Mathematical Methods Units 1 & 2

Chapter 7 — Transformations of the plane

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Contents

Theory

A **translation** slides a graph across the plane without changing its shape or size.

The rule. Starting from $y=f(x)$, the graph of

$$y=f(x-h)+k$$

is the translation of $y=f(x)$ by h **units horizontally** and k **units vertically** — right if $h>0$ (left if $h<0$), up if $k>0$ (down if $k<0$). To build the new rule, replace every x with $x-h$ and then add k.

Image of a point. Under this translation, each point (x,y) moves to its **image** $(x+h,y+k)$.

Applying it to the standard graphs. Every key feature moves by (h,k):

Base function	Translated form	Key feature (image)
$y=x^2$	$y=(x-h)^2+k$	turning point (h,k)
$y=x^3$	$y=(x-h)^3+k$	stationary point of inflection (h,k)
$y=\frac{1}{x}$	$y=\frac{1}{x-h}+k$	asymptotes $x=h$, $y=k$
$y=\sqrt{x}$	$y=\sqrt{x-h}+k$	endpoint (h,k); domain $[h,\infty)$

The domain and range also shift: for example $y=\sqrt{x-h}+k$ has domain $[h,\infty)$ and range $[k,\infty)$, and $y=\frac{1}{x-h}+k$ has domain $\mathbb{R}\setminus\{h\}$ and range $\mathbb{R}\setminus\{k\}$.

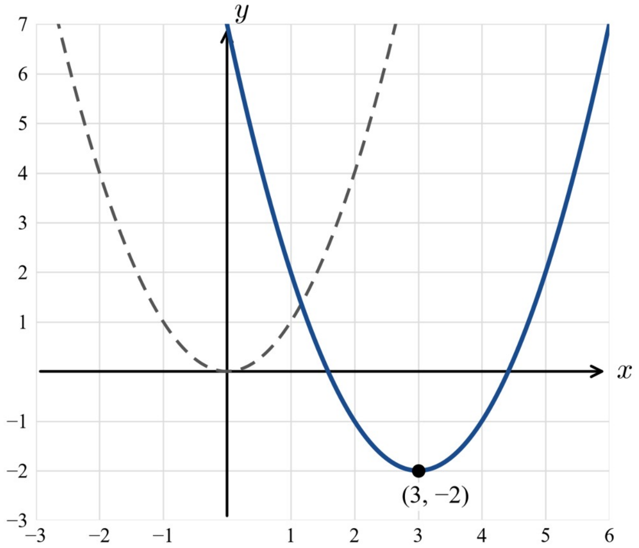
To sketch a translated graph: locate the image of the base graph's key feature (h,k), shift a couple of other points by (h,k), and draw the same shape through them. It is good practice to show the original graph as a dashed guide.

Worked examples

Example 1 — scaffolded

The graph of $y=x^2$ is translated 3 units right and 2 units down. Find the rule of the image and sketch it.

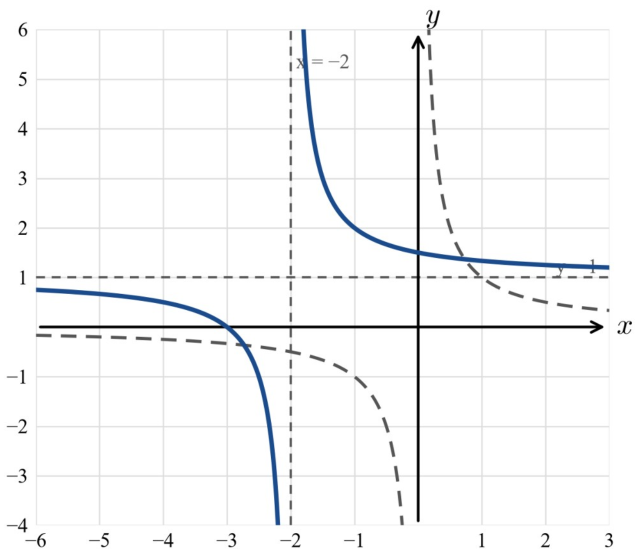
Working	Comment
Here $h=3$ (right) and $k=-2$ (down), so replace x with $x-3$ and subtract 2: $y=(x-3)^2-2$.	$y=f(x-h)+k$ with $f(x)=x^2$.
The turning point $(0,0)$ moves to its image $(0+3,0-2)=(3,-2)$.	Each point shifts by $(h,k)=(3,-2)$.
y-intercept: $x=0\Rightarrow y=(0-3)^2-2=9-2=7$, giving $(0,7)$.	Substitute $x=0$.
Draw the same parabola shape with turning point $(3,-2)$.	Shape is unchanged by a translation.



Example 2 — scaffolded

The graph of $y=\dfrac{1}{x}$ is translated 2 units left and 1 unit up. Find the rule, state the asymptotes, and sketch.

Working	Comment
$h=-2$ (left) and $k=1$ (up): $y=\dfrac{1}{x-(-2)}+1=\dfrac{1}{x+2}+1.$	Replace x with $x+2$, then add 1.
The asymptotes $x=0$ and $y=0$ move to $x=-2$ and $y=1$.	Asymptotes shift by $(h,k)=(-2,1)$.
y-intercept: $x=0\Rightarrow y=\dfrac{1}{2}+1=\dfrac{3}{2}$, giving $(0,3/2)$.	Substitute $x=0$.
x-intercept: $\dfrac{1}{x+2}+1=0\Rightarrow\dfrac{1}{x+2}=-1\Rightarrow x+2=-1$, so $x=-3$, giving $(-3,0)$.	Set $y=0$ and solve.
Domain $\mathbb{R}\setminus\{-2\}$, range $\mathbb{R}\setminus\{1\}$.	



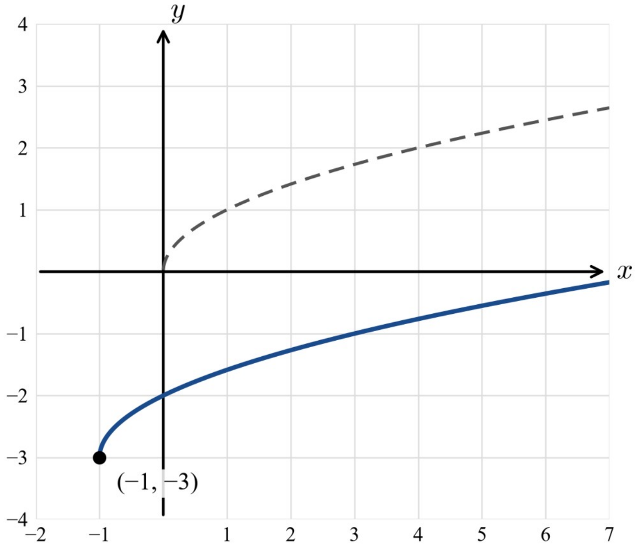
Example 3 — unscaffolded

Sketch $y=\sqrt{x+1}-3$, stating the transformation from $y=\sqrt{x}$, the endpoint, the domain and the range.

Comparing with $y=\sqrt{x-h}+k$: $h=-1$ and $k=-3$, so this is $y=\sqrt{x}$ translated 1 unit left and 3 units down.

Endpoint: $(0,0)\to(-1,-3)$. Domain $[-1,\infty)$, range $[-3,\infty)$.

y-intercept: $x=0\Rightarrow y=\sqrt{1}-3=-2$, giving $(0,-2)$. x-intercept: $\sqrt{x+1}=3\Rightarrow x+1=9$, so $x=8$, giving $(8,0)$.

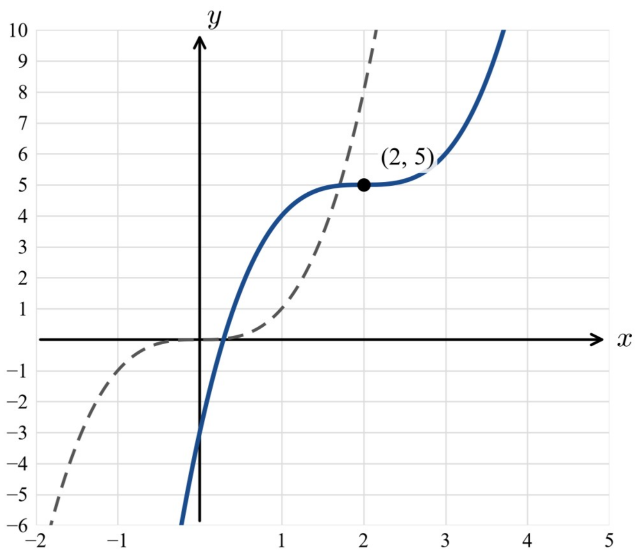


Example 4 — unscaffolded

State the translation that maps $y=x^3$ to $y=(x-2)^3+5$, and find the image of the point $(0,0)$.

Comparing with $y=(x-h)^3+k$: $h=2$, $k=5$ — a translation of 2 units right and 5 units up.

$\therefore$ the image of $(0,0)$ is $(0+2,0+5)=(2,5)$, which is the stationary point of inflection of the translated graph.



Practice questions

Scaffolded

Q1. The graph of $y=x^2$ is translated 1 unit right and 4 units down. (a) Write the rule of the image. (b) State the turning point. (c) Find the axis intercepts. (d) Sketch the image (show $y=x^2$ dashed).

Q2. The graph of $y=x^3$ is translated 2 units left. (a) Write the rule. (b) State the stationary point of inflection. (c) Find the y-intercept. (d) Sketch.

Q3. The graph of $y=\frac{1}{x}$ is translated 3 units up. (a) Write the rule. (b) State the equations of the two asymptotes. (c) Find the x-intercept. (d) Sketch.

Q4. The graph of $y=\sqrt{x}$ is translated 4 units right and 1 unit up. (a) Write the rule. (b) State the endpoint. (c) State the domain and range. (d) Sketch.

Q5. The graph of $y=x^3$ is translated 1 unit left and 8 units down. (a) Write the rule of the image. (b) State the stationary point of inflection. (c) Find the axis intercepts. (d) Sketch the image (show $y=x^3$ dashed).

Q6. A parabola of the same shape as $y=x^2$ has its turning point at $(3,-1)$. (a) Write its rule in the form $y=(x-h)^2+k$. (b) State the translation from $y=x^2$. (c) Sketch.

Unscaffolded

For each of the following, sketch the graph (showing the base graph dashed), and state the transformation from the relevant base function together with the labelled key feature.

Q7. $y=(x-2)^2-3$

Q8. $y=(x+1)^3+2$

Q9. $y=\frac{1}{x-3}$

Q10. $y=\sqrt{x}-4$

Q11. $y=\sqrt{x+3}-2$

Q12. $y=(x+2)^2-5$

Q13. $y=\frac{1}{x-1}+2$

Q14. $y=\sqrt{x-3}+1$

Q15. Find the image of the point $(2,4)$ (which lies on $y=x^2$) under a translation of 3 units left and 1 unit up, and write the rule of the translated parabola.

Q16. For $y=(x-1)^3-2$, state the transformation from $y=x^3$ and find the image of the inflection point $(0,0)$.

Q17. The point $(1,1)$ on $y=\frac{1}{x}$ has image $(4,3)$ under a translation. State the translation and write the rule of the translated graph.

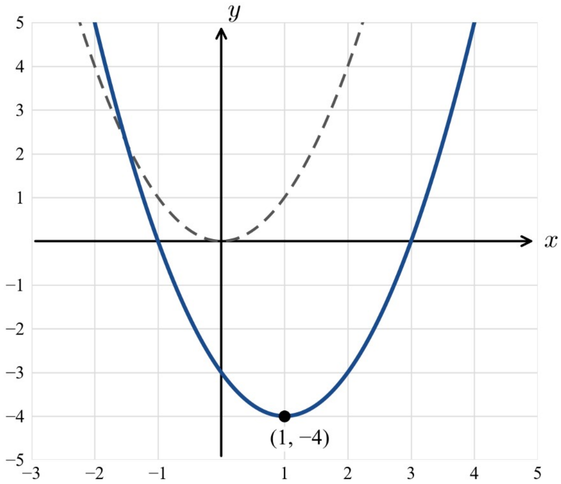
Q18. A graph of the same shape as $y=x^2$ has its turning point at $(-2,3)$. Write its rule.

Answers

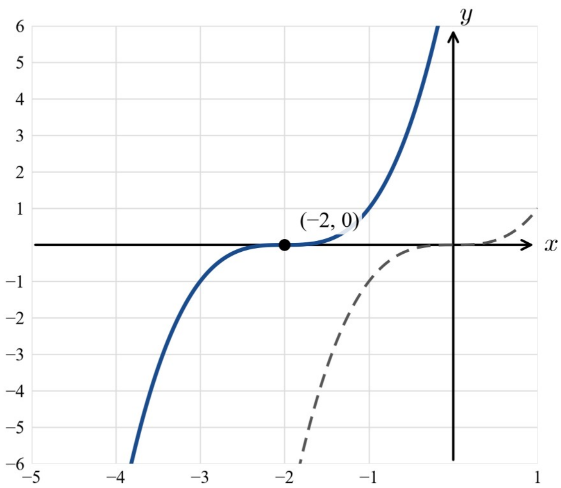
Q1. $y=(x-1)^2-4$; TP $(1,-4)$; x-int $(3,0),(-1,0)$; y-int $(0,-3)$. **Q2.** $y=(x+2)^3$; inflection $(-2,0)$; y-int $(0,8)$. **Q3.** $y=\frac{1}{x}+3$; asymptotes $x=0$, $y=3$; x-int $(-1/3,0)$. **Q4.** $y=\sqrt{x-4}+1$; endpoint $(4,1)$; domain $[4,\infty)$, range $[1,\infty)$. **Q5.** $y=(x+1)^3-8$; inflection $(-1,-8)$; x-int $(1,0)$; y-int $(0,-7)$. **Q6.** $y=(x-3)^2-1$; translated 3 right and 1 down. **Q7.** $y=x^2$ translated 2 right, 3 down; TP $(2,-3)$; y-int $(0,1)$; x-int $(2\pm\sqrt{3},0)$. **Q8.** $y=x^3$ translated 1 left, 2 up; inflection $(-1,2)$; y-int $(0,3)$. **Q9.** $y=\frac{1}{x}$ translated 3 right; asymptotes $x=3$, $y=0$; y-int $(0,-1/3)$. **Q10.** $y=\sqrt{x}$ translated 4 down; endpoint $(0,-4)$; domain $[0,\infty)$, range $[-4,\infty)$; x-int $(16,0)$. **Q11.** $y=\sqrt{x}$ translated 3 left, 2 down; endpoint $(-3,-2)$; domain $[-3,\infty)$, range $[-2,\infty)$; x-int $(1,0)$; y-int $(0,\sqrt{3}-2)$. **Q12.** $y=x^2$ translated 2 left, 5 down; TP $(-2,-5)$; x-int $(-2\pm\sqrt{5},0)$; y-int $(0,-1)$. **Q13.** $y=\frac{1}{x}$ translated 1 right, 2 up; asymptotes $x=1$, $y=2$; y-int $(0,1)$; x-int $(1/2,0)$. **Q14.** $y=\sqrt{x}$ translated 3 right, 1 up; endpoint $(3,1)$; domain $[3,\infty)$, range $[1,\infty)$. **Q15.** image $(-1,5)$; rule $y=(x+3)^2+1$. **Q16.** x^3 translated 1 right, 2 down; image of inflection $(1,-2)$. **Q17.** translation 3 right and 2 up; rule $y=\frac{1}{x-3}+2$. **Q18.** $y=(x+2)^2+3$.

Fully-worked solutions

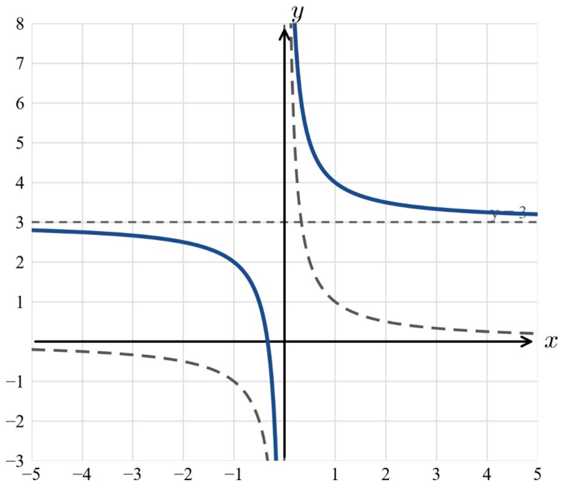
Q1. $h=1$, $k=-4$: $y=(x-1)^2-4$. TP $(0,0)\to(1,-4)$. y-int: $x=0\Rightarrow 1-4=-3$, so $(0,-3)$. x-int: $(x-1)^2=4\Rightarrow x-1=\pm 2$, so $x=3$ or -1.



