$.	1 woman, 2 men.
$Pr(X=2)=\dfrac{\binom{4}{2}\binom{2}{1}}{20}=\dfrac{12}{20}=\dfrac{3}{5}$.	2 women, 1 man.
$Pr(X=3)=\dfrac{\binom{4}{3}\binom{2}{0}}{20}=\dfrac{4}{20}=\dfrac{1}{5}$.	3 women.
Sum $=1$; mean $=\dfrac{3\times 4}{6}=2$.	Check against $\dfrac{nD}{N}$.



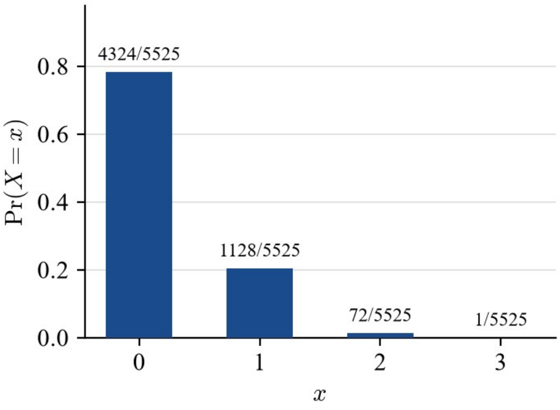
Example 3 — unscaffolded

Three cards are drawn without replacement from a standard deck of 52. Let X be the number of aces drawn. Find $Pr(X=2)$ and the mean of X.

$N=52$, $D=4$ (aces), $n=3$; total $=\binom{52}{3}=22\,100$.

$$Pr(X=2)=\frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}}=\frac{6\times 48}{22\,100}=\frac{288}{22\,100}=\frac{72}{5525}.$$

$\therefore$ mean $=\frac{nD}{N}=\frac{3\times 4}{52}=\frac{3}{13}$.



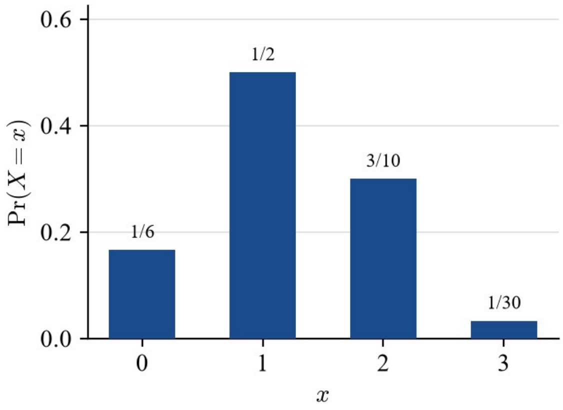
Example 4 — unscaffolded

A drawer holds 7 working and 3 faulty batteries. Four are chosen without replacement. Let X be the number of faulty batteries. Find the distribution of X and its mean.

$N=10$, $D=3$ (faulty), $n=4$; total $=\binom{10}{4}=210$.

$Pr(X=0)=\frac{\binom{7}{4}}{210}=\frac{35}{210}=\frac{1}{6}$; $Pr(X=1)=\frac{\binom{3}{1}\binom{7}{3}}{210}=\frac{105}{210}=\frac{1}{2}$; $Pr(X=2)=\frac{\binom{3}{2}\binom{7}{2}}{210}=\frac{63}{210}=\frac{3}{10}$;
$Pr(X=3)=\frac{\binom{3}{3}\binom{7}{1}}{210}=\frac{7}{210}=\frac{1}{30}$.

$\therefore$ the probabilities sum to 1 and the mean $= \frac{4 \times 3}{10} = \frac{6}{5}$.



Practice questions

Scaffolded

Q1. A jar holds 6 green and 4 yellow lollies. Two are taken without replacement; X is the number of green. (a) State the possible values of X. (b) Find the probability distribution. (c) Find the mean.

Q2. From 5 novels and 3 textbooks, 3 books are chosen. X is the number of textbooks. Find the distribution of X and its mean.

Q3. A team of 4 is chosen from 6 boys and 5 girls. X is the number of girls. (a) Find $Pr(X = 2)$. (b) Find the mean.

Q4. Three cards are drawn without replacement from a deck; X is the number of hearts. (a) State the possible values of X. (b) Find $Pr(X = 0)$. (c) Find $Pr(X = 1)$.

Q5. A batch of 8 phones contains 2 faulty ones. Three are inspected without replacement; X is the number faulty. Find the distribution of X and its mean.

Q6. A raffle has 10 tickets, of which 3 are winners. You buy 2 tickets; X is the number of winning tickets. (a) Find the distribution of X. (b) Find $Pr(X \geq 1)$. (c) Find the mean.

Unscaffolded

For each situation, let X be the stated count. Find its probability distribution and its mean, unless directed otherwise.

Q7. 5 red and 2 black counters; draw 2; X = number of red.

Q8. 4 women and 3 men; choose 2; X = number of women.

Q9. 6 working and 4 defective globes; choose 4; X = number defective.

Q10. A carton of 12 eggs has 4 cracked; take 3; X = number cracked.

Q11. Two cards drawn from a deck; X = number of kings.

Q12. A draw selects 4 numbers from 20, of which 5 are "winning"; X = number of winning numbers chosen.

Q13. From 7 boys and 4 girls, 3 are chosen; X = number of girls.

Q14. A box of 10 items has 2 defective; 4 are tested; X = number defective.

Q15. From 9 red and 6 blue balls, 4 are chosen; X = number of red. Find the **mean** and $Pr(\text{all 4 red})$.

Q16. A group of 10 items has 3 special ones; 3 are drawn; X = number special. Find $Pr(X \geq 1)$ and the mean.

Q17. From 9 tickets, of which 3 win, you buy 4; X = number of winning tickets.

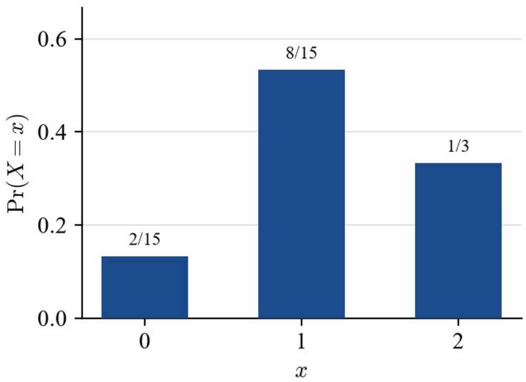
Q18. A committee of 5 is chosen from 7 senior and 5 junior members; X = number of juniors. Verify the distribution sums to 1 and find the mean.

