Mathematical Methods Units 1 & 2

Chapter 9 — Counting principles

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Chapter 9 Counting principles — 9A The addition and multiplication principles for counting

Theory

Counting techniques let us find the number of ways a task can be done **without listing every possibility**. Two principles do most of the work.

The multiplication principle. If a task is completed in a sequence of stages, and there are n_1 choices for the first stage, n_2 for the second, and so on, then the total number of ways to complete the task is

$$n_1 \times n_2 \times n_3 \times \cdots$$

A neat way to apply this is the **box (slot) method**: draw a box for each stage and write the number of choices in it, then multiply. For example, the number of three-letter arrangements from 5 different letters with **no repeats** is

$$\underline{5}\,\underline{4}\,\underline{3} = 60.$$

The addition principle. If a task can be carried out in one of several **mutually exclusive** ways (cases that cannot happen together), the total number of ways is the **sum** of the numbers of ways for each case.

"And" vs "or". As a rule of thumb, *and* (do this stage **and** then that stage) suggests **multiplying**, while *or* (do this **or** that, with no overlap) suggests **adding**.

Restrictions. When some positions are restricted (for example, the first digit cannot be 0, or a particular seat is fixed), deal with the **most restricted stage first**. Watch whether repetition is allowed: with repeats, each box keeps the full number of choices; with no repeats, the number of choices drops by one at each stage.

Worked examples

Example 1 — scaffolded

A café offers 4 entrées, 5 main courses and 3 desserts. How many different three-course meals are possible?

Working	Comment
A meal is chosen in three stages: entrée, then main, then dessert.	"And" between stages ⇒ multiply.
$\underline{4}\,\underline{5}\,\underline{3}$	Box method: choices at each stage.
Total $= 4 \times 5 \times 3 = 60$.	Multiplication principle.

∴ there are 60 possible meals.

Example 2 — scaffolded

Three-digit numbers are formed using the digits $1, 2, 3, 4, 5$. How many are possible (a) if digits may be repeated; (b) if no digit may be repeated?

Working	Comment
(a) Repeats allowed: each of the three positions has 5 choices.	The pool of digits does not shrink.
$\underline{5}\,\underline{5}\,\underline{5} = 5^3 = 125$.	
(b) No repeats: after using a digit it cannot be used again.	Choices drop by one each stage.
$\underline{5}\,\underline{4}\,\underline{3} = 60$.	

Example 3 — unscaffolded

To travel from town A to town C there are 3 roads from A to B and 2 roads from B to C, as well as 4 direct roads from A to C. In how many ways can the trip be made?

Via B (a road $A \to B$ **and** a road $B \to C$): $3 \times 2 = 6$ ways.

Direct $A \to C$: 4 ways.

These cases are mutually exclusive (a trip is either via B **or** direct), so add:

$$6 + 4 = 10.$$

$\therefore$ there are 10 ways to make the trip.

Example 4 — unscaffolded

A four-digit PIN is formed using the digits 0–9, with repetition allowed. (a) How many PINs are possible? (b) How many end in an even digit?

(a) Each of the four positions has 10 choices: $\underline{10}\,\underline{10}\,\underline{10}\,\underline{10} = 10^4 = 10\,000.$

(b) The last digit must be even ($0, 2, 4, 6, 8$ — 5 choices); the first three are free:

$$10 \times 10 \times 10 \times 5 = 5000.$$

$\therefore$ there are $10\,000$ PINs, of which 5000 end in an even digit.

Practice questions

Scaffolded

Q1. A set menu has 2 entrées, 4 mains and 3 desserts. How many three-course meals are possible?

Q2. Two-letter codes are formed from the letters A, B, C, D. How many are possible (a) if letters may be repeated; (b) if no letter may be repeated?

Q3. A coin is tossed and then a standard die is rolled. How many outcomes are in the sample space?

Q4. Three-digit numbers are formed from the digits $1, 2, 3, 4$ with no repetition. How many are possible?

Q5. A person may wear one of 5 shirts with one of 3 ties, **or** instead wear one of 4 dresses. In how many ways can they dress?

Q6. A code has two letters (from 26) followed by two digits (from 0–9), with repetition allowed. How many codes are possible?

Unscaffolded

Q7. A restaurant offers 4 starters, 6 mains, 3 desserts and 2 drinks. How many four-part meals are possible?