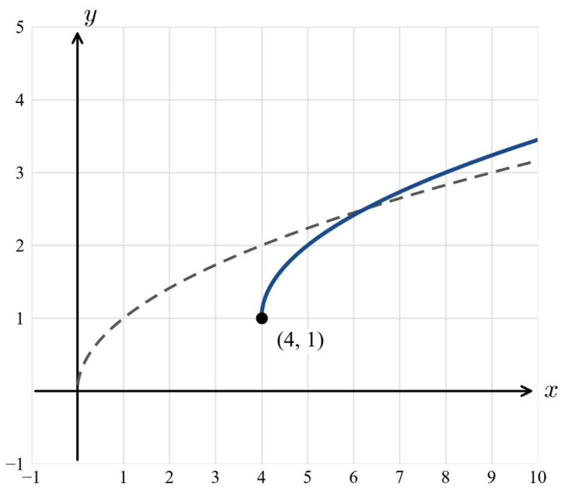
Q2. $y=(x+2)^3$. Inflection $(0,0)\to(-2,0)$. y-int: $x=0\Rightarrow 2^3=8$, so $(0,8)$.



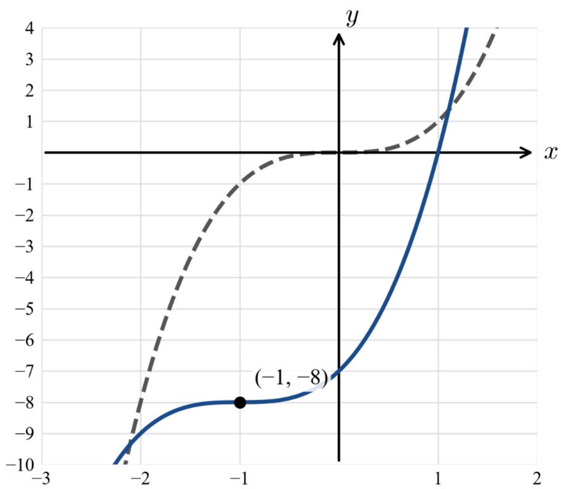
Q3. $y=\frac{1}{x}+3$. Asymptotes $x=0$ (unchanged) and $y=0 \rightarrow y=3$. x-int: $\frac{1}{x}=-3 \Rightarrow x=-1/3$, so $(-1/3,0)$. (No y-intercept, since $x=0$ is not in the domain.)



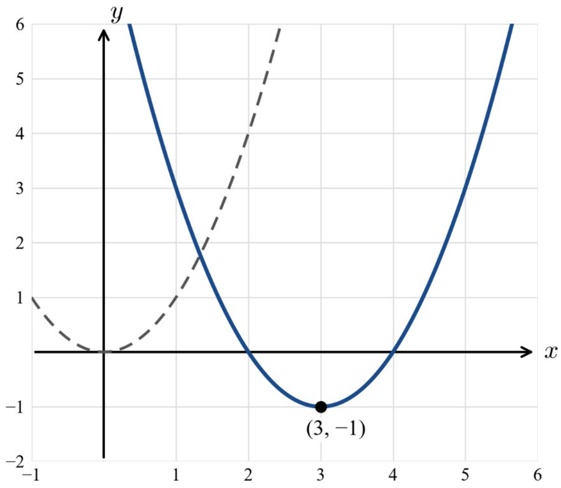
Q4. $y=\sqrt{x-4}+1$. Endpoint $(0,0) \rightarrow (4,1)$. Domain $[4,\infty)$, range $[1,\infty)$. (No axis intercepts: the graph stays in $x \geq 4$, $y \geq 1$.)



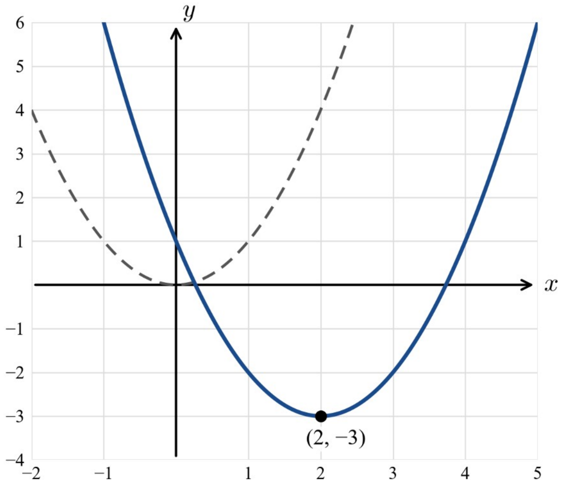
Q5. $h=-1$, $k=-8$: $y=(x+1)^3-8$. Inflection $(0,0) \rightarrow (-1,-8)$. y-int: $x=0 \Rightarrow 1-8=-7$, so $(0,-7)$. x-int: $(x+1)^3=8 \Rightarrow x+1=2$, so $x=1$, giving $(1,0)$.



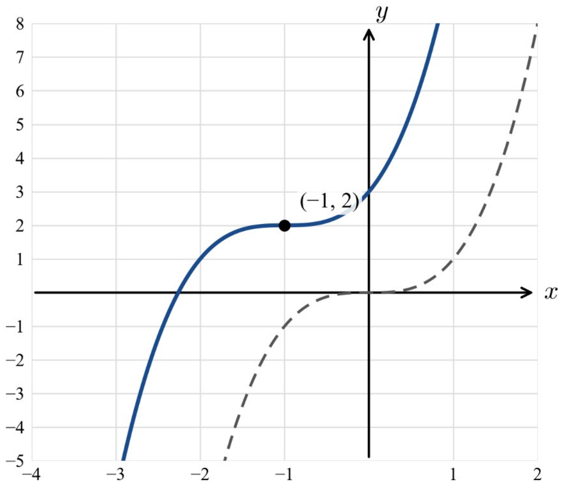
Q6. $y=(x-3)^2-1$. Comparing with $y=(x-h)^2+k$, the turning point is $(3,-1)$, so $y=x^2$ is translated 3 units right and 1 unit down.



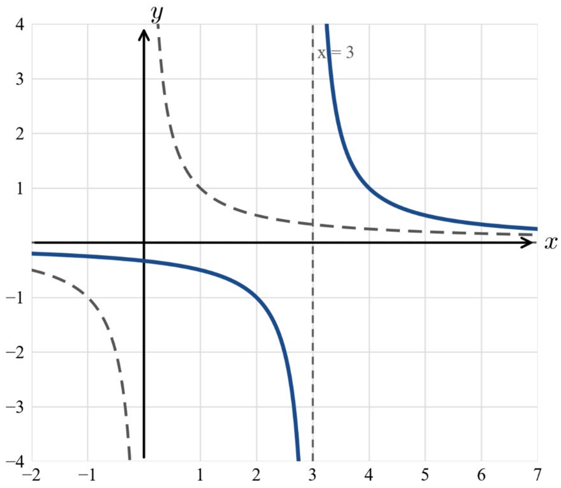
Q7. $y=x^2$ translated 2 right, 3 down. TP $(2,-3)$; y-int $(0,1)$; x-int $(x-2)^2=3 \Rightarrow x=2\pm\sqrt{3}$.



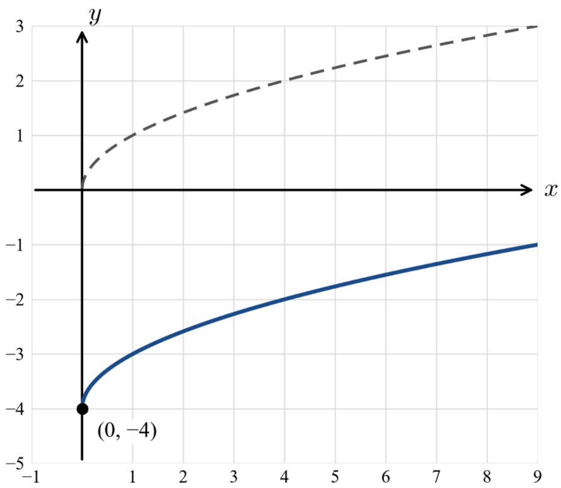
Q8. $y=x^3$ translated 1 left, 2 up. Inflection $(-1,2)$; y-int: $x=0 \Rightarrow 1+2=3$, so $(0,3)$.



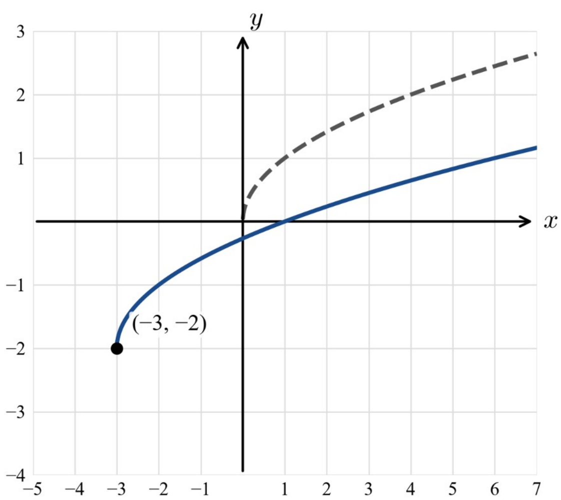
Q9. $y=\dfrac{1}{x}$ translated 3 right. Asymptotes $x=3$, $y=0$; y-int: $x=0 \Rightarrow \dfrac{1}{-3}=-1/3$, so $(0,-1/3)$; no x-intercept.



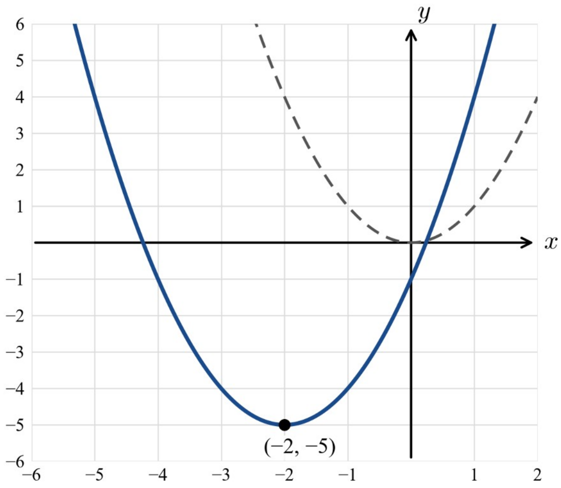
Q10. $y=\sqrt{x}$ translated 4 down. Endpoint $(0,-4)$; domain $[0,\infty)$, range $[-4,\infty)$; x-int: $\sqrt{x}=4 \Rightarrow x=16$, so $(16,0)$.



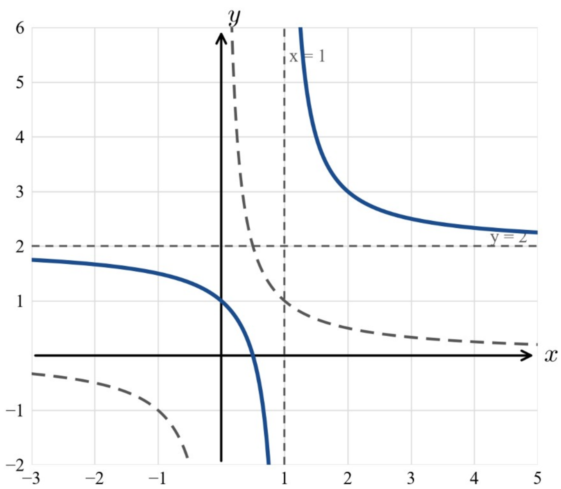
Q11. $y=\sqrt{x}$ translated 3 left, 2 down. Endpoint $(0,0)\to(-3,-2)$; domain $[-3,\infty)$, range $[-2,\infty)$. x-int: $\sqrt{x+3}=2 \Rightarrow x+3=4$, so $x=1$, giving $(1,0)$; y-int: $x=0 \Rightarrow \sqrt{3}-2$, so $(0,\sqrt{3}-2)$.



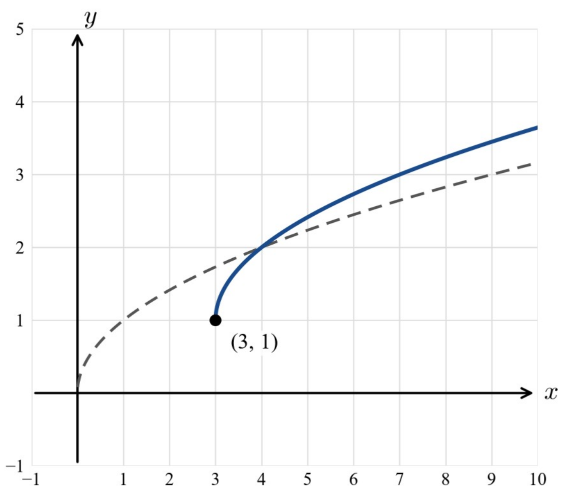
Q12. Comparing $y=(x+2)^2-5$ with $y=(x-h)^2+k$: $h=-2$, $k=-5$, so $y=x^2$ is translated 2 left and 5 down. TP $(-2,-5)$; x-int $(x+2)^2=5 \Rightarrow x=-2\pm\sqrt{5}$; y-int $(0,-1)$.



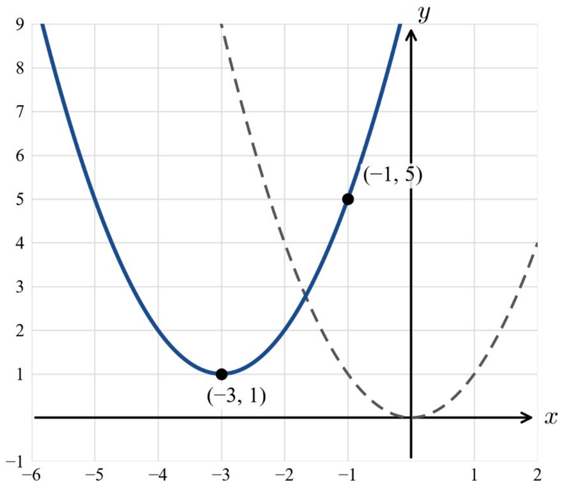
Q13. $y=\dfrac{1}{x-1}+2$ is $y=\dfrac{1}{x}$ translated 1 right and 2 up. Asymptotes $x=1$, $y=2$; y-int: $x=0\Rightarrow\dfrac{1}{-1}+2=1$, so $(0,1)$; x-int: $\dfrac{1}{x-1}=-2\Rightarrow x-1=-1/2$, so $(1/2,0)$.



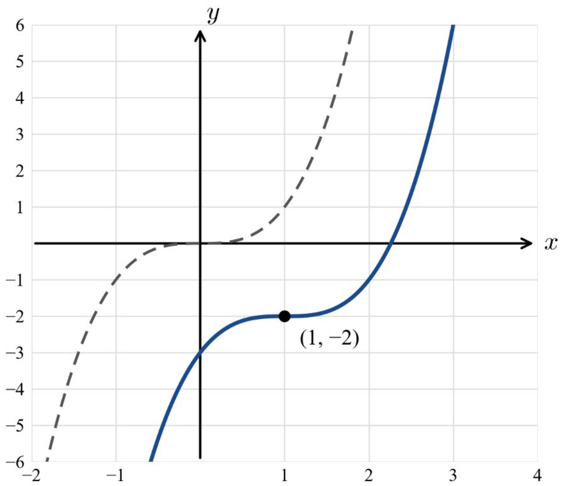
Q14. $y=\sqrt{x-3}+1$ is $y=\sqrt{x}$ translated 3 right and 1 up. Endpoint $(3,1)$; domain $[3,\infty)$, range $[1,\infty)$ (no axis intercepts).



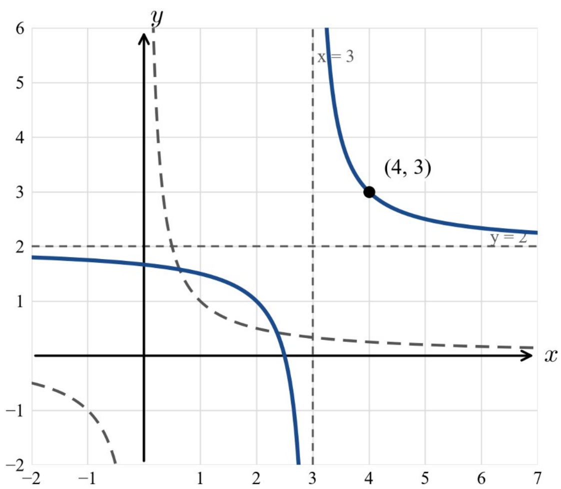
Q15. The image of $(2,4)$ under a translation 3 left and 1 up is $(2-3,4+1)=(-1,5)$. The parabola $y=x^2$ is translated the same way, giving $y=(x+3)^2+1$ (turning point $(-3,1)$, through $(-1,5)$).



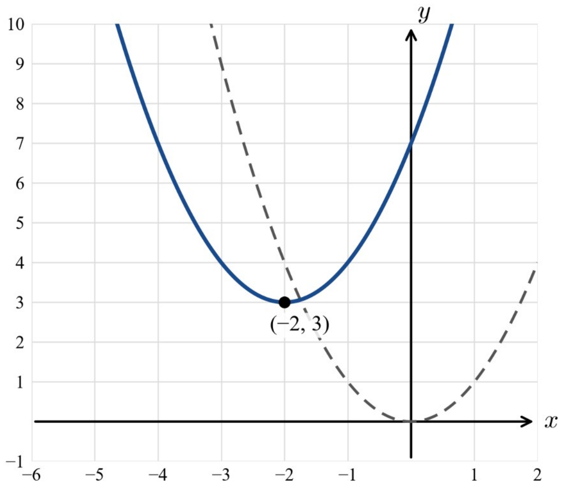
Q16. Comparing $y=(x-1)^3-2$ with $y=(x-h)^3+k$: $h=1$, $k=-2$, a translation of 1 right and 2 down. ∴ the inflection $(0,0)$ has image $(1,-2)$.



