Mathematical Methods Units 1 & 2

Chapter 8 — Probability and simulation

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Theory

A **random experiment** is a process whose outcome cannot be predicted with certainty (for example, rolling a die). The **sample space** ε is the set of all possible outcomes, and an **event** is a subset of the sample space.

Elementary and compound events. An **elementary** (or **simple**) **event** consists of a single outcome — for example $\{4\}$ when a die is rolled. A **compound event** consists of more than one outcome — for example "an even number" $=\{2,4,6\}$.

Equally likely outcomes. When every outcome is equally likely, the probability of an event A is

$$Pr(A)=\frac{|A|}{|\varepsilon|}=\frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}.$$

The probability scale. Every probability satisfies $0\leq Pr(A)\leq 1$. An impossible event has probability 0; a certain event has probability 1. The probabilities of all the outcomes in a sample space add to 1.

Complementary events. The **complement** A' is the event "A does not occur", and

$$Pr(A')=1-Pr(A).$$

This is often the quickest way to find "at least one" probabilities.

Listing a sample space. For two dice it helps to draw a 6×6 grid of the 36 ordered pairs; for tossing coins, list the sequences (e.g. for two coins, $\{HH, HT, TH, TT\}$). Always give probabilities as exact fractions in simplest form.

Worked examples

Example 1 — scaffolded

A fair six-sided die is rolled. Find the probability of rolling (a) a 4; (b) an even number; (c) a number greater than 4.

Working	Comment
$\varepsilon=\{1,2,3,4,5,6\}$, so $\vert\varepsilon\vert=6$.	List the sample space and count it.
(a) Only one outcome is a 4: $Pr(4)=\frac{1}{6}$.	One favourable outcome out of six.
(b) Even numbers are $\{2,4,6\}$: $Pr(\text{even})=\frac{3}{6}=\frac{1}{2}$.	Count the favourable outcomes, then simplify.
(c) Numbers greater than 4 are $\{5,6\}$: $Pr(>4)=\frac{2}{6}=\frac{1}{3}$.	

Example 2 — scaffolded

Two fair dice are rolled and the **sum** of the two numbers is recorded. Find (a) $Pr(\text{sum}=7)$; (b) $Pr(\text{sum}\geq 10)$; (c) $Pr(\text{sum is even})$.

The sample space is the 36 equally likely ordered pairs:

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10

	1	2	3	4	5	6
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Working	Comment
(a) A sum of 7 occurs 6 times (the diagonal from top-right to bottom-left): $Pr(\text{sum}=7)=\frac{6}{36}=\frac{1}{6}$.	Count the 7s in the grid.
(b) $\text{sum}\geq 10$: sums of 10 (3), 11 (2) and 12 (1) give 6 outcomes: $\frac{6}{36}=\frac{1}{6}$.	Add the favourable counts.
(c) Even sums occur 18 times: $Pr(\text{even})=\frac{18}{36}=\frac{1}{2}$.	

Example 3 — unscaffolded

A card is drawn at random from a standard deck of 52 playing cards. Find the probability that the card is (a) a king; (b) a heart; (c) a red card; (d) not a face card (J, Q, K).

(a) There are 4 kings: $Pr(\text{king})=\frac{4}{52}=\frac{1}{13}$.

(b) There are 13 hearts: $Pr(\text{heart})=\frac{13}{52}=\frac{1}{4}$.

(c) There are 26 red cards: $Pr(\text{red})=\frac{26}{52}=\frac{1}{2}$.

(d) There are 12 face cards, so 40 are not: $Pr(\text{not face})=\frac{40}{52}=\frac{10}{13}$.

$\therefore$ using the complement, $Pr(\text{not face})=1-\frac{12}{52}=\frac{10}{13}$.

Example 4 — unscaffolded

Three fair coins are tossed. List the sample space and find (a) $Pr(\text{exactly two heads})$; (b) $Pr(\text{at least one head})$; (c) $Pr(\text{no heads})$.

$\varepsilon=\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$, so $|\varepsilon|=8$.

(a) Exactly two heads: HHT, HTH, THH, so $Pr=\frac{3}{8}$.

(b) No heads occurs only with TTT, so $Pr(\text{no heads})=\frac{1}{8}$. Hence, by the complement,

$Pr(\text{at least one head})=1-\frac{1}{8}=\frac{7}{8}$.

(c) No heads: only TTT, so $Pr(\text{no heads})=\frac{1}{8}$.

Practice questions

Scaffolded

Q1. A letter is chosen at random from the word **MATHEMATICS**. (a) How many letters are there? (b) Find $Pr(\text{the letter M})$. (c) Find $Pr(\text{a vowel})$.

Q2. A fair die is rolled. Find (a) $Pr(\text{odd})$; (b) $Pr(\text{a multiple of } 3)$; (c) $Pr(\text{not a } 6)$.

Q3. A bag contains 5 red, 3 blue and 2 green marbles. One marble is drawn at random. (a) How many marbles in total? (b) Find $Pr(\text{red})$. (c) Find $Pr(\text{not green})$.

Q4. Two fair dice are rolled. Using a grid, find (a) $Pr(\text{a double})$; (b) $Pr(\text{sum} = 5)$; (c) $Pr(\text{at least one } 6)$.

Q5. A spinner has 8 equal sectors numbered 1 to 8. Find (a) $Pr(\text{a prime number})$; (b) $Pr(\text{a factor of } 8)$; (c) $Pr(\text{a number} \geq 6)$.

Q6. A card is drawn from a standard deck. Find (a) $Pr(\text{a spade})$; (b) $Pr(\text{an ace or a king})$; (c) $Pr(\text{a } 2, 3 \text{ or } 4)$.

Unscaffolded

Give each probability as an exact fraction in simplest form.

Q7. A fair die is rolled. Find $Pr(\text{a number} \leq 2)$.

Q8. A letter is chosen at random from **AUSTRALIA**. Find $Pr(\text{the letter A})$ and $Pr(\text{a vowel})$.

Q9. Two fair dice are rolled and the **product** is recorded. Find $Pr(\text{product} = 12)$.

Q10. Two fair dice are rolled. Find $Pr(\text{the sum is a multiple of } 4)$.

Q11. A card is drawn from a standard deck. Find $Pr(\text{a red face card})$.

Q12. Three fair coins are tossed. Find $Pr(\text{more heads than tails})$.

Q13. A whole number is chosen at random from 1 to 20. Find $Pr(\text{a multiple of } 3)$ and $Pr(\text{a prime number})$.

Q14. A bag holds 4 red, 6 white and 5 black counters. One is drawn. Find $Pr(\text{not white})$.

Q15. Two fair dice are rolled. Find $Pr(\text{the two numbers differ by } 2)$.

Q16. A card is drawn from a standard deck. Find $Pr(\text{a club or a queen})$.

Q17. A spinner has 12 equal sectors numbered 1 to 12. Find $Pr(\text{a factor of } 12)$ and $Pr(\text{an even number} \geq 8)$.

Q18. Two fair dice are rolled. Find $Pr(\text{the sum is a prime number})$.

Q19. Classify each event below for the stated experiment as **elementary** (a single outcome) or **compound** (more than one outcome). (a) Roll a die and get a 3. (b) Roll a die and get a prime number. (c) Draw a card and get the queen of hearts. (d) Draw a card and get a red card.

Answers

Q1. 11 letters; $Pr(M)=\frac{2}{11}$; $Pr(\text{vowel})=\frac{4}{11}$. **Q2.** $\frac{1}{2}$; $\frac{1}{3}$; $\frac{5}{6}$. **Q3.** 10; $\frac{1}{2}$; $\frac{4}{5}$. **Q4.** $\frac{1}{6}$; $\frac{1}{9}$; $\frac{11}{36}$. **Q5.** $\frac{1}{2}$; $\frac{1}{2}$; $\frac{3}{8}$. **Q6.** $\frac{1}{4}$; $\frac{2}{13}$; $\frac{3}{13}$. **Q7.** $\frac{1}{3}$. **Q8.** $Pr(A)=\frac{1}{3}$; $Pr(\text{vowel})=\frac{5}{9}$. **Q9.** $\frac{1}{9}$. **Q10.** $\frac{1}{4}$. **Q11.** $\frac{3}{26}$. **Q12.** $\frac{1}{2}$. **Q13.** $\frac{3}{10}$; $\frac{2}{5}$. **Q14.** $\frac{3}{5}$. **Q15.** $\frac{2}{9}$. **Q16.** $\frac{4}{13}$. **Q17.** $\frac{1}{2}$; $\frac{1}{4}$. **Q18.** $\frac{5}{12}$. **Q19.** (a) elementary; (b) compound; (c) elementary; (d) compound.

Fully-worked solutions

Q1. MATHEMATICS has 11 letters. (b) M appears twice: $Pr(M)=\frac{2}{11}$. (c) The vowels are A, A, E, I — that is 4 vowels: $Pr(\text{vowel})=\frac{4}{11}$.

Q2. $\varepsilon=\{1,2,3,4,5,6\}$. (a) Odd $\{1,3,5\}$: $\frac{3}{6}=\frac{1}{2}$. (b) Multiples of 3 are $\{3,6\}$: $\frac{2}{6}=\frac{1}{3}$. (c) Not a 6: $\frac{5}{6}$.

Q3. Total $=5+3+2=10$. (b) $Pr(\text{red})=\frac{5}{10}=\frac{1}{2}$. (c) Not green $=10-2=8$: $Pr(\text{not green})=\frac{8}{10}=\frac{4}{5}$.

Q4. There are 36 outcomes. (a) Doubles $(1,1),\ldots,(6,6)$: 6 outcomes, $\frac{6}{36}=\frac{1}{6}$. (b) Sum 5: $(1,4),(2,3),(3,2),(4,1)$ — 4 outcomes, $\frac{4}{36}=\frac{1}{9}$. (c) At least one 6: the last row and last column give $6+6-1=11$ outcomes (subtracting $(6,6)$ counted twice): $\frac{11}{36}$.

Q5. $\varepsilon=\{1,\ldots,8\}$. (a) Primes $\{2,3,5,7\}$: $\frac{4}{8}=\frac{1}{2}$. (b) Factors of 8 are $\{1,2,4,8\}$: $\frac{4}{8}=\frac{1}{2}$. (c) ≥ 6: $\{6,7,8\}$, $\frac{3}{8}$.

Q6. A deck has 52 cards. (a) 13 spades: $\frac{13}{52}=\frac{1}{4}$. (b) 4 aces and 4 kings: $\frac{8}{52}=\frac{2}{13}$. (c) Four each of $2,3,4$: $\frac{12}{52}=\frac{3}{13}$.

Q7. $\{1,2\}$: $Pr(\leq 2)=\frac{2}{6}=\frac{1}{3}$.

Q8. AUSTRALIA has 9 letters. A appears three times: $Pr(A)=\frac{3}{9}=\frac{1}{3}$. The vowels are A, U, A, A, I — 5 vowels: $Pr(\text{vowel})=\frac{5}{9}$.

Q9. Product $=12$: $(2,6),(6,2),(3,4),(4,3)$ — 4 outcomes, $\frac{4}{36}=\frac{1}{9}$.

Q10. Sum a multiple of 4 (i.e. $4,8$ or 12): $3+5+1=9$ outcomes, $\frac{9}{36}=\frac{1}{4}$.

Q11. Red face cards are J,Q,K of hearts and of diamonds: 6 cards, $\frac{6}{52}=\frac{3}{26}$.

Q12. More heads than tails means 2 or 3 heads: $3+1=4$ of the 8 outcomes, $\frac{4}{8}=\frac{1}{2}$.

Q13. From 1 to 20: (a) multiples of 3 are $3, 6, 9, 12, 15, 18$ — 6 numbers, $\frac{6}{20}=\frac{3}{10}$. (b) primes $2, 3, 5, 7, 11, 13, 17, 19$ — 8 numbers, $\frac{8}{20}=\frac{2}{5}$.

Q14. Total $=4+6+5=15$. Not white $=15-6=9$: $Pr(\text{not white})=\frac{9}{15}=\frac{3}{5}$.

Q15. Numbers differing by 2: $(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4)$ — 8 outcomes, $\frac{8}{36}=\frac{2}{9}$.

Q16. Clubs or queens: 13 clubs $+4$ queens -1 (queen of clubs counted twice) $=16$, $\frac{16}{52}=\frac{4}{13}$.

Q17. From 1 to 12: (a) factors of 12 are $1, 2, 3, 4, 6, 12$ — 6 numbers, $\frac{6}{12}=\frac{1}{2}$. (b) even numbers ≥ 8 are $8, 10, 12$ — 3 numbers, $\frac{3}{12}=\frac{1}{4}$.

Q18. Prime sums are $2, 3, 5, 7, 11$, occurring $1+2+4+6+2=15$ times, $\frac{15}{36}=\frac{5}{12}$.

Q19. (a) "a 3" is the single outcome $\{3\}$ → **elementary**. (b) primes on a die are $\{2, 3, 5\}$, more than one outcome → **compound**. (c) the queen of hearts is one specific card → **elementary**. (d) there are 26 red cards → **compound**.

Theory

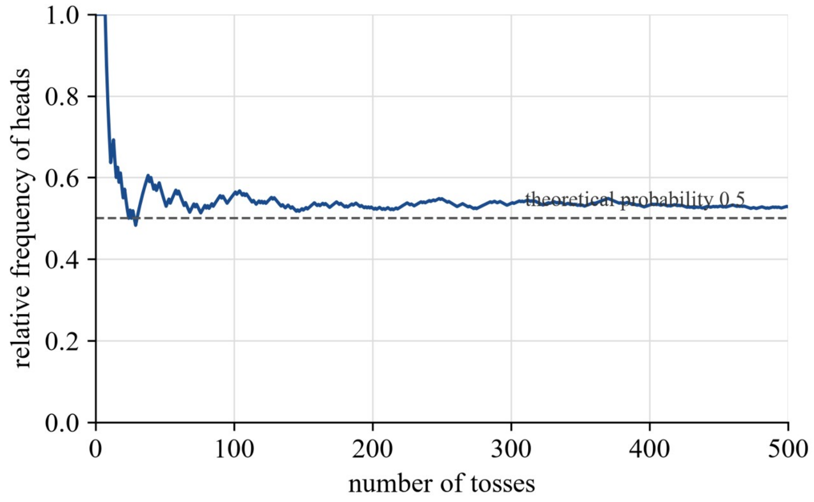
Sometimes the probability of an event cannot be calculated from equally likely outcomes — for example, the chance that a drawing pin lands point-up, or that a manufactured item is faulty. In these cases we **estimate** the probability from the results of an experiment or from collected data.

Experimental probability (relative frequency). If an event occurs in some of a number of trials, its **relative frequency** is

$$\text{relative frequency} = \frac{\text{number of times the event occurs}}{\text{total number of trials}}.$$

This relative frequency is an **estimate** of the true probability of the event.

Long-run behaviour. A relative frequency found from only a few trials can be unreliable, but as the number of trials increases the relative frequency tends to settle down close to the true (theoretical) probability. The graph below shows the relative frequency of "heads" for a fair coin settling towards 0.5 as the number of tosses increases.



Expected number of occurrences. If an event has probability $Pr(A)$, then in n independent trials the **expected number** of times it occurs is

$$\text{expected number} = Pr(A) \times n.$$

Point estimates from data. Given a frequency table, add the frequencies to find the total number of trials, then divide each frequency by the total to estimate the probability of each outcome.

Experimental vs theoretical. A **theoretical** probability is calculated from equally likely outcomes (e.g. $Pr(\text{six}) = 1/6$ for a fair die). An **experimental** probability is estimated from observed results. For a large number of trials the two are usually close, but they are rarely exactly equal.

Worked examples

Example 1 — scaffolded

A drawing pin is tossed 200 times and lands **point-up** 124 times.

Working	Comment
$Pr(\text{point-up}) \approx \frac{124}{200} = \frac{31}{50} = 0.62.$	Relative frequency = occurrences ÷ trials.

Working	Comment
$Pr(\text{point-down}) \approx \frac{200-124}{200} = \frac{76}{200} = \frac{19}{50} = 0.38.$	The pin lands point-down in the other 76 tosses.
Expected point-up in 500 tosses $= 0.62 \times 500 = 310.$	Expected number $= Pr \times n$.

Example 2 — scaffolded

A spinner is spun 100 times, giving the results below. Estimate the probability of each colour, and the expected number of greens in 300 spins.

Colour	Red	Blue	Green
Frequency	45	30	25

Working	Comment
Total trials $= 45 + 30 + 25 = 100.$	Add the frequencies.
$Pr(\text{red}) \approx \frac{45}{100} = \frac{9}{20}$, $Pr(\text{blue}) \approx \frac{30}{100} = \frac{3}{10}$, $Pr(\text{green}) \approx \frac{25}{100} = \frac{1}{4}.$	Divide each frequency by the total.
Expected greens in 300 spins $= \frac{1}{4} \times 300 = 75.$	Expected number $= Pr \times n$.

Example 3 — unscaffolded

In a survey of 250 people, 90 said they prefer brand A. Estimate the probability that a randomly chosen person prefers brand A, and hence estimate how many of a town of 8000 people prefer it.

$$Pr(\text{prefers A}) \approx \frac{90}{250} = \frac{9}{25} = 0.36.$$

Expected number in the town $= 0.36 \times 8000 = 2880.$

$\therefore$ about 2880 people would be expected to prefer brand A.

Example 4 — unscaffolded

A die is rolled 600 times and a "six" appears 88 times. Find the experimental and theoretical probabilities of rolling a six, and comment.