Answers

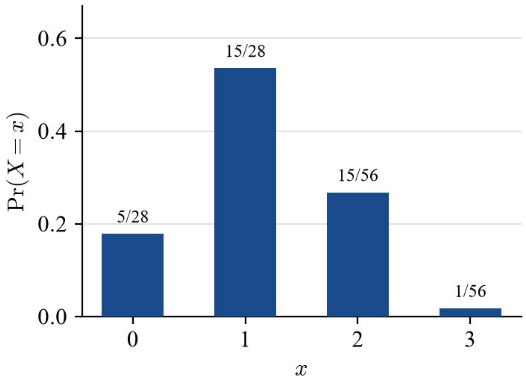
Q1. $X \in \{0,1,2\}$; $Pr = \frac{2}{15}, \frac{8}{15}, \frac{1}{3}$; mean $\frac{6}{5}$. **Q2.** $Pr(0-3) = \frac{5}{28}, \frac{15}{28}, \frac{15}{56}, \frac{1}{56}$; mean $\frac{9}{8}$. **Q3.** $Pr(X=2) = \frac{5}{11}$; mean $\frac{20}{11}$. **Q4.** $X \in \{0,1,2,3\}$; $Pr(X=0) = \frac{703}{1700}$; $Pr(X=1) = \frac{741}{1700}$. **Q5.** $Pr = \frac{5}{14}, \frac{15}{28}, \frac{3}{28}$; mean $\frac{3}{4}$. **Q6.** $Pr = \frac{7}{15}, \frac{7}{15}, \frac{1}{15}$; $Pr(X \geq 1) = \frac{8}{15}$; mean $\frac{3}{5}$. **Q7.** $Pr = \frac{1}{21}, \frac{10}{21}, \frac{10}{21}$; mean $\frac{10}{7}$. **Q8.** $Pr = \frac{1}{7}, \frac{4}{7}, \frac{2}{7}$; mean $\frac{8}{7}$. **Q9.** $Pr = \frac{1}{14}, \frac{8}{21}, \frac{3}{7}, \frac{4}{35}, \frac{1}{210}$; mean $\frac{8}{5}$. **Q10.** $Pr = \frac{14}{55}, \frac{28}{55}, \frac{12}{55}, \frac{1}{55}$; mean 1. **Q11.** $Pr = \frac{188}{221}, \frac{32}{221}, \frac{1}{221}$; mean $\frac{2}{13}$. **Q12.** $Pr = \frac{91}{323}, \frac{455}{969}, \frac{70}{323}, \frac{10}{323}, \frac{1}{969}$; mean 1. **Q13.** $Pr = \frac{7}{33}, \frac{28}{55}, \frac{14}{55}, \frac{4}{165}$; mean $\frac{12}{11}$. **Q14.** $Pr = \frac{1}{3}, \frac{8}{15}, \frac{2}{15}$; mean $\frac{4}{5}$. **Q15.** mean $\frac{12}{5}$; $Pr(\text{all 4 red}) = \frac{6}{65}$. **Q16.** $Pr(X \geq 1) = \frac{17}{24}$; mean $\frac{9}{10}$. **Q17.** $Pr = \frac{5}{42}, \frac{10}{21}, \frac{5}{14}, \frac{1}{21}$; mean $\frac{4}{3}$. **Q18.** sum $= 1$; mean $\frac{25}{12}$.

Fully-worked solutions

Q1. $N=10$, $D=6$, $n=2$; total $\binom{10}{2} = 45$. (a) $X \in \{0,1,2\}$. (b) $Pr(X=0) = \frac{\binom{6}{0}\binom{4}{2}}{45} = \frac{6}{45} = \frac{2}{15}$; $Pr(X=1) = \frac{\binom{6}{1}\binom{4}{1}}{45} = \frac{24}{45} = \frac{8}{15}$; $Pr(X=2) = \frac{\binom{6}{2}}{45} = \frac{15}{45} = \frac{1}{3}$. (c) Mean $= \frac{2 \times 6}{10} = \frac{6}{5}$.



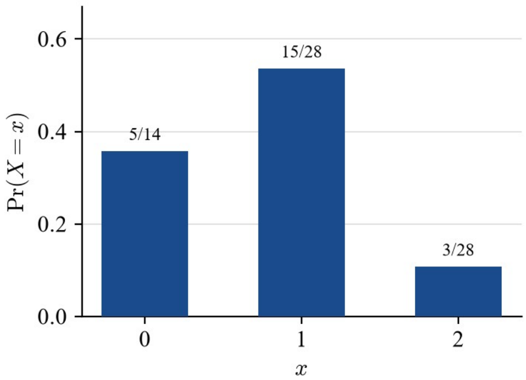
Q2. $N=8$, $D=3$, $n=3$; total $\binom{8}{3} = 56$. $Pr(X=0) = \frac{\binom{5}{3}}{56} = \frac{10}{56} = \frac{5}{28}$; $Pr(X=1) = \frac{\binom{3}{1}\binom{5}{2}}{56} = \frac{30}{56} = \frac{15}{28}$; $Pr(X=2) = \frac{\binom{3}{2}\binom{5}{1}}{56} = \frac{15}{56}$; $Pr(X=3) = \frac{\binom{3}{3}}{56} = \frac{1}{56}$. Mean $= \frac{3 \times 3}{8} = \frac{9}{8}$.



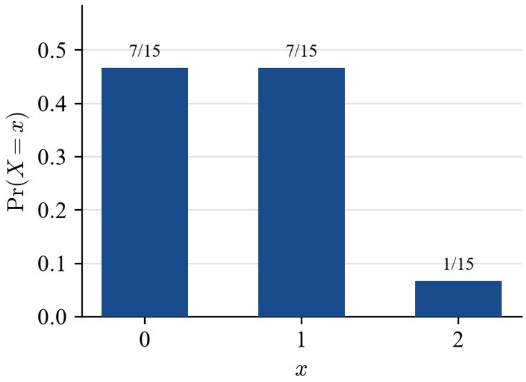
Q3. $N=11$, $D=5$, $n=4$; total $\binom{11}{4}=330$. (a) $Pr(X=2)=\frac{\binom{5}{2}\binom{6}{2}}{330}=\frac{10\times 15}{330}=\frac{150}{330}=\frac{5}{11}$. (b) Mean $=\frac{4\times 5}{11}=\frac{20}{11}$.

Q4. $N=52$, $D=13$, $n=3$; total $\binom{52}{3}=22\,100$. (a) $X\in\{0,1,2,3\}$. (b) $Pr(X=0)=\frac{\binom{39}{3}}{22\,100}=\frac{9139}{22\,100}=\frac{703}{1700}$. (c) $Pr(X=1)=\frac{\binom{13}{1}\binom{39}{2}}{22\,100}=\frac{13\times 741}{22\,100}=\frac{741}{1700}$.

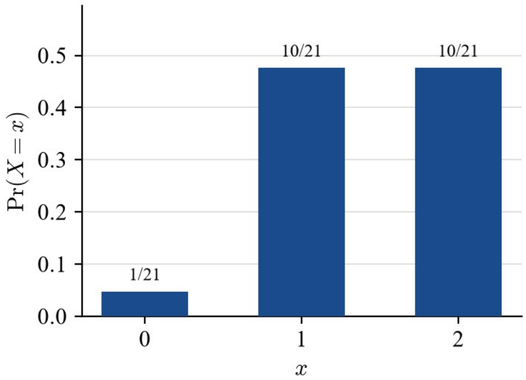
Q5. $N=8$, $D=2$, $n=3$; total $\binom{8}{3}=56$. $Pr(X=0)=\frac{\binom{6}{3}}{56}=\frac{20}{56}=\frac{5}{14}$; $Pr(X=1)=\frac{\binom{2}{1}\binom{6}{2}}{56}=\frac{30}{56}=\frac{15}{28}$; $Pr(X=2)=\frac{\binom{2}{2}\binom{6}{1}}{56}=\frac{6}{56}=\frac{3}{28}$. Mean $=\frac{3\times 2}{8}=\frac{3}{4}$.



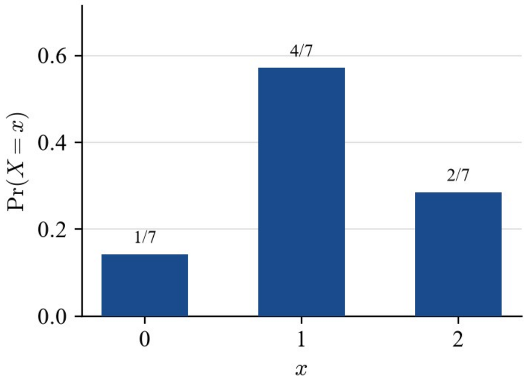
Q6. $N=10$, $D=3$, $n=2$; total $\binom{10}{2}=45$. (a) $Pr(X=0)=\frac{\binom{7}{2}}{45}=\frac{21}{45}=\frac{7}{15}$; $Pr(X=1)=\frac{\binom{3}{1}\binom{7}{1}}{45}=\frac{21}{45}=\frac{7}{15}$; $Pr(X=2)=\frac{\binom{3}{2}}{45}=\frac{3}{45}=\frac{1}{15}$. (b) $Pr(X\geq 1)=1-Pr(X=0)=1-\frac{7}{15}=\frac{8}{15}$. (c) Mean $=\frac{2\times 3}{10}=\frac{3}{5}$.