Q17. The image $(4,3)$ minus the original $(1,1)$ gives the shift $(4-1,3-1)=(3,2)$: a translation 3 right and 2 up. ∴ the rule is $y=\dfrac{1}{x-3}+2$.



Q18. A graph of the same shape as $y=x^2$ with turning point $(-2,3)$ has $h=-2$, $k=3$: ∴ $y=(x+2)^2+3$.


y
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2

Theory

A **transformation** moves every point of a graph to a new position. This section covers **dilations** (stretches) and **reflections** (flips). Throughout, $y=f(x)$ is the original graph and each rule below describes the image.

Dilation from the x-axis by factor a: $y=af(x)$. Each point (x,y) moves to (x,ay) — the graph is stretched vertically. If $a>1$ it is steeper (narrower); if $0<a<1$ it is flatter (wider). The x-intercepts are unchanged.

Dilation from the y-axis by factor b: $y=f\left(\frac{x}{b}\right)$. Each point (x,y) moves to (bx,y) — the graph is stretched horizontally. The y-intercept is unchanged.

Two ways of naming the same dilation. A dilation **from the** x-axis (the y-coordinates change) is also described as a dilation **parallel to the** y-axis, and a dilation **from the** y-axis (the x-coordinates change) as a dilation **parallel to the** x-axis. Both descriptions are used in examinations — read carefully which axis the dilation is *from* or *parallel to*.

Reflection in the x-axis: $y=-f(x)$. Each point (x,y) moves to $(x,-y)$ — the graph flips upside down.

Reflection in the y-axis: $y=f(-x)$. Each point (x,y) moves to $(-x,y)$ — the graph flips left–right.

Image of a point. To find where a single point moves, apply the coordinate rule above. For example, under $y=-f(x)$ the point $(3,5)$ has image $(3,-5)$.

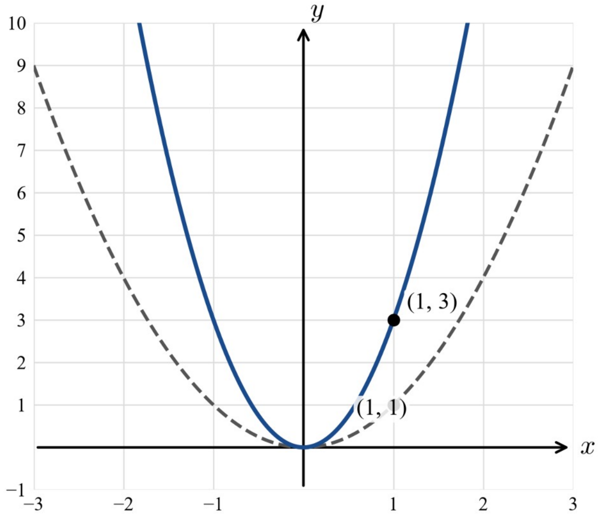
Note: for a parabola a dilation from the y-axis looks the same as a dilation from the x-axis (e.g. $y=(x/2)^2=1/4\,x^2$), but the **point rule** differs — from the y-axis the x-coordinates change, from the x-axis the y-coordinates change.

Worked examples

Example 1 — scaffolded

The graph of $f(x)=x^2$ is dilated from the x-axis by a factor of 3. Write the rule of the image and sketch both graphs.

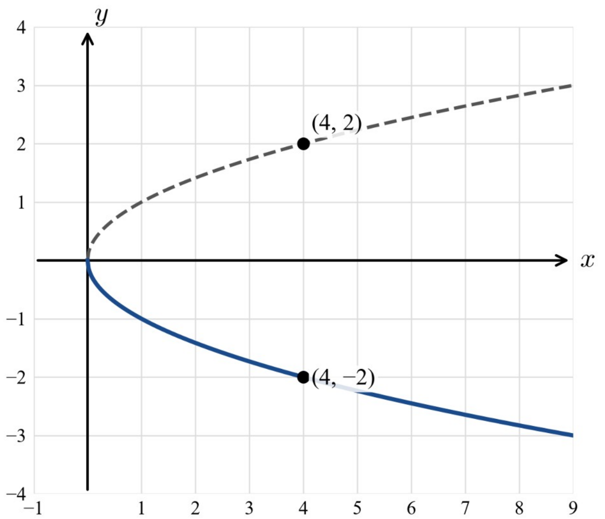
Working	Comment
Image rule: $y=3f(x)=3x^2$.	Dilation from the x-axis by factor a gives $y=af(x)$.
Point rule: $(x,y)\to(x,3y)$, so $(1,1)\to(1,3)$ and $(2,4)\to(2,12)$.	Multiply each y-coordinate by 3.
Turning point $(0,0)$ is unchanged; the parabola is steeper.	Points on the x-axis do not move.



Example 2 — scaffolded

Reflect the graph of $f(x)=\sqrt{x}$ in the x-axis. Write the rule of the image and sketch both graphs.

Working	Comment
Image rule: $y=-f(x)=-\sqrt{x}$.	Reflection in the x-axis gives $y=-f(x)$.
Point rule: $(x,y)\to(x,-y)$, so $(4,2)\to(4,-2)$.	Change the sign of each y-coordinate.
Domain $[0,\infty)$ is unchanged; the range becomes $(-\infty,0]$.	Reflecting does not change the domain here.



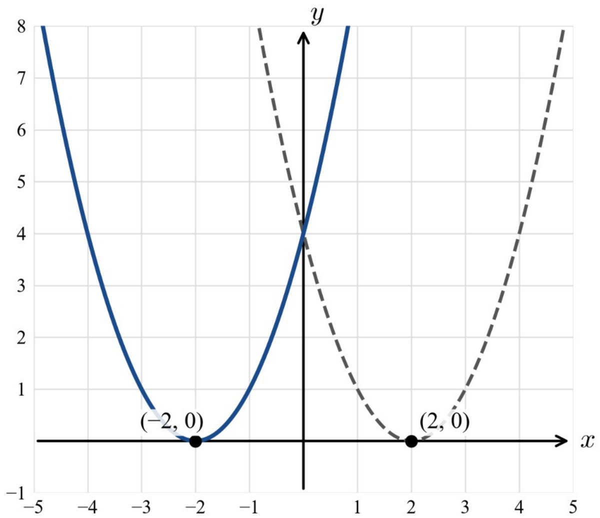
Example 3 — unscaffolded

Reflect the graph of $y=(x-2)^2$ in the y-axis and write the rule of the image.

Reflection in the y-axis replaces x with $-x$:

$$y=(-x-2)^2=(x+2)^2.$$

The turning point $(2,0)$ has image $(-2,0)$.

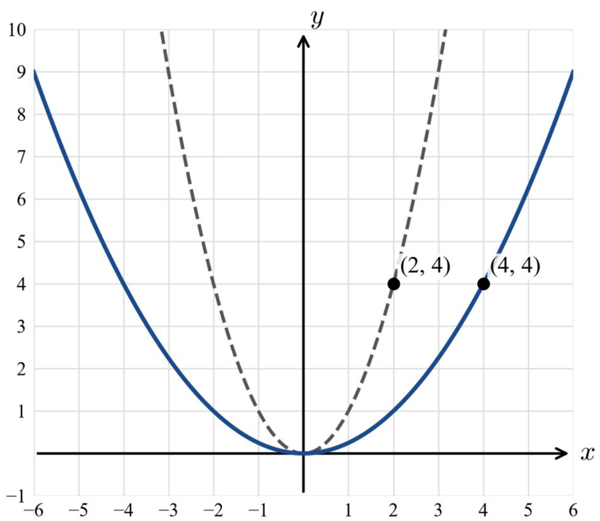


Example 4 — unscaffolded

The graph of $f(x)=x^2$ is dilated from the y-axis by a factor of 2. Write the rule of the image and sketch both graphs.

Image rule: $y=f\left(\frac{x}{2}\right)=\left(\frac{x}{2}\right)^2=\frac{x^2}{4}$.

Point rule: $(x,y)\to(2x,y)$, so $(1,1)\to(2,1)$ and $(2,4)\to(4,4)$. The parabola is wider. $\therefore y=\frac{1}{4}x^2$.



Practice questions

Scaffolded

Q1. The graph of $f(x)=x^2$ is dilated from the x-axis by a factor of 4. (a) Write the image rule. (b) Find the image of $(1,1)$. (c) Sketch both graphs.

Q2. Reflect $f(x)=x^2$ in the x-axis. (a) Write the image rule. (b) State the effect on the turning point. (c) Sketch both graphs.

Q3. For $f(x)=x^3$: (a) find the image of the point $(2,8)$ under a reflection in the y-axis; (b) write the image rule $y=f(-x)$; (c) sketch both graphs.

Q4. Reflect $f(x)=\sqrt{x}$ in the y-axis. (a) Write the image rule. (b) State the domain of the image. (c) Sketch both graphs.

Q5. Reflect $f(x)=\frac{1}{x}$ in the x-axis. (a) Write the image rule. (b) State the effect on the asymptotes. (c) Sketch both graphs.

Q6. The graph of $f(x)=x^2$ is dilated from the y-axis by a factor of 3. (a) Write the image rule. (b) Find the image of $(3,9)$. (c) Sketch both graphs.

Unscaffolded

For each: write the rule of the image and describe the transformation; sketch where a graph is shown in the solutions.

Q7. Dilate $y=x^2$ from the x-axis by a factor of 2.

Q8. Dilate $y=x^2$ from the x-axis by a factor of $1/2$.

Q9. Dilate $y=x^3$ from the x-axis by a factor of 2.

Q10. Dilate $y=\sqrt{x}$ from the y-axis by a factor of 4 (state the domain).

Q11. Dilate $y=\sqrt{x}$ from the x-axis by a factor of 3.

Q12. Dilate $y=\frac{1}{x}$ from the y-axis by a factor of 2.

Q13. Reflect $y=(x-1)^2$ in the y-axis.

Q14. Dilate $y=\frac{1}{x}$ from the x-axis by a factor of 3.

Q15. The point $(3,5)$ lies on $y=f(x)$. Find its image under a dilation from the x-axis by a factor of 4.

Q16. The point $(-2,7)$ lies on $y=f(x)$. Find its image under a reflection in the y-axis.

Q17. The point $(6,-4)$ lies on $y=f(x)$. Find its image on $y=f(3x)$ (a dilation from the y-axis by a factor of $1/3$).

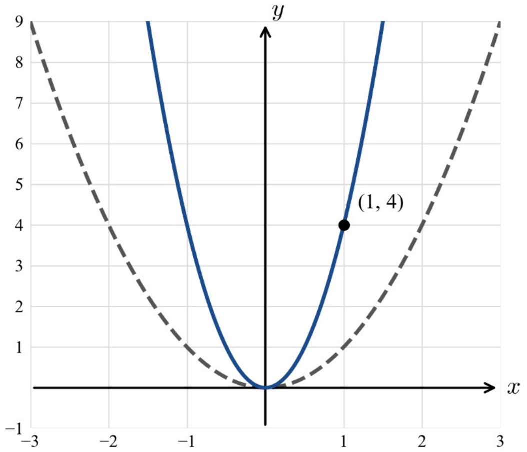
Q18. Describe $y=-2f(x)$ as a sequence of two transformations, and find the image of the point $(1,3)$.

Answers

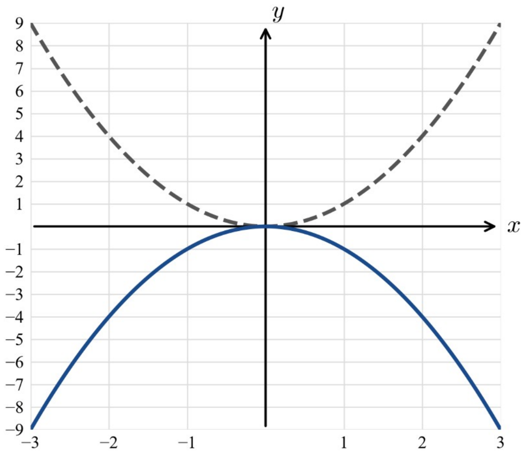
Q1. $y=4x^2$; $(1,4)$. **Q2.** $y=-x^2$; turning point $(0,0)$ unchanged, now a maximum. **Q3.** $(-2,8)$; $y=-x^3$. **Q4.** $y=\sqrt{-x}$; domain $(-\infty,0]$. **Q5.** $y=-\dfrac{1}{x}$; asymptotes $x=0$, $y=0$ unchanged. **Q6.** $y=\dfrac{1}{9}x^2$; $(9,9)$. **Q7.** $y=2x^2$ (dilation from x-axis, factor 2). **Q8.** $y=1/2\,x^2$ (dilation from x-axis, factor $1/2$). **Q9.** $y=2x^3$ (dilation from x-axis, factor 2). **Q10.** $y=1/2\sqrt{x}$, domain $[0,\infty)$ (dilation from y-axis, factor 4). **Q11.** $y=3\sqrt{x}$ (dilation from x-axis, factor 3). **Q12.** $y=\dfrac{2}{x}$ (dilation from y-axis, factor 2). **Q13.** $y=(x+1)^2$ (reflection in y-axis). **Q14.** $y=\dfrac{3}{x}$ (dilation from x-axis, factor 3). **Q15.** $(3,20)$. **Q16.** $(2,7)$. **Q17.** $(2,-4)$. **Q18.** dilation from the x-axis by factor 2, then reflection in the x-axis; image $(1,-6)$.

Fully-worked solutions

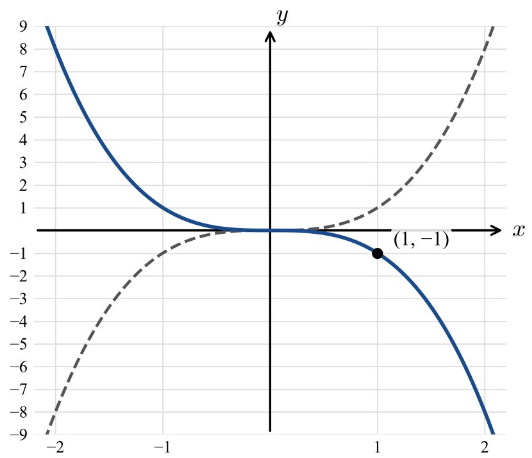
Q1. (a) $y=4f(x)=4x^2$. (b) $(x,y)\rightarrow(x,4y)$, so $(1,1)\rightarrow(1,4)$.



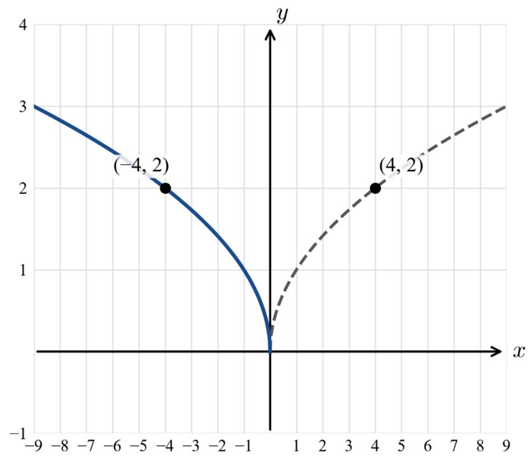
Q2. (a) $y=-f(x)=-x^2$. (b) The turning point $(0,0)$ is unchanged, but it is now a **maximum** (the parabola opens downward).



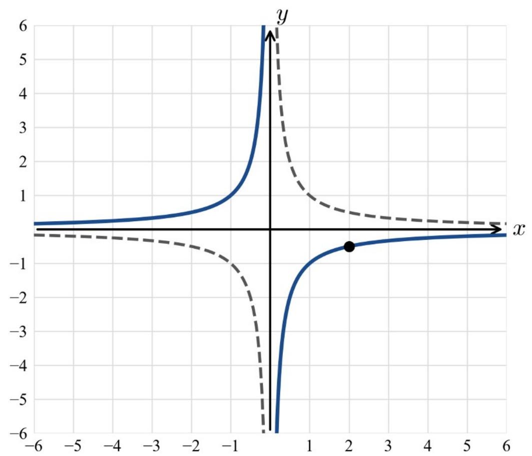
Q3. (a) Reflection in the y-axis: $(x,y)\rightarrow(-x,y)$, so $(2,8)\rightarrow(-2,8)$. (b) $y=f(-x)=(-x)^3=-x^3$.