Q8. How many three-digit numbers can be formed from the digits 1–9 if repetition is allowed?

Q9. How many three-digit numbers can be formed from the digits 1–9 with no repetition?

Q10. How many four-letter "words" (any arrangement) can be formed from the 26 letters if repetition is allowed?

Q11. How many four-digit PINs can be formed from 0–9 if no digit is repeated?

Q12. There are 3 roads from P to Q and 4 roads from Q to R. In how many ways can a person travel from P to R via Q?

Q13. There are 2 roads from A to B and 3 roads from B to C, as well as 5 direct roads from A to C. In how many ways can a person travel from A to C?

Q14. How many five-digit numbers can be formed from 0–9 (repetition allowed) if the first digit cannot be 0?

Q15. How many three-digit numbers (from 100 to 999) are even?

Q16. A licence plate consists of 3 letters followed by 3 digits, with no repetition among the letters and no repetition among the digits. How many plates are possible?

Q17. A multiple-choice test has 5 questions, each with 4 options. In how many different ways can the test be answered?

Q18. From a club of 8 members, a chairperson, a treasurer and a secretary are chosen (all different people). In how many ways can this be done?

Answers

Q1. 24. **Q2.** (a) 16; (b) 12. **Q3.** 12. **Q4.** 24. **Q5.** 19. **Q6.** $67\,600$. **Q7.** 144. **Q8.** 729. **Q9.** 504. **Q10.** $456\,976$. **Q11.** 5040. **Q12.** 12. **Q13.** 11. **Q14.** $90\,000$. **Q15.** 450. **Q16.** $11\,232\,000$. **Q17.** 1024. **Q18.** 336.

Fully-worked solutions

Q1. Three stages: $2 \times 4 \times 3 = 24$.

Q2. (a) Repeats allowed: $4 \times 4 = 16$. (b) No repeats: $4 \times 3 = 12$.

Q3. Coin **and** die: $2 \times 6 = 12$.

Q4. No repeats: $\underline{4}\,\underline{3}\,\underline{2} = 4 \times 3 \times 2 = 24$.

Q5. Shirt **and** tie: $5 \times 3 = 15$; **or** a dress: 4. Add: $15 + 4 = 19$.

Q6. $\underline{26}\,\underline{26}\,\underline{10}\,\underline{10} = 26^2 \times 10^2 = 676 \times 100 = 67\,600$.

Q7. $4 \times 6 \times 3 \times 2 = 144$.

Q8. Repeats allowed: $9 \times 9 \times 9 = 9^3 = 729$.

Q9. No repeats: $9 \times 8 \times 7 = 504$.

Q10. $26 \times 26 \times 26 \times 26 = 26^4 = 456\,976$.

Q11. No repeats: $10 \times 9 \times 8 \times 7 = 5040$.

Q12. Via Q: $3 \times 4 = 12$.

Q13. Via B: $2 \times 3 = 6$; **or** direct: 5. Add: $6 + 5 = 11$.

Q14. First digit: 9 choices (1–9); the other four: 10 each: $9 \times 10^4 = 90\,000$.

Q15. First digit 1–9 (9), second 0–9 (10), last digit even $\{0, 2, 4, 6, 8\}$ (5): $9 \times 10 \times 5 = 450$.

Q16. Letters (no repeats) $26 \times 25 \times 24 = 15\,600$; digits (no repeats) $10 \times 9 \times 8 = 720$. Multiply: $15\,600 \times 720 = 11\,232\,000$.

Q17. Each of 5 questions has 4 choices: $4^5 = 1024$.

Q18. Three distinct positions from 8 people: $8 \times 7 \times 6 = 336$.

Chapter 9 Counting principles — 9B Arrangements and permutations

Beyond the study design. This section goes beyond the Units 1 & 2 study design, which prescribes counting principles and combinations only. Arrangements and permutations belong to Specialist Mathematics rather than Mathematical Methods; the section is included as extension material and is not assessed in the graded topic tests or practice examinations in this series.

Theory

An **arrangement** (or **permutation**) is an ordering of objects. Order matters — AB and BA are different arrangements.