Experimental: $Pr(\text{six}) \approx \frac{88}{600} = \frac{11}{75} \approx 0.147.$

Theoretical (fair die): $Pr(\text{six}) = \frac{1}{6} \approx 0.167$ — so a fair die would be expected to give $\frac{1}{6} \times 600 = 100$ sixes.

$\therefore$ the experimental value (88 sixes) is a little lower than the theoretical 100, but reasonably close — consistent with a fair die.

Practice questions

Scaffolded

Q1. A coin is flipped 80 times and lands heads 44 times. (a) Estimate $Pr(\text{heads})$. (b) Estimate $Pr(\text{tails})$.

Q2. In a batch of 150 light globes, 6 are faulty. (a) Estimate $Pr(\text{faulty})$. (b) How many faulty globes would you expect in a batch of 1000?

Q3. The number of goals scored by a team in 20 matches is recorded below.

Goals	0	1	2	3
Matches	5	8	6	1

(a) Estimate $Pr(\text{exactly 2 goals})$. (b) Estimate $Pr(\text{at least 1 goal})$.

Q4. Of 300 seeds planted, 261 germinate. (a) Estimate $Pr(\text{germinates})$. (b) How many would you expect to germinate from 500 seeds?

Q5. A spinner numbered 1–4 is spun 200 times, giving the results below.

Number	1	2	3	4
Frequency	52	48	55	45

(a) Estimate $Pr(3)$. (b) Do these results suggest the spinner is fair? Explain.

Q6. A box of 40 chocolates contains 9 with soft centres. (a) Estimate $Pr(\text{soft centre})$. (b) How many soft-centred chocolates would you expect in a box of 24?

Unscaffolded

Q7. A coin is tossed 500 times and lands tails 285 times. Estimate $Pr(\text{tails})$.

Q8. Of 250 vehicles passing a point, 40 are trucks. (a) Estimate $Pr(\text{truck})$. (b) How many trucks would you expect among 900 vehicles?

Q9. The eye colours of 80 people are recorded below. Estimate $Pr(\text{blue eyes})$.

Eye colour	Brown	Blue	Green
Frequency	48	22	10

Q10. In a production run of 1200 items, 18 are defective. (a) Estimate $Pr(\text{defective})$. (b) How many defectives would you expect in 10 000 items?

Q11. A basketballer makes 34 of 40 free throws. Estimate the probability that she **misses** her next free throw.

Q12. In a survey of 400 people, 148 own a pet. (a) Estimate $Pr(\text{owns a pet})$. (b) Estimate how many people in a town of 25 000 own a pet.

Q13. A die is rolled 720 times and an even number appears 372 times. (a) Find the experimental $Pr(\text{even})$. (b) State the theoretical $Pr(\text{even})$. (c) Comment.

Q14. The number of pets owned by 200 households is recorded below.

Pets	0	1	2	3
Households	24	96	64	16

(a) Estimate $Pr(\text{exactly 2 pets})$. (b) Estimate $Pr(\text{at least 1 pet})$.

Q15. It rained on 219 of the 365 days of a year. (a) Estimate the probability that it rains on a given day. (b) How many rainy days would you expect in a 30-day month?

Q16. A bag of 90 counters is sampled with replacement, giving 40 red, 27 blue and 23 green. Estimate $Pr(\text{blue})$.

Q17. A spinner has regions A, B and C. In 300 spins it lands on A 63 times, B 135 times and C 102 times. Estimate $Pr(\text{A})$.

Q18. In a germination trial, 84 of 120 seeds sprout. (a) Estimate $Pr(\text{sprouts})$. (b) How many sprouts would you expect from 400 seeds?

Answers

Q1. (a) $11/20 = 0.55$; (b) $9/20 = 0.45$. **Q2.** (a) $1/25 = 0.04$; (b) 40. **Q3.** (a) $3/10$; (b) $3/4$. **Q4.** (a) $87/100 = 0.87$; (b) 435. **Q5.** (a) $11/40 = 0.275$; (b) roughly fair (each number close to the expected 50). **Q6.** (a) $9/40 = 0.225$; (b) 5.4, so about 5. **Q7.** $57/100 = 0.57$. **Q8.** (a) $4/25 = 0.16$; (b) 144. **Q9.** $11/40 = 0.275$. **Q10.** (a) $3/200 = 0.015$; (b) 150. **Q11.** $3/20 = 0.15$. **Q12.** (a) $37/100 = 0.37$; (b) 9250. **Q13.** (a) $31/60 \approx 0.517$; (b) $1/2$; (c) close, experimental slightly higher. **Q14.** (a) $8/25 = 0.32$; (b) $22/25 = 0.88$. **Q15.** (a) $3/5 = 0.6$; (b) 18. **Q16.** $3/10 = 0.3$. **Q17.** $21/100 = 0.21$. **Q18.** (a) $7/10 = 0.7$; (b) 280.

Fully-worked solutions

Q1. (a) $Pr(\text{heads}) \approx \frac{44}{80} = \frac{11}{20} = 0.55$. (b) $Pr(\text{tails}) \approx \frac{80-44}{80} = \frac{36}{80} = \frac{9}{20} = 0.45$.

Q2. (a) $Pr(\text{faulty}) \approx \frac{6}{150} = \frac{1}{25} = 0.04$. (b) Expected $= \frac{1}{25} \times 1000 = 40$ faulty globes.

Q3. Total $= 5+8+6+1 = 20$. (a) $Pr(2) \approx \frac{6}{20} = \frac{3}{10}$. (b) At least one goal occurs in $8+6+1 = 15$ matches, so $Pr(\geq 1) \approx \frac{15}{20} = \frac{3}{4}$.

Q4. (a) $Pr(\text{germinates}) \approx \frac{261}{300} = \frac{87}{100} = 0.87$. (b) Expected $= 0.87 \times 500 = 435$.

Q5. (a) $Pr(3) \approx \frac{55}{200} = \frac{11}{40} = 0.275$. (b) A fair spinner would give about $\frac{200}{4} = 50$ of each number. The observed frequencies $(52, 48, 55, 45)$ are all close to 50, so the results are consistent with a fair spinner.

Q6. (a) $Pr(\text{soft}) \approx \frac{9}{40} = 0.225$. (b) Expected $= \frac{9}{40} \times 24 = \frac{27}{5} = 5.4$, so about 5.

Q7. $Pr(\text{tails}) \approx \frac{285}{500} = \frac{57}{100} = 0.57$.

Q8. (a) $Pr(\text{truck}) \approx \frac{40}{250} = \frac{4}{25} = 0.16$. (b) Expected $= \frac{4}{25} \times 900 = 144$.

Q9. Total $= 48+22+10 = 80$, so $Pr(\text{blue}) \approx \frac{22}{80} = \frac{11}{40} = 0.275$.

Q10. (a) $Pr(\text{defective}) \approx \frac{18}{1200} = \frac{3}{200} = 0.015$. (b) Expected $= \frac{3}{200} \times 10\,000 = 150$.

Q11. Misses $= 40 - 34 = 6$, so $Pr(\text{miss}) \approx \frac{6}{40} = \frac{3}{20} = 0.15$.

Q12. (a) $Pr(\text{pet}) \approx \frac{148}{400} = \frac{37}{100} = 0.37$. (b) Expected $= 0.37 \times 25\,000 = 9250$.

Q13. (a) Experimental $Pr(\text{even}) \approx \frac{372}{720} = \frac{31}{60} \approx 0.517$. (b) Theoretical $Pr(\text{even}) = \frac{3}{6} = \frac{1}{2}$. (c) The experimental value 0.517 is close to the theoretical 0.5, slightly higher — consistent with a fair die over a large number of rolls.

Q14. Total $=24+96+64+16=200$. (a) $Pr\left(2\right)\approx\frac{64}{200}=\frac{8}{25}=0.32$. (b) At least one pet occurs in $96+64+16=176$ households, so $Pr\left(\geq 1\right)\approx\frac{176}{200}=\frac{22}{25}=0.88$.

Q15. (a) $Pr\left(\text{rain}\right)\approx\frac{219}{365}=\frac{3}{5}=0.6$. (b) Expected rainy days $=\frac{3}{5}\times 30=18$.

Q16. Total $=40+27+23=90$, so $Pr\left(\text{blue}\right)\approx\frac{27}{90}=\frac{3}{10}=0.3$.

Q17. Total $=63+135+102=300$, so $Pr\left(\text{A}\right)\approx\frac{63}{300}=\frac{21}{100}=0.21$.

Q18. (a) $Pr\left(\text{sprouts}\right)\approx\frac{84}{120}=\frac{7}{10}=0.7$. (b) Expected $=\frac{7}{10}\times 400=280$.

Theory

A **multi-stage experiment** is made up of more than one step — for example, drawing two balls from a bag, or tossing a coin several times. A **tree diagram** is the clearest way to organise the possibilities: each stage is drawn as a set of branches, and the probability of each outcome is written **on** its branch.

Two rules do all the work.

- **Multiplication (along the branches).** The probability of a particular path is the **product** of the branch probabilities along that path.
- **Addition (across paths).** The probability of an event is the **sum** of the path probabilities for all the paths that make up the event (these paths are mutually exclusive).

The branch probabilities leaving any one point always add to 1.

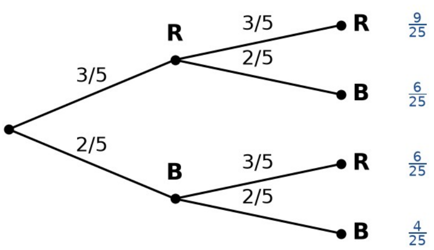
With or without replacement. If an item is **replaced** before the next stage, the probabilities are the **same** at each stage. If it is **not replaced**, the totals change, so the probabilities at the next stage **change**.

"At least one". It is usually quickest to use the complement: $Pr(\text{at least one}) = 1 - Pr(\text{none})$.

Worked examples

Example 1 — scaffolded

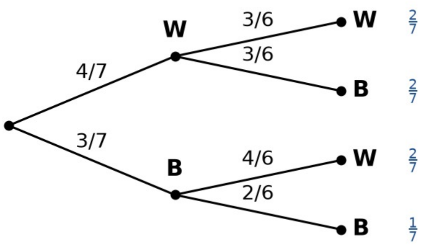
A bag holds 3 red and 2 blue counters. A counter is drawn and its colour noted, it is **replaced**, and a second counter is drawn. Find (a) $Pr(\text{both red})$, (b) $Pr(\text{one of each colour})$, (c) $Pr(\text{at least one red})$.



Working	Comment
Each draw: $Pr(R) = 3/5$, $Pr(B) = 2/5$.	The counter is replaced, so both draws have the same probabilities.
$Pr(RR) = 3/5 \times 3/5 = 9/25$.	Multiply along the R–R path.
$Pr(\text{one of each}) = Pr(RB) + Pr(BR) = 6/25 + 6/25 =$.	Add the two paths that give one of each colour.
$Pr(\text{at least one red}) = 1 - Pr(BB) = 1 - 4/25 = 21/25$.	Complement: the only way to get no red is BB.

Example 2 — scaffolded

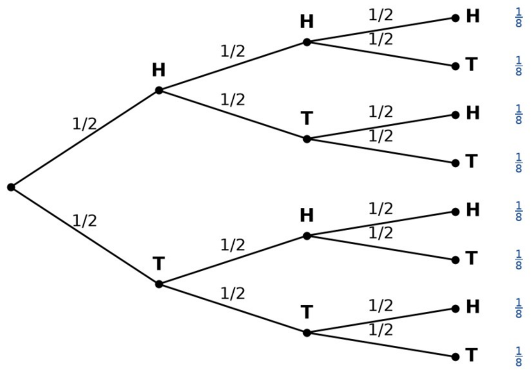
A bag holds 4 white and 3 black buttons. Two are drawn **without replacement**. Find (a) $Pr(\text{both white})$, (b) $Pr(\text{same colour})$, (c) $Pr(\text{different colours})$.



Working	Comment
First draw: $Pr(W)=4/7$, $Pr(B)=3/7$.	There are 7 buttons in total.
After a white, 3 white and 3 black remain (6 in total): $Pr(W\,W)=4/7\times3/6=2/7$.	Without replacement the totals change — one button is gone.
$Pr(B\,B)=3/7\times2/6=1/7$.	Multiply along the B–B path.
$Pr(\text{same colour})=2/7+1/7=3/7$; $Pr(\text{different})=1-3/7=4/7$.	Add the same-colour paths; use the complement for different colours.

Example 3 — unscaffolded

A fair coin is tossed three times. Find (a) $Pr(\text{exactly two heads})$, (b) $Pr(\text{at most one head})$.



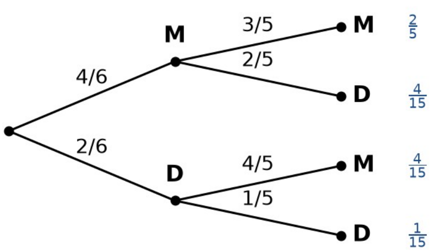
Each toss is independent with $Pr(H)=Pr(T)=1/2$, so every path has probability $(1/2)^3=1/8$.

(a) Exactly two heads occurs on the paths $H\,H\,T$, $H\,T\,H$ and $T\,H\,H$: $Pr(\text{exactly two heads})=3\times1/8=3/8$.

(b) At most one head occurs on the paths $T\,T\,T$, $H\,T\,T$, $T\,H\,T$ and $T\,T\,H$: $Pr(\text{at most one head})=4\times1/8=1/2$.

Example 4 — unscaffolded

A box holds 4 milk and 2 dark chocolates. Two are chosen **without replacement**. Find (a) $Pr(\text{both milk})$, (b) $Pr(\text{at least one dark})$.



$Pr(\text{both milk})=4/6\times3/5=12/30=2/5$.

$Pr(\text{at least one dark}) = 1 - Pr(\text{both milk}) = 1 - 2/5 = 3/5$. $\therefore$ there is a $3/5$ chance of getting at least one dark chocolate.

Practice questions

Scaffolded

Q1. A bag has 3 green and 1 yellow counter. One is drawn, its colour noted, and it is **replaced**; then a second is drawn. (a) Draw a tree diagram. (b) Find $Pr(\text{both green})$. (c) Find $Pr(\text{both yellow})$. (d) Find $Pr(\text{one of each colour})$.

Q2. A box has 5 apples and 3 oranges. Two are taken **without replacement**. (a) Draw a tree diagram. (b) $Pr(\text{both apples})$. (c) $Pr(\text{both oranges})$. (d) $Pr(\text{one of each})$.

Q3. On any day the probability of rain is $3/10$, independently of other days. (a) Draw a tree for two days. (b) $Pr(\text{rain on both days})$. (c) $Pr(\text{no rain on either day})$. (d) $Pr(\text{exactly one rainy day})$.

Q4. A fair coin is tossed three times. (a) Draw a tree diagram. (b) $Pr(\text{three heads})$. (c) $Pr(\text{at most two heads})$.

Q5. A bag has 2 red and 4 blue marbles. Two are drawn **without replacement**. (a) Draw a tree. (b) $Pr(\text{both red})$. (c) $Pr(\text{both blue})$. (d) $Pr(\text{one of each})$.

Q6. A jar has 3 red and 2 green lollies. Two are taken **without replacement**. (a) Draw a tree. (b) Find $Pr(\text{at least one green})$.

Unscaffolded

For each, draw a tree diagram if it helps, and give exact probabilities.

Q7. A fair coin is tossed twice. Find $Pr(\text{two tails})$ and $Pr(\text{at least one head})$.

Q8. A bag has 2 red and 3 blue balls. Two are drawn **with replacement**. Find $Pr(\text{both red})$, $Pr(\text{both blue})$ and $Pr(\text{one of each})$.

Q9. A bag has 4 red and 6 blue balls. Two are drawn **without replacement**. Find $Pr(\text{both red})$, $Pr(\text{both blue})$ and $Pr(\text{one of each})$.

Q10. A fair coin is tossed three times. Find $Pr(\text{exactly one head})$ and $Pr(\text{at least two heads})$.

Q11. A basketballer makes each free throw with probability $3/4$, independently, and takes three shots. Find $Pr(\text{all three made})$, $Pr(\text{exactly two made})$ and $Pr(\text{at least one miss})$.

Q12. Two cards are dealt **without replacement** from five cards, of which two are aces. Find $Pr(\text{both aces})$, $Pr(\text{no ace})$ and $Pr(\text{exactly one ace})$.

Q13. A train is late with probability $1/4$ each day, independently. Over two days, find $Pr(\text{late both days})$, $Pr(\text{on time both days})$ and $Pr(\text{exactly one late day})$.

Q14. A box has 3 white and 5 black buttons. Two are drawn **without replacement**. Find $Pr(\text{both white})$, $Pr(\text{both black})$ and $Pr(\text{different colours})$.

Q15. A fair coin is tossed three times. Find $Pr(\text{all three the same})$ and $Pr(\text{not all the same})$.

Q16. A bag has 4 blue and 2 yellow beads. Three are drawn **without replacement**. Find $Pr(\text{all blue})$ and $Pr(\text{at least one yellow})$.

Q17. An archer hits the target with probability $2/3$, independently. In two attempts, find $Pr(\text{two hits})$, $Pr(\text{no hits})$ and $Pr(\text{exactly one hit})$.

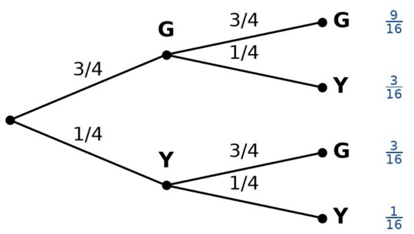
Q18. Box A contains 2 red and 1 blue ball; Box B contains 1 red and 3 blue balls. A box is chosen at random and one ball is drawn. Find $Pr(\text{red})$ and $Pr(\text{blue})$.