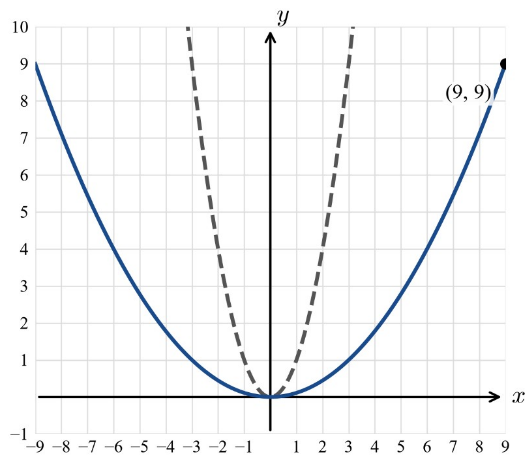
Q4. (a) $y = f(-x) = \sqrt{-x}$. (b) $\sqrt{-x}$ is defined when $-x \geq 0$, i.e. $x \leq 0$: domain $(-\infty, 0]$.



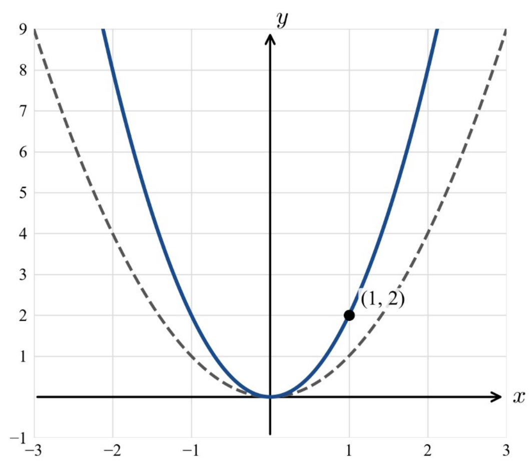
Q5. (a) $y = -f(x) = -\dfrac{1}{x}$. (b) The asymptotes $x = 0$ and $y = 0$ are unchanged; the branches move to the opposite quadrants.



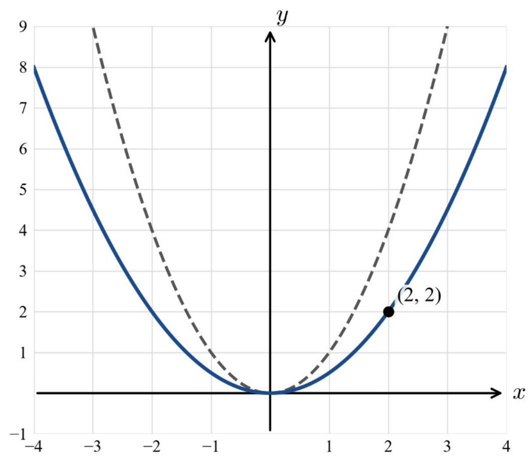
Q6. (a) $y = f\left(\dfrac{x}{3}\right) = \left(\dfrac{x}{3}\right)^2 = \dfrac{1}{9}x^2$. (b) $(x, y) \to (3x, y)$, so $(3, 9) \to (9, 9)$.



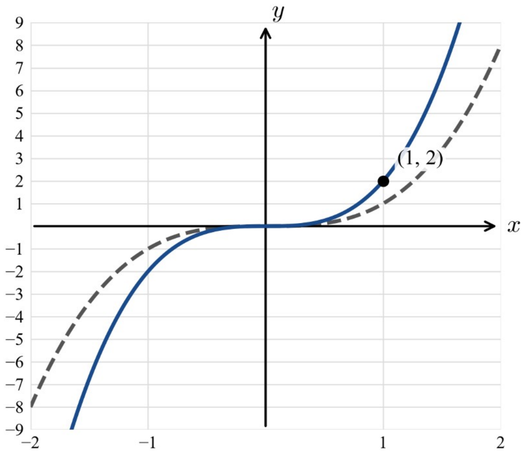
Q7. $y = 2x^2$. Dilation from the x-axis by a factor of 2: $(x, y) \to (x, 2y)$.



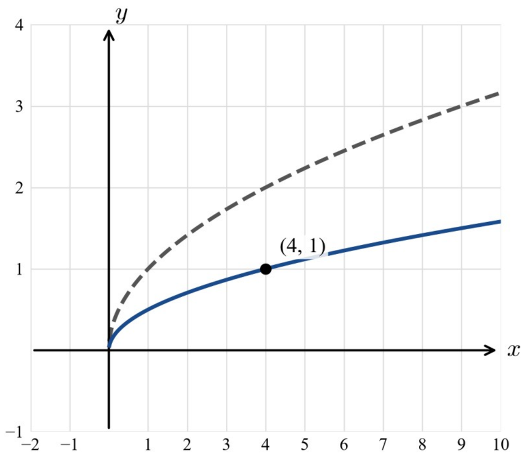
Q8. $y = 1/2\,x^2$. Dilation from the x-axis by a factor of $1/2$ (flatter).



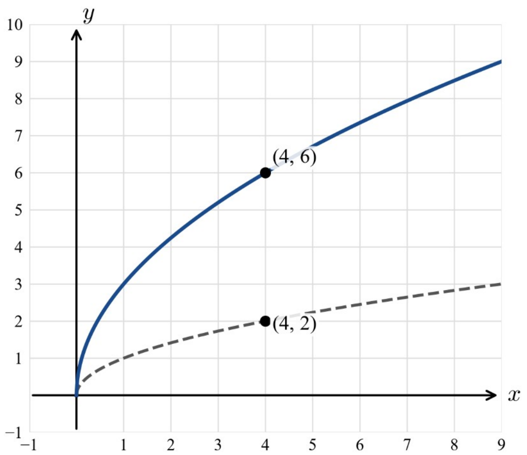
Q9. $y = 2f(x) = 2x^3$. Dilation from the x-axis by a factor of 2: $(x, y) \to (x, 2y)$.



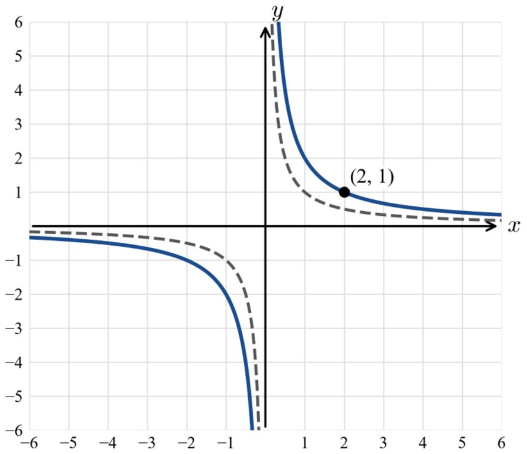
Q10. $y=f\left(\frac{x}{4}\right)=\sqrt{\frac{x}{4}}=1/2\sqrt{x}$. Dilation from the y-axis by a factor of 4: $(x,y)\to(4x,y)$, so $(1,1)\to(4,1)$. The domain $[0,\infty)$ is unchanged.



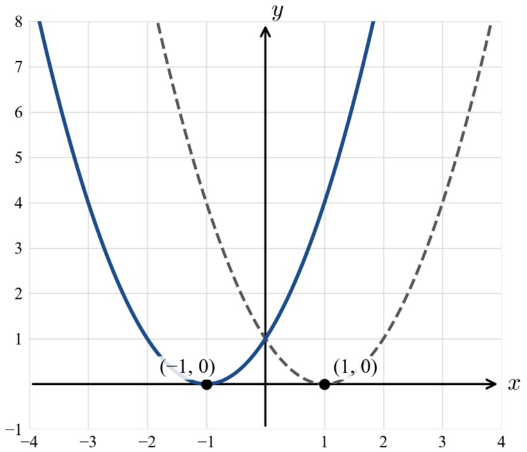
Q11. $y=3\sqrt{x}$. Dilation from the x-axis by a factor of 3: $(4,2)\to(4,6)$.



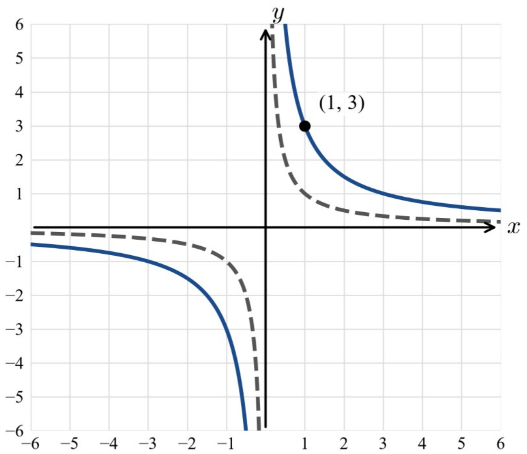
Q12. $y=f\left(\frac{x}{2}\right)=\frac{1}{x/2}=\frac{2}{x}$. Dilation from the y-axis by a factor of 2.



Q13. Reflection in the y-axis replaces x with $-x$: $y=(-x-1)^2=(x+1)^2$. The turning point $(1,0)\rightarrow(-1,0)$.



Q14. $y=3f(x)=\dfrac{3}{x}$ (dilation from the x-axis by a factor of 3).



Q15. Dilation from the x-axis by factor 4: $(x,y)\rightarrow(x,4y)$, so $(3,5)\rightarrow(3,20)$.

Q16. Reflection in the y-axis: $(x,y)\rightarrow(-x,y)$, so $(-2,7)\rightarrow(2,7)$.

Q17. $y=f(3x)$ is a dilation from the y-axis by factor $1/3$: $(x,y)\rightarrow(x/3,y)$, so $(6,-4)\rightarrow(2,-4)$.

Q18. $y=-2f(x)$ is a **dilation from the x-axis by a factor of** 2 followed by a **reflection in the x-axis**. The point rule is $(x,y)\rightarrow(x,-2y)$, so $(1,3)\rightarrow(1,-6)$.

Chapter 7 Transformations of the plane — 7C Combining transformations

Theory

A graph can be changed by combining **dilations**, **reflections** and **translations**. Applied to $y=f(x)$, the combined transformation gives

$$y=af(b(x-h))+k.$$

Order matters. Apply the dilations and reflections *first*, then the translations.

- a — a dilation of factor $|a|$ from the x-axis (multiplies y by a); if $a<0$ there is also a reflection in the x-axis.
- b — replacing x by bx is a dilation of factor $\frac{1}{|b|}$ from the y-axis; if $b<0$ there is also a reflection in the y-axis.
- h — a translation of h units horizontally (right if $h>0$, left if $h<0$).
- k — a translation of k units vertically (up if $k>0$, down if $k<0$).

Image of a point. If (x_0, y_0) lies on $y=f(x)$, then after the transformation its image is

$$\left(\frac{x_0}{b}+h, a y_0+k\right).$$

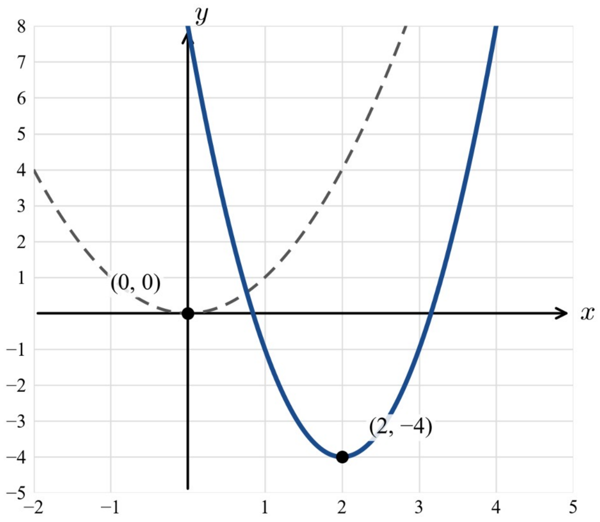
Describing a transformation. Given the rule of the image, compare it with $y=af(b(x-h))+k$ to read off a, b, h and k, then state the sequence in words.

Worked examples

Example 1 — scaffolded

The graph of $y=x^2$ is dilated by a factor of 3 from the x-axis, then translated 2 units to the right and 4 units down. Find the rule of the image and sketch it.

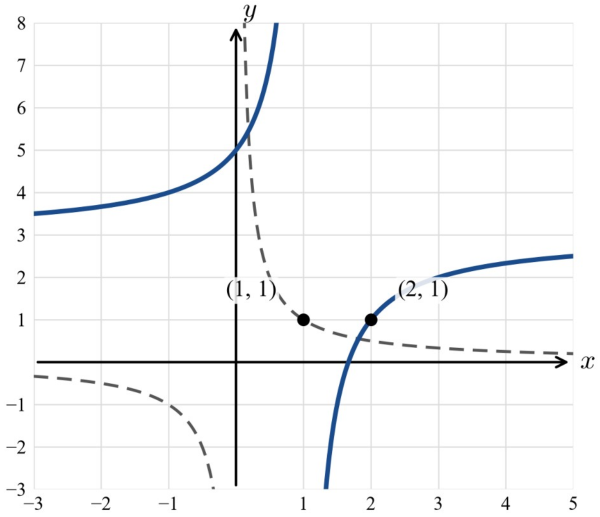
Working	Comment
Dilation of factor 3 from the x-axis: multiply the rule by 3, giving $y=3x^2$.	A dilation of factor a from the x-axis multiplies y by a.
Translate 2 right (replace x with $x-2$) and 4 down (subtract 4): $y=3(x-2)^2-4$.	Translations are applied after the dilation.
The turning point $(0,0)$ of $y=x^2$ maps to $(0+2,3(0)-4)=(2,-4)$.	Track a known point through the transformation.



Example 2 — scaffolded

The graph of $y=\frac{1}{x}$ is reflected in the x-axis, dilated by a factor of 2 from the x-axis, then translated 1 unit right and 3 units up. Find the rule of the image and the image of the point $(1,1)$.

Working	Comment
Reflection in the x-axis and dilation of factor 2 from the x-axis give $a=-2$: $y=-\frac{2}{x}$.	Combine both vertical effects into $a=-2$.
Translate 1 right and 3 up: $y=-\frac{2}{x-1}+3$.	Replace x with $x-1$, then add 3.
Image of $(1,1)$: $(1+1,-2(1)+3)=(2,1)$. Here $b=1$, so the point maps to $(x_0+h, a\,y_0+k)$.	Track the given point through the transformation.

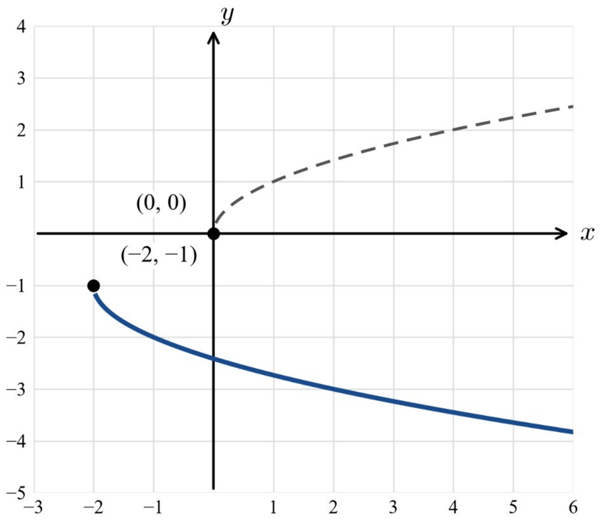


Example 3 — unscaffolded

Describe a sequence of transformations that maps $y=\sqrt{x}$ to $y=-\sqrt{x+2}-1$, and state the image of the endpoint $(0,0)$.

Comparing with $y=a\sqrt{x-h}+k$: $a=-1$, $h=-2$ and $k=-1$.

$\therefore$ a **reflection in the x-axis**, then a **translation of 2 units left and 1 unit down**. The endpoint $(0,0)$ maps to $(0-2,-1)=(-2,-1)$.



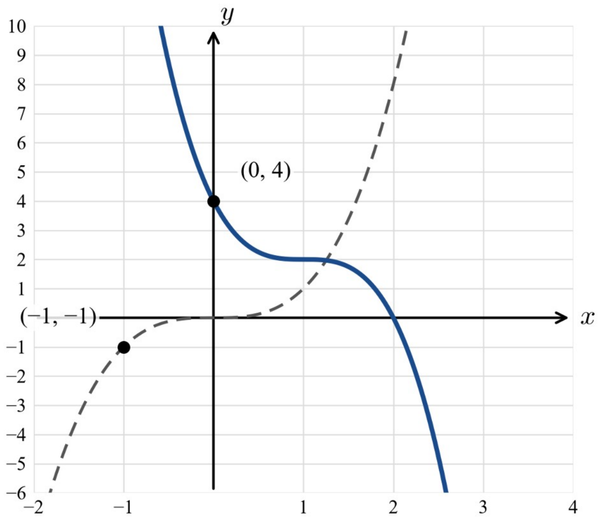
Example 4 — unscaffolded

The graph of $y = x^3$ is reflected in the x-axis, dilated by a factor of 2 from the x-axis, then translated 1 unit right and 2 units up. Find the rule of the image and the image of the point $(-1, -1)$.

The reflection and dilation give $a = -2$: $y = -2x^3$. Translating 1 right and 2 up:

$$y = -2(x-1)^3 + 2.$$

Image of $(-1, -1)$: $(-1+1, -2(-1)+2) = (0, 4)$. $\therefore$ $y = -2(x-1)^3 + 2$.



