

Mathematical Methods Units 1 & 2

Chapter 6 — Polynomial functions

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Chapter 6 Polynomial functions — 6A Polynomial terminology

Theory

A **polynomial** in x is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the **coefficients** $a_0, a_1, \dots, a_n$ are real numbers and n is a non-negative integer.

- The **degree** is the highest power of x with a non-zero coefficient.
- The **leading term** is the term of highest degree, $a_n x^n$; its coefficient a_n is the **leading coefficient**. A polynomial with leading coefficient 1 is **monic**.
- The **constant term** is a_0 (the term with no x); note $a_0 = P(0)$.

Low-degree polynomials have special names: degree 0 is a constant, degree 1 linear, degree 2 quadratic, degree 3 cubic, degree 4 quartic.

Function notation. Writing $P(x)$ lets us **evaluate** the polynomial at a value by substitution: $P(a)$ is the number obtained by replacing every x with a .

Adding and subtracting polynomials: collect like terms (terms with the same power of x). **Multiplying** polynomials: expand using the distributive law, then collect like terms. The degree of a product equals the **sum** of the degrees of the factors.

Equating coefficients. Two polynomials are equal **for all values of x** (written with $\equiv$, “is identically equal to”) if and only if the coefficients of corresponding powers of x are equal. This lets us find unknown coefficients: if $ax^2 + bx + c \equiv 2x^2 - 5x + 1$, then $a = 2$, $b = -5$ and $c = 1$.

An **identity** (using $\equiv$) is true for *all* x ; an ordinary **equation** (using $=$) is true only for particular values of x .

Worked examples

Example 1 — scaffolded

For $P(x) = 2x^3 - 5x^2 + 7$, state the degree, leading coefficient and constant term, and evaluate $P(2)$.

Working	Comment
The highest power is x^3 , so the degree is 3 .	Degree = highest power of x present.
Leading coefficient 2 (from $2x^3$); constant term 7.	The leading coefficient multiplies the highest power; the constant term has no x .
$P(2) = 2(2)^3 - 5(2)^2 + 7 = 16 - 20 + 7 = 3$.	Substitute $x = 2$ and evaluate.

Example 2 — scaffolded

For $P(x) = x^3 - 2x + 1$ and $Q(x) = 3x^2 - x - 4$, find $P(x) + Q(x)$ and $P(x) - Q(x)$.

Working	Comment
$P(x) + Q(x) = x^3 + 3x^2 + (-2x - x) + (1 - 4)$ $= x^3 + 3x^2 - 3x - 3$	Add, grouping like terms. Collect.
$P(x) - Q(x) = x^3 - 3x^2 + (-2x + x) + (1 + 4)$ $= x^3 - 3x^2 - x + 5$	Subtract — change the sign of <i>every</i> term of Q . Collect.

Example 3 — unscaffolded

Expand $(2x - 1)(x^2 + 3x - 2)$ and state its degree.

$$(2x - 1)(x^2 + 3x - 2) = 2x(x^2 + 3x - 2) - 1(x^2 + 3x - 2) = 2x^3 + 6x^2 - 4x - x^2 - 3x + 2.$$

$\therefore (2x - 1)(x^2 + 3x - 2) = 2x^3 + 5x^2 - 7x + 2$, which has **degree 3** (as expected, $1 + 2 = 3$).

Example 4 — unscaffolded

Find the values of a , b and c if $a(x + 1)^2 + b(x + 1) + c \equiv 2x^2 - 3x + 1$.

Expanding the left-hand side:

$$a(x^2 + 2x + 1) + b(x + 1) + c = ax^2 + (2a + b)x + (a + b + c).$$

Equating coefficients with $2x^2 - 3x + 1$:

$$a = 2, 2a + b = -3, a + b + c = 1.$$

From the first, $a = 2$. Then $2(2) + b = -3 \Rightarrow b = -7$. Then $2 + (-7) + c = 1 \Rightarrow c = 6$.

$\therefore a = 2, b = -7, c = 6$.

Practice questions

Scaffolded

Q1. For $P(x) = 4x^3 + 2x^2 - x + 5$, state (a) the degree, (b) the leading coefficient, (c) the constant term.

Q2. For $P(x) = x^4 - 3x^2 + 2x - 7$, evaluate $P(-1)$.

Q3. For $P(x) = 2x^2 - x + 3$ and $Q(x) = x^2 + 4x - 1$, find $P(x) + Q(x)$.

Q4. For $P(x) = x^3 + 2x$ and $Q(x) = x^3 - x^2 + 3$, find $P(x) - Q(x)$.

Q5. Expand $(x + 2)(x^2 - 3x + 1)$.

Q6. Find a and b if $2x^2 + ax + b \equiv 2(x - 1)(x + 3)$.

Unscaffolded

Q7. State the degree, leading coefficient and constant term of $P(x) = 7 - 2x + 5x^4 - x^3$.

Q8. Evaluate $P(3)$ for $P(x) = x^3 - 4x^2 + 2x - 1$.

Q9. Evaluate $P(-2)$ for $P(x) = 2x^4 + x^3 - 3x + 5$.

Q10. For $P(x) = 3x^2 - 2x + 1$ and $Q(x) = x^2 + 5x - 4$, find $P(x) + Q(x)$.

Q11. For $P(x) = x^3 - 2x^2 + x$ and $Q(x) = 2x^3 + x^2 - 3$, find $P(x) - Q(x)$.

Q12. Expand $(x^2 + 1)(x^2 - 4)$ and state its degree.

Q13. Expand $(2x - 3)(2x + 3)(x - 1)$.

Q14. Expand $(x - 2)^3$.

Q15. Find a , b and c if $a(x - 1)^2 + b(x - 1) + c \equiv x^2 + 4x + 7$.

Q16. State the coefficients a , b , c and d if $2x^3 - 3x^2 + 5 \equiv ax^3 + bx^2 + cx + d$.

Q17. Find a and b if $(x+a)(x+b) = x^2 + 7x + 12$ for all x .

Q18. The polynomial $P(x) = x^2 + bx + c$ satisfies $P(1) = 0$ and $P(-2) = 0$. Find b and c .

Answers

Q1. (a) 3 (b) 4 (c) 5. **Q2.** - 11. **Q3.** $3x^2 + 3x + 2$. **Q4.** $x^2 + 2x - 3$. **Q5.** $x^3 - x^2 - 5x + 2$. **Q6.** $a = 4, b = -6$. **Q7.** degree 4, leading coefficient 5, constant term 7. **Q8.** - 4. **Q9.** 35. **Q10.** $4x^2 + 3x - 3$. **Q11.** $-x^3 - 3x^2 + x + 3$. **Q12.** $x^4 - 3x^2 - 4$, degree 4. **Q13.** $4x^3 - 4x^2 - 9x + 9$. **Q14.** $x^3 - 6x^2 + 12x - 8$. **Q15.** $a = 1, b = 6, c = 12$. **Q16.** $a = 2, b = -3, c = 0, d = 5$. **Q17.** $a = 3, b = 4$ (or $a = 4, b = 3$). **Q18.** $b = 1, c = -2$.

Fully-worked solutions

Q1. $P(x) = 4x^3 + 2x^2 - x + 5$. (a) Highest power x^3 , so degree 3. (b) Leading coefficient 4. (c) Constant term 5.

Q2. $P(-1) = (-1)^4 - 3(-1)^2 + 2(-1) - 7 = 1 - 3 - 2 - 7 = -11$.

Q3. $P(x) + Q(x) = (2x^2 + x^2) + (-x + 4x) + (3 - 1) = 3x^2 + 3x + 2$.

Q4. $P(x) - Q(x) = (x^3 - x^3) + x^2 + 2x + (0 - 3) = x^2 + 2x - 3$ (the sign of every term of Q is changed).

Q5. $(x + 2)(x^2 - 3x + 1) = x^3 - 3x^2 + x + 2x^2 - 6x + 2 = x^3 - x^2 - 5x + 2$.

Q6. $2(x - 1)(x + 3) = 2(x^2 + 2x - 3) = 2x^2 + 4x - 6$. Equating with $2x^2 + ax + b$ gives $a = 4$ and $b = -6$.

Q7. Written in order, $P(x) = 5x^4 - x^3 - 2x + 7$: degree 4, leading coefficient 5, constant term 7.

Q8. $P(3) = 3^3 - 4(3)^2 + 2(3) - 1 = 27 - 36 + 6 - 1 = -4$.

Q9. $P(-2) = 2(-2)^4 + (-2)^3 - 3(-2) + 5 = 32 - 8 + 6 + 5 = 35$.

Q10. $P(x) + Q(x) = (3x^2 + x^2) + (-2x + 5x) + (1 - 4) = 4x^2 + 3x - 3$.

Q11. $P(x) - Q(x) = (x^3 - 2x^3) + (-2x^2 - x^2) + x + (0 + 3) = -x^3 - 3x^2 + x + 3$.

Q12. $(x^2 + 1)(x^2 - 4) = x^4 - 4x^2 + x^2 - 4 = x^4 - 3x^2 - 4$, degree 4.

Q13. First $(2x - 3)(2x + 3) = 4x^2 - 9$ (difference of squares). Then $(4x^2 - 9)(x - 1) = 4x^3 - 4x^2 - 9x + 9$.

Q14. $(x - 2)^3 = (x - 2)(x - 2)^2 = (x - 2)(x^2 - 4x + 4) = x^3 - 4x^2 + 4x - 2x^2 + 8x - 8 = x^3 - 6x^2 + 12x - 8$.

Q15. $a(x - 1)^2 + b(x - 1) + c = ax^2 + (-2a + b)x + (a - b + c)$. Equating with $x^2 + 4x + 7$: $a = 1$; $-2a + b = 4 \Rightarrow b = 6$; $a - b + c = 7 \Rightarrow 1 - 6 + c = 7 \Rightarrow c = 12$. $\therefore a = 1, b = 6, c = 12$.

Q16. Matching coefficients of x^3, x^2, x and the constant: $a = 2, b = -3, c = 0$ (there is no x term) and $d = 5$.

Q17. $(x + a)(x + b) = x^2 + (a + b)x + ab$. Equating with $x^2 + 7x + 12$: $a + b = 7$ and $ab = 12$, so $\{a, b\} = \{3, 4\}$.

Q18. $P(1) = 1 + b + c = 0$ and $P(-2) = 4 - 2b + c = 0$. Subtracting the first from the second: $3 - 3b = 0 \Rightarrow b = 1$, then $c = -2$. (Equivalently $P(x) = (x - 1)(x + 2) = x^2 + x - 2$.)

Chapter 6 Polynomial functions — 6B Polynomial division

Theory

Polynomial **long division** works just like long division of numbers. Dividing a polynomial $P(x)$ (the **dividend**) by $D(x)$ (the **divisor**) gives a **quotient** $Q(x)$ and a **remainder** $R(x)$, related by

$$P(x) = D(x)Q(x) + R(x),$$

where the remainder has lower degree than the divisor. When the divisor is linear, the remainder is a constant. If $R(x) = 0$ the division is **exact** and $D(x)$ is a **factor** of $P(x)$.

Method. Set the dividend out in descending powers of x , inserting a 0 for any missing term. Then repeat: **divide** the leading term of the current dividend by the leading term of the divisor; **multiply** the whole divisor by that result; **subtract**; and **bring down** the next term. Continue until the remainder has lower degree than the divisor. For example, dividing $x^3 + 2x^2 - 5x + 1$ by $x - 2$:

$$\begin{array}{r}
 x^2 + 4x + 3 \\
 \hline
 x - 2 \\
 x^3 - 5x + 1 \\
 \underline{x^3 - 2x^2} \\
 4x^2 - 5x \\
 \underline{4x^2 - 8x} \\
 3x + 1 \\
 \underline{3x - 6} \\
 7
 \end{array}$$

so $x^3 + 2x^2 - 5x + 1 = (x - 2)(x^2 + 4x + 3) + 7$.

Worked examples

Example 1 — scaffolded

Divide $x^3 + 2x^2 - 5x + 1$ by $x - 2$, and state the quotient and remainder.

Working	Comment
$x^3 \div x = x^2$. Then $x^2(x - 2) = x^3 - 2x^2$; subtract to get $4x^2$, bring down $-5x$: $4x^2 - 5x$.	Divide leading terms, multiply, subtract, bring down.
$4x^2 \div x = 4x$. Then $4x(x - 2) = 4x^2 - 8x$; subtract to get $3x$, bring down $+1$: $3x + 1$.	Repeat with the new leading term.
$3x \div x = 3$. Then $3(x - 2) = 3x - 6$; subtract to get 7.	The remainder 7 has lower degree than $x - 2$, so stop.
$\therefore$ quotient $x^2 + 4x + 3$, remainder 7:	Write the result in the form $P = DQ + R$.
$x^3 + 2x^2 - 5x + 1 = (x - 2)(x^2 + 4x + 3) + 7$.	

Example 2 — scaffolded

Divide $2x^3 - 3x^2 + x - 4$ by $x + 1$.

Working	Comment
$2x^3 \div x = 2x^2$; $2x^2(x + 1) = 2x^3 + 2x^2$; subtract: $-5x^2$,	Leading-term divide, multiply, subtract.

bring down $+x$.

$-5x^2 \div x = -5x$; $-5x(x+1) = -5x^2 - 5x$; subtract:
 $6x$, bring down -4 .

$6x \div x = 6$; $6(x+1) = 6x + 6$; subtract: -10 .

$\therefore$ quotient $2x^2 - 5x + 6$, remainder -10 :

$$2x^3 - 3x^2 + x - 4 = (x+1)(2x^2 - 5x + 6) - 10.$$

Watch the signs when subtracting.

Remainder -10 .

Result in the form $P = DQ + R$.

Example 3 — unscaffolded

Divide $x^3 - 1$ by $x - 1$.

Insert the missing terms: $x^3 + 0x^2 + 0x - 1$. Long division gives quotient $x^2 + x + 1$ with remainder 0:

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

$\therefore$ the division is exact, and $x - 1$ is a factor of $x^3 - 1$.

Example 4 — unscaffolded

Divide $x^4 - 3x^2 + 2x + 1$ by $x^2 - 1$.

Insert the missing terms: $x^4 + 0x^3 - 3x^2 + 2x + 1$. Dividing by the quadratic $x^2 - 1$: $x^4 \div x^2 = x^2$, giving $x^2(x^2 - 1) = x^4 - x^2$; subtract to get $-2x^2 + 2x$; then $-2x^2 \div x^2 = -2$, giving $-2(x^2 - 1) = -2x^2 + 2$; subtract to get $2x - 1$, which has lower degree than $x^2 - 1$.

$\therefore$ quotient $x^2 - 2$, remainder $2x - 1$: $x^4 - 3x^2 + 2x + 1 = (x^2 - 1)(x^2 - 2) + 2x - 1$.

Practice questions

Scaffolded

For each division, find the quotient and remainder, and write the result in the form $P(x) = D(x)Q(x) + R$.

Q1. $(x^2 + 5x + 6) \div (x + 2)$

Q2. $(x^3 + 3x^2 - 4x - 12) \div (x + 3)$

Q3. $(2x^2 - 7x + 3) \div (x - 3)$

Q4. $(x^3 - 2x^2 + 3x - 4) \div (x - 1)$

Q5. $(x^3 + x^2 - 10x + 8) \div (x - 2)$

Q6. $(3x^3 - 2x + 5) \div (x + 1)$

Unscaffolded

Divide, stating the quotient and remainder (and note when the division is exact).

Q7. $(x^2 + 7x + 10) \div (x + 2)$

Q8. $(x^3 - 6x^2 + 11x - 6) \div (x - 1)$

Q9. $(2x^3 + 3x^2 - 11x - 6) \div (x - 2)$

Q10. $(x^3 + 4x^2 + x - 6) \div (x + 3)$

Q11. $(x^4 - 1) \div (x - 1)$

Q12. $(x^3 - 27) \div (x - 3)$

Q13. $(2x^3 - 5x^2 + 8) \div (x - 2)$

Q14. $(x^3 + 2x^2 + 3x + 4) \div (x + 2)$

Q15. $(x^4 + x^3 - 2x^2 + 5) \div (x^2 + 1)$

Q16. $(x^4 - 5x^2 + 4) \div (x^2 - 4)$

Q17. $(2x^3 + x^2 - 8x - 4) \div (x^2 - 4)$

Q18. $(x^4 + 3x^3 - x + 2) \div (x^2 + x - 1)$

Answers

Q1. $Q = x + 3$, $R = 0$. **Q2.** $Q = x^2 - 4$, $R = 0$. **Q3.** $Q = 2x - 1$, $R = 0$. **Q4.** $Q = x^2 - x + 2$, $R = -2$. **Q5.** $Q = x^2 + 3x - 4$, $R = 0$. **Q6.** $Q = 3x^2 - 3x + 1$, $R = 4$. **Q7.** $Q = x + 5$, $R = 0$. **Q8.** $Q = x^2 - 5x + 6$, $R = 0$. **Q9.** $Q = 2x^2 + 7x + 3$, $R = 0$. **Q10.** $Q = x^2 + x - 2$, $R = 0$. **Q11.** $Q = x^3 + x^2 + x + 1$, $R = 0$. **Q12.** $Q = x^2 + 3x + 9$, $R = 0$. **Q13.** $Q = 2x^2 - x - 2$, $R = 4$. **Q14.** $Q = x^2 + 3$, $R = -2$. **Q15.** $Q = x^2 + x - 3$, $R = -x + 8$. **Q16.** $Q = x^2 - 1$, $R = 0$. **Q17.** $Q = 2x + 1$, $R = 0$. **Q18.** $Q = x^2 + 2x - 1$, $R = 2x + 1$.

Fully-worked solutions

Q1. $x + 2$ into $x^2 + 5x + 6$: $x^2 \div x = x \rightarrow x(x + 2) = x^2 + 2x \rightarrow$ remainder $3x + 6$; $3x \div x = 3 \rightarrow 3(x + 2) = 3x + 6 \rightarrow 0$. $\therefore x^2 + 5x + 6 = (x + 2)(x + 3)$ (exact).

Q2. $x^3 + 3x^2 - 4x - 12 \div (x + 3)$: quotient builds x^2 , then $0x$, then -4 ; each subtraction clears. $\therefore = (x + 3)(x^2 - 4)$, $R = 0$ (exact).

Q3. $2x^2 - 7x + 3 \div (x - 3)$: $2x^2 \div x = 2x \rightarrow 2x(x - 3) = 2x^2 - 6x \rightarrow -x + 3$; $-x \div x = -1 \rightarrow -1(x - 3) = -x + 3 \rightarrow 0$. $\therefore = (x - 3)(2x - 1)$ (exact).

Q4. $x^3 - 2x^2 + 3x - 4 \div (x - 1)$: quotient $x^2 - x + 2$, remainder -2 . $\therefore = (x - 1)(x^2 - x + 2) - 2$.

Q5. $x^3 + x^2 - 10x + 8 \div (x - 2)$: quotient $x^2 + 3x - 4$, remainder 0 . $\therefore = (x - 2)(x^2 + 3x - 4)$ (exact).

Q6. $3x^3 - 2x + 5 \div (x + 1)$ (insert $0x^2$): quotient $3x^2 - 3x + 1$, remainder 4 . $\therefore = (x + 1)(3x^2 - 3x + 1) + 4$.

Q7. $Q = x + 5$, $R = 0$: $x^2 + 7x + 10 = (x + 2)(x + 5)$ (exact).

Q8. $Q = x^2 - 5x + 6$, $R = 0$: $x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$ (exact).

Q9. $Q = 2x^2 + 7x + 3$, $R = 0$: $2x^3 + 3x^2 - 11x - 6 = (x - 2)(2x^2 + 7x + 3)$ (exact).

Q10. $Q = x^2 + x - 2$, $R = 0$: $x^3 + 4x^2 + x - 6 = (x + 3)(x^2 + x - 2)$ (exact).

Q11. insert $0x^3 + 0x^2 + 0x$: $Q = x^3 + x^2 + x + 1$, $R = 0$: $x^4 - 1 = (x - 1)(x^3 + x^2 + x + 1)$ (exact).

Q12. insert $0x^2 + 0x$: $Q = x^2 + 3x + 9$, $R = 0$: $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$ (exact).

Q13. insert $0x$: $Q = 2x^2 - x - 2$, $R = 4$: $2x^3 - 5x^2 + 8 = (x - 2)(2x^2 - x - 2) + 4$.

Q14. $Q = x^2 + 3$, $R = -2$: $x^3 + 2x^2 + 3x + 4 = (x + 2)(x^2 + 3) - 2$. (Check: the x^0 term $2 \cdot 3 = 6$; $6 - 2 = 4$. ✓)

Q15. dividing by the quadratic $x^2 + 1$: $Q = x^2 + x - 3$, $R = -x + 8$: $x^4 + x^3 - 2x^2 + 5 = (x^2 + 1)(x^2 + x - 3) + (-x + 8)$.

Q16. $Q = x^2 - 1$, $R = 0$: $x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1)$ (exact).

Q17. $Q = 2x + 1$, $R = 0$: $2x^3 + x^2 - 8x - 4 = (x^2 - 4)(2x + 1)$ (exact).

Q18. dividing by $x^2 + x - 1$: $Q = x^2 + 2x - 1$, $R = 2x + 1$: $x^4 + 3x^3 - x + 2 = (x^2 + x - 1)(x^2 + 2x - 1) + (2x + 1)$.

Chapter 6 Polynomial functions — 6C The remainder and factor theorems

Theory

When a polynomial $P(x)$ is divided by a linear expression $(x - a)$, we can write

$$P(x) = (x - a)Q(x) + R,$$

where $Q(x)$ is the **quotient** and the constant R is the **remainder**.

The remainder theorem. When $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$. (More generally, when $P(x)$ is divided by $(bx - c)$ the remainder is $P\left(\frac{c}{b}\right)$.)

The factor theorem. $(x - a)$ is a factor of $P(x)$ **if and only if** $P(a) = 0$. This follows from the remainder theorem: if $P(a) = 0$ the remainder is zero, so the division is exact.

Factorising a polynomial. To factorise a cubic (or higher) polynomial $P(x)$:

1. Find one linear factor by testing values: try $x = a$ where a is a factor of the constant term (for a non-monic polynomial, a factor of the constant divided by a factor of the leading coefficient — the **rational-root** idea). When $P(a) = 0$, $(x - a)$ is a factor.
2. Divide $P(x)$ by $(x - a)$ (long division or inspection) to obtain a quotient one degree lower.
3. Factorise the quotient further if possible.

Sum and difference of two cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2), a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Grouping. Some polynomials factorise by grouping pairs of terms with a common factor, e.g. $x^3 + 2x^2 - 9x - 18 = x^2(x + 2) - 9(x + 2) = (x + 2)(x^2 - 9)$.

Worked examples

Example 1 — scaffolded

Find the remainder when $P(x) = 2x^3 - 3x^2 + x - 5$ is divided by $(x - 2)$.

Working	Comment
By the remainder theorem, the remainder is $P(2)$.	Dividing by $(x - a)$ gives remainder $P(a)$; here $a = 2$.
$P(2) = 2(2)^3 - 3(2)^2 + 2 - 5 = 16 - 12 + 2 - 5$.	Substitute $x = 2$.
$\therefore$ the remainder is 1.	Evaluate.

Example 2 — scaffolded

Factorise $x^3 - 4x^2 + x + 6$ fully.

Working	Comment
Test factors of the constant term 6: $P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = -1 - 4 - 1 + 6 = 0$.	Since $P(-1) = 0$, by the factor theorem $(x + 1)$ is a factor.
Divide: $x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$.	Long division (or inspection) gives the quadratic quotient.
$x^2 - 5x + 6 = (x - 2)(x - 3)$.	Factorise the quotient.

$$\therefore x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3).$$

Write the full factorisation.

Example 3 — unscaffolded

Factorise $2x^3 + 3x^2 - 11x - 6$ fully.

Possible rational roots are $\pm \frac{1, 2, 3, 6}{1, 2}$. Testing $x=2$: $2(2)^3 + 3(2)^2 - 11(2) - 6 = 16 + 12 - 22 - 6 = 0$, so $(x-2)$ is a factor.

Dividing, $2x^3 + 3x^2 - 11x - 6 = (x-2)(2x^2 + 7x + 3)$, and $2x^2 + 7x + 3 = (2x+1)(x+3)$.

$$\therefore 2x^3 + 3x^2 - 11x - 6 = (x-2)(2x+1)(x+3).$$

Example 4 — unscaffolded

Factorise $8x^3 - 27$.

This is a difference of two cubes: $8x^3 - 27 = (2x)^3 - 3^3$. Using $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ with $a=2x$ and $b=3$:

$$8x^3 - 27 = (2x-3)((2x)^2 + (2x)(3) + 3^2) = (2x-3)(4x^2 + 6x + 9).$$

$$\therefore 8x^3 - 27 = (2x-3)(4x^2 + 6x + 9) \text{ (the quadratic factor does not factorise further).}$$

Practice questions

Scaffolded

Q1. Use the remainder theorem to find the remainder when $x^3 + x^2 - 4x + 2$ is divided by $(x-1)$.

Q2. Use the remainder theorem to find the remainder when $2x^3 - x^2 + 3x - 1$ is divided by $(x+2)$.

Q3. Show that $(x-2)$ is a factor of $x^3 - 3x^2 + 4$, and hence factorise it fully.

Q4. Factorise $x^3 + 6x^2 + 11x + 6$ fully. (Hint: test $x = -1$.)

Q5. Factorise $x^3 - 7x + 6$ fully. (Hint: test $x = 1$.)

Q6. Factorise $x^3 + 1$ using the sum of two cubes.

Unscaffolded

Q7. Find the remainder when $x^3 - 2x^2 + 3x - 4$ is divided by $(x-2)$.

Q8. Find the remainder when $3x^3 + 2x^2 - x + 5$ is divided by $(x+1)$.

Q9. Find the value of a for which $(x-2)$ is a factor of $x^3 - 3x^2 + ax - 2$.

Q10. Factorise $x^3 - x^2 - 10x - 8$ fully.

Q11. Factorise $x^3 + 5x^2 - 4x - 20$ fully by grouping.

Q12. Factorise $2x^3 - 5x^2 - 4x + 3$ fully.

Q13. Factorise $x^3 - 8$ using the difference of two cubes.

Q14. Factorise $27x^3 + 8$ using the sum of two cubes.

Q15. Factorise $x^4 - 5x^2 + 4$ fully.

Q16. Factorise $x^3 - 3x^2 - 4x + 12$ fully.

Q17. Factorise $x^4 + x^3 - 7x^2 - x + 6$ fully.

Q18. Given that $(x + 1)$ and $(x - 2)$ are both factors of $x^3 + ax^2 + bx + 2$, find a and b , and factorise the polynomial fully.

Answers

Q1. 0. **Q2.** -27. **Q3.** $(x-2)^2(x+1)$. **Q4.** $(x+1)(x+2)(x+3)$. **Q5.** $(x-1)(x-2)(x+3)$. **Q6.** $(x+1)(x^2-x+1)$. **Q7.** 2. **Q8.** 5. **Q9.** $a=3$. **Q10.** $(x+1)(x+2)(x-4)$. **Q11.** $(x+5)(x-2)(x+2)$. **Q12.** $(x-3)(2x-1)(x+1)$. **Q13.** $(x-2)(x^2+2x+4)$. **Q14.** $(3x+2)(9x^2-6x+4)$. **Q15.** $(x-1)(x+1)(x-2)(x+2)$. **Q16.** $(x-2)(x+2)(x-3)$. **Q17.** $(x-1)(x+1)(x-2)(x+3)$. **Q18.** $a=-2$, $b=-1$; $(x+1)(x-2)(x-1)$.

Fully-worked solutions

Q1. Remainder = $P(1) = 1^3 + 1^2 - 4(1) + 2 = 1 + 1 - 4 + 2 = 0$. (So $(x-1)$ is in fact a factor.)

Q2. Remainder = $P(-2) = 2(-2)^3 - (-2)^2 + 3(-2) - 1 = -16 - 4 - 6 - 1 = -27$.

Q3. $P(2) = 2^3 - 3(2)^2 + 4 = 8 - 12 + 4 = 0$, so by the factor theorem $(x-2)$ is a factor. Dividing, $x^3 - 3x^2 + 4 = (x-2)(x^2 - x - 2) = (x-2)(x-2)(x+1)$. $\therefore x^3 - 3x^2 + 4 = (x-2)^2(x+1)$.

Q4. $P(-1) = -1 + 6 - 11 + 6 = 0$, so $(x+1)$ is a factor. Dividing, $x^3 + 6x^2 + 11x + 6 = (x+1)(x^2 + 5x + 6) = (x+1)(x+2)(x+3)$.

Q5. $P(1) = 1 - 7 + 6 = 0$, so $(x-1)$ is a factor. Dividing, $x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = (x-1)(x-2)(x+3)$.

Q6. Sum of two cubes with $a=x$, $b=1$: $x^3 + 1 = (x+1)(x^2 - x + 1)$.

Q7. Remainder = $P(2) = 2^3 - 2(2)^2 + 3(2) - 4 = 8 - 8 + 6 - 4 = 2$.

Q8. Remainder = $P(-1) = 3(-1)^3 + 2(-1)^2 - (-1) + 5 = -3 + 2 + 1 + 5 = 5$.

Q9. $(x-2)$ is a factor $\Rightarrow P(2) = 0$: $2^3 - 3(2)^2 + 2a - 2 = 8 - 12 + 2a - 2 = 2a - 6 = 0$, so $a = 3$.

Q10. $P(-1) = -1 - 1 + 10 - 8 = 0$, so $(x+1)$ is a factor. Dividing, $x^3 - x^2 - 10x - 8 = (x+1)(x^2 - 2x - 8) = (x+1)(x+2)(x-4)$.

Q11. Group: $x^3 + 5x^2 - 4x - 20 = x^2(x+5) - 4(x+5) = (x+5)(x^2 - 4) = (x+5)(x-2)(x+2)$.

Q12. Testing $x=3$: $2(27) - 5(9) - 12 + 3 = 54 - 45 - 12 + 3 = 0$, so $(x-3)$ is a factor. Dividing, $2x^3 - 5x^2 - 4x + 3 = (x-3)(2x^2 + x - 1) = (x-3)(2x-1)(x+1)$.

Q13. Difference of two cubes with $a=x$, $b=2$: $x^3 - 8 = (x-2)(x^2 + 2x + 4)$.