Factorials. The product $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ is read "n factorial". By definition $0! = 1$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Arranging n distinct objects in a row. There are n choices for the first place, then $n-1$ for the second, and so on, giving

$$n! \text{ arrangements.}$$

Arranging r of n distinct objects. Filling r places from n objects gives ${}^{n}\mathrm{P}_{r} = \dfrac{n!}{(n-r)!} = n \times (n-1) \times \cdots \times (n-r+1) \, (r \text{ factors})$.

Restrictions.

- **Objects that must be together:** treat the group as a single **block**, arrange the blocks, then multiply by the arrangements *within* the block.
- **Objects that must not be together:** subtract the "together" arrangements from the total — (not together) = (total) − (together).
- **A fixed position:** place the restricted object first, then arrange the rest.

Arrangements around a circle. For n objects around a circle, fixing one object to remove equivalent rotations leaves

$$(n-1)! \text{ arrangements.}$$

Objects that are identical. If n objects include p alike of one kind, q alike of another, and so on, the number of distinct arrangements is

$$\frac{n!}{p!\,q!\cdots}.$$

Worked examples

Example 1 — scaffolded

(a) In how many ways can 6 different books be arranged in a row on a shelf?
(b) In how many ways can 3 of 7 people be arranged in a line?

Working	Comment
(a) 6 distinct objects in a row: $6! = 720$.	Arranging all n distinct objects gives $n!$.
(b) ${}^{7}\mathrm{P}_{3} = \dfrac{7!}{4!} = 7 \times 6 \times 5 = 210$.	Filling $r = 3$ places from $n = 7$: use ${}^{n}\mathrm{P}_{r}$.

Example 2 — scaffolded

Five people are arranged in a row. In how many ways can this be done if two particular people must sit together?

Working	Comment
Treat the two people as one **block**. There are now 4 units to arrange: $4!=24$.	Objects that must be together become a single block.
The two people can be ordered within the block in $2!=2$ ways.	Arrange within the block.
Total $=4!\times 2!=24\times 2=48$.	Multiply the two independent choices.

Example 3 — unscaffolded

How many distinct arrangements are there of the letters of the word **BANANA**?

BANANA has 6 letters: A appears 3 times, N appears 2 times, B appears once.

$$\frac{6!}{3!2!}=\frac{720}{6\times 2}=60.$$

∴ there are 60 distinct arrangements.

Example 4 — unscaffolded

Six people are seated around a circular table. (a) In how many ways can they be seated? (b) In how many ways if two particular people sit together? (c) In how many ways if those two people do **not** sit together?

(a) Around a circle: $(6-1)!=5!=120$.

(b) Treat the two as a block, leaving 5 units around the circle: $(5-1)!\times 2!=4!\times 2=48$.

(c) (not together) = (total) − (together) $=120-48=72$.

Practice questions

Scaffolded

Q1. Evaluate $7!$.

Q2. Evaluate $^{8}P_{3}$.

Q3. In how many ways can 5 different paintings be hung in a row?

Q4. In how many ways can 4 of 9 people be seated in a row?

Q5. How many arrangements are there of the letters of the word **NUMBER**?

Q6. In how many ways can the letters of **MATHS** be arranged if M and S must be together?

Unscaffolded

Q7. In how many ways can 8 different trophies be arranged in a row?

Q8. Evaluate $^{10}P_{4}$.

Q9. How many distinct arrangements are there of the letters of **KANGAROO**?

Q10. How many distinct arrangements are there of the letters of **MISSISSIPPI**?

Q11. Six people stand in a row. In how many ways can this be done if two particular people must stand together?

Q12. For the six people in Q11, in how many ways can they stand if those two particular people must **not** stand together?

Q13. In how many ways can 7 books be arranged in a row if 3 particular books must be kept together?

Q14. In how many ways can 5 boys and 3 girls be arranged in a row if all the girls must be together?

Q15. In how many ways can 8 people be seated around a circular table?

Q16. In how many ways can 8 people be seated around a circular table if two particular people must sit together?

Q17. How many 4-digit numbers can be formed from the digits $1, 2, 3, 4, 5, 6, 7$ if no digit is repeated?

Q18. How many distinct arrangements are there of the letters of **DECEMBER**?

Answers

Q1. 5040. **Q2.** 336. **Q3.** 120. **Q4.** 3024. **Q5.** 720. **Q6.** 48. **Q7.** 40 320. **Q8.** 5040. **Q9.** 10 080. **Q10.** 34 650. **Q11.** 240. **Q12.** 480. **Q13.** 720. **Q14.** 4320. **Q15.** 5040. **Q16.** 1440. **Q17.** 840. **Q18.** 6720.