Answers

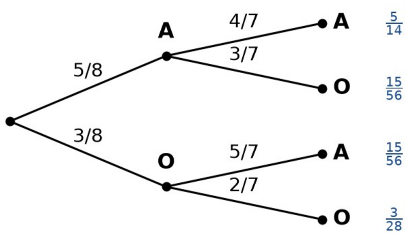
Q1. $9/16$; $1/16$; $3/8$. **Q2.** $5/14$; $3/28$; $15/28$. **Q3.** $9/100$; $49/100$; $21/50$. **Q4.** $1/8$; $7/8$. **Q5.** $1/15$; $2/5$; $8/15$. **Q6.** $7/10$. **Q7.** $1/4$; $3/4$. **Q8.** $4/25$; $9/25$; $12/25$. **Q9.** $2/15$; $1/3$; $8/15$. **Q10.** $3/8$; $1/2$. **Q11.** $27/64$; $27/64$; $37/64$. **Q12.** $1/10$; $3/10$; $3/5$. **Q13.** $1/16$; $9/16$; $3/8$. **Q14.** $3/28$; $5/14$; $15/28$. **Q15.** $1/4$; $3/4$. **Q16.** $1/5$; $4/5$. **Q17.** $4/9$; $1/9$; $4/9$. **Q18.** $11/24$; $13/24$.

Fully-worked solutions

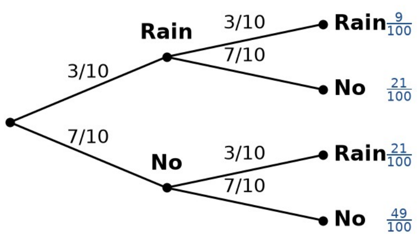
Q1. With replacement, $Pr(G)=3/4$, $Pr(Y)=1/4$ each draw. $Pr(GG)=3/4\times 3/4=9/16$; $Pr(YY)=1/16$; $Pr(\text{one of each})=2\times 3/4\times 1/4=6/16=3/8$.



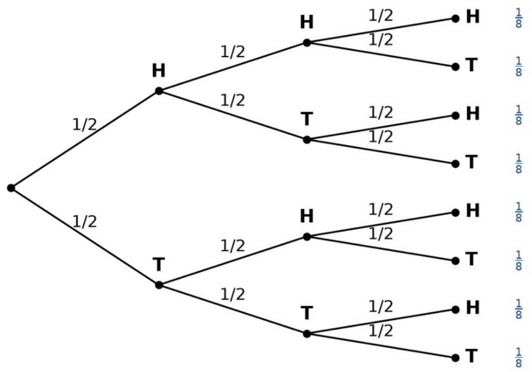
Q2. Without replacement (8 fruit). $Pr(AA)=5/8\times 4/7=5/14$; $Pr(OO)=3/8\times 2/7=3/28$; $Pr(\text{one of each})=1-5/14-3/28=15/28$.



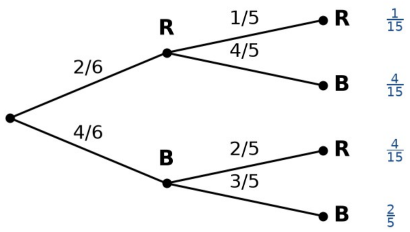
Q3. Independent days. $Pr(\text{both})=(3/10)^2=9/100$; $Pr(\text{neither})=(7/10)^2=49/100$; $Pr(\text{exactly one})=2\times 3/10\times 7/10=42/100=21/50$.



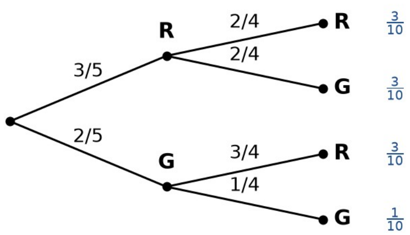
Q4. Every path has probability $1/8$. $Pr(HHH)=1/8$; $Pr(\text{at most two heads})=1-Pr(HHH)=1-1/8=7/8$.



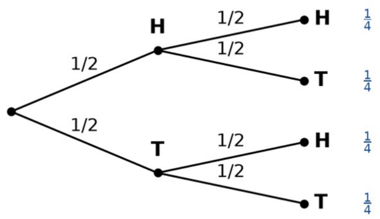
Q5. Without replacement (6 marbles). $Pr(RR) = 2/6 \times 1/5 = 1/15$; $Pr(BB) = 4/6 \times 3/5 = 2/5$; $Pr(\text{one of each}) = 1 - 1/15 - 2/5 = 8/15$.



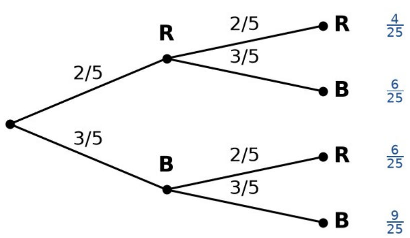
Q6. $Pr(\text{at least one green}) = 1 - Pr(RR) = 1 - 3/5 \times 2/4 = 1 - 3/10 = 7/10$.



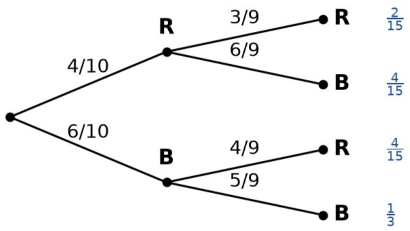
Q7. $Pr(TT) = 1/2 \times 1/2 = 1/4$; $Pr(\text{at least one head}) = 1 - 1/4 = 3/4$.



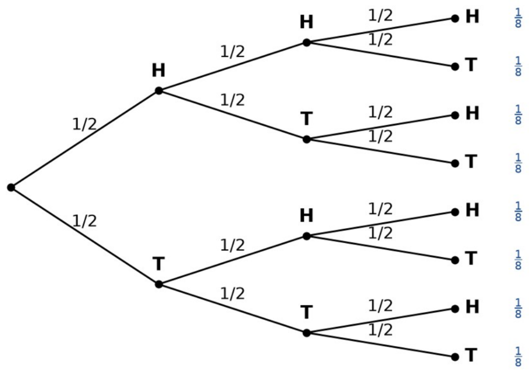
Q8. With replacement. $Pr(RR) = (2/5)^2 = 4/25$; $Pr(BB) = (3/5)^2 = 9/25$; $Pr(\text{one of each}) = 2 \times 2/5 \times 3/5 = 12/25$.



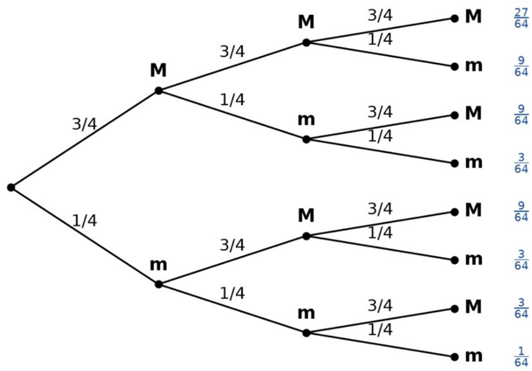
Q9. Without replacement (10 balls). $Pr(RR) = 4/10 \times 3/9 = 2/15$; $Pr(BB) = 6/10 \times 5/9 = 1/3$; $Pr(\text{one of each}) = 1 - 2/15 - 1/3 = 8/15$.



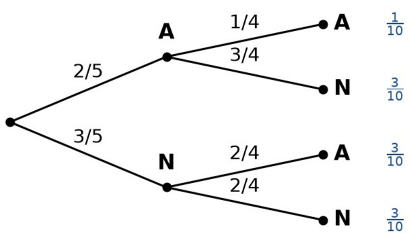
Q10. Each path $1/8$. Exactly one head (HTT, THT, TTH): $3 \times 1/8 = 3/8$. At least two heads (HHH, HHT, HTH, THH): $4 \times 1/8 = 1/2$.



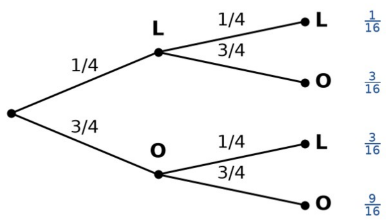
Q11. $Pr(\text{make}) = 3/4$, $Pr(\text{miss}) = 1/4$, independent. $Pr(\text{all three}) = (3/4)^3 = 27/64$; $Pr(\text{exactly two}) = 3 \times (3/4)^2 \times 1/4 = 27/64$; $Pr(\text{at least one miss}) = 1 - 27/64 = 37/64$.



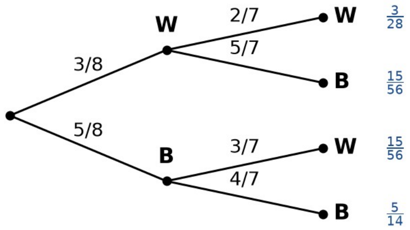
Q12. Without replacement (5 cards, 2 aces). $Pr(\text{both aces}) = 2/5 \times 1/4 = 1/10$; $Pr(\text{no ace}) = 3/5 \times 2/4 = 3/10$; $Pr(\text{exactly one ace}) = 1 - 1/10 - 3/10 = 3/5$.



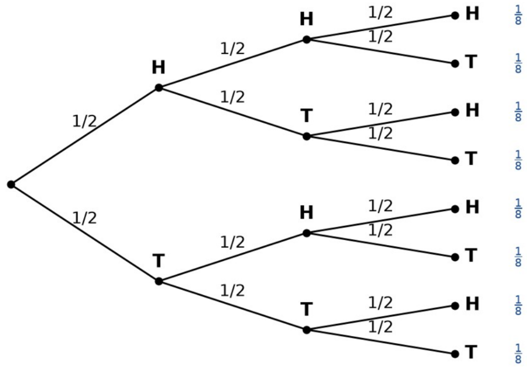
Q13. Independent days. $Pr(\text{late both}) = (1/4)^2 = 1/16$; $Pr(\text{on time both}) = (3/4)^2 = 9/16$; $Pr(\text{exactly one late}) = 2 \times 1/4 \times 3/4 = 6/16 = 3/8$.



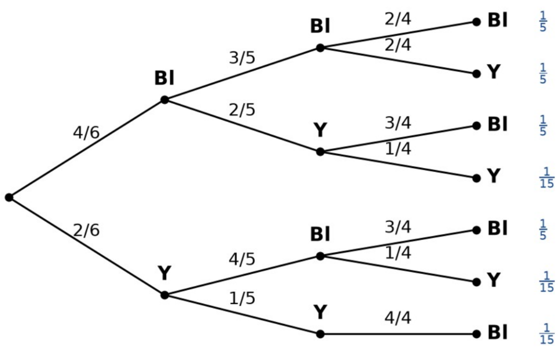
Q14. Without replacement (8 buttons). $Pr(W\,W) = 3/8 \times 2/7 = 3/28$; $Pr(B\,B) = 5/8 \times 4/7 = 5/14$; $Pr(\text{different}) = 1 - 3/28 - 5/14 = 15/28$.



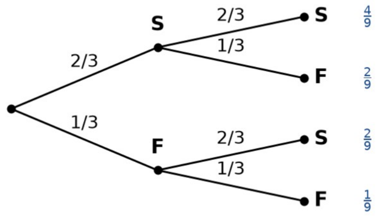
Q15. All the same means $H\,H\,H$ or $T\,T\,T$: $Pr = 2 \times 1/8 = 1/4$. $Pr(\text{not all the same}) = 1 - 1/4 = 3/4$.



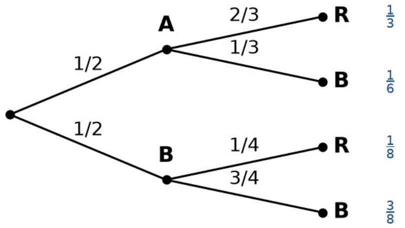
Q16. Without replacement (6 beads). $Pr(\text{all blue}) = 4/6 \times 3/5 \times 2/4 = 1/5$; $Pr(\text{at least one yellow}) = 1 - 1/5 = 4/5$.



Q17. $Pr(\text{hit}) = 2/3$, independent. $Pr(\text{two hits}) = (2/3)^2 = 4/9$; $Pr(\text{no hits}) = (1/3)^2 = 1/9$; $Pr(\text{exactly one hit}) = 2 \times 2/3 \times 1/3 = 4/9$.



Q18. The box is chosen with probability $1/2$ each. $Pr(\text{red}) = 1/2 \times 2/3 + 1/2 \times 1/4 = 1/3 + 1/8 = 11/24$; $Pr(\text{blue}) = 1 - 11/24 = 13/24$.



Theory

An **event** is a subset of the sample space ε. From two events A and B we build new events:

- the **complement** A' — "A does not occur";
- the **union** $A \cup B$ — "A or B (or both)";
- the **intersection** $A \cap B$ — "A and B".

The addition rule. For any two events,

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B).$$

The overlap $A \cap B$ is subtracted because it is counted in both $Pr(A)$ and $Pr(B)$.

Mutually exclusive events. If A and B cannot both occur then $A \cap B =$, so $Pr(A \cap B) = 0$ and the addition rule becomes $Pr(A \cup B) = Pr(A) + Pr(B)$.

Complements. $Pr(A') = 1 - Pr(A)$. The event "**at least one** of A, B" is $A \cup B$; "**neither**" is its complement $(A \cup B)'$, with $Pr(\text{neither}) = 1 - Pr(A \cup B)$.

Venn diagrams. A two-event Venn diagram splits ε into four regions — A **only** ($A \cap B'$), **both** ($A \cap B$), B **only** ($A' \cap B$) and **neither** ($(A \cup B)'$). The four values add to 1 (for probabilities) or to the total (for counts). Note

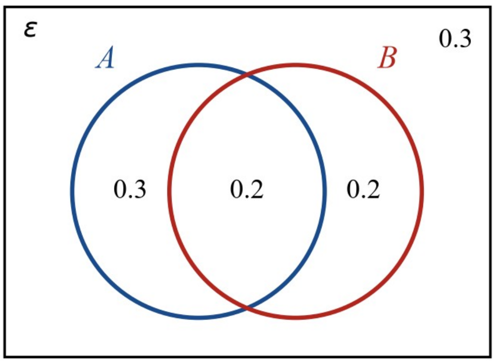
$$Pr(A\,\text{only}) = Pr(A \cap B') = Pr(A) - Pr(A \cap B).$$

Worked examples

Example 1 — scaffolded

Events A and B have $Pr(A) = 0.5$, $Pr(B) = 0.4$ and $Pr(A \cap B) = 0.2$. Find $Pr(A \cup B)$ and show the information on a Venn diagram.

Working	Comment
$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = 0.5 + 0.4$ · .	Addition rule — subtract the overlap.
A only $= 0.5 - 0.2 = 0.3$; B only $= 0.4 - 0.2 = 0.2$; neither $= 1 - 0.7 = 0.3$.	Fill the four regions; they sum to 1.



Example 2 — scaffolded

A card is drawn at random from a standard deck of 52. Let A = "a heart" and B = "a face card (J, Q or K)". Find $Pr(A \cup B)$.

Working	Comment
$Pr(A) = \frac{13}{52}$, $Pr(B) = \frac{12}{52}$.	3 face cards in each of 4 suits.
The heart face cards are J, Q, K of hearts, so $Pr(A \cap B) = \frac{3}{52}$.	The overlap.
$Pr(A \cup B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}$.	Addition rule, then simplify.

Example 3 — unscaffolded

Two **mutually exclusive** events have $Pr(A) = 0.35$ and $Pr(B) = 0.45$. Find $Pr(A \cup B)$ and the probability that neither occurs.

Because A and B are mutually exclusive, $Pr(A \cap B) = 0$, so

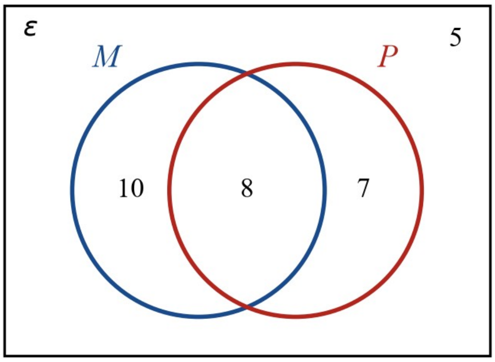
$$Pr(A \cup B) = 0.35 + 0.45 = 0.8.$$

$\therefore Pr(\text{neither}) = 1 - 0.8 = 0.2.$

Example 4 — unscaffolded

In a group of 30 students, 18 study Mathematics, 15 study Physics and 8 study both. Find the number who study neither, and the probability that a randomly chosen student studies Mathematics only.

Mathematics only $= 18 - 8 = 10$; Physics only $= 15 - 8 = 7$; neither $= 30 - (10 + 8 + 7) = 5$.



$\therefore Pr(\text{Mathematics only}) = \frac{10}{30} = \frac{1}{3}.$

Practice questions

Scaffolded

Q1. $Pr(A) = 0.6$, $Pr(B) = 0.3$, $Pr(A \cap B) = 0.1$. (a) Find $Pr(A \cup B)$. (b) Draw a Venn diagram. (c) Find $Pr(\text{neither})$.

Q2. A and B are mutually exclusive with $Pr(A) = 0.3$ and $Pr(B) = 0.45$. (a) State $Pr(A \cap B)$. (b) Find $Pr(A \cup B)$. (c) Find $Pr(\text{neither})$.