Q14. Sum of two cubes with $a=3x$, $b=2$: $27x^3 + 8 = (3x+2)(9x^2 - 6x + 4)$.

Q15. Treat as a quadratic in x^2 : $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x-1)(x+1)(x-2)(x+2)$.

Q16. Group: $x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x-3)(x^2 - 4) = (x-3)(x-2)(x+2)$.

Q17. $P(1) = 1 + 1 - 7 - 1 + 6 = 0$ and $P(-1) = 1 - 1 - 7 + 1 + 6 = 0$, so $(x-1)$ and $(x+1)$ are both factors, i.e. $(x^2 - 1)$ is a factor. Dividing, $x^4 + x^3 - 7x^2 - x + 6 = (x^2 - 1)(x^2 + x - 6) = (x-1)(x+1)(x-2)(x+3)$.

Q18. Since $(x+1)$ and $(x-2)$ are factors, $P(-1) = 0$ and $P(2) = 0$:

$$P(-1) = -1 + a - b + 2 = 0 \Rightarrow a - b = -1,$$

$$P(2) = 8 + 4a + 2b + 2 = 0 \Rightarrow 2a + b = -5.$$

Adding, $3a = -6$, so $a = -2$ and $b = a + 1 = -1$. Then $P(x) = x^3 - 2x^2 - x + 2 = (x + 1)(x - 2)(x - 1)$.

Chapter 6 Polynomial functions — 6D Cubic equations

Theory

To solve a polynomial equation, write it in the form $\dots = 0$ and use the **null factor law**: if a product of factors equals zero, then at least one of the factors is zero.

Always look for a common factor first. For example,

$$x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2).$$

The factor theorem. For a polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$. To solve a cubic with no common factor, test values of a that are factors of the constant term (divided by the leading coefficient) until $P(a) = 0$. This gives one linear factor $(x - a)$; divide it out to leave a quadratic, then factorise or use the quadratic formula.

Four-term cubics can often be factorised by **grouping** — for example

$$x^3 - 3x^2 - x + 3 = x^2(x - 3) - (x - 3) = (x - 3)(x^2 - 1).$$

Repeated factors give **repeated roots**: $x(x - 2)^2 = 0$ has solutions $x = 0$ and $x = 2$ (with $x = 2$ a repeated root). The quadratic factor may need the quadratic formula, giving **surd** solutions.

Worked examples

Example 1 — scaffolded

Solve $x^3 - 4x = 0$.

Working

$$x^3 - 4x = x(x^2 - 4)$$

$$= x(x - 2)(x + 2)$$

$$x(x - 2)(x + 2) = 0, \text{ so } x = 0, x - 2 = 0 \text{ or } x + 2 = 0.$$

$$\therefore x = 0, 2 \text{ or } -2.$$

Comment

Take out the common factor x first.

$x^2 - 4$ is a difference of two squares.

Apply the null factor law to the three factors.

Three solutions.

Example 2 — scaffolded

Solve $x^3 + 2x^2 - 13x + 10 = 0$.

Working

$$\text{Let } P(x) = x^3 + 2x^2 - 13x + 10.$$

$$P(1) = 1 + 2 - 13 + 10 = 0.$$

$$x^3 + 2x^2 - 13x + 10 = (x - 1)(x^2 + 3x - 10)$$

$$= (x - 1)(x - 2)(x + 5)$$

$$\therefore x = 1, 2 \text{ or } -5.$$

Comment

Test factors of the constant term 10: here $x = 1$ works, so $(x - 1)$ is a factor.

Divide $P(x)$ by $(x - 1)$ to find the quadratic factor.

Factorise the quadratic $x^2 + 3x - 10$.

Null factor law.

Example 3 — unscaffolded

Solve $2x^3 + 3x^2 - 8x + 3 = 0$.

Let $P(x) = 2x^3 + 3x^2 - 8x + 3$. $P(1) = 2 + 3 - 8 + 3 = 0$, so $(x - 1)$ is a factor.

Dividing gives $2x^3 + 3x^2 - 8x + 3 = (x - 1)(2x^2 + 5x - 3) = (x - 1)(2x - 1)(x + 3)$.

$$\therefore x = 1, \frac{1}{2} \text{ or } -3.$$

Example 4 — unscaffolded

Solve $x^3 - 3x^2 - x + 3 = 0$ by grouping.

$$x^3 - 3x^2 - x + 3 = x^2(x - 3) - (x - 3) = (x - 3)(x^2 - 1) = (x - 3)(x - 1)(x + 1).$$

$\therefore x = 3, 1$ or -1 .

Practice questions

Scaffolded

Q1. Solve $x^3 - 9x = 0$. (a) Take out the common factor. (b) Factorise fully. (c) State all solutions.

Q2. Solve $x^3 + x^2 - 6x = 0$. (a) Take out the common factor. (b) Factorise the quadratic. (c) State all solutions.

Q3. Solve $x^3 - 6x^2 + 11x - 6 = 0$. (a) Show that $x = 1$ is a solution. (b) Hence write a linear factor and find the quadratic factor. (c) Solve.

Q4. Solve $x^3 + 2x^2 - x - 2 = 0$ by grouping. (a) Group in pairs and factorise each pair. (b) Complete the factorisation. (c) State all solutions.

Q5. Solve $x^3 - 4x^2 + 4x = 0$. (a) Take out the common factor. (b) Factorise the quadratic (a perfect square). (c) State the solutions, noting the repeated root.

Q6. Solve $2x^3 - 5x^2 - 4x + 3 = 0$. (a) Show that $x = 3$ is a solution. (b) Find the quadratic factor and factorise it. (c) Solve.

Unscaffolded

Solve each of the following.

Q7. $x^3 = 16x$

Q8. $x^3 - 3x^2 = 0$

Q9. $x^3 - x^2 - 4x + 4 = 0$

Q10. $x^3 - 7x - 6 = 0$

Q11. $x^3 - 2x^2 - 9x + 18 = 0$

Q12. $2x^3 + x^2 - 13x + 6 = 0$

Q13. $x^3 + 6x^2 + 12x + 8 = 0$

Q14. $x^3 - 5x^2 + 2x + 8 = 0$

Q15. $x^3 + 2x^2 - 4x - 8 = 0$

Q16. $x^3 - x^2 - 5x - 3 = 0$

Q17. $x^3 - 2x^2 - 2x + 4 = 0$

Q18. $x^4 - 5x^2 + 4 = 0$

Answers

Q1. $x=0, 3, -3$. **Q2.** $x=0, 2, -3$. **Q3.** $x=1, 2, 3$. **Q4.** $x=1, -1, -2$. **Q5.** $x=0, 2$ ($x=2$ repeated). **Q6.** $x=3, 1/2, -1$. **Q7.** $x=0, 4, -4$. **Q8.** $x=0$ (repeated), 3 . **Q9.** $x=1, 2, -2$. **Q10.** $x=-1, 3, -2$. **Q11.** $x=2, 3, -3$. **Q12.** $x=2, 1/2, -3$. **Q13.** $x=-2$ (repeated three times). **Q14.** $x=2, 4, -1$. **Q15.** $x=2, -2$ ($x=-2$ repeated). **Q16.** $x=3, -1$ ($x=-1$ repeated). **Q17.** $x=2, \sqrt{2}, -\sqrt{2}$. **Q18.** $x=1, -1, 2, -2$.

Fully-worked solutions

Q1. $x^3 - 9x = x(x^2 - 9) = x(x-3)(x+3) = 0$. $\therefore x=0, 3, -3$.

Q2. $x^3 + x^2 - 6x = x(x^2 + x - 6) = x(x+3)(x-2) = 0$. $\therefore x=0, -3, 2$.

Q3. Let $P(x) = x^3 - 6x^2 + 11x - 6$. (a) $P(1) = 1 - 6 + 11 - 6 = 0$, so $x=1$ is a solution and $(x-1)$ is a factor. (b) Dividing, $P(x) = (x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3)$. (c) $\therefore x=1, 2, 3$.

Q4. $x^3 + 2x^2 - x - 2 = x^2(x+2) - (x+2) = (x+2)(x^2 - 1) = (x+2)(x-1)(x+1) = 0$. $\therefore x=1, -1, -2$.

Q5. $x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2 = 0$. $\therefore x=0$ or $x=2$ (a repeated root).

Q6. Let $P(x) = 2x^3 - 5x^2 - 4x + 3$. $P(3) = 54 - 45 - 12 + 3 = 0$, so $(x-3)$ is a factor. Dividing, $P(x) = (x-3)(2x^2 + x - 1) = (x-3)(2x-1)(x+1)$. $\therefore x=3, 1/2, -1$.

Q7. $x^3 = 16x \Rightarrow x^3 - 16x = 0 \Rightarrow x(x-4)(x+4) = 0$. $\therefore x=0, 4, -4$. (Do **not** divide both sides by x — that would lose the solution $x=0$.)

Q8. $x^3 - 3x^2 = x^2(x-3) = 0$. $\therefore x=0$ (repeated) or $x=3$.

Q9. By grouping: $x^2(x-1) - 4(x-1) = (x-1)(x^2 - 4) = (x-1)(x-2)(x+2) = 0$. $\therefore x=1, 2, -2$.

Q10. Let $P(x) = x^3 - 7x - 6$. $P(-1) = -1 + 7 - 6 = 0$, so $(x+1)$ is a factor. $P(x) = (x+1)(x^2 - x - 6) = (x+1)(x-3)(x+2)$. $\therefore x=-1, 3, -2$.

Q11. By grouping: $x^2(x-2) - 9(x-2) = (x-2)(x^2 - 9) = (x-2)(x-3)(x+3) = 0$. $\therefore x=2, 3, -3$.

Q12. Let $P(x) = 2x^3 + x^2 - 13x + 6$. $P(2) = 16 + 4 - 26 + 6 = 0$, so $(x-2)$ is a factor. $P(x) = (x-2)(2x^2 + 5x - 3) = (x-2)(2x-1)(x+3)$. $\therefore x=2, 1/2, -3$.

Q13. $x^3 + 6x^2 + 12x + 8 = (x+2)^3 = 0$. $\therefore x=-2$ (a triple root).

Q14. Let $P(x) = x^3 - 5x^2 + 2x + 8$. $P(2) = 8 - 20 + 4 + 8 = 0$, so $(x-2)$ is a factor. $P(x) = (x-2)(x^2 - 3x - 4) = (x-2)(x-4)(x+1)$. $\therefore x=2, 4, -1$.

Q15. By grouping: $x^2(x+2) - 4(x+2) = (x+2)(x^2 - 4) = (x+2)(x+2)(x-2) = (x+2)^2(x-2) = 0$. $\therefore x=-2$ (repeated) or $x=2$.

Q16. Let $P(x) = x^3 - x^2 - 5x - 3$. $P(3) = 27 - 9 - 15 - 3 = 0$, so $(x-3)$ is a factor. $P(x) = (x-3)(x^2 + 2x + 1) = (x-3)(x+1)^2$. $\therefore x=3$ or $x=-1$ (repeated).

Q17. Let $P(x) = x^3 - 2x^2 - 2x + 4$. $P(2) = 8 - 8 - 4 + 4 = 0$, so $(x-2)$ is a factor. $P(x) = (x-2)(x^2 - 2)$. Then $x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$. $\therefore x=2, \sqrt{2}, -\sqrt{2}$.

Q18. This quartic is quadratic in x^2 : $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x-1)(x+1)(x-2)(x+2) = 0$. $\therefore x=1, -1, 2, -2$.

Chapter 6 Polynomial functions — 6E Transformations of $y = x^3$

Theory

A **cubic function** has an x^3 term as its highest power. This section studies cubics written in the **turning-point-like form** $y = a(x - h)^3 + k$.

The basic cubic. The graph of $y = x^3$ passes through the origin, is increasing everywhere, and has a **stationary point of inflection** at $(0, 0)$ — the curve momentarily flattens there (gradient zero) but does **not** turn around. It is an odd function, symmetric about the origin.

The form $y = a(x - h)^3 + k$.

- The **point of inflection** is (h, k) .
- a controls the **dilation** from the x -axis: $|a| > 1$ makes the curve steeper, $0 < |a| < 1$ flatter. If $a < 0$ the curve is **reflected in the x -axis** (so it is decreasing).
- A cubic of this form has **no turning points** — it is **monotonic** (strictly increasing if $a > 0$, strictly decreasing if $a < 0$).

Transformations. $y = a(x - h)^3 + k$ is obtained from $y = x^3$ by a dilation of factor a from the x -axis (with a reflection in the x -axis when $a < 0$), then a translation of h units horizontally and k units vertically.

Intercepts.

- **y -intercept:** substitute $x = 0$: $y = a(-h)^3 + k = -ah^3 + k$.
- **x -intercept:** solve $a(x - h)^3 + k = 0$, giving $(x - h)^3 = -\frac{k}{a}$, so there is **exactly one** x -intercept, $x = h + \sqrt[3]{-\frac{k}{a}}$ (a cube root is defined for every real number, positive or negative).

To sketch: mark the point of inflection (h, k) , decide the direction from the sign of a , find the y -intercept and the single x -intercept, then draw a smooth cubic curve through them.

Worked examples

Example 1 — scaffolded

Sketch $y = (x - 1)^3 + 2$, labelling the point of inflection and the axis intercepts.

Working

Comment

$a = 1 > 0$: the curve is increasing, with a stationary point of inflection.

Read the shape from the sign of a .

Comparing with $y = a(x - h)^3 + k$: $h = 1$, $k = 2$. Point of inflection $(1, 2)$.

(h, k) is read straight off.

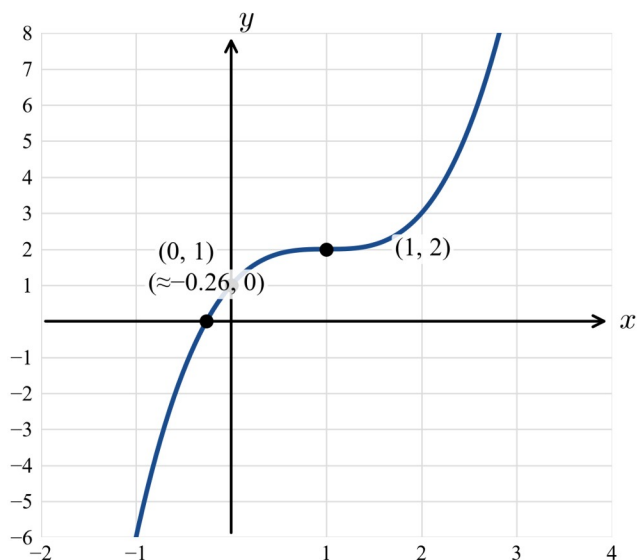
y -intercept: $x = 0 \Rightarrow y = (0 - 1)^3 + 2 = -1 + 2 = 1$, giving $(0, 1)$.

Substitute $x = 0$.

x -intercept:

Take the cube root of both sides.

$(x - 1)^3 + 2 = 0 \Rightarrow (x - 1)^3 = -2 \Rightarrow x - 1 = \sqrt[3]{-2} = -\sqrt[3]{2}$,
so $x = 1 - \sqrt[3]{2} (\approx -0.26)$.



Example 2 — scaffolded

Sketch $y = -2(x+1)^3$.

Working

$a = -2 < 0$: the curve is decreasing (reflected in the x -axis) and steeper than $y = x^3$.

$h = -1, k = 0$: point of inflection $(-1, 0)$.

y -intercept: $x = 0 \Rightarrow y = -2(1)^3 = -2$, giving $(0, -2)$.

x -intercept: $-2(x+1)^3 = 0 \Rightarrow x = -1$, giving $(-1, 0)$
— the same point as the inflection.

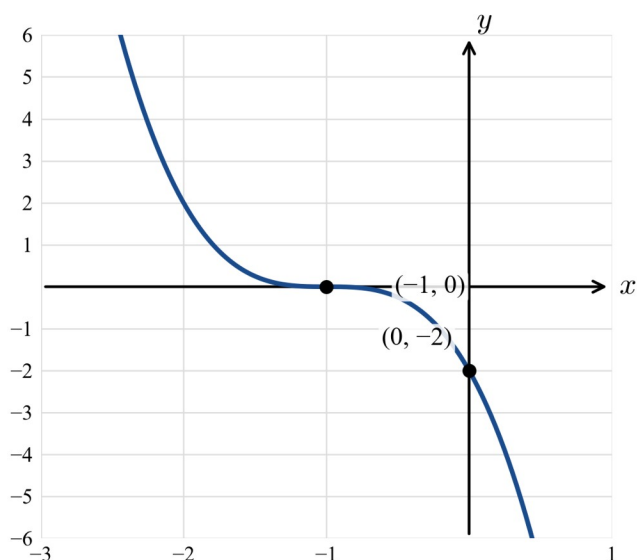
Comment

Sign and size of a .

This inflection sits **on** the x -axis.

Substitute $x = 0$.

The single root coincides with the inflection when $k = 0$.



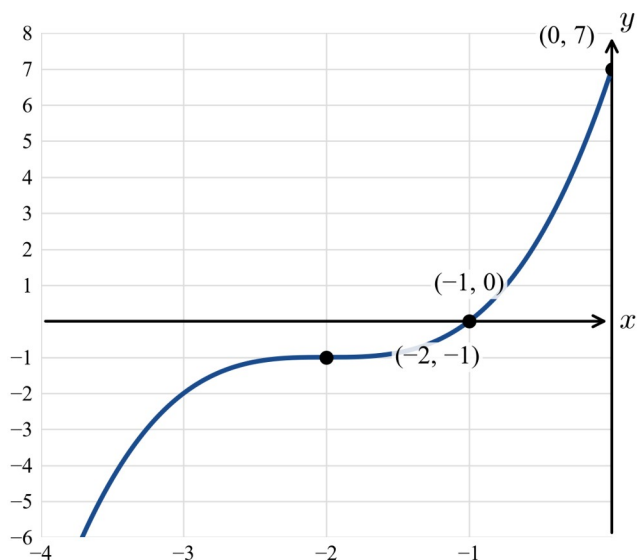
Example 3 — unscaffolded

Sketch $y = (x+2)^3 - 1$.

$a = 1 > 0$: increasing. Point of inflection $(-2, -1)$.

y -intercept: $x = 0 \Rightarrow y = (2)^3 - 1 = 8 - 1 = 7$, giving $(0, 7)$.

x -intercept: $(x+2)^3 - 1 = 0 \Rightarrow (x+2)^3 = 1 \Rightarrow x+2 = 1$, so $x = -1$, giving $(-1, 0)$.



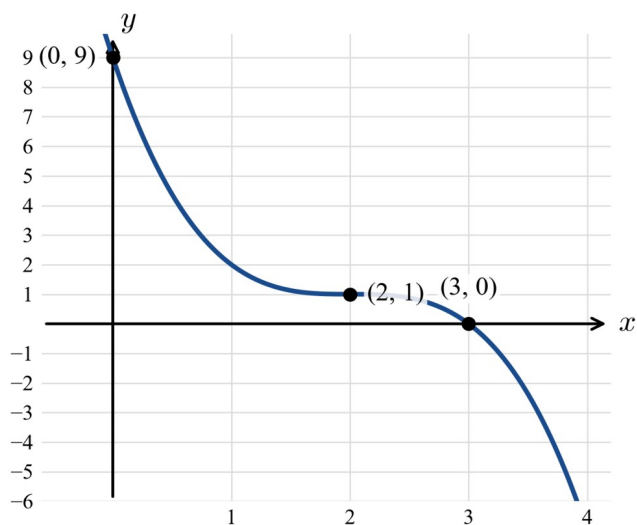
Example 4 — unscaffolded

Sketch $y = -(x - 2)^3 + 1$.

$a = -1 < 0$: decreasing. Point of inflection $(2, 1)$.

y -intercept: $x = 0 \Rightarrow y = -(-2)^3 + 1 = 8 + 1 = 9$, giving $(0, 9)$.

x -intercept: $-(x - 2)^3 + 1 = 0 \Rightarrow (x - 2)^3 = 1 \Rightarrow x - 2 = 1$, so $x = 3$, giving $(3, 0)$.



Practice questions

Scaffolded

Q1. For $y = (x - 2)^3$: (a) state the point of inflection; (b) find the y -intercept; (c) find the x -intercept; (d) sketch the graph.

Q2. For $y = x^3 + 1$: (a) state the point of inflection and the direction; (b) find the axis intercepts; (c) sketch the graph.

Q3. For $y = (x + 1)^3 - 8$: (a) state the point of inflection; (b) find the axis intercepts; (c) sketch the graph.

Q4. For $y = 2x^3$: (a) state the point of inflection; (b) describe how the graph differs from $y = x^3$; (c) sketch the graph.

Q5. For $y = -(x - 1)^3$: (a) state the direction and point of inflection; (b) find the axis intercepts; (c) sketch the graph.

Q6. For $y = (x - 1)^3 + 1$: (a) state the point of inflection; (b) find the axis intercepts; (c) sketch the graph.

Un scaffolded

For each of the following: sketch the graph, labelling the point of inflection and the axis intercepts, **and** describe the transformation(s) applied to $y = x^3$.

Q7. $y = x^3 - 8$

Q8. $y = (x - 3)^3$

Q9. $y = (x + 2)^3$

Q10. $y = -x^3$

Q11. $y = 3x^3$

Q12. $y = 1/2 x^3 - 4$

Q13. $y = (x - 1)^3 + 8$

Q14. $y = -(x + 1)^3 + 1$

Q15. $y = -2(x - 1)^3 - 2$

Q16. $y = -1/2(x + 2)^3 + 4$

Q17. $y = 2(x - 1)^3$

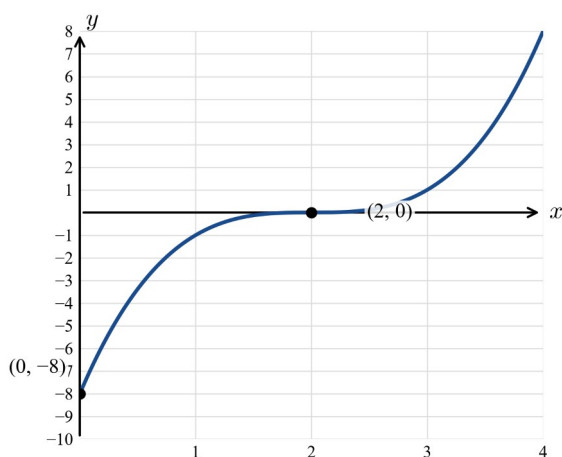
Q18. $y = (x - 2)^3 + 3$

Answers

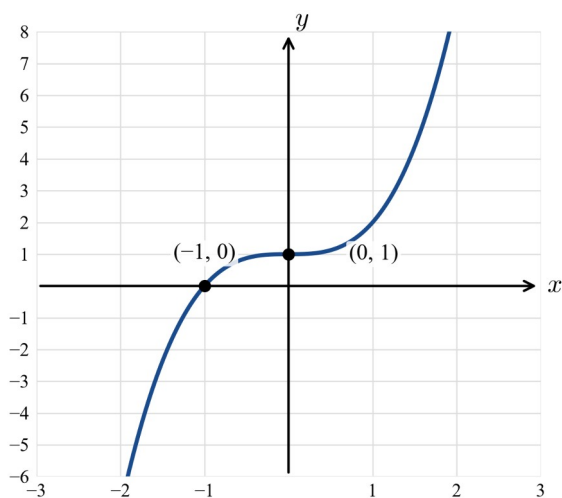
Q1. Inflection $(2, 0)$; x -int $(2, 0)$; y -int $(0, -8)$. **Q2.** Increasing; inflection $(0, 1)$; x -int $(-1, 0)$; y -int $(0, 1)$. **Q3.** Inflection $(-1, -8)$; x -int $(1, 0)$; y -int $(0, -7)$. **Q4.** Inflection $(0, 0)$; steeper than $y = x^3$ (dilation factor 2). **Q5.** Decreasing; inflection $(1, 0)$; x -int $(1, 0)$; y -int $(0, 1)$. **Q6.** Inflection $(1, 1)$; x -int $(0, 0)$; y -int $(0, 0)$. **Q7.** Inflection $(0, -8)$; x -int $(2, 0)$; y -int $(0, -8)$; translation 8 down. **Q8.** Inflection $(3, 0)$; x -int $(3, 0)$; y -int $(0, -27)$; translation 3 right. **Q9.** Inflection $(-2, 0)$; x -int $(-2, 0)$; y -int $(0, 8)$; translation 2 left. **Q10.** Inflection $(0, 0)$; through the origin; reflection in the x -axis. **Q11.** Inflection $(0, 0)$; through the origin; dilation factor 3 from the x -axis (steeper). **Q12.** Inflection $(0, -4)$; x -int $(2, 0)$; y -int $(0, -4)$; dilation factor $1/2$ (flatter), then 4 down. **Q13.** Inflection $(1, 8)$; x -int $(-1, 0)$; y -int $(0, 7)$; 1 right and 8 up. **Q14.** Inflection $(-1, 1)$; x -int $(0, 0)$; y -int $(0, 0)$; reflection in the x -axis, then 1 left and 1 up. **Q15.** Inflection $(1, -2)$; x -int $(0, 0)$; y -int $(0, 0)$; reflection, dilation factor 2, then 1 right and 2 down. **Q16.** Inflection $(-2, 4)$; x -int $(0, 0)$; y -int $(0, 0)$; reflection, dilation factor $1/2$, then 2 left and 4 up. **Q17.** Inflection $(1, 0)$; x -int $(1, 0)$; y -int $(0, -2)$; dilation factor 2, then 1 right. **Q18.** Inflection $(2, 3)$; x -int $(2 - \sqrt[3]{3}, 0) \approx (0.56, 0)$; y -int $(0, -5)$; 2 right and 3 up.

Fully-worked solutions

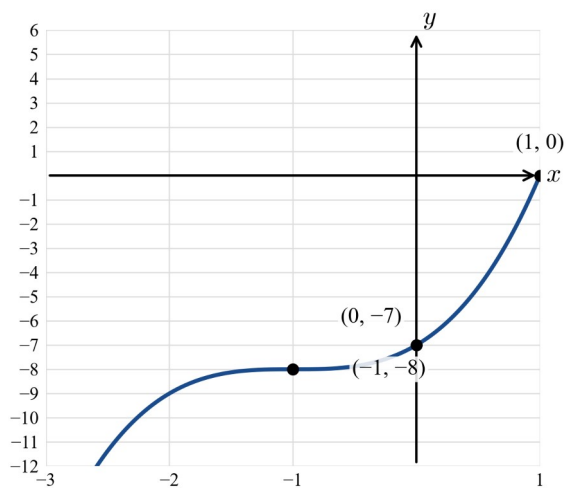
Q1. $y = (x - 2)^3$. Inflection $(2, 0)$. y -int $x = 0 \Rightarrow (-2)^3 = -8$, giving $(0, -8)$. x -int $(x - 2)^3 = 0 \Rightarrow x = 2$, giving $(2, 0)$.



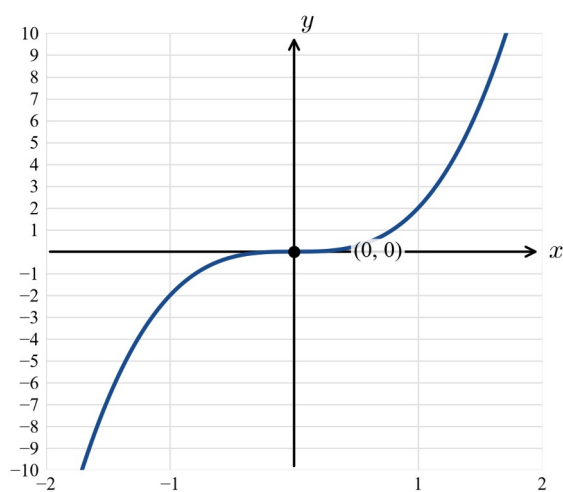
Q2. $y = x^3 + 1$. $a = 1 > 0$: increasing; inflection $(0, 1)$. y -int $(0, 1)$. x -int $x^3 = -1 \Rightarrow x = -1$, giving $(-1, 0)$.



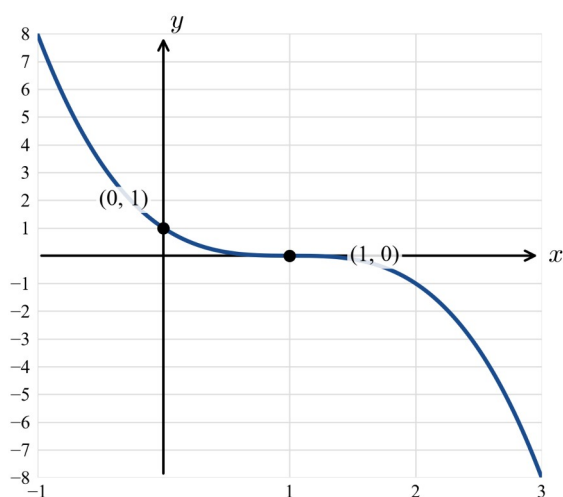
Q3. $y = (x + 1)^3 - 8$. Inflection $(-1, -8)$. y -int $x = 0 \Rightarrow 1 - 8 = -7$, giving $(0, -7)$. x -int $(x + 1)^3 = 8 \Rightarrow x + 1 = 2 \Rightarrow x = 1$, giving $(1, 0)$.



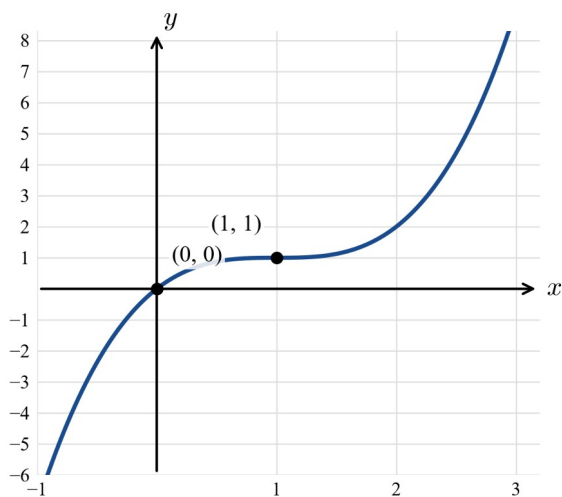
Q4. $y = 2x^3$. Inflection $(0, 0)$. The factor 2 is a dilation of factor 2 from the x -axis, so the curve is **twice as steep** as $y = x^3$; it still passes through the origin and is increasing.