Practice questions

Scaffolded

Q1. The graph of $y = x^2$ is dilated by a factor of 2 from the x-axis, then translated 3 units left and 1 unit up. Find the rule of the image and sketch it.

Q2. The graph of $y = \frac{1}{x}$ is reflected in the x-axis, then translated 2 units right and 1 unit down. Find the rule of the image and sketch it.

Q3. The graph of $y = \sqrt{x}$ is dilated by a factor of 3 from the x-axis, then translated 1 unit right. Find the rule of the image and sketch it.

Q4. The graph of $y = x^3$ is reflected in the x-axis, then translated 2 units up. Find the rule of the image and sketch it.

Q5. Describe the sequence of transformations that maps $y=x^2$ to $y=(x-4)^2+3$.

Q6. Find the image of the point $(2,4)$ on $y=x^2$ under the transformation to $y=1/2(x+1)^2-2$.

Unscaffolded

Q7. $y=x^2$ is dilated by a factor of 4 from the x-axis, then translated 2 right and 3 down. Find the rule and sketch the image.

Q8. $y=\frac{1}{x}$ is reflected in the x-axis, dilated by a factor of 3 from the x-axis, then translated 1 left and 2 up. Find the rule and sketch the image.

Q9. $y=\sqrt{x}$ is reflected in the x-axis, then translated 4 right and 1 up. Find the rule and sketch the image.

Q10. $y=x^3$ is dilated by a factor of 2 from the x-axis, then translated 1 right and 5 up. Find the rule and sketch the image.

Q11. Describe the transformations that map $y=x^2$ to $y=-2(x-1)^2+3$.

Q12. Describe the transformations that map $y=\frac{1}{x}$ to $y=\frac{1}{x+2}-4$.

Q13. Describe the transformations that map $y=\sqrt{x}$ to $y=3\sqrt{x-2}$.

Q14. Find the image of the point $(3,9)$ on $y=x^2$ under the transformation to $y=-(x-2)^2+1$.

Q15. $y=x^2$ is dilated by a factor of $1/2$ from the y-axis, then translated 3 up. Find the rule and the image of the point $(2,4)$.

Q16. $y=\sqrt{x}$ is dilated by a factor of 2 from the y-axis, then translated 1 down. Find the rule and the image of the point $(4,2)$.

Q17. $y=x^3$ is reflected in the x-axis, dilated by a factor of 2 from the x-axis, then translated 2 left and 3 down. Find the rule and the image of the point $(1,1)$.

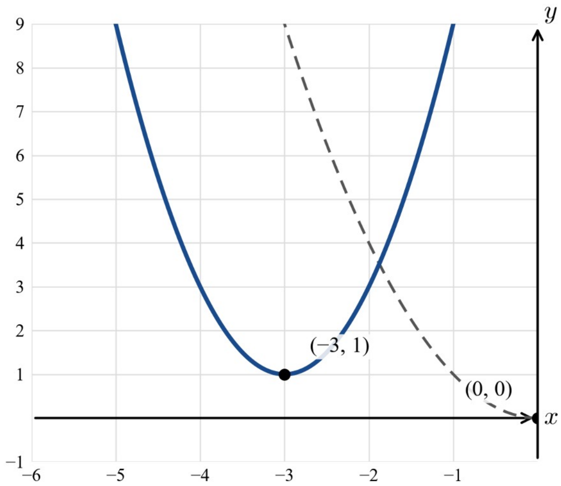
Q18. Describe the transformations that map $y=x^2$ to $y=2(x-3)^2-5$, and state the image of the turning point.

Answers

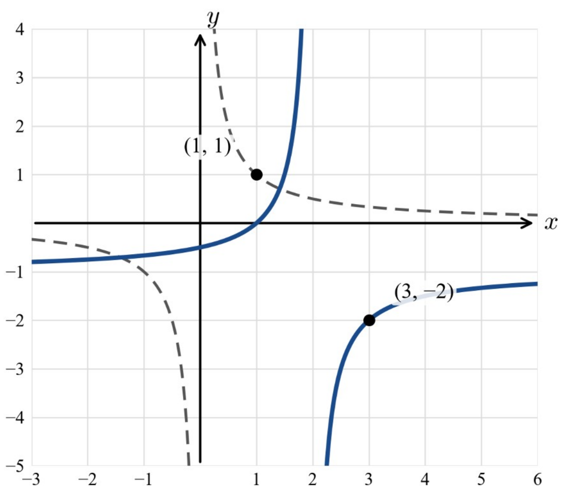
Q1. $y=2(x+3)^2+1$. **Q2.** $y=-\frac{1}{x-2}-1$. **Q3.** $y=3\sqrt{x-1}$. **Q4.** $y=-x^3+2$. **Q5.** translation 4 right and 3 up. **Q6.** $(1,0)$. **Q7.** $y=4(x-2)^2-3$. **Q8.** $y=-\frac{3}{x+1}+2$. **Q9.** $y=-\sqrt{x-4}+1$. **Q10.** $y=2(x-1)^3+5$. **Q11.** reflection in the x-axis, dilation of factor 2 from the x-axis, then 1 right and 3 up. **Q12.** translation 2 left and 4 down. **Q13.** dilation of factor 3 from the x-axis, then 2 right. **Q14.** $(5,-8)$. **Q15.** $y=4x^2+3$; image $(1,7)$. **Q16.** $y=\sqrt{x/2}-1$; image $(8,1)$. **Q17.** $y=-2(x+2)^3-3$; image $(-1,-5)$. **Q18.** dilation of factor 2 from the x-axis, then 3 right and 5 down; image of the turning point $(0,0)$ is $(3,-5)$.

Fully-worked solutions

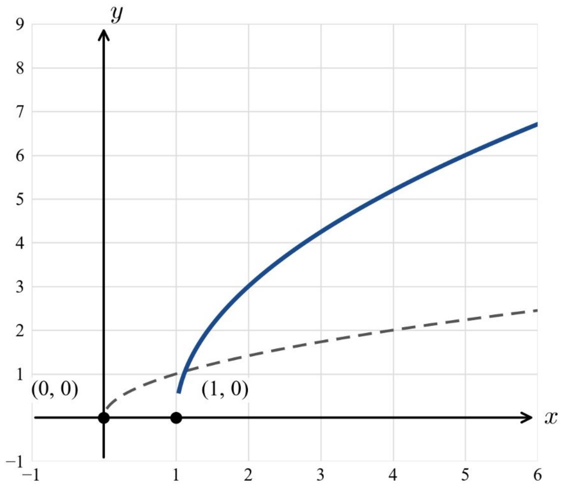
Q1. Dilate: $y=2x^2$; translate 3 left and 1 up: $y=2(x+3)^2+1$. Vertex $(0,0)\to(-3,1)$.



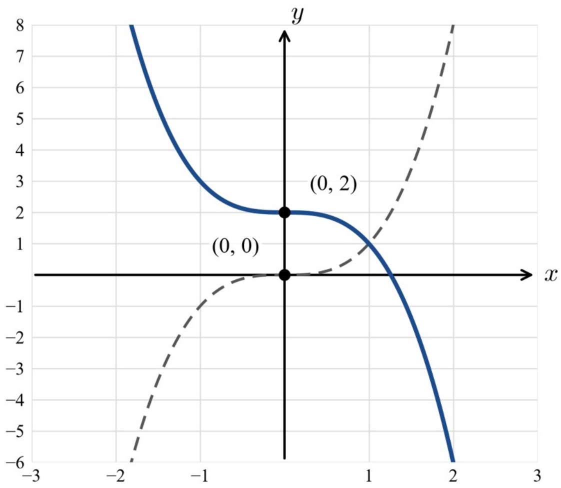
Q2. Reflect: $y=-\frac{1}{x}$; translate 2 right and 1 down: $y=-\frac{1}{x-2}-1$ (asymptotes $x=2$, $y=-1$). The point $(1,1)\to(3,-2)$.



Q3. Dilate: $y=3\sqrt{x}$; translate 1 right: $y=3\sqrt{x-1}$ (endpoint $(1,0)$, domain $[1,\infty)$).



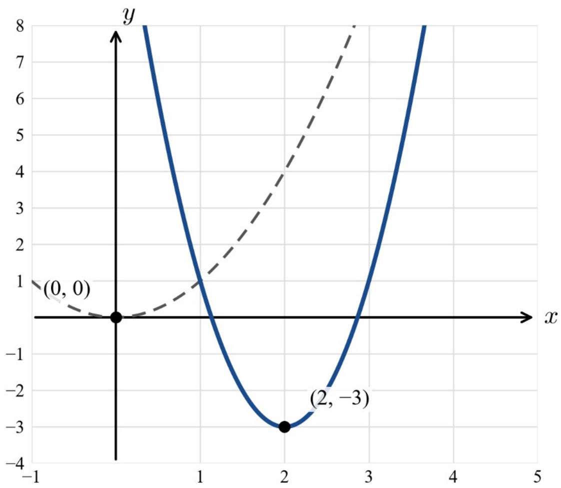
Q4. Reflect: $y=-x^3$; translate 2 up: $y=-x^3+2$ (point of inflection $(0,2)$).



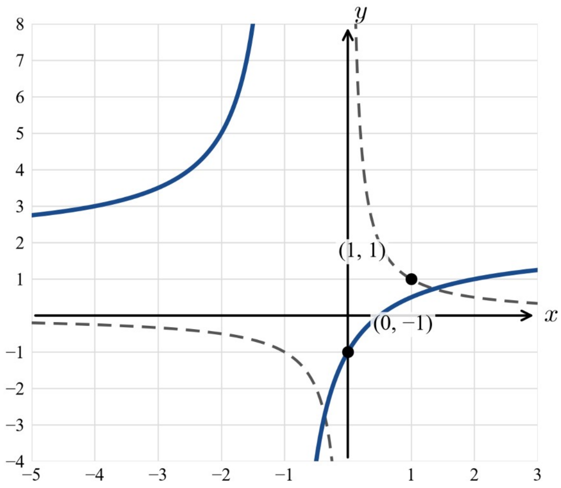
Q5. Comparing $y=(x-4)^2+3$ with $y=(x-h)^2+k$: $h=4$, $k=3$. $\therefore$ a translation of 4 units right and 3 units up.

Q6. Here $a=1/2$, $b=1$, $h=-1$, $k=-2$. Image of $(2,4)$: $(2+(-1),1/2(4)-2)=(1,0)$.

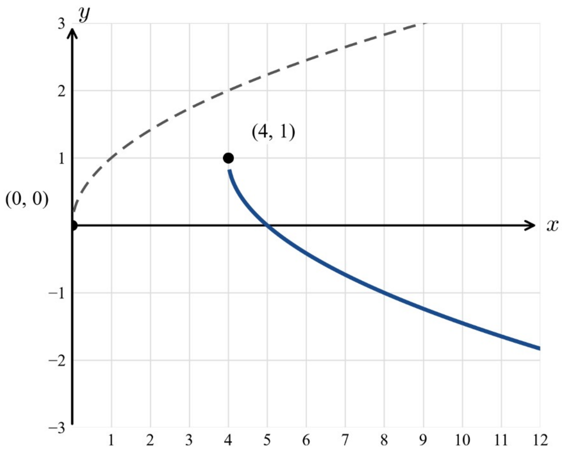
Q7. $y=4x^2\rightarrow y=4(x-2)^2-3$. Vertex $(0,0)\rightarrow(2,-3)$.



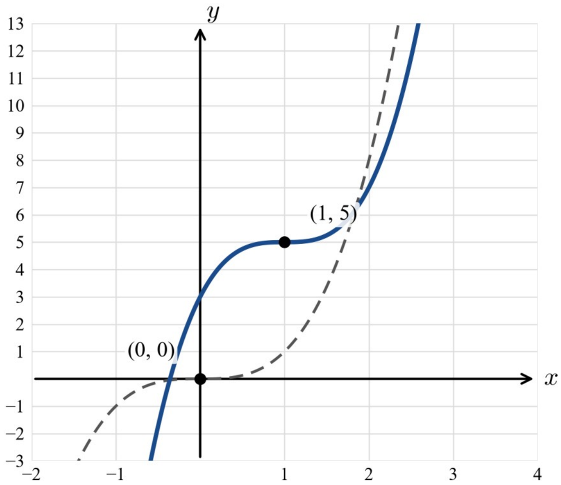
Q8. $a=-3$: $y=-\dfrac{3}{x}$; translate 1 left and 2 up: $y=-\dfrac{3}{x+1}+2$ (asymptotes $x=-1$, $y=2$). The point $(1,1)\rightarrow(0,-1)$.



Q9. Reflect: $y=-\sqrt{x}$; translate 4 right and 1 up: $y=-\sqrt{x-4}+1$ (endpoint $(4,1)$, domain $[4,\infty)$).



Q10. $y=2x^3 \to y=2(x-1)^3+5$ (point of inflection $(1,5)$).



Q11. Comparing $y=-2(x-1)^2+3$: $a=-2$, $h=1$, $k=3$. $\therefore$ a reflection in the x-axis, a dilation of factor 2 from the x-axis, then a translation of 1 right and 3 up.

Q12. Comparing $y=\dfrac{1}{x+2}-4$: $h=-2$, $k=-4$. $\therefore$ a translation of 2 left and 4 down.

Q13. Comparing $y=3\sqrt{x-2}$: $a=3$, $h=2$. $\therefore$ a dilation of factor 3 from the x-axis, then a translation of 2 right.

Q14. $a=-1$, $b=1$, $h=2$, $k=1$. Image of $(3,9)$: $(3+2,-(9)+1)=(5,-8)$.

Q15. A dilation of factor $1/2$ from the y-axis replaces x with $2x$: $y=(2x)^2=4x^2$; translate 3 up: $y=4x^2+3$. Image of $(2,4)$: $(2/2,4+3)=(1,7)$.

Q16. A dilation of factor 2 from the y-axis replaces x with $x/2$: $y=\sqrt{x/2}$; translate 1 down: $y=\sqrt{x/2}-1$. Image of $(4,2)$: $(4\times 2,2-1)=(8,1)$.

Q17. $a=-2$: $y=-2x^3$; translate 2 left and 3 down: $y=-2(x+2)^3-3$. Image of $(1,1)$: $(1-2,-2(1)-3)=(-1,-5)$.

Q18. Comparing $y=2(x-3)^2-5$: $a=2$, $h=3$, $k=-5$. ∴ a dilation of factor 2 from the x-axis, then a translation of 3 right and 5 down. The turning point $(0,0)\rightarrow(3,-5)$.

Chapter 7 Transformations of the plane — 7D Identifying transformations

Theory

Starting from a base graph $y=f(x)$, the graph of

$$y=af\big(n(x-h)\big)+k$$

is obtained by:

- a **dilation of factor a from the x-axis** (and a **reflection in the x-axis** if $a<0$);
- a **dilation of factor $\frac{1}{n}$ from the y-axis** (and a **reflection in the y-axis** if $n<0$);
- a **translation** of h units horizontally and k units vertically.

For the standard families in this course — the parabola $y=x^2$, the cubic $y=x^3$, the square root $y=\sqrt{x}$ and the reciprocal $y=\frac{1}{x}$ — a horizontal dilation can usually be absorbed into a, so we most often write $y=af(x-h)+k$.

Determining the transformations from a rule. Compare the image rule with the standard transformed form and read off a, h and k (and n if a horizontal dilation is present).

Finding unknown parameters. A key feature of the image fixes the translation:

- a parabola's **turning point**, a cubic's **stationary point of inflection**, or a square root's **endpoint** gives (h,k);
- a reciprocal's **asymptotes** $x=h$ and $y=k$ give h and k.

Then substitute a second **known point** into the rule and solve for a.

Working backwards. The key point of the parent — the origin $(0,0)$ for $y=x^2$, $y=x^3$ and $y=\sqrt{x}$ — maps to the corresponding key point (h,k) of the image.

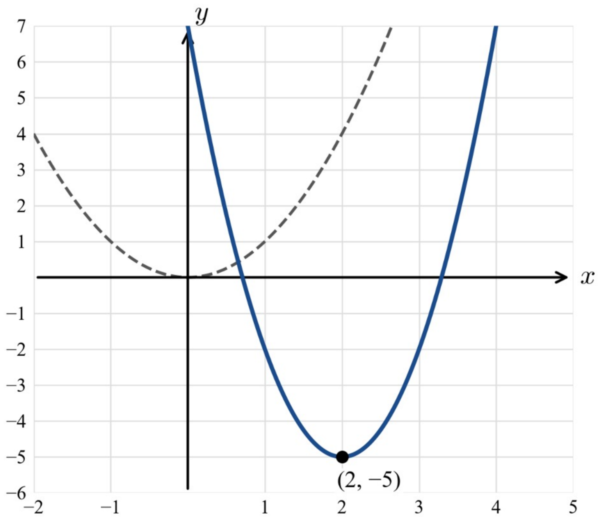
Worked examples

Example 1 — scaffolded

Describe the transformations that map $y=x^2$ to $y=3(x-2)^2-5$.

Working	Comment
Compare with $y=a(x-h)^2+k$: $a=3$, $h=2$, $k=-5$.	Read the parameters straight from the rule.
$a=3>0$: a **dilation of factor** 3 **from the x-axis** (no reflection).	The sign of a tells you whether there is a reflection.
$h=2$: translation 2 **units right**; $k=-5$: translation 5 **units down**.	h and k give the translation.