Q3. In a survey of 50 people, 28 like tea, 25 like coffee and 12 like both. (a) Complete a Venn diagram of the counts. (b) How many like neither?

Q4. A fair die is rolled. Let A = "an even number" and B = "a number greater than 3". Find $Pr(A \cup B)$.

Q5. $Pr(A \cup B) = 0.9$, $Pr(A) = 0.6$ and $Pr(B) = 0.5$. Find $Pr(A \cap B)$.

Q6. $Pr(A) = 0.7$ and $Pr(A \cap B) = 0.25$. Find $Pr(A \cap B')$ (the probability of A but not B).

Unscaffolded

Q7. $Pr(A) = 0.55$, $Pr(B) = 0.35$, $Pr(A \cap B) = 0.15$. Find $Pr(A \cup B)$ and draw a Venn diagram.

Q8. $Pr(A) = 0.4$, $Pr(B) = 0.5$ and $Pr(A \cup B) = 0.7$. Find $Pr(A \cap B)$.

Q9. A and B are mutually exclusive with $Pr(A) = 0.2$ and $Pr(B) = 0.35$. Find $Pr(A \cup B)$ and $Pr(\text{neither})$.

Q10. A card is drawn from a standard deck of 52. Find the probability that it is a club or an ace.

Q11. Of 40 students, 22 play a sport, 18 play a musical instrument and 10 do both. Find how many do neither, and the probability that a student does neither.

Q12. A fair die is rolled. Find the probability that the result is a multiple of 3 or an odd number.

Q13. $Pr(A') = 0.6$, $Pr(B) = 0.3$, and A and B are mutually exclusive. Find $Pr(A)$ and $Pr(A \cup B)$.

Q14. $Pr(A) = 0.45$, $Pr(B) = 0.35$, $Pr(A \cap B) = 0.15$. Find the probability that at least one of A, B occurs, and that neither occurs.

Q15. $Pr(A) = 0.5$ and $Pr(A \cap B) = 0.2$. Find $Pr(A \cap B')$.

Q16. $Pr(A) = 0.4$, $Pr(B) = 0.5$ and $Pr(A \cup B) = 0.9$. Find $Pr(A \cap B)$, and hence state whether A and B are mutually exclusive.

Q17. A Venn diagram shows $Pr(A\,\text{only}) = 0.3$, $Pr(\text{both}) = 0.2$ and $Pr(B\,\text{only}) = 0.25$. Find $Pr(\text{neither})$, $Pr(A)$, $Pr(B)$ and $Pr(A \cup B)$.

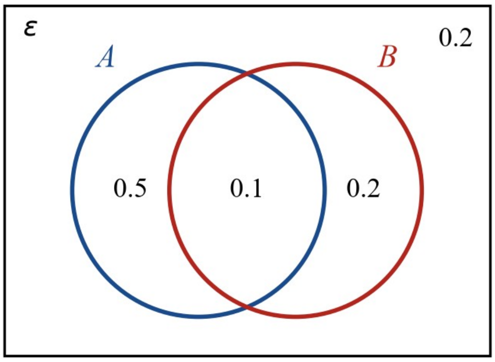
Q18. In a class, the probability that a student owns a dog is 0.4, owns a cat is 0.3, and owns both is 0.1. Find the probability that a student owns at least one, owns neither, and owns a dog but not a cat.

Answers

Q1. (a) 0.8 (c) 0.2. **Q2.** (a) 0 (b) 0.75 (c) 0.25. **Q3.** tea only 16, both 12, coffee only 13; (b) 9. **Q4.** $\frac{2}{3}$. **Q5.** 0.2. **Q6.** 0.45. **Q7.** 0.75. **Q8.** 0.2. **Q9.** $Pr(A \cup B) = 0.55$, neither $= 0.45$. **Q10.** $\frac{4}{13}$. **Q11.** 10 do neither; $Pr(\text{neither}) = \frac{1}{4}$. **Q12.** $\frac{2}{3}$. **Q13.** $Pr(A) = 0.4$; $Pr(A \cup B) = 0.7$. **Q14.** at least one $= 0.65$; neither $= 0.35$. **Q15.** 0.3. **Q16.** $Pr(A \cap B) = 0$, so A and B **are** mutually exclusive. **Q17.** neither $= 0.25$; $Pr(A) = 0.5$; $Pr(B) = 0.45$; $Pr(A \cup B) = 0.75$. **Q18.** at least one $= 0.6$; neither $= 0.4$; dog but not cat $= 0.3$.

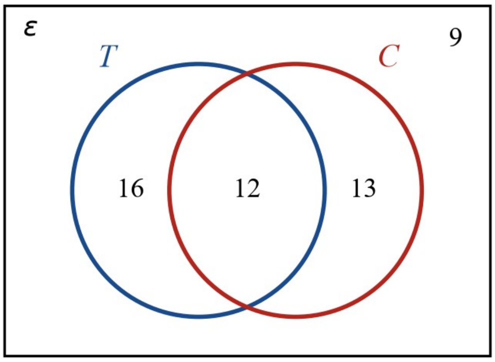
Fully-worked solutions

Q1. (a) $Pr(A \cup B) = 0.6 + 0.3 - 0.1 = 0.8$. (b) A only $= 0.5$, both $= 0.1$, B only $= 0.2$, neither $= 0.2$. (c) $Pr(\text{neither}) = 1 - 0.8 = 0.2$.



Q2. (a) Mutually exclusive $\Rightarrow Pr(A \cap B) = 0$. (b) $Pr(A \cup B) = 0.3 + 0.45 = 0.75$. (c) $Pr(\text{neither}) = 1 - 0.75 = 0.25$.

Q3. Tea only $= 28 - 12 = 16$; coffee only $= 25 - 12 = 13$; neither $= 50 - (16 + 12 + 13) = 9$.

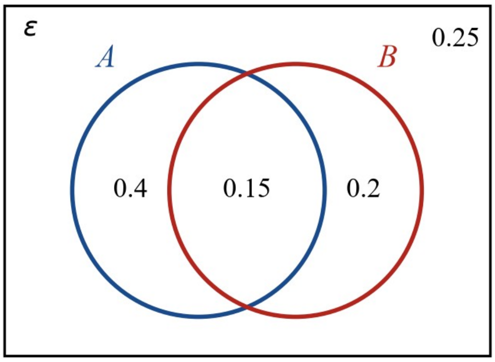


Q4. $A = \{2,4,6\}$, $B = \{4,5,6\}$, $A \cap B = \{4,6\}$. $Pr(A \cup B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$.

Q5. Rearranging the addition rule: $Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B) = 0.6 + 0.5 - 0.9 = 0.2$.

Q6. $Pr(A \cap B') = Pr(A) - Pr(A \cap B) = 0.7 - 0.25 = 0.45$.

Q7. $Pr(A \cup B) = 0.55 + 0.35 - 0.15 = 0.75$. Regions: A only 0.4, both 0.15, B only 0.2, neither 0.25.

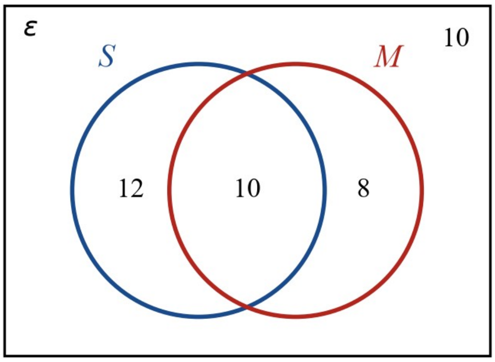


Q8. $Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B) = 0.4 + 0.5 - 0.7 = 0.2$.

Q9. Mutually exclusive: $Pr(A \cup B) = 0.2 + 0.35 = 0.55$; $Pr(\text{neither}) = 1 - 0.55 = 0.45$.

Q10. $Pr(\text{club}) = \frac{13}{52}$, $Pr(\text{ace}) = \frac{4}{52}$, $Pr(\text{club and ace}) = \frac{1}{52}$ (the ace of clubs).
$Pr(A \cup B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$.

Q11. Sport only $= 22 - 10 = 12$; music only $= 18 - 10 = 8$; neither $= 40 - (12 + 10 + 8) = 10$. $Pr(\text{neither}) = \frac{10}{40} = \frac{1}{4}$.



Q12. Multiple of 3: $\{3, 6\}$; odd: $\{1, 3, 5\}$; overlap $\{3\}$. $Pr = \frac{2}{6} + \frac{3}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$.

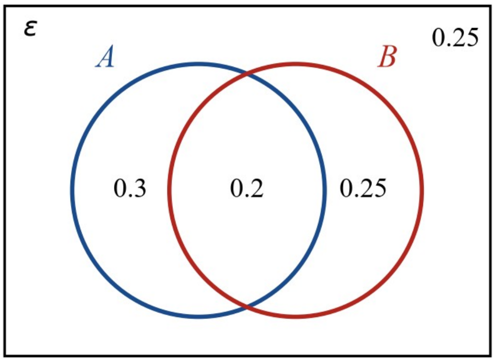
Q13. $Pr(A) = 1 - Pr(A') = 1 - 0.6 = 0.4$. Mutually exclusive, so $Pr(A \cup B) = 0.4 + 0.3 = 0.7$.

Q14. $Pr(\text{at least one}) = Pr(A \cup B) = 0.45 + 0.35 - 0.15 = 0.65$. $Pr(\text{neither}) = 1 - 0.65 = 0.35$.

Q15. $Pr(A \cap B') = Pr(A) - Pr(A \cap B) = 0.5 - 0.2 = 0.3$.

Q16. $Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B) = 0.4 + 0.5 - 0.9 = 0$. Since $Pr(A \cap B) = 0$, the events **are** mutually exclusive.

Q17. neither $= 1 - (0.3 + 0.2 + 0.25) = 0.25$; $Pr(A) = 0.3 + 0.2 = 0.5$; $Pr(B) = 0.2 + 0.25 = 0.45$;
$Pr(A \cup B) = 0.3 + 0.2 + 0.25 = 0.75$.



Q18. $Pr(\text{at least one}) = 0.4 + 0.3 - 0.1 = 0.6$; $Pr(\text{neither}) = 1 - 0.6 = 0.4$; $Pr(\text{dog but not cat}) = 0.4 - 0.1 = 0.3$.

Theory

A **two-way table** (or **Karnaugh table**) sets out the probabilities of two events A and B and their complements. The rows are A and A'; the columns are B and B'; a **Total** row and column give the marginal probabilities.

Pr	B	B'	Total
A	$Pr(A \cap B)$	$Pr(A \cap B')$	$Pr(A)$
A'	$Pr(A' \cap B)$	$Pr(A' \cap B')$	$Pr(A')$
Total	$Pr(B)$	$Pr(B')$	1

- The four inner cells are the **joint** probabilities. Each **row** adds to a row total ($Pr(A)$ or $Pr(A')$); each **column** adds to a column total ($Pr(B)$ or $Pr(B')$); the grand total is 1.
- **Filling in** a table: enter what you are given, then use "each row and column adds to its total" (and the grand total 1) to find the rest.
- **Reading** a table: $Pr(A \cap B)$ is a single cell; $Pr(A)$ and $Pr(B)$ are totals; $Pr(A')=1-Pr(A)$; and

$$Pr(A \cup B)=Pr(A \cap B)+Pr(A \cap B')+Pr(A' \cap B)=1-Pr(A' \cap B').$$

Link with a Venn diagram. The cells correspond to the Venn regions: $A \cap B$ is "both", $A \cap B'$ is "A only", $A' \cap B$ is "B only", and $A' \cap B'$ is "neither".

Worked examples

Example 1 — scaffolded

For two events, $Pr(A)=1/2$, $Pr(B)=2/5$ and $Pr(A \cap B)=1/5$. Complete a two-way table, then find $Pr(A \cup B)$ and $Pr(A' \cap B')$.

Working	Comment
Enter $Pr(A \cap B)=1/5$, the totals $Pr(A)=1/2$, $Pr(B)=2/5$, and grand total 1.	Place the given values first.
$Pr(A \cap B')=Pr(A)-Pr(A \cap B)=1/2-1/5=3/10$.	The A row adds to $Pr(A)$.
$Pr(A' \cap B)=Pr(B)-Pr(A \cap B)=2/5-1/5=1/5$.	The B column adds to $Pr(B)$.
$Pr(A' \cap B')=1-1/5-3/10-1/5=3/10$.	The four cells add to 1.

Pr	B	B'	Total
A	$1/5$	$3/10$	$1/2$
A'	$1/5$	$3/10$	$1/2$
Total	$2/5$	$3/5$	1

$Pr(A \cup B)=1-Pr(A' \cap B')=1-3/10=7/10$; and $Pr(A' \cap B')=3/10$.

Example 2 — scaffolded

For two events, $Pr(A)=3/5$, $Pr(B')=1/2$ and $Pr(A' \cap B')=1/5$. Complete the table and find $Pr(A \cup B)$.

Working	Comment
$Pr(A')=1-3/5=2/5$ and $Pr(B)=1-1/2=1/2$.	Complements of the given totals.
$Pr(A'\cap B)=Pr(A')-Pr(A'\cap B')=2/5-1/5=1/5$.	The A' row adds to $Pr(A')$.
$Pr(A\cap B')=Pr(B')-Pr(A'\cap B')=1/2-1/5=3/10$.	The B' column adds to $Pr(B')$.
$Pr(A\cap B)=Pr(A)-Pr(A\cap B')=3/5-3/10=3/10$.	The A row adds to $Pr(A)$.

Pr	B	B'	Total
A	$3/10$	$3/10$	$3/5$
A'	$1/5$	$1/5$	$2/5$
Total	$1/2$	$1/2$	1

$\therefore Pr(A\cup B)=1-Pr(A'\cap B')=1-1/5=4/5$.

Example 3 — unscaffolded

A two-way table has $Pr(A\cap B)=3/20$, $Pr(A\cap B')=1/4$ and $Pr(A'\cap B)=3/10$. Find $Pr(A'\cap B')$, the marginal probabilities, and $Pr(A\cup B)$.

$Pr(A'\cap B')=1-3/20-1/4-3/10=1-3/20-5/20-6/20=6/20=3/10$.

Pr	B	B'	Total
A	$3/20$	$1/4$	$2/5$
A'	$3/10$	$3/10$	$3/5$
Total	$9/20$	$11/20$	1

$\therefore Pr(A)=2/5$, $Pr(B)=9/20$, and $Pr(A\cup B)=1-3/10=7/10$.

Example 4 — unscaffolded

Of 100 students, 60 study Mathematics (A), 35 study Physics (B), and 20 study both. Set out a probability table and find the probability that a randomly chosen student studies neither subject, and the probability that a student studies Mathematics only.

Dividing each count by 100: $Pr(A\cap B)=20/100=1/5$; Mathematics only $=40/100=2/5$; Physics only $=15/100=3/20$; neither $=25/100=1/4$.

Pr	B	B'	Total
A	$1/5$	$2/5$	$3/5$
A'	$3/20$	$1/4$	$2/5$
Total	$7/20$	$13/20$	1

$\therefore Pr(\text{neither})=1/4$ and $Pr(\text{Mathematics only})=2/5$.

Practice questions

Scaffolded

Q1. $Pr(A)=3/5$, $Pr(B)=2/5$, $Pr(A\cap B)=1/4$. (a) Complete a two-way table. (b) Find $Pr(A\cup B)$. (c) Find $Pr(A'\cap B')$.

Q2. $Pr(A)=1/3$, $Pr(B)=1/2$, $Pr(A\cap B)=1/6$. (a) Complete a two-way table. (b) Find $Pr(A\cup B)$. (c) Find $Pr(A'\cap B)$.

Q3. A table has $Pr(A\cap B)=7/20$, $Pr(A\cap B')=3/20$, $Pr(A'\cap B)=1/5$, $Pr(A'\cap B')=3/10$. (a) Find the row and column totals. (b) State $Pr(A)$ and $Pr(B)$. (c) Find $Pr(A\cup B)$.

Q4. In a group of 100 people, 45 drink both coffee (A) and tea (B), 25 drink coffee only, 10 drink tea only, and 20 drink neither. (a) Complete a probability table. (b) Find $Pr(A)$. (c) Find $Pr(A\cup B)$.

Q5. $Pr(A)=2/5$, $Pr(B)=3/10$, $Pr(A\cap B)=1/10$. (a) Complete a two-way table. (b) Find $Pr(A\cup B)$. (c) Find $Pr(A'\cap B')$.

Q6. A Venn diagram shows $Pr(A\text{ only})=1/4$, $Pr(B\text{ only})=7/20$, $Pr(\text{both})=3/20$, $Pr(\text{neither})=1/4$. (a) Transfer this to a two-way table. (b) Find $Pr(A)$ and $Pr(B)$. (c) Find $Pr(A\cup B)$.

Unscaffolded

Complete a two-way table where needed, then answer each question.

Q7. $Pr(A)=1/2$, $Pr(B)=3/5$, $Pr(A\cap B)=3/10$. Find $Pr(A\cup B)$.

Q8. A table has $Pr(A\cap B)=0.18$, $Pr(A\cap B')=0.22$, $Pr(A'\cap B)=0.30$. Find $Pr(A'\cap B')$, $Pr(A)$, $Pr(B)$ and $Pr(A\cup B)$.

Q9. $Pr(A)=2/3$, $Pr(B)=1/2$, $Pr(A\cap B)=1/3$. Find $Pr(A\cup B)$ and $Pr(A'\cap B')$.

Q10. Of 120 people, 24 own a cat (A) and a dog (B), 36 own a cat only, 18 own a dog only, and the rest own neither. Find $Pr(A)$, $Pr(B)$ and $Pr(A\cup B)$.