Q7. $N=7$, $D=5$, $n=2$; total 21. $Pr=\dfrac{\binom{2}{2}}{21},\dfrac{\binom{5}{1}\binom{2}{1}}{21},\dfrac{\binom{5}{2}}{21}=\dfrac{1}{21},\dfrac{10}{21},\dfrac{10}{21}$. Mean $\dfrac{10}{7}$.

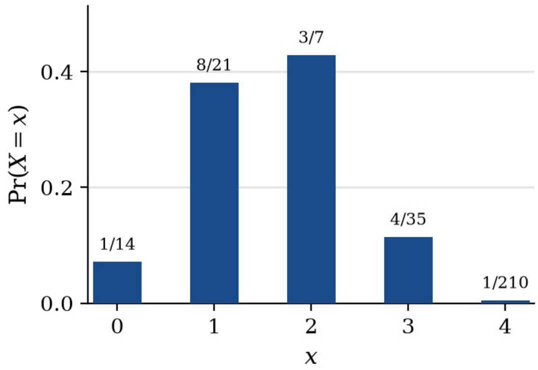


Q8. $N=7$, $D=4$, $n=2$; total 21. $Pr=\dfrac{1}{7},\dfrac{4}{7},\dfrac{2}{7}$. Mean $\dfrac{8}{7}$.

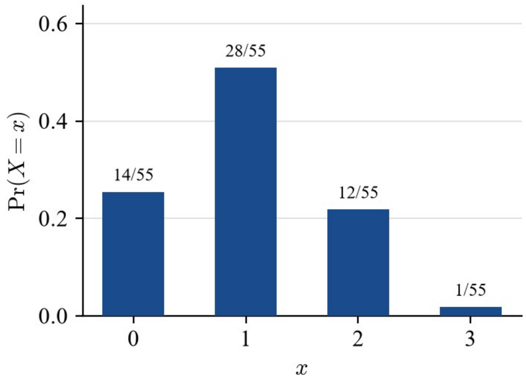


Q9. $N=10$, $D=4$, $n=4$; total $\binom{10}{4}=210$. $Pr=\dfrac{\binom{6}{4}}{210},\dfrac{\binom{4}{1}\binom{6}{3}}{210},\dfrac{\binom{4}{2}\binom{6}{2}}{210},\dfrac{\binom{4}{3}\binom{6}{1}}{210},\dfrac{\binom{4}{4}}{210}=\dfrac{1}{14},\dfrac{8}{21},\dfrac{3}{7},\dfrac{4}{35},\dfrac{1}{210}$.

Mean $\dfrac{8}{5}$.

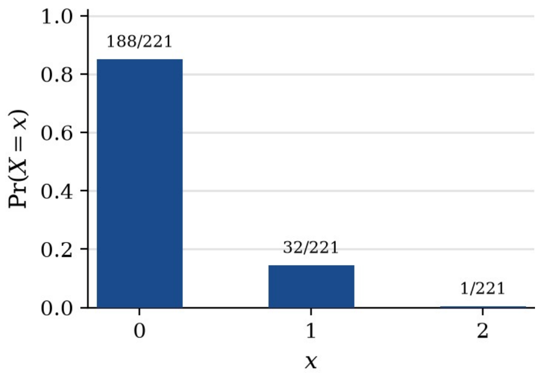


Q10. $N=12$, $D=4$, $n=3$; total 220. $Pr=\frac{14}{55},\frac{28}{55},\frac{12}{55},\frac{1}{55}$. Mean $=\frac{3\times 4}{12}=1$.

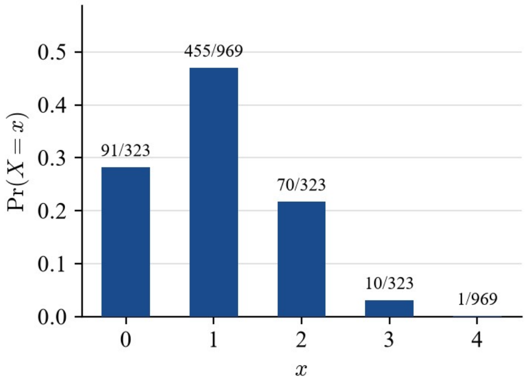


Q11. $N=52$, $D=4$, $n=2$; total $\binom{52}{2}=1326$. $Pr(X=0)=\frac{\binom{48}{2}}{1326}=\frac{1128}{1326}=\frac{188}{221}$;

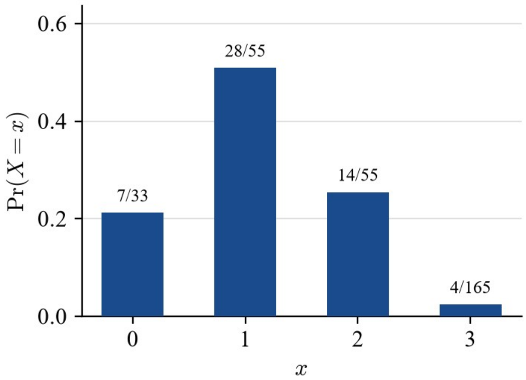
$Pr(X=1)=\frac{\binom{4}{1}\binom{48}{1}}{1326}=\frac{192}{1326}=\frac{32}{221}$; $Pr(X=2)=\frac{\binom{4}{2}}{1326}=\frac{6}{1326}=\frac{1}{221}$. Mean $\frac{2}{13}$.



Q12. $N=20$, $D=5$, $n=4$; total $\binom{20}{4}=4845$. $Pr(X=0)=\frac{91}{323}$, $Pr(X=1)=\frac{455}{969}$, $Pr(X=2)=\frac{70}{323}$, $Pr(X=3)=\frac{10}{323}$, $Pr(X=4)=\frac{1}{969}$. Mean $=\frac{4\times 5}{20}=1$.

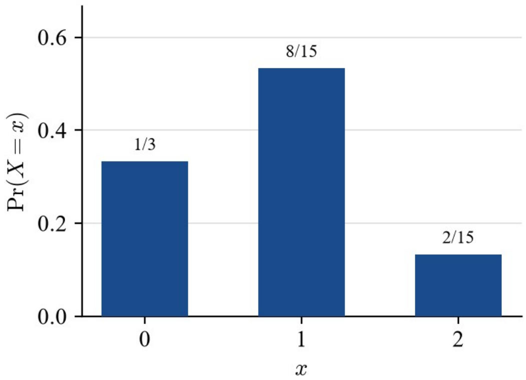


Q13. $N=11$, $D=4$, $n=3$; total $\binom{11}{3}=165$. $Pr=\frac{7}{33},\frac{28}{55},\frac{14}{55},\frac{4}{165}$. Mean $\frac{12}{11}$.



Q14. $N=10$, $D=2$, $n=4$; total 210. $Pr(X=0)=\frac{\binom{8}{4}}{210}=\frac{70}{210}=\frac{1}{3}$; $Pr(X=1)=\frac{\binom{2}{1}\binom{8}{3}}{210}=\frac{8}{15}$;

$Pr(X=2)=\frac{\binom{8}{2}}{210}=\frac{2}{15}$. Mean $\frac{4}{5}$.



Q15. $N=15$, $D=9$, $n=4$. Mean $=\frac{4\times 9}{15}=\frac{12}{5}$. $Pr(\text{all 4 red})=Pr(X=4)=\frac{\binom{9}{4}}{\binom{15}{4}}=\frac{126}{1365}=\frac{6}{65}$.