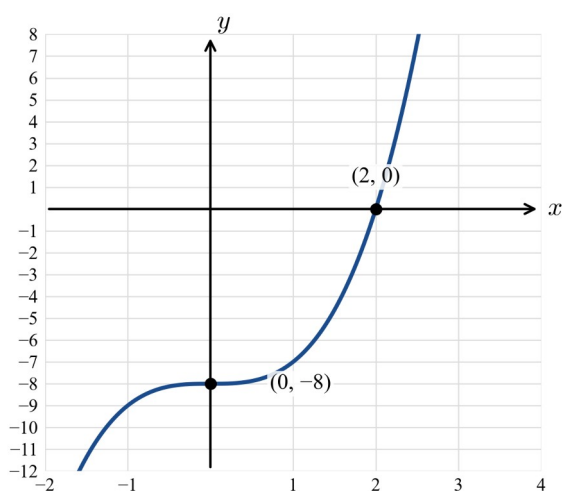
Q5. $y = -(x - 1)^3$. $a = -1 < 0$: decreasing; inflection $(1, 0)$. y -int $x = 0 \Rightarrow -(-1)^3 = 1$, giving $(0, 1)$. x -int $x = 1$, giving $(1, 0)$.



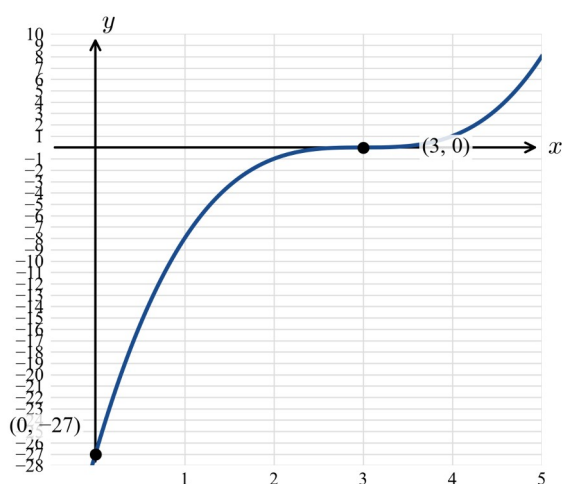
Q6. $y = (x - 1)^3 + 1$. Inflection $(1, 1)$. y -int $x = 0 \Rightarrow (-1)^3 + 1 = 0$, giving $(0, 0)$. x -int $(x - 1)^3 = -1 \Rightarrow x - 1 = -1 \Rightarrow x = 0$, giving $(0, 0)$.



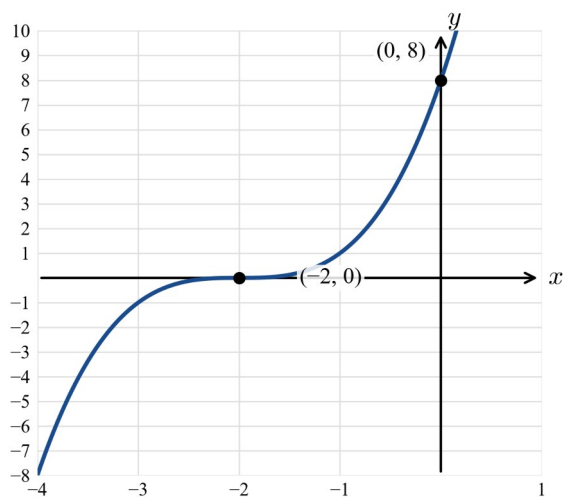
Q7. $y = x^3 - 8$. Inflection $(0, -8)$; y-int $(0, -8)$; x-int $x^3 = 8 \Rightarrow x = 2$, giving $(2, 0)$. *Transformation:* translation 8 units down.



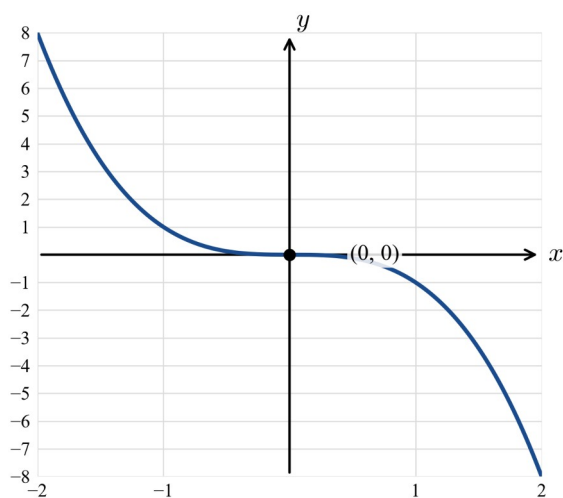
Q8. $y = (x - 3)^3$. Inflection $(3, 0)$; x-int $(3, 0)$; y-int $x = 0 \Rightarrow (-3)^3 = -27$, giving $(0, -27)$. *Transformation:* translation 3 units right.



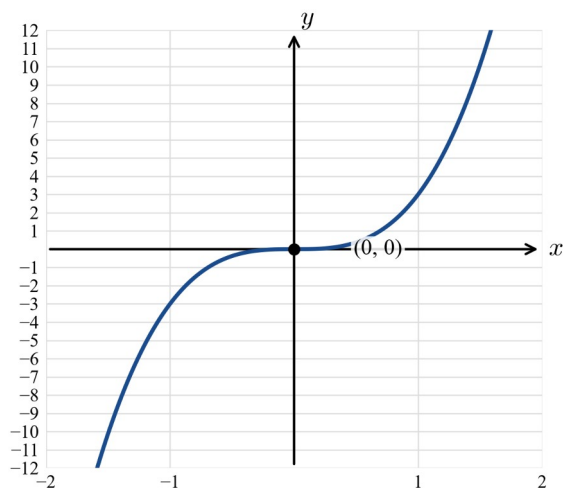
Q9. $y = (x + 2)^3$. Inflection $(-2, 0)$; x-int $(-2, 0)$; y-int $x = 0 \Rightarrow (2)^3 = 8$, giving $(0, 8)$. *Transformation:* translation 2 units left.



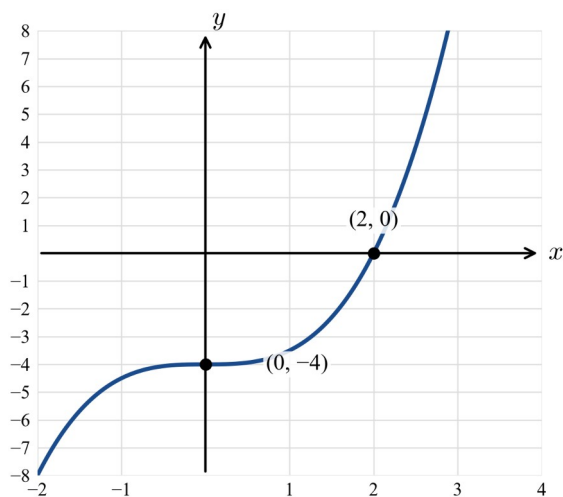
Q10. $y = -x^3$. Inflection $(0, 0)$; passes through the origin; decreasing. *Transformation:* reflection in the x -axis.



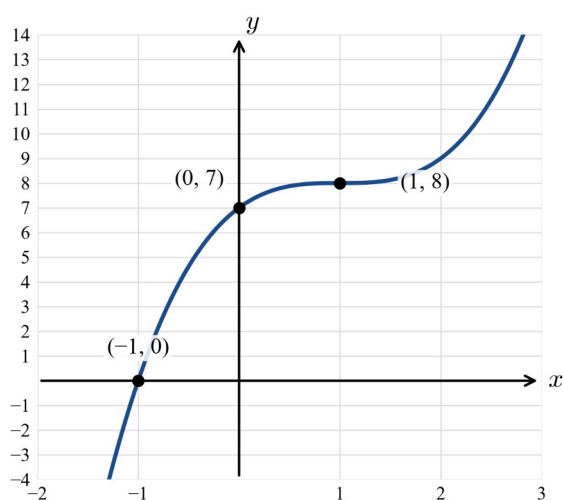
Q11. $y = 3x^3$. Inflection $(0, 0)$; through the origin. *Transformation:* dilation of factor 3 from the x -axis (three times as steep).



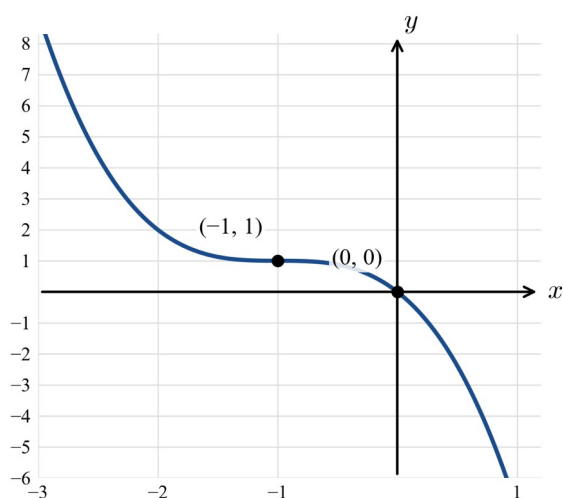
Q12. $y = \frac{1}{2}x^3 - 4$. Inflection $(0, -4)$; y -int $(0, -4)$; x -int $\frac{1}{2}x^3 = 4 \Rightarrow x^3 = 8 \Rightarrow x = 2$, giving $(2, 0)$. *Transformation:* dilation of factor $\frac{1}{2}$ from the x -axis (flatter), then 4 units down.



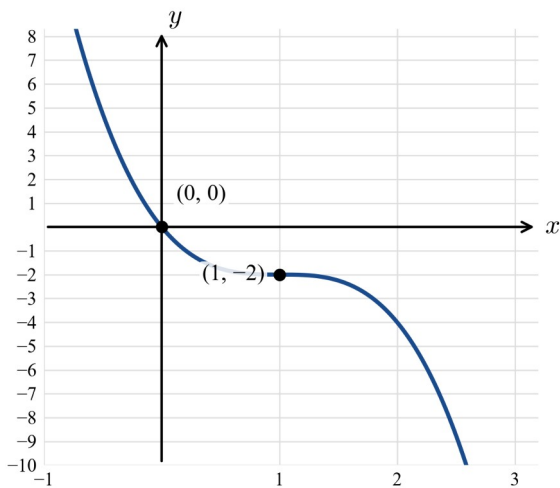
Q13. $y = (x - 1)^3 + 8$. Inflection $(1, 8)$; y -int $x = 0 \Rightarrow (-1)^3 + 8 = 7$, giving $(0, 7)$; x -int $(x - 1)^3 = -8 \Rightarrow x - 1 = -2 \Rightarrow x = -1$, giving $(-1, 0)$. *Transformation:* 1 right and 8 up.



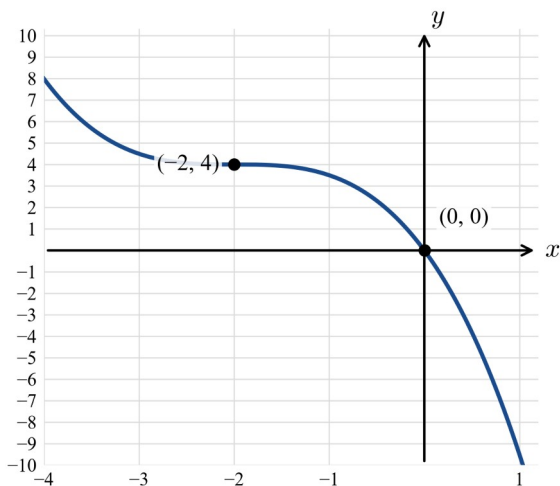
Q14. $y = -(x + 1)^3 + 1$. $a < 0$: decreasing; inflection $(-1, 1)$; y -int $x = 0 \Rightarrow -(1)^3 + 1 = 0$, giving $(0, 0)$; x -int $-(x + 1)^3 + 1 = 0 \Rightarrow (x + 1)^3 = 1 \Rightarrow x = 0$, giving $(0, 0)$. *Transformation:* reflection in the x -axis, then 1 left and 1 up.



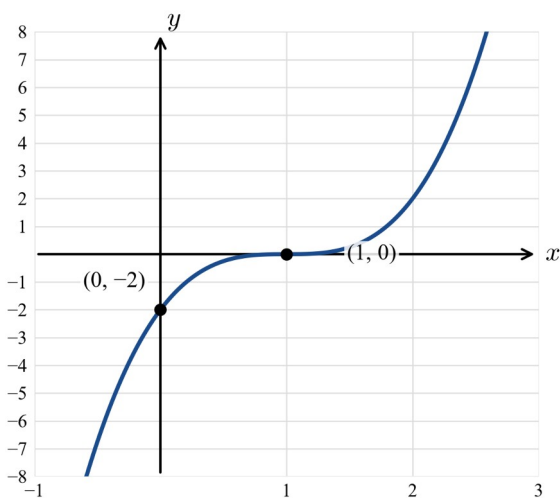
Q15. $y = -2(x - 1)^3 - 2$. $a = -2$: decreasing and steeper; inflection $(1, -2)$; y -int $x = 0 \Rightarrow -2(-1)^3 - 2 = 2 - 2 = 0$, giving $(0, 0)$; x -int $-2(x - 1)^3 - 2 = 0 \Rightarrow (x - 1)^3 = -1 \Rightarrow x = 0$, giving $(0, 0)$. *Transformation:* reflection, dilation of factor 2, then 1 right and 2 down.



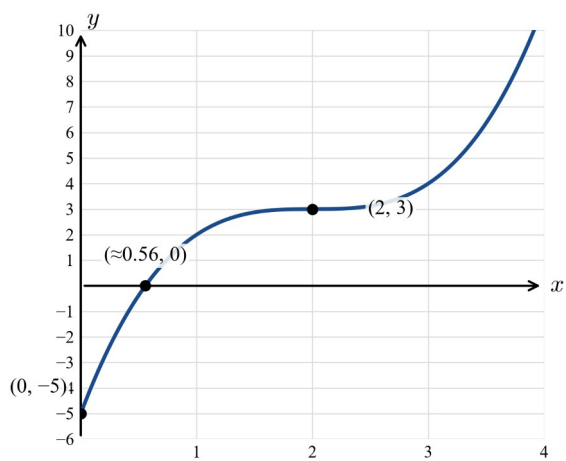
Q16. $y = -1/2(x+2)^3 + 4$. $a = -1/2$: decreasing and flatter; inflection $(-2, 4)$; y -int $x=0 \Rightarrow -1/2(2)^3 + 4 = -4 + 4 = 0$, giving $(0, 0)$; x -int $-1/2(x+2)^3 + 4 = 0 \Rightarrow (x+2)^3 = 8 \Rightarrow x+2 = 2 \Rightarrow x = 0$, giving $(0, 0)$. *Transformation:* reflection in the x -axis, dilation of factor $1/2$, then 2 left and 4 up.



Q17. $y = 2(x-1)^3$. Inflection $(1, 0)$; x -int $(1, 0)$; y -int $x=0 \Rightarrow 2(-1)^3 = -2$, giving $(0, -2)$. *Transformation:* dilation of factor 2, then 1 right.



Q18. $y = (x-2)^3 + 3$. Inflection $(2, 3)$; y -int $x=0 \Rightarrow (-2)^3 + 3 = -5$, giving $(0, -5)$; x -int $(x-2)^3 = -3 \Rightarrow x-2 = -\sqrt[3]{3}$, so $x = 2 - \sqrt[3]{3} (\approx 0.56)$, giving $(2 - \sqrt[3]{3}, 0)$. *Transformation:* 2 right and 3 up.



Chapter 6 Polynomial functions — 6F Sketching factorised cubics

Theory

A **cubic function** in factorised form

$$y = a(x - p)(x - q)(x - r)$$

can be sketched directly from its factors.

x-intercepts. Setting $y = 0$ and applying the null factor law gives the x -intercepts $x = p$, $x = q$ and $x = r$.

Shape and end behaviour. The sign of a (the coefficient of x^3 once expanded) fixes the long-run behaviour:

- if $a > 0$, the graph rises from bottom-left to top-right (as $x \rightarrow -\infty$, $y \rightarrow -\infty$; as $x \rightarrow \infty$, $y \rightarrow \infty$);
- if $a < 0$, it falls from top-left to bottom-right.

Behaviour at each intercept depends on the multiplicity of the factor:

- a **single** factor $(x - p)$ — the graph **crosses** the x -axis at $x = p$;
- a **squared** factor $(x - p)^2$ — the graph **touches** the x -axis at $x = p$ (a turning point sits on the axis there);
- a **cubed** factor $(x - p)^3$ — the graph has a **stationary point of inflection** at $x = p$ (it flattens, then crosses).

y-intercept. Substitute $x = 0$.

Turning points. A cubic with three distinct x -intercepts has **two** turning points (a local maximum and a local minimum), one between each pair of intercepts. Their exact coordinates are not needed for a sketch unless asked — the intercepts and shape are enough.

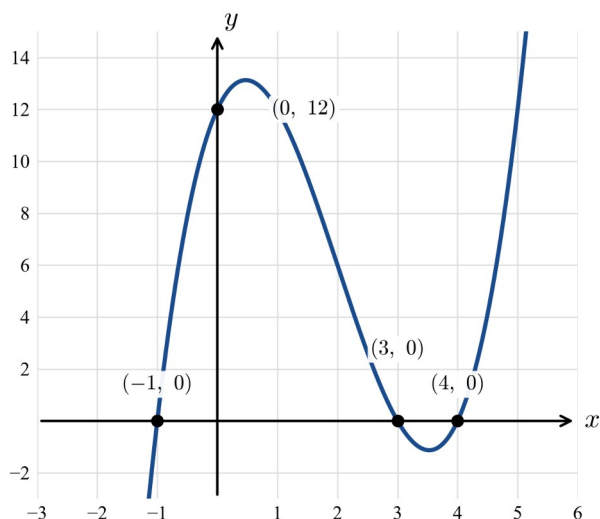
To sketch: find the x -intercepts and the behaviour at each (cross / touch / inflection), find the y -intercept, decide the end behaviour from the sign of a , then draw a smooth curve.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = (x + 1)(x - 3)(x - 4)$.

Working	Comment
Expanding, the leading term is x^3 , so $a = 1 > 0$: the graph rises to the right.	The sign of a gives the end behaviour.
x -intercepts: $x = -1, 3, 4$ — all single factors, so the graph crosses at each.	Null factor law; single factor $\Rightarrow$ cross.
y -intercept: $y = (1)(-3)(-4) = 12$, giving $(0, 12)$.	Substitute $x = 0$.
Three distinct intercepts $\Rightarrow$ two turning points (one between -1 and 3 , one between 3 and 4). Draw a smooth cubic.	Shape follows from the intercepts and end behaviour.



Example 2 — scaffolded

Sketch the graph of $y = (x - 2)^2(x + 1)$.

Working

Leading term x^3 , so $a = 1 > 0$.

x -intercepts: $x = 2$ (from $(x - 2)^2$, a **squared** factor $\Rightarrow$ **touches**) and $x = -1$ (single $\Rightarrow$ crosses).

y -intercept: $y = (-2)^2(1) = 4$, giving $(0, 4)$.

The graph touches the axis at $(2, 0)$ (a local minimum) and has a local maximum at $(0, 4)$.

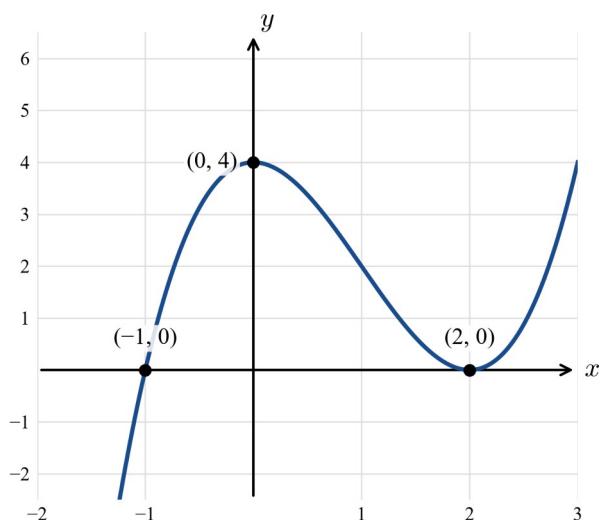
Comment

End behaviour: up to the right.

Multiplicity determines the behaviour at each intercept.

Substitute $x = 0$.

The touch point is a turning point on the axis.

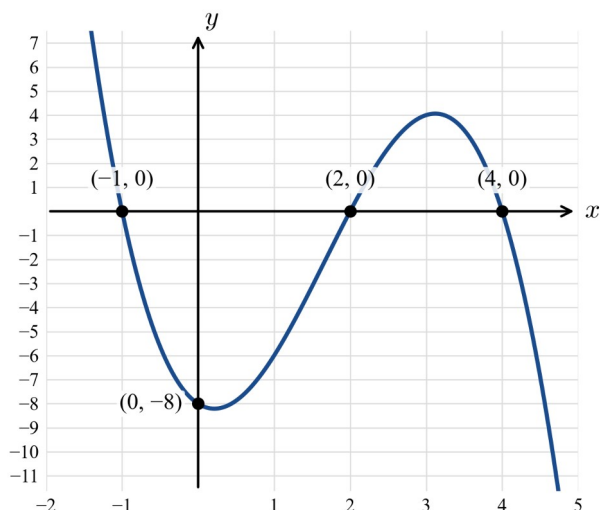


Example 3 — unscaffolded

Sketch the graph of $y = -(x + 1)(x - 2)(x - 4)$.

$a = -1 < 0$, so the graph falls to the right (up on the left, down on the right).

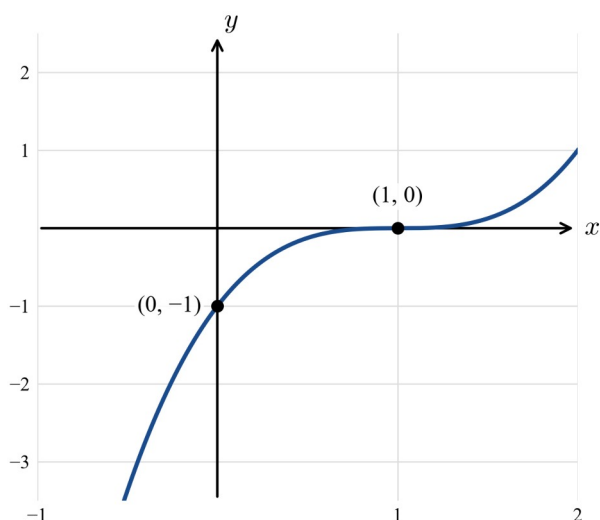
x -intercepts: $x = -1, 2, 4$ (all single $\Rightarrow$ crosses). y -intercept: $y = -(1)(-2)(-4) = -8$, giving $(0, -8)$.



Example 4 — unscaffolded

Sketch the graph of $y = (x - 1)^3$.

$a = 1 > 0$. The single cubed factor gives a **stationary point of inflection** at $x = 1$, so the graph flattens at $(1, 0)$ and crosses there. y -intercept: $y = (-1)^3 = -1$, giving $(0, -1)$.



Practice questions

Scaffolded

Q1. For $y = (x - 1)(x - 2)(x - 4)$: (a) state the sign of a and the end behaviour; (b) find the x -intercepts; (c) find the y -intercept; (d) sketch the graph.

Q2. For $y = (x + 3)(x + 1)(x - 2)$: (a) find the x -intercepts; (b) find the y -intercept; (c) sketch the graph.

Q3. For $y = (x + 2)^2(x - 1)$: (a) state which intercept is a touch and which is a cross; (b) find the y -intercept; (c) sketch the graph.

Q4. For $y = x(x - 3)(x + 3)$: (a) find the x -intercepts; (b) find the y -intercept; (c) sketch the graph.

Q5. For $y = -(x - 1)(x + 1)(x - 3)$: (a) state the end behaviour; (b) find all intercepts; (c) sketch the graph.

Q6. For $y = (x + 1)^3$: (a) describe the behaviour at $x = -1$; (b) find the y -intercept; (c) sketch the graph.

Un scaffolded

Sketch each of the following, labelling all axis intercepts and stating the behaviour (cross, touch or inflection) at each x -intercept.

Q7. $y = (x - 2)(x - 4)(x + 1)$

Q8. $y = (x + 3)(x - 1)(x - 4)$

Q9. $y = (x - 1)^2(x - 4)$

Q10. $y = (x + 2)^2(x - 2)$

Q11. $y = -(x + 1)(x - 1)(x - 2)$

Q12. $y = -x(x - 2)(x + 3)$

Q13. $y = (x - 2)^3$

Q14. $y = -(x + 1)^3$

Q15. $y = 2(x - 1)(x + 1)(x - 2)$

Q16. $y = x^2(x - 2)$

Q17. $y = (x + 2)(x - 1)^2$

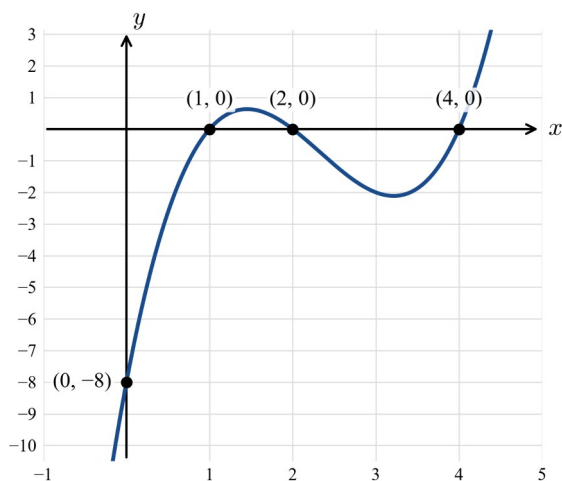
Q18. A cubic graph touches the x -axis at $x = -2$, crosses it at $x = 3$, and has y -intercept $(0, 12)$. Find its rule in the form $y = a(x - p)^2(x - q)$, then sketch it.

Answers

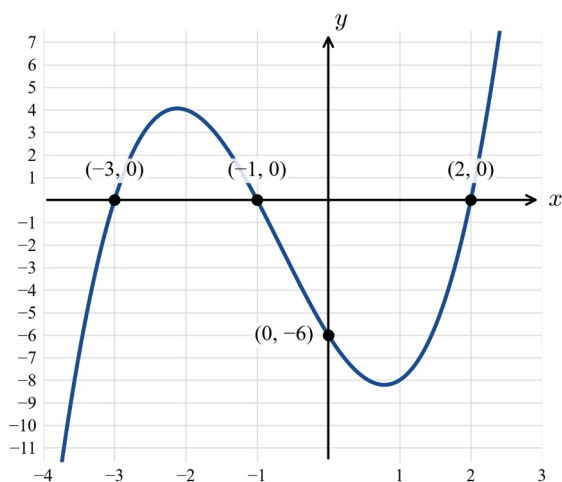
Q1. $a > 0$; x -int $(1, 0), (2, 0), (4, 0)$; y -int $(0, -8)$. **Q2.** x -int $(-3, 0), (-1, 0), (2, 0)$; y -int $(0, -6)$. **Q3.** Touch at $(-2, 0)$, cross at $(1, 0)$; y -int $(0, -4)$. **Q4.** x -int $(-3, 0), (0, 0), (3, 0)$; y -int $(0, 0)$. **Q5.** $a < 0$ (falls to the right); x -int $(-1, 0), (1, 0), (3, 0)$; y -int $(0, -3)$. **Q6.** Stationary point of inflection at $(-1, 0)$; y -int $(0, 1)$. **Q7.** Crosses at $(-1, 0), (2, 0), (4, 0)$; y -int $(0, 8)$. **Q8.** Crosses at $(-3, 0), (1, 0), (4, 0)$; y -int $(0, 12)$. **Q9.** Touch at $(1, 0)$, cross at $(4, 0)$; y -int $(0, -4)$. **Q10.** Touch at $(-2, 0)$, cross at $(2, 0)$; y -int $(0, -8)$. **Q11.** $a < 0$; crosses at $(-1, 0), (1, 0), (2, 0)$; y -int $(0, -2)$. **Q12.** $a < 0$; crosses at $(-3, 0), (0, 0), (2, 0)$; y -int $(0, 0)$. **Q13.** Inflection at $(2, 0)$; y -int $(0, -8)$. **Q14.** $a < 0$; inflection at $(-1, 0)$; y -int $(0, -1)$. **Q15.** $a > 0$ (steeper, dilation factor 2); crosses at $(-1, 0), (1, 0), (2, 0)$; y -int $(0, 4)$. **Q16.** Touch at $(0, 0)$, cross at $(2, 0)$; y -int $(0, 0)$. **Q17.** Cross at $(-2, 0)$, touch at $(1, 0)$; y -int $(0, 2)$. **Q18.** $y = -(x+2)^2(x-3)$; touch $(-2, 0)$, cross $(3, 0)$, y -int $(0, 12)$.

Fully-worked solutions

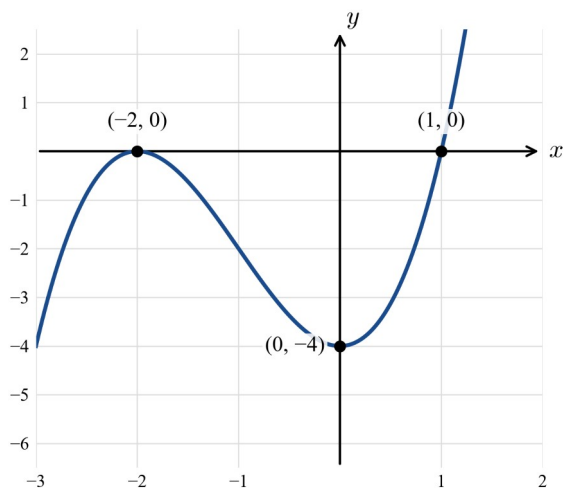
Q1. $y = (x-1)(x-2)(x-4)$. $a = 1 > 0$ (rises to the right). x -intercepts $x = 1, 2, 4$ (all cross). y -intercept $y = (-1)(-2)(-4) = -8$, giving $(0, -8)$.