Q11. $Pr(A\cap B)=1/8$, $Pr(A\cap B')=3/8$, $Pr(A'\cap B)=1/8$. Complete the table and find $Pr(A)$, $Pr(B)$ and $Pr(A\cup B)$.

Q12. $Pr(A)=1/2$, $Pr(B)=9/20$, $Pr(A\cap B)=1/5$. Find $Pr(A\cap B')$ and $Pr(A'\cap B')$.

Q13. $Pr(A)=7/10$, $Pr(B)=2/5$, $Pr(A\cap B)=3/10$. Find $Pr(A'\cap B')$ and $Pr(A\cup B)$.

Q14. In a class of 50, 12 students play both netball (A) and tennis (B), 8 play netball only, 15 play tennis only, and 15 play neither. Find $Pr(A)$, $Pr(B)$ and $Pr(A\cup B)$.

Q15. $Pr(A)=11/20$, $Pr(B)=9/20$, $Pr(A\cap B)=1/4$. Find $Pr(A'\cap B')$ and $Pr(A\cup B)$.

Q16. A table has $Pr(A\cap B)=1/4$, $Pr(A\cap B')=1/5$, $Pr(A'\cap B)=1/4$, $Pr(A'\cap B')=3/10$. Find $Pr(A)$, $Pr(B)$ and $Pr(A\cup B)$.

Q17. Of 80 students, 28 passed both tests (A and B), 12 passed test A only, 20 passed test B only, and 20 passed neither. Find $Pr(A)$, $Pr(B)$ and $Pr(A\cup B)$.

Q18. $Pr(A)=3/8$, $Pr(B)=1/2$, $Pr(A\cap B)=1/4$. Find $Pr(A\cap B')$, $Pr(A'\cap B')$ and $Pr(A\cup B)$.

Answers

Q1. cells $1/4, 7/20, 3/20, 1/4$; $Pr(A \cup B) = 3/4$; $Pr(A' \cap B') = 1/4$. **Q2.** cells $1/6, 1/6, 1/3, 1/3$; $Pr(A \cup B) = 2/3$; $Pr(A' \cap B) = 1/3$. **Q3.** $Pr(A) = 1/2$, $Pr(A') = 1/2$, $Pr(B) = 11/20$, $Pr(B') = 9/20$; $Pr(A \cup B) = 7/10$. **Q4.** cells $9/20, 1/4, 1/10, 1/5$; $Pr(A) = 7/10$; $Pr(A \cup B) = 4/5$. **Q5.** cells $1/10, 3/10, 1/5, 2/5$; $Pr(A \cup B) = 3/5$; $Pr(A' \cap B') = 2/5$. **Q6.** cells $3/20, 1/4, 7/20, 1/4$; $Pr(A) = 2/5$, $Pr(B) = 1/2$; $Pr(A \cup B) = 3/4$. **Q7.** $Pr(A \cup B) = 4/5$. **Q8.** $Pr(A' \cap B') = 0.30$; $Pr(A) = 0.40$; $Pr(B) = 0.48$; $Pr(A \cup B) = 0.70$. **Q9.** $Pr(A \cup B) = 5/6$; $Pr(A' \cap B') = 1/6$. **Q10.** $Pr(A) = 1/2$, $Pr(B) = 7/20$; $Pr(A \cup B) = 13/20$. **Q11.** $Pr(A' \cap B') = 3/8$; $Pr(A) = 1/2$, $Pr(B) = 1/4$; $Pr(A \cup B) = 5/8$. **Q12.** $Pr(A \cap B') = 3/10$; $Pr(A' \cap B') = 1/4$. **Q13.** $Pr(A' \cap B') = 1/5$; $Pr(A \cup B) = 4/5$. **Q14.** $Pr(A) = 2/5$, $Pr(B) = 27/50$; $Pr(A \cup B) = 7/10$. **Q15.** $Pr(A' \cap B') = 1/4$; $Pr(A \cup B) = 3/4$. **Q16.** $Pr(A) = 9/20$, $Pr(B) = 1/2$; $Pr(A \cup B) = 7/10$. **Q17.** $Pr(A) = 1/2$, $Pr(B) = 3/5$; $Pr(A \cup B) = 3/4$. **Q18.** $Pr(A \cap B') = 1/8$; $Pr(A' \cap B') = 3/8$; $Pr(A \cup B) = 5/8$.

Fully-worked solutions

Q1. $Pr(A \cap B) = 1/4$; $Pr(A \cap B') = 3/5 - 1/4 = 7/20$; $Pr(A' \cap B) = 2/5 - 1/4 = 3/20$; $Pr(A' \cap B') = 1 - 1/4 - 7/20 - 3/20 = 1/4$. $Pr(A \cup B) = 1 - 1/4 = 3/4$; $Pr(A' \cap B') = 1/4$.

Q2. $Pr(A \cap B') = 1/3 - 1/6 = 1/6$; $Pr(A' \cap B) = 1/2 - 1/6 = 1/3$; $Pr(A' \cap B') = 1 - 1/6 - 1/6 - 1/3 = 1/3$. $Pr(A \cup B) = 1 - 1/3 = 2/3$; $Pr(A' \cap B) = 1/3$.

Q3. Row totals: $Pr(A) = 7/20 + 3/20 = 1/2$; $Pr(A') = 1/5 + 3/10 = 1/2$. Column totals: $Pr(B) = 7/20 + 1/5 = 11/20$; $Pr(B') = 3/20 + 3/10 = 9/20$. $Pr(A \cup B) = 1 - Pr(A' \cap B') = 1 - 3/10 = 7/10$.

Q4. As probabilities: both $45/100 = 9/20$, coffee only $1/4$, tea only $1/10$, neither $1/5$. $Pr(A) = 9/20 + 1/4 = 7/10$; $Pr(A \cup B) = 1 - 1/5 = 4/5$.

Q5. $Pr(A \cap B') = 2/5 - 1/10 = 3/10$; $Pr(A' \cap B) = 3/10 - 1/10 = 1/5$; $Pr(A' \cap B') = 1 - 1/10 - 3/10 - 1/5 = 2/5$. $Pr(A \cup B) = 1 - 2/5 = 3/5$; $Pr(A' \cap B') = 2/5$.

Q6. Table cells: $A \cap B = 3/20$, $A \cap B' = 1/4$, $A' \cap B = 7/20$, $A' \cap B' = 1/4$. $Pr(A) = 3/20 + 1/4 = 2/5$; $Pr(B) = 3/20 + 7/20 = 1/2$; $Pr(A \cup B) = 1 - 1/4 = 3/4$.

Q7. $Pr(A' \cap B') = 1 - Pr(A) - Pr(B) + Pr(A \cap B) = 1 - 1/2 - 3/5 + 3/10 = 1/5$; $Pr(A \cup B) = 1 - 1/5 = 4/5$.

Q8. $Pr(A' \cap B') = 1 - 0.18 - 0.22 - 0.30 = 0.30$. $Pr(A) = 0.18 + 0.22 = 0.40$; $Pr(B) = 0.18 + 0.30 = 0.48$; $Pr(A \cup B) = 1 - 0.30 = 0.70$.

Q9. $Pr(A \cap B') = 2/3 - 1/3 = 1/3$; $Pr(A' \cap B) = 1/2 - 1/3 = 1/6$; $Pr(A' \cap B') = 1 - 1/3 - 1/3 - 1/6 = 1/6$. $Pr(A \cup B) = 1 - 1/6 = 5/6$.

Q10. Neither $= 120 - 24 - 36 - 18 = 42$. As probabilities: both $24/120 = 1/5$, cat only $3/10$, dog only $3/20$, neither $7/20$. $Pr(A) = 1/5 + 3/10 = 1/2$; $Pr(B) = 1/5 + 3/20 = 7/20$; $Pr(A \cup B) = 1 - 7/20 = 13/20$.

Q11. $Pr(A' \cap B') = 1 - 1/8 - 3/8 - 1/8 = 3/8$. $Pr(A) = 1/8 + 3/8 = 1/2$; $Pr(B) = 1/8 + 1/8 = 1/4$; $Pr(A \cup B) = 1 - 3/8 = 5/8$.

Q12. $Pr(A \cap B') = 1/2 - 1/5 = 3/10$; $Pr(A' \cap B) = 9/20 - 1/5 = 1/4$; $Pr(A' \cap B') = 1 - 1/5 - 3/10 - 1/4 = 1/4$.

Q13. $Pr(A' \cap B') = 1 - 7/10 - 2/5 + 3/10 = 1/5$; $Pr(A \cup B) = 1 - 1/5 = 4/5$.

Q14. As probabilities: both $12/50=6/25$, netball only $4/25$, tennis only $3/10$, neither $3/10$. $Pr(A)=6/25+4/25=2/5$; $Pr(B)=6/25+3/10=27/50$; $Pr(A\cup B)=1-3/10=7/10$.

Q15. $Pr(A\cap B')=11/20-1/4=3/10$; $Pr(A'\cap B)=9/20-1/4=1/5$; $Pr(A'\cap B')=1-1/4-3/10-1/5=1/4$. $Pr(A\cup B)=1-1/4=3/4$.

Q16. $Pr(A)=1/4+1/5=9/20$; $Pr(B)=1/4+1/4=1/2$; $Pr(A\cup B)=1-3/10=7/10$.

Q17. As probabilities: both $28/80=7/20$, A only $3/20$, B only $1/4$, neither $1/4$. $Pr(A)=7/20+3/20=1/2$; $Pr(B)=7/20+1/4=3/5$; $Pr(A\cup B)=1-1/4=3/4$.

Q18. $Pr(A\cap B')=3/8-1/4=1/8$; $Pr(A'\cap B)=1/2-1/4=1/4$; $Pr(A'\cap B')=1-1/4-1/8-1/4=3/8$. $Pr(A\cup B)=1-3/8=5/8$.

Theory

The **conditional probability** of A **given** that B has occurred is written $Pr(A/B)$ and defined by

$$Pr(A/B)=\frac{Pr(A\cap B)}{Pr(B)},\ Pr(B)\neq 0.$$

Rearranging gives the **multiplication rule**:

$$Pr(A\cap B)=Pr(B)Pr(A/B)=Pr(A)Pr(B/A).$$

Two-way tables. A conditional probability restricts attention to one row or column of a two-way (contingency) table: $Pr(A/B)$ is found by dividing the count in the $A\cap B$ cell by the **total for** B (not the grand total).

Tree diagrams. In a probability tree, the labels on the **second-stage** branches are already conditional probabilities — the probability of the second outcome *given* the first. Multiplying along a path (first branch × second branch) applies the multiplication rule and gives the probability of that combined outcome.

Law of total probability. When A can occur along either branch of a partition into B and its complement B', its probability is the **weighted sum** over the two branches:

$$Pr(A)=Pr(A/B)Pr(B)+Pr(A/B')Pr(B').$$

That is, add the two complete tree paths that end in A (see Example 4, where the "defective" paths through machines A and B are added).

Independence. Events A and B are **independent** when $Pr(A/B)=Pr(A)$ — knowing B occurred does not change the probability of A. Equivalently $Pr(A\cap B)=Pr(A)Pr(B)$.

Order matters. In general $Pr(A/B)\neq Pr(B/A)$.

Worked examples

Example 1 — scaffolded

The table shows 100 students by gender and whether they play a sport.

	Plays sport	Does not	Total
Male	30	20	50
Female	25	25	50
Total	55	45	100

Find (a) $Pr(\text{plays sport}/\text{male})$ and (b) $Pr(\text{male}/\text{plays sport})$.

Working	Comment
(a) Restrict to the **Male** row (total 50): $Pr(\text{sport}/\text{male})=\frac{30}{50}=\frac{3}{5}$.	Divide the $\text{sport}\cap\text{male}$ cell by the male total.
(b) Restrict to the **Plays sport** column (total 55): $Pr(\text{male}/\text{sport})=\frac{30}{55}=\frac{6}{11}$.	Same cell, different condition — divide by the column total.

Note $\frac{3}{5} \neq \frac{6}{11}$: the two conditionals are different.

Example 2 — scaffolded

For events A and B, $Pr(A)=\frac{1}{2}$, $Pr(B)=\frac{1}{3}$ and $Pr(A \cap B)=\frac{1}{6}$.

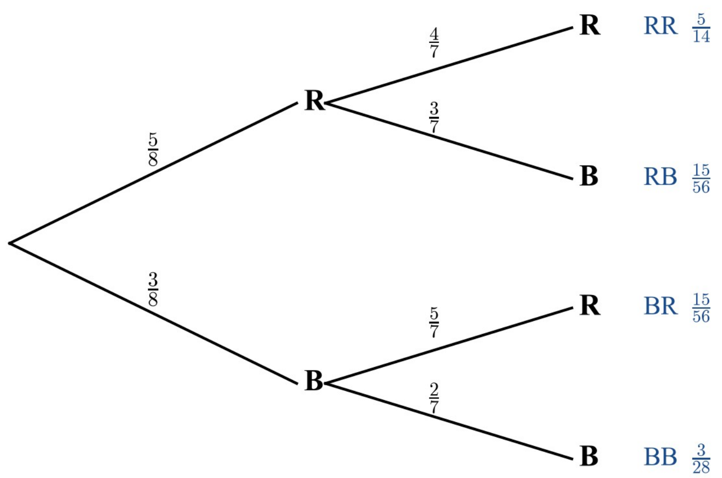
Working	Comment
$Pr(A/B)=\frac{Pr(A \cap B)}{Pr(B)}=\frac{1/6}{1/3}=\frac{1}{2}$.	Apply the definition.
$Pr(B/A)=\frac{Pr(A \cap B)}{Pr(A)}=\frac{1/6}{1/2}=\frac{1}{3}$.	Divide by $Pr(A)$ instead.
Since $Pr(A/B)=\frac{1}{2}=Pr(A)$, the events are **independent**.	Knowing B did not change $Pr(A)$.

Example 3 — unscaffolded

A bag holds 5 red and 3 blue marbles. Two are drawn **without replacement**. Find $Pr(\text{2nd red} / \text{1st red})$ and $Pr(\text{both red})$.

After a red is drawn, 7 marbles remain, 4 red, so $Pr(\text{2nd red} / \text{1st red})=\frac{4}{7}$.

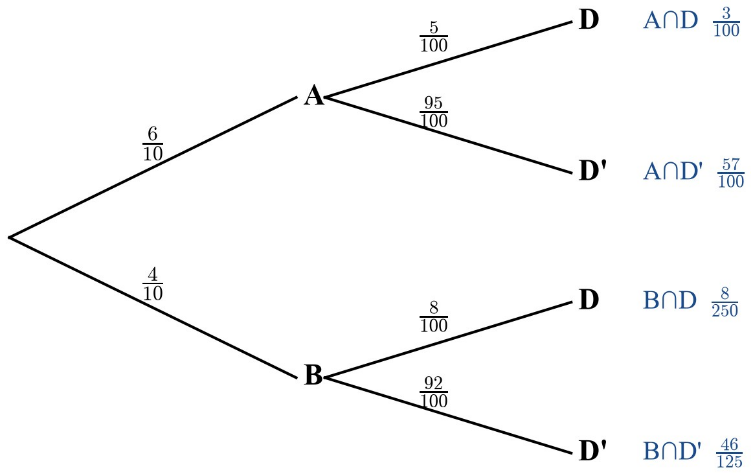
By the multiplication rule, $Pr(\text{both red})=\frac{5}{8} \times \frac{4}{7}=\frac{20}{56}=\frac{5}{14}$.



$\therefore Pr(\text{both red})=\frac{5}{14}$.

Example 4 — unscaffolded

In a factory, machine A makes 60% of the items and 5% of them are defective; machine B makes the other 40% and 8% of those are defective. An item is chosen at random. Find the probability it is defective, and hence $Pr(A / \text{defective})$.



$$Pr(\text{defective}) = Pr(A)Pr(D/A) + Pr(B)Pr(D/B) = \frac{6}{10}\cdot\frac{5}{100} + \frac{4}{10}\cdot\frac{8}{100} = \frac{3}{100} + \frac{8}{250} = \frac{31}{500}.$$

$$Pr(A/\text{defective}) = \frac{Pr(A \cap \text{defective})}{Pr(\text{defective})} = \frac{3/100}{31/500} = \frac{15}{31}.$$

$\therefore Pr(A/\text{defective}) = \frac{15}{31}$.

Practice questions

Scaffolded

Q1. The table shows 200 people by home type and whether they own a pet.

	Pet	No pet	Total
House	90	40	130
Apartment	30	40	70
Total	120	80	200

(a) Find $Pr(\text{pet}/\text{apartment})$. (b) Find $Pr(\text{apartment}/\text{pet})$.

Q2. For events A and B, $Pr(A) = 0.5$, $Pr(B) = 0.4$ and $Pr(A \cap B) = 0.3$. (a) Find $Pr(A/B)$. (b) Find $Pr(B/A)$. (c) Are A and B independent? Justify.

Q3. A box has 4 white and 6 black counters. Two are drawn without replacement. (a) Find $Pr(\text{2nd white}/\text{1st white})$. (b) Find $Pr(\text{both white})$.

Q4. On any day $Pr(\text{rain}) = 0.3$. If it rains, $Pr(\text{late}/\text{rain}) = 0.6$; if not, $Pr(\text{late}/\text{no rain}) = 0.1$. (a) Draw a tree diagram. (b) Find $Pr(\text{late})$.

Q5. A gym surveys its members.