Q16. $N=10$, $D=3$, $n=3$; total 120. $Pr(X=0)=\frac{\binom{7}{3}}{120}=\frac{35}{120}=\frac{7}{24}$, so $Pr(X\geq 1)=1-\frac{7}{24}=\frac{17}{24}$. Mean $=\frac{3\times 3}{10}=\frac{9}{10}$

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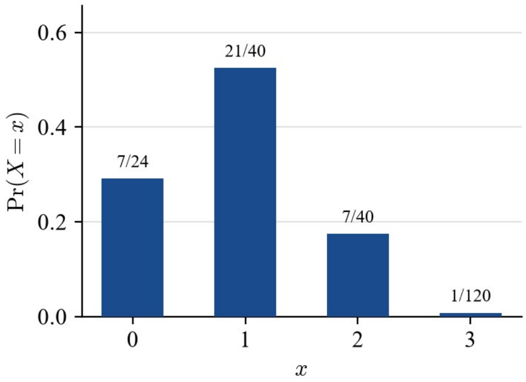
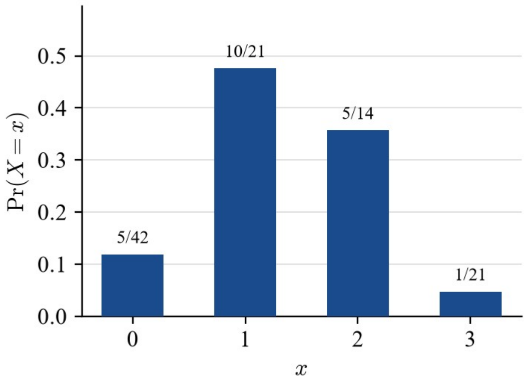
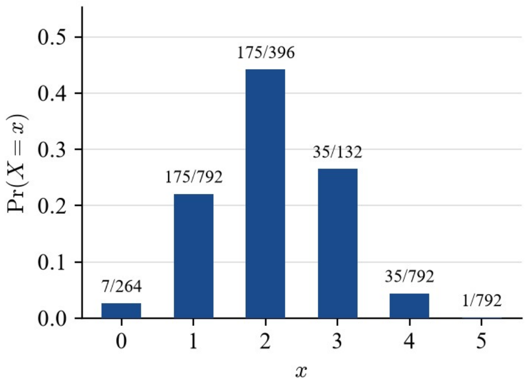


Q17. $N=9$, $D=3$, $n=4$; total $\binom{9}{4}=126$. $Pr(X=0)=\dfrac{\binom{6}{4}}{126}=\dfrac{15}{126}=\dfrac{5}{42}$; $Pr(X=1)=\dfrac{\binom{3}{1}\binom{6}{3}}{126}=\dfrac{60}{126}=\dfrac{10}{21}$; $Pr(X=2)=\dfrac{\binom{3}{2}\binom{6}{2}}{126}=\dfrac{45}{126}=\dfrac{5}{14}$; $Pr(X=3)=\dfrac{\binom{3}{3}\binom{6}{1}}{126}=\dfrac{6}{126}=\dfrac{1}{21}$. Mean $\dfrac{4}{3}$.



Q18. $N=12$, $D=5$, $n=5$; total $\binom{12}{5}=792$. $Pr(X=x)=\dfrac{\binom{5}{x}\binom{7}{5-x}}{792}$ gives $\dfrac{7}{264},\dfrac{175}{792},\dfrac{175}{396},\dfrac{35}{132},\dfrac{35}{792},\dfrac{1}{792}$ for $x=0,1,2,3,4,5$. These sum to 1, and the mean $=\dfrac{5\times 5}{12}=\dfrac{25}{12}$.



Chapter 10 Discrete random variables — 10C Selection with replacement and the binomial distribution

Units 3 & 4 preview. This section goes beyond the Units 1 & 2 study design. It is included as preparation for Units 3 & 4 and is not assessed in the graded topic tests or practice examinations in this series.

Theory

Many situations consist of **repeated independent trials**, each with only two outcomes — a **success** (probability p) or a **failure** (probability $1-p$). When the number of trials n is fixed and p stays constant, the number of successes follows a **binomial distribution**.

When is a distribution binomial? All four conditions must hold:

- a **fixed** number of trials n;
- each trial has exactly **two** outcomes (success / failure);
- the trials are **independent**;
- the probability of success p is **constant** from trial to trial.

Sampling **with replacement** keeps p constant, so it is binomial; sampling *without* replacement changes p each draw, so it is **not** binomial.

The distribution. If X is the number of successes in n trials, we write $X \sim Bi(n, p)$, and

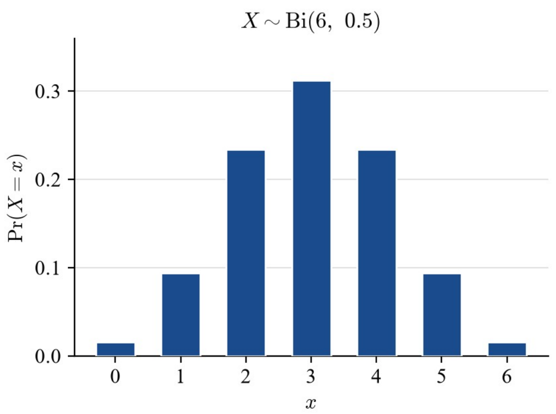
$$Pr(X=x)=\binom{n}{x}p^x(1-p)^{n-x}, x=0,1,2,\dots,n.$$

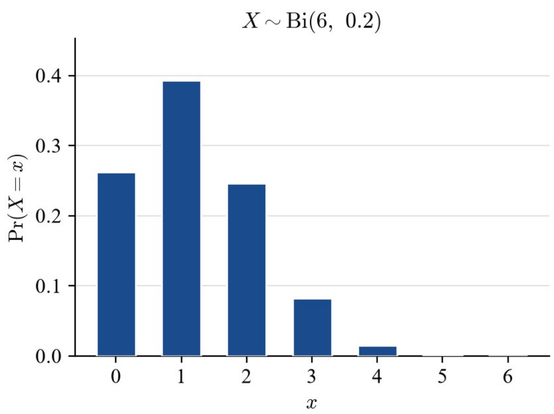
Here $\binom{n}{x}$ counts the ways of choosing which x of the n trials are the successes, p^x is the probability of those successes, and $(1-p)^{n-x}$ the probability of the failures.

Useful results.

- $Pr(X \geq 1)=1-Pr(X=0)$ ("at least one" via the complement);
- $Pr(X \leq k)=Pr(X=0)+Pr(X=1)+\cdots+Pr(X=k)$;
- **mean** $E(X)=np$; **variance** $Var(X)=np(1-p)$; **standard deviation** $sd(X)=\sqrt{np(1-p)}$.

The graph of the distribution is a bar graph of $Pr(X=x)$ against x. It is **symmetric** when $p=0.5$ and **skewed** otherwise.



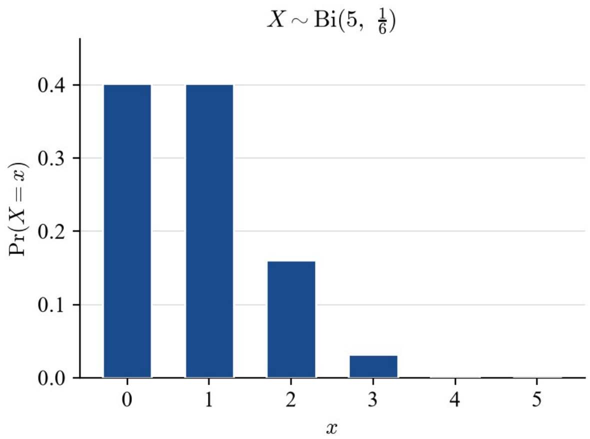


Worked examples

Example 1 — scaffolded

A fair die is rolled 5 times. Let X be the number of sixes. Find $Pr(X=2)$ and $Pr(X=0)$.