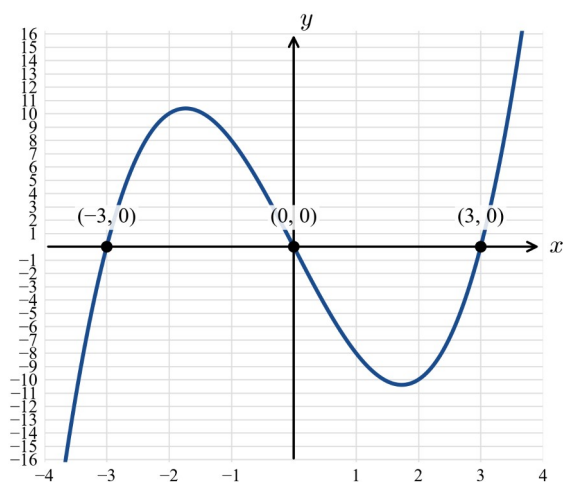
Q2. $y = (x+3)(x+1)(x-2)$. x -intercepts $x = -3, -1, 2$. y -intercept $y = (3)(1)(-2) = -6$, giving $(0, -6)$.



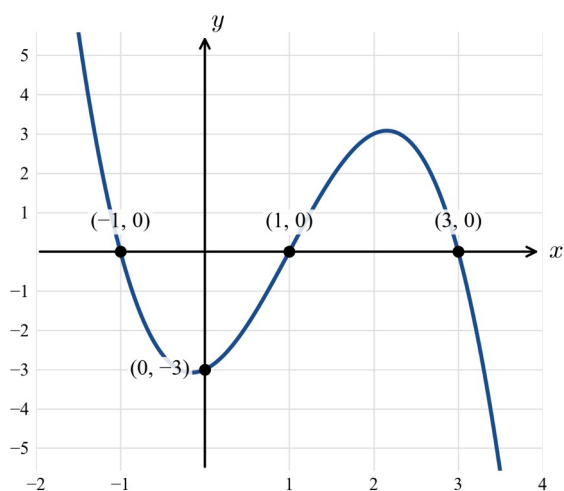
Q3. $y = (x+2)^2(x-1)$. The squared factor $(x+2)^2$ gives a **touch** at $(-2, 0)$; $(x-1)$ gives a **cross** at $(1, 0)$. y -intercept $y = (2)^2(-1) = -4$, giving $(0, -4)$.



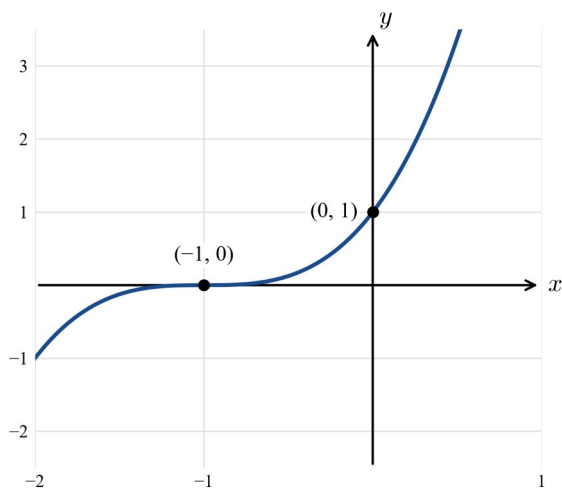
Q4. $y = x(x - 3)(x + 3)$. x -intercepts $x = 0, 3, -3$ (all cross). y -intercept $(0, 0)$.



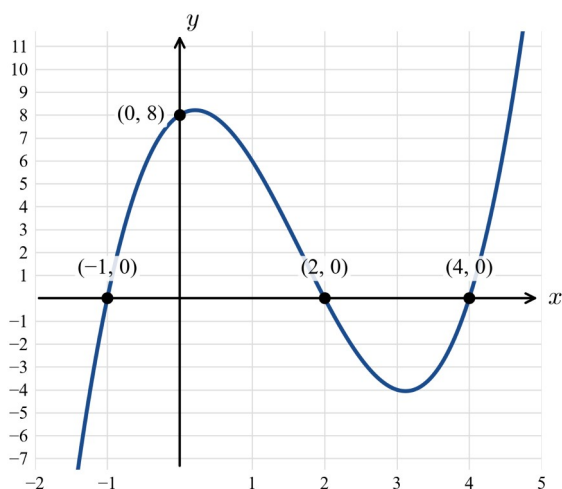
Q5. $y = -(x - 1)(x + 1)(x - 3)$. $a = -1 < 0$, so the graph falls to the right. x -intercepts $x = -1, 1, 3$. y -intercept $y = -(-1)(1)(-3) = -3$, giving $(0, -3)$.



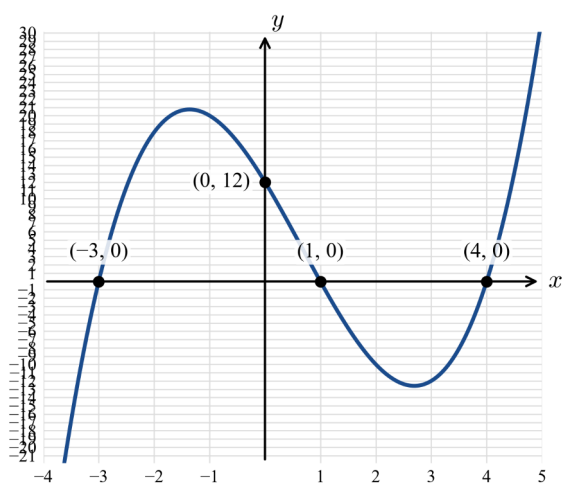
Q6. $y = (x + 1)^3$. The cubed factor gives a **stationary point of inflection** at $(-1, 0)$. y -intercept $y = (1)^3 = 1$, giving $(0, 1)$.



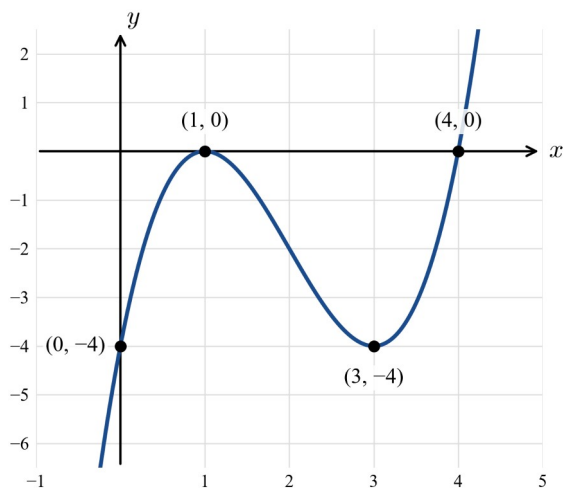
Q7. $y = (x - 2)(x - 4)(x + 1)$. Crosses at $x = -1, 2, 4$; $a > 0$. y -intercept $y = (-2)(-4)(1) = 8$, giving $(0, 8)$.



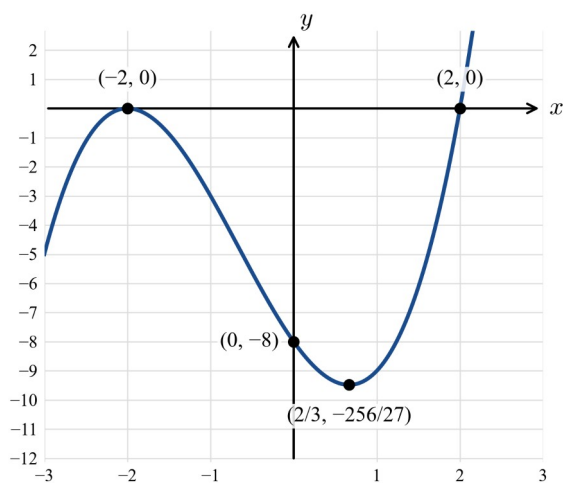
Q8. $y = (x + 3)(x - 1)(x - 4)$. Crosses at $x = -3, 1, 4$; $a > 0$. y -intercept $y = (3)(-1)(-4) = 12$, giving $(0, 12)$.



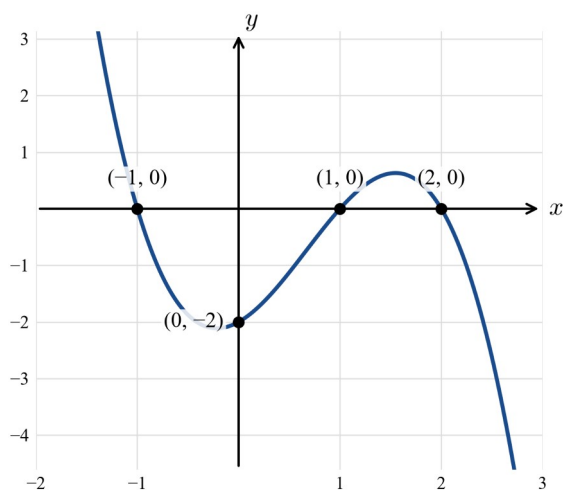
Q9. $y = (x - 1)^2(x - 4)$. Touch at $(1, 0)$, cross at $(4, 0)$. y -intercept $y = (-1)^2(-4) = -4$, giving $(0, -4)$.



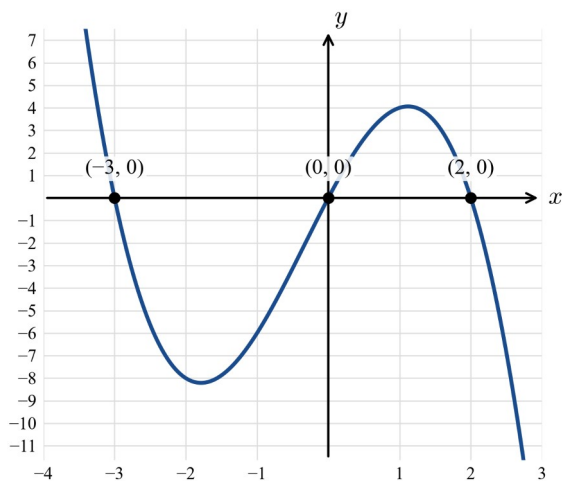
Q10. $y = (x+2)^2(x-2)$. Touch at $(-2, 0)$, cross at $(2, 0)$. y -intercept $y = (2)^2(-2) = -8$, giving $(0, -8)$.



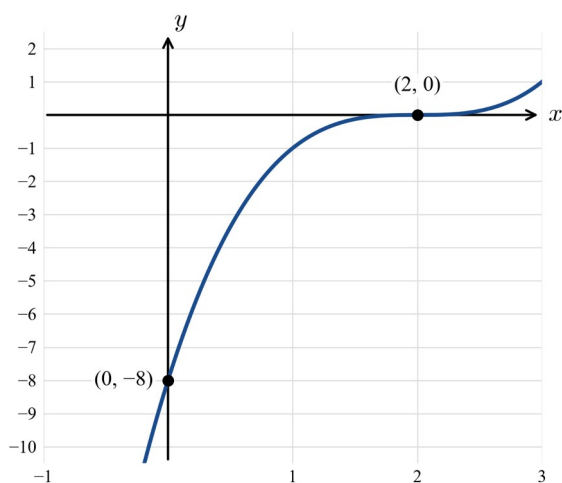
Q11. $y = -(x+1)(x-1)(x-2)$. $a = -1 < 0$ (falls to the right). Crosses at $x = -1, 1, 2$. y -intercept $y = -(1)(-1)(-2) = -2$, giving $(0, -2)$.



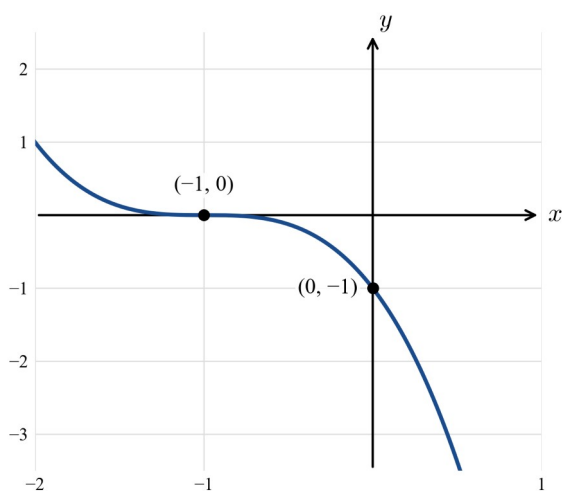
Q12. $y = -x(x-2)(x+3)$. $a = -1 < 0$. Crosses at $x = -3, 0, 2$. y -intercept $(0, 0)$.



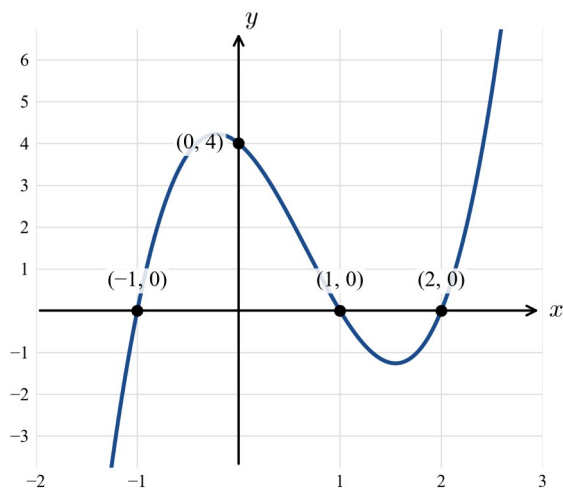
Q13. $y = (x - 2)^3$. Stationary point of inflection at $(2, 0)$. y -intercept $y = (-2)^3 = -8$, giving $(0, -8)$.



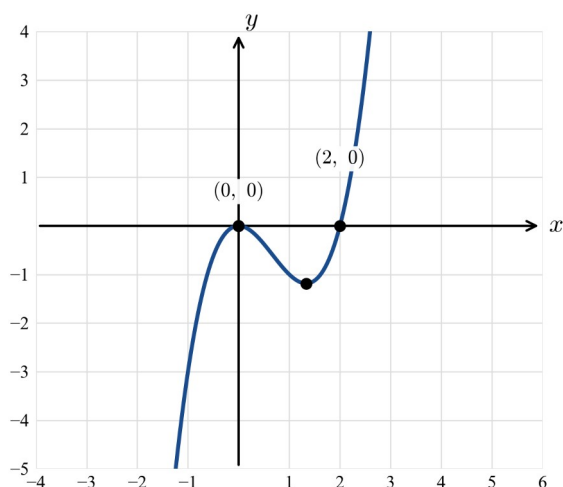
Q14. $y = -(x + 1)^3$. $a = -1 < 0$. Stationary point of inflection at $(-1, 0)$. y -intercept $y = -(1)^3 = -1$, giving $(0, -1)$.



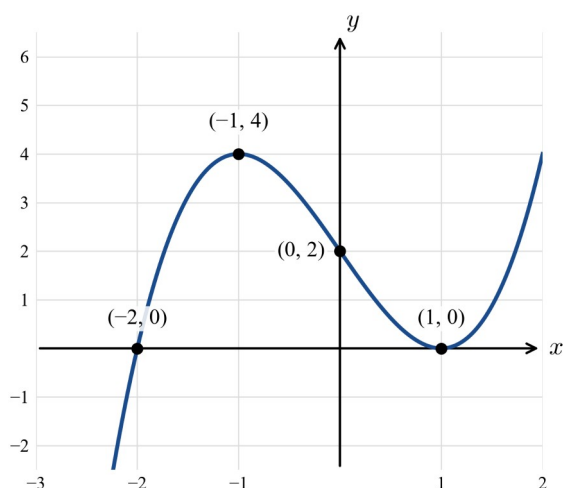
Q15. $y = 2(x - 1)(x + 1)(x - 2)$. $a = 2 > 0$ — same intercepts as an $a = 1$ cubic but **steeper** (dilated by factor 2 from the x -axis). Crosses at $x = -1, 1, 2$. y -intercept $y = 2(-1)(1)(-2) = 4$, giving $(0, 4)$.



Q16. $y = x^2(x - 2)$. The factor x^2 gives a **touch** at $(0, 0)$; $(x - 2)$ gives a **cross** at $(2, 0)$. y -intercept $(0, 0)$.



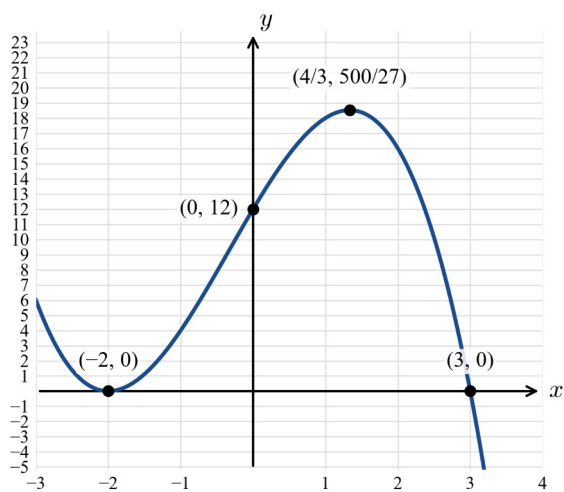
Q17. $y = (x + 2)(x - 1)^2$. Cross at $(-2, 0)$; the squared factor gives a **touch** at $(1, 0)$. y -intercept $y = (2)(-1)^2 = 2$, giving $(0, 2)$.



Q18. A touch at $x = -2$ needs the squared factor $(x + 2)^2$; a cross at $x = 3$ needs $(x - 3)$. So $y = a(x + 2)^2(x - 3)$. Using the y -intercept $(0, 12)$:

$$12 = a(2)^2(-3) = -12a \Rightarrow a = -1.$$

$$\therefore y = -(x + 2)^2(x - 3).$$



Chapter 6 Polynomial functions — 6G Cubic inequalities

Theory

A **cubic inequality** such as $P(x) > 0$ is solved by finding where the graph of $y = P(x)$ lies **above** (or **below**) the x -axis.

Method:

1. Rearrange so that one side is 0.
2. **Factorise** $P(x)$ and find the x -intercepts (the roots).
3. **Sketch** the cubic. The shape is set by the sign of the leading coefficient: a **positive** cubic rises from bottom-left to top-right; a **negative** cubic falls from top-left to bottom-right. At a **single** root the graph *crosses* the axis; at a **double (repeated)** root it *touches* the axis without crossing.
4. **Read** the solution from the graph: $P(x) > 0$ where the curve is above the axis, $P(x) < 0$ where it is below.
Use $\geq$ or $\leq$ to **include** the roots.

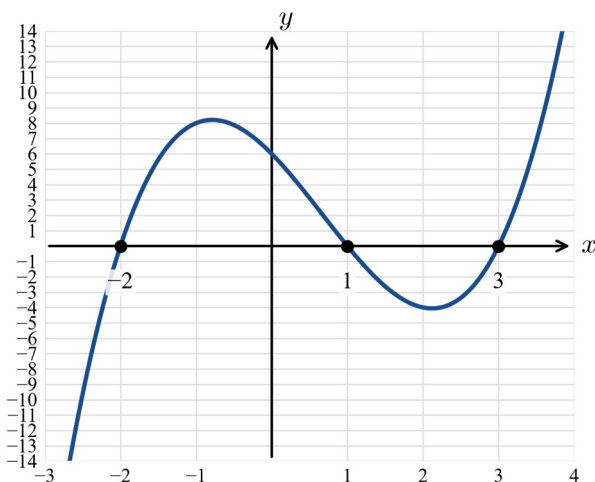
Write the answer in set-builder colon form $\{x : \dots\}$ or in interval notation, joining separate pieces with $\cup$.

Worked examples

Example 1 — scaffolded

Solve $(x - 1)(x + 2)(x - 3) > 0$.

Working	Comment
The expression is already factorised, so the x -intercepts are $x = -2, 1, 3$.	Set each factor equal to 0.
The leading term is $x \times x \times x = x^3$, so the cubic is positive — it rises from bottom-left to top-right.	Shape from the leading coefficient.
The curve changes sign at each single root: below the axis for $x < -2$, above on $(-2, 1)$, below on $(1, 3)$, above for $x > 3$.	Read the sign region by region.
$\therefore (x - 1)(x + 2)(x - 3) > 0$ for $x \in (-2, 1) \cup (3, \infty)$.	Take the intervals where the curve is above the axis.



Example 2 — scaffolded

Solve $x^3 - 4x \leq 0$.

Factorise: $x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$.

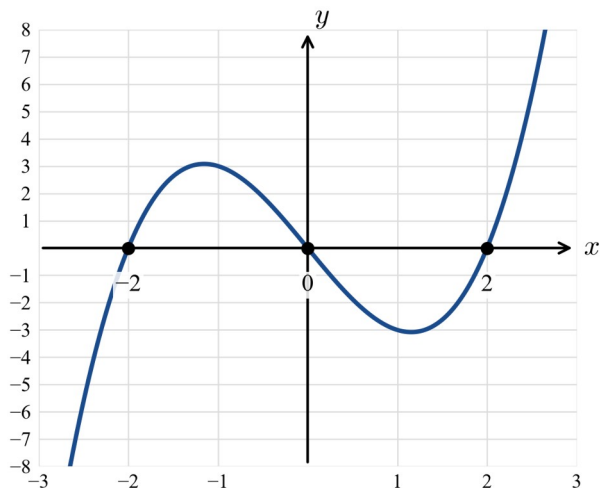
Common factor, then difference of two squares.

x -intercepts $x = -2, 0, 2$; positive cubic.

We want ≤ 0 : the curve **on or below** the axis, which occurs for $x \leq -2$ and for $0 \leq x \leq 2$.

Include the roots because of $\leq$.

$\therefore x^3 - 4x \leq 0$ for $x \in (-\infty, -2] \cup [0, 2]$.



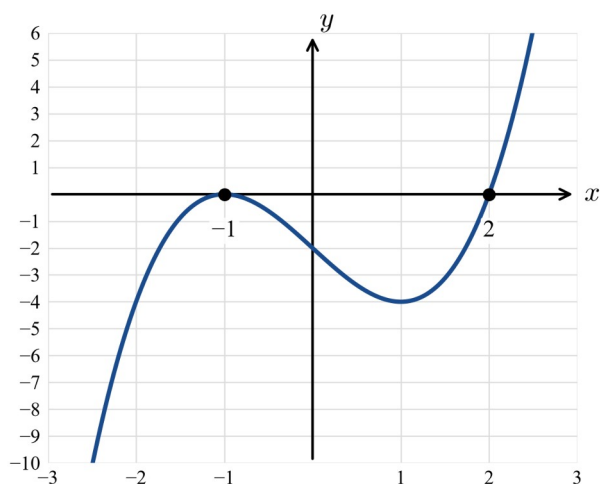
Example 3 — unscaffolded

Solve $(x + 1)^2(x - 2) < 0$.

The x -intercepts are $x = -1$ (a **repeated** root — the graph touches the axis here) and $x = 2$. The cubic is positive.

Since $(x + 1)^2 \geq 0$ for all x , the sign of the product is the sign of $(x - 2)$, except at $x = -1$ where the product equals 0. So the product is negative for every $x < 2$ **except** $x = -1$.

$\therefore (x + 1)^2(x - 2) < 0$ for $x \in (-\infty, -1) \cup (-1, 2)$ — that is, $x < 2$ with $x \neq -1$.



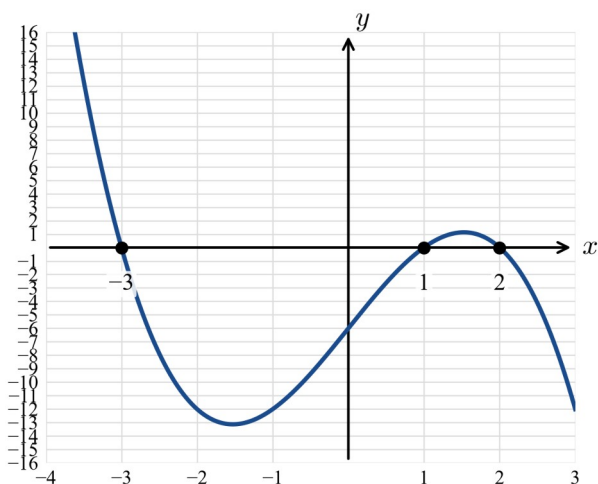
Example 4 — unscaffolded

Solve $-x^3 + 7x - 6 \geq 0$.

Factorise: $-x^3 + 7x - 6 = -(x^3 - 7x + 6) = -(x - 1)(x - 2)(x + 3)$, so the x -intercepts are $x = -3, 1, 2$.

The leading coefficient is **negative**, so the cubic falls from top-left to bottom-right. Reading where the curve is on or above the axis gives $x \leq -3$ and $1 \leq x \leq 2$.

$$\therefore -x^3 + 7x - 6 \geq 0 \text{ for } x \in (-\infty, -3] \cup [1, 2].$$



Practice questions

Scaffolded

Q1. Solve $(x-1)(x-2)(x-3) > 0$.

Q2. Solve $(x+1)(x-2)(x+4) < 0$.

Q3. Solve $x(x-3)(x+2) \geq 0$.

Q4. Solve $x^3 - x \leq 0$. (*Factorise first.*)

Q5. Solve $(x-2)^2(x+1) > 0$. (*Note the repeated factor.*)

Q6. Solve $x(2x-1)(x+2) < 0$.

Un scaffolded

Solve each of the following.

Q7. $(x-1)(x-4)(x+2) > 0$

Q8. $(x+3)(x-1)(x-5) < 0$

Q9. $x(x+4)(x-2) \geq 0$

Q10. $x(x-1)(x-5) \leq 0$

Q11. $x^3 - 9x > 0$

Q12. $x^2(x-4) \leq 0$

Q13. $(x-2)^2(x-5) < 0$

Q14. $(x+1)^2(x-3) \geq 0$

Q15. $-x(x-2)(x+2) > 0$

Q16. $-(x-1)(x-3)(x+2) \leq 0$

Q17. $x(2x+1)(x-3) \geq 0$

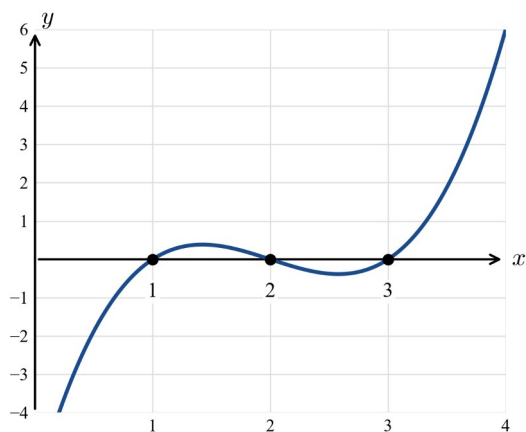
Q18. $x^3 + 2x^2 - 5x - 6 > 0$ (Use the factor theorem.)

Answers

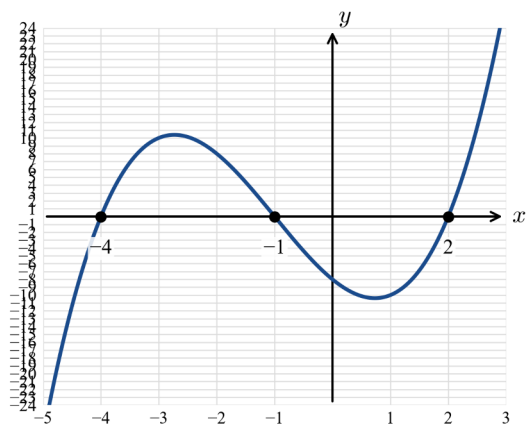
Q1. $(1, 2) \cup (3, \infty)$. **Q2.** $(-\infty, -4) \cup (-1, 2)$. **Q3.** $[-2, 0] \cup [3, \infty)$. **Q4.** $(-\infty, -1] \cup [0, 1]$. **Q5.** $(-1, 2) \cup (2, \infty)$. **Q6.** $(-\infty, -2) \cup (0, 1/2)$. **Q7.** $(-2, 1) \cup (4, \infty)$. **Q8.** $(-\infty, -3) \cup (1, 5)$. **Q9.** $[-4, 0] \cup [2, \infty)$. **Q10.** $(-\infty, 0] \cup [1, 5]$. **Q11.** $(-3, 0) \cup (3, \infty)$. **Q12.** $(-\infty, 4]$. **Q13.** $(-\infty, 2) \cup (2, 5)$. **Q14.** $\{-1\} \cup [3, \infty)$. **Q15.** $(-\infty, -2) \cup (0, 2)$. **Q16.** $[-2, 1] \cup [3, \infty)$. **Q17.** $[-1/2, 0] \cup [3, \infty)$. **Q18.** $(-3, -1) \cup (2, \infty)$.

Fully-worked solutions

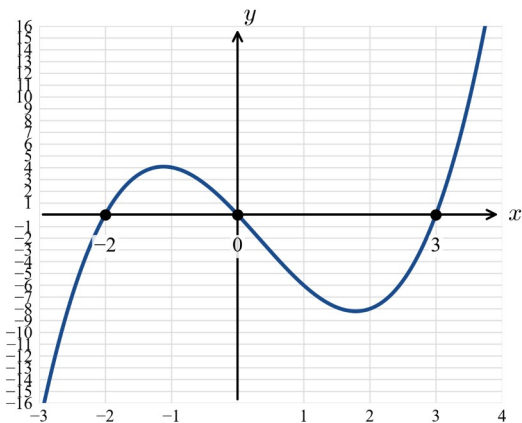
Q1. $(x-1)(x-2)(x-3) > 0$. Roots 1, 2, 3; positive cubic. Above the axis on $(1, 2)$ and for $x > 3$.
 $\therefore x \in (1, 2) \cup (3, \infty)$.



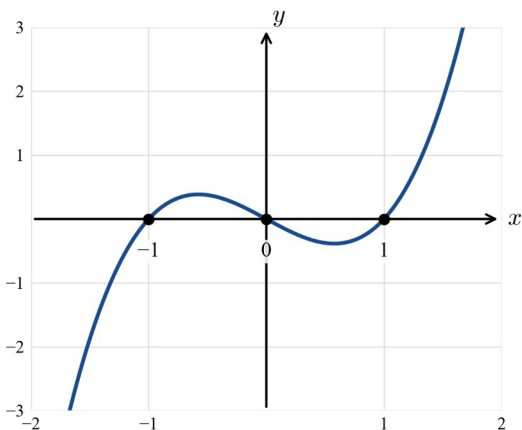
Q2. $(x+1)(x-2)(x+4) < 0$. Roots $-4, -1, 2$; positive cubic. Below the axis for $x < -4$ and on $(-1, 2)$.
 $\therefore x \in (-\infty, -4) \cup (-1, 2)$.