	Morning	Evening	Total
Full	24	16	40
Casual	27	9	36
Total	51	25	76

(a) Find $Pr(\text{morning}/\text{full})$. (b) Find $Pr(\text{full}/\text{morning})$.

Q6. For events A and B, $Pr(A)=0.6$, $Pr(B)=0.5$ and $Pr(A \cap B)=0.3$. By finding $Pr(A / B)$, decide whether A and B are independent.

Unscaffolded

Q7. A survey of 100 students records who likes maths and who likes science.

	Likes science	Does not	Total
Likes maths	40	10	50
Does not	20	30	50
Total	60	40	100

Find $Pr(\text{likes science} / \text{likes maths})$ and $Pr(\text{likes maths} / \text{likes science})$.

Q8. $Pr(A)=0.7$, $Pr(B)=0.5$, $Pr(A \cap B)=0.35$. Find $Pr(A / B)$ and $Pr(B / A)$, and state whether A and B are independent.

Q9. One card is drawn from a standard deck of 52. Find $Pr(\text{king} / \text{face card})$ (the face cards are the jacks, queens and kings).

Q10. Two fair dice are rolled. Find $Pr(\text{sum} > 9 / \text{first die is } 5)$.

Q11. A bag has 3 red and 2 green discs. Two are drawn without replacement. Find $Pr(\text{2nd green} / \text{1st red})$ and $Pr(\text{red then green})$.

Q12. A bag has 5 red and 3 blue balls. Two are drawn **with replacement**. Find $Pr(\text{2nd red} / \text{1st red})$, and hence explain why the two draws are independent.

Q13. A disease affects 2% of a population. A test is positive for 95% of people who have the disease and for 10% of people who do not. Find $Pr(\text{positive})$ and hence $Pr(\text{disease} / \text{positive})$.

Q14. For events A and B, $Pr(B)=0.4$ and $Pr(A / B)=0.25$. Find $Pr(A \cap B)$.

Q15. The table shows 100 people by gender and handedness.

	Left	Right	Total
Male	5	45	50
Female	5	45	50
Total	10	90	100

By comparing $Pr(\text{left} / \text{male})$ with $Pr(\text{left})$, decide whether handedness and gender are independent.

Q16. A bag has 6 red and 4 black balls. Three are drawn without replacement. Find $Pr(\text{all three red})$.

Q17. A café records 100 orders.

	Cake	No cake	Total
Coffee	30	50	80
Tea	10	10	20
Total	40	60	100

Find $Pr(\text{cake} / \text{coffee})$ and $Pr(\text{coffee} / \text{cake})$, and comment.

Q18. Bag A holds 3 red and 2 white balls; bag B holds 1 red and 4 white. A bag is chosen at random and one ball is drawn. Given the ball is red, find $Pr(\text{bag } A / \text{red})$.

Answers

Q1. (a) $3/7$ (b) $1/4$. **Q2.** (a) $3/4$ (b) $3/5$ (c) not independent. **Q3.** (a) $1/3$ (b) $2/15$. **Q4.** (b) $1/4$. **Q5.** (a) $3/5$ (b) $8/17$. **Q6.** $Pr(A/B)=3/5=Pr(A)$, so independent. **Q7.** $4/5$ and $2/3$. **Q8.** $7/10$, $1/2$; independent. **Q9.** $1/3$. **Q10.** $1/3$. **Q11.** $1/2$ and $3/10$. **Q12.** $5/8$; independent. **Q13.** $117/1000$ and $19/117$. **Q14.** $1/10$. **Q15.** $1/10=1/10$, so independent. **Q16.** $1/6$. **Q17.** $3/8$ and $3/4$; the two conditionals differ. **Q18.** $3/4$.

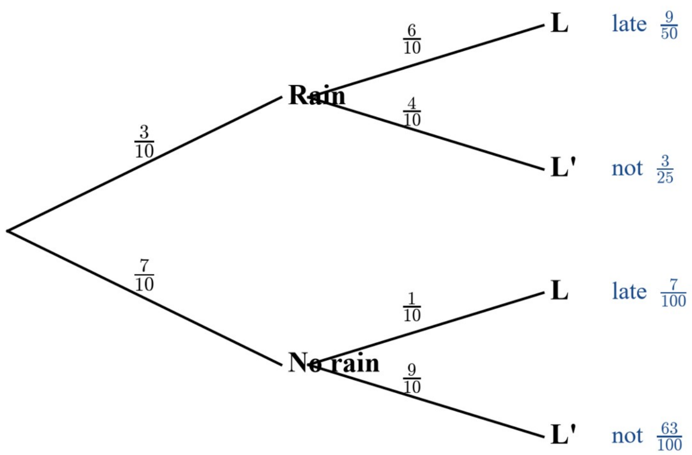
Fully-worked solutions

Q1. (a) Apartment row total 70: $Pr(\text{pet}/\text{apartment})=\frac{30}{70}=\frac{3}{7}$. (b) Pet column total 120: $Pr(\text{apartment}/\text{pet})=\frac{30}{120}=\frac{1}{4}$.

Q2. (a) $Pr(A/B)=\frac{0.3}{0.4}=\frac{3}{4}$. (b) $Pr(B/A)=\frac{0.3}{0.5}=\frac{3}{5}$. (c) $Pr(A/B)=\frac{3}{4}\neq\frac{1}{2}=Pr(A)$, so A and B are **not independent**.

Q3. (a) After a white is removed, 9 counters remain, 3 white: $Pr(\text{2nd white}/\text{1st white})=\frac{3}{9}=\frac{1}{3}$. (b) $Pr(\text{both white})=\frac{4}{10}\times\frac{3}{9}=\frac{12}{90}=\frac{2}{15}$.

Q4. (b) From the tree, $Pr(\text{late})=\frac{3}{10}\cdot\frac{6}{10}+\frac{7}{10}\cdot\frac{1}{10}=\frac{18}{100}+\frac{7}{100}=\frac{25}{100}=\frac{1}{4}$.



Q5. (a) Full row total 40: $Pr(\text{morning}/\text{full})=\frac{24}{40}=\frac{3}{5}$. (b) Morning column total 51: $Pr(\text{full}/\text{morning})=\frac{24}{51}=\frac{8}{17}$.

Q6. $Pr(A/B)=\frac{0.3}{0.5}=\frac{3}{5}=0.6=Pr(A)$. Since $Pr(A/B)=Pr(A)$, the events are **independent**.

Q7. Maths row total 50: $Pr(\text{science}/\text{maths})=\frac{40}{50}=\frac{4}{5}$. Science column total 60: $Pr(\text{maths}/\text{science})=\frac{40}{60}=\frac{2}{3}$.

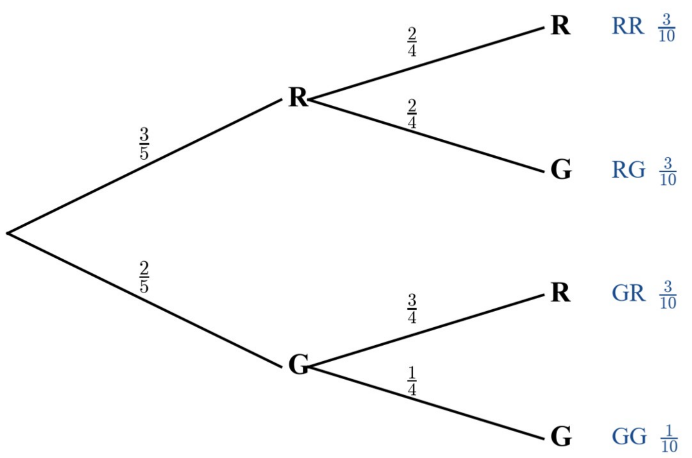
Q8. $Pr(A/B)=\frac{0.35}{0.5}=\frac{7}{10}$; $Pr(B/A)=\frac{0.35}{0.7}=\frac{1}{2}$. Since $Pr(A/B)=\frac{7}{10}=Pr(A)$, A and B are **independent**.

Q9. There are 12 face cards, of which 4 are kings: $Pr(\text{king} / \text{face card}) = \frac{4}{12} = \frac{1}{3}$.

Q10. Given the first die is 5, the six equally likely outcomes are $(5,1), \ldots, (5,6)$, with sums 6 to 11. The sum exceeds 9 for $(5,5)$ and $(5,6)$, so the probability is $\frac{2}{6} = \frac{1}{3}$.

Q11. After a red is drawn, 4 discs remain, 2 green: $Pr(\text{2nd green} / \text{1st red}) = \frac{2}{4} = \frac{1}{2}$.

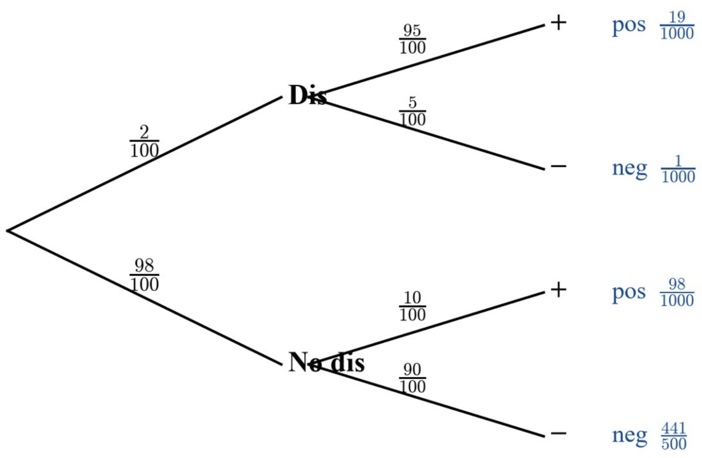
$Pr(\text{red then green}) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$.



Q12. With replacement the bag is restored, so $Pr(\text{2nd red} / \text{1st red}) = \frac{5}{8}$. Because this equals $Pr(\text{red}) = \frac{5}{8}$, the first draw does not affect the second — the draws are **independent**.

Q13. $Pr(\text{positive}) = \frac{2}{100} \cdot \frac{95}{100} + \frac{98}{100} \cdot \frac{10}{100} = \frac{19}{1000} + \frac{98}{1000} = \frac{117}{1000}$.

$$Pr(\text{disease} / \text{positive}) = \frac{Pr(\text{disease} \cap \text{positive})}{Pr(\text{positive})} = \frac{19/1000}{117/1000} = \frac{19}{117}.$$



Only about 16 % of positive tests are true positives, because the disease is rare.

Q14. By the multiplication rule, $Pr(A \cap B) = Pr(B)\,Pr(A / B) = 0.4 \times 0.25 = \frac{1}{10}$.

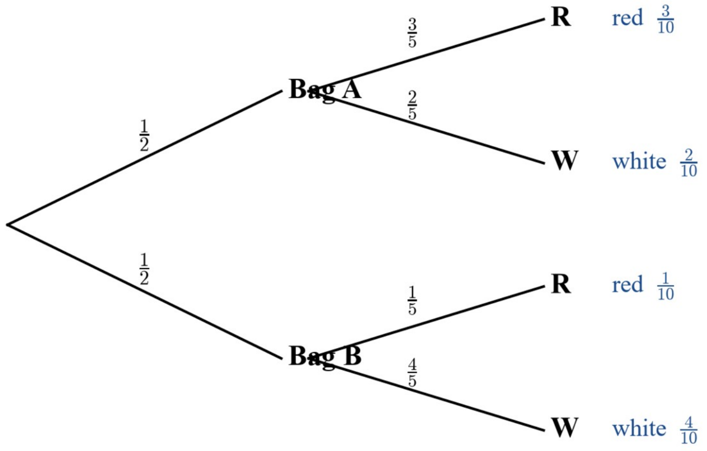
Q15. $Pr(\text{left} / \text{male}) = \frac{5}{50} = \frac{1}{10}$ and $Pr(\text{left}) = \frac{10}{100} = \frac{1}{10}$. They are equal, so handedness and gender are **independent**.

Q16. $Pr(\text{all three red}) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720} = \frac{1}{6}$.

Q17. Coffee row total 80: $Pr(\text{cake} / \text{coffee}) = \frac{30}{80} = \frac{3}{8}$. Cake column total 40: $Pr(\text{coffee} / \text{cake}) = \frac{30}{40} = \frac{3}{4}$. The two conditionals are different — $Pr(A / B) \neq Pr(B / A)$ in general.

Q18. $Pr(\text{red}) = \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{1}{5} = \frac{3}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$.

$$Pr(\text{bag } A / \text{red}) = \frac{Pr(\text{bag } A \cap \text{red})}{Pr(\text{red})} = \frac{3/10}{2/5} = \frac{3}{4}.$$



Chapter 8 Probability and simulation — 8G Independence

Theory

Two events A and B are **independent** if the occurrence of one does not change the probability of the other. Formally, A and B are independent when

$$Pr(A \cap B) = Pr(A)Pr(B).$$

Equivalently (provided $Pr(B) > 0$), A and B are independent when $Pr(A / B) = Pr(A)$ — knowing that B has occurred leaves the probability of A unchanged.

Testing for independence. Compute $Pr(A)Pr(B)$ and compare it with $Pr(A \cap B)$. If they are **equal**, the events are independent; if they **differ**, the events are dependent. The same test works when the probabilities come from a two-way table.

Independent is not the same as mutually exclusive. Mutually exclusive events cannot both occur, so $Pr(A \cap B) = 0$. Independent events satisfy $Pr(A \cap B) = Pr(A)Pr(B)$. Two events that each have a non-zero probability therefore **cannot** be both mutually exclusive and independent.

Sequences of independent trials. If events are independent, the probability that they **all** occur is the product of their probabilities. For the probability of **"at least one"**, use the complement:

$$Pr(\text{at least one}) = 1 - Pr(\text{none}).$$

Worked examples

Example 1 — scaffolded

A fair six-sided die is rolled and a fair coin is tossed. Let A be the event "the die shows a 6" and B the event "the coin shows heads". Explain why A and B are independent, and find $Pr(A \cap B)$.

Working	Comment
The die roll and the coin toss do not affect each other, so A and B are **independent**.	Physically separate actions are independent.
$Pr(A) = \frac{1}{6}$, $Pr(B) = \frac{1}{2}$.	Each outcome is equally likely.
$Pr(A \cap B) = Pr(A)Pr(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.	For independent events, multiply.

Example 2 — scaffolded

For two events, $Pr(A) = 0.4$, $Pr(B) = 0.5$ and $Pr(A \cap B) = 0.2$. Determine whether A and B are independent.

Working	Comment
$Pr(A)Pr(B) = 0.4 \times 0.5 = 0.2$.	Compute the product.
$Pr(A \cap B) = 0.2$.	Compare with the given intersection.
Since $Pr(A)Pr(B) = Pr(A \cap B)$, the events **are independent**.	Equal ⇒ independent.

Example 3 — unscaffolded

For two events, $Pr(A) = 0.3$, $Pr(B) = 0.6$ and $Pr(A \cup B) = 0.72$. Are A and B independent?

First find the intersection using the addition rule:

$$Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B) = 0.3 + 0.6 - 0.72 = 0.18.$$

Now test: $Pr(A)Pr(B) = 0.3 \times 0.6 = 0.18 = Pr(A \cap B)$.

$\therefore$ A and B are independent.

Example 4 — unscaffolded

A fair coin is tossed three times. Find the probability of obtaining at least one head.

The tosses are independent. Use the complement — the only way to get no heads is TTT:

$$Pr(\text{no heads}) = (1/2)^3 = 1/8.$$

$$Pr(\text{at least one head}) = 1 - 1/8 = 7/8.$$

$\therefore$ the probability of at least one head is $\frac{7}{8}$.

Practice questions

Scaffolded

Q1. A and B are independent with $Pr(A) = 0.5$ and $Pr(B) = 0.3$. Find $Pr(A \cap B)$.

Q2. For two events, $Pr(A) = 0.2$, $Pr(B) = 0.5$ and $Pr(A \cap B) = 0.1$. (a) Find $Pr(A)Pr(B)$. (b) Hence state whether A and B are independent.

Q3. Two fair dice are rolled. Find the probability that **both** show a 5.

Q4. A spinner lands on red with probability 0.4. It is spun twice. Find the probability that it lands on red **both** times.

Q5. For two events, $Pr(A) = 0.6$, $Pr(B) = 0.5$ and $Pr(A \cap B) = 0.35$. Determine whether A and B are independent.

Q6. A component works with probability 0.9. Two such components operate independently. (a) Find the probability that **both** work. (b) Find the probability that **at least one** works.

Unscaffolded

Q7. A and B are independent with $Pr(A) = 0.7$ and $Pr(B) = 0.4$. Find $Pr(A \cap B)$ and $Pr(A \cup B)$.

Q8. A and B are independent with $Pr(A) = 1/3$ and $Pr(B) = 1/4$. Find $Pr(A \cap B)$ and $Pr(A \cup B)$.

Q9. For two events, $Pr(A) = 0.5$, $Pr(B) = 0.4$ and $Pr(A \cap B) = 0.25$. Are A and B independent?

Q10. The two-way table below records 100 people by whether they own a car (A) and whether they own a bike (B). Determine whether owning a car and owning a bike are independent.

B	B'	
A	12	18
A'	28	42

Q11. For the two-way table below (again 100 people), determine whether A and B are independent.

B	B'	
A	20	10
A'	20	50

Q12. A fair die is rolled twice. Find the probability of rolling at least one 6.

Q13. A basketball player makes each free throw independently with probability 0.8. She takes three shots. Find (a) the probability she makes all three, and (b) the probability she misses at least one.

Q14. For two events, $Pr(A)=0.6$ and $Pr(B)=0.3$, and A and B are **mutually exclusive**. Are A and B independent? Justify your answer.