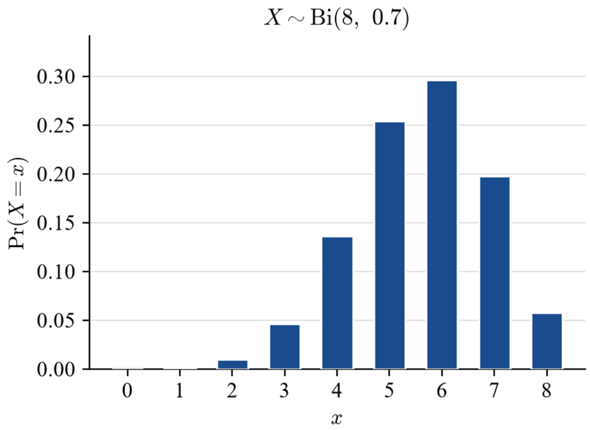
Working	Comment
Each roll is independent with $Pr(\text{six})=1/6$, and there are 5 rolls, so $X \sim Bi(5,1/6)$.	Fixed $n=5$, constant $p=1/6$, independent trials.
$Pr(X=2)=\binom{5}{2}(1/6)^2(5/6)^3=10\cdot 1/36\cdot 125/216=\frac{(}{3}$.	Apply the formula with $x=2$.
$Pr(X=0)=\binom{5}{0}(1/6)^0(5/6)^5=(5/6)^5=\frac{3125}{7776}\approx 0.4019$.	No sixes — all five rolls fail.



Example 2 — scaffolded

In a test, 70 % of students pass. A random sample of 8 students is taken. Let X be the number who pass. Find $Pr(X=6)$ and $Pr(X\geq 1)$, and state the mean of X.

Working	Comment
$X \sim Bi(8,0.7)$.	$n=8$, $p=0.7$ constant.
$Pr(X=6)=\binom{8}{6}(0.7)^6(0.3)^2=28\cdot 0.117649\cdot 0.09\approx ($.	$\binom{8}{6}=28$.
$Pr(X\geq 1)=1-Pr(X=0)=1-(0.3)^8\approx 1-0.0001=0.$.	Use the complement for “at least one”.
Mean $E(X)=np=8\times 0.7=5.6$.	On average 5.6 of the 8 pass.

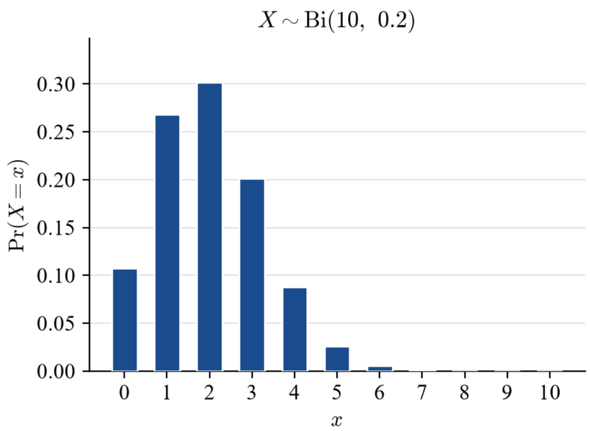


Example 3 — unscaffolded

$X \sim Bi(10, 0.2)$. Find $Pr(X=3)$ and the mean of X.

$$Pr(X=3)=\binom{10}{3}(0.2)^3(0.8)^7=120 \cdot 0.008 \cdot 0.2097152 \approx 0.2013.$$

$\therefore$ mean $E(X)=n\,p=10\times 0.2=2$.



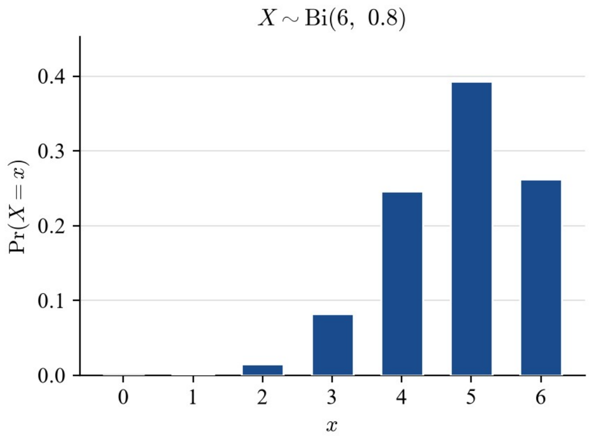
Example 4 — unscaffolded

A basketballer has a 0.8 chance of making each free throw and takes 6 shots. Let X be the number made. Find $Pr(X=6)$ and $Pr(X\geq 5)$.

$X \sim Bi(6, 0.8)$.

$$Pr(X=6)=(0.8)^6=\frac{4096}{15625}\approx 0.2621.$$

$$Pr(X\geq 5)=Pr(X=5)+Pr(X=6)=\binom{6}{5}(0.8)^5(0.2)+(0.8)^6=\frac{2048}{3125}\approx 0.6554.$$



Practice questions

Scaffolded

Q1. A fair coin is tossed 4 times; X is the number of heads, so $X \sim Bi(4, 1/2)$. (a) Find $Pr(X=2)$. (b) Find $Pr(X=4)$. (c) Find $Pr(X \geq 1)$.

Q2. For $X \sim Bi(6, 0.5)$: (a) find $Pr(X=3)$; (b) state the mean; (c) draw the bar graph of the distribution.

Q3. A machine produces items, 10% of which are faulty. A sample of 5 is taken; $X \sim Bi(5, 0.1)$. (a) Find $Pr(X=0)$. (b) Hence find $Pr(X \geq 1)$.

Q4. For $X \sim Bi(10, 0.9)$: (a) find $Pr(X=10)$; (b) state the mean.

Q5. A fair die is rolled 4 times; X is the number of sixes. Find $Pr(X=1)$.

Q6. For $X \sim Bi(8, 0.25)$: (a) find the mean np; (b) find the variance $np(1-p)$.

Unscaffolded

Give each probability correct to four decimal places (or as an exact fraction).

Q7. $X \sim Bi(7, 0.3)$. Find $Pr(X=2)$.

Q8. $X \sim Bi(10, 0.5)$. Find $Pr(X=5)$ and draw the bar graph of the distribution.

Q9. A fair coin is tossed 5 times. Find the probability of exactly 3 heads.

Q10. 20% of chocolates in a large batch are dark. Six are chosen with replacement. Find the probability that none are dark.

Q11. For the situation in **Q10**, find the probability that at least one is dark.

Q12. A student guesses each of 5 multiple-choice questions, each with 4 options. Find the probability of exactly 2 correct.

Q13. $X \sim Bi(8, 0.6)$. Find $Pr(X=5)$.

Q14. $X \sim Bi(20, 0.15)$. Find the mean and the standard deviation (to 4 d.p.).

Q15. $X \sim Bi(5, 0.4)$. Find $Pr(X \leq 1)$.

Q16. $X \sim Bi(6, 0.7)$. Find $Pr(X \geq 4)$.

Q17. State, with reasons, whether each is a binomial situation: (a) drawing 3 cards from a pack **without** replacement and counting the aces; (b) rolling a die 10 times and counting the fives.

Q18. $X \sim Bi(n, 0.5)$ has a mean of 4. Find n.