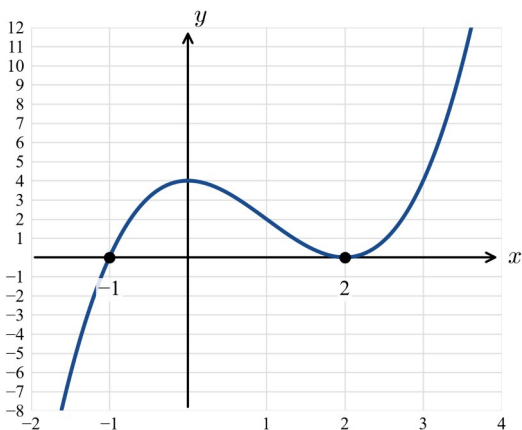
Q3. $x(x-3)(x+2) \geq 0$. Roots $-2, 0, 3$; positive cubic. On or above the axis on $[-2, 0]$ and for $x \geq 3$.
 $\therefore x \in [-2, 0] \cup [3, \infty)$.



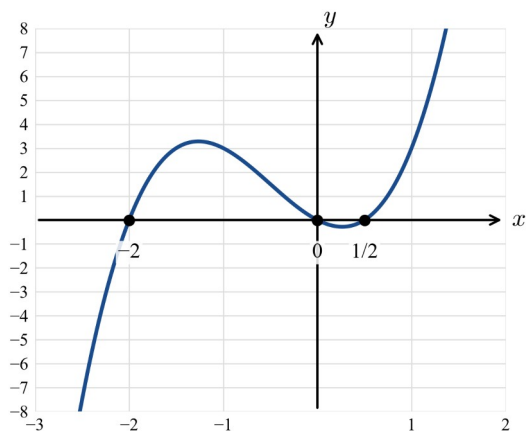
Q4. $x^3 - x \leq 0$. Factorise $x^3 - x = x(x-1)(x+1)$; roots $-1, 0, 1$; positive cubic. On or below the axis for $x \leq -1$ and on $[0, 1]$. $\therefore x \in (-\infty, -1] \cup [0, 1]$.



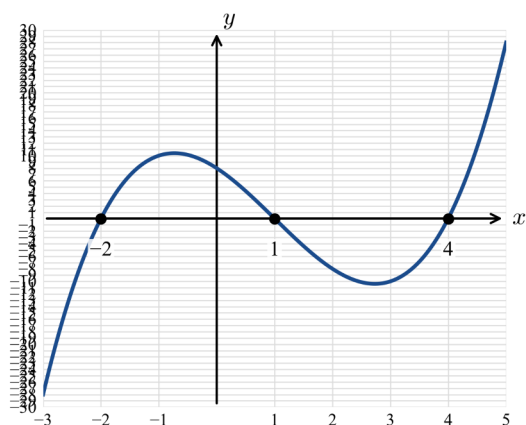
Q5. $(x-2)^2(x+1) > 0$. Repeated root $x=2$ (touch), single root $x=-1$; positive cubic. Since $(x-2)^2 \geq 0$, the sign follows $(x+1)$, except at $x=2$ where the value is 0. So it is positive for $x > -1$ with $x \neq 2$. $\therefore x \in (-1, 2) \cup (2, \infty)$.



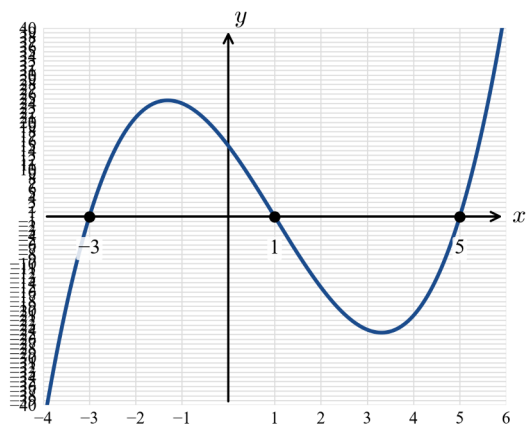
Q6. $x(2x-1)(x+2) < 0$. Roots $-2, 0, 1/2$; positive cubic. Below the axis for $x < -2$ and on $(0, 1/2)$. $\therefore x \in (-\infty, -2) \cup (0, 1/2)$.



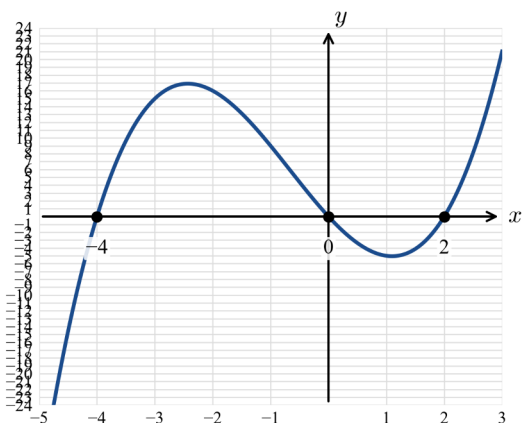
Q7. $(x-1)(x-4)(x+2) > 0$. Roots $-2, 1, 4$; positive cubic. Above the axis on $(-2, 1)$ and for $x > 4$.
 $\therefore x \in (-2, 1) \cup (4, \infty)$.



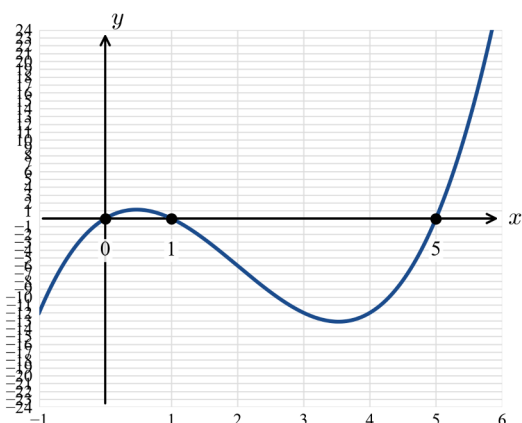
Q8. $(x+3)(x-1)(x-5) < 0$. Roots $-3, 1, 5$; positive cubic. Below the axis for $x < -3$ and on $(1, 5)$.
 $\therefore x \in (-\infty, -3) \cup (1, 5)$.



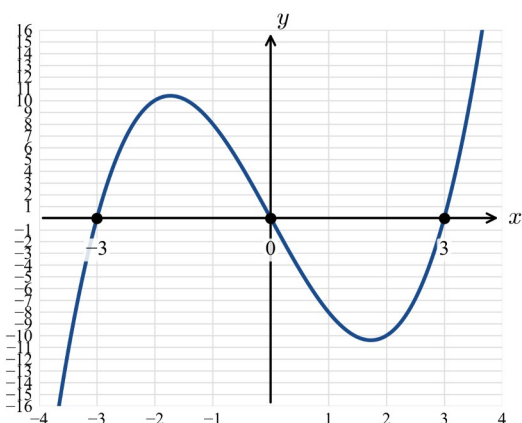
Q9. $x(x+4)(x-2) \geq 0$. Roots $-4, 0, 2$; positive cubic. On or above the axis on $[-4, 0]$ and for $x \geq 2$.
 $\therefore x \in [-4, 0] \cup [2, \infty)$.



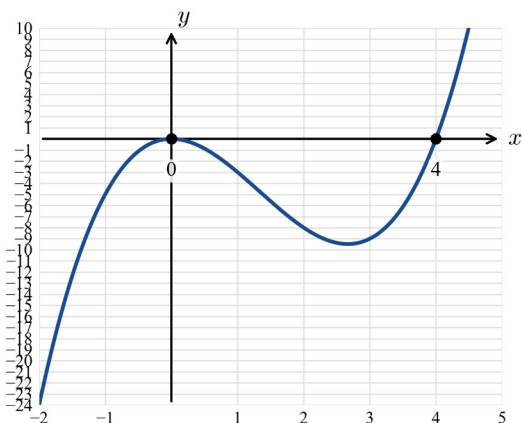
Q10. $x(x-1)(x-5) \leq 0$. Roots 0, 1, 5; positive cubic. On or below the axis for $x \leq 0$ and on $[1, 5]$.
 $\therefore x \in (-\infty, 0] \cup [1, 5]$.



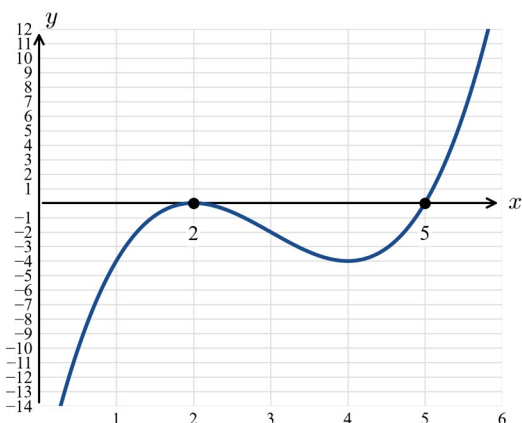
Q11. $x^3 - 9x > 0$. Factorise $x(x-3)(x+3)$; roots -3, 0, 3; positive cubic. Above the axis on $(-3, 0)$ and for $x > 3$
 $\therefore x \in (-3, 0) \cup (3, \infty)$.



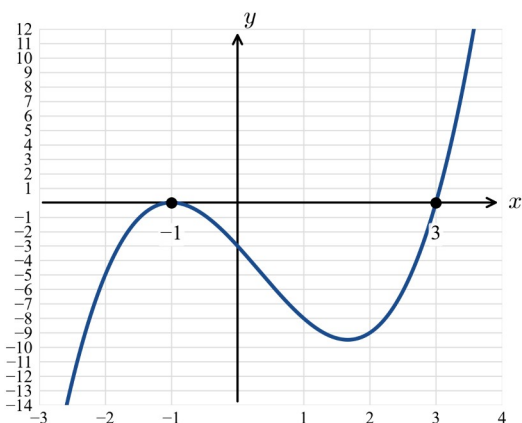
Q12. $x^2(x-4) \leq 0$. Repeated root $x=0$ (touch), single root $x=4$. Since $x^2 \geq 0$, the sign follows $(x-4)$, so the expression is ≤ 0 for all $x \leq 4$ (and $=0$ at $x=0$ and $x=4$). $\therefore x \in (-\infty, 4]$.



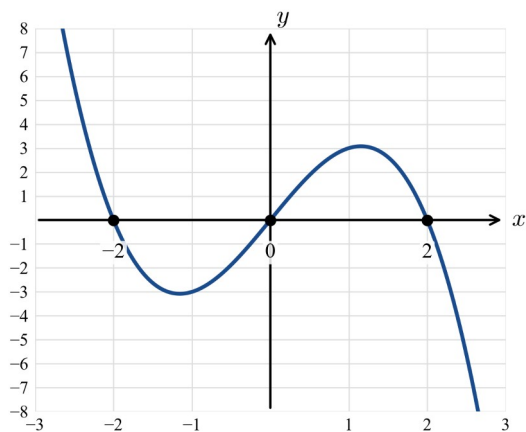
Q13. $(x - 2)^2(x - 5) < 0$. Repeated root $x = 2$, single root $x = 5$; positive cubic. The sign follows $(x - 5)$ except at $x = 2$ (value 0), so it is negative for $x < 5$ with $x \neq 2$. $\therefore x \in (-\infty, 2) \cup (2, 5)$.



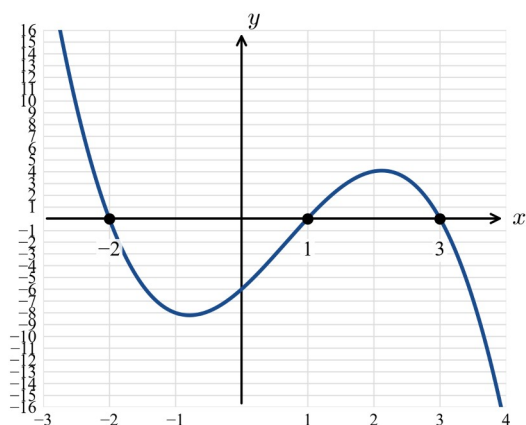
Q14. $(x + 1)^2(x - 3) \geq 0$. Repeated root $x = -1$, single root $x = 3$. The sign follows $(x - 3)$, so it is positive for $x > 3$; it also equals 0 at $x = -1$ and $x = 3$ (both allowed by $\geq$). $\therefore x = -1$ or $x \geq 3$, i.e. $x \in \{-1\} \cup [3, \infty)$.



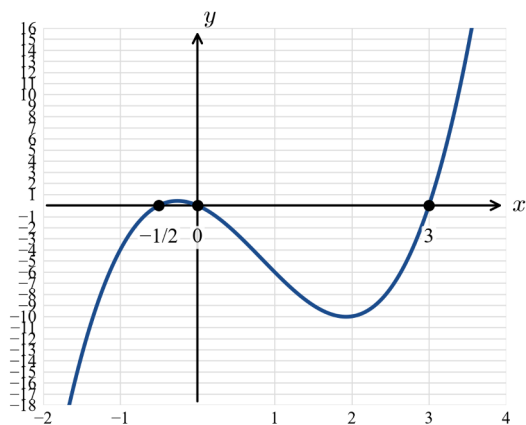
Q15. $-x(x - 2)(x + 2) > 0$. Roots $-2, 0, 2$; **negative** cubic (falls from top-left). Above the axis for $x < -2$ and on $(0, 2)$. $\therefore x \in (-\infty, -2) \cup (0, 2)$.



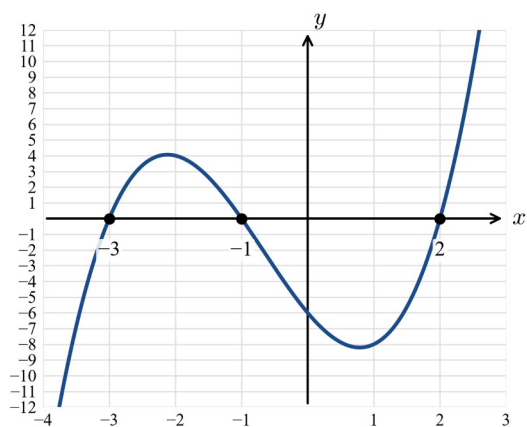
Q16. $-(x-1)(x-3)(x+2) \leq 0$. Roots $-2, 1, 3$; negative cubic. On or below the axis on $[-2, 1]$ and for $x \geq 3$.
 $\therefore x \in [-2, 1] \cup [3, \infty)$.



Q17. $x(2x+1)(x-3) \geq 0$. Roots $-1/2, 0, 3$; positive cubic. On or above the axis on $[-1/2, 0]$ and for $x \geq 3$.
 $\therefore x \in [-1/2, 0] \cup [3, \infty)$.



Q18. $x^3 + 2x^2 - 5x - 6 > 0$. By the factor theorem, $x=2$ gives $8+8-10-6=0$, so $(x-2)$ is a factor:
 $x^3 + 2x^2 - 5x - 6 = (x-2)(x+1)(x+3)$. Roots $-3, -1, 2$; positive cubic. Above the axis on $(-3, -1)$ and for $x > 2$.
 $\therefore x \in (-3, -1) \cup (2, \infty)$.



Chapter 6 Polynomial functions — 6H Families of cubic functions

Theory

Just as with quadratics, the rule of a cubic can be found once enough information is given. Choose the form that best matches the information.

Stationary-point-of-inflection form. A cubic with a stationary point of inflection at (h, k) has the rule

$$y = a(x - h)^3 + k.$$

The turning-point-like point (h, k) is read straight off; one further point gives a .

Factored form. A cubic with x -intercepts p , q and r has the rule

$$y = a(x - p)(x - q)(x - r).$$

One further point (often the y -intercept) gives a . A **repeated factor** $(x - p)^2$ means the graph **touches** the x -axis at $x = p$; a single factor means it **crosses**.

General (polynomial) form. $y = ax^3 + bx^2 + cx + d$ has four unknowns, so **four** independent conditions (usually four points) are needed. Substitute each point and solve the resulting system of simultaneous equations.

Families of cubics. A *family* shares one feature and differs by a parameter — for example, every cubic with a stationary point of inflection at $(1, 2)$ has the form $y = a(x - 1)^3 + 2$, and every cubic with x -intercepts -1 , 0 and 2 has the form $y = ax(x + 1)(x - 2)$. A single extra condition selects one member of the family.

Worked examples

Example 1 — scaffolded

A cubic has a stationary point of inflection at $(2, -1)$ and passes through $(3, 2)$. Find its rule and sketch its graph.

Working

Comment

Stationary point of inflection $\Rightarrow y = a(x - h)^3 + k$ with
 $h = 2, k = -1$: $y = a(x - 2)^3 - 1$.

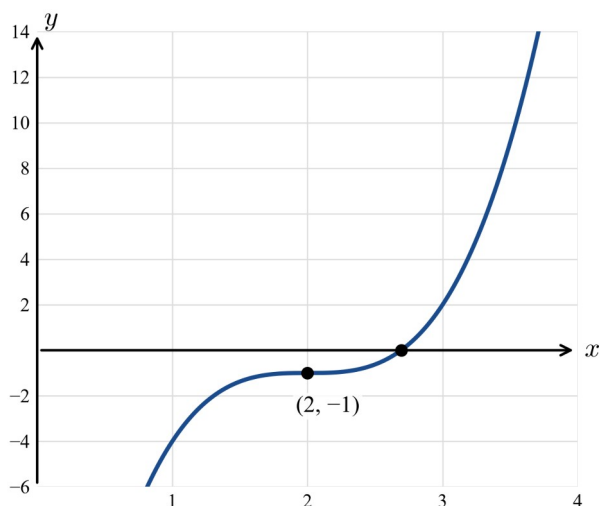
Choose inflection form and read off (h, k) .

Substitute $(3, 2)$: $2 = a(3 - 2)^3 - 1 = a - 1$.

Use the given point.

$\therefore a = 3$, so $y = 3(x - 2)^3 - 1$.

Solve for a .



Example 2 — scaffolded

A cubic has x -intercepts -2 , 1 and 3 , and a y -intercept of 12 . Find its rule and sketch its graph.

Working

x -intercepts given $\Rightarrow y = a(x+2)(x-1)(x-3)$.

Substitute the y -intercept $(0, 12)$:

$$12 = a(2)(-1)(-3) = 6a.$$

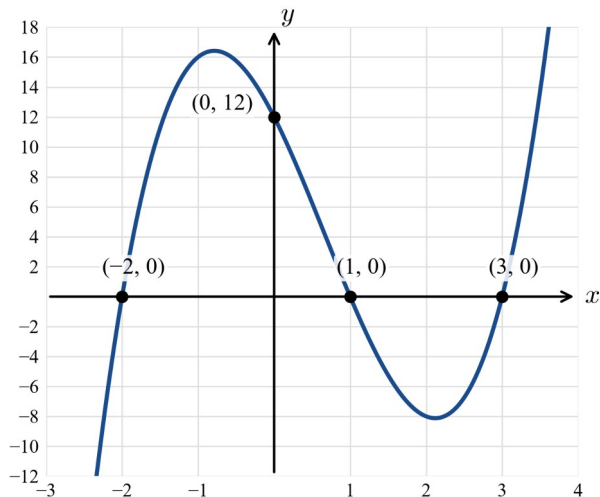
$$\therefore a = 2, \text{ so } y = 2(x+2)(x-1)(x-3).$$

Comment

Choose factored form, one factor per intercept.

Put $x = 0$.

Solve for a .



Example 3 — unscaffolded

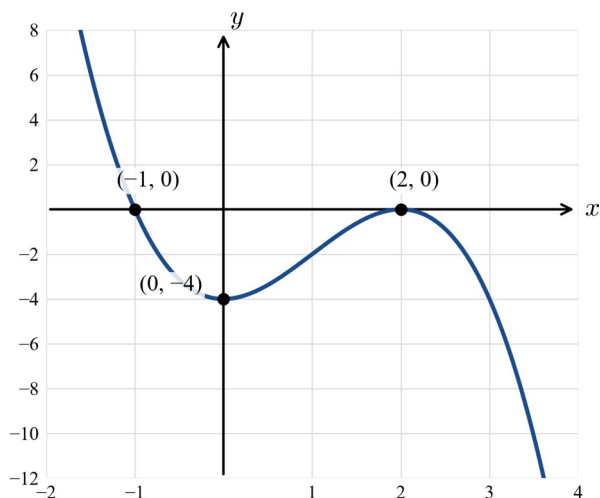
A cubic touches the x -axis at $x = 2$, crosses it at $x = -1$, and passes through $(0, -4)$. Find its rule and sketch its graph.

Touching at $x = 2$ gives a repeated factor $(x - 2)^2$; crossing at $x = -1$ gives $(x + 1)$:

$$y = a(x - 2)^2(x + 1).$$

Substitute $(0, -4)$: $-4 = a(-2)^2(1) = 4a$, so $a = -1$.

$$\therefore y = -(x - 2)^2(x + 1).$$



Example 4 — unscaffolded

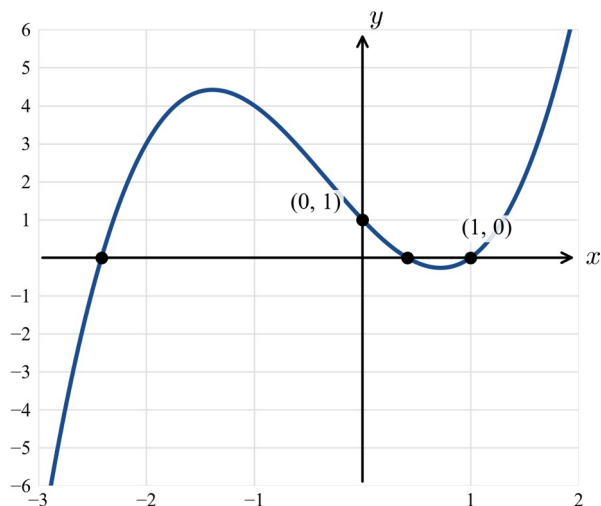
Find the rule of the cubic $y = ax^3 + bx^2 + cx + d$ that passes through $(0, 1)$, $(1, 0)$, $(-1, 4)$ and $(2, 7)$.

From $(0, 1)$: $d = 1$. Substituting the other three points:

$$a + b + c + 1 = 0, -a + b - c + 1 = 4, 8a + 4b + 2c + 1 = 7.$$

That is $a + b + c = -1$, $-a + b - c = 3$, $4a + 2b + c = 3$. Adding the first two gives $2b = 2$, so $b = 1$; then $a + c = -2$ and $4a + c = 1$, giving $a = 1$ and $c = -3$.

$$\therefore y = x^3 + x^2 - 3x + 1.$$



Practice questions

Scaffolded

Q1. A cubic has a stationary point of inflection at $(0, 0)$ and passes through $(1, 2)$. (a) Write the form of the rule. (b) Find a . (c) Sketch the graph.

Q2. A cubic has a stationary point of inflection at $(-1, 3)$ and passes through $(1, -5)$. (a) Write the form of the rule. (b) Find a . (c) Sketch the graph.

Q3. A cubic has x -intercepts $-3, 0$ and 2 , and passes through $(1, -8)$. (a) Write the factored form. (b) Find a . (c) Sketch the graph.

Q4. A cubic touches the x -axis at $x = -1$, crosses it at $x = 3$, and has y -intercept 6 . (a) Write the form using a repeated factor. (b) Find a . (c) Sketch the graph.

Q5. (a) Describe the family of cubics with x -intercepts $-2, 1$ and 4 . (b) Find the member that passes through $(2, 8)$. (c) Sketch it.

Q6. Find the rule of the cubic $y = ax^3 + bx^2 + cx + d$ passing through $(0, 0)$, $(1, 6)$, $(-1, -6)$ and $(2, 6)$, then sketch it.

Un scaffolded

Find the rule of each cubic and sketch its graph.

Q7. A stationary point of inflection at $(1, 2)$; passes through $(0, 1)$.

Q8. A stationary point of inflection at $(-2, 0)$; passes through $(-1, 5)$.

Q9. x -intercepts $-1, 2$ and 3 ; y -intercept 6 .

Q10. x -intercepts $-4, -1$ and 2 ; passes through $(1, 20)$.

Q11. Touches the x -axis at $x = 1$, crosses at $x = -2$; passes through $(0, 4)$.

Q12. Touches the x -axis at $x = -3$, crosses at $x = 1$; y -intercept -9 .

Q13. $y = ax^3 + bx^2 + cx + d$ through $(0, 1)$, $(1, 1)$, $(2, 3)$ and $(-1, -3)$.

Q14. (a) Describe the family of cubics with a stationary point of inflection at $(2, -1)$. (b) Find the member that passes through $(4, 7)$.

Q15. (a) Describe the family of cubics with x -intercepts $0, 3$ and -2 . (b) Find the member that passes through $(1, 12)$.

Q16. Touches the x -axis at $x = 2$, crosses at $x = 0$; passes through $(1, 3)$.

Q17. $y = ax^3 + bx^2 + cx + d$ through $(0, 0)$, $(1, -3)$, $(2, 0)$ and $(3, 15)$.

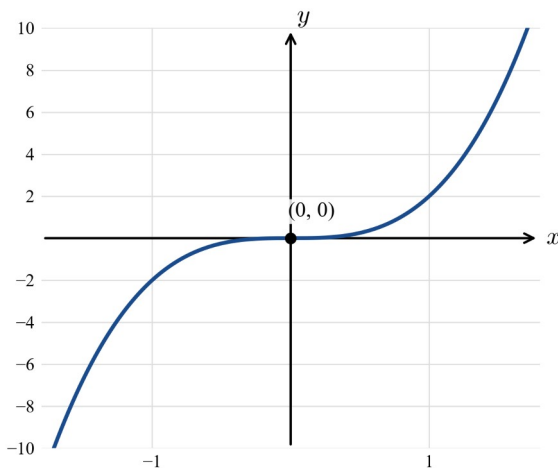
Q18. A stationary point of inflection at $(0, 4)$; passes through $(2, -4)$. Also state the exact x -intercept.

Answers

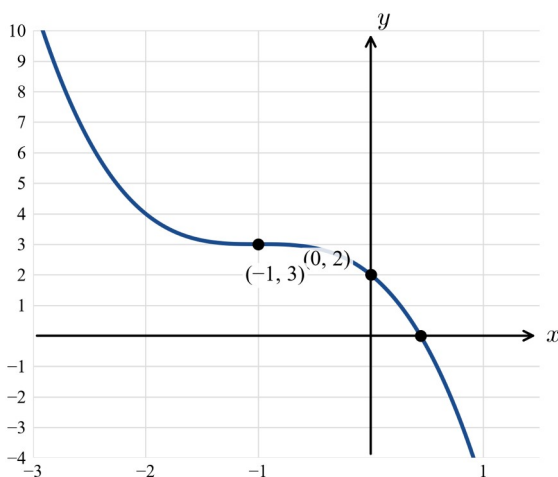
Q1. $y = 2x^3$. **Q2.** $y = -(x+1)^3 + 3$. **Q3.** $y = 2x(x+3)(x-2)$. **Q4.** $y = -2(x+1)^2(x-3)$. **Q5.** $y = a(x+2)(x-1)(x-4)$; member $y = -(x+2)(x-1)(x-4)$. **Q6.** $y = -x^3 + 7x$. **Q7.** $y = (x-1)^3 + 2$. **Q8.** $y = 5(x+2)^3$. **Q9.** $y = (x+1)(x-2)(x-3)$. **Q10.** $y = -2(x+4)(x+1)(x-2)$. **Q11.** $y = 2(x-1)^2(x+2)$. **Q12.** $y = (x+3)^2(x-1)$. **Q13.** $y = x^3 - 2x^2 + x + 1$. **Q14.** $y = a(x-2)^3 - 1$; member $y = (x-2)^3 - 1$. **Q15.** $y = ax(x-3)(x+2)$; member $y = -2x(x-3)(x+2)$. **Q16.** $y = 3x(x-2)^2$. **Q17.** $y = x^3 - 4x$. **Q18.** $y = -x^3 + 4$; x -intercept $x = \sqrt[3]{4}$.

Fully-worked solutions

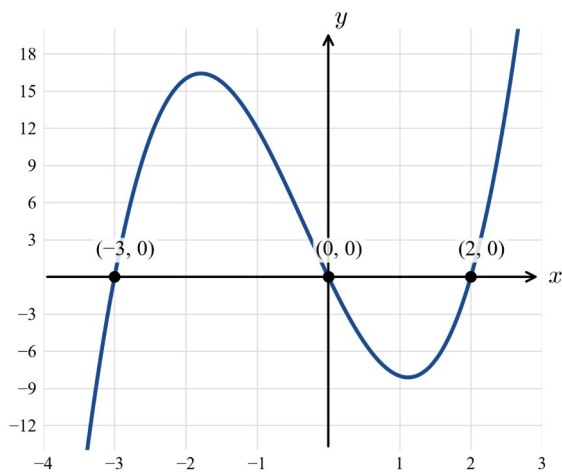
Q1. Inflection at $(0, 0)$: $y = ax^3$. Through $(1, 2)$: $2 = a(1)^3$, so $a = 2$. $\therefore y = 2x^3$.



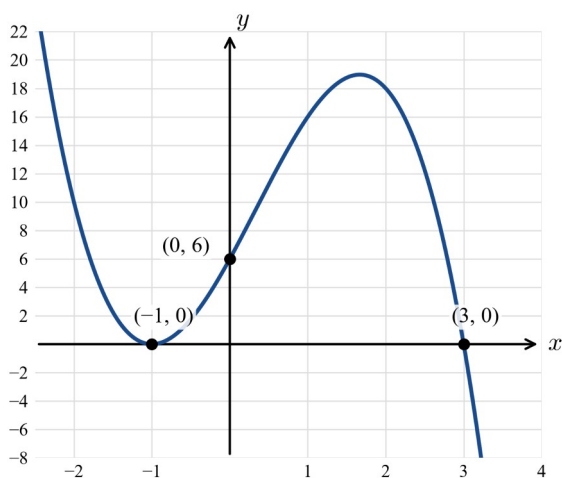
Q2. Inflection at $(-1, 3)$: $y = a(x+1)^3 + 3$. Through $(1, -5)$: $-5 = a(2)^3 + 3 = 8a + 3$, so $a = -1$. $\therefore y = -(x+1)^3 + 3$.