Fully-worked solutions

Q1. $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

Q2. ${}^{8}P_{3} = \dfrac{8!}{5!} = 8 \times 7 \times 6 = 336$.

Q3. 5 distinct paintings in a row: $5! = 120$.

Q4. ${}^{9}P_{4} = 9 \times 8 \times 7 \times 6 = 3024$.

Q5. NUMBER has 6 distinct letters: $6! = 720$.

Q6. Treat M and S as a block: 4 units give $4!$, and the block orders in $2!$ ways, so $4! \times 2! = 24 \times 2 = 48$.

Q7. $8! = 40\,320$.

Q8. ${}^{10}P_{4} = 10 \times 9 \times 8 \times 7 = 5040$.

Q9. KANGAROO: $A \times 2$, $O \times 2$, so $\dfrac{8!}{2!2!} = \dfrac{40\,320}{4} = 10\,080$.

Q10. MISSISSIPPI has 11 letters: $I \times 4$, $S \times 4$, $P \times 2$, $M \times 1$, so

$$\frac{11!}{4!4!2!} = \frac{39\,916\,800}{24 \times 24 \times 2} = 34\,650.$$

Q11. Two particular people together: block gives 5 units, so $5! \times 2! = 120 \times 2 = 240$.

Q12. (not together) $= 6! - 240 = 720 - 240 = 480$.

Q13. Three books together: block gives 5 units, so $5! \times 3! = 120 \times 6 = 720$.

Q14. All 3 girls together: block gives 6 units (5 boys + 1 block), so $6! \times 3! = 720 \times 6 = 4320$.

Q15. Around a circle: $(8-1)! = 7! = 5040$.

Q16. Two people together around a circle: block gives 7 units, so $(7-1)! \times 2! = 6! \times 2 = 1440$.

Q17. Choosing and ordering 4 of 7 digits: ${}^{7}P_{4} = 7 \times 6 \times 5 \times 4 = 840$.

Q18. DECEMBER has 8 letters with $E \times 3$: $\dfrac{8!}{3!} = \dfrac{40\,320}{6} = 6720$.

Chapter 9 Counting principles — 9C Selections and combinations

Theory

A **selection** (or **combination**) is a choice of objects in which the **order does not matter**. Choosing Anna, Ben and Cara for a committee is the *same* selection as choosing Cara, Ben and Anna.

The number of selections. The number of ways of selecting r objects from n distinct objects is ${}^nC_r$ $=\binom{n}{r}=\frac{n!}{r!(n-r)!}$.

Selections versus arrangements. An arrangement counts different orders separately. Each selection of r objects can be ordered in $r!$ ways, so a count that treats the orderings as different counts each selection $r!$ times; dividing by $r!$ removes the orderings and leaves the number of selections.

Useful values and properties.

- $\binom{n}{0}=1$, $\binom{n}{n}=1$, and $\binom{n}{1}=n$.
- **Symmetry:** $\binom{n}{r}=\binom{n}{n-r}$ — choosing the r to include is the same as choosing the $n-r$ to leave out.
- **Pascal's rule:** $\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$, which generates Pascal's triangle.

Selections with restrictions.

- **Must include** certain objects: place them in the selection first, then count the ways to choose the remaining places from the objects that are left.
- **Must exclude** certain objects: remove them from the pool, then select as usual.
- **From several groups:** count the selections from each group and **multiply**.
- **"At least" / "at most":** either add the counts of the separate allowed cases, or subtract the unwanted cases from the total.

Worked examples

Example 1 — scaffolded

In how many ways can a committee of 3 be chosen from 8 people?

Working	Comment
Order does not matter, so this is a selection of 3 from 8.	A committee has no ranked positions.
$\binom{8}{3}=\frac{8!}{3!5!}$.	Use $\binom{n}{r}=\frac{n!}{r!(n-r)!}$ with $n=8$, $r=3$.
$=\frac{8\times7\times6}{3\times2\times1}=\frac{336}{6}=56$.	Cancel $5!$, then evaluate.

∴ there are 56 ways.