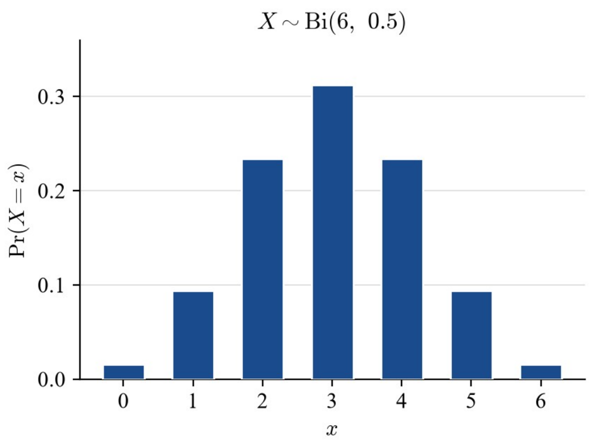
Answers

Q1. (a) $3/8 = 0.375$; (b) $1/16 = 0.0625$; (c) $15/16 = 0.9375$. **Q2.** (a) $5/16 = 0.3125$; (b) mean 3; (c) symmetric bar graph. **Q3.** (a) 0.5905; (b) 0.4095. **Q4.** (a) 0.3487; (b) mean 9. **Q5.** $125/324 \approx 0.3858$. **Q6.** (a) mean 2; (b) variance 1.5. **Q7.** 0.3177. **Q8.** $63/256 \approx 0.2461$. **Q9.** $5/16 = 0.3125$. **Q10.** $4096/15625 \approx 0.2621$. **Q11.** 0.7379. **Q12.** $135/512 \approx 0.2637$. **Q13.** 0.2787. **Q14.** mean 3, sd 1.5969. **Q15.** 0.3370. **Q16.** 0.7443. **Q17.** (a) not binomial (p changes without replacement); (b) binomial ($n = 10$, $p = 1/6$, independent). **Q18.** $n = 8$.

Fully-worked solutions

Q1. $X \sim Bi(4, 1/2)$. (a) $Pr(X=2) = \binom{4}{2}(1/2)^2(1/2)^2 = 6 \cdot 1/16 = 3/8$. (b) $Pr(X=4) = (1/2)^4 = 1/16$. (c) $Pr(X \geq 1) = 1 - Pr(X=0) = 1 - (1/2)^4 = 15/16$.

Q2. (a) $Pr(X=3) = \binom{6}{3}(1/2)^6 = 20 \cdot 1/64 = 20/64 = 5/16$. (b) Mean $= np = 6 \times 0.5 = 3$. (c) The distribution is symmetric about $x = 3$:



Q3. $X \sim Bi(5, 0.1)$. (a) $Pr(X=0) = (0.9)^5 = 59049/100000 \approx 0.5905$. (b) $Pr(X \geq 1) = 1 - 0.5905 = 0.4095$.

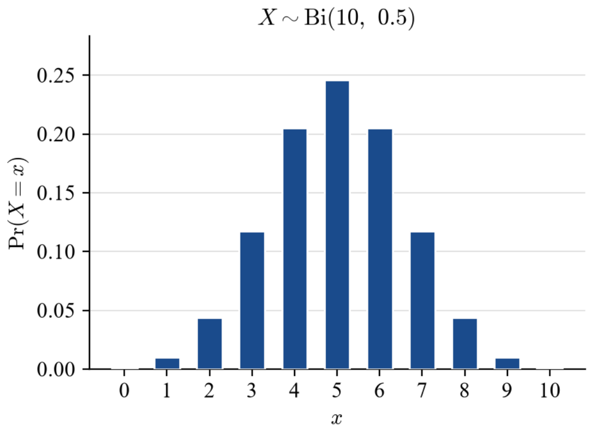
Q4. $X \sim Bi(10, 0.9)$. (a) $Pr(X=10) = (0.9)^{10} \approx 0.3487$. (b) Mean $= np = 10 \times 0.9 = 9$.

Q5. $X \sim Bi(4, 1/6)$. $Pr(X=1) = \binom{4}{1}(1/6)^1(5/6)^3 = 4 \cdot 1/6 \cdot 125/216 = 125/324 \approx 0.3858$.

Q6. $X \sim Bi(8, 0.25)$. (a) Mean $= np = 8 \times 0.25 = 2$. (b) Variance $= np(1-p) = 8 \times 0.25 \times 0.75 = 1.5$.

Q7. $Pr(X=2) = \binom{7}{2}(0.3)^2(0.7)^5 = 21 \cdot 0.09 \cdot 0.16807 \approx 0.3177$.

Q8. $Pr(X=5) = \binom{10}{5}(1/2)^{10} = 252 \cdot 1/1024 = 63/256 \approx 0.2461$. The distribution is symmetric about $x = 5$:



Q9. $X \sim Bi(5, 1/2)$; $Pr(X=3) = \binom{5}{3}(1/2)^5 = 10 \cdot 1/32 = 5/16 = 0.3125$.

Q10. $X \sim Bi(6, 0.2)$; $Pr(X=0) = (0.8)^6 = 4096/15625 \approx 0.2621$.

Q11. $Pr(X \ge 1) = 1 - Pr(X=0) = 1 - 0.2621 = 0.7379$.

Q12. $X \sim Bi(5, 1/4)$; $Pr(X=2) = \binom{5}{2}(1/4)^2(3/4)^3 = 10 \cdot 1/16 \cdot 27/64 = 135/512 \approx 0.2637$.

Q13. $Pr(X=5) = \binom{8}{5}(0.6)^5(0.4)^3 = 56 \cdot 0.07776 \cdot 0.064 \approx 0.2787$.

Q14. Mean $= np = 20 \times 0.15 = 3$; $sd = \sqrt{np(1-p)} = \sqrt{20 \times 0.15 \times 0.85} = \sqrt{2.55} \approx 1.5969$.

Q15. $Pr(X \le 1) = Pr(X=0) + Pr(X=1) = (0.6)^5 + \binom{5}{1}(0.4)(0.6)^4 = 0.07776 + 0.2592 = 0.3370$.

Q16. $Pr(X \ge 4) = Pr(4) + Pr(5) + Pr(6)$ where $X \sim Bi(6, 0.7)$:
$\binom{6}{4}(0.7)^4(0.3)^2 + \binom{6}{5}(0.7)^5(0.3) + (0.7)^6 = 0.324135 + 0.302526 + 0.117649 \approx 0.7443$.

Q17. (a) **Not binomial** — drawing without replacement changes the probability of an ace on each draw, so p is not constant. (b) **Binomial** — $n = 10$ fixed independent rolls, each a five/not-five with constant $p = 1/6$.

Q18. Mean $= np = n \times 0.5 = 4 \Rightarrow n = 8$.

Topic 2 Test A — Technology-active

Topic 2 covers Chapters 8–10 (probability and simulation, counting principles, discrete random variables). Reading time: 5 minutes. Writing time: 50 minutes. Total: 50 marks. One bound reference, one approved CAS calculator or CAS software, and one scientific calculator are permitted. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (4 marks)

One card is drawn at random from a standard 52-card deck.

a. Find $Pr(\text{queen})$. 1 mark

b. Find $Pr(\text{spade})$. 1 mark

c. Find $Pr(\text{queen or spade})$. 2 marks

Question 2 (5 marks)

A bag contains 6 red and 3 blue counters. Two counters are drawn at random **without replacement**.

a. Draw a tree diagram showing all the probabilities. 2 marks

b. Find $Pr(\text{both red})$. 1 mark

c. Find $Pr(\text{one of each colour})$. 2 marks

Question 3 (6 marks)

In a group of 40 students, 25 play tennis (T), 18 play netball (N), and 10 play both.

a. Represent this information in a Venn diagram. 2 marks

b. Find $Pr(T \cup N)$. 1 mark

c. Find $Pr(\text{neither sport})$. 1 mark

d. Find $Pr(\text{tennis only})$. 1 mark

e. Determine whether T and N are independent. 1 mark

Question 4 (6 marks)

200 people were surveyed about a product. The results are:

	Like (L)	Dislike (L')	Total
Male (M)	84	36	120
Female (M')	40	40	80
Total	124	76	200

a. Find $Pr(L)$. 1 mark

b. Find $Pr(M \cap L)$. 1 mark

c. Find $Pr(L / M)$. 2 marks

d. Are the events "Male" and "Like" independent? Justify your answer. 2 marks

Question 5 (5 marks)

A basketball player scores on 70% of her free-throw attempts. To simulate a sequence of attempts, single random digits (0–9) are used: the digits 0, 1, 2, 3, 4, 5 and 6 represent a score, and the digits 7, 8 and 9 represent a miss.

a. Explain why this assignment of digits correctly models a scoring rate of 70%. 1 mark

b. Using the random digits 4, 8, 2, 9, 7, 5 in order, simulate two trials of three attempts each, and state the number of scores in each trial. 2 marks

Twenty simulated trials of three attempts each gave the following numbers of scores.