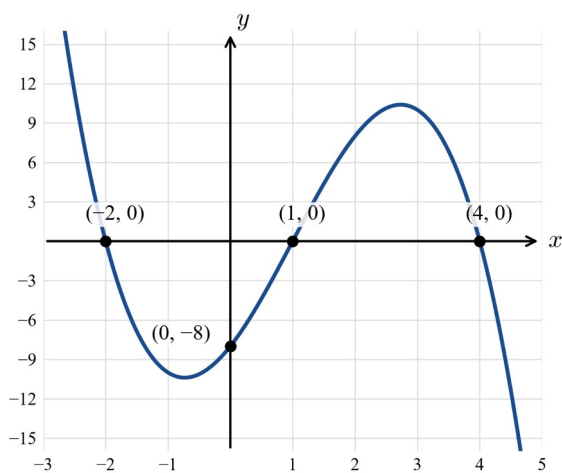
Q3. Intercepts $-3, 0, 2$: $y = ax(x+3)(x-2)$. Through $(1, -8)$: $-8 = a(1)(4)(-1) = -4a$, so $a = 2$. $\therefore y = 2x(x+3)(x-2)$.



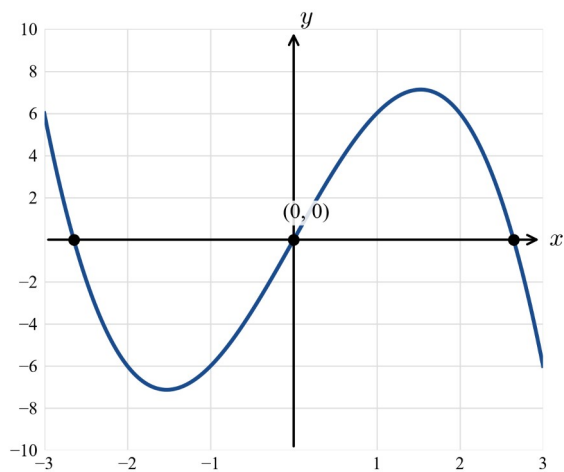
Q4. Touch at -1 , cross at 3 : $y = a(x+1)^2(x-3)$. y -intercept 6 : $6 = a(1)(-3) = -3a$, so $a = -2$.
 $\therefore y = -2(x+1)^2(x-3)$.



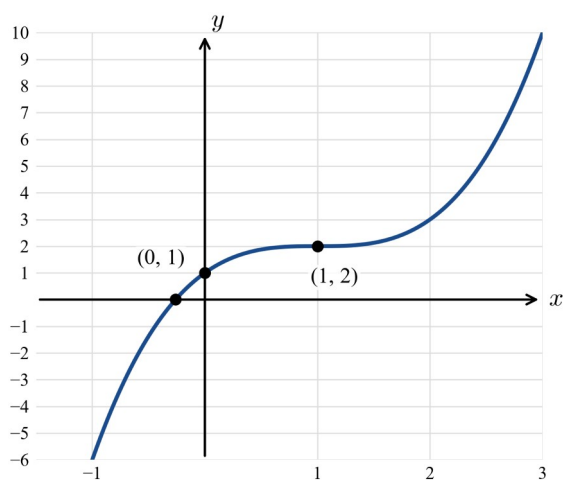
Q5. (a) Intercepts $-2, 1, 4$ give the family $y = a(x+2)(x-1)(x-4)$. (b) Through $(2, 8)$:
 $8 = a(4)(1)(-2) = -8a$, so $a = -1$. $\therefore y = -(x+2)(x-1)(x-4)$.



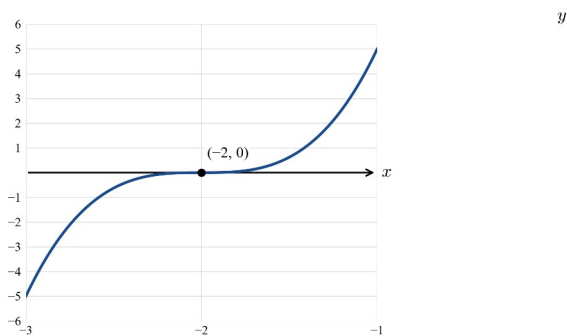
Q6. $(0, 0) \Rightarrow d = 0$. Then $a + b + c = 6$, $-a + b - c = -6$ and $8a + 4b + 2c = 6$. Adding the first two: $2b = 0$, so $b = 0$; then $a + c = 6$ and $4a + c = 3$, giving $a = -1$, $c = 7$. $\therefore y = -x^3 + 7x$.



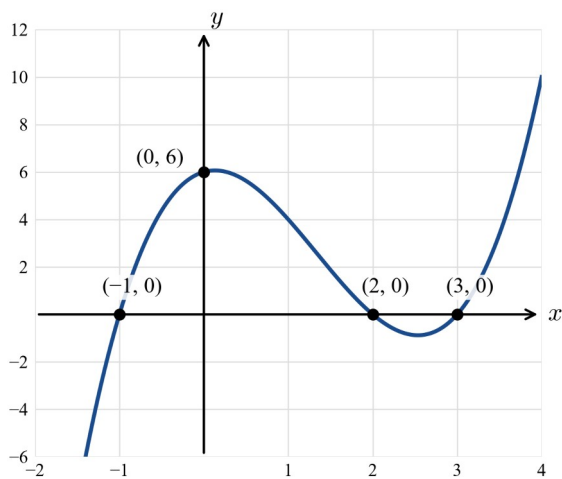
Q7. $y = a(x-1)^3 + 2$. Through $(0, 1)$: $1 = a(-1)^3 + 2 = -a + 2$, so $a = 1$. $\therefore y = (x-1)^3 + 2$.



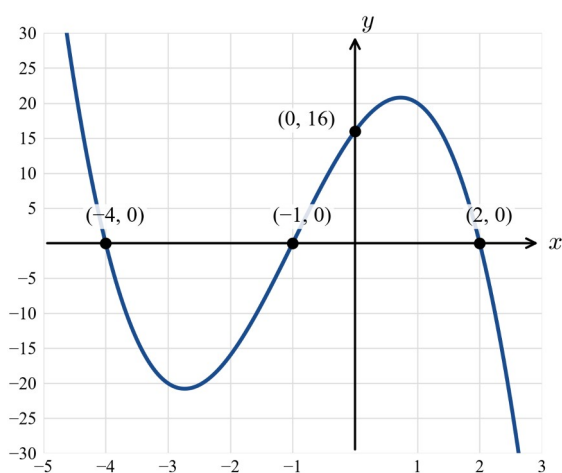
Q8. $y = a(x+2)^3$. Through $(-1, 5)$: $5 = a(1)^3$, so $a = 5$. $\therefore y = 5(x+2)^3$.



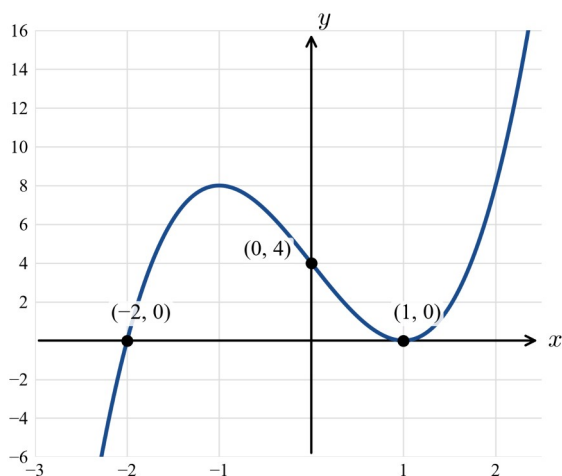
Q9. $y = a(x+1)(x-2)(x-3)$. y-intercept 6: $6 = a(1)(-2)(-3) = 6a$, so $a = 1$. $\therefore y = (x+1)(x-2)(x-3)$.



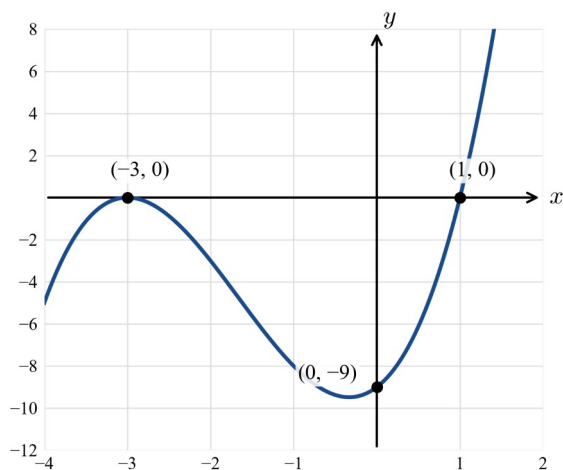
Q10. $y = a(x+4)(x+1)(x-2)$. Through $(1, 20)$: $20 = a(5)(2)(-1) = -10a$, so $a = -2$.
 $\therefore y = -2(x+4)(x+1)(x-2)$.



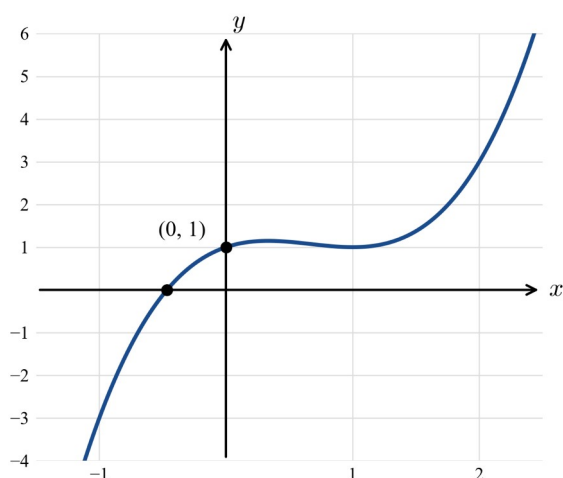
Q11. $y = a(x-1)^2(x+2)$. Through $(0, 4)$: $4 = a(1)(2) = 2a$, so $a = 2$. $\therefore y = 2(x-1)^2(x+2)$.



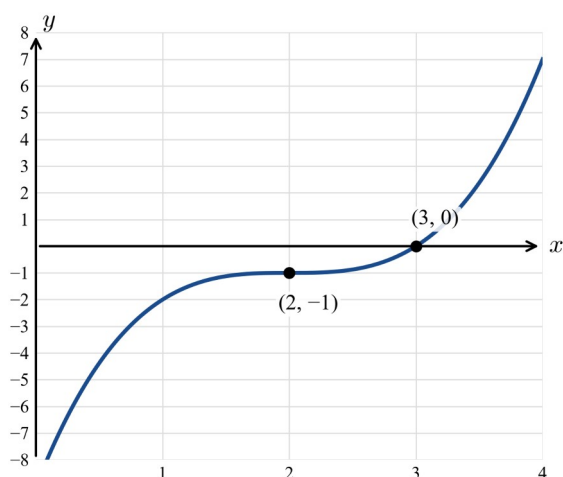
Q12. $y = a(x+3)^2(x-1)$. y-intercept -9 : $-9 = a(9)(-1) = -9a$, so $a = 1$. $\therefore y = (x+3)^2(x-1)$.



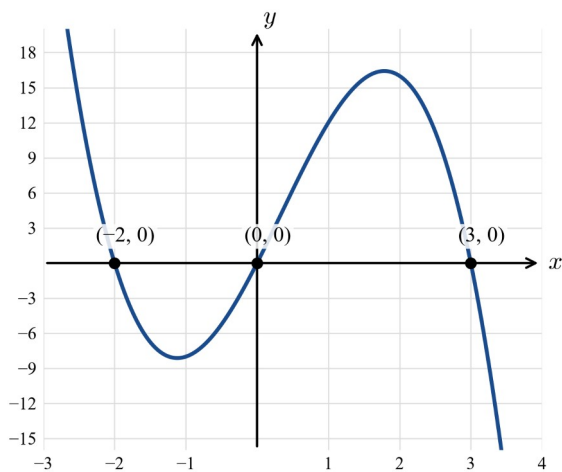
Q13. $(0, 1) \Rightarrow d = 1$. Then $a + b + c = 0$, $8a + 4b + 2c = 2$ and $-a + b - c = -4$. Adding the first and third: $2b = -4$, so $b = -2$; then $a + c = 2$ and $4a + c = 3$, giving $a = 1$, $c = 1$. $\therefore y = x^3 - 2x^2 + x + 1$.



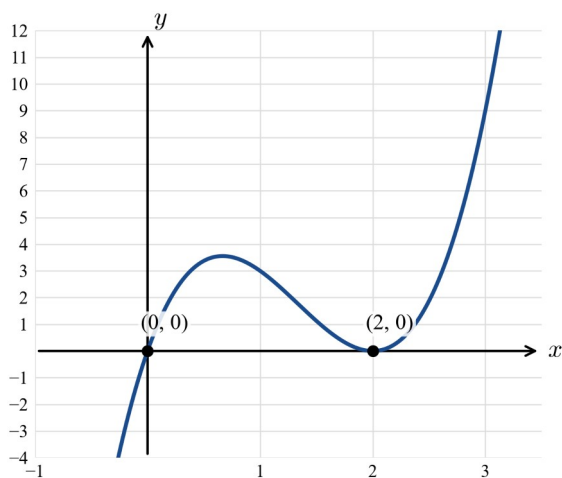
Q14. (a) A stationary point of inflection at $(2, -1)$ gives the family $y = a(x - 2)^3 - 1$. (b) Through $(4, 7)$: $7 = a(2)^3 - 1 = 8a - 1$, so $a = 1$. $\therefore y = (x - 2)^3 - 1$.



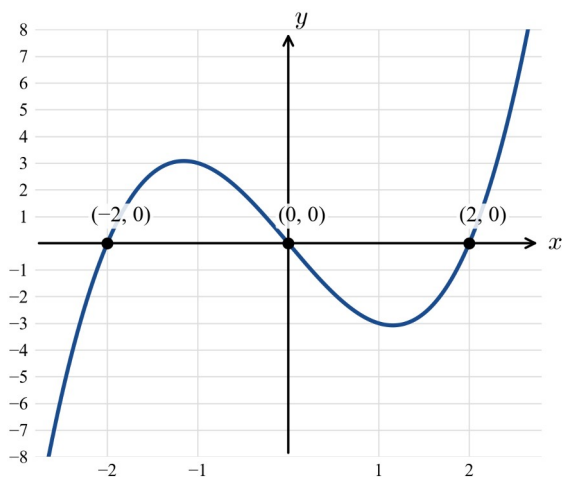
Q15. (a) Intercepts $0, 3, -2$ give the family $y = ax(x - 3)(x + 2)$. (b) Through $(1, 12)$: $12 = a(1)(-2)(3) = -6a$, so $a = -2$. $\therefore y = -2x(x - 3)(x + 2)$.



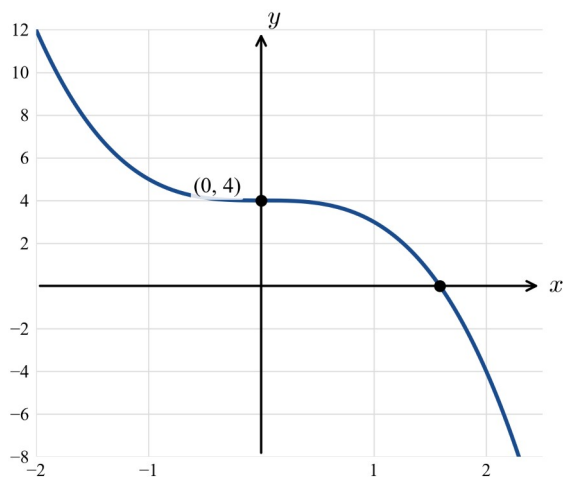
Q16. Touch at 2, cross at 0: $y = ax(x-2)^2$. Through $(1, 3)$: $3 = a(1)(1) = a$, so $a = 3$. $\therefore y = 3x(x-2)^2$.



Q17. $(0, 0) \Rightarrow d = 0$. Then $a + b + c = -3$, $8a + 4b + 2c = 0$ and $27a + 9b + 3c = 15$, i.e. $4a + 2b + c = 0$ and $9a + 3b + c = 5$. Subtracting in turn gives $3a + b = 3$ and $5a + b = 5$, so $a = 1$, $b = 0$, $c = -4$. $\therefore y = x^3 - 4x$.



Q18. Inflection at $(0, 4)$: $y = ax^3 + 4$. Through $(2, -4)$: $-4 = 8a + 4$, so $a = -1$, giving $y = -x^3 + 4$. For the x -intercept, $-x^3 + 4 = 0 \Rightarrow x^3 = 4 \Rightarrow x = \sqrt[3]{4}$.



Chapter 6 Polynomial functions — 6I Quartics and higher-degree polynomials

Theory

A **quartic** is a polynomial of degree 4. This section shows how to sketch quartics and how the ideas extend to polynomials of any degree.

The basic quartic. The graph of $y = x^4$ is symmetric about the y -axis with a minimum turning point at the origin. Near the origin it is flatter than $y = x^2$, but for $|x| > 1$ it rises much more steeply.

Turning-point form. $y = a(x - h)^4 + k$ has its turning point at (h, k) : a minimum if $a > 0$, a maximum if $a < 0$. $|a|$ dilates the graph from the x -axis.

Factorised form and multiplicity. If a quartic is written as a product of linear factors, the x -intercepts are the values that make each factor zero. The **multiplicity** of a factor controls the behaviour there:

- factor to the power **1** (e.g. $(x - p)$): the graph **crosses** the x -axis straight through;
- power **2** (e.g. $(x - p)^2$): the graph **touches** the axis (a turning point on the axis);
- power **3** (e.g. $(x - p)^3$): the graph crosses with a **stationary point of inflection** (it flattens as it crosses);
- power **4**: the graph touches, even flatter.

End behaviour. For large $|x|$ the leading term dominates.

- **Even degree** (e.g. quartic): both ends point the **same** way — both up if the leading coefficient is positive, both down if negative.
- **Odd degree** (e.g. cubic, quintic): the ends point in **opposite** directions — down on the left and up on the right if the leading coefficient is positive.

Turning points. A polynomial of degree n has **at most** $n - 1$ turning points (a quartic has at most 3; a quintic at most 4).

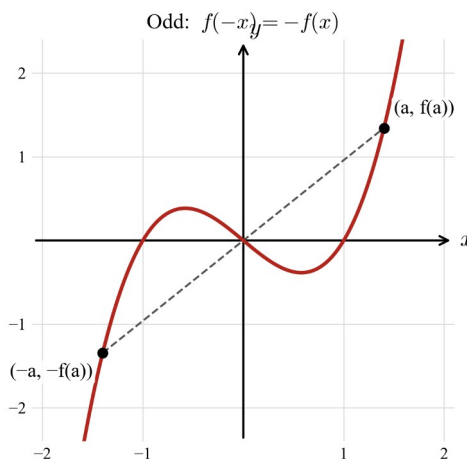
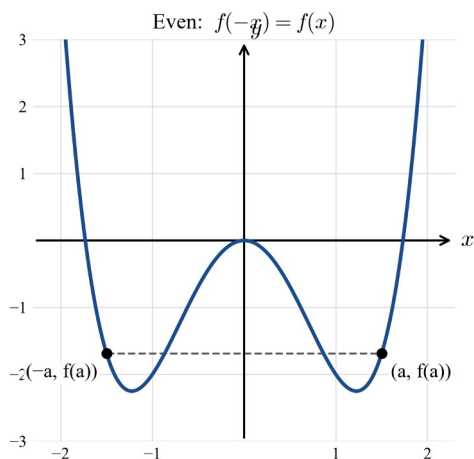
Even and odd functions (symmetry). Some graphs have a special symmetry that can be read from the rule alone.

- A function is **even** if $f(-x) = f(x)$ for every x . Its graph is **symmetric about the y -axis** (a left–right mirror image). Examples: x^2 , x^4 , $|x|$ and $\frac{1}{x^2}$.
- A function is **odd** if $f(-x) = -f(x)$ for every x . Its graph has **point symmetry about the origin** — a 180° rotation about $(0, 0)$ maps the graph onto itself. Examples: x , x^3 , x^5 and $\frac{1}{x}$.

The algebraic test. Substitute $-x$ into the rule and simplify, then compare with the original:

- if the result is identical to $f(x)$, then $f(-x) = f(x)$ and the function is **even**;
- if every term has changed sign, then $f(-x) = -f(x)$ and the function is **odd**;
- otherwise it is **neither**.

Most functions are neither even nor odd. A quick check for polynomials: the function is even when *every* term has an **even power**, and odd when *every* term has an **odd power**; a mix of even and odd powers gives neither (a non-zero constant term counts as an even power, since $c = cx^0$).



Do not confuse an even/odd **function** with an even/odd **degree**: the *degree* is just the highest power and fixes the end behaviour above, while *even/odd function* names a symmetry of the whole graph. For example $y = x^3 - x$ has odd **degree** **and** is an odd function, but $y = x^2 + x$ has even degree yet is neither even nor odd.

To sketch a factorised polynomial: decide the end behaviour from the degree and leading coefficient; find the x -intercepts and their multiplicity (cross / touch / inflect); find the y -intercept; then draw a smooth curve consistent with these features.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y = (x - 1)^4 - 16$.

Working

$a = 1 > 0$, so this is a quartic with a **minimum** turning point.

Turning point $(1, -16)$.

y -intercept: $x = 0 \Rightarrow y = (-1)^4 - 16 = 1 - 16 = -15$, giving $(0, -15)$.

x -intercepts: $(x - 1)^4 = 16 \Rightarrow (x - 1)^2 = 4 \Rightarrow x - 1 = \pm 2$, so $x = 3$ or $x = -1$; points $(3, 0)$ and $(-1, 0)$.

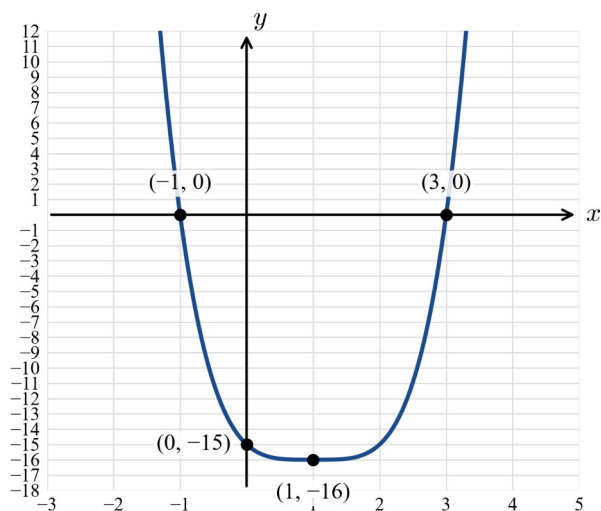
Comment

Turning-point form $y = a(x - h)^4 + k$.

Read (h, k) from the rule.

Substitute $x = 0$.

Take the positive square root first (a square cannot be negative), then solve.



Example 2 — scaffolded

Sketch the graph of $y = (x + 1)(x - 2)^2(x - 3)$.

Working

Comment

Degree 4, positive leading coefficient, so **both ends point up**.

Even degree, leading coefficient +1.

x -intercepts: $x = -1$ (**crosses**), $x = 2$ (**touches**, factor squared), $x = 3$ (**crosses**).

Read the factors and their multiplicities.

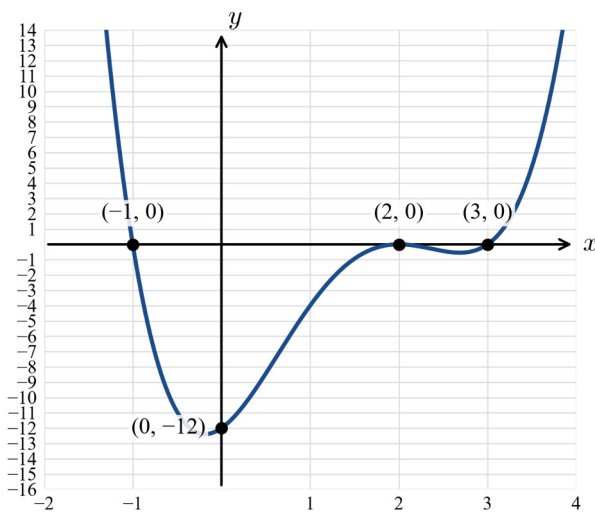
y -intercept:

Substitute $x = 0$.

$x = 0 \Rightarrow y = (1)(-2)^2(-3) = (1)(4)(-3) = -12$,
giving $(0, -12)$.

Draw a smooth curve: down from top left, up through $(-1, 0)$, touching at $(2, 0)$, then up through $(3, 0)$.

Consistent with the intercepts and end behaviour.



Example 3 — unscaffolded

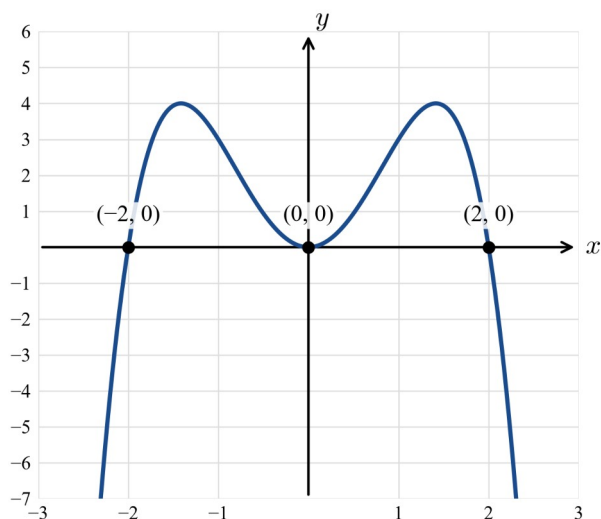
Sketch the graph of $y = -x^4 + 4x^2$.

Factorise: $y = -x^2(x^2 - 4) = -x^2(x - 2)(x + 2)$.

Leading coefficient -1 , even degree, so **both ends point down**.

x -intercepts: $x = 0$ (**touches**, factor x^2) and $x = 2$, $x = -2$ (**cross**). y -intercept $(0, 0)$.

$\therefore$ the graph rises from bottom left, crosses at $(-2, 0)$, touches the origin, crosses at $(2, 0)$, and falls to bottom right.

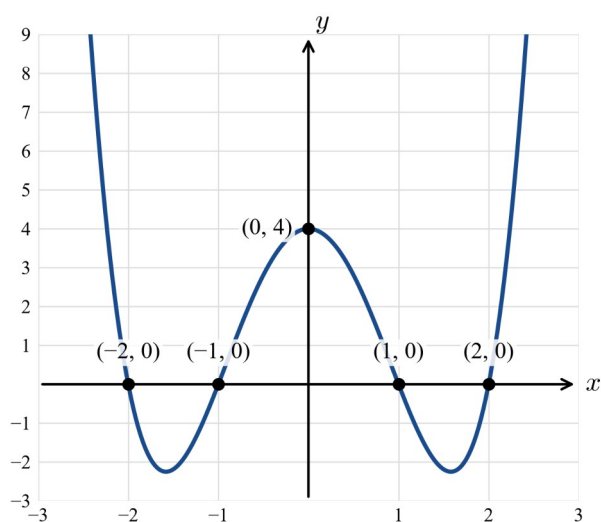


Example 4 — unscaffolded

Sketch the graph of $y = x^4 - 5x^2 + 4$.

Factorise: $y = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$.

Leading coefficient +1, even degree: **both ends up**. x -intercepts $x = \pm 1, \pm 2$ (all crossing). y -intercept $(0, 4)$.



Practice questions

Scaffolded

Q1. For $y = x^4 - 1$: (a) state the shape (min/max) and turning point; (b) find the y -intercept; (c) find the x -intercepts; (d) sketch.

Q2. For $y = (x - 2)^4$: (a) state the turning point; (b) explain why the graph touches the x -axis at exactly one point; (c) find the y -intercept; (d) sketch.

Q3. For $y = (x + 2)(x - 1)(x - 2)(x - 3)$: (a) state the end behaviour; (b) find the x -intercepts and say whether each is a cross or a touch; (c) find the y -intercept; (d) sketch.

Q4. For $y = x^2(x - 3)^2$: (a) state the end behaviour; (b) find the x -intercepts and their type; (c) find the y -intercept; (d) sketch.

Q5. For $y = -(x + 1)^4 + 16$: (a) state the shape and turning point; (b) find all intercepts; (c) sketch.

Q6. For $y = (x - 1)^3(x + 2)$: (a) state the end behaviour; (b) describe the behaviour at $x = 1$ and at $x = -2$; (c) find the y -intercept; (d) sketch.

Unscaffolded

Sketch each of the following, labelling the axis intercepts (with cross/touch behaviour) and the y -intercept.

Q7. $y = x^4 - 4x^2$

Q8. $y = -(x - 1)(x + 1)(x - 2)(x + 2)$

Q9. $y = (x + 2)^2(x - 2)^2$

Q10. $y = (x - 1)^3(x - 3)$

Q11. $y = 2(x - 1)^2(x + 2)^2$

Q12. $y = -2(x - 1)^4 + 2$

Q13. $y = -3x^4 + 2x^2 - 5$: describe the end behaviour, state the maximum possible number of turning points, and explain why the graph has no x -intercepts. Then sketch.

Q14. $y = x^5 - 5x^3 + 4x$ (a quintic): factorise, state the end behaviour, and sketch.

Q15. $y = (x - 1)^4 + 2$

Q16. $y = x^3(x - 2)$

Q17. A quartic has x -intercepts at -2 , 0 , 1 and 3 and passes through the point $(2, -8)$. Find its rule and sketch it.

Q18. $y = (x^2 - 1)(x^2 - 9)$

Classify each of the following as an **even function**, an **odd function**, or **neither**, using the algebraic test (compute $f(-x)$ and compare with $f(x)$ and $-f(x)$).