Example 2 — scaffolded

A committee of 4 is to be chosen from 6 men and 4 women, and must contain exactly 2 men and 2 women. In how many ways can this be done?

Working	Comment
Choose the 2 men from 6: $\binom{6}{2}=15$.	Selection from the group of men.
Choose the 2 women from 4: $\binom{4}{2}=6$.	Selection from the group of women.
Multiply the independent choices: $15\times 6=90$.	Selections from separate groups multiply.

$\therefore$ there are 90 ways.

Example 3 — unscaffolded

A group of 5 is to be selected from 13 people, but 2 particular people must both be included. In how many ways can the group be formed?

Place the 2 required people in the group first. This leaves 3 places to fill from the remaining 11 people:

$$\binom{11}{3}=\frac{11\times 10\times 9}{3\times 2\times 1}=165.$$

$\therefore$ there are 165 ways.

Example 4 — unscaffolded

How many 5-card hands dealt from a standard deck of 52 cards contain exactly 3 hearts?

Choose the 3 hearts from the 13 hearts, and the other 2 cards from the 39 non-hearts:

$$\binom{13}{3}\times\binom{39}{2}=286\times 741=211\,926.$$

$\therefore$ there are $211\,926$ such hands.

Practice questions

Scaffolded

Q1. Evaluate $\binom{7}{2}$.

Q2. Evaluate $\binom{10}{4}$.

Q3. In how many ways can 3 books be chosen from 9 different books?

Q4. A team of 5 is chosen from 12 players. In how many ways can this be done?

Q5. From 5 red balls and 4 blue balls, in how many ways can 2 red and 2 blue balls be chosen?

Q6. Show that $\binom{9}{4}=\binom{9}{5}$, and state the property this illustrates.

Unscaffolded

Q7. Evaluate $\binom{11}{9}$.

Q8. In how many ways can 4 people be selected from 11?

Q9. A committee of 6 is chosen from 15 people. In how many ways can this be done?

Q10. A committee of 4 is chosen from 8 women and 5 men and must contain 2 women and 2 men. In how many ways can it be formed?

Q11. A team of 5 is chosen from 9 players, and the captain (one particular player) must be included. In how many ways can the team be chosen?

Q12. From 10 people, a group of 4 is chosen. In how many ways can this be done if two particular people cannot both be in the group?

Q13. How many 5-card hands from a standard deck contain all four aces?

Q14. How many 5-card hands contain at least 4 hearts?

Q15. From 6 mathematics books and 5 science books, 4 books are chosen. In how many ways can this be done if at least 3 of them are mathematics books?

Q16. How many diagonals does a regular decagon (10 sides) have?

Q17. Twelve points are marked on a page, with no three in a straight line. How many triangles can be formed using these points as vertices?

Q18. A group of 10 friends will split into a committee of 3 and a remaining group. In how many ways can the committee be chosen?

Answers

Q1. 21. **Q2.** 210. **Q3.** 84. **Q4.** 792. **Q5.** 60. **Q6.** Both equal 126 (symmetry $\binom{n}{r}=\binom{n}{n-r}$). **Q7.** 55. **Q8.** 330. **Q9.** 5005. **Q10.** 280. **Q11.** 70. **Q12.** 182. **Q13.** 48. **Q14.** 29 172. **Q15.** 115. **Q16.** 35. **Q17.** 220. **Q18.** 120.

Fully-worked solutions

Q1. $\binom{7}{2}=\frac{7\times 6}{2\times 1}=21$.

Q2. $\binom{10}{4}=\frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}=\frac{5040}{24}=210$.

Q3. Order does not matter: $\binom{9}{3}=\frac{9\times 8\times 7}{3\times 2\times 1}=84$.

Q4. $\binom{12}{5}=\frac{12\times 11\times 10\times 9\times 8}{5!}=\frac{95\,040}{120}=792$.

Q5. Choose 2 red from 5 and 2 blue from 4: $\binom{5}{2}\times\binom{4}{2}=10\times 6=60$.

Q6. $\binom{9}{4}=\frac{9!}{4!5!}=126$ and $\binom{9}{5}=\frac{9!}{5!4!}=126$, so they are equal. This illustrates the symmetry property $\binom{n}{r}=\binom{n}{n-r}$.

Q7. $\binom{11}{9}=\binom{11}{2}=\frac{11\times 10}{2\times 1}=55$ (using symmetry to simplify).