Number of scores	0	1	2	3
Frequency	1	4	9	6

c. Use these simulation results to estimate the probability that the player scores on at least two of three attempts. 2 marks

Question 6 (5 marks)

A committee of 4 is chosen from 7 women and 5 men.

a. In how many ways can the committee be chosen? 1 mark

b. In how many ways can it be chosen with exactly 2 women and 2 men? 2 marks

c. In how many ways can it be chosen with at least one man? 2 marks

Question 7 (5 marks)

Five cards are dealt from a standard 52-card deck.

a. State the total number of possible 5-card hands. 1 mark

b. Find the probability that all five cards are hearts. 2 marks

c. Find the probability of a hand containing exactly two kings. 2 marks

Question 8 (7 marks)

The discrete random variable X has the distribution:

x	0	1	2	3
$Pr(X=x)$	0.15	k	0.25	0.2

a. Find the value of k. 1 mark

b. Find $Pr(X \geq 2)$. 1 mark

c. Find $Pr(X \geq 1 / X \leq 2)$. 2 marks

d. The value of X is recorded on two separate days, and the two recorded values are independent of each other. Find the probability that $X \geq 2$ on exactly one of the two days. 3 marks

Question 9 (7 marks)

A factory has two machines. Machine A produces 70% of the factory's components and Machine B produces the remaining 30%. Of the components produced by Machine A, 4% are faulty; of the components produced by Machine B, 9% are faulty. One component is selected at random from the factory's output. Let A be the event that it was produced by Machine A, and let F be the event that it is faulty.

a. Find $Pr(A \cap F)$. 1 mark

b. Show that $Pr(F)=0.055$. 2 marks

c. Given that the selected component is faulty, find the probability that it was produced by Machine B. 2 marks

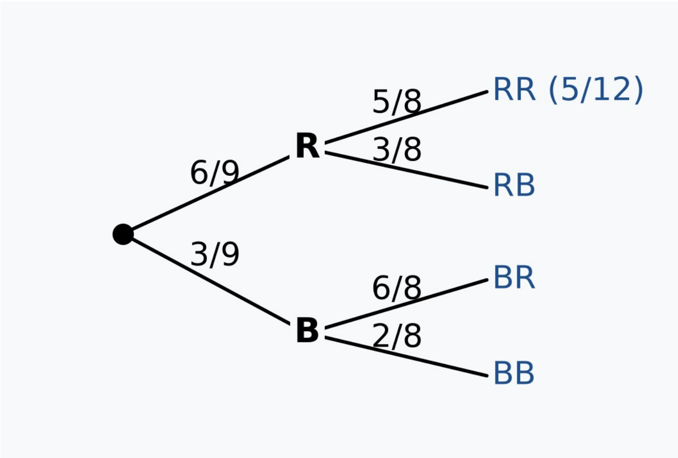
d. Determine, with justification, whether the events A and F are independent. 2 marks

End of Topic 2 Test A

Topic 2 Test A — solutions

Q1. (a) $Pr(\text{queen}) = \frac{4}{52} = \frac{1}{13}$. (b) $Pr(\text{spade}) = \frac{13}{52} = \frac{1}{4}$. (c) $Pr(\text{queen or spade}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$.

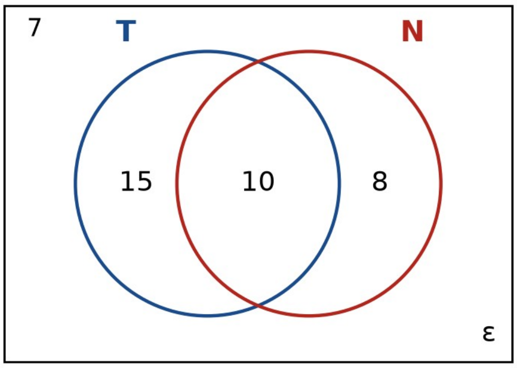
Q2. (a)



(b) $Pr(\text{both red}) = \frac{6}{9} \times \frac{5}{8} = \frac{5}{12}$.

(c) $Pr(\text{one of each}) = \frac{6}{9} \times \frac{3}{8} + \frac{3}{9} \times \frac{6}{8} = \frac{18}{72} + \frac{18}{72} = \frac{1}{2}$.

Q3. (a) Tennis only 15, both 10, netball only 8, neither 7:



(b) $Pr(T \cup N) = \frac{15+10+8}{40} = \frac{33}{40}$. (c) $Pr(\text{neither}) = \frac{7}{40}$. (d) $Pr(T\text{ only}) = \frac{15}{40} = \frac{3}{8}$.

(c) $Pr(T)Pr(N) = \frac{25}{40} \times \frac{18}{40} = \frac{9}{32}$, but $Pr(T \cap N) = \frac{10}{40} = \frac{1}{4} = \frac{8}{32}$. Since $\frac{9}{32} \neq \frac{8}{32}$, T and N are **not independent**.

Q4. (a) $Pr(L) = \frac{124}{200} = \frac{31}{50}$. (b) $Pr(M \cap L) = \frac{84}{200} = \frac{21}{50}$. (c) $Pr(L/M) = \frac{84}{120} = \frac{7}{10}$. (d) $Pr(L) = \frac{31}{50} = 0.62$ but $Pr(L/M) = \frac{7}{10} = 0.7$. Since $Pr(L/M) \neq Pr(L)$, the events are **not independent**.

Q5. (a) Seven of the ten equally likely digits (0, 1, 2, 3, 4, 5, 6) represent a score, so the model gives $Pr(\text{score}) = \frac{7}{10} = 0.7$, which matches the 70% scoring rate. (b) Trial 1 uses the digits 4, 8, 2: score, miss, score, so **2 scores**. Trial 2 uses the digits 9, 7, 5: miss, miss, score, so **1 score**. (c) At least two scores occurred in $9+6=15$ of the 20 trials, $\therefore$ the estimated probability is $\frac{15}{20} = \frac{3}{4}$.

Q6. (a) $\binom{12}{4}=495$. (b) $\binom{7}{2}\binom{5}{2}=21\times 10=210$. (c) at least one man = total − all women $=495-\binom{7}{4}=495-35=460$.

Q7. (a) $\binom{52}{5}=2\,598\,960$. (b) $Pr(\text{all hearts})=\frac{\binom{13}{5}}{\binom{52}{5}}=\frac{1287}{2\,598\,960}=\frac{33}{66\,640}$. (c)
$Pr(\text{exactly 2 kings})=\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}=\frac{6\times 17\,296}{2\,598\,960}=\frac{2162}{54\,145}$.

Q8. (a) $0.15+k+0.25+0.2=1\Rightarrow k=0.4$. (b) $Pr(X\geq 2)=0.25+0.2=0.45$. (c)
$Pr(X\geq 1/X\leq 2)=\frac{Pr(1\leq X\leq 2)}{Pr(X\leq 2)}=\frac{0.4+0.25}{0.15+0.4+0.25}=\frac{0.65}{0.8}=\frac{13}{16}$. (d) On each day, $Pr(X\geq 2)=0.25+0.2=0.45$ and $Pr(X\leq 1)=0.55$. The days are independent, so
$Pr(\text{exactly one day with } X\geq 2)=0.45\times 0.55+0.55\times 0.45=0.2475+0.2475=0.495$.

Q9. (a) $Pr(A\cap F)=0.7\times 0.04=0.028$. (b)
$Pr(F)=Pr(A\cap F)+Pr(A'\cap F)=0.7\times 0.04+0.3\times 0.09=0.028+0.027=0.055$. (c)
$Pr(A'/F)=\frac{Pr(A'\cap F)}{Pr(F)}=\frac{0.027}{0.055}=\frac{27}{55}$. (d) $Pr(A)Pr(F)=0.7\times 0.055=0.0385$, but $Pr(A\cap F)=0.028$. Since $0.028\neq 0.0385$, the events A and F are **not independent**.