The parent $y=x^2$ (dashed) and the image $y=3(x-2)^2-5$, with turning point $(2,-5)$:



Example 2 — scaffolded

Describe the transformations that map $y=\sqrt{x}$ to $y=-2\sqrt{x+1}$.

Working	Comment
Compare with $y=a\sqrt{x-h}+k$: $a=-2$, $h=-1$, $k=0$.	Note $x+1=x-(-1)$, so $h=-1$.
$a=-2$: a **dilation of factor** 2 **from the** x**-axis** and a **reflection in the** x**-axis** (since $a<0$).	$\lvert a\rvert=2$ is the dilation; the negative sign is the reflection.
$h=-1$: translation 1 **unit left**; $k=0$: no vertical translation.	

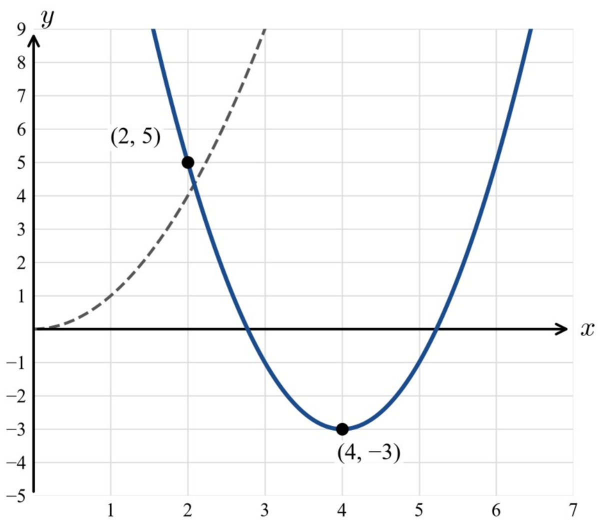
Example 3 — unscaffolded

The graph of $y=a(x-h)^2+k$ is the image of $y=x^2$. It has turning point $(4,-3)$ and passes through $(2,5)$. Find a, h and k, and describe the transformations.

The turning point gives $h=4$ and $k=-3$.

Substituting $(2,5)$: $5=a(2-4)^2-3=4a-3\Rightarrow 4a=8\Rightarrow a=2$.

$\therefore$ $y=2(x-4)^2-3$: a dilation of factor 2 from the x-axis, then a translation 4 units right and 3 units down.



Example 4 — unscaffolded

The image of $y=\dfrac{1}{x}$ is $y=\dfrac{a}{x-h}+k$. It has vertical asymptote $x=3$, horizontal asymptote $y=-2$, and passes through $(4,1)$. Find the rule and describe the transformations.

The asymptotes give $h=3$ and $k=-2$.

Substituting $(4,1)$: $1=\frac{a}{4-3}-2=a-2\Rightarrow a=3$.

$\therefore\ y=\frac{3}{x-3}-2$: a dilation of factor 3 from the x-axis, then a translation 3 units right and 2 units down.

Practice questions

Scaffolded

Describe the transformations that map the parent graph to the given image.

Q1. $y=x^2\rightarrow y=(x-5)^2+2$

Q2. $y=x^2\rightarrow y=-(x+3)^2$

Q3. $y=\sqrt{x}\rightarrow y=\sqrt{x-4}+1$

Q4. $y=\frac{1}{x}\rightarrow y=\frac{1}{x+2}-3$

Q5. $y=x^3\rightarrow y=2(x-1)^3$

Q6. $y=x^2\rightarrow y=1/2(x+2)^2-1$

Unscaffolded

Q7. Describe the transformations that map $y=x^2$ to $y=4(x-1)^2+3$.

Q8. Describe the transformations that map $y=\sqrt{x}$ to $y=-3\sqrt{x+2}-1$.

Q9. Describe the transformations that map $y=x^3$ to $y=-(x-2)^3+4$.

Q10. Describe the transformations that map $y=\frac{1}{x}$ to $y=-\frac{2}{x-1}+3$.

Q11. The graph of $y=a(x-h)^2+k$ is the image of $y=x^2$, with turning point $(-2,1)$, passing through $(0,9)$. Find a, h and k.

Q12. The graph of $y=a(x-h)^3+k$ is the image of $y=x^3$, with stationary point of inflection $(1,-2)$, passing through $(2,1)$. Find the rule.

Q13. The graph of $y=a\sqrt{x-h}+k$ is the image of $y=\sqrt{x}$, with endpoint $(3,-1)$, passing through $(7,3)$. Find the rule.

Q14. Find the rule of the image of $y=\frac{1}{x}$ after a reflection in the x-axis, then a translation 2 units right and 1 unit up.

Q15. Find the rule of the image of $y=x^2$ after a dilation of factor 3 from the x-axis, a reflection in the x-axis, then a translation 1 unit left and 2 units up.

Q16. The image is $y=(x-3)^2-4$. State the transformations applied to $y=x^2$, and state the coordinates of the point on the image corresponding to the vertex of $y=x^2$.

Q17. Describe the transformation that maps $y=\sqrt{x}$ to $y=\sqrt{2x}$.

Q18. A function $y = f(x)$ is transformed to $y = 2f(x-3)+1$. Describe the transformations, and given that f passes through $(1, 4)$, find the coordinates of the corresponding point on the image.

Answers

Q1. 5 right, 2 up. **Q2.** Reflection in the x-axis, 3 left. **Q3.** 4 right, 1 up. **Q4.** 2 left, 3 down. **Q5.** Dilation factor 2 from the x-axis, 1 right. **Q6.** Dilation factor $1/2$ from the x-axis, 2 left, 1 down. **Q7.** Dilation factor 4 from the x-axis, 1 right, 3 up. **Q8.** Dilation factor 3 from the x-axis, reflection in the x-axis, 2 left, 1 down. **Q9.** Reflection in the x-axis, 2 right, 4 up. **Q10.** Dilation factor 2 from the x-axis, reflection in the x-axis, 1 right, 3 up. **Q11.** $a=2$, $h=-2$, $k=1$; $y=2(x+2)^2+1$. **Q12.** $a=3$; $y=3(x-1)^3-2$. **Q13.** $a=2$; $y=2\sqrt{x-3}-1$. **Q14.** $y=-\dfrac{1}{x-2}+1$. **Q15.** $y=-3(x+1)^2+2$. **Q16.** 3 right, 4 down; the vertex maps to $(3,-4)$. **Q17.** Dilation of factor $1/2$ from the y-axis. **Q18.** Dilation factor 2 from the x-axis, 3 right, 1 up; image point $(4,9)$.

Fully-worked solutions

Q1. $y=(x-5)^2+2$: compare with $(x-h)^2+k$, so $h=5$, $k=2$ — translation 5 units right and 2 units up ($a=1$, no dilation or reflection).

Q2. $y=-(x+3)^2$: $a=-1$, $h=-3$, $k=0$ — a reflection in the x-axis and a translation 3 units left.

Q3. $y=\sqrt{x-4}+1$: $h=4$, $k=1$ — translation 4 units right and 1 unit up.

Q4. $y=\dfrac{1}{x+2}-3$: $h=-2$, $k=-3$ — translation 2 units left and 3 units down.

Q5. $y=2(x-1)^3$: $a=2$, $h=1$ — a dilation of factor 2 from the x-axis and a translation 1 unit right.

Q6. $y=1/2(x+2)^2-1$: $a=1/2$, $h=-2$, $k=-1$ — a dilation of factor $1/2$ from the x-axis (wider), and a translation 2 units left and 1 unit down.

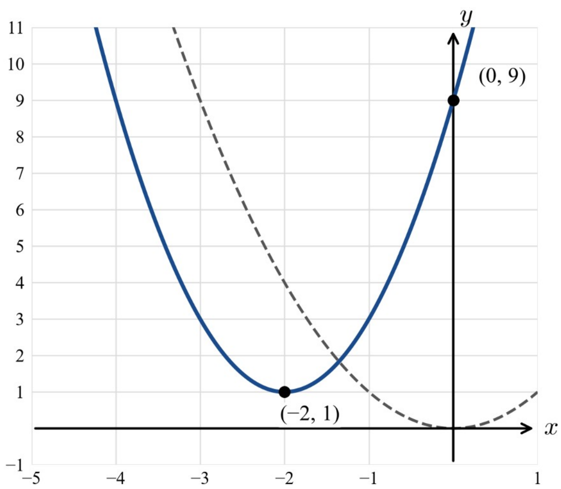
Q7. $y=4(x-1)^2+3$: $a=4$, $h=1$, $k=3$ — dilation of factor 4 from the x-axis, translation 1 right and 3 up.

Q8. $y=-3\sqrt{x+2}-1$: $a=-3$, $h=-2$, $k=-1$ — dilation of factor 3 from the x-axis with a reflection in the x-axis, translation 2 left and 1 down.

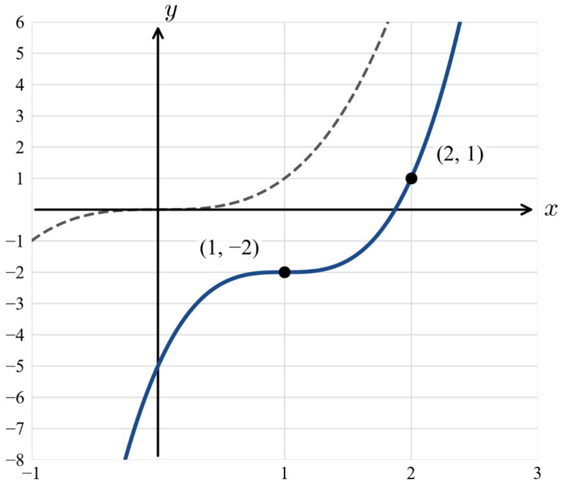
Q9. $y=-(x-2)^3+4$: $a=-1$, $h=2$, $k=4$ — reflection in the x-axis, translation 2 right and 4 up.

Q10. $y=-\dfrac{2}{x-1}+3$: $a=-2$, $h=1$, $k=3$ — dilation of factor 2 from the x-axis with a reflection in the x-axis, translation 1 right and 3 up.

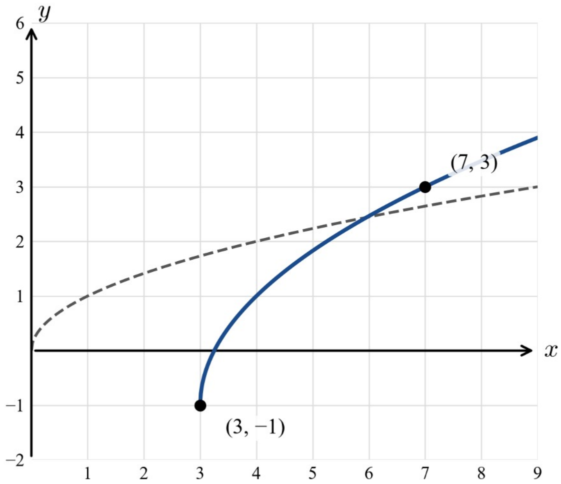
Q11. The turning point gives $h=-2$, $k=1$. Substitute $(0,9)$: $9=a(0+2)^2+1=4a+1 \Rightarrow a=2$. $\therefore y=2(x+2)^2+1$.



Q12. The stationary point of inflection gives $h=1$, $k=-2$. Substitute $(2,1)$: $1=a(2-1)^3-2=a-2 \Rightarrow a=3$.
$\therefore y=3(x-1)^3-2$.

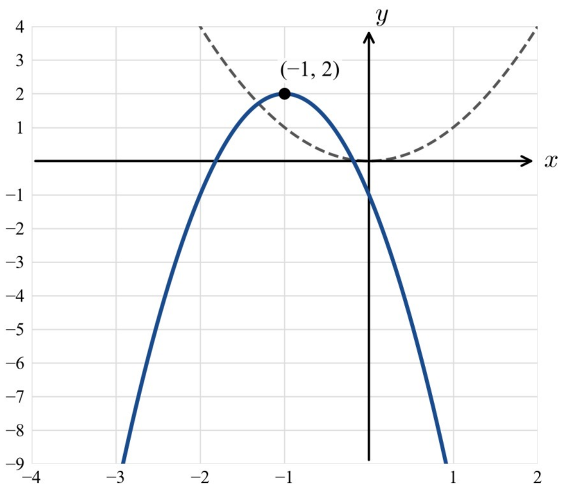


Q13. The endpoint gives $h=3$, $k=-1$. Substitute $(7,3)$: $3=a\sqrt{7-3}-1=2a-1 \Rightarrow 2a=4 \Rightarrow a=2$.
$\therefore y=2\sqrt{x-3}-1$.

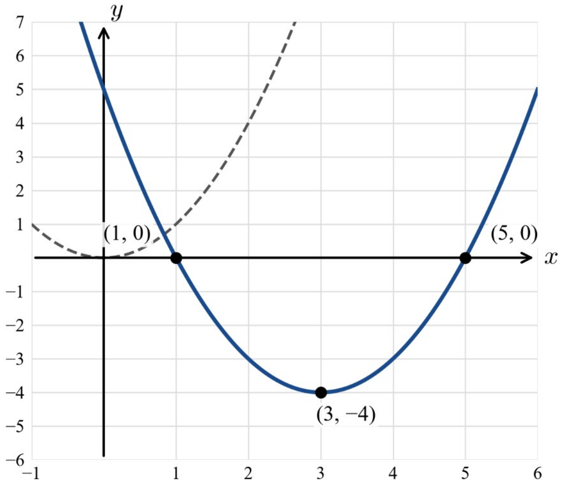


Q14. Reflection in the x-axis: $y=-\frac{1}{x}$. Then 2 right and 1 up: replace x by $x-2$ and add 1: $\therefore y=-\frac{1}{x-2}+1$.

Q15. Dilation of factor 3 then reflection in the x-axis: $y=-3x^2$. Then 1 left and 2 up: $\therefore y=-3(x+1)^2+2$.



Q16. $y=(x-3)^2-4$: $h=3$, $k=-4$ — a translation 3 units right and 4 units down. The vertex $(0,0)$ of $y=x^2$ maps to $(0+3,0-4)=(3,-4)$.



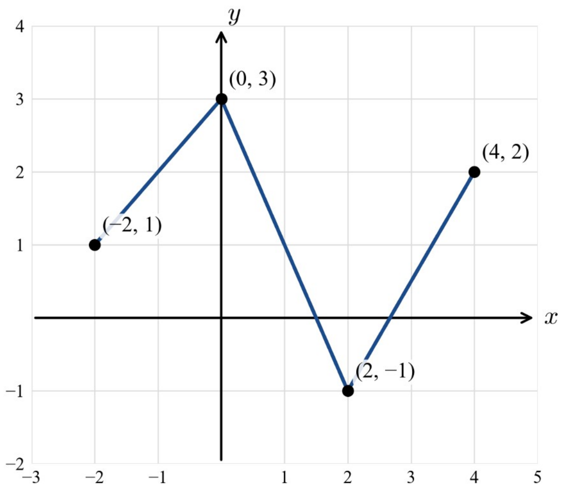
Q17. $y=\sqrt{2x}=\sqrt{2(x)}$, i.e. x is replaced by $2x$ ($n=2$). This is a dilation of factor $1/2$ from the y-axis (a horizontal compression). *(For a square root this is equivalent to a dilation of factor $\sqrt{2}$ from the x-axis, since $\sqrt{2x}=\sqrt{2}\sqrt{x}$.)*

Q18. $y=2f(x-3)+1$: $a=2$, $h=3$, $k=1$ — a dilation of factor 2 from the x-axis, then a translation 3 units right and 1 unit up. A point (x_0,y_0) on $y=f(x)$ maps to $(x_0+3,2y_0+1)$. So $(1,4)$ maps to $(1+3,2(4)+1)=(4,9)$.

Theory

Often we know the **graph** of $y=f(x)$ — its key points, domain and range — without a formula. We can still transform it, by transforming each **key point** and then joining the images in the same way.

Throughout this section, f is the function whose graph passes through the key points $(-2,1)$, $(0,3)$, $(2,-1)$ and $(4,2)$, with domain $[-2,4]$ and range $[-1,3]$:



The transformations and their point mappings. Each rule moves a point (x,y) on $y=f(x)$ to a new image point:

Rule	Effect	Point mapping $(x,y)\to$
$y=f(x)+c$	translation c units up	$(x,y+c)$
$y=f(x-h)$	translation h units right	$(x+h,y)$
$y=af(x)$	dilation by factor a from the x-axis	(x,ay)
$y=f(nx)$	dilation by factor $\frac{1}{n}$ from the y-axis	$\left(\frac{x}{n},y\right)$
$y=-f(x)$	reflection in the x-axis	$(x,-y)$
$y=f(-x)$	reflection in the y-axis	$(-x,y)$

Combinations. For $y=af(x-h)+k$ the combined mapping is $(x,y)\to(x+h,ay+k)$. Apply the horizontal change to the x-coordinate and the vertical changes to the y-coordinate.