Q8. $\binom{11}{4}=\frac{11\times 10\times 9\times 8}{4!}=\frac{7920}{24}=330$.

Q9. $\binom{15}{6}=\frac{15!}{6!9!}=5005$.

Q10. Choose 2 women from 8 and 2 men from 5: $\binom{8}{2}\times\binom{5}{2}=28\times 10=280$.

Q11. The captain is fixed, so choose the other 4 from the remaining 8: $\binom{8}{4}=70$.

Q12. Total groups $\binom{10}{4}=210$. Groups containing **both** particular people: fix the two, choose 2 more from 8, giving $\binom{8}{2}=28$. Subtract: $210-28=182$.

Q13. All four aces are fixed; choose the remaining 1 card from the other 48: $\binom{4}{4}\times\binom{48}{1}=1\times 48=48$.

Q14. "At least 4 hearts" means exactly 4 or exactly 5:

$$\binom{13}{4}\binom{39}{1}+\binom{13}{5}=715\times 39+1287=27\,885+1287=29\,172.$$

Q15. "At least 3 mathematics" means 3 maths $+1$ science, or 4 maths $+0$ science:

$$\binom{6}{3}\binom{5}{1}+\binom{6}{4}\binom{5}{0}=20\times5+15\times1=100+15=115.$$

Q16. Each pair of vertices gives a line; $\binom{10}{2}=45$ lines. Of these, 10 are the sides of the decagon, so the number of diagonals is $45-10=35$.

Q17. Each triangle is determined by a selection of 3 of the 12 points (no three collinear, so every selection gives a genuine triangle): $\binom{12}{3}=220$.

Q18. Choosing the committee of 3 automatically determines the remaining group, so there are $\binom{10}{3}=\frac{10\times9\times8}{3\times2\times1}=120$ ways.

Chapter 9 Counting principles — 9D Applying counting techniques to probability

Units 3 & 4 preview (in part). Items in this section that use arrangements or permutations go beyond the Units 1 & 2 study design, which prescribes counting principles and combinations only; they are included as preparation for Units 3 & 4 and are not assessed in the graded topic tests or practice examinations.

Theory

When every outcome of an experiment is **equally likely**, the probability of an event is

$$Pr(A) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}.$$

The power of this section is that both counts can be found using the **counting techniques** of this chapter — the multiplication principle, **arrangements** (permutations ${}^{n}P_{r}$) and **selections** (combinations $\binom{n}{r} = {}^{n}C_{r}$). Order matters for arrangements (${}^{n}P_{r}$), not for selections.

Choosing which count to use.

- When the **order does not matter** (a committee, a hand of cards, a set of lotto numbers), count with **combinations** $\binom{n}{r}$.
- When the **order matters** (an arrangement in a row, a first/second/third placing), count with **arrangements** ${}^{n}P_{r}$ or factorials.
- Use the **same kind of count** on the top and the bottom of the fraction.

Selecting from groups. To choose a group containing a required mix — say k from one type and the rest from another — multiply the separate selections, e.g. the number of ways to choose 2 women and 2 men from 6 women and 5 men is $\binom{6}{2}\binom{5}{2}$.

“At least one”. It is almost always shorter to use the complement:

$$Pr(\text{at least one}) = 1 - Pr(\text{none}).$$

Always give probabilities as exact fractions in simplest form.

Worked examples

Example 1 — scaffolded

A committee of 3 people is chosen at random from 5 women and 4 men. Find the probability that the committee consists of all women.

Working	Comment
Order does not matter, so count with combinations. Total $= \binom{9}{3} = 84$.	3 people chosen from 9; the sample space size.
Favourable $= \binom{5}{3} = 10$.	Choose all 3 from the 5 women.
$Pr(\text{all women}) = \frac{10}{84} = \frac{5}{42}$.	Divide, then simplify.

Example 2 — scaffolded

Four cards are dealt from a standard pack of 52. Find the probability that all four are hearts.

Working	Comment
Total $=\binom{52}{4}=270\,725$.	A hand of 4 from 52; order does not matter.
Favourable $=\binom{13}{4}=715$.	Choose all 4 from the 13 hearts.
$Pr\left(4 \text{ hearts}\right)=\dfrac{715}{270\,725}=\dfrac{11}{4165}$.	Simplify (divide by 65).