Topic 2 Test B — Technology-free

Topic 2 covers Chapters 8–10 (probability and simulation, counting principles, discrete random variables). Reading time: 5 minutes. Writing time: 45 minutes. Total: 40 marks. Students are not permitted to bring any technology (calculators or software), or notes of any kind. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (4 marks)

Two fair six-sided dice are rolled.

a. Find $Pr(\text{sum} = 8)$. 1 mark

b. Find $Pr(\text{a double})$. 1 mark

c. Find $Pr(\text{sum is a multiple of } 3)$. 2 marks

Question 2 (5 marks)

A box contains 3 white and 2 black balls. Two balls are drawn at random **without replacement**.

a. Draw a tree diagram. 2 marks

b. Find $Pr(\text{both white})$. 1 mark

c. Find $Pr(\text{the two balls are different colours})$. 2 marks

Question 3 (5 marks)

For two events A and B, $Pr(A) = 3/5$, $Pr(B) = 1/3$ and $Pr(A \cap B) = 1/5$.

a. Find $Pr(A \cup B)$. 2 marks

b. Find $Pr(A \cap B')$. 1 mark

c. Determine whether A and B are independent. 2 marks

Question 4 (5 marks)

The two-way table gives probabilities for events A and B.

Pr	B	B'
A	0.3	0.1
A'	0.2	0.4

a. Find $Pr(A)$. 1 mark

b. Find $Pr(A / B)$. 2 marks

c. Find $Pr(B / A)$. 2 marks

Question 5 (5 marks)

A book club has 5 members.

a. In how many ways can 2 of the 5 members be chosen to host the next meeting? 1 mark

b. Each of the 5 members votes for one of 3 shortlisted novels. In how many different ways can the five votes be cast? 2 marks

c. In how many ways can 3 of the 5 members be chosen if a particular member must be included? 2 marks

Question 6 (6 marks)

A gift box contains 7 chocolates: 4 dark and 3 milk. Three chocolates are selected at random from the box.

a. In how many ways can the three chocolates be selected? 1 mark

b. Find the probability that all three chocolates are dark. 2 marks

c. Find the probability that exactly one of the chocolates is milk. 2 marks

d. Find the probability that at least one of the chocolates is milk. 1 mark

Question 7 (5 marks)

The discrete random variable X has the distribution:

x	1	2	3	4
$Pr(X=x)$	$1/8$	$3/8$	$1/4$	$1/4$

a. Show that the probabilities sum to 1. 1 mark

b. Find $Pr(X \leq 2)$. 1 mark

c. Find $Pr(X \geq 3 / X \geq 2)$. 2 marks

d. Find the probability that the value of X is odd. 1 mark

Question 8 (5 marks)

Box P contains 2 red discs and 3 green discs. Box Q contains 4 red discs and 1 green disc. A fair coin is tossed: if it lands heads, one disc is drawn at random from Box P; if it lands tails, one disc is drawn at random from Box Q.

a. Find the probability that a green disc is drawn, given that the coin lands tails. 1 mark

b. Find the probability that a red disc is drawn. 2 marks

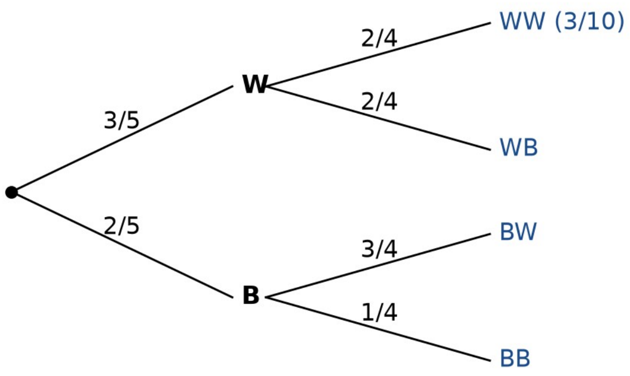
c. Given that a red disc is drawn, find the probability that the coin landed heads. 2 marks

End of Topic 2 Test B

Topic 2 Test B — solutions

Q1. (a) The favourable outcomes are $(2,6),(3,5),(4,4),(5,3),(6,2)$: $Pr=\frac{5}{36}$. (b) Doubles: $Pr=\frac{6}{36}=\frac{1}{6}$. (c) Sums that are multiples of 3 are $3,6,9,12$, occurring $2+5+4+1=12$ times: $Pr=\frac{12}{36}=\frac{1}{3}$.

Q2. (a)



(b) $Pr(\text{both white})=\frac{3}{5}\times\frac{2}{4}=\frac{3}{10}$.

(c) $Pr(\text{different})=\frac{3}{5}\times\frac{2}{4}+\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}+\frac{6}{20}=\frac{3}{5}$.

Q3. (a) $Pr(A\cup B)=\frac{3}{5}+\frac{1}{3}-\frac{1}{5}=\frac{9}{15}+\frac{5}{15}-\frac{3}{15}=\frac{11}{15}$. (b) $Pr(A\cap B')=Pr(A)-Pr(A\cap B)=\frac{3}{5}-\frac{1}{5}=\frac{2}{5}$. (c) $Pr(A)Pr(B)=\frac{3}{5}\times\frac{1}{3}=\frac{1}{5}=Pr(A\cap B)$. $\therefore$ A and B **are independent**.

Q4. (a) $Pr(A)=0.3+0.1=0.4$. (b) $Pr(A/B)=\frac{Pr(A\cap B)}{Pr(B)}=\frac{0.3}{0.5}=\frac{3}{5}$. (c) $Pr(B/A)=\frac{Pr(A\cap B)}{Pr(A)}=\frac{0.3}{0.4}=\frac{3}{4}$.

Q5. (a) $\binom{5}{2}=10$. (b) By the multiplication principle, each of the five members independently has 3 choices: $3\times3\times3\times3\times3=3^5=243$. (c) The particular member is included, so the remaining 2 are chosen from the other 4: $\binom{4}{2}=6$.

Q6. (a) $\binom{7}{3}=35$. (b) $Pr(\text{all dark})=\frac{\binom{4}{3}}{\binom{7}{3}}=\frac{4}{35}$. (c) $Pr(\text{exactly one milk})=\frac{\binom{3}{1}\binom{4}{2}}{\binom{7}{3}}=\frac{3\times6}{35}=\frac{18}{35}$. (d) $Pr(\text{at least one milk})=1-Pr(\text{all dark})=1-\frac{4}{35}=\frac{31}{35}$.

Q7. (a) $\frac{1}{8}+\frac{3}{8}+\frac{1}{4}+\frac{1}{4}=\frac{1}{8}+\frac{3}{8}+\frac{2}{8}+\frac{2}{8}=\frac{8}{8}=1$. ✓ (b) $Pr(X\le2)=\frac{1}{8}+\frac{3}{8}=\frac{4}{8}=\frac{1}{2}$. (c) $Pr(X\ge3/X\ge2)=\frac{Pr(X\ge3)}{Pr(X\ge2)}=\frac{\frac{1}{4}+\frac{1}{4}}{\frac{3}{8}+\frac{1}{4}+\frac{1}{4}}=\frac{\frac{1}{2}}{\frac{7}{8}}=\frac{4}{7}$. (d) $Pr(X\text{ is odd})=Pr(X=1)+Pr(X=3)=\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$.

Q8. (a) Box Q contains 1 green disc out of 5, so $Pr(\text{green} / \text{tails}) = \frac{1}{5}$. (b)

$Pr(\text{red}) = Pr(\text{heads})Pr(\text{red} / \text{heads}) + Pr(\text{tails})Pr(\text{red} / \text{tails}) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{5} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$. (c)

$$Pr(\text{heads} / \text{red}) = \frac{Pr(\text{heads} \cap \text{red})}{Pr(\text{red})} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}.$$