Domain and range. Horizontal transformations change the **domain**; vertical transformations change the **range**. For example, $y=f(x-h)$ shifts the domain by h; $y=af(x)+k$ transforms each range value y to $ay+k$.

To sketch the image: map each key point, then join the images the same way as the original, and state the new domain and range.

Worked examples

Example 1 — scaffolded

The point $(2,-1)$ lies on the graph of $y=f(x)$. Find the coordinates of its image on each of the following graphs.

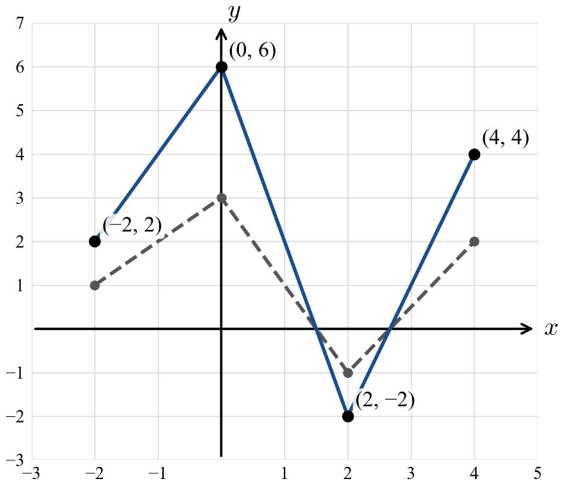
Working	Comment
(a) $y=f(x)+3$: $(x,y)\to(x,y+3)$, so	Add 3 to the y-coordinate only.

Working	Comment
$(2,-1)\to(2,2)$.	
(b) $y=f(x-1)$: $(x,y)\to(x+1,y)$, so $(2,-1)\to(3,-1)$.	$f(x-1)$ shifts **right** 1: add to x.
(c) $y=-f(x)$: $(x,y)\to(x,-y)$, so $(2,-1)\to(2,1)$.	Reflection in the x-axis negates y.
(d) $y=2f(x)$: $(x,y)\to(x,2y)$, so $(2,-1)\to(2,-2)$.	Dilation factor 2 from the x-axis doubles y.

Example 2 — scaffolded

Sketch the graph of $y=2f(x)$, labelling the images of the key points.

Working	Comment
$y=2f(x)$ is a dilation of factor 2 from the x-axis: $(x,y)\to(x,2y)$.	Double every y-coordinate; x is unchanged.
$(-2,1)\to(-2,2)$; $(0,3)\to(0,6)$; $(2,-1)\to(2,-2)$; $(4,2)\to(4,4)$.	Map each key point.
New domain $[-2,4]$ (unchanged); new range $[-2,6]$.	Vertical dilation changes the range only.



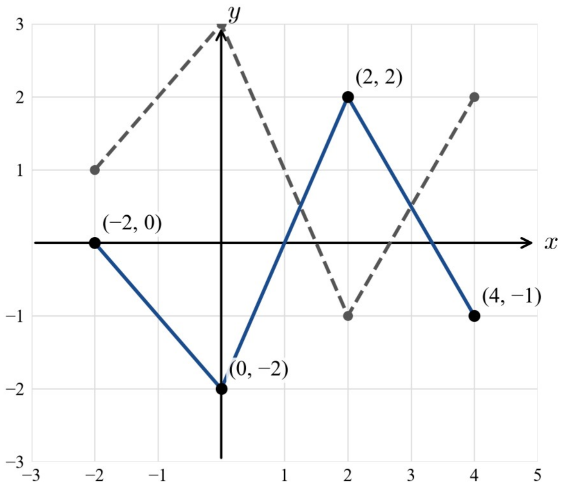
Example 3 — unscaffolded

Sketch the graph of $y=-f(x)+1$.

This is a reflection in the x-axis followed by a translation 1 unit up: $(x,y)\to(x,-y+1)$.

Mapping the key points: $(-2,1)\to(-2,0)$; $(0,3)\to(0,-2)$; $(2,-1)\to(2,2)$; $(4,2)\to(4,-1)$.

$\therefore$ domain $[-2,4]$, range $[-2,2]$.



Example 4 — unscaffolded

Under the transformation from $y=f(x)$ to $y=f(x-3)+2$, a point P on $y=f(x)$ maps to $(5,1)$. Find the coordinates of P.

The mapping is $(x,y)\to(x+3,y+2)$. Setting the image equal to $(5,1)$:

$$x+3=5\Rightarrow x=2,\ y+2=1\Rightarrow y=-1.$$

$\therefore P=(2,-1)$.

Practice questions

In Questions that refer to f, use the function with key points $(-2,1)$, $(0,3)$, $(2,-1)$, $(4,2)$, domain $[-2,4]$, range $[-1,3]$.

Scaffolded

Q1. The point $(-2,1)$ lies on $y=f(x)$. Find its image on: (a) $y=f(x)+3$; (b) $y=f(x-1)$; (c) $y=-f(x)$; (d) $y=2f(x)$.

Q2. Write the point mapping $(x,y)\to(\dots)$ for: (a) $y=f(x-2)+1$; (b) $y=3f(x)$; (c) $y=f(-x)$; (d) $y=-f(x)-2$.

Q3. State the domain and range of: (a) $y=f(x)+2$; (b) $y=f(x-3)$; (c) $y=2f(x)$; (d) $y=f(-x)$.

Q4. Sketch $y=f(x)+2$, labelling the images of the key points.

Q5. Sketch $y=-f(x)$, labelling the images of the key points.

Q6. Sketch $y=f(x-2)$, labelling the images of the key points.

Unscaffolded

Q7. Find the image of $(0,3)$ on $y=f(x)-4$.

Q8. Find the image of $(4,2)$ on $y=f(x+1)$.

Q9. Find the image of $(-2,1)$ on $y=3f(x)$.

Q10. Find the image of $(2,-1)$ on $y=f(2x)$.

Q11. Find the image of $(2,-1)$ on $y=f(-x)$.

Q12. Write the point mapping for $y=2f(x-1)+3$, and hence the image of a general point (a,b).

Q13. Sketch $y=2f(x)-1$, labelling the images of the key points.

Q14. Sketch $y=f(-x)$, labelling the images of the key points.

Q15. Sketch $y=-2f(x)$, labelling the images of the key points.

Q16. Sketch $y=f(x+2)-1$, labelling the images of the key points.

Q17. The range of $y=f(x)$ is $[-1,3]$. Find the range of $y=-2f(x)+1$.

Q18. Under the transformation from $y=f(x)$ to $y=3f(x)-1$, a point maps to $(0,8)$. Find the original point on $y=f(x)$.

Answers

Q1. (a) $(-2,4)$; (b) $(-1,1)$; (c) $(-2,-1)$; (d) $(-2,2)$. **Q2.** (a) $(x+2,y+1)$; (b) $(x,3y)$; (c) $(-x,y)$; (d) $(x,-y-2)$. **Q3.** (a) domain $[-2,4]$, range $[1,5]$; (b) domain $[1,7]$, range $[-1,3]$; (c) domain $[-2,4]$, range $[-2,6]$; (d) domain $[-4,2]$, range $[-1,3]$. **Q4.** Images $(-2,3),(0,5),(2,1),(4,4)$. **Q5.** Images $(-2,-1),(0,-3),(2,1),(4,-2)$. **Q6.** Images $(0,1),(2,3),(4,-1),(6,2)$. **Q7.** $(0,-1)$. **Q8.** $(3,2)$. **Q9.** $(-2,3)$. **Q10.** $(1,-1)$. **Q11.** $(-2,-1)$. **Q12.** $(x,y)\to(x+1,2y+3)$; image $(a+1,2b+3)$. **Q13.** Images $(-2,1),(0,5),(2,-3),(4,3)$. **Q14.** Images $(2,1),(0,3),(-2,-1),(-4,2)$. **Q15.** Images $(-2,-2),(0,-6),(2,2),(4,-4)$. **Q16.** Images $(-4,0),(-2,2),(0,-2),(2,1)$. **Q17.** $[-5,3]$. **Q18.** $(0,3)$.

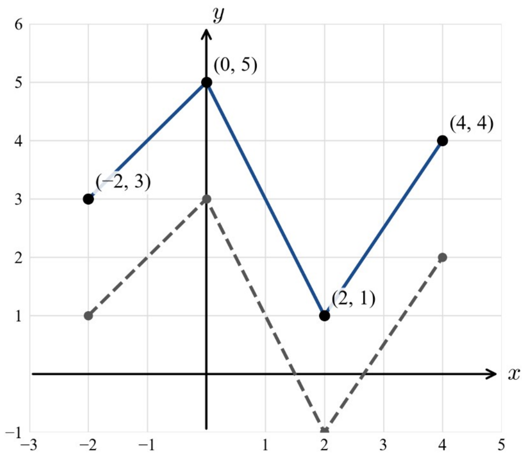
Fully-worked solutions

Q1. $(-2,1)$ under each rule: (a) $y=f(x)+3\Rightarrow(x,y+3)$: $(-2,4)$. (b) $y=f(x-1)\Rightarrow(x+1,y)$: $(-1,1)$. (c) $y=-f(x)\Rightarrow(x,-y)$: $(-2,-1)$. (d) $y=2f(x)\Rightarrow(x,2y)$: $(-2,2)$.

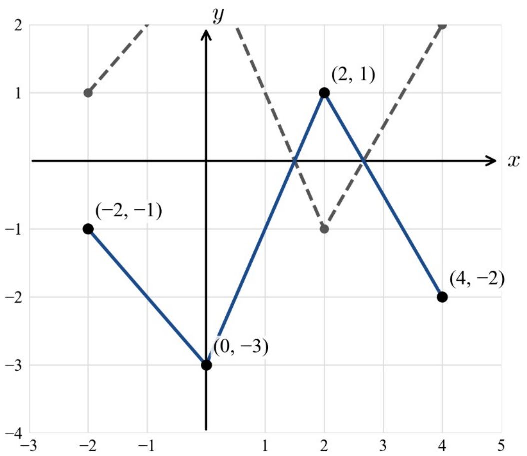
Q2. (a) right 2, up 1: $(x+2,y+1)$. (b) dilation 3 from x-axis: $(x,3y)$. (c) reflection in y-axis: $(-x,y)$. (d) reflection in x-axis then down 2: $(x,-y-2)$.

Q3. (a) vertical shift $+2$: domain $[-2,4]$, range $[1,5]$. (b) horizontal shift right 3: domain $[1,7]$, range $[-1,3]$. (c) vertical dilation 2: domain $[-2,4]$, range $[2\times(-1),2\times3]=[-2,6]$. (d) reflection in y-axis: domain $[-4,2]$, range $[-1,3]$.

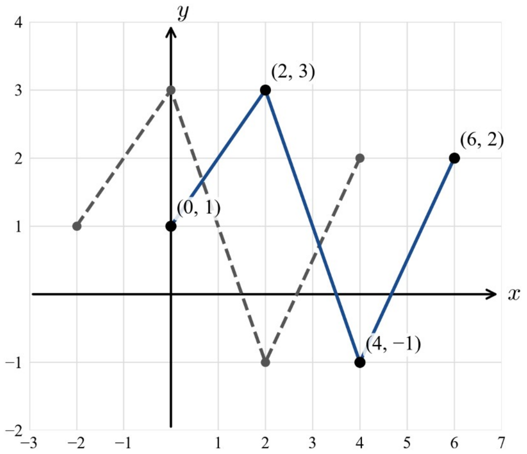
Q4. $y=f(x)+2$: $(x,y)\to(x,y+2)$, giving $(-2,3),(0,5),(2,1),(4,4)$.



Q5. $y=-f(x)$: $(x,y)\to(x,-y)$, giving $(-2,-1),(0,-3),(2,1),(4,-2)$.



Q6. $y=f(x-2)$: $(x,y)\to(x+2,y)$, giving $(0,1),(2,3),(4,-1),(6,2)$.



Q7. $y=f(x)-4 \Rightarrow (x, y-4)$: $(0,3) \to (0,-1)$.

Q8. $y=f(x+1) \Rightarrow (x-1, y)$ (shift **left** 1): $(4,2) \to (3,2)$.

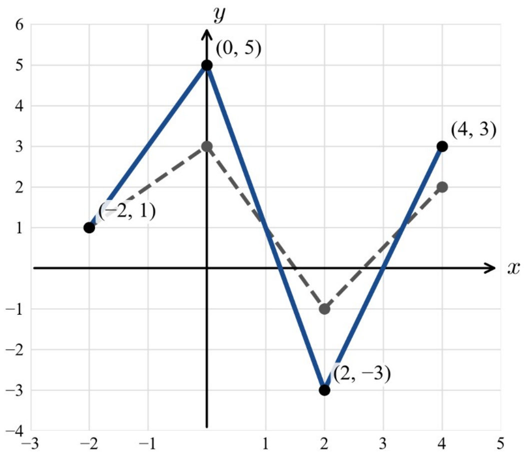
Q9. $y=3f(x) \Rightarrow (x, 3y)$: $(-2,1) \to (-2,3)$.

Q10. $y=f(2x) \Rightarrow \left(\frac{x}{2}, y\right)$: $(2,-1) \to (1,-1)$.

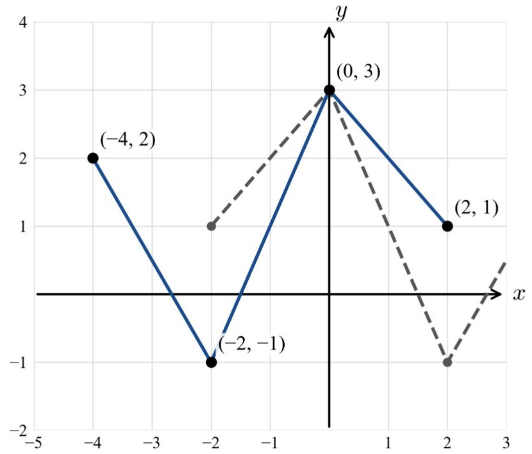
Q11. $y=f(-x) \Rightarrow (-x, y)$: $(2,-1) \to (-2,-1)$.

Q12. $y=2f(x-1)+3$: horizontal shift right 1, vertical dilation 2 then up 3: $(x, y) \to (x+1, 2y+3)$. A general point $(a, b) \to (a+1, 2b+3)$.

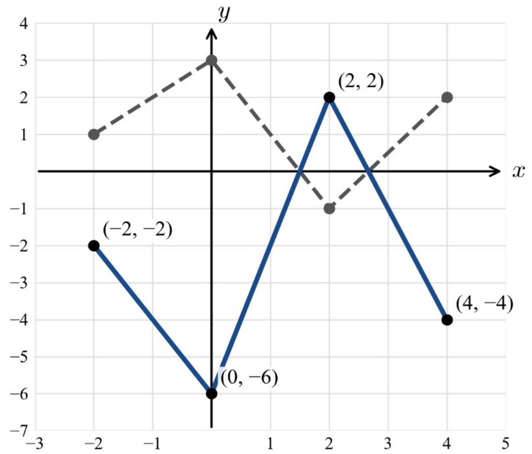
Q13. $y=2f(x)-1$: $(x, y) \to (x, 2y-1)$, giving $(-2,1),(0,5),(2,-3),(4,3)$. Range $[-3,5]$.



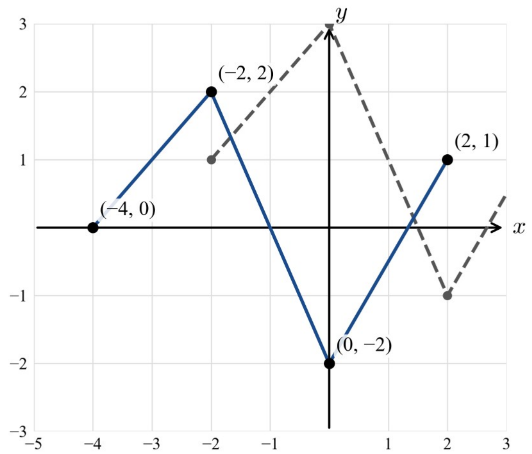
Q14. $y=f(-x)$: $(-x, y)$, giving $(2,1),(0,3),(-2,-1),(-4,2)$. Domain $[-4,2]$.



Q15. $y=-2f(x)$: $(x,y)\to(x,-2y)$, giving $(-2,-2),(0,-6),(2,2),(4,-4)$. Range $[-6,2]$.



Q16. $y=f(x+2)-1$: shift left 2, down 1: $(x-2,y-1)$, giving $(-4,0),(-2,2),(0,-2),(2,1)$. Domain $[-4,2]$, range $[-2,2]$.



Q17. $y=-2f(x)+1$ transforms each range value y to $-2y+1$. As y ranges over $[-1,3]$: at $y=3$, $-2(3)+1=-5$; at $y=-1$, $-2(-1)+1=3$. $\therefore$ range $[-5,3]$.

Q18. $y=3f(x)-1$ has mapping $(x,y)\to(x,3y-1)$. Setting $(x,3y-1)=(0,8)$: $x=0$ and $3y-1=8\Rightarrow y=3$. $\therefore$ the original point is $(0,3)$.