Q19. $f(x) = x^4 - 3x^2$

Q20. $g(x) = x^3 - x$

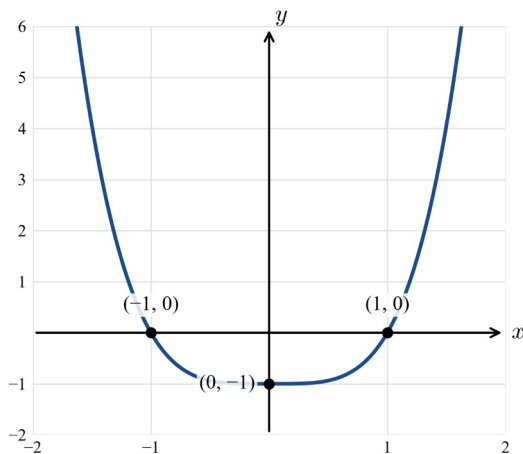
Q21. $h(x) = x^2 + x$

Answers

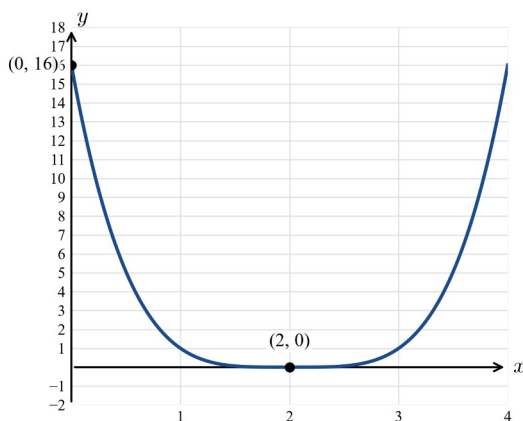
Q1. Min; TP $(0, -1)$; y-int $(0, -1)$; x-int $(\pm 1, 0)$. **Q2.** TP $(2, 0)$; touches at $(2, 0)$ (factor to the power 4); y-int $(0, 16)$. **Q3.** Both ends up; x-int $(-2, 0), (1, 0), (2, 0), (3, 0)$ all crossing; y-int $(0, -12)$. **Q4.** Both ends up; x-int $(0, 0)$ and $(3, 0)$, both touches; y-int $(0, 0)$. **Q5.** Max; TP $(-1, 16)$; x-int $(1, 0), (-3, 0)$; y-int $(0, 15)$. **Q6.** Both ends up; crosses with a stationary point of inflection at $x=1$, crosses at $x=-2$; y-int $(0, -2)$. **Q7.** Both ends up; x-int $(0, 0)$ touch, $(\pm 2, 0)$ cross; y-int $(0, 0)$. **Q8.** Both ends down; x-int $(\pm 1, 0), (\pm 2, 0)$ all cross; y-int $(0, -4)$. **Q9.** Both ends up; x-int $(\pm 2, 0)$, both touches; y-int $(0, 16)$. **Q10.** Both ends up; inflection-cross at $(1, 0)$, cross at $(3, 0)$; y-int $(0, 3)$. **Q11.** Both ends up; x-int $(1, 0), (-2, 0)$ both touches; y-int $(0, 8)$. **Q12.** Max; TP $(1, 2)$; x-int $(0, 0), (2, 0)$; y-int $(0, 0)$. **Q13.** Even degree, negative leading coefficient $\rightarrow$ both ends down; at most 3 turning points; the maximum value is $-14/3 < 0$ so the graph lies entirely below the x-axis (no x-intercepts); y-int $(0, -5)$. **Q14.** $y = x(x-1)(x+1)(x-2)(x+2)$; odd degree, positive leading coefficient $\rightarrow$ down on the left, up on the right; x-int $(0, 0), (\pm 1, 0), (\pm 2, 0)$. **Q15.** Min; TP $(1, 2)$; no x-intercepts (graph lies above the axis); y-int $(0, 3)$. **Q16.** Both ends up; inflection-cross at $(0, 0)$, cross at $(2, 0)$; y-int $(0, 0)$. **Q17.** $y = x(x+2)(x-1)(x-3)$; x-int $(-2, 0), (0, 0), (1, 0), (3, 0)$. **Q18.** $y = (x-1)(x+1)(x-3)(x+3)$; both ends up; x-int $(\pm 1, 0), (\pm 3, 0)$; y-int $(0, 9)$. **Q19.** Even. **Q20.** Odd. **Q21.** Neither.

Fully-worked solutions

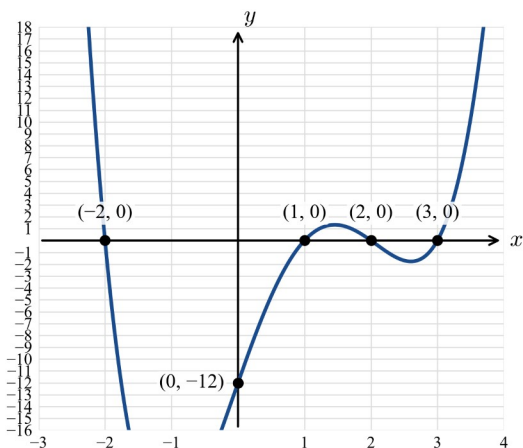
Q1. $y = x^4 - 1$. (a) $a = 1 > 0$: **minimum** at $(0, -1)$. (b) y-int $(0, -1)$. (c) $x^4 = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$, giving $(\pm 1, 0)$.



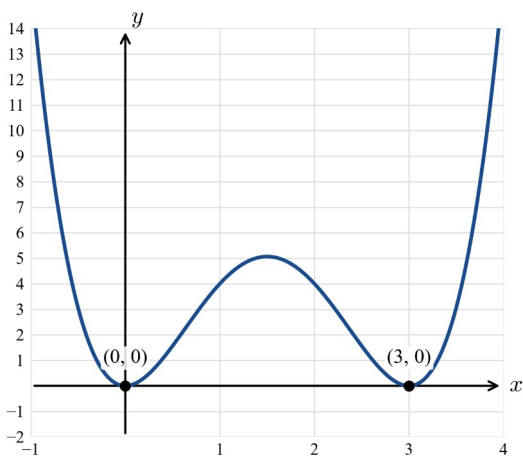
Q2. $y = (x - 2)^4$. (a) Minimum turning point $(2, 0)$. (b) $y = (x - 2)^4 = 0$ only when $x = 2$ (a single repeated solution), so the graph **touches** the axis there. (c) y-int $(0, 16)$.



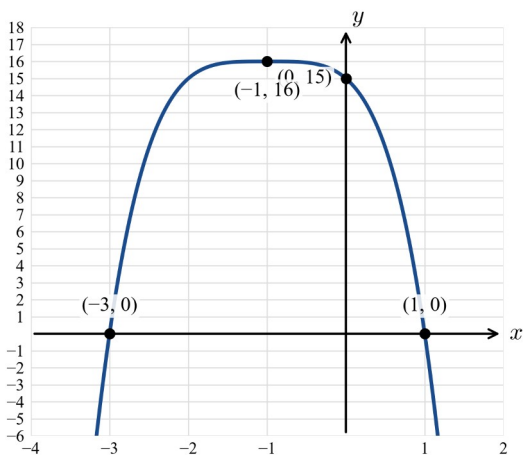
Q3. $y = (x + 2)(x - 1)(x - 2)(x - 3)$. (a) Degree 4, leading coefficient +1: **both ends up**. (b) $x = -2, 1, 2, 3$, each a single factor, so all **cross**. (c) $y = (2)(-1)(-2)(-3) = -12$, giving $(0, -12)$.



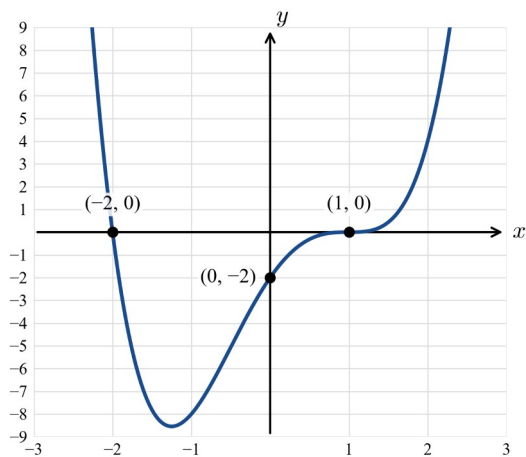
Q4. $y = x^2(x-3)^2$. (a) Both ends up. (b) $x=0$ and $x=3$, each squared, so both are **touches** (turning points on the axis). (c) y-int $(0, 0)$.



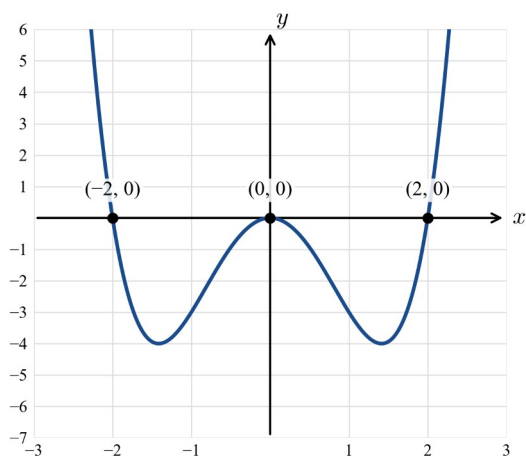
Q5. $y = -(x+1)^4 + 16$. (a) $a = -1 < 0$: **maximum** at $(-1, 16)$. (b) y-int $x=0 \Rightarrow y = -(1)^4 + 16 = 15$, giving $(0, 15)$; x-int $(x+1)^4 = 16 \Rightarrow (x+1)^2 = 4 \Rightarrow x+1 = \pm 2$, so $x=1$ or $x=-3$.



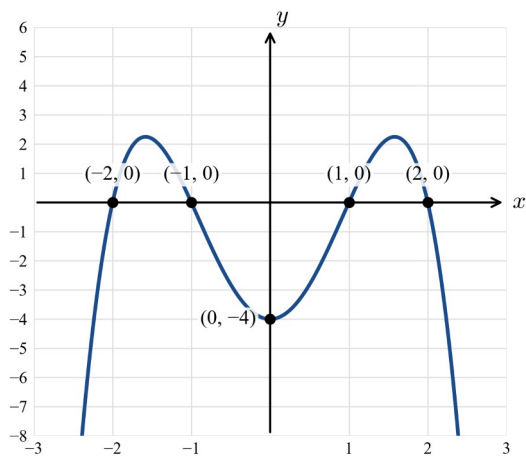
Q6. $y = (x-1)^3(x+2)$. (a) Degree 4, leading coefficient $+1$: **both ends up**. (b) At $x=1$ (factor cubed) the graph crosses with a **stationary point of inflection**; at $x=-2$ (single factor) it **crosses**. (c) $y = (-1)^3(2) = -2$, giving $(0, -2)$.



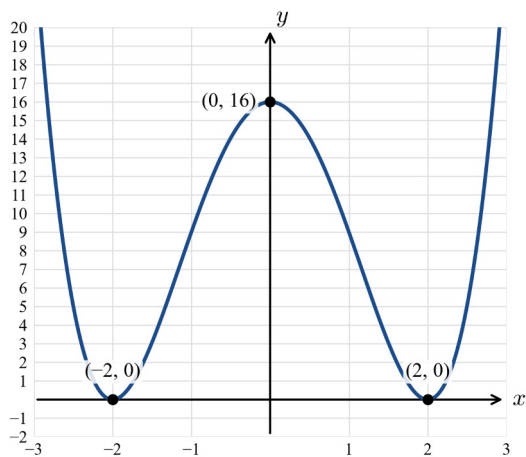
Q7. $y = x^4 - 4x^2 = x^2(x - 2)(x + 2)$. Both ends up. $x = 0$ **touches** (factor x^2), $x = \pm 2$ **cross**. y -int $(0, 0)$.



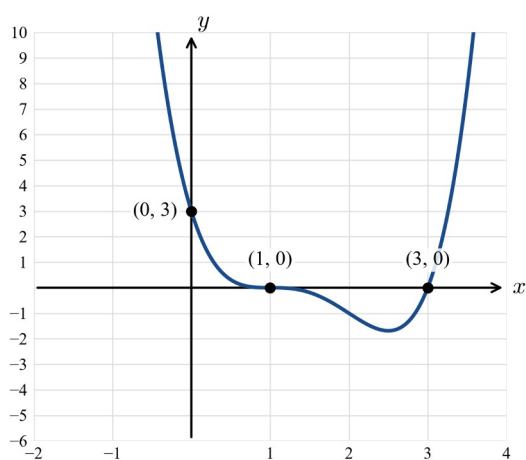
Q8. $y = -(x - 1)(x + 1)(x - 2)(x + 2)$. Leading coefficient -1 , even degree: **both ends down**. $x = \pm 1, \pm 2$ all **cross**. y -int $y = -(-1)(1)(-2)(2) = -4$, giving $(0, -4)$.



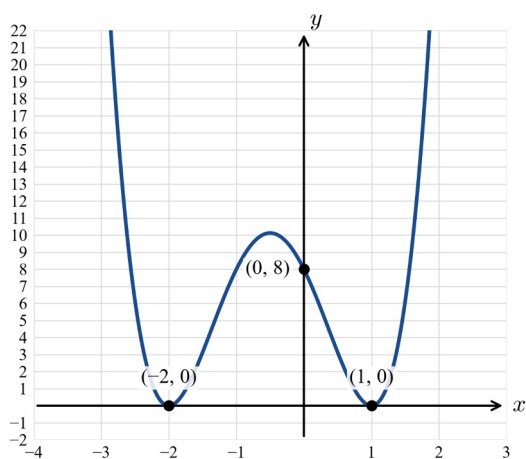
Q9. $y = (x + 2)^2(x - 2)^2$. Both ends up. $x = -2$ and $x = 2$, each squared, so both **touch**. y -int $(2)^2(-2)^2 = 16$, giving $(0, 16)$.



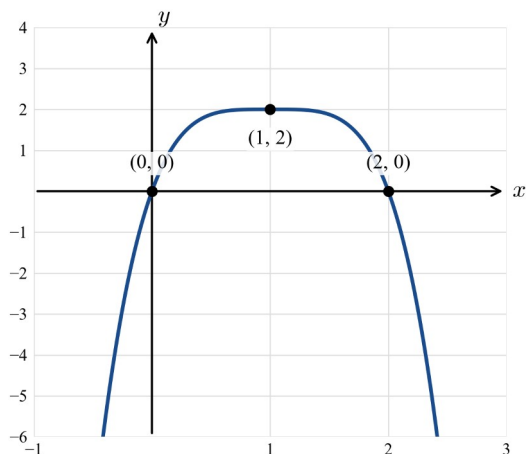
Q10. $y = (x - 1)^3(x - 3)$. Both ends up. At $x = 1$ (cubed) it crosses with a **stationary inflection**; at $x = 3$ it **crosses**. y -int $(-1)^3(-3) = 3$, giving $(0, 3)$.



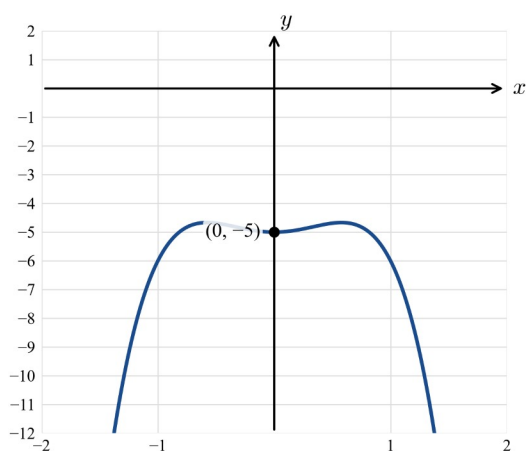
Q11. $y = 2(x - 1)^2(x + 2)^2$. Both ends up. $x = 1$ and $x = -2$ both **touch**. y -int $2(1)(4) = 8$, giving $(0, 8)$.



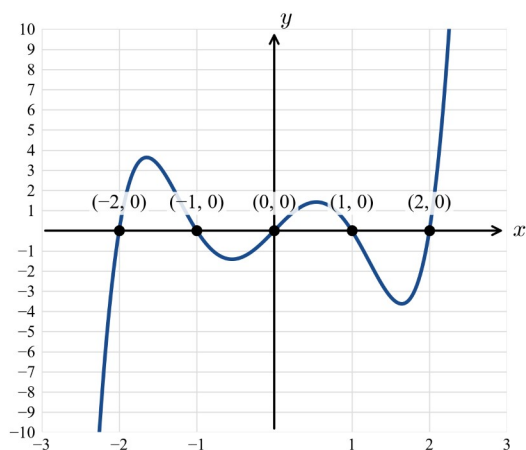
Q12. $y = -2(x - 1)^4 + 2$. $a = -2 < 0$: **maximum** at $(1, 2)$. x -int $-2(x - 1)^4 + 2 = 0 \Rightarrow (x - 1)^4 = 1 \Rightarrow (x - 1)^2 = 1 \Rightarrow x = 0$ or $x = 2$. y -int $-2(1) + 2 = 0$, giving $(0, 0)$.



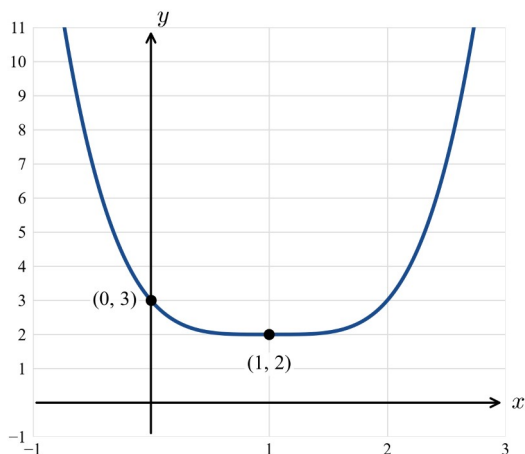
Q13. $y = -3x^4 + 2x^2 - 5$. Even degree with a negative leading coefficient, so **both ends point down**; a quartic has **at most 3 turning points**. The maximum value of the graph is $-14/3$ (at $x = \pm\sqrt{3}/3$), which is negative, so the whole graph lies **below the x-axis** and there are **no x-intercepts**. y-int $(0, -5)$.



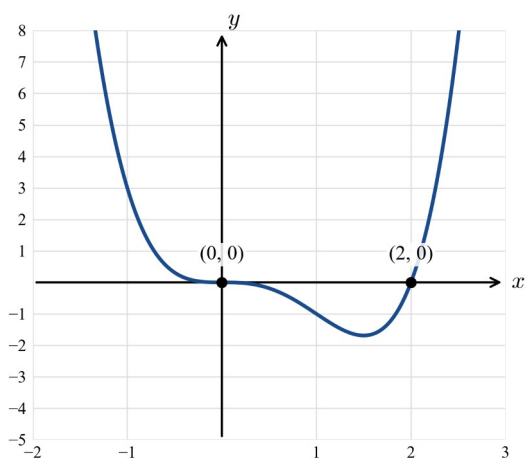
Q14. $y = x^5 - 5x^3 + 4x = x(x^4 - 5x^2 + 4) = x(x-1)(x+1)(x-2)(x+2)$. Degree 5 (odd) with a positive leading coefficient, so the graph goes **down on the left and up on the right**, crossing at $x = 0, \pm 1, \pm 2$.



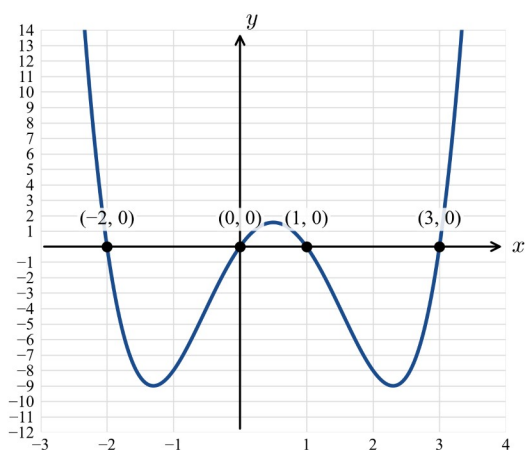
Q15. $y = (x-1)^4 + 2$. $a = 1 > 0$: **minimum** at $(1, 2)$. Since the minimum value is $2 > 0$, the graph lies **above the x-axis** — no x-intercepts. y-int $(0, 3)$.



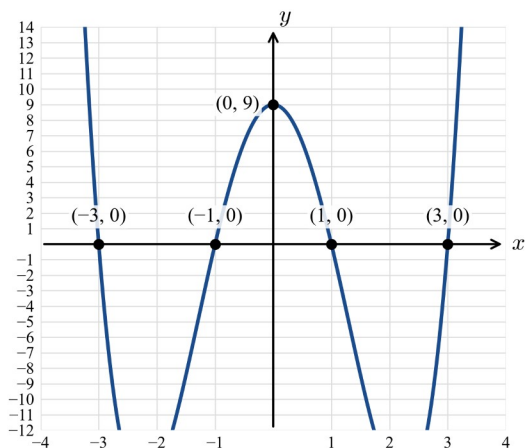
Q16. $y = x^3(x - 2)$. Degree 4, both ends up. At $x=0$ (factor cubed) it crosses with a **stationary inflection**; at $x=2$ it **crosses**. y-int $(0, 0)$.



Q17. The four x -intercepts give factors x , $(x+2)$, $(x-1)$, $(x-3)$, so $y = ax(x+2)(x-1)(x-3)$. Substituting $(2, -8)$: $-8 = a(2)(4)(1)(-1) = -8a \Rightarrow a = 1$. $\therefore y = x(x+2)(x-1)(x-3)$.



Q18. $y = (x^2 - 1)(x^2 - 9) = (x - 1)(x + 1)(x - 3)(x + 3)$. Both ends up; crosses at $x = \pm 1, \pm 3$. y-int $(-1)(-9) = 9$, giving $(0, 9)$.



Q19. $f(x) = x^4 - 3x^2$. Substitute $-x$: $f(-x) = (-x)^4 - 3(-x)^2 = x^4 - 3x^2 = f(x)$. Since $f(-x) = f(x)$, f is **even** (its graph is symmetric about the y -axis). Every term has an even power, confirming the result.

Q20. $g(x) = x^3 - x$. Substitute $-x$: $g(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -g(x)$. Since $g(-x) = -g(x)$, g is **odd** (its graph has point symmetry about the origin). Every term has an odd power, confirming the result.

Q21. $h(x) = x^2 + x$. Substitute $-x$: $h(-x) = (-x)^2 + (-x) = x^2 - x$. This is not equal to $h(x) = x^2 + x$ (the x -term differs), so h is **not even**; and it is not equal to $-h(x) = -x^2 - x$ (the x^2 -term differs), so h is **not odd**. Therefore h is **neither** — its rule mixes an even power (x^2) with an odd power (x).

Chapter 6 Polynomial functions — 6J Polynomial modelling

Theory

Many practical quantities are modelled by **polynomial functions** — especially **cubics**, which arise whenever a volume depends on a single length.

The open-box model. If squares of side x are cut from each corner of a rectangular sheet a cm by b cm and the sides are folded up, the base measures $(a - 2x)$ by $(b - 2x)$ and the height is x , so the volume is

$$V(x) = x(a - 2x)(b - 2x).$$

For a square sheet ($a = b$) this becomes $V(x) = x(a - 2x)^2$.

Sensible domain. Both side lengths must be positive, so $0 < x < \frac{1}{2} \min(a, b)$. Only values of x in this domain describe a real box.

Working with the model:

- **Evaluate** $V(x)$ at a value of x to find a particular volume.
- **Maximum volume:** the greatest volume occurs at the turning point inside the domain (find it with calculus or a CAS). Reject any turning point outside the domain.
- **Solve** $V(x) = k$ to find the value(s) of x giving a required volume — this is a cubic equation. **Reject** solutions outside the sensible domain.
- Always **state units** and interpret the answer in context.

Other quantities (profit, strength, geometric measures) are modelled the same way: form the rule, restrict the domain to what is physically sensible, then evaluate, optimise or solve.

Worked examples

Example 1 — scaffolded

An open box is made from a square sheet of card 12 cm by 12 cm by cutting a square of side x cm from each corner and folding up the sides.

Working

Comment

The base is $(12 - 2x)$ by $(12 - 2x)$ and the height is x ,
so $V(x) = x(12 - 2x)^2$.

Two corners are removed from each side length.

Sensible domain: $0 < x < 6$.

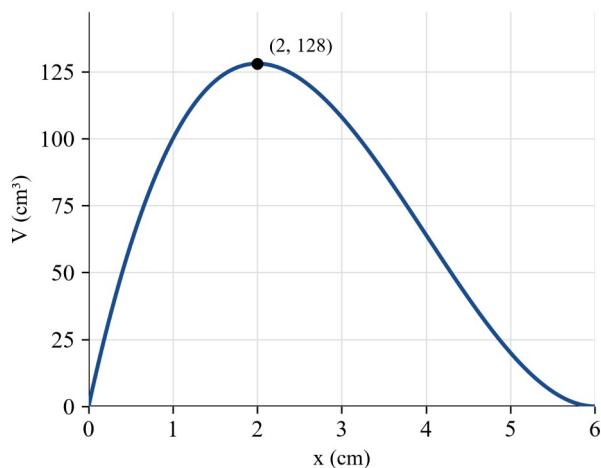
Need $12 - 2x > 0$ and $x > 0$.

$V(2) = 2(12 - 4)^2 = 2 \times 64 = 128$.

Substitute $x = 2$.

The box of height 2 cm has volume 128 cm³.

State units.



Example 2 — scaffolded

An open box is made from a rectangular sheet 16 cm by 10 cm in the same way.

Working

$$V(x) = x(16 - 2x)(10 - 2x).$$

Sensible domain: $0 < x < 5$.

$$V(2) = 2(12)(6) = 144.$$

The volume is 144 cm^3 when $x = 2 \text{ cm}$.

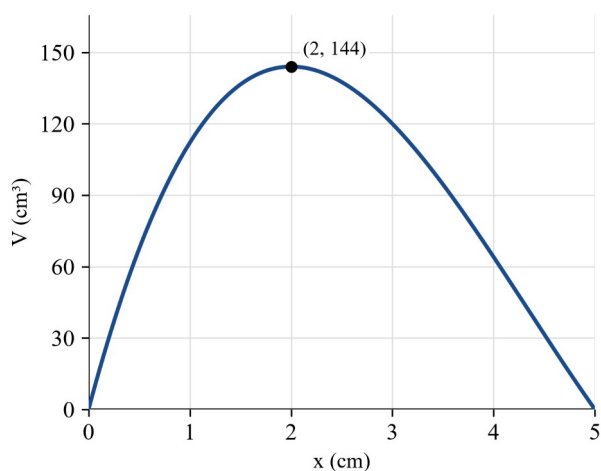
Comment

Base $(16 - 2x)$ by $(10 - 2x)$, height x .

The smaller side gives $10 - 2x > 0$.

Substitute $x = 2$.

Interpret with units.



Example 3 — unscaffolded

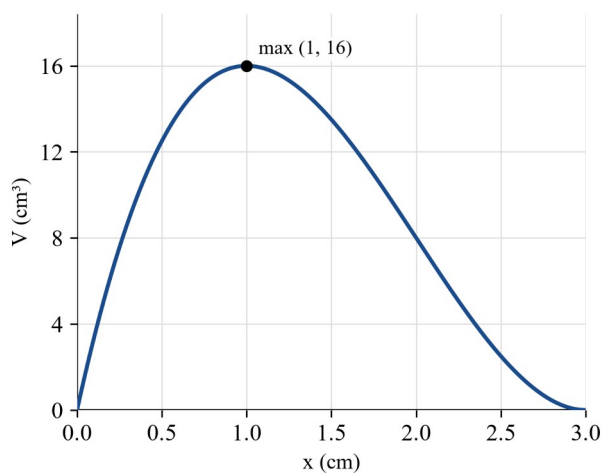
An open box is made from a square sheet 6 cm by 6 cm. Find the value of x that gives the maximum volume, and state that maximum volume.

$$V(x) = x(6 - 2x)^2 = 4x^3 - 24x^2 + 36x, \text{ with domain } 0 < x < 3.$$

$$V'(x) = 12x^2 - 48x + 36 = 12(x - 1)(x - 3), \text{ so } V'(x) = 0 \text{ at } x = 1 \text{ or } x = 3.$$

$$\text{Only } x = 1 \text{ lies in the domain. } V(1) = 1(6 - 2)^2 = 16.$$

$\therefore$ the maximum volume is 16 cm^3 , when $x = 1 \text{ cm}$.



Example 4 — unscaffolded

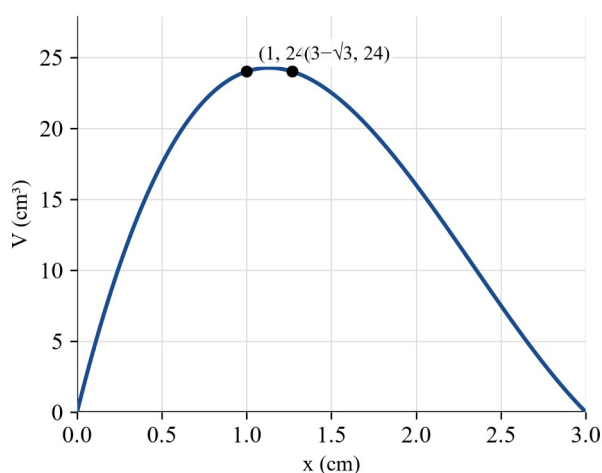
An open box is made from an 8 cm by 6 cm sheet. Find the value(s) of x for which the volume is 24 cm^3 .

$V(x) = x(8 - 2x)(6 - 2x) = 4x^3 - 28x^2 + 48x$, domain $0 < x < 3$. Solve $V(x) = 24$:

$$4x^3 - 28x^2 + 48x - 24 = 0 \Rightarrow x^3 - 7x^2 + 12x - 6 = 0 \Rightarrow (x - 1)(x^2 - 6x + 6) = 0.$$

So $x = 1$ or $x = 3 \pm \sqrt{3}$. In the domain $0 < x < 3$: $x = 1$ and $x = 3 - \sqrt{3}$.

$\therefore$ the volume is 24 cm^3 when $x = 1 \text{ cm}$ or $x = 3 - \sqrt{3} \approx 1.27 \text{ cm}$.



Practice questions

Scaffolded

Q1. An open box is made from a 10 cm by 10 cm square sheet. (a) Write a rule for $V(x)$. (b) State the sensible domain. (c) Find $V(1)$. (d) Find $V(2)$.