Q15. Machine A works with probability 0.95 and machine B with probability 0.90, and they operate independently. Find (a) the probability that both work, and (b) the probability that at least one works.

Q16. A card is drawn from a standard pack, replaced, and a second card is drawn. Find the probability that both cards are hearts.

Q17. A and B are independent, $Pr(B)=0.5$ and $Pr(A\cap B)=0.35$. Find $Pr(A)$.

Q18. The probability of rain on Saturday is 0.3 and on Sunday is 0.4, and these are independent. Find the probability that it rains (a) on both days, (b) on neither day, and (c) on at least one day.

Answers

Q1. 0.15. **Q2.** (a) 0.1; (b) independent. **Q3.** $1/36$. **Q4.** 0.16. **Q5.** Not independent ($0.30 \neq 0.35$). **Q6.** (a) 0.81; (b) 0.99. **Q7.** $Pr(A \cap B) = 0.28$, $Pr(A \cup B) = 0.82$. **Q8.** $1/12$; $1/2$. **Q9.** Not independent ($0.20 \neq 0.25$). **Q10.** Independent. **Q11.** Not independent. **Q12.** $11/36$. **Q13.** (a) 0.512; (b) 0.488. **Q14.** Not independent (see below). **Q15.** (a) 0.855; (b) 0.995. **Q16.** $1/16$. **Q17.** 0.7. **Q18.** (a) 0.12; (b) 0.42; (c) 0.58.

Fully-worked solutions

Q1. Independent, so $Pr(A \cap B) = Pr(A)Pr(B) = 0.5 \times 0.3 = 0.15$.

Q2. (a) $Pr(A)Pr(B) = 0.2 \times 0.5 = 0.1$. (b) This equals $Pr(A \cap B) = 0.1$, so A and B **are independent**.

Q3. The two rolls are independent: $Pr(\text{both } 5) = 1/6 \times 1/6 = 1/36$.

Q4. Independent spins: $Pr(\text{red twice}) = 0.4 \times 0.4 = 0.16$.

Q5. $Pr(A)Pr(B) = 0.6 \times 0.5 = 0.30$, but $Pr(A \cap B) = 0.35$. Since $0.30 \neq 0.35$, the events are **not independent**.

Q6. (a) $Pr(\text{both work}) = 0.9 \times 0.9 = 0.81$. (b) $Pr(\text{at least one}) = 1 - Pr(\text{both fail}) = 1 - (0.1)^2 = 1 - 0.01 = 0.99$.

Q7. $Pr(A \cap B) = 0.7 \times 0.4 = 0.28$; $Pr(A \cup B) = 0.7 + 0.4 - 0.28 = 0.82$.

Q8. $Pr(A \cap B) = 1/3 \times 1/4 = 1/12$; $Pr(A \cup B) = 1/3 + 1/4 - 1/12 = 4 + 3 - 1/12 = 6/12 = 1/2$.

Q9. $Pr(A)Pr(B) = 0.5 \times 0.4 = 0.20 \neq 0.25 = Pr(A \cap B)$, so **not independent**.

Q10. From the table (totals 100): $Pr(A) = 30/100 = 0.3$, $Pr(B) = 40/100 = 0.4$, $Pr(A \cap B) = 12/100 = 0.12$. Since $Pr(A)Pr(B) = 0.3 \times 0.4 = 0.12 = Pr(A \cap B)$, A and B are **independent**.

Q11. Here $Pr(A) = 0.3$, $Pr(B) = 0.4$ but $Pr(A \cap B) = 20/100 = 0.20$. Since $0.3 \times 0.4 = 0.12 \neq 0.20$, A and B are **not independent**.

Q12. $Pr(\text{at least one } 6) = 1 - Pr(\text{no } 6) = 1 - (5/6)^2 = 1 - 25/36 = 11/36$.

Q13. (a) $Pr(\text{all three}) = (0.8)^3 = 0.512$. (b) $Pr(\text{at least one miss}) = 1 - 0.512 = 0.488$.

Q14. Mutually exclusive events have $Pr(A \cap B) = 0$. But $Pr(A)Pr(B) = 0.6 \times 0.3 = 0.18 \neq 0$. Since $Pr(A \cap B) \neq Pr(A)Pr(B)$, the events are **not independent** — in fact mutually exclusive events with non-zero probabilities are always dependent.

Q15. (a) $Pr(\text{both work}) = 0.95 \times 0.90 = 0.855$. (b) $Pr(\text{at least one}) = 1 - Pr(\text{both fail}) = 1 - (0.05)(0.10) = 1 - 0.005 = 0.995$.

Q16. With replacement the draws are independent and $Pr(\text{heart}) = 13/52 = 1/4$, so $Pr(\text{both hearts}) = 1/4 \times 1/4 = 1/16$.

Q17. Independent, so $Pr(A \cap B) = Pr(A)Pr(B)$: $0.35 = Pr(A) \times 0.5$, giving $Pr(A) = \dfrac{0.35}{0.5} = 0.7$.

Q18. Independent days. (a) $Pr(\text{both}) = 0.3 \times 0.4 = 0.12$. (b) $Pr(\text{neither}) = (1 - 0.3)(1 - 0.4) = 0.7 \times 0.6 = 0.42$. (c) $Pr(\text{at least one}) = 1 - 0.42 = 0.58$.

Theory

A **simulation** uses a random device to imitate a real situation so that a probability can be **estimated** from the results. Simulation is useful when a probability is difficult to calculate exactly, or to model how a random process behaves over many repetitions.

Steps in a simulation.

1. **Choose a random device** whose outcomes match the probabilities in the situation.
2. **Assign** the device's outcomes to the real outcomes *in proportion* to their probabilities.
3. **Define one trial** (one repetition of the situation) and state what counts as a **success**.
4. **Carry out many trials** and count the successes.
5. **Estimate** the probability:

$$Pr(\text{event}) \approx \frac{\text{number of successes}}{\text{number of trials}}.$$

Common devices. A **coin** gives two equally likely outcomes ($Pr = 1/2$ each); a **die** gives six ($1/6$ each); **random digits** 0–9 give ten ($1/10$ each) and suit probabilities written in tenths; **two-digit numbers** 00–99 give hundredths.

Assigning in proportion. Match the number of device outcomes to the probability. For example, if $Pr(\text{success}) = 0.4$, assign 4 of the 10 digits (say $0, 1, 2, 3$) to a success and the remaining 6 digits to a failure. For $Pr = 0.35$, assign the two-digit numbers 00–34 (that is 35 of the 100 numbers) to a success.

Reliability. A simulation gives an **estimate**, not the exact value. By the long-run stability of relative frequencies, the estimate generally gets **closer to the true probability as the number of trials increases**. An estimate from 10 trials can differ noticeably from the true value; an estimate from 1000 trials is usually much more reliable.

Worked examples

Example 1 — scaffolded

A netball goal shooter scores 60 % of her shots. Describe how to use the random digits 0–9 to simulate one shot, stating what one trial is and what counts as a success.

Working	Comment
$Pr(\text{goal}) = 0.6 = \frac{6}{10}$, so assign 6 of the 10 digits to a goal: digits $0, 1, 2, 3, 4, 5$ → **goal**; digits $6, 7, 8, 9$ → **miss**.	Match the proportion — 6 digits out of 10 models a 0.6 chance.
One **trial** = read one random digit (one shot).	A trial is one repetition of the situation.
A **success** = the digit is 0–5 (a goal).	State clearly what is being counted.

Example 2 — scaffolded

Using the assignment from Example 1, these 20 random digits are generated:

$$38159264072851934607.$$

Estimate the probability that she scores a goal.

Working	Comment
Mark each digit: a **goal** if it is 0–5. Goals: $3, 1, 5, 2, 4, 0, 2, 5, 1, 3, 4, 0$.	Only the digits 0–5 count as a goal.

Working	Comment
Number of goals $= 12$ out of 20 trials.	Count the successes and the trials.
$Pr(\text{goal}) \approx \frac{12}{20} = 0.6.$	Estimate $= \frac{\text{successes}}{\text{trials}}$.

Example 3 — unscaffolded

A student guesses the answers to 3 true/false questions. Use coin tosses (H = correct, T = wrong) to estimate the probability of getting **at least** 2 **correct**, then compare with the theoretical value. The results of 10 trials (three tosses each) are:

$$HHT, THH, TTT, HTH, HHH, TTH, HTT, THT, HHT, HTH.$$

One trial = three tosses; a success = at least 2 heads. Counting heads per trial, the successes are $HHT, THH, HTH, HHH, HHT, HTH$ — that is 6 **successes out of** 10.

$$Pr(\text{at least 2 correct}) \approx \frac{6}{10} = 0.6.$$

Theoretically, $Pr(\text{at least 2 of 3}) = \dfrac{\binom{3}{2} + \binom{3}{3}}{8} = \dfrac{4}{8} = 0.5$. $\therefore$ the estimate 0.6 is close to the true value 0.5; more trials would be expected to bring the estimate closer.

Example 4 — unscaffolded

A spinner lands on red with probability 0.35. Explain how to use two-digit random numbers 00–99 to simulate one spin, and use the numbers

$$07, 42, 91, 33, 88, 15, 60, 29, 04, 77$$

to estimate $Pr(\text{red})$.

$Pr(\text{red}) = 0.35 = \frac{35}{100}$, so assign 00–34 (that is 35 numbers) to **red** and 35–99 to **not red**. One trial = one two-digit number.

The numbers that are 00–34 (red) are $07, 33, 15, 29, 04$ — that is 5 reds out of 10.

$$Pr(\text{red}) \approx \frac{5}{10} = 0.5.$$

$\therefore$ the estimate is 0.5. It differs from the true value 0.35 because only 10 trials were used; many more trials would be needed for a reliable estimate.

Example 5 — unscaffolded

A bag holds 3 red marbles (labelled by digits 1–3) and 2 blue marbles (digits 4–5). Two marbles are drawn **without replacement**. Describe a simulation using random digits, then estimate $Pr(\text{both red})$ from the digit string

$$3815264021375542934150322145,$$

comparing with the exact value.

Because the draw is **without replacement**, no marble can be chosen twice, so a digit already used in a trial is rejected. Read the digits in order: a digit 1–5 names the marble drawn; the digits 0 and 6–9 are **ignored** (they name no marble), and a **repeat** of a digit already drawn in the current trial is also ignored. One **trial** = read digits until 2 *distinct* marbles are drawn; a **success** = both are red (both digits 1–3).

Splitting the string into trials (ignored digits in parentheses):

Trial	Digits read	Marbles drawn	Both red?
1	$3,(8),1$	$\{3,1\}$	yes
2	$5,2$	$\{5,2\}$	no
3	$(6),4,(0),2$	$\{4,2\}$	no
4	$1,3$	$\{1,3\}$	yes
5	$(7),5,(5),4$	$\{5,4\}$	no
6	$2,(9),3$	$\{2,3\}$	yes
7	$4,1$	$\{4,1\}$	no
8	$5,(0),3$	$\{5,3\}$	no
9	$2,(2),1$	$\{2,1\}$	yes
10	$4,5$	$\{4,5\}$	no

There are 4 successes ("both red") in 10 trials, so

$$Pr(\text{both red}) \approx \frac{4}{10} = 0.4.$$

The exact value is $Pr(\text{both red}) = \dfrac{\binom{3}{2}}{\binom{5}{2}} = \dfrac{3}{10} = 0.3$ (equivalently $\dfrac{3}{5} \times \dfrac{2}{4} = \dfrac{3}{10}$). $\therefore$ the estimate 0.4 is close to the true value 0.3; many more trials would bring the estimate closer.

Practice questions

Scaffolded

Q1. A basketball player scores 70% of her free throws. (a) How many of the digits 0–9 should be assigned to a score? (b) Give a suitable assignment. (c) State what one trial is. (d) State what counts as a success.

Q2. Using the assignment "0–6 = score, 7–9 = miss", the digits $4\,2\,9\,1\,7\,5\,8\,0\,3\,6$ are generated. Estimate the probability of scoring.

Q3. A fair die is rolled and a "6" wins. (a) Explain why one roll of a die models this directly. (b) In 12 rolls $3\,6\,1\,6\,4\,2\,6\,5\,1\,6\,3\,2$, estimate $Pr(\text{win})$.

Q4. The probability of rain on any day is 0.5; use a coin (H = rain). One trial = three tosses (a three-day period); success = rain on all three days. From the trials $HHT, HTH, THH, HHH, TTH, THT, HTT, HHT$, estimate the probability of rain on all three days.

Q5. For an event with $Pr(\text{success}) = 0.2$, the digits $0, 1$ are assigned to a success. From $5\,3\,0\,8\,1\,6\,0\,9\,2\,7$, estimate the probability of a success.

Q6. Explain why a simulation using 1000 trials is likely to give a better estimate of a probability than one using only 10 trials.

Unscaffolded

Q7. A goalkeeper saves 40% of penalties. Give a suitable assignment of the digits 0–9 to simulate one penalty.

Q8. Using "0–3 = save", estimate $Pr(\text{save})$ from $2\,7\,5\,1\,9\,3\,8\,0\,4\,6$.

Q9. A fair coin is tossed and heads is a success. From the 15 tosses $H\,T\,H\,H\,T\,T\,H\,T\,H\,H\,H\,T\,T\,H\,T$, estimate $Pr(\text{heads})$.

Q10. A die is rolled and "more than 4" (a 5 or 6) is a success. From the 12 rolls $5\,2\,6\,3\,1\,4\,5\,6\,2\,3\,6\,1$, estimate the probability, and state the theoretical value.

Q11. For $Pr(\text{defective}) = 0.15$, the two-digit numbers 00–14 are assigned to a defective item. From $21\,04\,56\,88\,13\,47\,09\,72\,30\,99$, estimate $Pr(\text{defective})$.

Q12. A four-option multiple-choice question is answered by guessing, so $Pr(\text{correct}) = 0.25$. Explain how to use two-digit random numbers to simulate one guess and state what a success is.

Q13. Using the assignment from Q12 (00–24 = correct), estimate $Pr(\text{correct})$ from $18\,63\,07\,91\,24\,55\,30\,88\,12\,46$.

Q14. A family has 3 children. Using coins (H = boy, T = girl), one trial = three tosses and success = all three children the same sex. From $H\,H\,T, T\,T\,T, H\,T\,H, H\,H\,H, T\,H\,T, T\,T\,H, H\,H\,H, T\,T\,T, H\,T\,T, H\,H\,T$, estimate the probability, and state the theoretical value.

Q15. Explain the difference between the **estimated** probability from a simulation and the **theoretical** probability, and describe how the estimate can be made more reliable.

Q16. The probability of a sunny day is 0.7; assign the digits 0–6 to a sunny day. From the 20 digits $4\,9\,1\,7\,3\,0\,8\,5\,2\,6\,1\,9\,4\,7\,0\,3\,8\,6\,2\,5$, estimate $Pr(\text{sunny})$.

Q17. A machine's items pass inspection with probability 0.9; assign 0–8 = pass, 9 = fail. From the 15 digits $3\,7\,1\,9\,5\,0\,8\,2\,9\,4\,6\,1\,7\,3\,9$, estimate $Pr(\text{pass})$.

Q18. In a survey, $Pr(\text{owns a pet}) = 0.55$; assign the two-digit numbers 00–54 to owning a pet. From $23\,71\,08\,55\,42\,90\,17\,63\,39\,04$, estimate the probability, and state the theoretical value being estimated.

Q19. A jar holds 4 red beads (digits 1–4) and 2 white beads (digits 5–6). Two beads are drawn **without replacement**, so digits 0 and 7–9 are ignored and any digit already drawn in a trial is ignored; one trial ends when 2 distinct beads have been drawn. Using the digits

$$1\,3\,5\,2\,7\,4\,4\,2\,6\,5\,0\,3\,1\,2\,6\,8\,1\,4\,5\,9\,3\,4\,4\,2\,6\,1,$$

estimate $Pr(\text{both red})$, and state the exact theoretical value for comparison.

Answers

Q1. (a) 7; (b) e.g. 0–6 = score, 7–9 = miss; (c) one trial = one digit (one shot); (d) success = a digit 0–6. **Q2.** $\frac{7}{10}=0.7$. **Q3.** (a) a die has 6 equally likely outcomes, matching the six faces; (b) $\frac{4}{12}=\frac{1}{3}\approx 0.33$. **Q4.** $\frac{1}{8}=0.125$. **Q5.** $\frac{3}{10}=0.3$. **Q6.** More trials → the relative frequency is more stable and closer to the true probability. **Q7.** e.g. 0–3 = save, 4–9 = no save. **Q8.** $\frac{4}{10}=0.4$. **Q9.** $\frac{8}{15}\approx 0.53$. **Q10.** $\frac{5}{12}\approx 0.42$; theoretical $\frac{2}{6}=\frac{1}{3}$. **Q11.** $\frac{3}{10}=0.3$. **Q12.** 00–24 = correct, 25–99 = wrong; success = a number 00–24. **Q13.** $\frac{4}{10}=0.4$. **Q14.** $\frac{4}{10}=0.4$; theoretical $\frac{2}{8}=\frac{1}{4}$. **Q15.** See solutions. **Q16.** $\frac{14}{20}=0.7$. **Q17.** $\frac{12}{15}=0.8$. **Q18.** $\frac{6}{10}=0.6$; estimating the true value 0.55. **Q19.** $\frac{5}{10}=0.5$; exact value $\frac{\binom{4}{2}}{\binom{6}{2}}=\frac{2}{5}=0.4$.