Topic 1 Test A — Technology-active

Topic 1 covers Chapters 1–7. Reading time: 5 minutes. Writing time: 50 minutes. Total: 50 marks. One bound reference, one approved CAS calculator or CAS software, and one scientific calculator are permitted. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (4 marks)

Solve the simultaneous equations $3x-2y=8$ and $x+y=1$.

Question 2 (5 marks)

The points $A(-1,2)$ and $B(5,-6)$ are given. Find:

a. the distance AB; 1 mark

b. the midpoint of AB; 1 mark

c. the gradient of AB; 1 mark

d. the equation of the perpendicular bisector of AB. 2 marks

Question 3 (5 marks)

Consider $y=2x^2+4x-5$.

a. Express the rule in turning-point form and state the turning point. 2 marks

b. Find the axis intercepts (exact values). 2 marks

c. Sketch the graph. 1 mark

Question 4 (3 marks)

Find the value(s) of k for which $x^2+kx+36=0$ has exactly one solution.

Question 5 (4 marks)

A circle has equation $x^2+y^2-10x+4y+20=0$.

a. Find its centre and radius. 3 marks

b. Sketch the circle. 1 mark

Question 6 (5 marks)

Consider $y=\frac{2}{x-1}+3$.

a. State the equations of the asymptotes. 2 marks

b. Find the axis intercepts. 2 marks

c. Sketch the graph. 1 mark

Question 7 (5 marks)

Let $f(x)=\sqrt{x-2}$.

a. State the implied domain and the range. 2 marks

b. Find the rule for $f^{-1}(x)$ and state its domain. 3 marks

Question 8 (4 marks)

A function is defined by $f(x)=\begin{cases} x+1 & x\le 0 \\ x^2 & x>0 \end{cases}$

a. Find $f(-2)$, $f(0)$ and $f(3)$. 2 marks

b. Is f continuous at $x=0$? Justify your answer. 2 marks

Question 9 (6 marks)

Let $P(x)=x^3-2x^2-5x+6$.

a. Show that $(x-1)$ is a factor. 1 mark

b. Factorise $P(x)$ fully. 3 marks

c. Hence solve $P(x)=0$. 1 mark

d. Sketch the graph of $y=P(x)$. 1 mark

Question 10 (5 marks)

The graph of $y=x^2$ is transformed to $y=-2(x-3)^2+4$.

a. Describe the sequence of transformations. 3 marks

b. State the turning point. 1 mark

c. Find the x-intercepts (exact values). 1 mark

Question 11 (4 marks)

Find the coordinates of the points of intersection of $y=x^2+x-4$ and $y=2x+2$.

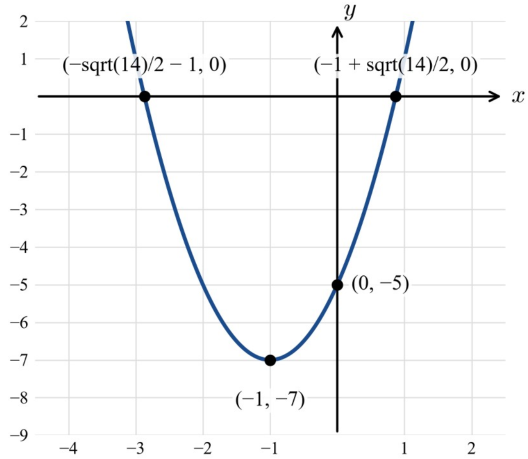
End of Topic 1 Test A

Topic 1 Test A — solutions

Q1. Substitute $x = 1 - y$ into $3x - 2y = 8$: $3(1-y) - 2y = 8 \Rightarrow 3 - 5y = 8 \Rightarrow y = -1$, so $x = 2$. $\therefore (x, y) = (2, -1)$. *(4)*

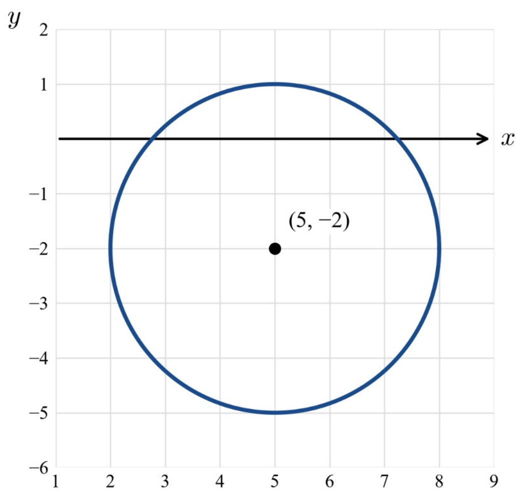
Q2. (a) $AB = \sqrt{(5-(-1))^2 + (-6-2)^2} = \sqrt{36+64} = \sqrt{100} = 10$. (b) Midpoint $= \left(\frac{-1+5}{2}, \frac{2+(-6)}{2}\right) = (2, -2)$. (c) Gradient $= \frac{-6-2}{5-(-1)} = \frac{-8}{6} = -\frac{4}{3}$. (d) Perpendicular gradient $= \frac{3}{4}$; through $(2, -2)$: $y + 2 = \frac{3}{4}(x-2)$, so $y = \frac{3}{4}x - \frac{7}{2}$.

Q3. (a) $2x^2 + 4x - 5 = 2(x^2 + 2x) - 5 = 2((x+1)^2 - 1) - 5 = 2(x+1)^2 - 7$. Turning point $(-1, -7)$. (b) y-intercept $(0, -5)$; x-intercepts: $2(x+1)^2 = 7 \Rightarrow (x+1)^2 = \frac{7}{2} \Rightarrow x = -1 \pm \frac{\sqrt{14}}{2}$.

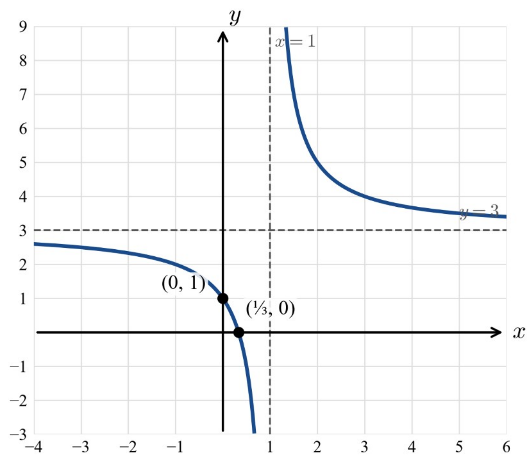


Q4. One solution $\Rightarrow \Delta = 0$: $k^2 - 4(1)(36) = 0 \Rightarrow k^2 = 144 \Rightarrow k = \pm 12$.

Q5. (a) Complete the square: $(x^2 - 10x + 25) + (y^2 + 4y + 4) = -20 + 25 + 4$, so $(x-5)^2 + (y+2)^2 = 9$. Centre $(5, -2)$, radius 3. (b)



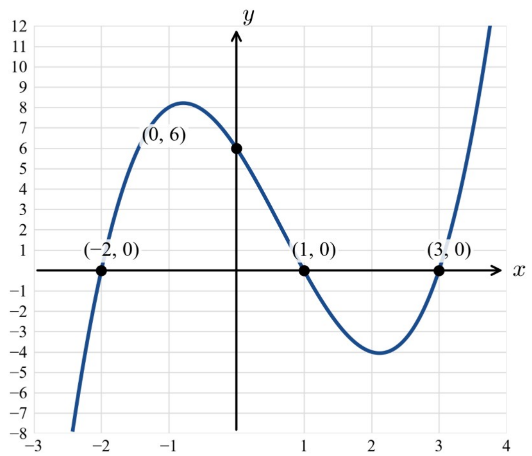
Q6. (a) Vertical asymptote $x = 1$, horizontal asymptote $y = 3$. (b) y-intercept: $x = 0 \Rightarrow y = \frac{2}{-1} + 3 = 1$, giving $(0, 1)$; x-intercept: $\frac{2}{x-1} + 3 = 0 \Rightarrow x - 1 = -\frac{2}{3} \Rightarrow x = \frac{1}{3}$, giving $\left(\frac{1}{3}, 0\right)$.



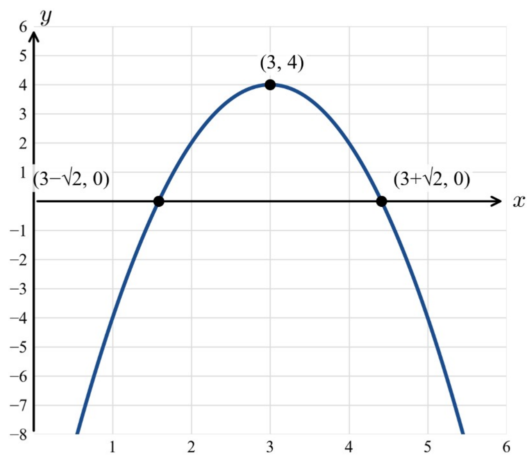
Q7. (a) $\sqrt{x-2}$ requires $x-2\geq 0$, so the domain is $[2,\infty)$ and the range is $[0,\infty)$. (b) Let $y=\sqrt{x-2}$; swap: $x=\sqrt{y-2}\Rightarrow x^2=y-2\Rightarrow y=x^2+2$. $\therefore f^{-1}(x)=x^2+2$, with domain $[0,\infty)$ (the range of f).

Q8. (a) $f(-2)=-2+1=-1$; $f(0)=0+1=1$ (use $x\leq 0$); $f(3)=3^2=9$. (b) As $x\to 0^-$, $f(x)\to 1$; as $x\to 0^+$, $f(x)\to 0$. The one-sided limits differ, so f is **not continuous** at $x=0$ (there is a jump).

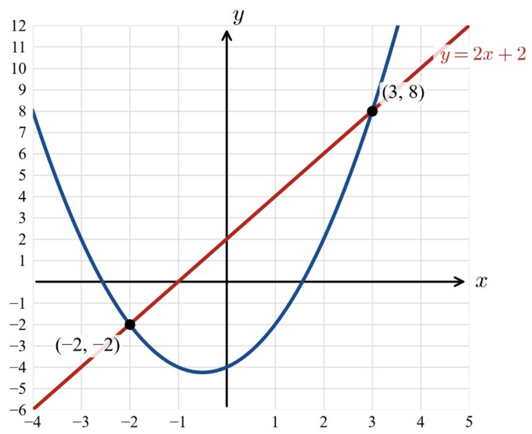
Q9. (a) $P(1)=1-2-5+6=0$, so $(x-1)$ is a factor. (b) Dividing, $P(x)=(x-1)(x^2-x-6)=(x-1)(x-3)(x+2)$. (c) $P(x)=0\Rightarrow x=1,3,-2$. (d)



Q10. (a) From $y=x^2$: a reflection in the x-axis, then a dilation of factor 2 from the x-axis, then a translation 3 units right and 4 units up. (b) Turning point $(3,4)$ (maximum). (c) $-2(x-3)^2+4=0\Rightarrow(x-3)^2=2\Rightarrow x=3\pm\sqrt{2}$. *(The y-intercept $(0,-14)$ lies below the window shown.)*



Q11. $x^2+x-4=2x+2\Rightarrow x^2-x-6=0\Rightarrow(x-3)(x+2)=0$, so $x=3$ or $x=-2$. Then $y=2x+2$ gives $(3,8)$ and $(-2,-2)$.


y
x
$y = 2x + 2$
(3, 8)
(−2, −2)
12
11
10
9
8
7
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−4
−3
−2
−1
1
2
3
4
5

Topic 1 Test B — Technology-free

Topic 1 covers Chapters 1–7. Reading time: 5 minutes. Writing time: 45 minutes. Total: 40 marks. Students are not permitted to bring any technology (calculators or software), or notes of any kind. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. In questions where more than one mark is available, appropriate working must be shown.

Question 1 (3 marks)

Solve $3(x-2)=2x+5$.

Question 2 (3 marks)

Solve the inequality $5-2x\geq 11$, and express the solution in set-builder notation.

Question 3 (4 marks)

Solve $3x^2-x-2=0$ by factorising.

Question 4 (3 marks)

Express x^2-6x+2 in turning-point form and state the turning point.

Question 5 (5 marks)

Sketch the graph of $y=(x+1)^2-4$, labelling the turning point and all axis intercepts.

Question 6 (4 marks)

Find the equation of the straight line passing through $(2,1)$ and $(4,9)$.

Question 7 (4 marks)

Factorise $x^3+3x^2-16x-48$ fully.

Question 8 (3 marks)

State the centre and radius of the circle $(x-4)^2+(y+1)^2=9$.

Question 9 (4 marks)

For $f(x)=\dfrac{1}{x-5}$, state the implied domain and the equations of the two asymptotes, and sketch the graph.

Question 10 (4 marks)

Describe the transformation of $y=\sqrt{x}$ that gives $y=\sqrt{x-4}+3$, and state the coordinates of the endpoint.

Question 11 (3 marks)

Show that $x^2+6x+11>0$ for all real x.

End of Topic 1 Test B

Topic 1 Test B — solutions

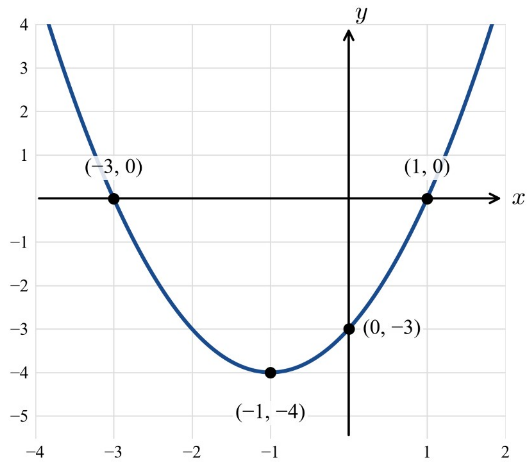
Q1. $3(x-2)=2x+5 \Rightarrow 3x-6=2x+5 \Rightarrow x=11$.

Q2. $5-2x \geq 11 \Rightarrow -2x \geq 6 \Rightarrow x \leq -3$ (the inequality reverses on dividing by -2). Solution: $\{x : x \leq -3\}$.

Q3. $3x^2-x-2=(3x+2)(x-1)=0$, so $x=1$ or $x=-\dfrac{2}{3}$.

Q4. $x^2-6x+2=(x^2-6x+9)-9+2=(x-3)^2-7$. Turning point $(3,-7)$.

Q5. Turning point $(-1,-4)$; y-intercept $x=0 \Rightarrow y=1-4=-3$, giving $(0,-3)$; x-intercepts $(x+1)^2=4 \Rightarrow x=1$ or $x=-3$, giving $(1,0)$ and $(-3,0)$.

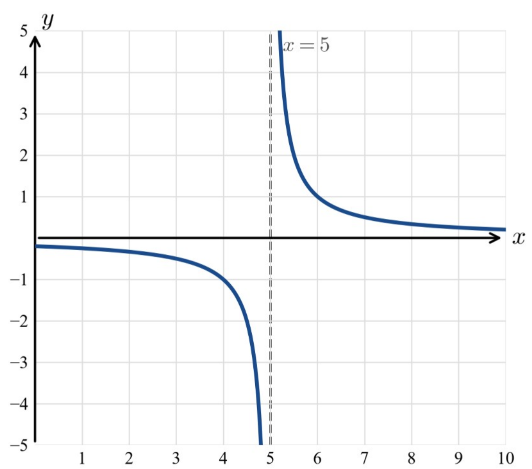


Q6. Gradient $=\dfrac{9-1}{4-2}=4$; through $(2,1)$: $y-1=4(x-2)$, so $y=4x-7$.

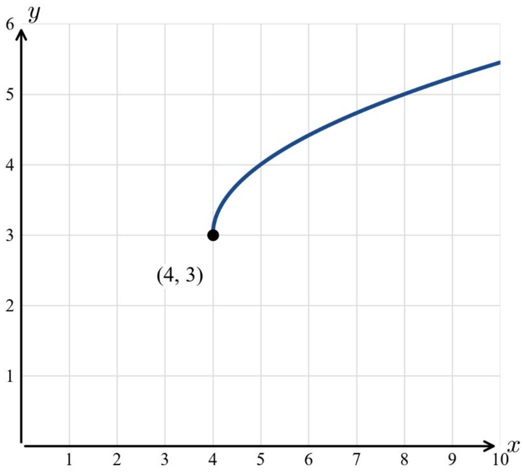
Q7. Group: $x^3+3x^2-16x-48=x^2(x+3)-16(x+3)=(x+3)(x^2-16)=(x+3)(x-4)(x+4)$.

Q8. Centre $(4,-1)$, radius 3.

Q9. $x-5 \neq 0$, so the domain is $\mathbb{R} \setminus \{5\}$. Vertical asymptote $x=5$; horizontal asymptote $y=0$.



Q10. From $y=\sqrt{x}$: a translation 4 units right and 3 units up. Endpoint $(0,0) \rightarrow (4,3)$.



Q11. For $x^2+6x+11$, $\Delta=6^2-4(1)(11)=36-44=-8<0$, so there are no real zeros, and since the leading coefficient $a=1>0$ the parabola lies entirely above the x-axis. $\therefore x^2+6x+11>0$ for all real x.