Q2. An open box is made from a 12 cm by 8 cm sheet. (a) Write a rule for $V(x)$. (b) State the domain. (c) Find the volume when $x = 1 \text{ cm}$.

Q3. An open box is made from an 18 cm by 18 cm square sheet. (a) Write $V(x)$ and its domain. (b) Show $V'(x) = 0$ at $x = 3$ and $x = 9$. (c) Hence find the maximum volume.

Q4. An open box is made from a 20 cm by 12 cm sheet. (a) Write $V(x)$. (b) State the domain. (c) Find $V(2)$.

Q5. An open box is made from a 12 cm by 12 cm sheet, so $V(x) = x(12 - 2x)^2$. (a) Solve $V(x) = 128$. (b) Explain why one solution must be rejected.

Q6. An open box is made from a 10 cm by 6 cm sheet. (a) Write $V(x)$ and its domain. (b) Find $V(1)$. (c) Find $V(2)$.

Unscaffolded

Q7. A box is made from a 20 cm by 20 cm square sheet. Find $V(3)$, and find the maximum volume and the value of x at which it occurs.

Q8. Find the volume when $x = 4$ cm for a box made from a 24 cm by 18 cm sheet.

Q9. A box is made from a 10 cm by 8 cm sheet. Find the value(s) of x for which the volume is 48 cm^3 .

Q10. Find $V(2)$ for a box made from a 14 cm by 10 cm sheet.

Q11. A box is made from a 30 cm by 30 cm square sheet. Find the maximum volume and the value of x at which it occurs.

Q12. A box is made from a 24 cm by 24 cm square sheet. Find $V(3)$ and the maximum volume.

Q13. A box is made from a 16 cm by 12 cm sheet. State the sensible domain and find $V(2)$.

Q14. A box is made from an 8 cm by 8 cm square sheet. Find $V(1)$ and $V(2)$.

Q15. A rectangular garden bed of length x m has a strength model $P(x) = x(15 - x)^2$ ($0 < x < 15$). Find $P(5)$, and the value of x that maximises P .

Q16. A sphere has volume $V = \frac{4}{3}\pi r^3$. (a) Find the volume when $r = 3$ cm. (b) Find r if the volume is $288\pi \text{ cm}^3$.

Q17. A company's weekly profit is modelled by $P(x) = -x^3 + 9x^2 - 15x$ (thousands of dollars), where x is the number of thousands of items made. Find the production level that gives a local maximum profit, and state that profit.

Q18. A box is made from a 40 cm by 40 cm square sheet. Find the maximum volume and the value of x at which it occurs (give exact values).

Answers

Q1. $V(x) = x(10 - 2x)^2$; $0 < x < 5$; $V(1) = 64 \text{ cm}^3$; $V(2) = 72 \text{ cm}^3$. **Q2.** $V(x) = x(12 - 2x)(8 - 2x)$; $0 < x < 4$; $V(1) = 60 \text{ cm}^3$. **Q3.** $V(x) = x(18 - 2x)^2$, $0 < x < 9$; max 432 cm^3 at $x = 3 \text{ cm}$. **Q4.** $V(x) = x(20 - 2x)(12 - 2x)$; $0 < x < 6$; $V(2) = 256 \text{ cm}^3$. **Q5.** $x = 2$ or $x = 8$; reject $x = 8$ (outside $0 < x < 6$), so $x = 2 \text{ cm}$. **Q6.**
 $V(x) = x(10 - 2x)(6 - 2x)$, $0 < x < 3$; $V(1) = 32 \text{ cm}^3$; $V(2) = 24 \text{ cm}^3$. **Q7.** $V(3) = 588 \text{ cm}^3$; max $\frac{16000}{27} \approx 592.6$
 cm^3 at $x = \frac{10}{3} \text{ cm}$. **Q8.** $V(4) = 640 \text{ cm}^3$. **Q9.** $x = 1 \text{ cm}$ or $x = 2 \text{ cm}$. **Q10.** $V(2) = 120 \text{ cm}^3$. **Q11.** Max 2000 cm^3 at
 $x = 5 \text{ cm}$. **Q12.** $V(3) = 972 \text{ cm}^3$; max 1024 cm^3 at $x = 4 \text{ cm}$. **Q13.** $0 < x < 6$; $V(2) = 192 \text{ cm}^3$. **Q14.** $V(1) = 36 \text{ cm}^3$;
 $V(2) = 32 \text{ cm}^3$. **Q15.** $P(5) = 500$; maximum at $x = 5$. **Q16.** (a) $36\pi \text{ cm}^3$; (b) $r = 6 \text{ cm}$. **Q17.** Local maximum at $x = 5$
(5000 items), profit \$25 000. **Q18.** Max $\frac{128000}{27} \approx 4740.7 \text{ cm}^3$ at $x = \frac{20}{3} \text{ cm}$.

Fully-worked solutions

Q1. $V(x) = x(10 - 2x)^2$, domain $0 < x < 5$. $V(1) = 1(8)^2 = 64$; $V(2) = 2(6)^2 = 72 \text{ cm}^3$.

Q2. $V(x) = x(12 - 2x)(8 - 2x)$, domain $0 < x < 4$. $V(1) = 1(10)(6) = 60 \text{ cm}^3$.

Q3. $V(x) = x(18 - 2x)^2 = 4x^3 - 72x^2 + 324x$, domain $0 < x < 9$. $V'(x) = 12x^2 - 144x + 324 = 12(x - 3)(x - 9)$,
so $V'(x) = 0$ at $x = 3$ or $x = 9$. Only $x = 3$ is in the domain: $V(3) = 3(12)^2 = 432$. $\therefore$ maximum volume 432 cm^3 at
 $x = 3 \text{ cm}$.

Q4. $V(x) = x(20 - 2x)(12 - 2x)$, domain $0 < x < 6$. $V(2) = 2(16)(8) = 256 \text{ cm}^3$.

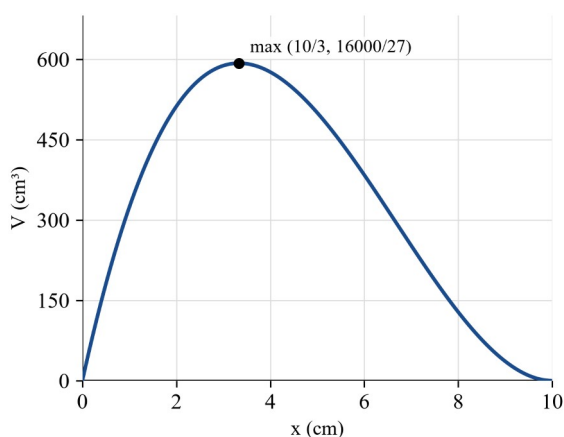
Q5. $x(12 - 2x)^2 = 128 \Rightarrow 4x^3 - 48x^2 + 144x - 128 = 0 \Rightarrow x^3 - 12x^2 + 36x - 32 = 0 \Rightarrow (x - 2)^2(x - 8) = 0$, so $x = 2$
or $x = 8$. Since the domain is $0 < x < 6$, reject $x = 8$; $\therefore x = 2 \text{ cm}$. (The repeated root $x = 2$ is where the volume reaches
its maximum, 128 cm^3 .)

Q6. $V(x) = x(10 - 2x)(6 - 2x)$, domain $0 < x < 3$. $V(1) = 1(8)(4) = 32$; $V(2) = 2(6)(2) = 24 \text{ cm}^3$.

Q7. $V(x) = x(20 - 2x)^2 = 4x^3 - 80x^2 + 400x$, domain $0 < x < 10$. $V(3) = 3(14)^2 = 588$.

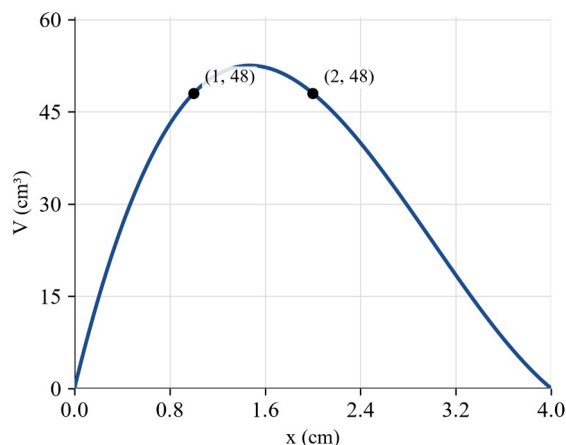
$V'(x) = 12x^2 - 160x + 400 = 4(3x - 10)(x - 10)$, so the turning point in the domain is $x = \frac{10}{3}$:

$$V\left(\frac{10}{3}\right) = \frac{16000}{27} \approx 592.6 \text{ cm}^3.$$



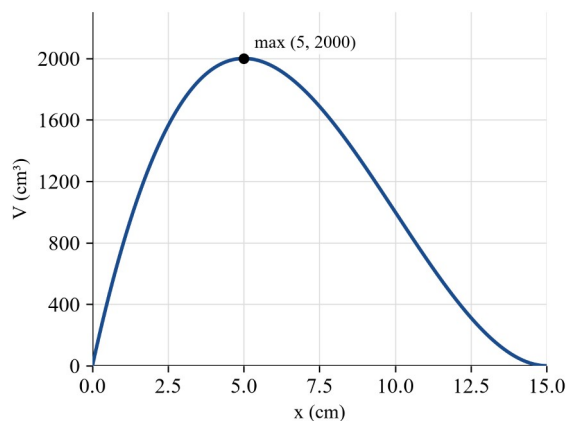
Q8. $V(4) = 4(24 - 8)(18 - 8) = 4(16)(10) = 640 \text{ cm}^3$.

Q9. $x(10-2x)(8-2x)=48 \Rightarrow 4x^3-36x^2+80x-48=0 \Rightarrow x^3-9x^2+20x-12=0 \Rightarrow (x-1)(x-2)(x-6)=0$
 . In $0 < x < 4$: $x=1$ cm or $x=2$ cm.

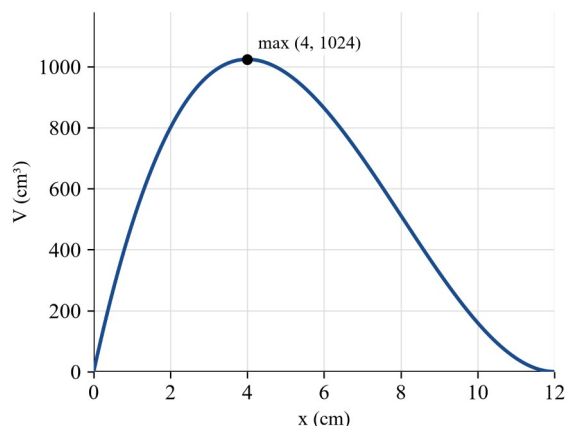


Q10. $V(2)=2(14-4)(10-4)=2(10)(6)=120$ cm³.

Q11. $V(x)=x(30-2x)^2$, domain $0 < x < 15$. $V'(x)=0$ at $x=5$ (in domain) and $x=15$ (rejected).
 $V(5)=5(20)^2=2000$. $\therefore$ maximum volume 2000 cm³ at $x=5$ cm.



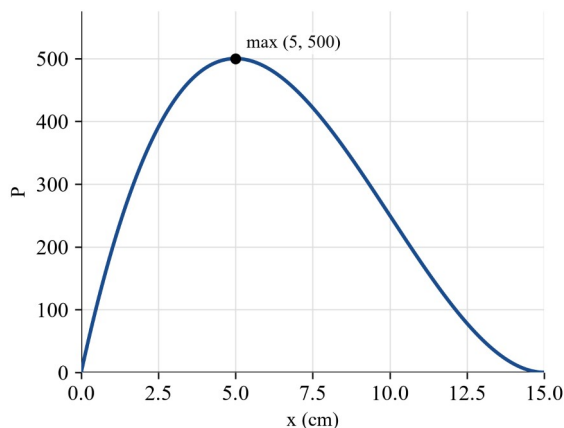
Q12. $V(x)=x(24-2x)^2$, domain $0 < x < 12$. $V(3)=3(18)^2=972$. The turning point in the domain is $x=4$:
 $V(4)=4(16)^2=1024$ cm³ (the maximum).



Q13. $V(x)=x(16-2x)(12-2x)$, domain $0 < x < 6$. $V(2)=2(12)(8)=192$ cm³.

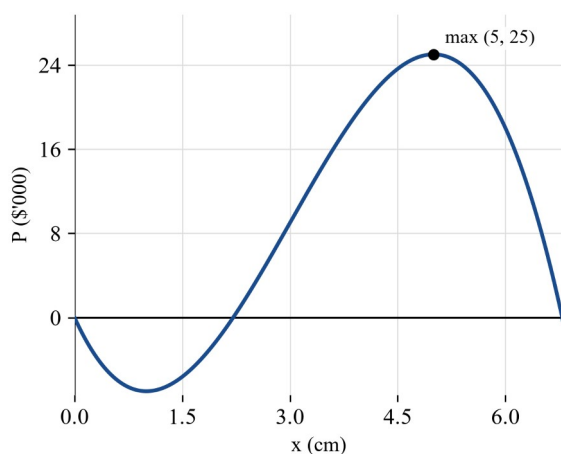
Q14. $V(x)=x(8-2x)^2$, domain $0 < x < 4$. $V(1)=1(6)^2=36$; $V(2)=2(4)^2=32$ cm³.

Q15. $P(5)=5(15-5)^2=5(100)=500$. $P(x)=x(15-x)^2=x^3-30x^2+225x$;
 $P'(x)=3x^2-60x+225=3(x-5)(x-15)$, so the maximum in $0 < x < 15$ is at $x=5$.



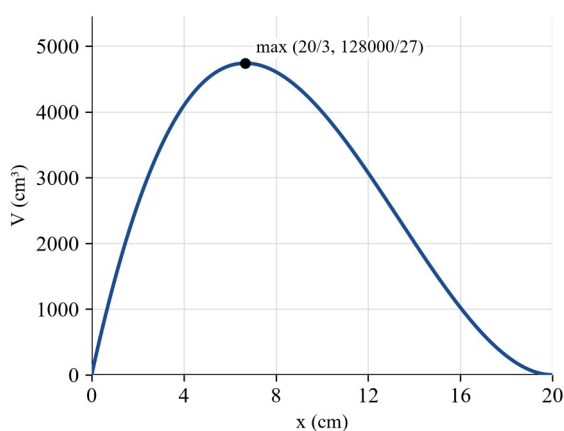
Q16. (a) $V = \frac{4}{3}\pi(3)^3 = \frac{4}{3}\pi(27) = 36\pi \text{ cm}^3$. (b) $\frac{4}{3}\pi r^3 = 288\pi \Rightarrow r^3 = 216 \Rightarrow r = 6 \text{ cm}$.

Q17. $P'(x) = -3x^2 + 18x - 15 = -3(x-1)(x-5)$, so $P'(x) = 0$ at $x=1$ and $x=5$. $P(5) = -125 + 225 - 75 = 25$, a local maximum (and $P(1) = -7$, a local minimum). $\therefore$ the local maximum profit is \$25 000, at $x=5$ (i.e. 5000 items).



Q18. $V(x) = x(40 - 2x)^2$, domain $0 < x < 20$. The turning point in the domain is at $x = \frac{20}{3}$:

$$V\left(\frac{20}{3}\right) = \frac{128000}{27} \approx 4740.7 \text{ cm}^3.$$

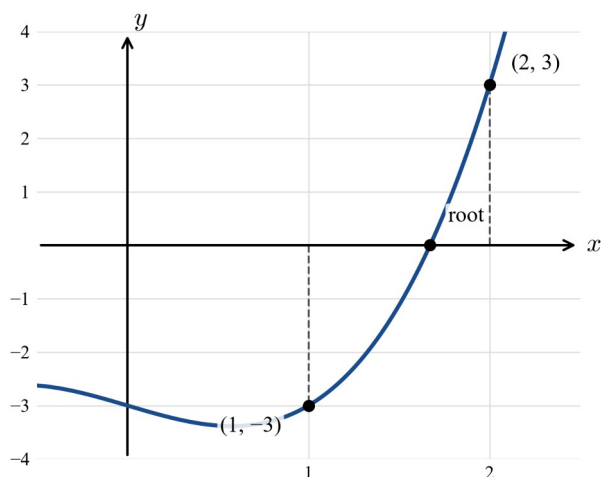


Chapter 6 Polynomial functions — 6K The bisection method algorithm

Theory

Most polynomial equations of degree three or higher cannot be solved exactly by factorising. The **bisection method** is a simple numerical procedure that locates a root to any required accuracy.

The sign-change principle. Every polynomial $P(x)$ is **continuous** — its graph has no breaks. So if $P(a)$ and $P(b)$ have **opposite signs**, the graph must cross the x -axis somewhere between $x=a$ and $x=b$; that is, there is **at least one root** in the interval (a, b) .



For $P(x) = x^3 - x - 3$ above, $P(1) = -3 < 0$ and $P(2) = 3 > 0$, so a root lies between 1 and 2.

The method. Starting with an interval (a, b) on which P changes sign:

1. Find the **midpoint** $m = \frac{a+b}{2}$ and evaluate $P(m)$.
2. The root lies in whichever half shows the sign change:
 - if $P(a)$ and $P(m)$ have opposite signs, the root is in (a, m) — replace b with m ;
 - otherwise the root is in (m, b) — replace a with m .
3. Repeat. Each step **halves** the interval, trapping the root more tightly.

Stopping. Stop when the interval is short enough — for example when its width is below a set tolerance, or when both endpoints round to the same value at the required number of decimal places. The midpoint of the final interval is then the estimate.

Worked examples

Example 1 — scaffolded

Show that $P(x) = x^3 - x - 3$ has a root between $x=1$ and $x=2$, then apply two steps of the bisection method.

Working

$$P(1) = 1 - 1 - 3 = -3 \text{ and } P(2) = 8 - 2 - 3 = 3.$$

The signs are opposite ($-$ then $+$), so a root lies in $(1, 2)$.

$$m_1 = \frac{1+2}{2} = 1.5: P(1.5) = -1.125 (-), \text{ same sign as } P(1),$$

so the root is in $(1.5, 2)$.

Comment

Evaluate at each end.

Sign change $\Rightarrow$ a root between.

Keep the half with the sign change.

Working	Comment
$m_2 = \frac{1.5+2}{2} = 1.75$: $P(1.75) = 0.6094$ (+), so the root is in $(1.5, 1.75)$.	Halve again.

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	-1.125	$(1.5, 2)$
2	1.5	2	1.75	0.6094	$(1.5, 1.75)$

$\therefore$ the root lies in $(1.5, 1.75)$ (the exact root is $x \approx 1.672$).

Example 2 — scaffolded

Apply three steps of the bisection method to $P(x) = x^3 + 2x - 5$ on $[1, 2]$.

Working	Comment
$P(1) = -2$ (-), $P(2) = 7$ (+): a root lies in $(1, 2)$.	Confirm the sign change first.
$m_1 = 1.5$: $P(1.5) = 1.375$ (+), so the root is in $(1, 1.5)$.	Opposite sign to $P(1)$, keep the left half.
$m_2 = 1.25$: $P(1.25) = -0.5469$ (-), so the root is in $(1.25, 1.5)$.	
$m_3 = 1.375$: $P(1.375) = 0.3496$ (+), so the root is in $(1.25, 1.375)$.	

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	1.375	$(1, 1.5)$
2	1	1.5	1.25	-0.5469	$(1.25, 1.5)$
3	1.25	1.5	1.375	0.3496	$(1.25, 1.375)$

$\therefore$ the root lies in $(1.25, 1.375)$ (exact root $x \approx 1.328$).

Example 3 — unscaffolded

Use the bisection method to find the root of $P(x) = x^3 - 3x - 1$ between 1 and 2, correct to one decimal place.

$P(1) = -3$ (-), $P(2) = 1$ (+): a root lies in $(1, 2)$.

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	-2.125	$(1.5, 2)$
2	1.5	2	1.75	-0.8906	$(1.75, 2)$
3	1.75	2	1.875	-0.0332	$(1.875, 2)$
4	1.875	2	1.9375	0.4607	$(1.875, 1.9375)$

Both endpoints 1.875 and 1.9375 round to 1.9. $\therefore$ the root is $x \approx 1.9$ (to 1 decimal place).

Example 4 — unscaffolded

Apply four steps of the bisection method to $P(x) = x^3 + x - 1$ on $[0, 1]$.

$P(0) = -1$ (-), $P(1) = 1$ (+): a root lies in $(0, 1)$.

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	0	1	0.5	-0.375	$(0.5, 1)$
2	0.5	1	0.75	0.1719	$(0.5, 0.75)$
3	0.5	0.75	0.625	-0.1309	$(0.625, 0.75)$
4	0.625	0.75	0.6875	0.0125	$(0.625, 0.6875)$

$\therefore$ the root lies in $(0.625, 0.6875)$ (exact root $x \approx 0.682$).

Practice questions

Scaffolded

Q1. For $P(x) = x^3 - x - 1$: (a) show that a root lies between 1 and 2; (b) apply two steps of the bisection method and state the interval containing the root.

Q2. For $P(x) = x^3 + x - 3$ on $[1, 2]$: (a) confirm the sign change; (b) complete three steps of bisection.

Q3. For $P(x) = x^3 - 4x + 1$: (a) evaluate $P(0)$ and $P(1)$ to show a root lies in $(0, 1)$; (b) apply two steps of bisection.

Q4. For $P(x) = 2x^3 - 4x - 1$ on $[1, 2]$, copy and complete this table for three steps.

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2			
2					
3					

Q5. Apply four steps of the bisection method to $P(x) = x^3 + 3x - 5$ on $[1, 2]$, and state the interval containing the root.

Q6. Explain, using the word **continuous**, why a change in the sign of $P(x)$ between $x = a$ and $x = b$ guarantees that the equation $P(x) = 0$ has a solution between a and b .

Un scaffolded

Q7. Apply three steps of the bisection method to $x^3 - 2x - 5 = 0$ on $[2, 3]$.

Q8. Find the root of $x^3 - x - 2 = 0$ between 1 and 2, correct to one decimal place.

Q9. Apply three steps of the bisection method to $x^3 + 2x^2 - 1 = 0$ on $[0, 1]$.

Q10. Apply two steps of the bisection method to $x^3 - 7x + 2 = 0$ on $[0, 1]$.

Q11. Apply three steps of the bisection method to $x^4 - x - 2 = 0$ on $[1, 2]$.

Q12. The equation $x^3 - 5 = 0$ has the root $x = \sqrt[3]{5}$. Use bisection on $[1, 2]$ to find it correct to one decimal place.

Q13. Apply three steps of the bisection method to $x^3 + x^2 - 3 = 0$ on $[1, 2]$.

Q14. Find the root of $2x^3 + x - 4 = 0$ between 1 and 2, correct to one decimal place.

Q15. The equation $x^3 - 6 = 0$ has the root $x = \sqrt[3]{6}$. Apply three steps of bisection on $[1, 2]$.

Q16. Apply three steps of the bisection method to $x^3 - 2x^2 - 1 = 0$ on $[2, 3]$.

Q17. A student tries to use the bisection method on $P(x) = x^2 - 4$ over the interval $[-3, 3]$. Explain why the sign-change test does **not** apply here, even though $P(x) = 0$ has solutions in this interval. Suggest an interval on which the method would work.

Q18. A root of $x^3 + x - 1 = 0$ lies in $(0, 1)$. How many steps of the bisection method are needed to locate it within an interval of width less than 0.05?

Answers

Q1. $P(1) = -1$, $P(2) = 5$ (sign change); root in $(1.25, 1.5)$. **Q2.** Root in $(1.125, 1.25)$. **Q3.** Root in $(0.25, 0.5)$. **Q4.** Root in $(1.5, 1.625)$. **Q5.** Root in $(1.125, 1.1875)$. **Q6.** A sign change forces a continuous graph to cross the x -axis between a and b . **Q7.** Root in $(2, 2.125)$. **Q8.** $x \approx 1.5$. **Q9.** Root in $(0.5, 0.625)$. **Q10.** Root in $(0.25, 0.5)$. **Q11.** Root in $(1.25, 1.375)$. **Q12.** $x \approx 1.7$. **Q13.** Root in $(1.125, 1.25)$. **Q14.** $x \approx 1.1$. **Q15.** Root in $(1.75, 1.875)$. **Q16.** Root in $(2.125, 2.25)$. **Q17.** $P(-3) = 5$ and $P(3) = 5$ have the **same** sign, so the test gives no information; e.g. on $[0, 3]$, $P(0) = -4$ and $P(3) = 5$ change sign and bracket the root $x = 2$. **Q18.** 5 steps.

Fully-worked solutions

Q1. $P(1) = 1 - 1 - 1 = -1$ (-), $P(2) = 8 - 2 - 1 = 5$ (+): a root lies in $(1, 2)$.

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	0.875	$(1, 1.5)$
2	1	1.5	1.25	-0.2969	$(1.25, 1.5)$

$\therefore$ the root lies in $(1.25, 1.5)$.

Q2. $P(1) = -1$ (-), $P(2) = 7$ (+): a root lies in $(1, 2)$.

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	1.875	$(1, 1.5)$
2	1	1.5	1.25	0.2031	$(1, 1.25)$
3	1	1.25	1.125	-0.4512	$(1.125, 1.25)$

$\therefore$ the root lies in $(1.125, 1.25)$.

Q3. $P(0) = 1$ (+), $P(1) = -2$ (-): a root lies in $(0, 1)$.

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	0	1	0.5	-0.875	$(0, 0.5)$
2	0	0.5	0.25	0.0156	$(0.25, 0.5)$

$\therefore$ the root lies in $(0.25, 0.5)$.

Q4. $P(1) = -3$ (-), $P(2) = 7$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	-0.25	$(1.5, 2)$
2	1.5	2	1.75	2.7188	$(1.5, 1.75)$
3	1.5	1.75	1.625	1.0820	$(1.5, 1.625)$

$\therefore$ the root lies in $(1.5, 1.625)$.

Q5. $P(1) = -1$ (-), $P(2) = 9$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	2.875	(1, 1.5)
2	1	1.5	1.25	0.7031	(1, 1.25)
3	1	1.25	1.125	-0.2012	(1.125, 1.25)
4	1.125	1.25	1.1875	0.2371	(1.125, 1.1875)

$\therefore$ the root lies in (1.125, 1.1875).

Q6. A polynomial is **continuous**, so its graph is an unbroken curve. If $P(a)$ and $P(b)$ have opposite signs, one of the points $(a, P(a))$, $(b, P(b))$ is below the x -axis and the other above it. To join them without lifting the pen, the curve must cross the x -axis at some x between a and b — and at that point $P(x)=0$.

Q7. $P(2)=-1$ (-), $P(3)=16$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	2	3	2.5	5.625	(2, 2.5)
2	2	2.5	2.25	1.8906	(2, 2.25)
3	2	2.25	2.125	0.3457	(2, 2.125)

$\therefore$ the root lies in (2, 2.125).

Q8. $P(1)=-2$ (-), $P(2)=4$ (+). Bisection:

$(1, 2) \rightarrow (1.5, 2) \rightarrow (1.5, 1.75) \rightarrow (1.5, 1.625) \rightarrow (1.5, 1.5625) \rightarrow (1.5, 1.53125)$. The endpoints agree to one decimal place at 1.5. $\therefore x \approx 1.5$.

Q9. $P(0)=-1$ (-), $P(1)=2$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	0	1	0.5	-0.375	(0.5, 1)
2	0.5	1	0.75	0.5469	(0.5, 0.75)
3	0.5	0.75	0.625	0.0254	(0.5, 0.625)

$\therefore$ the root lies in (0.5, 0.625).

Q10. $P(0)=2$ (+), $P(1)=-4$ (-).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	0	1	0.5	-1.375	(0, 0.5)
2	0	0.5	0.25	0.2656	(0.25, 0.5)

$\therefore$ the root lies in (0.25, 0.5).

Q11. $P(1)=-2$ (-), $P(2)=12$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	1.5625	(1, 1.5)
2	1	1.5	1.25	-0.8086	(1.25, 1.5)

n	a_n	b_n	m_n	$P(m_n)$	New interval
3	1.25	1.5	1.375	0.1995	(1.25, 1.375)

$\therefore$ the root lies in (1.25, 1.375).

Q12. $P(1) = -4$ (-), $P(2) = 3$ (+). Bisectioning $[1, 2]$ five times gives the interval (1.6875, 1.7188); both endpoints round to 1.7. $\therefore \sqrt[3]{5} \approx 1.7$.

Q13. $P(1) = -1$ (-), $P(2) = 9$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	2.625	(1, 1.5)
2	1	1.5	1.25	0.5156	(1, 1.25)
3	1	1.25	1.125	-0.3105	(1.125, 1.25)

$\therefore$ the root lies in (1.125, 1.25).

Q14. $P(1) = -1$ (-), $P(2) = 14$ (+). Bisectioning $[1, 2]$ six times gives the interval (1.125, 1.1406); both endpoints round to 1.1. $\therefore x \approx 1.1$.

Q15. $P(1) = -5$ (-), $P(2) = 2$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	1	2	1.5	-2.625	(1.5, 2)
2	1.5	2	1.75	-0.6406	(1.75, 2)
3	1.75	2	1.875	0.5918	(1.75, 1.875)

$\therefore$ the root lies in (1.75, 1.875) (and $\sqrt[3]{6} \approx 1.817$).

Q16. $P(2) = -1$ (-), $P(3) = 8$ (+).

n	a_n	b_n	m_n	$P(m_n)$	New interval
1	2	3	2.5	2.125	(2, 2.5)
2	2	2.5	2.25	0.2656	(2, 2.25)
3	2	2.25	2.125	-0.4355	(2.125, 2.25)

$\therefore$ the root lies in (2.125, 2.25).

Q17. $P(-3) = 9 - 4 = 5$ and $P(3) = 9 - 4 = 5$ have the **same** sign, so the sign-change test gives no information — it cannot detect that the interval actually contains the two roots $x = -2$ and $x = 2$. The method needs an interval whose endpoints give opposite signs. For example, on $[0, 3]$, $P(0) = -4$ (-) and $P(3) = 5$ (+), which brackets the root $x = 2$.

Q18. The starting interval (0, 1) has width 1, and each step halves it: 0.5, 0.25, 0.125, 0.0625, 0.03125. After 5 steps the width is $\frac{1}{2^5} = 0.03125 < 0.05$. $\therefore$ 5 steps are needed.