Example 3 — unscaffolded

In a lottery, 6 numbers are drawn at random from 45. Find the probability that a single ticket matches all 6 drawn numbers.

The number of possible draws is $\binom{45}{6}=8\,145\,060$, and each is equally likely. Exactly one of these matches the ticket.

$$\therefore Pr\left(\text{all } 6\right)=\frac{1}{8\,145\,060}.$$

Example 4 — unscaffolded

The six letters of the word **NUMBER** are arranged at random in a row. Find the probability that the arrangement begins with a vowel.

The letters are all different, so the total number of arrangements is $6!=720$. The vowels are U and E: choose one of the 2 vowels for the first position, then arrange the remaining 5 letters in $5!$ ways, giving $2\times 5!=240$ favourable arrangements.

$$\therefore Pr\left(\text{begins with a vowel}\right)=\frac{240}{720}=\frac{1}{3}.$$

Practice questions

Scaffolded

Q1. A team of 2 is chosen at random from 7 students, one of whom is Sam. (a) How many teams are possible? (b) How many of these include Sam? (c) Find the probability that Sam is chosen.

Q2. A bag holds 4 red and 3 blue balls. Three are drawn together. (a) Find the total number of selections. (b) Find the number that are all red. (c) Find the probability that all three are red.

Q3. Five cards are dealt from a standard pack. (a) State the total number of 5-card hands. (b) Find the number that are all spades. (c) Find the probability of a hand of five spades.

Q4. The letters of the word **CAT** are arranged at random in a row. (a) How many arrangements are there? (b) How many begin with C? (c) Find the probability that the arrangement begins with C.

Q5. A committee of 4 is chosen from 6 women and 5 men. (a) Find the total number of committees. (b) Find the number with exactly 2 women and 2 men. (c) Find the probability of exactly 2 women.

Q6. Two people are chosen at random from a group of five: A, B, C, D, E. (a) List or count the total number of pairs. (b) Find the number that include A. (c) Find the probability that A is chosen.

Unscaffolded

Give each probability as an exact fraction in simplest form.

Q7. In a 6-from-45 lottery, find the probability that a single ticket matches exactly 5 of the 6 drawn numbers.

Q8. A box contains 10 batteries, of which 4 are flat. Three are chosen at random. Find the probability that none is flat.

Q9. For the box in Q8, find the probability that at least one battery is flat.

Q10. Five cards are dealt from a standard pack. Find the probability of getting exactly two aces.

Q11. The letters of the word **PRIME** are arranged at random in a row. Find the probability that the arrangement begins with a vowel.

Q12. A committee of 5 is chosen from 7 men and 5 women. Find the probability that the committee is all men.

Q13. Five people stand in a row in random order. Find the probability that two particular people, A and B, stand next to each other.

Q14. Two balls are drawn together from a bag of 3 red and 5 blue. Find the probability of one red and one blue.

Q15. In a 6-from-45 lottery, find the probability that a single ticket matches exactly 4 of the 6 drawn numbers.

Q16. Five cards are dealt from a standard pack. Find the probability that all five are the same colour.

Q17. The letters of the word **GARDEN** are arranged at random in a row. Find the probability that the arrangement begins with G and ends with N.

Q18. Three numbers are chosen at random from 1 to 9. Find the probability that all three are odd.

Answers

Q1. (a) 21 (b) 6 (c) $\frac{2}{7}$. **Q2.** (a) 35 (b) 4 (c) $\frac{4}{35}$. **Q3.** (a) $2\,598\,960$ (b) 1287 (c) $\frac{33}{66\,640}$. **Q4.** (a) 6 (b) 2 (c) $\frac{1}{3}$. **Q5.** (a) 330 (b) 150 (c) $\frac{5}{11}$. **Q6.** (a) 10 (b) 4 (c) $\frac{2}{5}$. **Q7.** $\frac{39}{1\,357\,510}$. **Q8.** $\frac{1}{6}$. **Q9.** $\frac{5}{6}$. **Q10.** $\frac{2162}{54\,145}$. **Q11.** $\frac{2}{5}$. **Q12.** $\frac{7}{264}$. **Q13.** $\frac{2}{5}$. **Q14.** $\frac{15}{28}$. **Q15.** $\frac{741}{543\,004}$. **Q16.** $\frac{253}{4998}$. **Q17.** $\frac{1}{30}$. **Q18.** $\frac{5}{42}$.