Fully-worked solutions

Q1. (a) $Pr(\text{score})=0.7=\frac{7}{10}$, so 7 digits are assigned to a score. (b) e.g. $0,1,2,3,4,5,6\rightarrow$ score; $7,8,9\rightarrow$ miss. (c) One trial = reading one random digit (one free throw). (d) A success = the digit is 0–6.

Q2. The digits that are 0–6 (a score) are $4,2,1,5,0,3,6$ — that is 7 scores out of 10. $\therefore Pr(\text{score})\approx\frac{7}{10}=0.7$.

Q3. (a) A die has 6 equally likely outcomes, one for each face, so rolling it directly models the situation with no assignment needed. (b) There are 4 sixes among the 12 rolls, so $Pr(\text{win})\approx\frac{4}{12}=\frac{1}{3}\approx 0.33$ (the true value is $\frac{1}{6}$; the estimate is high because only 12 trials were used).

Q4. Success = $H\,H\,H$ (rain all three days). Only 1 trial is $H\,H\,H$, out of 8 trials. $\therefore Pr\approx\frac{1}{8}=0.125$, which matches the theoretical $(1/2)^3=\frac{1}{8}$.

Q5. The digits 0 or 1 (a success) are $0,1,0$ — that is 3 out of 10. $\therefore Pr(\text{success})\approx\frac{3}{10}=0.3$ (true value 0.2).

Q6. By the long-run stability of relative frequencies, the estimate $\frac{\text{successes}}{\text{trials}}$ settles closer to the true probability as the number of trials grows. With only 10 trials a single result changes the estimate by 0.1, so the estimate is unstable; with 1000 trials individual results have far less effect and the estimate is much more reliable.

Q7. $Pr(\text{save})=0.4=\frac{4}{10}$, so assign 4 digits, e.g. $0,1,2,3\rightarrow$ save; $4,5,6,7,8,9\rightarrow$ no save.

Q8. The digits 0–3 (a save) are $2,1,3,0$ — that is 4 out of 10. $\therefore Pr(\text{save})\approx\frac{4}{10}=0.4$.

Q9. There are 8 heads among the 15 tosses, so $Pr(\text{heads})\approx\frac{8}{15}\approx 0.53$ (true value 0.5).

Q10. A "5" or "6" occurs at $5, 6, 5, 6, 6$ — that is 5 successes out of 12. $\therefore Pr \approx \frac{5}{12} \approx 0.42$; the theoretical value is $\frac{2}{6} = \frac{1}{3} \approx 0.33$.

Q11. The numbers 00–14 (defective) are $04, 13, 09$ — that is 3 out of 10. $\therefore Pr(\text{defective}) \approx \frac{3}{10} = 0.3$ (true value 0.15).

Q12. $Pr(\text{correct}) = 0.25 = \frac{25}{100}$, so assign the two-digit numbers 00–24 (that is 25 numbers) to a correct guess and 25–99 to a wrong guess. One trial = one two-digit number; a success = a number in 00–24.

Q13. The numbers 00–24 (correct) are $18, 07, 24, 12$ — that is 4 out of 10. $\therefore Pr(\text{correct}) \approx \frac{4}{10} = 0.4$ (true value 0.25).

Q14. Success = HHH or TTT (all the same sex). These occur at TTT, HHH, HHH, TTT — that is 4 out of 10. $\therefore Pr \approx \frac{4}{10} = 0.4$; the theoretical value is $\frac{2}{8} = \frac{1}{4} = 0.25$.

Q15. The **theoretical** probability is the exact value found by analysing all equally likely outcomes; the **estimated** (experimental) probability is $\frac{\text{successes}}{\text{trials}}$ obtained from a simulation or experiment. The estimate is subject to random variation and rarely equals the theoretical value exactly, but it becomes more reliable — closer to the theoretical value — as the number of trials increases (and by using a well-designed, unbiased device).

Q16. The digits 0–6 (sunny) number 14 out of the 20 digits. $\therefore Pr(\text{sunny}) \approx \frac{14}{20} = 0.7$, matching the given probability.

Q17. A "fail" is the digit 9, which occurs 3 times, so a "pass" occurs $15 - 3 = 12$ times out of 15. $\therefore Pr(\text{pass}) \approx \frac{12}{15} = 0.8$ (true value 0.9).

Q18. The numbers 00–54 (owns a pet) are $23, 08, 42, 17, 39, 04$ — that is 6 out of 10. $\therefore Pr(\text{owns a pet}) \approx \frac{6}{10} = 0.6$; this estimates the true value 0.55.

Q19. Reading the digits (ignoring 0, 7–9 and repeats) gives the 10 trials $\{1,3\}, \{5,2\}, \{4,2\}, \{6,5\}, \{3,1\}, \{2,6\}, \{1,4\}, \{5,3\}, \{4,2\}, \{6,1\}$, of which "both red" (both digits 1–4) are $\{1,3\}, \{4,2\}, \{3,1\}, \{1,4\}, \{4,2\}$ — that is 5 out of 10. $\therefore Pr(\text{both red}) \approx \frac{5}{10} = 0.5$. The exact value is $\frac{\binom{4}{2}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5} = 0.4$ (equivalently $\frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$); the estimate is close, and more trials would bring it closer.

Theory

Pseudocode is a structured, plain-language way of writing an algorithm — precise enough to follow step by step, but not tied to any programming language. In Mathematical Methods it is used to describe **simulations** that estimate probabilities.

Assignment. The symbol ← means "becomes" (assign the value on the right to the variable on the left). It is **not** an equals sign: `count ← count + 1` increases `count` by 1. The = sign is kept for testing equality inside conditions.

Sequence, selection, iteration. An algorithm is built from three structures:

- **selection** — `if condition then … else … end if` chooses between actions;
- **for-loop** — `for i from 1 to n do … end for` repeats a fixed number of times;
- **while-loop** — `while condition do … end while` repeats until the condition fails.

Randomness. `randomInteger(a, b)` returns a whole number from a to b inclusive, each equally likely (e.g. `randomInteger(1, 6)` models one roll of a fair die; `randomInteger(0, 1)` models a coin, with 1 for heads).

The shape of a simulation. To estimate a probability by simulation:

```
success ← 0
for trial from 1 to n do
    … generate the random outcome …
    if the outcome is a success then
        success ← success + 1
    end if
end for
estimate ← success / n
```

As the number of trials n increases, the **relative frequency** `success / n` tends towards the true probability.

Tracing. To *trace* pseudocode, keep a table of the variables and update them line by line using the given random values.

Worked examples

Example 1 — scaffolded

Trace the following, given that the calls to `randomInteger(1, 6)` return $3, 5, 2, 6, 4$ in order. State the value printed.

```
count ← 0
for i from 1 to 5 do
    r ← randomInteger(1, 6)
    if r ≥ 5 then
        count ← count + 1
    end if
end for
print count
```

i	r	$r \geq 5$?	`count`
start	—	—	0
1	3	no	0

i	r	$r \geq 5$?	`count`
2	5	yes	1
3	2	no	1
4	6	yes	2
5	4	no	2

∴ the value printed is 2 — the number of rolls that were 5 or more.

Example 2 — scaffolded

Trace the following, given that `randomInteger(1, 6)` returns $3, 1, 6$ in order.

```
rolls ← 0
d ← 0
while d ≠ 6 do
    d ← randomInteger(1, 6)
    rolls ← rolls + 1
end while
print rolls
```

`d` (before)	`d ← randomInteger(1,6)`	`rolls`	continue? ($d \neq 6$)
0	3	1	yes
3	1	2	yes
1	6	3	no — stop

∴ the value printed is 3: the simulation counts **the number of rolls needed to get the first six**.

Example 3 — unscaffolded

Write pseudocode that estimates the probability of tossing a head, using n tosses of a fair coin.

```
heads ← 0
for i from 1 to n do
    c ← randomInteger(0, 1)          // 0 = tail, 1 = head
    if c = 1 then
        heads ← heads + 1
    end if
end for
estimate ← heads / n
print estimate
```

∴ `estimate` is the relative frequency of heads, which approaches $1/2$ as n increases.

Example 4 — unscaffolded

Write pseudocode that estimates the probability that the sum of two fair dice is 7, using 1000 trials.

```
success ← 0
for trial from 1 to 1000 do
    d1 ← randomInteger(1, 6)
    d2 ← randomInteger(1, 6)
    if d1 + d2 = 7 then
        success ← success + 1
    end if
end for
estimate ← success / 1000
print estimate
```

$\therefore$ `estimate` approaches the true probability $\frac{6}{36} = \frac{1}{6} \approx 0.167$.

Practice questions

Scaffolded

Q1. Trace and state the value printed:

```
x ← 3
x ← x + 4
x ← 2 × x
print x
```

Q2. Trace and state the value printed:

```
s ← 0
for i from 1 to 3 do
    s ← s + i
end for
print s
```

Q3. Trace the following, given that `randomInteger(1, 10)` returns $2, 7, 1, 9$ in order:

```
c ← 0
for i from 1 to 4 do
    r ← randomInteger(1, 10)
    if r ≤ 3 then
        c ← c + 1
    end if
end for
print c
```

Q4. The simulation in **Q3** is run to estimate a probability. Write the extra line that should be added after the loop to output the estimate, and state which probability it estimates.

Q5. In one sentence, state what the algorithm in **Example 2** computes.

Q6. State the probability that the following simulation estimates, and its exact value:

```
success ← 0
for trial from 1 to n do
    d ← randomInteger(1, 6)
    if d = 6 then
        success ← success + 1
    end if
end for
estimate ← success / n
```

Unscaffolded

Q7. Trace and state the final values of `a` and `b`:

```
a ← 10
b ← 4
a ← a − b
b ← a − b
```

Q8. Trace and state the value printed:

```
p ← 1
for k from 1 to 4 do
    p ← p × k
end for
print p
```

Q9. Trace the following, given that `randomInteger(1, 6)` returns $4, 6, 2, 6, 5$ in order. State the value of `success`, and hence the estimate `success / 5`.

```
success ← 0
for i from 1 to 5 do
    d ← randomInteger(1, 6)
    if d = 6 then
        success ← success + 1
    end if
end for
```

Q10. Trace and state the value printed, given `randomInteger(1, 6)` returns $4, 2, 6$ in order:

```
n ← 0
d ← 0
while d ≠ 6 do
    d ← randomInteger(1, 6)
    n ← n + 1
end while
print n
```

Q11. Trace and state the value of `result`, given `randomInteger(1, 6)` returns 2 then 3:

```
a ← randomInteger(1, 6)
b ← randomInteger(1, 6)
if a > b then
    result ← "A"
else if a < b then
    result ← "B"
else
    result ← "tie"
end if
```

Q12. Copy and complete the pseudocode so that it estimates the probability of rolling an even number on a fair die, using n trials. Fill in the three blanks.

```
success ← 0
for i from 1 to n do
    d ← randomInteger(1, 6)
    if ______ then
        ______
    end if
end for
estimate ← ______
```

Q13. The pseudocode below is meant to estimate the probability of rolling a 6, but it contains an error. Identify the error and write the corrected pseudocode.

```
for i from 1 to n do
    d ← randomInteger(1, 6)
    if d = 6 then
        success ← success + 1
    end if
```

```
end for
estimate ← success / n
```

Q14. Write pseudocode that estimates the probability of rolling a number greater than 4 on a fair die, using n trials. State the exact probability it approaches.

Q15. Write pseudocode that estimates the probability of getting **at least one head** in three tosses of a fair coin, using 1000 trials. State the exact probability it approaches.

Q16. Write pseudocode that estimates the probability that the sum of two fair dice is at least 10, using n trials. State the exact probability it approaches.

Q17. Trace and state the value of `best`, given `randomInteger(1, 6)` returns $6, 1, 6, 6, 3, 6$ in order:

```
run ← 0
best ← 0
for i from 1 to 6 do
    d ← randomInteger(1, 6)
    if d = 6 then
        run ← run + 1
        if run > best then
            best ← run
        end if
    else
        run ← 0
    end if
end for
```

Q18. State in words what probability the following simulation estimates, and give its exact value.

```
success ← 0
for trial from 1 to n do
    d1 ← randomInteger(1, 6)
    d2 ← randomInteger(1, 6)
    if (d1 is even) and (d2 is even) then
        success ← success + 1
    end if
end for
estimate ← success / n
```

Answers

Q1. 14. **Q2.** 6. **Q3.** 2. **Q4.** `estimate ← c / 4`; it estimates $P(\text{roll} \le 3) = 3/10$. **Q5.** It counts the number of rolls of a fair die needed to obtain the first six. **Q6.** $P(\text{rolling a } 6) = 1/6$. **Q7.** $a = 6$, $b = 2$. **Q8.** 24. **Q9.** `success` $= 2$; estimate $= 2/5$. **Q10.** 3. **Q11.** `"B"`. **Q12.** `if d = 2 or d = 4 or d = 6 then` / `success ← success + 1` / `estimate ← success / n`. **Q13.** `success` is never initialised; add `success ← 0` before the loop. **Q14.** approaches $2/6 = 1/3$. **Q15.** approaches $1 - 1/8 = 7/8$. **Q16.** approaches $6/36 = 1/6$. **Q17.** 2. **Q18.** $P(\text{both dice even}) = 9/36 = 1/4$.

Fully-worked solutions

Q1. $x \leftarrow 3$; then $x \leftarrow 3 + 4 = 7$; then $x \leftarrow 2 \times 7 = 14$. $\therefore 14$.

Q2. `s` runs $0 \to 1 \to 3 \to 6$ as $i = 1, 2, 3$ are added. $\therefore 6$.

Q3.

i	r	$r \le 3$?	`c`
1	2	yes	1
2	7	no	1
3	1	yes	2
4	9	no	2

$\therefore c = 2$.

Q4. The loop counts, in `c`, how many of the 4 trials gave $r \le 3$, so the estimate is `estimate ← c / 4`. It estimates $P(r \le 3) = \frac{3}{10}$ (the values $1, 2, 3$ out of 1–10).

Q5. It simulates rolling a die until a six appears and outputs the number of rolls required (the number of rolls to get the first six).

Q6. Each trial scores a success when the die shows 6, so `estimate` approaches $P(\text{six}) = \frac{1}{6}$.

Q7. $a \leftarrow 10 - 4 = 6$; then $b \leftarrow a - b = 6 - 4 = 2$. $\therefore a = 6$, $b = 2$ (the two values have been swapped-and-changed; note b uses the *new* value of a).

Q8. `p` runs $1 \to 1 \to 2 \to 6 \to 24$ (it computes $1 \times 2 \times 3 \times 4 = 4!$). $\therefore 24$.

Q9.

i	d	$d = 6$?	`success`
1	4	no	0
2	6	yes	1
3	2	no	1
4	6	yes	2
5	5	no	2

$\therefore$ `success` $= 2$ and the estimate is $\frac{2}{5}$.

Q10. $d = 0$ initially, so the loop runs: $d \leftarrow 4$ ($n = 1$), $d \leftarrow 2$ ($n = 2$), $d \leftarrow 6$ ($n = 3$, and $d \ne 6$ is now false). $\therefore 3$.

Q11. $a = 2$, $b = 3$. Since $a > b$ is false but $a < b$ is true, `result ← "B"`. $\therefore$ `"B"`.

Q12. A die shows an even number when $d \in \{2, 4, 6\}$:

```
if d = 2 or d = 4 or d = 6 then
    success ← success + 1
end if
…
estimate ← success / n
```

∴ the estimate approaches $P(\text{even}) = 3/6 = 1/2$.

Q13. The variable `success` is used inside the loop but is **never initialised**, so the algorithm has no defined starting value to add to. Add `success ← 0` before the loop:

```
success ← 0
for i from 1 to n do
    d ← randomInteger(1, 6)
    if d = 6 then
        success ← success + 1
    end if
end for
estimate ← success / n
```

Q14. "Greater than 4" means $d = 5$ or $d = 6$:

```
success ← 0
for i from 1 to n do
    d ← randomInteger(1, 6)
    if d > 4 then
        success ← success + 1
    end if
end for
estimate ← success / n
```

∴ `estimate` approaches $\frac{2}{6} = \frac{1}{3}$.

Q15. Toss three coins per trial and count a success if at least one is a head:

```
success ← 0
for trial from 1 to 1000 do
    heads ← 0
    for j from 1 to 3 do
        c ← randomInteger(0, 1)
        if c = 1 then
            heads ← heads + 1
        end if
    end for
    if heads ≥ 1 then
        success ← success + 1
    end if
end for
estimate ← success / 1000
```

The only failure is three tails, with probability $(1/2)^3 = 1/8$, so `estimate` approaches $1 - 1/8 = \frac{7}{8}$.

Q16.

```
success ← 0
for i from 1 to n do
```

```
    d1 ← randomInteger(1, 6)
    d2 ← randomInteger(1, 6)
    if d1 + d2 ≥ 10 then
        success ← success + 1
    end if
end for
estimate ← success / n
```

The sums of 10 or more are $(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)$ — 6 of the 36 equally likely outcomes — so `estimate` approaches $\frac{6}{36}=\frac{1}{6}$.

Q17. `run` tracks the current run of sixes and `best` the longest so far.

i	*d*	`run`	`best`
1	6	1	1
2	1	0	1
3	6	1	1
4	6	2	2
5	3	0	2
6	6	1	2

$\therefore$ `best` $=2$ (the longest run of consecutive sixes).

Q18. Each trial rolls two dice and scores a success when **both** are even. A die is even with probability $1/2$ and the dice are independent, so the estimate approaches $1/2\times1/2=\frac{9}{36}=\frac{1}{4}$ (the even faces are $2,4,6$, so $3\times3=9$ of the 36 outcomes).