Fully-worked solutions

Q1. (a) $\binom{7}{2}=21$. (b) A team with Sam needs 1 more from the other 6: $\binom{6}{1}=6$. (c) $Pr(\text{Sam})=\frac{6}{21}=\frac{2}{7}$.

Q2. (a) $\binom{7}{3}=35$. (b) $\binom{4}{3}=4$. (c) $Pr(\text{all red})=\frac{4}{35}$.

Q3. (a) $\binom{52}{5}=2\,598\,960$. (b) $\binom{13}{5}=1287$. (c) $Pr(5\text{ spades})=\frac{1287}{2\,598\,960}=\frac{33}{66\,640}$.

Q4. (a) $3!=6$. (b) Fix C first, arrange the other 2: $2!=2$. (c) $Pr=\frac{2}{6}=\frac{1}{3}$.

Q5. (a) $\binom{11}{4}=330$. (b) $\binom{6}{2}\binom{5}{2}=15\times 10=150$. (c) $Pr(2\text{ women})=\frac{150}{330}=\frac{5}{11}$.

Q6. (a) $\binom{5}{2}=10$. (b) Pairs with A: A with one of the other 4, so 4. (c) $Pr(A)=\frac{4}{10}=\frac{2}{5}$.

Q7. Matching exactly 5 means 5 of the ticket's numbers are among the 6 drawn and the sixth is one of the 39 not drawn: $\binom{6}{5}\binom{39}{1}=6\times 39=234$ ways out of $\binom{45}{6}=8\,145\,060$. $\therefore Pr=\frac{234}{8\,145\,060}=\frac{39}{1\,357\,510}$.

Q8. Choose 3 from the 6 good batteries: $\binom{6}{3}=20$, out of $\binom{10}{3}=120$. $Pr(\text{none flat})=\frac{20}{120}=\frac{1}{6}$.

Q9. $Pr(\text{at least one flat})=1-Pr(\text{none flat})=1-\frac{1}{6}=\frac{5}{6}$.

Q10. Choose 2 of the 4 aces and 3 of the other 48 cards: $\binom{4}{2}\binom{48}{3}=6\times 17\,296=103\,776$, out of $\binom{52}{5}=2\,598\,960$. $\therefore Pr(\text{exactly 2 aces})=\frac{103\,776}{2\,598\,960}=\frac{2162}{54\,145}$.

Q11. 5 different letters give $5!=120$ arrangements. Vowels I, E: $2\times 4!=48$ begin with a vowel. $Pr=\frac{48}{120}=\frac{2}{5}$.

Q12. $\binom{7}{5}=21$ all-men committees out of $\binom{12}{5}=792$. $Pr(\text{all men})=\frac{21}{792}=\frac{7}{264}$.

Q13. Treat A and B as a single block: the block plus the other 3 people give $4!$ arrangements, and A, B can be ordered 2 ways, so $2\times 4!=48$ favourable, out of $5!=120$. $Pr=\frac{48}{120}=\frac{2}{5}$.

Q14. One red and one blue: $\binom{3}{1}\binom{5}{1}=15$ out of $\binom{8}{2}=28$. $Pr=\dfrac{15}{28}$.

Q15. Matching exactly 4: $\binom{6}{4}\binom{39}{2}=15\times 741=11\,115$ out of $\binom{45}{6}=8\,145\,060$. $\therefore Pr=\dfrac{11\,115}{8\,145\,060}=\dfrac{741}{543\,004}$.

Q16. All same colour = all red or all black: $2\binom{26}{5}=2\times 65\,780=131\,560$ out of $\binom{52}{5}=2\,598\,960$.

$\therefore Pr=\dfrac{131\,560}{2\,598\,960}=\dfrac{253}{4998}$.

Q17. Fix G first and N last, then arrange the remaining 4 letters in the middle: $4!=24$ favourable, out of $6!=720$.
$Pr=\dfrac{24}{720}=\dfrac{1}{30}$.

Q18. The odd numbers $1,3,5,7,9$ number 5. Choose 3 of them: $\binom{5}{3}=10$ out of $\binom{9}{3}=84$. $Pr(\text{all odd})=\dfrac{10}{84}=\dfrac{5}{42}$.